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LARGE DEFLECTION OF A SEMI-CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER A UNIFORM LOAD

By

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Large deflection of a semi-circular plate on elastic foundation under a uniform load

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ABSTRACT

Large deflection of a simply supported semi-circular plate placed on elastic foundation and subjected to a uniform load has been investigated following Berger's approximate method. Expressions for the deflections and bending moments are obtained and the theoretical results have been presented in the form of graphs.

1. INTRODUCTION

SMALL deflections of thin plates on elastic foundation have been examined by Timoshenko and Woinowsky-Krieger[1] and several other authors on the assumption that the strains of the middle plane of the plate can be neglected. But when the deflection is moderately large, that is, of the order of the thickness of the plate, then the strain of the middle plane should be considered. Thus for moderately large deflections, solutions of differential equations for deflections and displacements become difficult because of their non-linear character. Way[2] and many other authors have examined moderately large deflections of plates not resting on elastic foundations and the methods used by them involve considerable computation. Berger[3] suggested that the strain energy due to the second strain invariant of the middle surface strains may be neglected in analysing moderately large deflection of plates having axisymmetric deformation. Berger's technique of neglecting the second strain invariant in the expression corresponding to the total potential energy of the system reduces computation and although no complete explanation of this method is offered, the stresses and deflections obtained for both rectangular and circular plates are in good agreement with those found in practical analysis. Berger's method has been applied successfully by Nowinski[4] to his different plate problems. Nash and Modeer[5] investigated the problems having no axial symmetry using Berger's technique.

Berger's technique of neglecting the second strain invariant in the middle plane has been applied by Sinha[6] to determine large deflection of circular and rectangular plates placed on elastic foundation and under uniform lateral load.

In this paper large static deflection of a simply supported semi-circular isotropic plate placed on elastic foundation and under a uniform load has been solved. Foundation is assumed to be such that its reaction is proportional to the deflection of the plate.

2. Formulation of Problem

For moderately large deflections, the strain displacement relationships and the strain energy of the middle plane of the plate are:

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{1}$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{3}$$

$$V_1 = \frac{D}{2}\iint\left[(\nabla^2 w)^2 + \frac{12}{h^2}e_1{}^2 - 2(1-\nu)\left\{\frac{12}{h^2}e_2 + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}\right] dxdy \tag{4}$$

Neglecting e_2 and by adding the potential energy of the transverse load and of the foundation reaction the modified energy equation becomes

$$V = \frac{D}{2}\iint\left[(\nabla^2 w)^2 + \frac{12}{h^2}e_1{}^2 - 2(1-\nu)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\} + \frac{K}{D}w^2 - \frac{2qw}{D}\right] dxdy \tag{5}$$

Applying Euler's variational method to (5) the following differential equations in polar co-ordinates are obtained:

$$\nabla^4 w - \alpha_1^2 \nabla_1^2 w + \frac{K}{D}w = \frac{q}{D} \tag{6}$$

where α is a constant given by

$$\frac{\alpha^2 h^2}{12} = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \tag{7}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

3. Solution of Problem

Let the plate be in the form of a semicircle (figure 1) and simply supported. Let the centre be the origin, the bounding diameter be the initial line and the plate be uniformly loaded. To solve equation (6) let us put it in following form:

$$(\nabla^2 - P_1^2)(\nabla^2 - P_2^2)\, w = \frac{q}{D} \tag{8}$$

where

$$P_1^2 + P_2^2 = \alpha^2 \tag{9}$$

$$P_1^2 P_2^2 = \frac{K}{D} \tag{10}$$

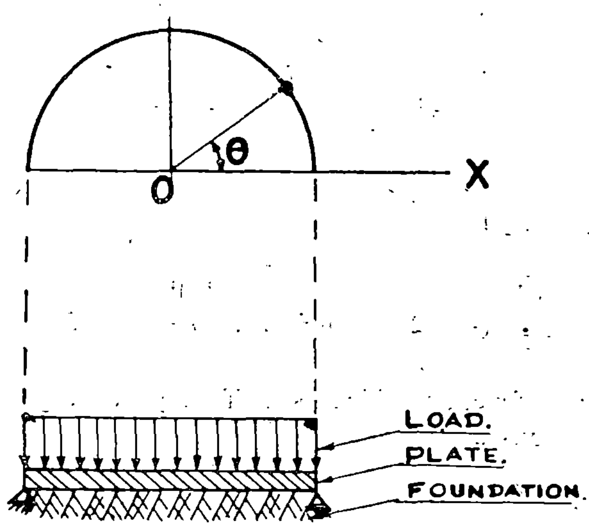

Figure 1. Semi-circular plate on foundation.

Expanding the load into the appropriate Fourier series,

$$q = \frac{4q}{\pi} \sum_{m=1,3,5}^{\infty} \frac{\sin m\theta}{m} \tag{11}$$

and assuming

$$w = \sum_{m=1,3,5}^{\infty} R_m \sin m\theta \tag{12}$$

where R_m is a function of r only, and substituting (11) and (12) in (8) one gets

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - P_1^{\,2}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - P_2^{\,2}\right) R_m = \frac{4q}{\pi D m} \tag{13}$$

The appropriate solution of (8) is given by

$$w = \sum_{m=1,3,5}^{\infty} \Big| A_m I_m(P_1 r) + B_m I_m(P_2 r)$$

$$+ \frac{4q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} S_{3+2s,\,m}(iP_2 r)\Big] \sin m\theta \tag{14}$$

where

$$\lambda_s = \frac{(-1)^s (iP_1)^{2s}}{(2^2 - m^2)(4^2 - m^2)\ldots\{(2+2s)^2 - m^2\}}$$

and

$$S_{3+2s,\,m}(iP_2 r)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^n (iP_2 r)^{4+2s+2n}}{\{(4+2s)^2 - m^2\}\ \ldots\{(4+2s+2n)^2 - m^2\}}$$

is the Lommel's function which is uniformly convergent. The required boundary conditions are

$$(w)_{r=a} = 0 \tag{15}$$

$$\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right]_{r=a} = 0. \tag{16}$$

Considering equations (14), (15) and (16) and solving for the constants A_m and B_m one gets

$$A_m = \frac{4q}{\pi Dm} \cdot \frac{\left[\{aP_2^2 I''_m(P_2a) + \nu P_2 I'_m(P_2a)\} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} S_{3+2s,\,m}(iP_2a) - I_m(P_2a)\left\{a \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} S''_{3+2s,\,m}(iP_2a) + \nu \sum \frac{\lambda_s}{(iP_2)^{4+2s}} S'_{3+2s,\,m}(iP_2a)\right\}\right]}{[I_m(P_2a)\{aP_1^2 I''_m(P_1a) + \nu P_1 I'_m(P_1a)\} - I_m(P_1a)\{aP_2^2 I''_m(P_2a) + \nu P_2 I'_m(P_2a)\}]} \tag{17}$$

$$B_m = -\frac{4q}{\pi Dm} \cdot \frac{\left[\{aP_1^2 I''_m(P_1a) + \nu P_1 I'_m(P_1a)\} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} S_{3+2s,\,m}(iP_2a) - I_m(P_1a)\left\{a \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} S''_{3+2s,\,m}(iP_2a) + \nu \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} S'_{3+2s,\,m}(iP_2a)\right\}\right]}{[I_m(P_2a)\{aP_1^2 I''_m(P_1a) + \nu P_1 I'_m(P_1a)\} - I_m(P_1a)\{aP_2^2 I''_m(P_2a) + \nu P_2 I'_m(P_2a)\}]} \tag{18}$$

To determine α, let

$$u = \Sigma\; U(r) \cos m\theta \tag{19}$$

$$v = \Sigma\; V(r) \sin m\theta \tag{20}$$

subject to the boundary conditions $U(a) = V(a) = 0$.

Multiplying (7) by $rd\theta dr$ and integrating within the limits 0 to a and 0 to π one gets

$$\int_0^a \int_0^\pi r \sum U'(r) \cos m\theta \, d\theta dr + \frac{1}{2} \int_0^a \int_0^\pi r \left(\frac{\partial w}{dr}\right)^2 d\theta dr + \int_0^a \int_0^\pi \sum U(r) \cos m\theta \, d\theta dr + \int_0^a \int_0^\pi \sum mV(r) \cos m\theta \, d\theta dr + \frac{1}{2} \int_0^a \int_0^\pi \frac{1}{r} \left(\frac{\partial w}{\partial \theta}\right)^2 d\theta dr = \frac{\alpha^2 h^2}{12} \int_0^a \int_0^\pi r d\theta dr.$$

After evaluating the integrals the following equation leading to a is obtained:

$$\sum_{m=1,3,5,\ldots}^{\infty} \Big[A_m^2 P_1^2 \Big(-\frac{a^2}{8}\Big\{\frac{1}{4}[I_{m-2}(P_1a) + I_m(P_1a)]^2$$

$$-\Big[1 + \frac{(m-1)^2}{P_1^2 a^2}\Big] I_{m-1}^2(P_1a)\Big\} - \frac{a^2}{8}\Big\{\frac{1}{4}[I_m(P_1a) + I_{m+2}(P_1a)]^2$$

$$-\Big[1 + \frac{(m+1)^2}{P_1^2 a^2}\Big] I_{m+1}^2(P_1a)\Big\} + \frac{1}{2}\sum_{\substack{n=0\\ n\neq t}}^{\infty}\sum_{t=0}^{\infty}\phi\Big(\frac{P_1}{2}\Big)^{2m+2n+2t}\Big)$$

$$+ B_m^2 P_2^2 \Big(\frac{1}{2}\sum_{\substack{n=0\\ n\neq t}}^{\infty}\sum_{t=0}^{\infty}\phi\Big(\frac{P_2}{2}\Big)^{2m+2n+2t} - \frac{a^2}{8}\Big\{\frac{1}{4}[I_{m-2}(P_2a)$$

$$+ I_m(P_2a)]^2 + \Big[1 + \frac{(m-1)^2}{P_2^2 a^2}\Big] I_{m-1}^2(P_2a)\Big\}$$

$$-\frac{a^2}{2}\Big\{[I_m(P_2a) + I_{m+2}(P_2a)]^2 + \Big[1 + \frac{(m+1)^2}{P_2^2 a^2}\Big] I_{m+1}^2(P_2a)\Big\}\Big)$$

$$+\frac{16q^2}{\pi^2 D^2 m^2}\Big\{\sum_{s=0}^{\infty}\frac{\lambda_s^2}{(iP_2)^{8+4s}}\Big[\sum_{n=0}^{\infty}\frac{\mu_n^2(4+2s+2n)^2 a^{8+4s+4n}}{8+4s+4n}$$

$$+\sum_{\substack{n=0\\ n\neq t}}^{\infty}\sum_{t=0}^{\infty}\frac{\mu_n\mu_t(4+2s+2n)(4+2s+2t)a^{8+4s+2n+2t}}{8+4s+2n+2t}\Big]\Big\}$$

$$+\frac{1}{2}A_mB_mP_1P_2\Big\{\sum_{\substack{n=0\\ n\neq t}}^{\infty}\sum_{t=0}^{\infty}\Big[\frac{\big(\frac{P_1}{2}\big)^{m-1+2n}\big(\frac{P_2}{2}\big)^{m-1+2t}a^{2m+2n+2t}}{(2m+2n+2t)\,\lfloor n\,\lfloor t\,\Gamma(m+n)\Gamma(m+t)}$$

$$+\frac{\big(\frac{P_1}{2}\big)^{m+1+2n}\big(\frac{P_2}{2}\big)^{m+1+2t}a^{2m+2n+2t+4}}{(2m+2n+2t+4)\,\lfloor n\,\lfloor t\,\Gamma(m+n+2)\Gamma(m+t+2)}$$

$$+\phi\Big(\frac{P_1}{2}\Big)^{m-1+2n}\Big(\frac{P_2}{2}\Big)^{m+1+2t} + \phi\Big(\frac{P_1}{2}\Big)^{m+1+2n}\Big(\frac{P_2}{2}\Big)^{m-1+2t}\Big]\Big\}$$

$$+4A_mP_1\frac{q}{\pi Dm}\sum_{s=0}^{\infty}\frac{\lambda_s}{(iP_2)^{4+2s}}\Big\{\sum_{\substack{n=0\\ n\neq t}}^{\infty}\sum_{t=0}^{\infty}\Big[\mu_n\psi\Big(\frac{P_1}{2}\Big)^{m-1+2t}$$

$$+\mu_n\psi_1\Big(\frac{P_1}{2}\Big)^{m+1+2t}\Big]\Big\}$$

$$+ 4B_m P_2 \frac{q}{\pi Dm} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \Bigg\{ \sum_{\substack{n=0 \\ n\neq t}}^{\infty} \sum_{t=0}^{\infty} \Bigg[\mu_n \psi \left(\frac{P_2}{2}\right)^{m-1+2t}$$

$$+ \mu_n \psi_1 \left(\frac{P_2}{2}\right)^{m+1+2t} \Bigg]\Bigg\} + m^2 \Bigg(A_m^2 \sum_{n=0}^{\infty} \psi_2 \left(\frac{P_1}{2}\right)^{2m+4n}$$

$$+ A_m^2 \sum_{\substack{n=0 \\ n\neq t}}^{\infty} \sum_{t=0}^{\infty} \psi_3 \left(\frac{P_1}{2}\right)^{2m+2n+2t} + B_m^2 \sum_{n=0}^{\infty} \psi_2 \left(\frac{P_2}{2}\right)^{2m+4n}$$

$$+ B_m^2 \sum_{\substack{n=0 \\ n\neq t}}^{\infty} \sum_{t=0}^{\infty} \psi_3 \left(\frac{P_2}{2}\right)^{2m+2n+2t} + \frac{16q^2}{\pi^2 D^2 m^2} \sum_{s=0}^{\infty} \frac{\lambda_s{}^2}{(iP_2)^{8+4s}}$$

$$\times \left\{ \sum_{n=0}^{\infty} \frac{\mu_n{}^2 a^{8+4s+4n}}{8+4s+4n} + \sum_{\substack{n=0 \\ n\neq t}}^{\infty} \sum_{t=0}^{\infty} \frac{\mu_n \mu_t a^{8+4s+2n+2t}}{8+4s+2n+2t} \right\}$$

$$+ 2A_m B_m \sum_{\substack{n=0 \\ n\neq t}}^{\infty} \sum_{t=0}^{\infty} \frac{\left(\frac{P_1}{2}\right)^{m+2n} \left(\frac{P_2}{2}\right)^{m+2t} a^{2m+2n+2t}}{(2m+2n+2t)\, \underline{|n}\, \underline{|t}\, \Gamma(m+n+1)\, \Gamma(m+t+1)}$$

$$+ 8A_m \frac{q}{\pi Dm} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \sum_{\substack{n=0 \\ n\neq t}}^{\infty} \sum_{t=0}^{\infty} \mu_n \phi_1 \left(\frac{P_1}{2}\right)^{m+2t}$$

$$+ 8B_m \frac{q}{\pi Dm} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \sum_{\substack{n=0 \\ n\neq t}}^{\infty} \sum_{t=0}^{\infty} \mu_n \phi_1 \left(\frac{P_2}{2}\right)^{m+2t} \Bigg)\Bigg] = \frac{\alpha^2 h^2 a^2}{6} \tag{21}$$

where

$$\mu_n = \frac{(-1)^n (iP_2)^{4+2s+2n}}{\{(4+2s)^2 - m^2\} \cdots \{(4+2s+2n)^2 - m^2\}}$$

$$\mu_t = \frac{(-1)^n (iP_2)^{4+2s+2t}}{\{(4+2s)^2 - m^2\} \ldots \{(4+2s+2t)^2 - m^2\}}$$

$$\phi = \frac{a^{2m+2n+2t+2}}{(2m+2n+2t+2)\, \underline{|n}\, \underline{|t}\, \Gamma(m+n)\, \Gamma(m+t+2)}$$

$$\phi_1 = \frac{a^{4+2s+2n+2t+m}}{(4+2s+2n+2t+m)\,\underline{|t}\;\Gamma(m+t+1)}$$

$$\psi = \frac{(4+2s+2n)\,a^{4+2s+2n+2t+m}}{(4+2s+2n+2t+m)\,\underline{|t}\;\Gamma(m+t)}$$

$$\psi_1 = \frac{(4+2s+2n)\,a^{6+2s+2n+2t+m}}{(6+2s+2n+2t+m)\,\underline{|t}\;\Gamma(m+t+2)}$$

$$\psi_2 = \frac{a^{2m+4n}}{(2m+4n)\,\{\underline{|n}\;\Gamma(m+n+1)\}^2}$$

$$\psi_3 = \frac{a^{2m+2n+2t}}{(2m+2n+2t)\,\underline{|n}\;\underline{|t}\;\Gamma(m+n+1)\,\Gamma(m+t+1)}\cdot$$

As

$$P_1 \to 0, \quad P_2 \to 0$$

Equation (14) reduces to

$$w = \frac{qa^4}{D}\sum_{m=1,3,5,\ldots}^{\infty}\left\{\frac{4r^4}{a^4}\,\frac{1}{m\pi(16-m^2)(4-m^2)} + \frac{r^m}{a^m}\,\frac{m+5+\nu}{m\pi(16-m^2)(2+m)[m+\frac{1}{2}(1+\nu)]} - \frac{r^{m+2}}{a^{m+2}}\,\frac{m+3+\nu}{m\pi(4+m)(4-m^2)[m+\frac{1}{2}(1+\nu)]}\right\}\sin m\theta$$

as obtained by Timoshenko[1] for the corresponding problem of small deflection without any elastic foundation.

4. NUMERICAL CALCULATION

To obtain deflection for a given value of plate radius 'a' and foundation modulus 'K_F' one has to start from (21) with an assumed value of 'α' in order to obtain the corresponding value of the load function (qa^4/Dh). Once this relationship is obtained the corresponding deflection w/h can be calculated by (14) with the help of (17) and (18). For $a = 80$ mm, $\nu = 0{\cdot}3$, and $K_F = 350$ deflections have been plotted in figure 2 for various values of load function (qa^4/Dh).

5. DISCUSSION

On examination of eq. (14), it is clear that the radius of symmetry of the plate undergoes the maximum deflection with respect to other radii.

The expression for the deflection at a given point on the radius of symmetry can be expressed in the form $w = \beta\,(qa^4/D)$, where β is a numerical factor. Deflections at various points on the radius of symmetry are plotted in figure 3 for a given value of load function. From figure 3 it is observed that maximum deflection occurs at the centre of gravity of the plate.

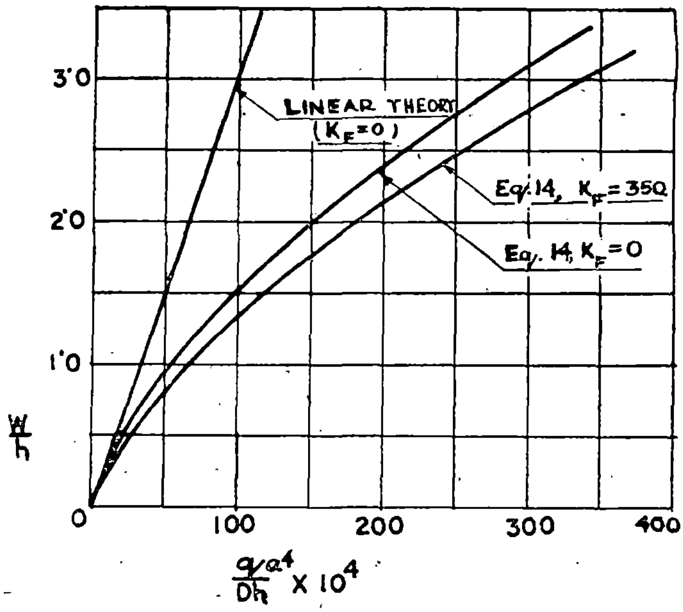

Figure 2. Deflection curve.

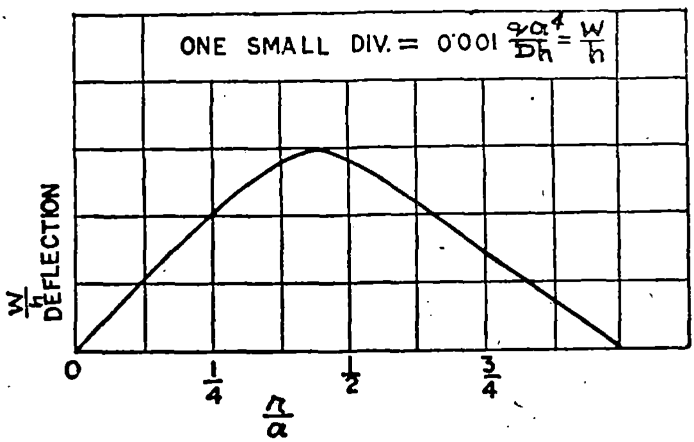

Figure 3. Deflection curve.

The plate is subjected to bending moments in radial and tangential directions as well as to a twisting moment. The expressions for bending and twisting moments are

$$M_r = \sum_{m=1,3,5,\ldots}^{\infty} -D\Big[\frac{A_m P_1^2}{4}\{I_{m+2}(P_1 r) + 2I_m(P_1 r) + I_{m-2}(P_1 r)\}$$

$$+ \frac{B_m P_2^2}{4}\{I_{m+2}(P_2 r) + 2I_m(P_2 r) + I_{m-2}(P_2 r)\}$$

$$+ \nu\Big\{\frac{A_m P_1}{2r}[I_{m-1}(P_1 r) + I_{m+1}(P_1 r)] + \frac{B_m P_2}{2r}[I_{m-1}(P_2 r)$$

$$+ I_{m+1}(P_2 r)] - \frac{m^2}{r^2}[A_m I_m(P_1 r) + B_m I_m(P_2 r)]\Big\}$$

$$+ \frac{4q}{\pi D m}\sum_{s=0}^{\infty}\frac{\lambda_s}{(iP_2)^{4+2s}}\Big\{S''_{3+2s,\,m}(iP_2 r) + \frac{\nu}{r}S'_{3+2s,\,m}(iP_2 r)$$

$$- \frac{m^2\,\nu}{r^2}S_{3+2s,\,m}(iP_2 r)\Big\}\Big]\sin m\theta \tag{22}$$

$$M_\theta = \sum_{m=1,3,5,\ldots}^{\infty} -D\Big[\frac{A_m P_1^2\,\nu}{4}\{I_{m+2}(P_1 r) + 2I_m(P_1 r) + I_{m-2}(P_1 r)\}$$

$$+ \frac{B_m P_2^2\,\nu}{4}\{I_{m+2}(P_2 r) + 2I_m(P_2 r) + I_{m-2}(P_2 r)\}$$

$$+ \frac{A_m P_1}{2r}\{I_{m-1}(P_1 r) + I_{m+1}(P_1 r)\}$$

$$+ \frac{B_m P_2}{2r}\{I_{m-1}(P_2 r) + I_{m+1}(P_2 r)\}$$

$$- \frac{m^2}{r^2}\{A_m I_m(P_1 r) + B_m I_m(P_2 r)\} + \frac{4q}{\pi D m}\sum_{s=0}^{\infty}\frac{\lambda_s}{(iP_2)^{4+2s}}$$

$$\times\Big\{\nu S''_{3+2s,\,m}(iP_2 r) + \frac{1}{r}S'_{3+2s,\,m}(iP_2 r)$$

$$- \frac{m^2}{r^2}S_{3+2s,\,m}(iP_2 r)\Big\}\Big]\sin m\theta \tag{23}$$

$$M_{r\theta} = (1 - \nu) D \sum_{m=1,3,5,\ldots}^{\infty} m \left[\frac{A_m P_1}{2r} \{I_{m-1}(P_1 r) + I_{m+1}(P_1 r)\} \right.$$

$$+ \frac{B_m P_2}{2r} \{I_{m-1}(P_2 r) + I_{m+1}(P_2 r)\}$$

$$- \frac{1}{r^2} \{A_m I_m (P_1 r) + B_m I_m (P_2 r)\} + \frac{4q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}}$$

$$\left. \times \left\{ \frac{1}{r} S'_{3+2s,m}(iP_2 R) - \frac{1}{r^2} S_{3+2s,m}(iP_2 r) \right\} \right] \cos m\theta. \qquad (24)$$

Eqs (22), (23) and (24) show that the bending moments M_r and M_θ are both maximum on the radius of symmetry and the twisting moment $M_{r\theta}$ is maximum on the bounding diameter.

The bending moments can be expressed in the form

$$M_r = \beta_1 q a^2; \qquad M_\theta = \beta_2 q a^2 \qquad (25)$$

Bending stresses can be calculated from the expressions

$$\sigma_r = \frac{6M_r}{h^2} \qquad \sigma_\theta = \frac{6M_\theta}{h^2} \qquad (26)$$

Expressions for shearing forces can be obtained with the help of (14)

$$Q_r = -D \sum_{m=1,3,5,\ldots}^{\infty} \left\{ \left[A_m P_1^3 I_m'''(P_1 r) + B_m P_2^3 I_m'''(P_2 r) \right. \right.$$

$$\left. + \frac{4q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} S'''_{3+2s,m}(iP_2 r) \right]$$

$$+ \frac{1}{r} \left[A_m P_1^2 I_m''(P_1 r) + B_m P_2^2 I_m''(P_2 r) \right.$$

$$\left. + \frac{4q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda s}{(iP_2)^{4+2s}} S''_{3+2s,m}(iP_2 r) \right]$$

$$- \left(a^2 + \frac{m^2}{r^2} \right) \left[A_m P_1 I'_m (P_1 r) + B_m P_2 I'_m (P_2 r) \right.$$

$$\left. \left. + \frac{4q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda s}{(iP_2)^{4+2s}} \times S'_{3+2s,m}(iP_2 r) \right] \right\} \sin m\theta \qquad (27)$$

$$Q_\theta = -\frac{D}{r}\sum_{m=1,3,5,\ldots}^{\infty} m\Big[A_m P_1^2 I''_m(P_1 r) + B_m P_2^2 I''_m(P_2 r)$$

$$+\frac{4q}{\pi D m}\sum_{s=0}^{\infty}\frac{\lambda s}{(iP_2)^{4+2s}} S''_{3+2s,\,m}(iP_2 r)$$

$$+\frac{1}{r}\Big(A_m P_1 I'_m(P_1 r) + B_m P_2 I'_m(P_2 r)$$

$$+\frac{4q}{\pi D m}\sum_{s=0}^{\infty}\frac{\lambda s}{(iP_2)^{4+2s}} S'_{3+2s,\,m}(iP_2 r)\Big)$$

$$-\left(\alpha^2+\frac{m^2}{r^2}\right)\Big(A_m I_m(P_1 r) + B_m I_m(P_2 r)$$

$$+\frac{4q}{\pi D m}\sum_{s=0}^{\infty}\frac{\lambda_s}{(iP_2)^{4+2s}} S_{3+2s,\,m}(iP_2 r)\Big)\Big]\cos m\theta \qquad (28)$$

The shearing stresses can be calculated from the expressions

$$\tau_{r\theta} = \frac{6M_{r\theta}}{h^2};\ \tau_{rz} = \frac{3}{2}\frac{Q_r}{h};\ \tau_{\theta z} = \frac{3}{2}\frac{Q_\theta}{h}. \qquad (29)$$

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LARGE DEFLECTION OF A CIRCULAR PLATE ON ELASTIC FOUNDATION AND SUPPORTED AT SEVERAL POINTS

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ABSTRACT

Large deflections of a uniformly loaded circular plate placed on elastic foundation and supported at several points along the boundary have been analysed following Berger's method. A particular case, where the number of supports is two, has been treated fully. Numerical results have been presented in the form of graphs.

Key words : Large deflections, elastic foundation

INTRODUCTION

Small deflections of thin plates placed on elastic foundations have been examined by S. Timoshenko and S. Woinowsky Krieger [1] and several other authors on the assumption that strain due to stretching of the middle surface of the plate is negligible. When the deflections are moderately large, that is, on the order of thickness of the plate, then the forces in the middle surface of the plate must be taken into account. In the case of such large deflections of plates placed on elastic foundations, three differential equations for displacement and deflection may be written, but it is usually difficult to obtain the solutions of these equations because of their nonlinear character.

On the other hand, various problems of large deflections of plates not resting on elastic foundations have been examined by S. Way [2], S. Levy [3] and many other authors. But the methods used by them involve and require considerable computation. A simple and approximate, yet fairly accurate, method of analysing large deflections of plates was suggested by H. M. Berger [4]. The method uses the technique of neglecting the strain energy due to the second strain invariant of the middle surface strains in analysing large deflection of plates having axisymmetric deformation. Berger's method reduces computation and although no complete explanation of this method is offered in, the stresses and deflections obtained for both rectangular and circular plates are in good agreement with those found in practical analysis. Berger's method has been applied successfully by Nowinsk [5] to his plate problems and Nash and Modeer [6] investigated the problems having no axial symmetry.

The technique of neglecting the second strain invariant in the expression corresponding to the total potential energy of the system has been successfully applied by Sinha [7] to determine large deflection of circular and rectangular plates placed on elastic foundations and under uniform latera loads.

In this paper large deflection of a circular plate placed on elastic foundation and supported at several points along the boundary has been solved. The load is assumed to be uniformly distributed and the foundation is of the Winkler type. A complete analysis of a particular case, where the number of supports is two is given.

FORMULATION OF PROBLEM

For moderately large deflections, the strain displacement relationships and the strain energy of the middle plane of the plate are

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{1}$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{3}$$

$$V_1 = \frac{D}{2}\int\int\left[(\nabla^2 w)^2 + \frac{12}{h^2}e_1{}^2 - 2(1-v)\right.$$
$$\left.\times\left\{\frac{12}{h^2}e_2 + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}\right]dxdy \tag{4}$$

in which e_1 and e_2 are the first and second middle surface strain invariants, respectively.

Neglecting e_2 and by adding the potential energy of the transverse load and of the foundation reaction, K, the modified energy equation becomes

$$V = \frac{D}{2}\int\int\Big[(\nabla^2 w)^2 + \frac{12}{h^2}e_1{}^2 - 2(1-v)$$
$$\times\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\} + \frac{K}{D}w^2 - \frac{2qw}{D}\Big]\,dxdy. \qquad (5)$$

Applying Euler's variational method to eq. 5 the following differential equations in polar co-ordinates are obtained [7]

$$\nabla^4 w - \alpha^2 \nabla^2 w + \frac{K}{D} w = \frac{q}{D} \qquad (6)$$

where α is a constant given by

$$\frac{\alpha^2 h^2}{12} = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \qquad (7)$$

Solution of Problem

Let the circular plate (Fig. 1) be of radius a, supported at several points along the boundary and placed on the elastic foundation. Let the centre of the plate be the origin and a diameter as the initial line, $\theta = 0$. The general solution of eq. 6 is

$$w = w_0 + w_1 \qquad (8)$$

in which w_0 is the large deflection of a plate placed on elastic foundation and simply supported along the entire boundary and w_1 satisfies the equation

$$\nabla^4 w_1 - \alpha^2 \nabla^2 w_1 + \frac{K}{D} w_1 = 0 \qquad (9)$$

Eq. (9) can be written in the form

$$(\nabla^2 - P_1{}^2)(\nabla^2 - P_2{}^2)\, w_1 = 0 \qquad (10)$$

where

$$P_1{}^2 + P_2{}^2 = \alpha^2 \qquad (11)$$

$$P_1{}^2 P_2{}^2 = \frac{K}{D}. \qquad (12)$$

Considering the number of points of support is i, and denoting the concentrated reactions at these points $N_1, N_2 \ldots N_i$, the expression for each reaction N_i is (1, P. 293)

$$\frac{N_i}{\pi a}\left[\tfrac{1}{2} + \sum_{m=1}^{\infty} \cos m\theta_i\right] \tag{13}$$

where $\theta_i = \theta - \psi_i$, ψ_i is the angle defining the position of the support i.

The intensity of the reactive forces at any point of the boundary is then given by the expression.

$$\sum_{i=1}^{i} \frac{N_i}{\pi a}\left[\tfrac{1}{2} + \sum_{m=1}^{\infty} \cos m\theta_i\right] \tag{13 a}$$

in which the summation is extended over all the concentrated reactions. Assuming that the plate is solid and considering that deflections and moments

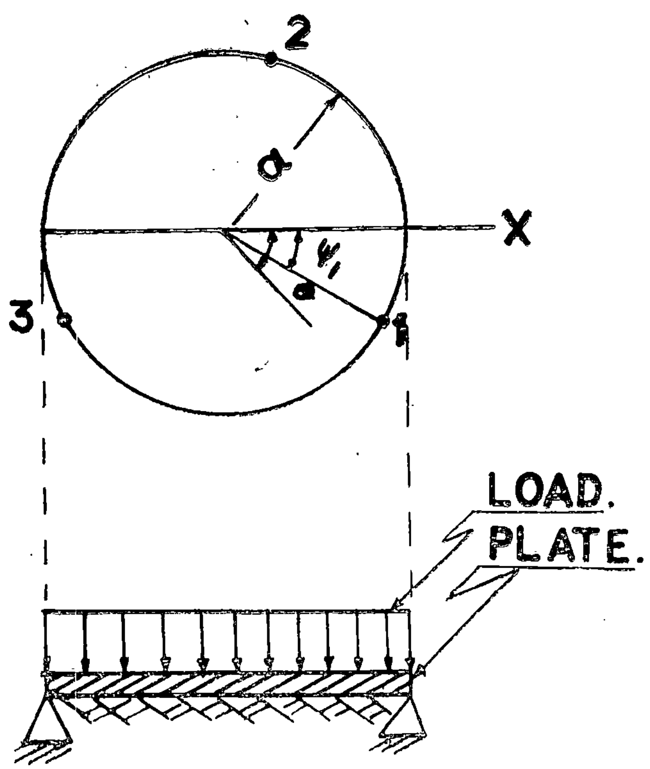

FIG. 1. Circular plate on foundation.

at the centre must be finite, the appropriate solution of eq. 9 can be taken in the form

$$w_1 = A_0 I_0 (P_1 r) + B_0 I_0 (P_2 r) + \sum_{m=1}^{\infty} [A_m I_m (P_1 r) + B_m I_m (P_2 r)]$$

$$\times \cos m\theta + \sum_{m=1}^{\infty} [A'_m I_m (P_1 r) + B'_m I_m (P_2 r)] \sin m\theta \qquad (14)$$

in which I_0 is the modified Bessel function of the first kind and zero order, and I_m is of the first kind and mth order. For determining the constants we have the following conditions at the boundary:

$$w \Big|_{\substack{r = a \\ \theta = 0,\, \pi}} = 0 \qquad (15)$$

$$\left[\frac{\partial^2 w_1}{\partial r^2} + \frac{\nu}{r}\frac{\partial w_1}{\partial r} + \frac{\nu}{r^2}\frac{\partial^2 w_1}{\partial \theta^2}\right]_{r=a} = 0 \qquad (16)$$

$$\left[Q_r - \frac{1}{r}\frac{\partial}{\partial \theta} M_{r\theta}\right]_{r=a} = -\sum_{i=1}^{i} \frac{N_i}{\pi a}\left[\tfrac{1}{2} + \sum_{m=1}^{\infty} \cos m\theta_i\right] \qquad (17)$$

where

$$Q_r = -D\frac{\partial}{\partial r}[(\nabla^2 - a^2) w_1] \qquad (17\,a)$$

$$M_{r\theta} = (1 - v) D\left[\frac{1}{r}\frac{\partial^2 w_1}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w_1}{\partial \theta}\right]. \qquad (17\,b)$$

Consider a particular case when the plate is supported at two points which are the two end points of the diameter taken as the initial line from which θ is measured. Then

$$\psi_1 = 0, \quad \psi_2 = \pi.$$

Considering the above boundary conditions one gets after solving for the constants

$$A_0 = \frac{P}{\pi D a} \beta \psi_0 (a) \qquad (18)$$

$$B_0 = -\frac{P}{\pi D a} \beta \phi_0 (a) \qquad (19)$$

$$A_m = -\frac{P}{\pi Da} \frac{\mu_m(a)}{\{\beta_m(a)\mu_m(a) - \lambda_m(a)\eta_m(a)\}} \tag{20}$$

$$B_m = \frac{P}{\pi Da} \frac{\lambda_m(a)}{\{\beta_m(a)\mu_m(a) - \lambda_m(a)\eta_m(a)\}} \tag{21}$$

$$A'_m = 0 = B'_m \tag{22}$$

where $P = \pi a^2 q =$ total load on the plate

$$\beta = \frac{[\lambda_m(a) I_m(P_2a) - \mu_m(a) I_m(P_1a)]}{[\beta_m(a)\mu_m - ((a)\lambda_m(a)\eta_m a)] \times [I_0(P_2a)\phi_0(a) - I_0(P_1 a)\psi_0(a)]} \tag{23}$$

$$\psi_0(a) = P_2^2 I_0''(P_2a) + \frac{\nu}{a} P_2 I_1(P_2a) \tag{24}$$

$$\phi_0(a) = P_1^2 I_0''(P_1a) + \frac{\nu}{a} P_1 I_1(P_1a) \tag{25}$$

$$\mu_m(a) = P_2^2 I''_m(P_2a) + \frac{\nu}{a} P_2 I'_m(P_2a) - \frac{\nu m^2}{a^2} I_m(P_2a) \tag{26}$$

$$\beta_m(a) = P_2^2 P_1 I'_m(P_1a) - (1-\nu)\left\{\frac{m^2}{a^3} I_m(P_1a) - \frac{P_1 m^2}{a^2} I'_m(P_1a)\right\} \tag{27}$$

$$\lambda_m(a) = P_1^2 I''_m(P_1a) + \frac{\nu}{a} P_1 I'_m(P_1a) - \frac{\nu m^2}{a^2} I_m(P_1a) \tag{28}$$

$$\eta_m(a) = P_1^2 P_2 I'_m(P_2a) - (1-\nu)\left\{\frac{m^2}{a^3} I_m(P_2a) - \frac{P_2 m^2}{a^2} I'_m(P_2a)\right\} \tag{29}$$

Thus the complete solution of eq. 6 is obtained in the following form

$$w = w_0 + A_0 I_0(P_1r) + B_0 I_0(P_2r) + \sum_{m=2,4,6,\ldots}^{\infty} [A_m I_m(P_1r) + B_m I_m(P_2r)] \cos m\theta \tag{30}$$

where

$$w_0 = \frac{q}{K} + A_0' I_0(P_1r) + B_0' I_0(P_2r) \tag{31}$$

$$A_0 = -\frac{q}{K}\left[\frac{P_2^2 I_0''(P_2a) + P_2 \frac{v}{a} I_1(P_2a)}{\phi(Pa)}\right] \tag{31 a}$$

$$B_0' = \frac{q}{K}\left[\frac{P_1^2 I_0''(P_1a) + P_1 \frac{v}{a} I_1(P_1a)}{\phi(Pa)}\right] \tag{31 b}$$

$$\phi(Pa) = \{I_0(P_1a) P_2^2 I_0''(P_2a) - I_0(P_2a) P_1^2 I_0''(P_1a)\}$$
$$+ \frac{v}{a}\{P_2 I_1(P_2a) I_0(P_1a) - P_1 I_1(P_1a) I_0(P_2a)\} \tag{31 c}$$

Substitution of the values of the constants A_0', B_0', A_0, B_0, A_m and B_m into eq. 30 yields

$$\frac{\bar{w}}{h} = \left(\frac{qa^4}{Dh}\right)\Bigg[\frac{1}{K_F}\Bigg\{1 + \frac{\left[P_1^2 I_0''(P_1a) + P_1 \frac{v}{a} I_1(P_1a)\right] I_0(P_2r)}{\phi(Pa)}$$
$$- \frac{\left[P_2^2 I_0''(P_2a) + P_2 \frac{v}{a} I_1(P_2a)\right] I_0(P_1r)}{\phi(Pa)}\Bigg\}$$
$$+ \frac{1}{a^3}\Bigg\{\beta\psi_0(a) I_0(P_1r) - \beta\phi_0(a) I_0(P_2r)$$
$$- \sum_{m=2,4,6,\ldots}^{\infty} \left[\frac{\mu_m(a) I_m(P_1r) - \lambda_m(a) I_m(P_2r)}{\beta_m(a)\mu_m(a) - \lambda_m(a)\eta_m(a)}\right]\cos m\theta\Bigg\}\Bigg]. \tag{32}$$

As $P_1 \to 0$ and $P_2 \to 0$, eq. 32 reduces to

$$w = w_0 + \frac{Pa^2}{2\pi D}\frac{1}{(3+\nu)}\Bigg\{2\log 2 - 1 + \frac{1+\nu}{1-\nu}\left(2\log 2 - \frac{\pi^2}{12}\right)$$
$$- \sum_{m=2,4,6,\ldots}^{\infty}\Bigg[\frac{1}{m(m-1)} + \frac{2(1+\nu)}{m^2(m-1)(1-\nu)}$$
$$- \frac{(r/a)^2}{m(m+1)}\Bigg]\left(\frac{r}{a}\right)^m \cos m\theta\Bigg\} \tag{33}$$

as obtained by Timoshenko [1] in the corresponding small deflection problem for a plate supported at two points on the boundary.

The normalised constant α can be determined from Eqs. 7 and 30. Since we are interested only in the lateral displacement w, the radial and cross-radial displacements u and v have been eliminated by choosing suitable expressions for u and v, compatible with their boundary conditions and integrating over the whole area of the plate. The radial and cross-radial displacements have been assumed in the forms

$$u = \Sigma\ U(r) \cos m\theta \tag{34}$$

$$v = \Sigma\ V(r) \sin m\theta \tag{35}$$

subject to the boundary conditions $U(a) = V(a) = 0$. Multiplying both sides of the equation 7 by $rdrd\theta$ and integrating between the limits 0 to a and 0 to 2π, one gets

$$\int_0^a \int_0^{2\pi} rU'(r) \cos m\theta\, drd\theta + \int_0^a \int_0^{2\pi} U(r) \cos m\theta\, drd\theta$$

$$+ \int_0^a \int_0^{2\pi} mV(r) \cos m\theta\, drd\theta + \frac{1}{2} \int_0^a \int_0^{2\pi} r \left(\frac{\partial w}{\partial r}\right)^2 drd\theta$$

$$+ \frac{1}{2} \int_0^a \int_0^{2\pi} \frac{1}{r} \left(\frac{\partial w}{\partial \theta}\right)^2 drd\theta = \int_0^a \int_0^{2\pi} \frac{\alpha^2 h^2}{12}\, r\, drd\theta.$$

After evaluating the integrals the following equation leading to α is obtained.

$$\frac{\alpha^2 h^2 a^2}{12} = -\frac{1}{2} A_0'^2 P_1^2 a^2 \left\{\frac{1}{4} [I_0 (P_1a) + I_2 (P_1a)]^2 \right.$$

$$\left. - \left[1 + \frac{1}{P_1^2 a^2}\right] I_1^2 (P_1a)\right\} - \frac{1}{2} B_0'^2 P_2^2 a^2$$

$$\times \left\{\frac{1}{4} [I_0 (P_2a) + I_2 (P_2a)]^2 - \left[1 + \frac{1}{P_2^2 a^2}\right] I_1^2 (P_2a)\right\}$$

$$+ 2A_0' B_0' P_1 P_2 \frac{a}{P_2^2 - P_1^2} [- \tfrac{1}{2} P_1 I_1 (P_2a) \{I_0 (P_1a)$$

$$+ I_2 (P_1a)\} + \tfrac{1}{2} P_2 I_1 (P_1a) \{I_0 (P_2a) + I_2 (P_2a)\}]$$

$$+ \sum_{m=2,4,6,\dots}^{\infty} \left[A^2{}_m P_1^2 \left\{- \frac{1}{8} a^2 \left[\frac{1}{4} \{I_{m-2} (P_1a) + I_m (P_1a)\}^2\right.\right.\right.$$

$$\left. \left\{1 + \frac{(m-1)^2}{P_1^2 a^2}\right\} I_m^2 (P_1a)\right]$$

$$- \frac{1}{8} a^2 \left[\frac{1}{4} \{I_m(P_1a) + I_{m+2} (P_1a)\}^2 - \left\{1 + \frac{(m+1)^2}{P_1^2 a^2}\right\}\right.$$

$$\times I^2_{m+1}(P_1 a)\Big] + \frac{1}{2}\sum_{\substack{n=0\\n\neq t}}^{\infty}\sum_{t=0}^{\infty}\left(\frac{P_1}{2}\right)^{2m+2n+2t}\cdot\phi\Big\}\cdot\frac{1}{2}$$

$$+ B_m{}^2 P_2{}^2\Big\{-\frac{1}{8}a^2\Big[\frac{1}{4}\{I_{m-2}(P_2 a) + I_m(P_2 a)\}^2$$

$$+\left\{1+\frac{(m-1)^2}{P_2{}^2 a^2}\right\} I^2_{m-1}(P_2 a)\Big] - \frac{1}{8}a^2\Big[\frac{1}{4}\{I_m(P_2 a)$$

$$+ I_{m+2}(P_2 a)\}^2 + \left\{1+\frac{(m+1)^2}{P_2{}^2 a^2}\right\} I^2_{m+1}(P_2 a)\Big]$$

$$+\frac{1}{2}\sum_{\substack{n=0\\n\neq t}}^{\infty}\sum_{t=0}^{\infty}\left(\frac{P_2}{2}\right)^{2m+2n+2t}\cdot\phi\Big\}\frac{1}{2} + \frac{1}{2}A_m B_m P_1 P_2$$

$$\times\Big\{\sum_{\substack{n=0\\n\neq t}}^{\infty}\sum_{t=0}^{\infty}\Big[\frac{\left(\frac{P_1}{2}\right)^{m+2n-1}\left(\frac{P_2}{2}\right)^{m+2t-1} a^{2m+2n+2t}}{(2m+2n+2t)\,\lfloor n\,\lfloor t\,\Gamma(m+n)\,\Gamma(m+t)}$$

$$+\left(\frac{P_2}{2}\right)^{m+2n-1}\left(\frac{P_2}{2}\right)^{m+2t+1}\phi$$

$$+\frac{\left(\frac{P_1}{2}\right)^{m+2n+1}\left(\frac{P_2}{2}\right)^{m+2t-1} a^{2m+2n+2t+2}}{(2m+2n+2t+2)\,\lfloor n\,\lfloor t\,\Gamma(m+t)\,\Gamma(m+n+2)}$$

$$+\frac{\left(\frac{P_1}{2}\right)^{m+2n+1}\left(\frac{P_2}{2}\right)^{m+2t+1} a^{2m+2n+2t+4}}{(2m+2n+2t+4)\,\lfloor n\,\lfloor t\,\Gamma(m+n+2)\,\Gamma(m+t+2)}\Big]\Big\}\frac{1}{2}$$

$$+ A_m{}^2\Big\{\sum_{n=0}^{\infty}\left(\frac{P_1}{2}\right)^{2m+4n}\phi_1 + \sum_{\substack{n=0\\n\neq t}}^{\infty}\sum_{t=0}^{\infty}\left(\frac{P_1}{2}\right)^{2m+2n+2t}\psi\Big\}\frac{m^2}{2}$$

$$+ B_m{}^2\Big\{\sum_{n=0}^{\infty}\left(\frac{P_2}{2}\right)^{2m+4n}\phi_1 + \sum_{\substack{n=0\\n\neq t}}^{\infty}\sum_{t=0}^{\infty}\left(\frac{P_2}{2}\right)^{2m+2n+2t}\psi\Big\}\frac{m^2}{2}$$

$$+ 2A_m B_m\Big\{\sum_{\substack{n=0\\n\neq t}}^{\infty}\sum_{t=0}^{\infty}\left(\frac{P_1}{2}\right)^{m+2n}\left(\frac{P_2}{2}\right)^{m+2t}\psi\Big\}\frac{m^2}{2}\Big] \quad (36)$$

where

$$\phi = \frac{a^{2m+2n+2t+2}}{(2m+2n+2t+2)\,\lfloor n \,\lfloor t\, \Gamma(m+n)\,\Gamma(m+t+2)}$$

$$\phi_1 = \frac{a^{2m+4n}}{(2m+4n)\{\lfloor n\, \Gamma(m+n+1)\}^2}$$

$$\psi = \frac{a^{2m+2n+2t}}{(2m+2n+2t)\,\lfloor n \,\lfloor t\, \Gamma(m+n+1)\,\Gamma(m+t+1)}.$$

Thus the deflection, w is completely determined. The expressions for the bending and twisting moment can now be determined.

$$\begin{aligned} M_r = -D\Big[& P_1^2 (A_0' + A_0) I_0''(P_1 r) + P_2^2 (B_0' + B_0) I_0''(P_2 r) \\ & + \sum_{m=2,4,6,\ldots}^{\infty} \{P_1^2 A_m I''_m (P_1 r) + P_2^2 B_m I''_m (P_2 r)\} \cos m\theta \\ & + \nu \Big\{ \frac{P_1}{r} (A'_0 + A_0) I_1'(P_1 r) + \frac{P_2}{r} (B_0' + B_0) I_1'(P_2 r) \\ & + \frac{1}{r} \sum_{m=2,4,6,\ldots}^{\infty} [P_1 A_m I'_m (P_1 r) + P_2 B_m I'_m (P_2 r)] \cos m\theta \\ & - \frac{1}{r^2} \sum_{m=2,4,6,\ldots}^{\infty} m^2 [A_m I_m (P_1 r) + B_m I_m (P_2 r)] \cos m\theta) \Big\} \Big] \end{aligned} \tag{37}$$

$$\begin{aligned} M_\theta = -D\Big[& \frac{P_1}{r} (A_0' + A_0) I_1'(P_1 r) + \frac{P_2}{r} (B_0' + B_0) I_1'(P_2 r) \\ & + \frac{1}{r} \sum_{m=2,4,6,\ldots}^{\infty} \{P_1 A_m I_m'(P_1 r) + P_2 B_m I_m'(P_2 r)\} \cos m\theta \\ & - \frac{1}{r^2} \sum_{m=2,4,6,\ldots}^{\infty} m^2 \{A_m I_m (P_1 r) + B_m I_m (P_2 r)\} \cos m\theta \\ & + \nu \Big\{ P_1^2 (A_0' + A_0) I_0''(P_1 r) + P_2^2 (B_0' + B_0) I_0''(P_2 r) \end{aligned}$$

$$+ \sum_{m=2,4,6,\ldots}^{\infty} [P_1^2 A_m I_m''(P_1 r) + P_2^2 B_m I_m''(P_2 r)] \cos m\theta \Big\} \Big] \tag{38}$$

$$M_{r\theta} = (1-\nu) D \Big[-\frac{1}{r} \sum_{m=2,4,6,\ldots}^{\infty} m \{P_1 A_m I_m'(P_1 r) + P_2 B_m I_m'(P_2 r)\}$$

$$\times \sin m\theta + \frac{1}{r^2} \sum_{m=2,4,6,\ldots}^{\infty} m \{A_m I_m(P_1 r) + B_m I_m(P_2 r)\}$$

$$\times \sin m\theta \Big]. \tag{39}$$

The stresses can be calculated from the expressions

$$\sigma_r \frac{6M_r}{h^2}\ ; \quad \sigma_\theta = \frac{6M_\theta}{h^2}\ ; \quad \tau_{r\theta} = \frac{6M_{r\theta}}{h^2} \tag{40}$$

Numerical Calculation

To obtain deflection for a given value of plate radius 'a' and foundation modulus 'K_F' one has to start from the equation (36) with an assumed value of 'α' in order to obtain the corresponding value of the load function qa^4/Dh. Once this relationship is obtained the corresponding deflection w/h can be calculated by eq. 32. For $a = 50$ mm, $h = 0 \cdot 75$ mm, $v = 0 \cdot 3$ and $K_F = 80$ deflections have been presented in Fig 2.

Concluding Remarks

An examination of the eq. 32 will reveal that the deflection (w/h) depends on K_F, the plate radius 'a' and on the value of the angle, θ. For a given value of the load function eq. 32 can be written as

$$\left(\frac{w}{h}\right)_{\substack{r=0\\ \theta=0}} = K_1\left(\frac{qa^4}{Dh}\right)\ ; \quad \left(\frac{w}{h}\right)_{\substack{r=a\\ \theta=\pi/2}} = K_2\left(\frac{qa^4}{Dh}\right) \tag{41}$$

where K_1 and K_2 are two numerical constants, K_2 being greater than K_1. Because of the reactive forces at the two points of support, deflections on the diameter at $\theta = 0$ will be less than those on the diameter at $\theta = \pi/2$. Maximum deflection will occur at the boundary at $\theta = \pm \pi/2$. Deflections according to the linear theory have also been plotted in Fig. 2 and it is

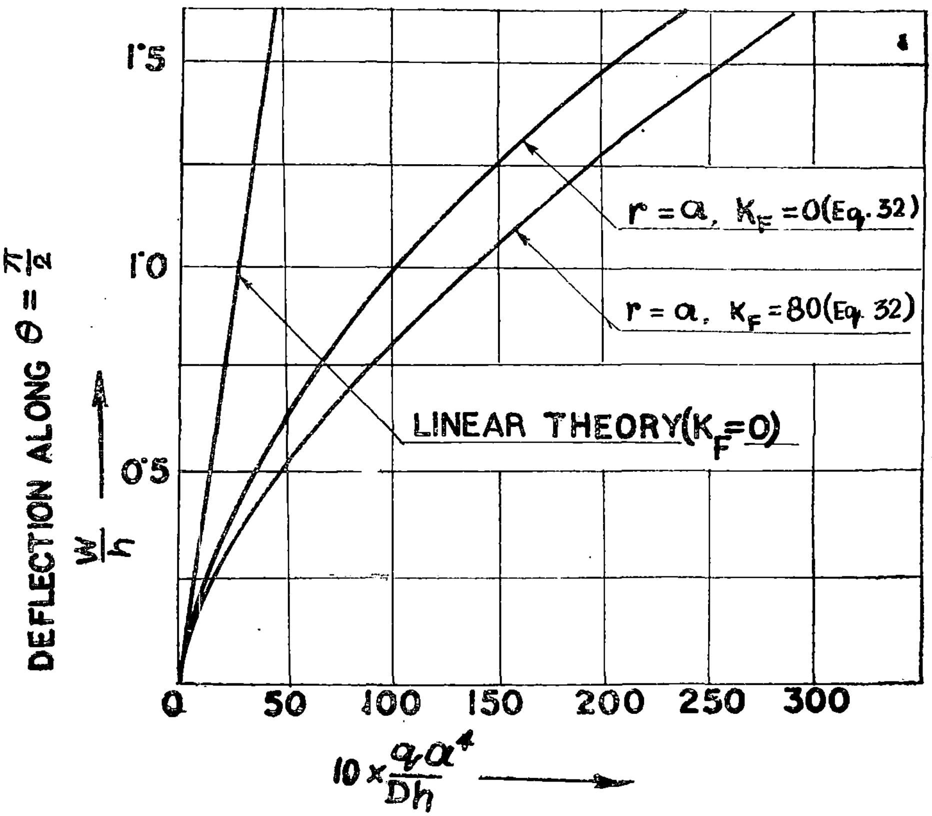

FIG. 2. Load deflection curve.

clear that the errors of the linear theory increases as the load increases. In order to study the variation of moments, eqs. 37, 38 and 39 are plotted in Fig. 3 for various values of (r/a) and for the angles at which they become maximum. It is observed that the maximum bending moments, their magnitudes being unequal, are developed at $r = 3a/4$, $\theta = \pm\pi/2$ and the twisting moment is maximum at $r = a$, $\theta = \pm\pi/4$, $\pm 3\pi/4$.

As the plate must be in equilibrium on the supports, the foregoing analysis for two simple supports represents the worst condition when the deflections and stresses are maximum for a given load function. With the increase in the number of supports, w_1 in eq. 8 decreases. For an infinitely large number of supports, w in eq. 8 will approach to w_0 in the limit and the point of maximum bending moments will shift to the centre of the plate, $(M_r)_{max}$ being equal to $(M_\theta)_{max}$ in that case.

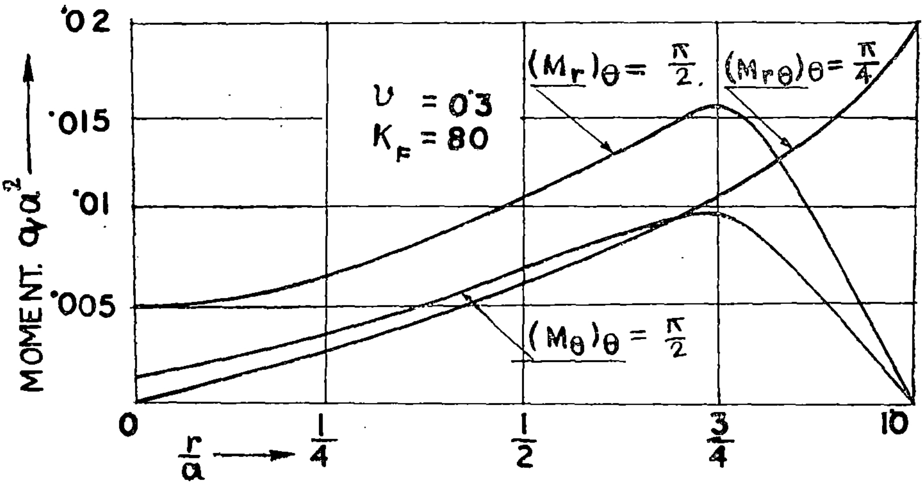

FIG. 3. Moment curve.

The present study can be extended to any number of supports, provided the supports are so chosen as not to disturb the equilibrium of the plate. For example, if three equidistant supports are chosen, $\psi_1 = 0$, $\psi_2 = 2\pi/3$, $\psi_3 = 4\pi/3$, the differential equations together with the boundary condition remaining unchanged. If the plate is clamped on the supports, the boundary conditions and the concentrated reactions at the supports will change totally demanding a separate investigation.

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NOTATION

The following symbols have been used in this paper:

a = plate radius

A_0', A_0, B_0', B_0, Am, Bm = Constants

D = flexural rigidity of the plate $= \dfrac{Eh^3}{12(1-\nu^2)}$

E = Young's modulus

e_1 = first invariant of middle surface strains
$= \epsilon_x + \epsilon_y$ in rectangular co-ordinates
$= \epsilon_r + \epsilon_\theta$ in cylindrical co-ordinates

e_2 = second invariant of middle surface strains
$= \epsilon_x \epsilon_y - \frac{1}{4}\gamma_{xy}^2$ in rectangular co-ordinates
$= \epsilon_r \epsilon_\theta$ in cylindrical co-ordinates in case oı circular symmetry

h = plate thickness

I_0, I_m = Modified Bessel's function of the first kind and of the zero order and mth order respectively.

K = foundation reaction per unit area per unit deflection

K_F	= dimensionless foundation modulus $= \frac{K}{D} a^4$.
M	= moment
q	= uniform lateral load
r, θ	= polar co-ordinates
u, v	= radial and crossradial displacements
V_1	= strain energy
w	= deflection in z-direction
σ	= direct stress
τ	= shear stress
ϵ	= direct strain
γ	= shear strain
ν	= Poisson's ratio
Γ	= Gamma function.

for a bonded composite material, it should be assumed that interactions between constituents occur between congruent particles. Since the constituents can undergo individual motions, this assumption implies that during any motion of the material, interactions between constituents occur between material particles which can occupy different spatial positions. This is in contrast to the usual continuum theories of mixtures, in which interactions between constituents are assumed to occur between material particles which are spatially superimposed [3].

In [2], the interaction force vector $\mathbf{b}$ between constituents was assumed to be a function of the relative displacement $\mathbf{u}_{(pm)} = \mathbf{x}_{(p)} - \mathbf{x}_{(m)}$ between congruent particles. Assuming that it also depends upon the relative velocity $\dot{\mathbf{u}}_{(pm)} = \dot{\mathbf{x}}_{(p)} - \dot{\mathbf{x}}_{(m)}$ introduces an interesting difficulty in that the component of the relative velocity normal to the line joining the congruent particles is not invariant under imposed rigid body rotations and thus does not satisfy the principle of material-frame indifference (PMI) [4].

Formally, consider motions of the constituents

$$\bar{\mathbf{x}}_{(m)} = \bar{\mathbf{x}}_{(m)}(X_{(m)}, t), \quad \bar{\mathbf{x}}_{(p)} = \bar{\mathbf{x}}_{(p)}(X_{(p)}, t) \tag{4}$$

related to the motions (3) by

$$\bar{\mathbf{x}}_{(m)} = \mathbf{Q}\mathbf{x}_{(m)} + \mathbf{c}, \quad \bar{\mathbf{x}}_{(p)} = \mathbf{Q}\mathbf{x}_{(p)} + \mathbf{c} \tag{5}$$

where $\mathbf{Q}$ is an arbitrary orthogonal time-dependent linear transformation and $\mathbf{c}$ is an arbitrary time-dependent vector. The PMI requires that a vector-valued constitutive variable $\mathbf{v}$ associated with the motion given by equations (3) be related to its value $\bar{\mathbf{v}}$ associated with the motion given by equations (4) by $\bar{\mathbf{v}} = \mathbf{Q}\,\mathbf{v}$. If this is satisfied, $\mathbf{v}$ is said to be indifferent.

The relative displacement between congruent particles is indifferent,

$$\bar{\mathbf{u}}_{(pm)} = \mathbf{Q}\mathbf{u}_{(pm)}, \tag{6}$$

but the relative velocity is not.

$$\dot{\bar{\mathbf{u}}}_{(pm)} = \mathbf{Q}\dot{\mathbf{u}}_{(pm)} + \dot{\mathbf{Q}}\mathbf{u}_{(pm)}. \tag{7}$$

Let a decomposition of $\dot{\mathbf{u}}_{(pm)}$ into indifferent $(\dot{\mathbf{u}}_{(pm)}{}^T)$ and nonindifferent $(\dot{\mathbf{u}}_{(pm)}{}^N)$ parts be sought,[3] $\dot{\mathbf{u}}_{(pm)} = \dot{\mathbf{u}}_{(pm)}{}^T + \dot{\mathbf{u}}_{(pm)}{}^N$, such that

$$\dot{\bar{\mathbf{u}}}_{(pm)}{}^T = \mathbf{Q}\dot{\mathbf{u}}_{(pm)}{}^T, \quad \dot{\bar{\mathbf{u}}}_{(pm)}{}^N = \mathbf{Q}\dot{\mathbf{u}}_{(pm)}{}^N + \dot{\mathbf{Q}}\mathbf{u}_{(pm)} \tag{8}$$

It is easy to show that equations (8) are satisfied if $\dot{\mathbf{u}}_{(pm)}{}^T$ and $\dot{\mathbf{u}}_{(pm)}{}^N$ are the components of $\dot{\mathbf{u}}_{(pm)}$ tangential and normal to $\mathbf{u}_{(pm)}$. For, defining

$$\dot{\mathbf{u}}_{(pm)}{}^T = \begin{cases} \left[\dfrac{\dot{\mathbf{u}}_{(pm)}\cdot\mathbf{u}_{(pm)}}{\mathbf{u}_{(pm)}\cdot\mathbf{u}_{(pm)}}\right]\mathbf{u}_{(pm)} \text{ if } \mathbf{u}_{(pm)} \neq 0 \\ \dot{\mathbf{u}}_{(pm)} \text{ if } \mathbf{u}_{(pm)} = 0 \end{cases} \tag{9}$$

the transformation equation is

$$\dot{\bar{\mathbf{u}}}_{(pm)}{}^T = \left[\frac{\dot{\bar{\mathbf{u}}}_{(pm)}\cdot\bar{\mathbf{u}}_{(pm)}}{\bar{\mathbf{u}}_{(pm)}\cdot\bar{\mathbf{u}}_{(pm)}}\right]\bar{\mathbf{u}}_{(pm)} = \mathbf{Q}\dot{\mathbf{u}}_{(pm)}{}^T \tag{10}$$

and then

$$\dot{\bar{\mathbf{u}}}_{(pm)}{}^N = \dot{\bar{\mathbf{u}}}_{(pm)} - \dot{\bar{\mathbf{u}}}_{(pm)}{}^T = \mathbf{Q}\dot{\mathbf{u}}_{(pm)}{}^N + \dot{\mathbf{Q}}\mathbf{u}_{(pm)}. \tag{11}$$

It is also easy to see that the decomposition is unique. Thus interactions between congruent particles can depend only on the component $\dot{\mathbf{u}}_{(pm)}{}^T$ of $\dot{\mathbf{u}}_{(pm)}$.

A physical picture which helps to clarify this result is to imagine the congruent particles of the two constituents to be connected by springs and dashpots. The dashpot (relative velocity dependent) forces do not depend on the relative velocity components normal to the lines joining the particles, but depend only on the tangential components.

Assuming that the interaction force vector $\mathbf{b}$ and the stress tensor in the elastic material $\mathbf{t}_{(m)}$ depend upon $\dot{\mathbf{u}}_{(pm)}$, $\dot{\mathbf{u}}_{(pm)}{}^T$ and the strain of the elastic material, the linearized isotropic three-dimensional forms of equations (1) and (2) are, in Cartesian tensor form,

$$\rho_{(m)}\frac{\partial^2 u_{(m)k}}{\partial t^2} = t_{(m)kj,j} + b_k, \quad \rho_{(m)}\frac{\partial^2 u_{(p)k}}{\partial t^2} = -b_k, \tag{12}$$

where

$$t_{(m)kj} = \lambda_{(m)}\delta_{kj}e_{(m)nn} + 2\mu_{(m)}e_{(m)kj},$$
$$b_k = \nu u_{(pm)k} + \sigma\dot{u}_{(pm)k}{}^T, \tag{13}$$

and

$$e_{(m)kj} = \frac{1}{2}(u_{(m)k,j} + u_{(m)j,k}). \tag{14}$$

For one-dimensional compressional or shear wave motion, equations (12)–(14) reduce to the form of equations (1) and (2). However, for more complicated motions, including combined compression and shear, it is interesting to note that equations (12)–(14) are intrinsically nonlinear due to the term $\dot{u}_{(pm)k}{}^T$, constituting what might be called a geometric material nonlinearity.

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[3] This is similar to the familiar decomposition of the velocity gradient into indifferent (deformation rate) and nonindifferent (vorticity) parts.

Large Deflection of a Circular Plate on Elastic Foundation Under a Concentrated Load at the Center

S. Datta[1]

Introduction

For moderately large deflections of thin plates, solutions of the differential equations for deflections and displacements become difficult because of their nonlinear character. Neglecting the second strain invariant of the middle surface strains, Berger [1][2] solved the large deflection of circular and rectangular plates under uniform load with ease and sufficient accuracy. Following Berger's method many investigators [2–6] have solved various large deflection problems and have obtained satisfactory results.

Following Berger's method the large deflection of a clamped cir-

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[2] Numbers in brackets designate References at end of Note.

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cular isotropic plate resting on elastic foundation of the Winkler type has been investigated in this study for a concentrated load applied at the center. The results have been experimentally verified.

Analysis

Using polar coordinates the governing differential equation for a circular plate of radius a, with a concentrated load P, at the center is given by

$$\nabla^2(\nabla^2 - \alpha^2)w + \frac{K}{D}w = 0 \text{ (except at the load point)} \quad (1)$$

where α is a constant determined from

$$\frac{u}{r} + \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 = \frac{\alpha^2 h^2}{12} \quad (2)$$

and

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\cdot\frac{d}{dr}; \quad w = \text{displacement}$$

along z-direction; K = foundation reaction per unit area per unit deflection; $D = (Eh^3/12(1 - \nu^2)$, E being the modulus of elasticity; h is the plate thickness, and ν is Poisson's ratio.

Considering the radial stress and shearing stress on a concentric circular area of radius r, the concentrated load P at the center, and since u and dw/dr are both zero at the center one gets

$$\lim_{r\to 0} Dr\frac{d}{dr}[(\nabla^2 - \alpha^2)]w = \frac{P}{2\pi} \quad (3)$$

Solution of equation (1) can be taken in the following convenient form

$$w = AI_0(P_1r) + BI_0(P_2r) + C[K_0(P_1r) - K_0(P_2r)] \quad (4)$$

in which I_0 is the modified Bessel function of the first kind and zero order and K_0 is the modified Bessel function of the second kind and zero order

$$P_1^2 + P_2^2 = \alpha^2; \; P_1^2P_2^2 = \frac{K}{D}$$

A, B, and C are constants to be determined from the boundary conditions. Clamped edge boundary conditions are

$$(w)_{r=a} = 0 = \left(\frac{dw}{dr}\right)_{r=a} \quad (5)$$

Considering equations (3) and (4) one gets

$$C = \frac{P}{2\pi D(P_2^2 - P_1^2)}$$

Considering equations (4) and (5) one gets

$$A = C\left[\frac{\frac{1}{a} - \psi}{\phi}\right]; \; B = C\left[\frac{\frac{1}{a} - \psi_1}{\phi}\right]$$

where

$\psi = K_1(P_1a)\,P_1I_0(P_2a) - K_0(P_1a)\,P_2I_1(P_2a)$
$\phi = P_2I_1(P_2a)I_0(P_1a) - P_1I_1(P_1a)I_0P_2a)$
$\psi_1 = K_0(P_2a)P_1I_1(P_1a) - K_1(P_2a)P_2I_0(P_1a)$

Thus

$$w = C\left[\left\{\frac{\frac{1}{a} - \psi}{\phi}\right\} I_0(P_1r) + \left\{\frac{\frac{1}{a} - \psi_1}{\phi}\right\} I_0(P_2r) + \{K_0(P_1r) - K_0(P_2r)\}\right] \quad (6)$$

is determined completely.

Setting $r \to 0$ in equation (4) one gets the maximum deflection w_0, at the center of the plate,

$$w_0 = A + B + C\log_e\left(\frac{P_2}{P_1}\right) \quad (7)$$

To determine the displacement u, one gets from equation (2)

$$ur = \frac{\alpha^2h^2r^2}{24} - \frac{1}{2}\int\left(\frac{dw}{dr}\right)^2 rdr + K' \quad (8)$$

where K' is the constant of integration. After evaluating the integrals and using the boundary condition $u \to 0$ as $r \to a$, the equation determining u may be obtained. Also as $r \to 0$, $u \to 0$ from symmetry. Thus the equation for α is obtained as

$$\begin{aligned}
\frac{\alpha^2h^2a^2}{12}
&= A^2P_1^2\left[\frac{aI_0(P_1a)I_1(P_1a)}{P_1} + \frac{1}{2}I_1^2(P_1a)a^2 - \frac{1}{2}I_0^2(P_1a)a^2\right] \\
&+ B^2P_2^2\left[\frac{aI_0(P_2a)I_1(P_2a)}{P_2} + \frac{1}{2}I_1^2(P_2a)a^2 - \frac{1}{2}I_0^2(P_2a)a^2\right] \\
&+ C^2P_1^2\left[\frac{1}{2}a^2K_1^2(P_1a) - \frac{1}{2}K_0^2(P_1a)a^2 - \frac{a}{P_1}K_0(P_1a)K_1(P_1a)\right] \\
&+ C^2P_2^2\left[\frac{1}{2}a^2K_1^2(P_2a) - \frac{1}{2}K_0^2(P_2a)a^2 - \frac{a}{P_2}K_0(P_2a)K_1(P_2a)\right] \\
&+ \frac{2ABP_1P_2a}{P_2^2 - P_1^2}[P_2I_1(P_1a)I_0(P_2a) - P_1I_1(P_2a)I_0(P_1a)] \\
&- \frac{2C^2P_1P_2a}{P_2^2 - P_1^2}[P_1K_1(P_2a)K_0(P_1a) - P_2K_1(P_1a)K_0(P_2a)] \\
&- \frac{2ACP_1P_2a}{P_2^2 - P_1^2}[P_2I_1(P_1a)K_0(P_2a) + P_1K_1(P_2a)I_0(P_1a)] \\
&- \frac{2BCP_1P_2a}{P_2^2 - P_1^2}[P_1I_1(P_2a)K_0(P_1a) + P_2K_1(P_1a)I_0(P_2a)] \\
&+ BC[P_2^2a^2\{I_1(P_2a)K_1(P_2a) + I_0(P_2a)K_0(P_2a)\} \\
&\quad - P_2a\{I_1(P_2a)K_0(P_2a) - I_0(P_2a)K_1(P_2a)\}] \\
&- AC[P_1^2a^2\{I_1(P_1a)K_1(P_1a) + I_0(P_1a)K_0(P_1a)\} \\
&\quad - P_1a\{I_1(P_1a)K_0(P_1a) - I_0(P_1a)K_1(P_1a)\}] \\
&+ \frac{2ACP_1^2}{P_2^2 - P_1^2} + \frac{2BCP_2^2}{P_2^2 - P_1^2} + AC - BC \\
&- C^2\left[I + \frac{P_2^2 + P_1^2}{P_2^2 - P_1^2}\log_e\frac{P_1}{P_2}\right] \quad (9)
\end{aligned}$$

If $P_1 \to 0$, $P_2 \to \alpha$ or $P_1 \to \alpha$, $P_2 \to 0$, one gets from equation (6)

$$\begin{aligned}
w = -\frac{P}{\pi 2D\alpha^3 aI_1(\alpha a)}\Big[&I_0(\alpha a) - 1 - I_0(\alpha r) \\
&+ I_0(\alpha r)\alpha aK_1(\alpha a) + \alpha aI_1(\alpha a)\log_e\left(\frac{r}{a}\right) \\
&+ K_0(\alpha r)\alpha aI_1(\alpha a)\Big] \quad (10)
\end{aligned}$$

and equation (9) also reduces to

$$\left[\frac{Pa^2}{\pi Dh}\right]^2 = \left[\frac{\frac{1}{3}(\alpha a)^6}{\gamma + \log_e\frac{\alpha a}{2} - \frac{I_0(\alpha a) + \alpha aK_1(\alpha a) - 2}{\alpha aI_1(\alpha a)} - \frac{1}{2}\left\{\frac{I_0(\alpha a) - 1}{I_1(\alpha a)}\right\}^2}\right] \quad (11)$$

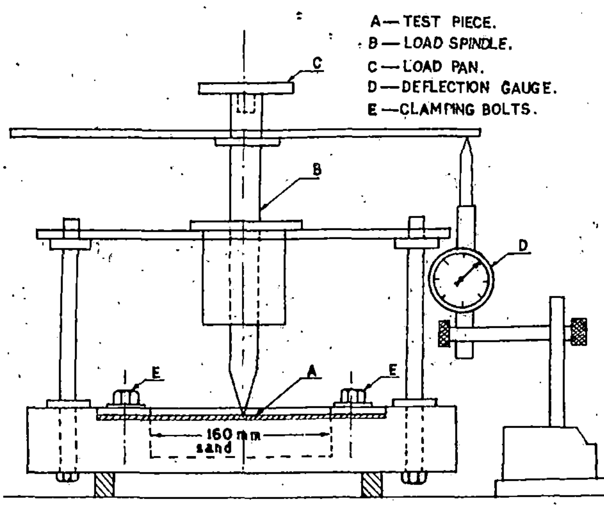

Fig. 1 Experimental apparatus for load deflection test

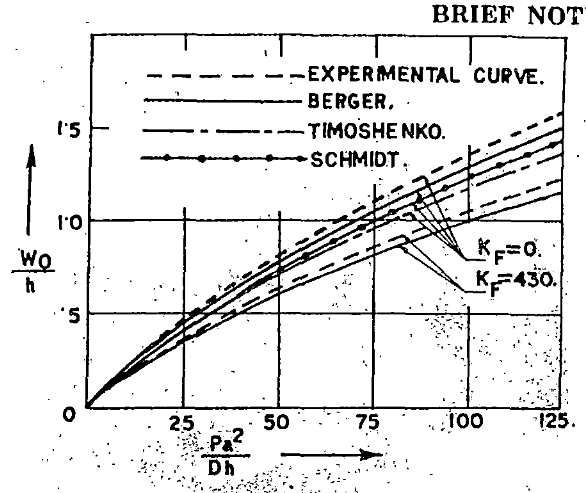

Fig. 2 Deflection curve

Equations (10) and (11) are the results obtained by Basuli [5].

Experimental and Theoretical Results

Deflections of a 0.75-mm thick mild steel plate were measured experimentally using sand as foundation material. The sketch of the apparatus used is shown in Fig. 1 and is self-explanatory. The value of the nondimensional foundation modulus, $K_F = (K/D)a^4$ for sand used was determined experimentally to be 430.

The theoretical results according to Berger's approximate method and experimental results both for $K_F = 0$ and $K_F = 430$ have been presented in Fig. 2. Results according to Timoshenko and Krieger [7] and Schmidt [8] corresponding to $K_F = 0$ have also been presented for comparison in the same graph. It is observed that results obtained from Berger's method approach more closely toward the practical values and the error increases progressively with the increase of load function.

Acknowledgment

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Motion of a Stretched String Loaded by an Accelerating Force[1]

M. J. Sagartz[2] and M. J. Forrestal[2]

The use of explosives to simulate distributed impulse loads on various structural shapes has recently motivated several analytical solutions for the response of structures to moving forces traveling at a constant velocity. Solutions are available for stretched strings [1–3],[3] beams [4, 5], and circular rings [6]. A recently developed rocket-propelled trolley facility at Sandia Laboratories, Albuquerque motivated this analysis. Unlike the problems which have forces moving at constant velocity, the trolley is propelled at nearly constant acceleration along a steel cable suspended between two mountain peaks. In order to estimate the cable motion, a wave solution for the response of a pinned end, semi-infinite stretched string loaded by a force moving at constant acceleration is derived.

[1] This work was supported by the U. S. Atomic Energy Commission.

[2] Shock Simulation Department, Sandia Laboratories, Albuquerque, New Mexico. Mems. ASME

[3] Numbers in brackets designate References at end of Note.

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Analysis

A wave solution is obtained by employing Laplace and Fourier transform techniques, and the definitions for the integral transforms given in [7] are adopted. For purposes of analysis the string is considered infinite, and an image load is included in the equation of motion to satisfy the pinned end boundary condition. The equation of motion is

$$\rho\partial^2 y/\partial t^2 - T\partial^2 y/\partial x^2 = P\delta(x - at^2/2) - P\delta(x + at^2/2) \tag{1}$$

where ρ is mass per unit length, T is tension, y is deflection, t is time, x is the axial coordinate measured positive to the right, P is the force magnitude, a is the constant acceleration, and δ is the Dirac delta function. The response is evaluated for positive x, and the second term on the right-hand side of equation (1) is the image load. The transformed solution is

$$\bar{y}^*(\xi, s) = \frac{P}{\rho}\int_0^\infty \frac{\exp[-s(2\alpha/a)](e^{-i\xi\alpha} - e^{i\xi\alpha})}{(2\alpha/a)^{1/2}(s^2 + c^2\xi^2)}\, d\alpha \tag{2}$$

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THERMAL BUCKLING OF SOME HEATED PLATES PLACED ON ELASTIC FOUNDATION

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Thermal buckling of a heated equilateral triangular plate and a clamped elliptic plate placed on elastic foundation has been investigated. The boundary of the plate is transformed conformally onto the unit circle. The critical buckling temperature is obtained with the help of error function.

Thermal buckling of thin elastic plates is of much practical importance in modern engineering. Nowacki[1] has discussed the thermal buckling of a rectangular plate under different boundary conditions. Mansfield[2] has investigated the buckling and curling of a heated thin circular plate of constant thickness. Klosner & Forray[3] have studied the thermal buckling of simply supported plates under symmetrical temperature distribution.

Stability of thin elastic plates having exotic boundaries subjected to hydrostatic in-plane loading can easily be investigated with the help of approximate techniques such as collocation, finite difference, finite elements, etc. Laura & Shahady[4] have shown that it is convenient to conformally transform the given domain onto a simpler one, i.e., the unit circle and the boundary conditions can then be satisfied identically.

In this paper thermal buckling of a heated equilateral triangular plate and a clamped elliptic plate placed on elastic foundation has been investigated. The foundation is assumed to be of the Winkler type. The boundary has been transformed conformally onto the unit circle and solution has been obtained with the help of error function.

NOTATIONS

The following notations have been used in this paper :

B_n, B_m = constants ;

D = flexural rigidity of the plate = $\dfrac{Eh^3}{12(1-\nu^2)}$,

E = Young's modulus ;

h = plate thickness ;

$$N_T = \alpha E \int_{-\frac{h}{2}}^{\frac{h}{2}} T dz ;$$

T = temperature ;

u, v = displacement in x and y direction respectively ;

W = deflection normal to the middle plane of the plate ;

K_1 = foundation reaction per unit area per unit deflection ;

α = coefficient of linear thermal expansion.

THEORY

Let us consider a plate of thickness, h, subjected to a temperature distribution T which is independent of x and y, but varies arbitrarily through the thickness, i.e.,

$$T = T(z)$$

The plate is subjected to no external load and motion of all supports in the plane of the plate is prevented. It justifies then, that under the above condition there are no displacements in the plane of the plate, i.e.,

$$u = v = 0$$

On the above propositions the differential equation for the displacement[5] is

$$D \nabla^4 W + \frac{N_T}{1-\nu} \nabla^2 W = 0 \tag{1}$$

For a plate placed on elastic foundation having the foundation reaction, K_1, (1) becomes

$$D \nabla^4 W + \frac{N_T}{1-\nu} \nabla^2 W + K_1 W = 0 \tag{2}$$

Eq. (2) may be written as

$$(\nabla^2 + P_1^2)(\nabla^2 + P_2^2) W = 0 \tag{3}$$

in which

$$P_1^2 P_2^2 = \frac{K_1}{D} \tag{4}$$

$$P_1^2 + P_2^2 = \frac{N_T}{D(1-\nu)} \tag{5}$$

If $z = x + iy$, $\bar{z} = x - iy$ Eq. (3) changes into

$$\left(4 \frac{\partial^2}{\partial z \, \partial \bar{z}} + P_1^2\right)\left(4 \frac{\partial^2}{\partial z \, \partial \bar{z}} + P_2^2\right) W = 0 \tag{6}$$

Let $z = f(\xi)$ be the analytic function which maps the given shape in the ξ-plane onto a unit circle. Thus (6) transforms into complex co-ordinates as

$$\left(\nabla^2 + P_1^2 \left[\frac{dz}{d\xi}\right]^2\right)\left(\nabla^2 + P_2^2 \left[\frac{dz}{d\xi}\right]^2\right) W(\xi, \bar{\xi}) = 0 \tag{7}$$

Eq. (7) is written as

$$(\nabla^2 + \lambda_1^2)(\nabla^2 + \lambda_2^2) W(\xi, \bar{\xi}) = 0 \tag{8}$$

in which

$$\lambda_1^2 = P_1^2 (dz/d\xi)^2, \; \lambda_2^2 = P_2^2 (dz/d\xi)^2$$

Let

$$W = \sum_{n=1}^{\infty} B_n \left[1 - (\xi \bar{\xi})^n\right] \tag{9}$$

Clearly the above form of W satisfies the edge condition $W = 0$ at $r = 1$. Putting (9) in (8) one gets the error function, $\epsilon_{r,\theta}$. Galerkin's procedure requires that the error function to be orthogonal over the domain, i.e.,

$$\int_C \epsilon_{r,\theta}(\xi, \bar{\xi}) \, W(\xi, \bar{\xi}) \, dC = 0 \qquad (n = 1, 2, \ldots\ldots K) \tag{10}$$

This generates $(K \times K)$ determinantal equation. The lowest root of this gives the critical buckling temperature.

APPLICATIONS

(*I*) Let us consider an equilateral triangular plate of side, $2a$, and placed on an elastic foundation. To solve the differential (8) let us put

$$W = W_1 + W_2 \tag{11}$$

From (8) one gets

$$(\nabla^2 + \lambda_1^2)\, W_1 = 0 \tag{12}$$

$$(\nabla^2 + \lambda_2^2)\, W_2 = 0 \tag{13}$$

For the edge condition $W = 0$ along the boundary, let

$$W_2 \approx \sum_{n=1}^{K} B_n \left[1 - (\xi\, \bar{\xi})^n \right] = \sum_{n=1}^{K} B_n \,(1 - r^{2n}) \tag{14}$$

It is sufficient to solve either (12) or (13). The mapping function

$$z = 1{\cdot}1352\, a \left[\xi + \frac{1}{6}\,\xi^4 + \frac{5}{63}\,\xi^7 + \frac{4}{81}\,\xi^{10} \right] \tag{15}$$

maps an equilateral triangular plate a unit circle in the ξ-plane.

With this mapping function putting (14) in (13) and remembering $\xi = re^{i\theta}$ one gets the required error function. After evaluating the integral given by (10) and taking $K=2$, the following determinant is obtained.

$$\begin{vmatrix} \dfrac{\lambda_2^2}{6} - 1 & \dfrac{5\lambda_2^2}{24} - \dfrac{4}{3} \\[2ex] \dfrac{5\lambda_2^2}{24} - \dfrac{4}{3} & \dfrac{4\lambda_2^2}{15} - 2 \end{vmatrix} = 0 \tag{16}$$

Solving (16) for the lowest root, one gets the critical buckling temperature.

$$(N_T)_{Cr} = D\,(1 - \nu) \left[\frac{5{\cdot}8}{(1{\cdot}1352\, a)^2} - \frac{K_1\,(1{\cdot}1352\, a)^2}{5{\cdot}8\, D} \right] \tag{17}$$

(*II*) Let us consider an elliptic plate having centre at the origin. Let h be the thickness of the plate. For clamped edge boundary condition let us take W in the following form.

$$W = \sum_{n=1}^{K} B_n \left[1 - (\xi\,\bar{\xi})^n \right]^2 \tag{18}$$

Clearly

$$W = \frac{\partial W}{\partial r} = 0 \qquad \text{at } r = 1$$

For the ellipse

$$\left(\frac{x^2}{4/3} + \frac{y^2}{4/5} \right) = 1$$

mapping function

$$z = 0{\cdot}99\, b\, (\xi + 0{\cdot}12\, \xi^3 + 0{\cdot}03\, \xi^5 + 0{\cdot}01\, \xi^7) \tag{19}$$

maps the above ellipse a unit circle in the ξ-plane. With this mapping function putting (18) in (8) and remembering $\xi = re^{i\theta}$ one gets the required error function. After evaluating the integral given by (10) and taking $K = 2$, the following determinant is obtained.

$$\begin{vmatrix} \dfrac{32}{3} - \dfrac{2}{3}\left(\lambda_1^2 + \lambda_2^2\right) + \dfrac{\lambda_1^2\,\lambda_2^2}{10} & \dfrac{256}{15} - \dfrac{4}{5}\left(\lambda_1^2 + \lambda_2^2\right) + \dfrac{33}{70}\,\lambda_1^2\,\lambda_2^2 \\[2ex] \dfrac{256}{15} - \dfrac{4}{5}\left(\lambda_1^2 + \lambda_2^2\right) + \dfrac{29}{576}\,\lambda_1^2\,\lambda_2^2 & \dfrac{5632}{105} - \dfrac{4}{3}\left(\lambda_1^2 + \lambda_2^2\right) + \dfrac{127}{315}\,\lambda_1^2\,\lambda_2^2 \end{vmatrix} = 0 \tag{20}$$

Solving (20) for the lowest root, the critical buckling temperature is obtained as

$$(N_T)_{Cr} = D\,(1-\nu)\left[\frac{44\cdot6}{b^2} + 0\cdot094\; b^2\,\frac{K_1}{D} - \left\{\left(\frac{29\cdot2}{b^2} + 0\cdot094\; b^2\,\frac{K_1}{D}\right)^2 + 0\cdot106\; b^4\left(\frac{K_1}{D}\right)^2\right\}^{\frac{1}{2}}\right] \qquad (21)$$

CONCLUSION

Solutions obtained in this study are only approximate, because only the first term of the mapping function is considered and K is taken to be 2. More accurate results are obtained by considering the remaining terms of the mapping function and taking K more than 2. Solution of the eigenvalue problem governing the stability of the thin elastic plates having various configurations, such as regular polygonal shape, circular boundary with flat sides, epitrochoidal boundary etc. is easily accomplished with the help of the complex variable theory applied in this study.

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LARGE DEFLECTION OF A TRIANGULAR ORTHOTROPIC PLATE ON ELASTIC FOUNDATION

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Large deflection of an equilateral triangular orthotropic plate, resting on elastic foundation has been solved for a uniform load throughout the plate. General expressions for deflection and bending moment at a particular point have been obtained and the limiting values of the theoretical results have been verified with the known results for small deflection and without any elastic foundation of the corresponding isotropic plate. Theoretical results have also been presented in the form of graphs.

Triangular reinforced concrete slabs are sometimes used as bottom slabs of bunkers. Thus the design of this type of structure is of practical interest for Defence. These slabs may rest freely on soil or sand and generally are subjected to a uniform load. If the thickness of the slab is small compared to the other dimensions, then it may be regarded as a thin orthotropic plate resting on elastic foundation and subjected to a uniform load. Within the elastic limit, the deflection of such plates may be large, i.e., the deflection is on the order of the thickness of the plate. When a plate undergoes large deflection, three differential equations for displacement and deflection may be written, but it is usually difficult to obtain solutions of these equations because of their non-linear character.

Various problems of large deflections of thin plates not resting on elastic foundation have been examined by Way[1], Levy[2] and many other authors. But the methods used by them involve and require considerable computation. Berger[3] suggested that the strain energy due to the second strain invariant of the middle surface strains may be neglected in analysing large deflection of plates having axis symmetric deformation. Berger's method reduces computation and although no complete explanation of this method is offered in, Berger has shown that the deflections and stresses obtained for circular plates under uniform load are in good agreement with those found in practical analysis. Since then numerous problems have been solved with remarkable case and satisfactory approximation by using this method. Iwinski and Nowinski[4] generalised the procedure of Berger to orthotropic plates and found out the deflections of circular and rectangular plates under uniform load and various boundary conditions. By using this approximate method Banerjee[5] obtained deflections of a circular orthotropic plate under a concentrated load at the centre.

Berger's technique of neglecting the second strain invariant in the middle plane has been applied by Sinha[6] to determine large deflection of circular and rectangular plates under uniform load and resting on elastic foundation.

In this paper large deflection of an equilateral triangular orthotropic plate, such as reinforced concrete, resting on elastic foundation has been solved for a uniform load throughout the plate. Foundation is assumed to be such that its reaction is proportional to the deflection of the plate.

NOTATIONS

a = one-half of the length of each side of the plate

e_1 = first invariant of middle surface strains

= $\epsilon_x + E_y$ in rectangular coordinates

h = plate thickness

K = foundation reaction per unit area per unit deflection

K_F = non-dimensional foundation modulus = $\dfrac{K}{D}\, a^4$

q = uniform lateral load

u, v = displacement along x and y direction respectively

V, V_1 = strain energy

w = deflection in z-direction

ϵ = direct strain

γ = shear strain

FORMULATION OF PROBLEM

For moderately large deflections, the strain displacement relationships are

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \qquad (1)$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad (2)$$

and

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y} \qquad (3)$$

Neglecting the second middle surface strain invariant, the strain energy due to bending and stretching of the middle surface of the plate of thickness, h, can be written as

$$V_1 = \frac{1}{2}\iint\left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_1\frac{\partial^2 w}{\partial x^2} + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4\,D_{xy}\left(\frac{\partial^2 w}{\partial x.\,\partial y}\right)^2 + \right.$$
$$\left. + D_x\frac{12}{h^2}e_1^2\right]dxdy \qquad (4)$$

in which

$$D_x = \frac{E'_x h^3}{12},\ D_y = \frac{E'_y h^3}{12},\ D_1 = \frac{E'' h^3}{12},\ D_{xy} = \frac{G\,h^3}{12} \qquad (5)$$

$$e_1 = \frac{\partial u}{\partial x} + K_1\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{K_1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad (6)$$

$$K_1^2 = \frac{D_y}{D_x} \qquad (7)$$

and E'_x, E'_y, E'', and G are constants to characterise the elastic properties of the material.

By adding the potential energy of the uniform normal load, 'q' and of the foundation reaction, K to the energy expression (4), the modified energy expression is obtained as follows:

$$V = \frac{1}{2}\iint\left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_1\frac{\partial^2 w}{\partial x^2} + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_{xy}\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 + \right.$$
$$\left. + D_x\frac{12}{h^2}e_1^2\right]dxdy - \iint q w\,dxdy + \frac{1}{2}\iint K\,w^2\,dxdy \qquad (8)$$

According to the principle of minimum potential energy, the displacements satisfying the equilibrium conditions make the potential energy V minimum. In order for the integral of equation (8) to be an extremum, its integrand F, must satisfy the following Euler's variational principle:

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial w_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial w_{xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial w_{yy}}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial w_{xy}}\right) = 0 \qquad (9)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) = 0 \qquad (10)$$

and

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right) = 0 \qquad (11)$$

Applying (10) and (11) to (8) respectively, we get

$$\frac{\partial}{\partial x}(e_1) = 0 \qquad (12)$$

Thus

$$\frac{\partial}{\partial y}\ (e_1) = 0 \qquad (13)$$

$$e_1 = C, \qquad (14)$$

a normalised constant of integration to be determined. Applying (9) to (8) and considering (14), we get

$$\frac{\partial^4 w}{\partial x^4} + K_1^2 \frac{\partial^4 w}{\partial y^4} + \frac{2(D_1 + 2D_{xy})}{D_x} \frac{\partial^4 w}{\partial x^2 \partial y^2} - \frac{12C}{h^2}\left(\frac{\partial^2 w}{\partial x^2} + K_1 \frac{\partial^2 w}{\partial y^2}\right) + \frac{K}{D_x} = \frac{q}{D_x} \qquad (15)$$

Introducing the notation

$$H = D_1 + 2\, D_{xy}$$

Equation (15) can be written as

$$\frac{\partial^4 w}{\partial x^4} + K_1^2 \frac{\partial^4 w}{\partial y^4} + 2\,\frac{H'}{D_x} \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} - \frac{12\, C}{h^2}\left(\frac{\partial^2 w}{\partial x^2} + K_1 \frac{\partial^2 w}{\partial y^2}\right) + \frac{K}{D_x} = \frac{q}{D_x} \qquad (16)$$

For a slab with two-way reinforcement in the directions x and y, H can be taken as[7]

$$H = (D_x\ D_y)^{\frac{1}{2}}$$

Introducing now

$$x_1 = x$$

$$y_1 = y\left(\frac{D_x}{D_y}\right)^{\frac{1}{4}} \qquad (17)$$

Equation (16) is reduced to the form

$$(\nabla^2 - \alpha^2)\,\nabla^2\ w + \frac{K}{D_x}\, w = \frac{q}{D_x} \qquad (18)$$

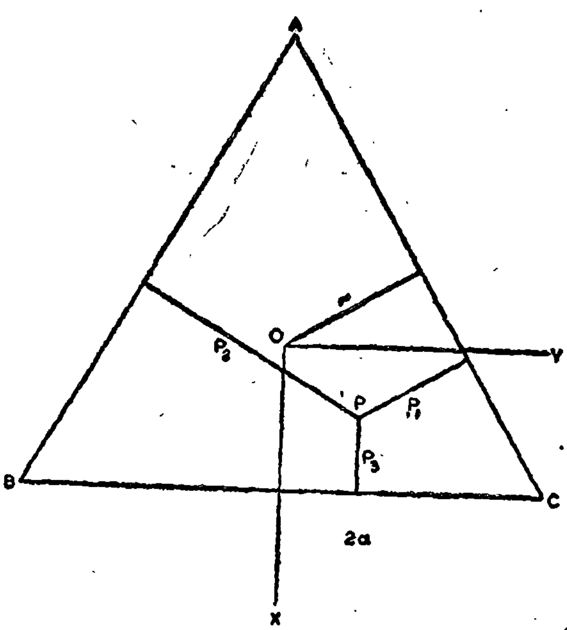

Fig. 1– Equilateral triangular orthotropic plate.

in which

$$\alpha^2 = \frac{12\, C}{h^2}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}$$

SOLUTION OF PROBLEM

Let the plate be in the form of an equilateral triangle, ABC (Fig. 1) having each side of length $2a$. Let the centroid O be the origin, X-axis and Y-axis perpendicular and parallel to the base BC respectively. If x_1, y_1 be the cartesian coordinates of any point, p, within the triangle $p_1\ p_2\ p_3$ be the three perpendiculars from P on CA, AB and BC respectively, and r be the radius of the inscribed circle, then

$$P_1 = r + \frac{x_1}{2} - \frac{y_1\sqrt{3}}{2},$$

$$P_2 = r + \frac{x_1}{2} + \frac{y_1\sqrt{3}}{2},$$

$$P_3 = r - x_1 \,,$$

$$P_1 + P_2 + P_3 = 3r = \sqrt{3}a = K_2 = \text{constant},$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial y_1{}^2}$$

$$= \frac{\partial^2}{\partial P_1{}^2} + \frac{\partial^2}{\partial P_2{}^2} + \frac{\partial^2}{\partial P_3{}^2} - \frac{\partial^2}{\partial P_1 \partial P_2} - \frac{\partial^2}{\partial P_2 \partial P_3} - \frac{\partial^2}{\partial P_3 \partial P_1}$$

Using the trilinear Coordinates[8] p_1, p_2, p_3 the deflection w can be taken in the form

$$w = \sum_{n=1}^{\infty} A_n \left[\sin \frac{2n\pi P_1}{K_2} + \sin \frac{2n\pi P_2}{K_2} + \sin \frac{2n\pi P_3}{K_2} \right] \tag{19}$$

where $A_n =$ a constant.

The above form of w satisfies the following boundary conditions of simply supported edges :

$$w = 0 \text{ at } P_1 = 0, \; P_2 = 0, \; P_3 = 0$$

Expanding the transverse uniform load q, into Fourier Sine series

$$q = \sum_{n=1}^{\infty} \frac{2q}{n\pi} \left[\sin \frac{2n\pi P_1}{K_2} + \sin \frac{2n\pi P_2}{K_2} + \sin \frac{2n\pi P_3}{K_2} \right] \tag{20}$$

and substituting (19) and (20) into (18), we get

$$A_n = \sum_{n=1}^{\infty} \frac{2q}{n\pi D_x} \cdot \frac{1}{\left[\left(\frac{2n\pi}{K_2} \right) + \alpha^2 \left(\frac{2n\pi}{K_2} \right) + \frac{K}{D_x} \right]} \tag{21}$$

To determine α, Equation (6) is transformed into x_1, y_1 coordinates in the following form

$$\frac{\alpha^2 h^2}{12} = \frac{\partial u}{\partial x_1} + (K_1)^{\frac{1}{2}} \frac{\partial v}{\partial y_1} + \frac{1}{2} \left(\frac{\partial w}{\partial x_1} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y_1} \right) \tag{22}$$

The boundary conditions on u and v are

$$u = 0 \text{ at } P_3 = 0 \tag{23}$$

$$\sqrt{3}\, v + u = 0 \text{ at } P_2 = 0 \tag{24}$$

$$\sqrt{3}\, v - u = 0 \text{ at } P_1 = 0 \tag{25}$$

The following forms of u and v satisfy the above boundary conditions.

$$u = \sum_{m=1}^{\infty} \sqrt{3}\, B_m \left[\sin \frac{2_m \pi (P_2 + P_3)}{K_2} + \sin \frac{2_m \pi (P_1 + P_3)}{K_2} \right] \tag{26}$$

$$v = \sum_{m=1}^{\infty} \frac{1}{\sqrt{K_1}} B_m \left[\sin \frac{2_m \pi (P_1 + P_3)}{K_2} - \sin \frac{2_m \pi (P_2 + P_3)}{K_2} \right] \tag{27}$$

in which B_m is a constant.

Substituting the expressions for u, v and w into (22) and integrating over the whole area of the plate, the following equation determining α is obtained:

$$\frac{\alpha^2 h^2}{12} = \sum_{n=1}^{\infty} \frac{3 A_n^2 n^2 \pi^2}{K_2^2} \tag{28}$$

Thus w is completely determined in the following form in x, y coordinates

$$w = A_n \left[2 \sin 2n\pi \left(\frac{1}{3} + \frac{x}{2\sqrt{3}a} \right) \cos \frac{2n\pi y}{2\sqrt{K_1}\, a} + \sin 2n\pi \left(\frac{1}{3} - \frac{x}{\sqrt{3}a} \right) \right] \tag{29}$$

If $D_x = D_y = D$, $\alpha \to 0$, and $K = 0$, (19) and (21) give the deflection equation for an isotropic plate not resting on the elastic foundation in the following form :

$$w = \sum_{n=1}^{\infty} \frac{q K_2^4}{8 n^5 \pi^5 D} \left[\sin \frac{2n\pi P_1}{K_2} + \sin \frac{2n\pi P_2}{K_2} + \sin \frac{2n\pi P_3}{K_2} \right] \tag{30}$$

The corresponding equation as obtained by S. Woinowsky-Krieger for a plate having each side of length $\frac{2a}{\sqrt{3}}$ is

$$w = \frac{q}{64aD} \left[x^3 - 3y^2x - a(x^2+y^2) + \frac{4}{27} a^3 \right] \left(\frac{4}{9} a^2 - x^2 - y^2 \right) \tag{31}$$

At the origin ($P_1 = P_2 = P_3$), w is given by (30) as

$$w = \frac{27\, q\, a^4}{8\pi^5 D} \sum_{n=1}^{\infty} \frac{1}{n^5} \sin \frac{2n\pi}{3} = \cdot 009 \frac{qa^4}{D} \tag{32}$$

which is numerically equal to that obtained from (31) for the plate having each side of length $2a$ as

$$(w)_{x=y=0} = \frac{qa^4}{108D} = \cdot 009 \frac{qa^4}{D}$$

NUMERICAL CALCULATION

To calculate deflection at any point within the plate, we have to start from (28) with an assumed value of ($\alpha\, a$) leading to the corresponding value of the load function $\frac{qa^4}{D_x h}$. Once this relationship is obtained, the corresponding deflection can be obtained from (19) with the help of (21).

At the origin maximum deflection is obtained and is given by

$$\frac{w_{max}}{h} = \frac{6}{\pi} \left(\frac{qa^4}{D_x h} \right) \sum_{n=1}^{\infty} \frac{\sin \frac{2n\pi}{3}}{n \left[\frac{16\pi^4 n^4}{9} + \frac{4\pi^2 n^2 \alpha^2 a^2}{3} + K_F \right]} \tag{33}$$

in which the non-dimensional foundation modulus

$$K = \frac{K a^4}{D_x} \tag{34}$$

For $K_F = 0$ and $K_F = 100$ graphs are plotted in Fig. 2 showing the deflection $\frac{w}{h}$ at the centroid of the plate against the loads. Fig 2. also contains a graph plotted according to the linear theory.

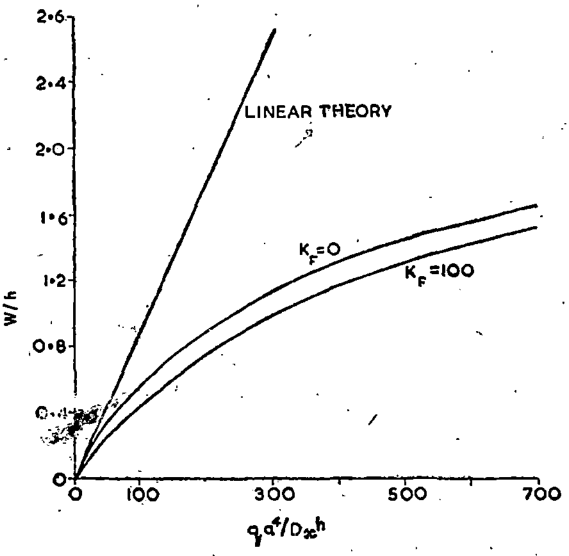

Fig. 2— Deflection curve.

CONCLUSION

From Fig. 2 it is clear that design calculations should be made according to the non-linear theory, because deflections calculated according to small deflection theory will be far from the actual values for higher values of load function. The effect of the foundation is to reduce the deflection for a given value of load function.

Because the deflection, w, has been determined, bending moments and stresses can be computed easily. The bending moments M_x and M_y at the centroid of the plate are obtained as

$$M_x = 4\,(1+\nu_c)\,qa^2 \sum_{n=1}^{\infty} \frac{n \sin \dfrac{2n\pi}{3}}{\left[\dfrac{16\,\pi^4 n^4}{9} + \dfrac{4\,\pi^2 n^2 \alpha^2 a^2}{3} + K\right]} \tag{35}$$

$$M_y = K_1\, M_x \tag{36}$$

ν_c is the Poisson's ratio for concrete.

For isotropic plate without elastic foundation and undergoing small deflection $\nu_c = \nu$, $K_1 = 1$, $K_F = 0$, $\alpha \to 0$ and for a plate having each side of length $\dfrac{2a}{\sqrt{3}}$, (35) and (36) lead to

$$M_x = M_y = (1+\nu)\,\frac{q\,a^2}{54} \tag{37}$$

which is the same result obtained by Timoshenko[7].

ACKNOWLEDGEMENT

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Buckling of a non-homogeneous rectangular plate on elastic foundation

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Critical buckling conditions of a non-homogeneous simply supported rectangular plate under the action of combined bending and compression and placed on an elastic foundation are investigated with the help of the error function. Results obtained are presented in the form of graphs.

Critical buckling conditions of homogeneous thin rectangular plates subjected to combined bending and compression were investigated by Timoshenko and Gere[1], Johnson and Noel[2], and many others. The object of this paper is to use error function to obtain the approximate solutions in the case of buckling of a non-homogeneous thin rectangular plate under the action of combined bending and compression in the middle plane of the plate. The plate is placed on an elastic foundation and is simply supported. Bradley[3] used finite difference approximations to the governing differential equations to investigate stability of equilateral triangular plates. There are other numerical methods for the solutions of these types of buckling problems. But these methods are time consuming.

Since the governing differential equation obtained in this paper cannot be exactly solved, approximate solutions have been sought with the help of error function and by applying Galerkin's principle. It is observed that the results obtained from this method for the homogeneous plate not resting on foundation are in good agreement with the known results obtained by strain energy method. Flexural rigidity of the plate is assumed to vary exponentially and the foundation is taken of the Winkler[4] type. Results obtained have been presented in the form of graphs.

ANALYSIS

Consider a simply supported rectangular plate of varying flexural rigidity, and along whose sides $x = 0$ and $x = a$ (Fig. 1) distributed forces, acting in the middle plane of the plate are applied. The intensity of the forces are given by the equation

$$N_x = N_0 \left(1 - \alpha \frac{y}{b}\right) \tag{1}$$

where N_0 is the intensity of compressive force at edge $y = 0$ and α is a numerical factor. The plate is placed on an elastic foundation having the reaction, K_1 per unit area per unit deflection and is subjected to a uniform transverse load, q.

The governing differential equation of equilibrium of an element of the plate not resting on foundation is (Timoshenko and Krieger[5])

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\left(+q N_x \frac{\partial^2 w}{\partial x^2}\right) \tag{2}$$

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where $$\left.\begin{aligned} M_x &= -D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\\ M_y &= -D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\\ M_{xy} &= D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}\end{aligned}\right\} \quad (3)$$

D being the flexural rigidity, ν the Poisson's ratio, and w the deflection in z-direction. Substituting Eq. (3) in Eq. (2) and observing that the flexural rigidity is a function of the co-ordinates x and y, one gets the differential equation of equilibrium for a plate resting on elastic foundation

$$\nabla^2(D\nabla^2 w) - (1-\nu)\left\{\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 D}{\partial x\partial y}\cdot\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right\} + K_1 w = q + N_x\frac{\partial^2 w}{\partial x^2} \quad (4)$$

For simply supported edges the deflection can be represented by the double series :

$$W = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (5)$$

As the flexural rigidity is variable, let

$$D = D_0\, e^{-2\alpha_1 x/a} \quad (6)$$

where D_0 and α_1 are constants.

Eq. (5) is an approximate solution of Eq. (4) and therefore substitution of Eq. (5) into Eq. (4) results in the following error function, $E_{(x,y)}$

$$E_{(x,y)} = C_{mn} D_0\left[\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}^2 - \frac{4\alpha_1^2}{a^2}\left\{\left(\frac{m\pi}{a}\right)^2+\nu\left(\frac{n\pi}{b}\right)^2\right\}\right] e^{-2\alpha_1 x/a}$$
$$\times\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} + C_{mn}K_1\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} - q$$
$$- C_{mn}N_0\left(\frac{m\pi}{a}\right)^2\left(1-\frac{\alpha}{b}y\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (7)$$

According to Galerkin's principle, the following condition is imposed on the error function, $E_{(x,y)}$

$$\int_0^a\int_0^b E_{(x,y)}\, W(x,y)\, dx\, dy = 0 \quad (8)$$

Substituting Eq. (5) into Eq. (8) and observing that

$$\int_0^b y\sin\frac{i\pi y}{b}\sin\frac{j\pi y}{b}dy = \frac{b^2}{4}\text{ for } i=j=0 \text{ for } i\neq j \text{ and } i\pm j \text{ an even number}$$
$$= -\frac{4b^2}{\pi^2}\frac{ij}{(i^2-j^2)^2}\text{ for } i\neq j \text{ and } i\pm j \text{ an odd number}$$

one gets the following

$$\frac{\pi^6}{2\alpha_1}(1-e^{-2\alpha_1})\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}\left[\pi^2\left(m^2+\frac{n^2a^2}{b^2}\right)^2 - 4\alpha_1^2\left(m^2+\nu\frac{n^2a^2}{b^2}\right)\right]\frac{m^2}{(\pi^2m^2+\alpha_1^2)}$$
$$+\pi^2K_F\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn} - \frac{16qa^4}{D_0}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}$$
$$-\frac{N_0}{D_0}\pi^4a^2\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^2C_{mn} - \frac{\alpha}{2}\sum_{m=1}^{\infty}m^2\left\{\sum_{n=1}^{\infty}C_{mn} - \frac{16}{\pi^2}\sum_{n=1}^{\infty}\sum_{i}C_{mi}\frac{ni}{(n^2-i^2)^2}\right\}\right] = 0 \quad (9)$$

Where the nondimensional foundation modulus,

$K_F = \dfrac{K_1 a^4}{D_0}$ and $n \pm i$ is always odd.

Taking $n = i$, the deflection, W is obtained from Eq. (9)

$$W = \frac{16qa^4}{D_0}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\times\frac{1}{\lambda_{mn}}$$

where,

$$\lambda_{mn} = \left[\frac{\pi^6(1-e^{-2\alpha_1})}{2\alpha_1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\pi^2\left(m^2+\frac{n^2a^2}{b^2}\right)^2 - 4\alpha_1^2\left(m^2+\nu\frac{n^2a^2}{b^2}\right)\right]\frac{m^2}{(\pi^2m^2+\alpha_1^2)} + \pi^2K_F - \frac{N_0\pi^4a^2}{D_0}\left\{\sum_{m=1}^{\infty}m^2\left(1-\frac{\alpha}{2}\right)\right\}\right] \quad (10)$$

From Eq. (9) the critical buckling condition is obtained when

$$\frac{N_0 a^2}{D_0}\Bigg[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^2 C_{mn} - \frac{\alpha}{2}\sum_{m=1}^{\infty} m^2 \Bigg\{\sum_{n=1}^{\infty} C_{mn}$$

$$-\frac{16}{\pi^2}\sum_{n=1}^{\infty}\sum_{i}^{\infty} C_{mi}\ \frac{ni}{(n^2-i^2)^2}\Bigg\}\Bigg]$$

$$= \frac{\pi^2(1-e^{-2\alpha_1})}{2\alpha_1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\Bigg[\pi^2\left(m^2+\frac{n^2a^2}{b^2}\right)^2$$

$$-4\alpha_1^2\left(m^2+\nu\frac{n^2a^2}{b^2}\right)\Bigg]\frac{m^2}{(\pi^2 m^2+\alpha_1^2)}$$

$$+\frac{1}{\pi^2}K_F\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \tag{11}$$

The plate may buckle in such a way that there can be several half-waves in the direction of compression but only one half-wave in the perpendicular direction. For one half wave buckling $m = 1$, for two half-waves buckling $m = 2$ and so on.

If the plate buckles in one half-wave, one gets from Eq. (11) by taking $m=1$ a system of equations of the following kind

$$C_{1n}\Bigg[\frac{(1-e^{-2\alpha_1})}{2\alpha_1(\pi^2+\alpha_1^2)}\Bigg\{\pi^2\left(1+\frac{n^2a^2}{b^2}\right)^2-4\alpha_1^2$$

$$\times\left(1+\nu\frac{n^2a^2}{b^2}\right)\Bigg\}+\frac{K_F}{\pi^4}-\sigma_{cr}\frac{a^2h}{\pi^2 D_0}\left(1-\frac{\alpha}{2}\right)\Bigg]$$

$$-8\,\alpha\,\sigma_{cr}\frac{a^2h}{\pi^4 D_0}\sum_{i}^{\infty} C_{1i}\ \frac{ni}{(n^2-i^2)^2}=0 \tag{12}$$

where $\sigma_{cr} = (N_0)_{cr}/h$, h being the plate thickness.

The lowest root of the determinental equation thus formed will yield the critical buckling load. From the first approximate lowest root, one gets by taking $n = 1$

$$\sigma_{cr} = K\frac{\pi^2 D_0}{b^2 h} \tag{13}$$

where

$$K=\frac{1}{1-\alpha/2}\Bigg[\frac{1-e^{-2\alpha_1}}{2\alpha_1(\pi^2+\alpha_1^2)}\Bigg\{\pi^2\left(\frac{b}{a}+\frac{a}{b}\right)^2$$

$$-4\alpha_1^2\left(\frac{b^2}{a^2}+\nu\right)\Bigg\}+\frac{K_F}{\pi^4}\ \frac{b^2}{a^2}\Bigg]$$

Thus the buckling load is a function of a/b and the foundation modulus, K_F.

For $\alpha = 0$, the critical buckling load, σ_{cr} is obtained from Eq. (11) by taking $n = 1$

$$\sigma_{cr} = K\frac{\pi^2 D_0}{b^2 h} \tag{14}$$

where

$$K=\frac{1-e^{-2\alpha_1}}{2\alpha_1(\pi^2m^2+\alpha_1^2)}\Bigg\{\pi^2\left(m^2\frac{b}{a}+\frac{a}{b}\right)^2$$

$$-4\alpha_1^2\left(m^2\frac{b^2}{a^2}+\nu\right)\Bigg\}+\frac{K_F}{\pi^4 m^2}\ \frac{b^2}{a^2}$$

$$(m = 1, 2, 3,\ldots) \tag{15}$$

For homogeneous material, $D_0 \rightarrow D$ when $\alpha_1 \rightarrow 0$. Setting $\alpha_1 \rightarrow 0$ in Eq. (13) one gets the critical buckling load for a homogeneous plate on elastic foundation for one half-wave buckling

$$\sigma_{cr} = K\frac{\pi^2 D}{b^2 h} \tag{16}$$

where $K=\frac{1}{1-\alpha/2}\left[\left(\frac{b}{a}+\frac{a}{b}\right)^2+\frac{K_F}{\pi^4}\frac{b^2}{a^2}\right]$

For $K_F=0$, Eq. (16) is the result obtained by Timoshenko and Gere[1]. For m half-waves buckling, K in Eq. (16) can be expressed

$$\text{as } K=\frac{1}{m^2}\left(m^2\frac{b}{a}+\frac{a}{b}\right)^2+\frac{K_F}{\pi^4}\frac{1}{m^2}\frac{b^2}{a^2} \tag{17}$$

The ratio a/b for which σ_{cr} becomes a minimum for uniform compression is obtained from Eq. (15) and denoting this ratio by $(a/b)_{cr}$, one gets for a homogeneous plate

$$\left(\frac{a}{b}\right)_{cr}=\frac{1}{\pi}(K_F+\pi^4 m^4)^{1/4} \tag{18}$$

and for a nonhomogeneous plate

$$\left(\frac{a}{b}\right)_{cr}=\left[m^4+\frac{K_F}{\pi^6 m^2}\cdot\frac{2\alpha_1(\pi^2m^2+\alpha_1^2)}{(1-e^{-2\alpha_1})}-\frac{4\alpha_1^2 m^2}{\pi^2}\right]^{1/4} \tag{19}$$

The ratio a/b at which the transition from m to $m+1$ half-waves buckling occur can also be computed from Eq. (15). For homogeneous plate under uniform compression, transition from one to two half-waves occurs when

$$\frac{a}{b}=\left(4-\frac{K_F}{\pi^4}\right)^{1/4}$$

and transition from two to three half-waves occurs when

$$\frac{a}{b} = \left(36 - \frac{K_F}{\pi^4}\right)^{1/4}$$

Eq. (13) gives satisfactory results for small values of α. An improved result is obtained by taking two equations of the system Eq. (12) with coefficients C_{11} and C_{12} and setting the determinant equal to zero. Thus for one half-wave buckling

$$\sigma_{cr} = K\frac{\pi^2 D_0}{b^2 h} \tag{20}$$

where

$$K = \left[\frac{1}{\{\cdot0065\,\alpha^2 - \cdot2\,(1-\alpha/2)^2\}} \times \left\{\cdot01\left(1-\frac{\alpha}{2}\right)^2 \times \left(B + \frac{2K_F}{\pi^4}\right)^2 + \left[\cdot0013\,\alpha^2 - \cdot041\left(1-\frac{\alpha}{2}\right)^2\right] \times \left[A + B\frac{K_F}{\pi^4} + \frac{K_F^2}{\pi^8}\right]\right\}^{1/2} - \frac{(1-\alpha/2)\,(B + 2K_F/\pi^4)}{\cdot064\,\alpha^2 - 2\,(1-\alpha/2)^2}\right] \times \frac{b^2}{a^2}$$

and

$$A = \left\{\frac{1-e^{-2\alpha_1}}{2\alpha_1(\pi^2+\alpha_1^2)}\right\}^2 \left\{\pi^2\left(1 + \frac{a^2}{b^2}\right)^2 - 4\alpha_1^2\left(1 + \nu\frac{a^2}{b^2}\right)\right\}\left\{\pi^2\left(1 + \frac{4a^2}{b^2}\right)^2 - 4\alpha_1^2\left(1 + 4\nu\frac{a^2}{b^2}\right)\right\}$$

$$B = \frac{1-e^{-2\alpha_1}}{2\alpha_1(\pi^2+\alpha_1^2)}\left\{\pi^2\left(2 + 10\,\frac{a^2}{b^2} + 17\frac{a^4}{b^4}\right) - 4\,\alpha_1^2\left(2 + 5\,\nu\frac{a^2}{b^2}\right)\right\}$$

For pure bending when $\alpha = 2$ Eq. (20) reduces to

$$\sigma_{cr} = \frac{\pi^2 D_0}{b^2 h}\left[2.77\,\frac{b^2}{a^2}\left(A + B\,\frac{K_F}{\pi^4} + \frac{K_F^2}{\pi^8}\right)^{1/2}\right] \tag{21}$$

Setting $\alpha_1 \rightarrow 0$ in Eq. 21 one gets the buckling load under pure bending for a homogeneous plate for one half-wave buckling

$$\sigma_{cr} = K\frac{\pi^2 D}{b^2 h} \tag{22}$$

where

$$K = 2{\cdot}77\,\frac{b^2}{a^2}\left[\left(1 + \frac{a^2}{b^2}\right)^2\left(1 + \frac{4a^2}{b^2}\right)^2 + \left(2 + 10\,\frac{a^2}{b^2} + 17\,\frac{a^4}{b^4}\right)\frac{K_F}{\pi^4} + \frac{K_F^2}{\pi^8}\right]^{1/2}$$

For m half-wave buckling K in Eq. (22) can be written as

$$K = \frac{1}{m^2} \times 2.77\frac{b^2}{a^2}\left[\left(m^2 + \frac{a^2}{b^2}\right)^2\left(m^2 + \frac{4a^2}{b^2}\right)^2 + \left(2m^4 + 10m^2\,\frac{a^2}{b^2} + 17\,\frac{a^4}{b^4}\right)\frac{K_F}{\pi^4} + \frac{K_F^2}{\pi^8}\right]^{1/2} \tag{23}$$

The presence of the foundation modulus, K_F in Eq. (10) reduces the deflection, w and hence the bending stress. Therefore Eqs. (20), (21) and (22) resulting from two term approximation of Eq. (12) may be taken as fairly accurate.

RESULTS

Numerical results are obtained for two cases (a) when the plate is under uniform compression, and (b) when it is under pure bending. For uniform compression of a homogeneous plate, the values of K are calculated for different values of a/b with the help of Eq. (16) for one half-wave buckling taking $K_F = \pi^4$, and $\alpha = 0$. For two half-waves and three half-waves buckling Eq. (17) is used with the same value of K_F. These results are presented in the form of graphs in Fig. (2). For a non homogeneous plate under uniform compression, the values of K are calculated for one half-wave, two, and three half-waves buckling by taking the same value of K_F and $\alpha_1 = 0{\cdot}1$ in Eq. (15). These results are presented in Fig. 3. The values of K when the plates are not on the foundation are also presented in Figs. 2 and 3 for comparison.

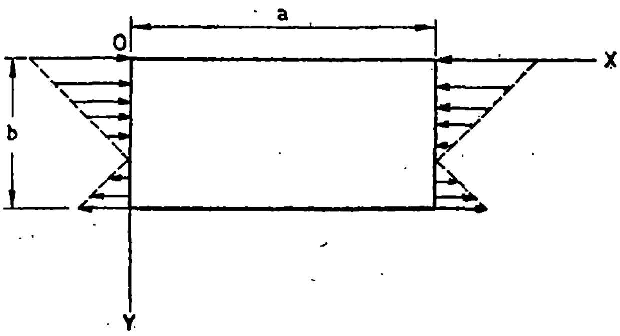

FIG. 1 RECTANGULAR PLATE

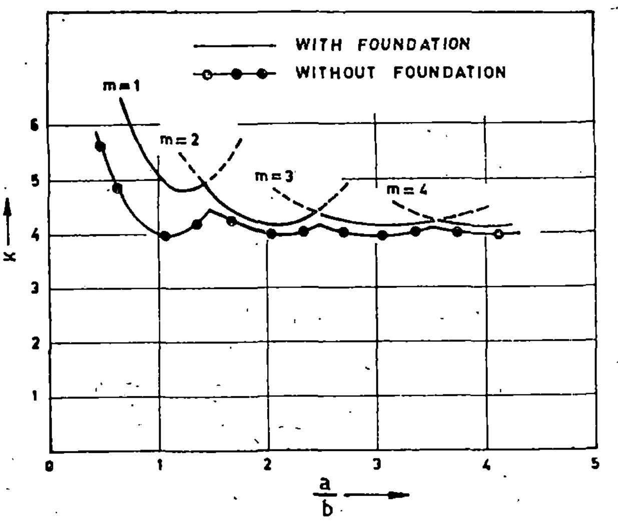

FIG. 2 HOMOGENEOUS PLATE UNDER UNIFORM COMPRESSION

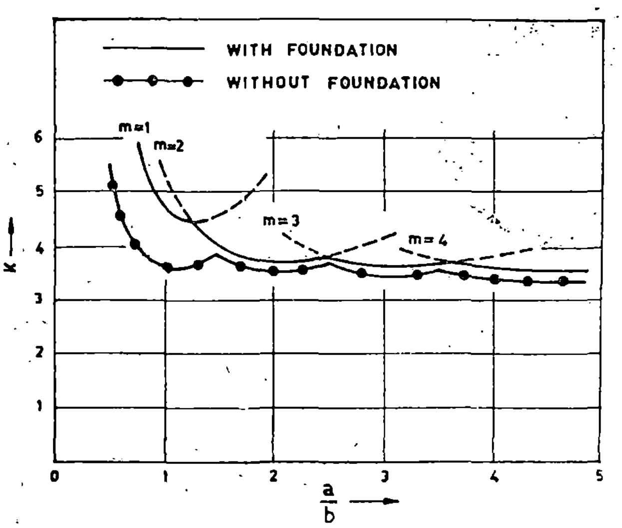

FIG. 3 NON-HOMOGENEOUS PLATE UNDER UNIFORM COMPRESSION

For pure bending of a homogeneous plate Eq. (23) is used for calculation of K for different values of a/b taking $K_F = \pi^4$ and the results are presented in Fig. 4. In the same figure the corresponding results for K without foundation are also presented for comparison.

CONCLUSIONS

From the foregoing analysis and from Figs. 2, 3 and 4 the following conclusions may be drawn:

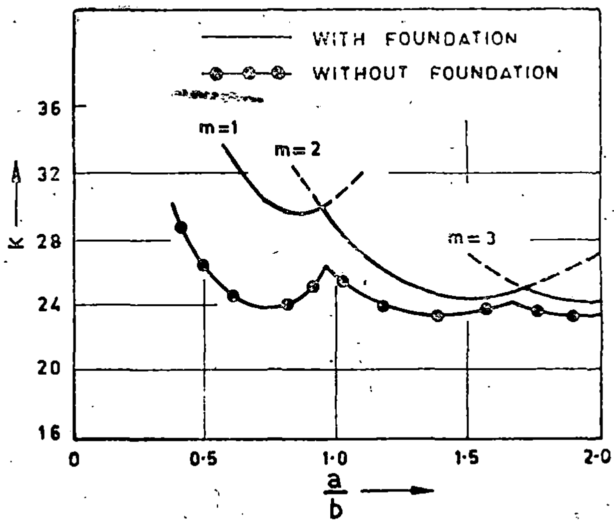

FIG. 4 HOMOGENEOUS PLATE UNDER PURE BENDING

(i) Foundation increases the buckling load

(ii) Resistance offered by the foundation is more for one half-wave buckling compared to multiple half-waves buckling. When buckling is in more than one half-wave, the foundation resistance remains practically constant.

(iii) Foundation increases the $(a/b)_{cr}$ ratio and reduces the a/b ratio at which transition takes place as compared to a plate not resting on foundation.

(iv) A non-homogeneous plate will have lower buckling load as compared to a homogeneous plate.

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NOTATION

a, b	Length and breadth of the plate respectively
D	Flexural rigidity of the plate $= \dfrac{Eh^3}{12(1-\nu^2)}$
E	Modulus of elasticity
h	Plate thickness
K	A numerical factor
q	Lateral distributed load
W	Deflection in z-direction

x, y	Cartesian co-ordinates
D_0	Flexural rigidity of the plate at the edge $x=0$
K_1	Foundation reaction per unit area per unit deflection
K_F	Non-dimensional foundation modulus $=\frac{K_1}{D_0}a^4$
N_x	Normal force resultants per unit length in middle plane of the plate
N_0	Intensity of compressive force at the edge $y=0$
α, α_1	Numerical constants
$(N_0)_{cr}$	Critical buckling load
σ_{cr}	Critical buckling stress
ν	Poisson's ratio

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Large Deflection of a Circular Plate on Elastic Foundation under Symmetrical Load

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ABSTRACT

The large deflection of a clamped circular plate on elastic foundation under nonuniform but symmetrical loads has been investigated following Berger's approximate method. The deflections are obtained in the form of an infinite series involving Bessel functions. Graphs are plotted for deflections, bending moments, and bending stresses for various values of foundation modulus and load functions.

INTRODUCTION

Timoshenko and Woinowsky-Krieger [1] and several other authors have examined small deflections of thin plates on elastic foundations on the assumption that the strains of the middle plane of the plate can be neglected. When the deflection is moderately large, that is, on the order of the thickness of the plate, then the strain of the middle plane of the plate must be considered. In that case the analytical solution of the differential equations becomes difficult because of their nonlinear character. Way [2] and many other authors have examined moderately large deflections of plates not resting on elastic foundations, and the methods used by them involve considerable computation.

Berger [3] has suggested that the strain energy due to the second strain invariant of the middle surface strains may be neglected in analyzing moderately large deflection of plates having axisymmetric deformation. Berger's technique reduces the computational effort considerably, yet the stresses and deflections obtained for both rectangular and circular plates are in good agreement with those found by exact analysis. Berger's method has been extended by Nowinski [4] to the case of orthotropic plates. Nash and Modeer [5] have investigated problems without axial symmetry by using Berger's technique. The same approximate method has also been applied by Sinha [6] to determine the moderately large static deflections of circular and rectangular plates resting on elastic foundations and under uniform load distribution.

In this paper we study moderately large static deflections of circular plates on elastic foundation and subjected to special classes of symmetrical transverse loads, which are distributed over a concentric circular portion of the plate. Deflections, bending moments, and bending stresses are calculated for different values of foundation modulus, and these are presented in the form of graphs.

FORMULATION OF THE PROBLEM

For moderately large deflections the total potential energy of the system is given by

$$V = \frac{D}{2}\iint\left[(\nabla^2 W)^2 + \frac{12}{h^2}e_1^2 - 2(1-\nu)\left(W_{xx}W_{yy} - W_{xy}^2 + \frac{12}{h^2}e_2\right) + \frac{K}{D}W^2 - \frac{2qW}{D}\right]dx\,dy \tag{1}$$

in which the last two terms in the integrand represent the potential energy of the foundation and of the applied load, respectively, and e_1 and e_2 are the first and second invariants of the membrane strains. If, following Berger [3], e_2 is neglected, then the variation of V with respect to the in-plane displacements leads to the drastic simplification that e_1 is constant.

In polar coordinates, and under the assumption of circular symmetry, the governing equations become

$$e_1 \equiv \frac{du}{dr} + \frac{u}{r} + \frac{1}{2}\left(\frac{dW}{dr}\right)^2 = \frac{\alpha^2 h^2}{12} = \text{const} \tag{2}$$

$$\nabla^4 W - \alpha^2 \nabla^2 W + \frac{K}{D} W = \frac{q}{D} \tag{3}$$

in which

$$\nabla^2 \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \tag{4}$$

To these equations must be added a suitable set of boundary conditions. As has been observed in a recent note by Nowinsky and Ohnabe [10], the present simplified method leads to acceptable results if these boundary conditions involve fixity against in-plane displacements. This assumption has been adopted in the following examples.

SOLUTION OF PROBLEM

Let us consider a clamped circular plate of radius a, with the center of the plate taken as the origin. Let there be a symmetrical distribution of transverse load varying as $(b^2 - r^2)^\lambda$, $(\lambda > -1)$, over a concentric circular area of radius $b < a$. Hence

$$\frac{q}{D} = \begin{cases} f(r) = C(b^2 - r^2)^\lambda & (r < b < a) \\ 0 & (b < r < a) \end{cases} \tag{5}$$

and Eq. (3) now becomes

$$(\nabla^2 - \alpha^2)\nabla^2 W + \frac{K}{D} W = f(r) \tag{6}$$

The boundary conditions for clamped edges are

$$(W)_{r=a} = 0 = \left(\frac{dW}{dr}\right)_{r=a} \tag{7}$$

Let us now assume the deflection W in the form

$$W = \sum_{S=1}^{\infty} A_s[J_0(P_s r) - J_0(P_s a)] \tag{8}$$

where J_0 is the Bessel function of the first kind and zero order and P_s is the Sth root of $J_1\ (Pa) = 0$, J_1 being the Bessel function of the first kind and first order.

This automatically satisfies the boundary conditions for clamped edges. Since

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right] J_0(P_s r) = -P_s^2 J_0(P_s r)$$

substitution of Eq. (8) in Eq. (6) leads to

$$\sum_{S=1}^{\infty} A_s\left[P_s^4 J_0(P_s r) + \alpha^2 P_s^2 J_0(P_s r) + \frac{K}{D}\{J_0(P_s r) - J_0(P_s a)\}\right] = f(r) \tag{9}$$

or, by expanding $f(r)$ in a series of Bessel functions,

$$A_s \frac{a^2}{2}\left[P_s^2(P_s^2 + \alpha^2) + \frac{K}{D}\right] J_0^2(P_s a) = \int_0^a f(r) J_0(P_s r) r\, dr \tag{10}$$

Setting $r = b \sin\theta$ and $f(r) = C(b^2 - r^2)^\lambda$ in the integral of (10) one obtains

$$\begin{aligned}\int_0^a f(r) J_0(P_s r) r\, dr &= \int_0^b Cr(b^2 - r^2)^\lambda J_0(P_s r)\, dr \\ &= Cb^{2(\lambda+1)} \int_0^{\pi/2} \sin\theta \cos^{2\lambda+1}\theta\, J_0(P_s b \sin\theta)\, d\theta \\ &= \frac{Cb^{2(\lambda+1)} J_{\lambda+1}(P_s b)\, 2^\lambda(\lambda + 1)}{(P_s b)^{\lambda+1}}\end{aligned} \tag{11}$$

This is a special form of Sonine's first definite integral containing Bessel function [7], where $\lambda > -1$. Finally, with the value obtained from Eq. (11) used in Eq. (10) one gets, after simplification,

$$A_s = \frac{C(2b)^{\lambda+1} J_{\lambda+1}(P_s b)\Gamma(\lambda + 1)}{a^2\left[P_s^{\lambda+3}(P_s^2 + \alpha^2) + P_s^{\lambda+1}\dfrac{K}{D}\right] J_0^2(P_s a)} \tag{12}$$

and hence

$$W = \frac{C(2b)^{\lambda+1}\Gamma(\lambda+1)}{a^2} \sum_{S=1}^{\infty} \frac{J_{\lambda+1}(P_s b)[J_0(P_s r) - J_0(P_s a)]}{\left[P_s^{\lambda+3}(P_s^2+\alpha^2) + P_s^{\lambda+1}\dfrac{K}{D}\right] J_0^2(P_s a)} \tag{13}$$

Except for the as yet unknown value of α this determines the deflection curve $W(r)$, including the maximum deflection

$$\begin{aligned} W_{\max} = W(0) &= \frac{C(2b)^{\lambda+1}\Gamma(\lambda+1)}{a^2} \\ &\times \sum_{S=1}^{\infty} \frac{J_{\lambda+1}(P_s b)[1 - J_0(P_s a)]}{\left[P_s^{\lambda+3}(P_s^2+\alpha^2) + P_s^{\lambda+1}\dfrac{K}{D}\right] J_0^2(P_s a)} \end{aligned} \tag{14}$$

To determine the displacement u we obtain, from Eqs. (2) and (8),

$$\begin{aligned} \frac{du}{dr} + \frac{u}{r} &= \frac{\alpha^2 h^2}{12} - \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \\ &= \frac{\alpha^2 h^2}{12} - \frac{1}{2}\sum_{S=1}^{\infty} A_s^2 P_s^2 J_1^2(P_s r) \\ &\quad - \frac{1}{2}\sum_{S=1}^{\infty}\sum_{\substack{m=1,\\ S\neq m}}^{\infty} A_s A_m P_s P_m J_1(P_s r) J_1(P_m r) \end{aligned} \tag{15}$$

and, by multiplying Eq. (15) by r and integrating with respect to r,

$$\begin{aligned} ru = {} & \frac{\alpha^2 h^2 r^2}{24} - \frac{1}{2}\sum_{S=1}^{\infty} A_s^2 P_s^2 \left[\frac{r^2}{2}\left\{\left(1 - \frac{1}{P_s^2 r^2}\right) J_1'^2(P_s r) + J_1'^2(P_s r)\right\}\right] \\ & - \frac{1}{2}\sum_{S=1}^{\infty}\sum_{\substack{m=1\\ S\neq m}}^{\infty} A_s A_m P_s P_m \\ & \times \left[r\left\{\frac{P_s J_2(P_s r) J_1(P_m r) - P_m J_1(P_s r) J_2(P_m r)}{P_s^2 - P_m^2}\right\}\right] + K' \end{aligned} \tag{16}$$

where K' is the constant of integration, whose value is determined from the boundary condition

$$(u)_{r=a} = 0 \tag{17}$$

Thus

$$K' = \frac{1}{4}\sum_{S=1}^{\infty} A_s^2 P_s^2\, a^2 J_0^2(P_s a) - \frac{\alpha^2 h^2 a^2}{24} = 0 \tag{18}$$

where the second equality follows from the regularity condition

$$(u)_{r=0} = 0 \tag{19}$$

Hence

$$\frac{\alpha^2 h^2}{6} = \sum_{S=1}^{\infty} A_s^2 P_s^2 J_0^2(P_s a) \tag{20}$$

determines the value of α. For example, let

$$\lambda = \tfrac{1}{2}, \qquad K = 0 \tag{21}$$

then the deflection W is given by

$$\begin{aligned} W &= \sum_{S=1}^{\infty} A_s[J_0(P_s r) - J_0(P_s a)] \\ &= \frac{2b^3 C}{a^2}\sum_{S=1}^{\infty} \frac{Q(P_s b)}{P_s^2(P_s^2 + \alpha^2)J_0^2(P_s a)}[J_0(P_s r) - J_0(P_s a)] \end{aligned} \tag{22}$$

in which

$$Q(P_s b) = \frac{1}{3}\left[1 - \frac{P_s^2 b^2}{2 \times 5} + \frac{P_s^4 b^4}{2 \times 4 \times 5 \times 7} - \cdots\right]$$

As is common in Berger's approximation the large deflection effect is contained entirely in the value of α. With $\alpha = 0$, Eq. (22) agrees with the result obtained by Sen [8] for the corresponding small deflection problem.

Another type of transverse load function to be considered is given by

$$f(r) = \begin{cases} C(r^4 - b^4) & (0 \leqq r \leqq b < a) \\ 0 & (b \leqq r \leqq a) \end{cases} \tag{23}$$

Expanding $f(r)$ in a series of Bessel functions and proceeding in the same manner we find

$$A_s = \frac{32bC(4 - P_s^2b^2)J_1(P_sb)}{a^2\left[P_s^7(P_s^2 + \alpha^2) + P_s^5\frac{K}{D}\right]J_0^2(P_sa)} \tag{24}$$

or

$$W = \frac{32bC}{a^2}\sum_{S=1}^{\infty}\frac{(4 - P_s^2b^2)J_1(P_sb)[J_0(P_sr) - J_0(P_sa)]}{\left[P_s^7(P_s^2 + \alpha^2) + P_s^5\frac{K}{D}\right]J_0^2(P_sa)} \tag{25}$$

The central deflection is obtained putting $r = 0$, that is,

$$W_{\max} = \frac{32bC}{a^2}\sum_{S=1}^{\infty}\frac{(4 - P_s^2b^2)J_1(P_sb)[1 - J_0(P_sa)]}{\left[P_s^7(P_s^2 + \alpha^2) + P_s^5\frac{K}{D}\right]J_0^2(P_sa)} \tag{26}$$

Once again, $u(r)$ and α are found by substituting Eq. (24) in Eqs. (16), (18), and (20).

With W a function of r only, the radial bending moment is

$$M_r = -D\left[\frac{d^2W}{dr^2} + \nu\left(\frac{1}{r}\frac{dW}{dr}\right)\right] \tag{27}$$

Considering Eqs. (13) and (27) the value for the bending moment for the type of loading in Eq. (5) then becomes

$$M_r = \frac{DC(2b)^{\lambda+1}\Gamma(\lambda + 1)}{a^2}\sum_{S=1}^{\infty}\frac{P_sJ_{\lambda+1}(P_sb)}{\left[P_s^{\lambda+3}(P_s^2 + \alpha^2) + P_s^{\lambda+1}\frac{K}{D}\right]J_0^2(P_sa)}$$
$$\times\left[P_sJ_0(P_sr) + (\nu - 1)\frac{1}{r}J_1(P_sr)\right] \tag{28}$$

For clamped edges the bending moment is a maximum at the center, that is,

$$(M_r)_{\max} = \frac{DC(2b)^{\lambda+1}\Gamma(\lambda + 1)}{a^2} \times \frac{1 + \nu}{2}$$
$$\times \sum_{S=1}^{\infty}\frac{P_s^2J_{\lambda+1}(P_sb)}{\left[P_s^{\lambda+3}(P_s^2 + \alpha^2) + P_s^{\lambda+1}\frac{K}{D}\right]J_0^2(P_sa)} \tag{29}$$

The maximum bending stress is given by

$$(\sigma_r)_{\max} = -\frac{6}{h^2}(M_r)_{\max} \tag{30}$$

Hence, in summary, the maximum deflection, bending moment, and bending stress for $\lambda = 1$ are as follows:

$$W_{\max} = \frac{4b^2C}{a^2} \sum_{S=1}^{\infty} \frac{J_2(P_sb)[1 - J_0(P_sa)]}{\left[P_s^4(P_s^2 + \alpha^2) + P_s^2 \frac{K}{D}\right] J_0^2(P_sa)} \tag{31}$$

$$(M_r)_{\max} = \frac{2DCb^2}{a^2}(1 + \nu) \sum_{S=1}^{\infty} \frac{J_2(P_sb)}{\left[P_s^2(P_s^2 + \alpha^2) + \frac{K}{D}\right] J_0^2(P_sa)} \tag{32}$$

$$(\sigma r)_{\max} = -\frac{12DCb^2}{a^2h^2}(1 + \nu) \sum_{S=1}^{\infty} \frac{J_2(P_sb)}{\left[P_s^2(P_s^2 + \alpha^2) + \frac{K}{D}\right] J_0^2(P_sa)} \tag{33}$$

For small deflections ($\alpha = 0$) and for $K = 0$:

$$W_{\max} = \frac{4b^2C}{a^2} \sum_{S=1}^{\infty} \frac{J_2(P_sb)[1 - J_0(P_sa)]}{P_s^6 J_0^2(P_sa)} \tag{34}$$

$$(M_r)_{\max} = \frac{2DCb^2}{a^2}(1 + \nu) \sum_{S=1}^{\infty} \frac{J_2(P_sb)}{P_s^4 J_0^2(P_sa)} \tag{35}$$

$$(\sigma r)_{\max} = -\frac{12DCb^2}{a^2h^2}(1 + \nu) \sum_{S=1}^{\infty} \frac{J_2(P_sb)}{P_s^4 J_0^2(P_sa)} \tag{36}$$

The results are now used for the numerical computations and evaluation.

NUMERICAL RESULTS

Numerical results are presented here for the case of the circular plate with clamped edge. The type of load function considered is as in Eq. (5), with $\lambda = 1$ and $a = 2b$. The maximum deflections and bending stresses are calculated for various values of the load and for various values of the foundation modulus. These are presented in the form of graphs. Central deflection and maximum bending stresses are also calculated for small deflections, and these are also presented in the form of graphs for comparison. Variation of the bending moment along the radius is also calculated both for small deflection and large deflection.

In calculating the central deflection we start from Eq. (20), with an assumed value of (αa) leading to a particular value of the load. Once this relationship is obtained the maximum value of the deflection can be obtained from Eq. (31) for various values of the foundation modulus. These results are presented in Fig. 2. An examination of Eq. (31) reveals that as the radius of the plate in-

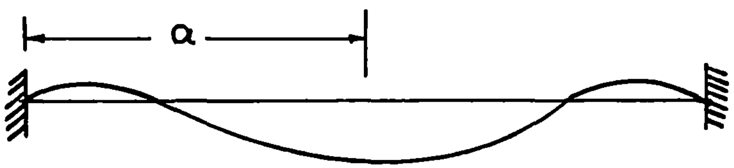

Fig. 1 Deflected plate shape.

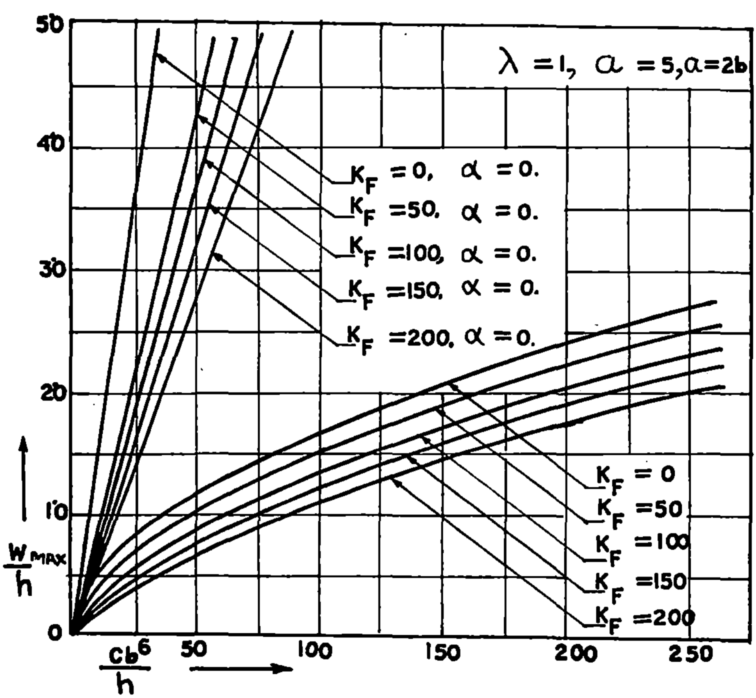

Fig. 2 Deflection.

creases, the central deflection also increases for given load. For small deflection Eq. (34) is to be used for the calculation of the central deflection.

In calculating the bending moment for various values of (r/a), Eq. (28) is used, with $\lambda = 1$. The variation of the bending moment along the radius of the plate is presented in Fig. 3. Variation of the bending moment along the radius according to the linear theory can be calculated with the help of Eq. (28) by letting $\alpha = 0$ and $\lambda = 1$. The maximum bending stresses both for large and small deflection and for various values of foundation modulus can be calculated with the help of Eqs. (33) and (36). These values are presented in Fig. 4.

For the type of loading in Eq. (23) the central deflection for various values of the load and foundation modulus are calculated with the help of Eq. (26) in conjunction with the corresponding equation for α. Values of the bending

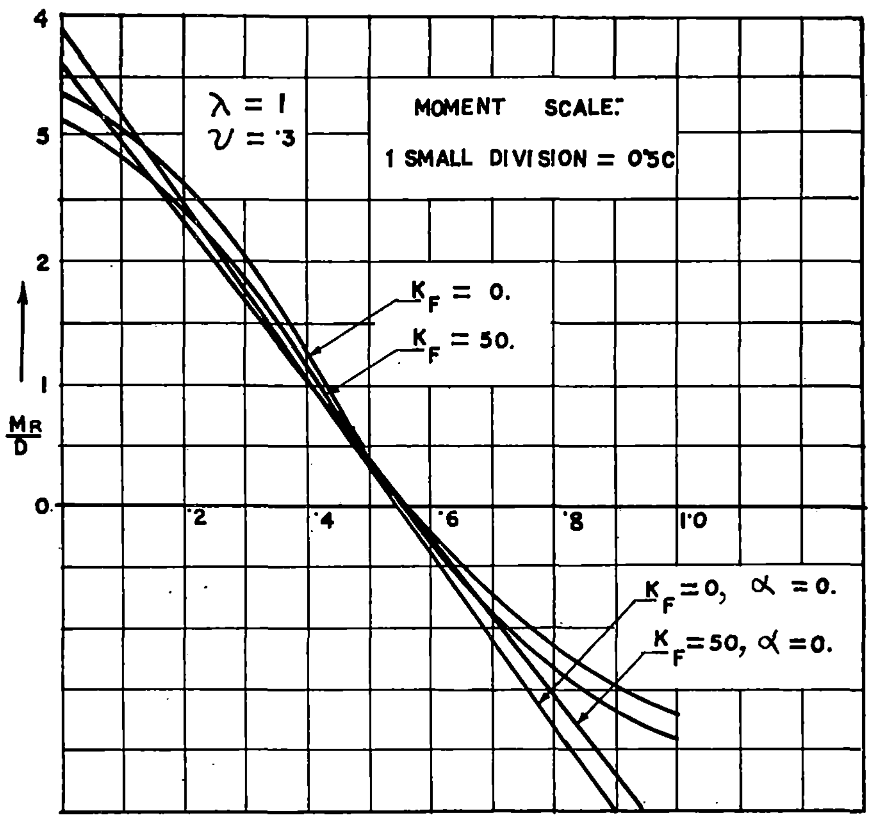

Fig. 3 Bending moment.

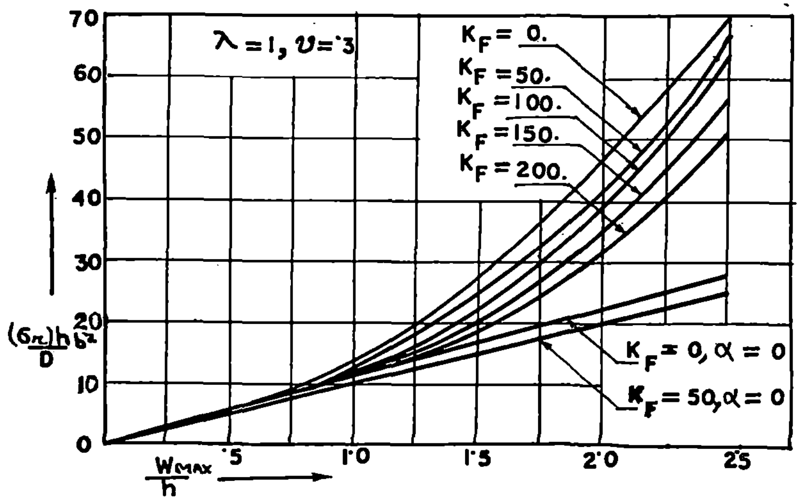

Fig. 4 Stresses.

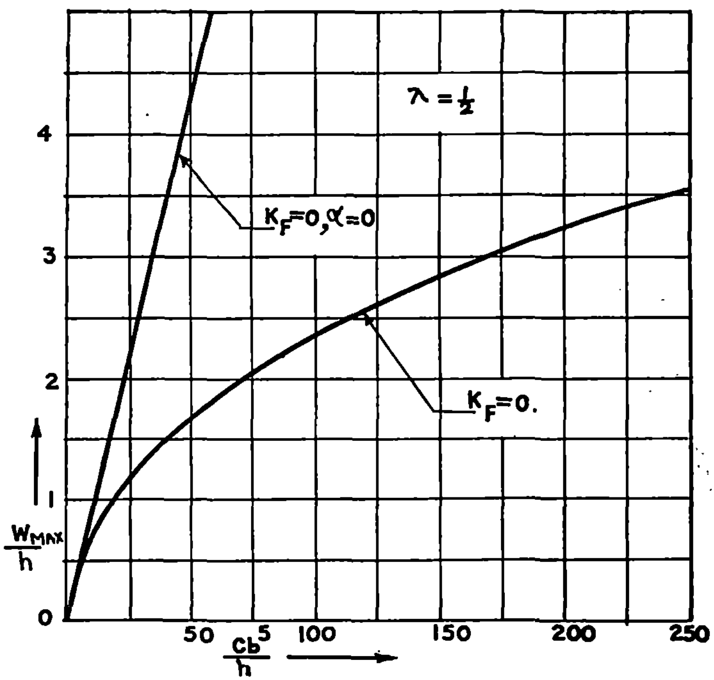

Fig. 5 Deflection.

moment and bending stresses can also be calculated from equations similar to (28), (32), and (33).

The results obtained are in good agreement with those found by other authors (where applicable). This includes the case of $\lambda = 1/2$ (see Eq. (5)) and $K_F = 0$ as shown in Fig. 5, which was previously treated by Banerjee [9].

APPENDIX—NOTATION

The following symbols have been adopted:

a radius of the plate
b a constant less than a
C a constant
D flexural rigidity of the plate $= \dfrac{Eh^3}{12(1 - \nu^2)}$
E Young's modulus
e_1 first invariant of middle surface strains
e_2 second invariant of middle surface strains
h thickness of plate
J_0 Bessel function of the first kind and zero order
J_1 Bessel function of the first kind and first order
J_2 Bessel function of the first kind and second order
K foundation reaction per unit area per unit deflection
q load
r, θ polar coordinates
u radial displacement
V strain energy
W deflection of plate in Z-direction
x, y rectangular coordinates
ε strain in middle surface
K_F foundation modulus $= \dfrac{K}{D} a^4$
ν Poisson's ratio
Γ gamma function

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LARGE AMPLITUDE FREE VIBRATIONS OF IRREGULAR PLATES PLACED ON AN ELASTIC FOUNDATION

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Abstract—A unified method for determining the lowest natural frequency of large amplitude free vibrations of thin elastic plates of any shape and placed on elastic foundation is given. The conformal mapping technique is introduced and Galerkin's method is used to calculate approximate values of the lowest natural frequency. Time periods for circular, square and cornered plates placed on elastic foundation have been determined for simply supported and clamped edge boundary conditions. Practical values have also been determined experimentally. The results are presented in the form of graphs and they are compared with other known results.

INTRODUCTION

An approximate method for investigating the large deflection of initially flat isotropic plates has been proposed by Berger [1]. Essentially, this method is based on the neglect of the second invariant of the middle surface strains in the expression corresponding to the total potential energy of the system. An application of this technique to the case of orthotropic plates has been offered by Iwinski and Nowinski [2] and further boundary value problems associated with circular and rectangular plates have been investigated by Nowinski [3]. Sinha [4] applied this method to investigate large deflections of circular and rectangular plates placed on elastic foundation. Nash and Modeer [5] found the large amplitude free vibrations of rectangular and circular plates by applying the technique offered by Berger.

In this paper a unified method for determining the lowest natural frequency of large amplitude free vibrations of thin elastic plates of any shape and placed on an elastic foundation is given. Following Berger's method a simple fourth-order differential equation coupled with a second-order non-linear equation is obtained. If the boundary of the plate is a curve natural to any of the common coordinate systems, the solution of the differential equation can be expressed in terms of known functions. For more unusual boundaries, the natural coordinates must first be determined and after this is done, the solution would inevitably involve some unfamiliar functions. The determination of natural frequencies in this case will then be very complicated. Therefore a common coordinate system and its associated functions is used for the case of plates with complicated boundaries.

In order to satisfy the prescribed boundary conditions, the domain is conformally transformed on to a unit circle. Once the transformation function is known, the problem is reduced to the solution of the transformed differential system. In this paper Galerkin's method is used to solve the transformed equation.

The ratio of time periods for circular, square and cornered plates placed on an elastic foundation have been determined for simply supported and clamped edge boundary conditions. The foundation is assumed to be of the Winkler type. Experimental values are also obtained for circular and square plates under both boundary conditions. The results are presented in the form of graphs and they are compared with other known results.

THEORY

Let us consider the large amplitude free vibrations of a thin elastic plate placed on an elastic foundation having the reaction k' per unit area per unit deflection.

The strain energy V_1, due to bending and stretching of the middle surface of the deflected plate, may be written in the Cartesian coordinates in the form [6]

$$V_1 = \frac{D}{2}\int\int\left[\left\{(\nabla^2 w)^2 + \frac{12}{h^2}e_1^2\right\} - 2(1-\nu)\left\{\frac{12}{h^2}e_2 + \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}\right]\mathrm{d}x\,\mathrm{d}y \tag{1}$$

in which D is the flexural rigidity of the plate given by $Eh^3/12(1-\nu^2)$, E being the modulus of elasticity, h the thickness of the plate, ν Poisson's ratio, and w the deflection in the direction normal to the middle plane. Also, e_1, the first invariant of the middle surface strains, is defined by the relation

$$e_1 = \varepsilon_x + \varepsilon_y = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{2}$$

$$\left.\begin{aligned}\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\ \varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\end{aligned}\right\} \tag{3}$$

e_2, the second invariant, is defined by

$$e_2 = \varepsilon_x \varepsilon_y - \tfrac{1}{4}\gamma_{xy}^2 \tag{4}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}; \tag{5}$$

and u, v are the displacements in the x and y directions, respectively.

By adding the potential energy of the foundation reaction to equation (1) and neglecting e_2 one gets

$$V = \frac{D}{2}\int\int\left[\left\{(\nabla^2 w)^2 + \frac{12}{h^2}e_1^2\right\} - 2(1-\nu)\left\{\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\} + \frac{k'}{D}w^2\right]\mathrm{d}x\,\mathrm{d}y. \tag{6}$$

The kinetic energy, T, of the vibrating plate is

$$T = \frac{\rho h}{2}\int\int(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\,\mathrm{d}x\,\mathrm{d}y \tag{7}$$

in which ρ is the density of the plate material and $\dot{u}, \dot{v}, \dot{w}$ are derivations with respect to time.

Neglecting the inertia effects in the plane of the plate and applying Euler's variational equations to equations (6) and (7), one gets the following differential equation for w [5]

$$\nabla^4 w - \alpha^2 F^2(t)\nabla^2 w + \frac{12}{h^2 c_\rho^2}\cdot\frac{\partial^2 w}{\partial t^2} + \frac{k'}{D}w = 0 \tag{8}$$

in which

$$c_\rho^{-2} = \frac{\rho h^3}{12D}; \qquad \frac{\alpha^2 h^2}{12}f(t) = e_1, \qquad \text{and} \qquad f(t) = F^2(t). \tag{9}$$

Let

$$w = w(x, y)F(t). \tag{10}$$

Combining equations (8) and (10) one finds

$$F(t)\nabla^4 w - \alpha^2 F^3(t)\nabla^2 w + \frac{12}{h^2 c_\rho^2}\cdot\frac{\mathrm{d}^2 F}{\mathrm{d}t^2}w + \frac{k'}{D}wF(t) = 0. \tag{11}$$

Equation (11) may be written as

$$\left(\frac{\nabla^4 w}{w} + \frac{k'}{D}\right)F(t) - \alpha^2 F^3(t)\frac{\nabla^2 w}{w} + \frac{12}{h^2 c_\rho^2}\cdot\frac{\mathrm{d}^2 F}{\mathrm{d}t^2} = 0. \tag{12}$$

A solution of equation (12) is possible if

$$\frac{\nabla^4 w}{w} = k^4 \tag{13a}$$

and

$$\frac{\nabla^2 w}{w} = -k^2 \tag{13b}$$

in which k is a constant. From equation (13a)

$$(\nabla^2 - k^2)(\nabla^2 + k^2)w = 0. \tag{14a}$$

From equation (13b)

$$(\nabla^2 + k^2)w = 0. \tag{14b}$$

Therefore a solution of equation (12) can be obtained by satisfying equation (14b). To satisfy the prescribed boundary conditions, let the domain be conformally transformed on to a unit circle. If $z = x+iy$, $\bar{z} = x-iy$, equation (14b) becomes

$$\frac{\partial^2 w}{\partial z \partial \bar{z}} + k^2 w = 0. \tag{15}$$

Let $z = f(\xi)$ be the analytic function which maps the boundary under consideration in the ξ-plane on to a unit circle. Thus equation (15) transforms into complex coordinates as

$$\left[\nabla^2 + k^2 \left(\frac{\mathrm{d}z}{\mathrm{d}\xi}\right)^2\right] w(\xi\bar{\xi}) = 0 \tag{16}$$

in which $\xi = re^{i\theta}, \bar{\xi} = re^{-i\theta}$.

The solution of equation (16) can be expressed in the form

$$w \approx \sum_{n=1}^{\infty} Bn[1-(\xi\bar{\xi})]^n \tag{17a}$$

or

$$w \approx \sum_{n=1}^{\infty} Bn[1-(\xi\bar{\xi})^n]^2 \tag{17b}$$

according to the prescribed boundary conditions. Equation (17a) is an admissible function for the simply supported edge condition in the sense that this satisfies the kinematic boundary condition $w = 0$ at $r = 1$, but does not satisfy the force boundary condition $M_n = 0$. The form of w in equation (17b) satisfies $w = 0 = \mathrm{d}w/\mathrm{d}r$ at $r = 1$ and can be taken as an admissible function for the clamped edge condition. Substituting equation (17a) or equation (17b) into equation (16) yields the error function, $\varepsilon_{\pi,\theta}$, which does not vanish, in general, since equation (17a) or equation (17b) is not an exact solution. Galerkin's procedure requires that the error function, $\varepsilon_{\pi,\theta}$, be orthogonal over the domain under consideration, i.e.

$$\int_C \varepsilon_{\pi,\theta}(\xi\bar{\xi}) w(\xi\bar{\xi})\,\mathrm{d}c = 0 \qquad (n = 1,2,3,\ldots,N). \tag{18}$$

From equation (18) a homogeneous system of linear equations is obtained. Such a system can have nontrivial solutions only if the determinant of the coefficients of the unknowns vanishes identically. From this equation, the values of $k_1^2, k_2^2 \ldots k_N^2$ can be found. For the fundamental frequency the lowest value of k^2 is to be taken.

Combining equations (12), (13a), and (13b) the following differential equation for determining $F(t)$ is obtained:

$$\ddot{F}(t) + \lambda_1 F(t) + \mu F^3(t) = 0 \tag{19}$$

in which

$$\lambda_1 = \frac{1}{12}\left(k^4 + \frac{k'}{D}\right) h^2 c_\rho^2 \tag{20}$$

$$\mu = \tfrac{1}{12}\alpha^2 k^2 h^2 c_\rho^2. \tag{21}$$

Equation (19) is to be solved subject to the initial conditions

$$F(0)=1, \qquad \dot{F}(0)=0. \tag{22}$$

The solution of equation (19) can be taken in the form

$$F(t)=c_n(\omega_1 t, \lambda_2) \tag{23}$$

in which ω_1 and λ_2 are positive constants given by

$$\omega_1^2=\frac{1}{12}\left(1+\frac{\alpha^2}{k^2}+\frac{k'}{Dk^4}\right)h^2c_\rho^2k^4 \tag{24}$$

$$\lambda_2^2=\frac{1}{2\left(1+\dfrac{k^2}{\alpha^2}+\dfrac{k'}{D\alpha^2k^2}\right)} \tag{25}$$

and c_n is Jacobi's elliptic function. To determine α, equation (9) is transformed into complex coordinates by the transformation $z=x+iy$, $\bar{z}=x-iy$. Thus one finds

$$\frac{\alpha^2h^2}{12}f(t)=\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}\right)u+i\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)v+2\frac{\partial w}{\partial z}\cdot\frac{\partial w}{\partial \bar{z}}. \tag{26}$$

If the mapping function $z=f(\xi)$ be introduced, equation (26) reduces to

$$\frac{\alpha^2h^2}{12}\frac{\mathrm{d}z}{\mathrm{d}\xi}\cdot\frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{\xi}}f(t)=\frac{\partial u}{\partial \xi}\cdot\frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{\xi}}+\frac{\partial u}{\partial \bar{\xi}}\cdot\frac{\mathrm{d}z}{\mathrm{d}\xi}+i\left\{\frac{\partial v}{\partial z}\cdot\frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{\xi}}-\frac{\partial v}{\partial \bar{z}}\cdot\frac{\mathrm{d}z}{\mathrm{d}\xi}\right\}+2\frac{\partial w}{\partial z}\cdot\frac{\partial w}{\partial \bar{z}}. \tag{27}$$

Now the normalised constant α can be determined from equation (17a) or (17b), and (27) by integrating equation (27) over the cycle 2π. The terms involving u and v can be eliminated (since u and v are of little importance in the case of large amplitude vibration) by considering suitable expressions for u and v, compatible with the boundary conditions. Finally the following integral will determine α:

$$\int_S\int\frac{\alpha^2h^2}{12}\frac{\mathrm{d}z}{\mathrm{d}\xi}\cdot\frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{\xi}}\mathrm{d}S=2\int_S\int\frac{\partial w}{\partial \xi}\cdot\frac{\partial w}{\partial \bar{\xi}}\mathrm{d}S. \tag{28}$$

Thus having determined k and α, the non-linear frequency, ω_1 is completely determined. The non-linear period, T_1, is given by

$$T_1=\frac{4K}{\omega_1}, \tag{29}$$

K being the complete elliptic integral of the first kind. The linear period, T_2, is given by

$$T_2=\frac{2\pi}{\omega_2} \tag{30}$$

in which ω_2 is to be determined from the equation

$$\ddot{F}(t)+\lambda_1F(t)=0 \tag{31}$$

in the form $\omega_2^2=\lambda_1$. Thus the ratio of the periods, T_1/T_2, is obtained as

$$\frac{T_1}{T_2}=\frac{2K}{\pi}\cdot 1\Bigg/\sqrt{\left(1+\frac{k^2\alpha^2}{k^4+\dfrac{k'}{D}}\right)}. \tag{32}$$

APPLICATIONS

(a). Let us apply the procedure explained above to the case of a clamped cornered plate. The mapping function is given by

$$z=\tfrac{25}{48}a(\xi-\tfrac{1}{25}\xi^5+\ldots). \tag{33}$$

Using equation (17b) with $n = 1$ an approximate value of k^2 is obtained from equation (18), viz.:

$$k^2 = \frac{24.55}{a^2}. \tag{34}$$

With $n = 2$, an improved lower value of k^2 is obtained

$$k^2 = \frac{21.71}{a^2}. \tag{35}$$

To determine α the following functions for u and v are taken,

$$u = \sum_{m=1,3,5,\ldots}^{\infty} Um(r)\cos m\theta F^2(t) \tag{36}$$

$$v = \sum_{m=1,3,5,\ldots}^{\infty} Vm(r)\sin m\theta F^2(t). \tag{37}$$

Substituting equations (36) and (37) in equation (27) one gets equation (28) for determining α. To determine the value of α for the fundamental frequency the value of n in equation (17b) is taken to be 1. Substituting equation (17b) with $n = 1$, and equation (33) in equation (28) the following value of α corresponding to the lowest frequency is obtained:

$$\alpha^2 = 29.28\frac{B_1^2}{a^2h^2}. \tag{38}$$

Thus T_1/T_2 is obtained from equation (32) as

$$\frac{T_1}{T_2} = \frac{2K}{\pi}\cdot\frac{1}{\left[1+\frac{B_1^2}{h^2}\left(\frac{636}{471+K_F}\right)\right]^{1/2}} \tag{39}$$

in which the nondimensional foundation modulus, K_F, is given by $K_F = (K'/D)a^4$.

The mapping function of a square plate is given by

$$Z = 1.08a[\xi - \tfrac{1}{10}\xi^5 + \ldots]. \tag{40}$$

Using equation (17b) with $n = 1$ and proceeding in the same manner as before, one gets for a clamped square plate

$$\frac{T_1}{T_2} = \frac{2K}{\pi}\cdot\frac{1}{\left[1+\frac{B_1^2}{h^2}\left(\frac{457}{400+12.3K_F}\right)\right]^{1/2}}. \tag{41}$$

The mapping function for a circular plate is given by

$$Z = a\xi \tag{42}$$

and for a clamped circular plate one finds

$$\frac{T_1}{T_2} = \frac{2K}{\pi}\cdot\frac{1}{\left[1+\frac{B_1^2}{h^2}\left(\frac{480}{400+9K_F}\right)\right]^{1/2}}. \tag{43}$$

(b) Let us consider the case of a simply supported circular plate. Using equation (17a) with $n = 1$ and proceeding in the same manner as before, one gets the ratio T_1/T_2,

$$\frac{T_1}{T_2} = \frac{2K}{\pi}\cdot\frac{1}{\left[1+\frac{B_1^2}{h^2}\left(\frac{72}{36+K_F}\right)\right]^{1/2}}. \tag{44}$$

For $K_F = 0$, equation (44) becomes

$$\frac{T_1}{T_2} = \frac{2K}{\pi}\cdot\frac{1}{\left[1+2\frac{B_1^2}{h^2}\right]^{1/2}}. \tag{45}$$

The corresponding result for the circular plate obtained by Nash and Modeer [5] is

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + 6\frac{A^2}{h^2} J_1^2(KR)\right]^{1/2}}, \tag{46}$$

where $J_0(KR) = 0$, R being the radius of the circle. For a simply supported square plate one finds

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + \frac{B_1^2}{h^2}\left(\frac{67.5}{36 + 1.37K_F}\right)\right]^{1/2}}. \tag{47}$$

EXPERIMENTAL VERIFICATION

Experimental verifications were made with circular and square plates having either simply supported or clamped boundary conditions. The circular plates were 150 mm dia and the square plates had 150 mm side. The plate material was mild steel 0.75 mm thickness. Free transverse vibrations of different amplitudes and frequencies were initiated by the apparatus shown in Fig. 1. The test piece, T, was statically deflected by the load spindle, L,

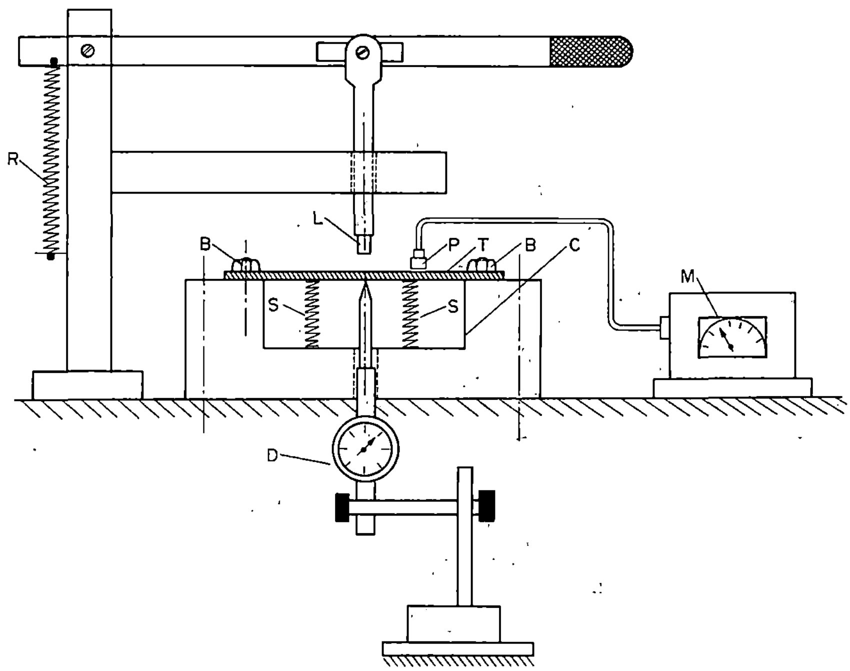

Fig. 1

and the central deflection was measured by the dial indicator, D. After giving a predetermined central deflection the spindle, L, was lifted quickly by the release spring, R, and the corresponding frequency was measured in a vibration meter, M, with the help of a noncontact type of vibration pick-up, P. Simply supported edge conditions were realised by placing the edges of the plates over a knife edge placed around the periphery of the cavity, C, the shape of which conformed to the shape of the plate used. Clamped edge conditions were achieved by clamping the edges of the plates rigidly by means of eight bolts, B, with the base of the apparatus. Experiments were carried out first with the cavity empty and next by placing the plates over eight free helical springs, S, each spring being located at the centre of eight equal areas of the plates. The combined reaction of the springs used was determined experimentally to be $K_F \doteq 6.2$. Care was taken in selecting the stiffness of the spring, R, so that the spindle, L, was released quickly from the plate without obstructing the upward motion of the plates.

RESULTS

Numerical as well as experimental results for the case of simply supported circular and square plates without any foundation have been presented in Figs. 2 and 3 respectively.

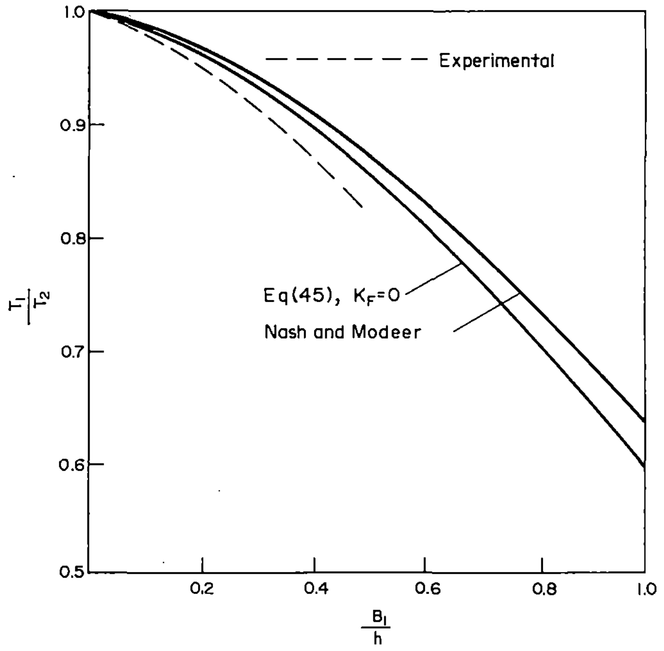

Fig. 2. Simply supported circular plate.

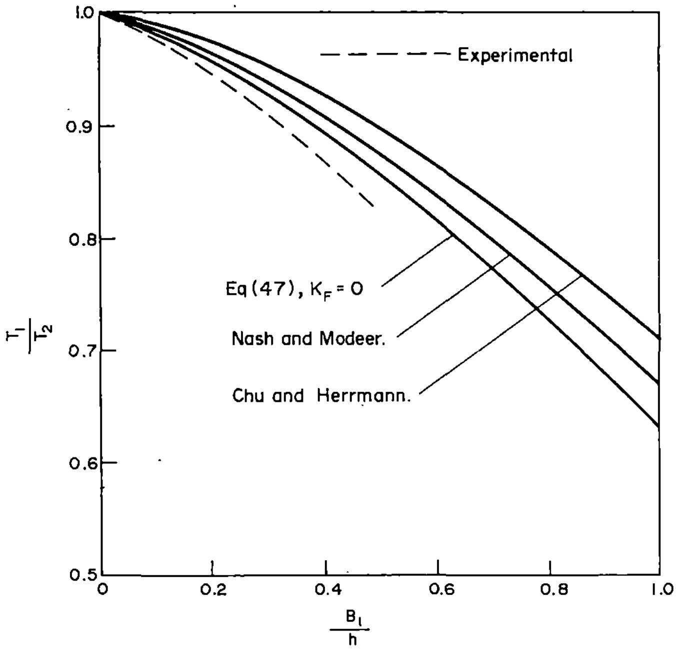

Fig. 3. Simply supported square plate.

The corresponding results obtained by Nash and Modeer [5] for the circular and square plates and the results obtained by Chu and Herrmann [7] for the square plates have also been presented for comparison. Numerical and experimental results for clamped circular and square plates both with and without foundation have been presented in Figs. 4 and 5 respectively.

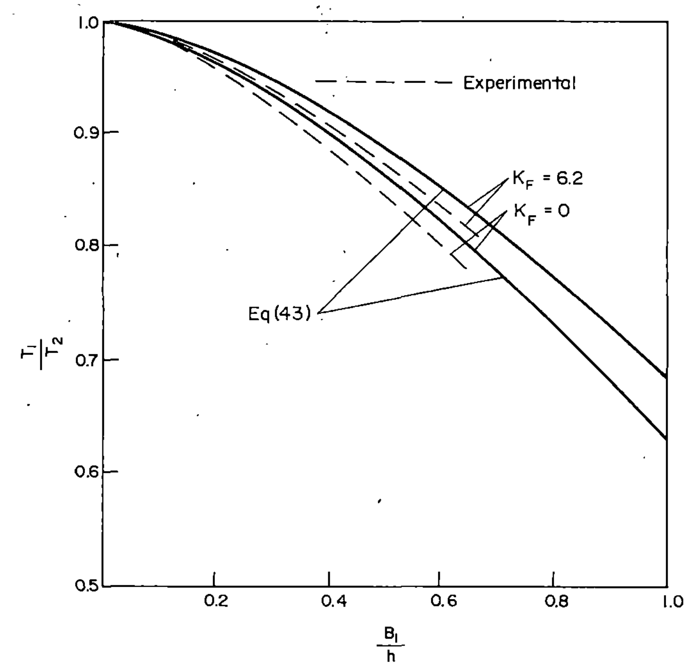

Fig. 4. Clamped circular plate.

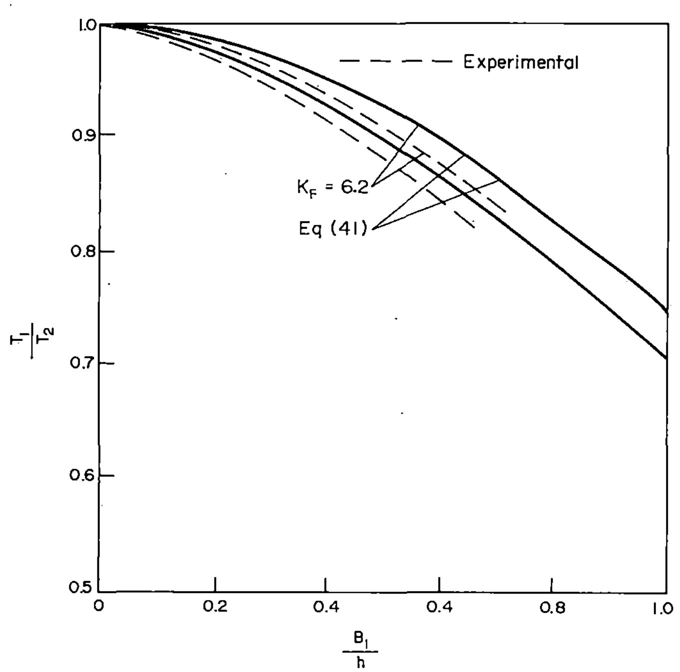

Fig. 5. Clamped square plate.

CONCLUSIONS

Lowest natural frequencies of large amplitude free vibrations of thin plates of any shape can readily be calculated by conformal mapping techniques used in this study if the mapping functions are known. From Figs. 4 and 5 it is observed that the results obtained with a one-term approximation of the trial function, equation (17b), for the clamped edge boundary conditions are in excellent agreement with the practical values. For the simply supported edge conditions the theoretical results given in Figs. 2 and 3 are in somewhat poorer agreement with the values obtained experimentally. By using higher approximations of the trial functions, equation (17a) and (17b) and with smoothed mapping functions the results for both simply supported and clamped edge boundary conditions will be refined.

The periods for rectangular plates obtained by Chu and Herrmann [7] is dependent on the aspect ratio of the plate, whereas the corresponding results obtained by Nash and Modeer [5] are independent of that ratio. The mapping functions for rectangular plates with different aspect ratios will be different and therefore the present study indicates that the periods will depend on the aspect ratio. It should be pointed out that the theory used in this study allows the solution of the eigenvalue problem under consideration from a unified point of view since the trial functions used are the same for all shapes. For a one-term approximation the results obtained in this study are considered satisfactory for practical purposes.

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RÉSUMÉ

On donne une méthode unifiée pour déterminer la plus petite fréquence propre pour des vibrations libres de grande amplitude de plaques élastiques minces de forme quelconque placées sur un support élastique. On introduit la technique de transformation conforme et on utilise la méthode de Galerkin pour calculer la valeur approchée de la plus petite fréquence propre. On détermine les périodes de plaques circulaires, carrées et en coin placées sur un support élastique pour des conditions aux limites en appui simple et encastrées. Les valeurs pratiques ont également été déterminées expérimentalement. On présente les résultats sous forme graphique et on les compare avec d'autres résultats connus.

Zusammenfassung:

Eine einheitliche Methode für die Bestimmung der niedrigsten Eignefrequenz von frei, mit grosser Amplitude schwingenden dünnen elastischen Scheiben beliebiger Form mit elastischer Lagerung wird gegeben. Das Verfahren der konformen Abbildung wird eingeführt und die Galerkinsche Methode wird zur Berechnung von Näherungswerten der niedrigsten Eigenfrequenz benutzt. Die Schwingungsdauern für kreisförmige, quadratische und eckige Scheiben auf elastischer Lagerung wurden für die Randbedingungen der frei aufliegenden und eingespannten Kanten bestimmt. Praktische Werte wurden experimentell bestimmt. Die Resultate werden in Form von Diagrammen dargestellt und mit anderen bekannten Ergebnissen verglichen.

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LARGE DEFLECTION OF A HEATED ELLIPTIC PLATE ON ELASTIC FOUNDATION

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Abstract

Following Berger's method the large deflection of a heated elliptic plate with clamped edges and placed on elastic foundation has been investigated under stationary temperature distribution. The deflection is obtained in terms of Mathieu function of the first kind and of zero order.

Keywords: Berger's method; Mathieu function; Elliptic plate; Elastic foundation.

Introduction

In recent years there has been a rapid development of thermoelasticity stimulated by various engineering sciences. In the field of machine structures, mainly with aircraft, steam and gasturbines and in chemical and nuclear engineering, thermal stresses play an important and frequently even a primary role. Determination of thermal deflections of plates, especially of thin plates, is of vital importance in the design of machine structures, because excessive deflections may cause heavy undesirable thermal stresses.

The classical large deflection of thin plate problems usually lead to non-linear differential equations which cannot be exactly solved. H. M. Berger [1] has shown that if, in deriving the differential equations from the expressions for strain energy, the strain energy due to second invariant in the middle plane of the plate is neglected, a simple fourth order differential equation coupled with a non-linear second order equation is obtained. Although no complete explanation of the method is set forth, the stresses and deflections obtained by Berger himself for rectangular and circular plates agree well with those found from more precise analysis. This approximate method has been extended to orthotropic plates by Iwinski and Nowinski [2] and further boundary value problems associated with rectangular and circular plates have been solved by Nowinski [3]. Thein Wah and Robert Schmidt [4] and Nash and Modeer [5] obtained satisfactory results following this method Basuli [6] has extended this approximate method of Berger to problems under uniform load and heating under stationary temperature distribution.

Berger's technique of neglecting the second invariant of the middle surface strains has been applied by Sinha [7] to circular and rectangular plates placed on elastic foundation and under uniform transverse load.

In this paper the author has applied the method of Berger to investigate the large deflection of an elliptic plate placed on elastic foundation and heated under stationary temperature distribution. The foundation is assumed to be such that its reaction is proportional to the deflection. The deflection is obtained in terms of Mathieu function of the first kind and of zero order.

NOTATIONS

The following notations have been used in the paper:

D = Flexural rigidity of the plate $= \dfrac{Eh^3}{12(1-\nu^2)}$

E, ν, α = Young's modulus, Poisson's ratio and Coefficient of thermal expansion respectively.

h = Thickness of plate.

u, v = Displacement along the x and y axis respectively.

w = Lateral displacement

e_1 = First strain invariant;

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

e_2 = Second strain invariant.

K = Foundation reaction per unit area per unit deflection.

∇ = Laplacian operator.

FORMULATION OF PROBLEM

The strain energy due to bending and stretching of the middle surface of the plate is given by:

$$V_1 = \frac{D}{2} \int\int \left[(\nabla^2 W)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu) \left\{ \frac{12}{h^2} e_2 \right.\right.$$

$$+\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\Big\}\Big]\,dxdy \tag{1}$$

Combining the potential energy of the foundation reaction and also the potential energy due to heating with Eq. 1 and neglecting e_2, the modified energy expression for the total energy becomes:

$$V=\frac{D}{2}\iint\Big[(\nabla^2 w)^2+\frac{12}{h^2}e_1^2-2(1-\nu)\Big\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\Big\}+\frac{K}{D}w^2\Big]\,dxdy-\iint\int_{-h/2}^{h/2}\frac{E\alpha T'}{1-\nu}\times(e_1-z\nabla^2 w)\,dxdydz \tag{2}$$

in which T' is the temperature distribution at any point given by (Basuli [6])

$$T'(x,y,z)=T_0(x,y)+g(z)\,T(x,y) \tag{3}$$

and

$$\int_{-h/2}^{h/2} zg(z)\,dz=f(h);\qquad \int_{-h/2}^{h/2} g(z)\,dz=0 \tag{4}$$

Combining Equations 2, 3 and 4 one gets

$$V=\frac{D}{2}\iint\Big[(\nabla^2 w)^2+\frac{12}{h^2}e_1^2-2(1-\nu)\Big\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\Big\}+\frac{K}{D}w^2\Big]\,dxdy-\iint\frac{E\alpha}{1-\nu}\left(T_0e_1h-f(h)\,T\nabla^2 w\right)\,dxdy. \tag{5}$$

According to the principle of minimum potential energy, the displacements that satisfy the equilibrium conditions make the potential energy, V, minimum. In order for the integral of Eq. 5 to be an extremum, the integrand, F, must satisfy the following Euler's equations of the calculus of variation:

$$\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)=0 \tag{6 a}$$

$$\frac{\partial F}{\partial v}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial V_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial V_y}\right)=0 \tag{6 b}$$

$$\frac{\partial F}{\partial w}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial w_y}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial w_{xx}}\right)$$
$$+\frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial w_{yy}}\right)+\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial w_{xy}}\right)=0. \tag{6 c}$$

Application on the Eqs. 6 *a*, 6 *b*, and 6 *c* to Eq. 5 yields:

$$\frac{\partial}{\partial x}\{e_1-(1+\nu)\,\alpha T_0\}=0 \tag{7 a}$$

$$\frac{\partial}{\partial y}\{e_1-(1+\nu)\,\alpha T_0\}=0. \tag{7 b}$$

$$\nabla^4 w-\frac{12}{h^2}\{e_1-(1+\nu)\,\alpha\, T_0\}\,\nabla^2 w+\frac{K}{D}\,w$$
$$+\frac{E\alpha f(h)}{D(1-\nu)}\,\nabla^2 T=0. \tag{7 c}$$

Eqs. 7 *a*, and 7 *b* prove that:

$$\{e_1-(1+\nu)\,\alpha T_0\}$$

is independent of x and y and therefore

$$e_1-(1+\nu)\,\alpha T_0=\text{constant}=\frac{\beta^2 h^2}{12} \tag{8 a}$$

in which β is a normalised constant of integration, and

$$e_1=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{8 b}$$

Considering Eq. 8 *a*, Eq. 7*c* reduces to

$$\nabla^2(\nabla^2-\beta^2)\,w+\frac{K}{D}\,w=-\frac{E\alpha\, f(h)}{D(1-\nu)}\,\nabla^2 T \tag{9}$$

SOLUTION OF PROBLEM

Let us take an elliptic plate of thickness, h. The centre of the plate in the middle surface is taken as the origin and the Z-axis downwards.

If there is no source of heat inside the plate the following differential equations must be satisfied for stationary temperature distribution (Nowacki [8])

$$\nabla^2 T_0 - \epsilon T_0 = -\frac{\epsilon_0}{2}(\theta_1 + \theta_2)$$

$$\nabla^2 T - \frac{12}{h^2}(1 + \epsilon) T = -\frac{12\epsilon}{h^3}(\theta_1 - \theta_2) \tag{11}$$

in which θ_1 and θ_2 denote temperatures at the upper and lower media of the plate respectively.

If $\theta_1 = \theta_2$, Eq. (11) becomes

$$\nabla^2 T - \beta_1{}^2 T = 0 \tag{12}$$

In which

$$\beta_1{}^2 = (1 + \epsilon)\frac{12}{h^2} \tag{13}$$

Transferring to elliptic co-ordinates (ξ, η) defined by $x + iy = d \cosh(\xi + i\eta)$, where $2d$ is the interfocal distance of the ellipse, Eq. 12 reduces to

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2} - \frac{\beta_1{}^2 d^2}{2}(\cosh\ 2\xi - \cos 2\eta)\, T = 0. \tag{14}$$

Solution of Eq. 14 can be taken in the following form

$$T = \sum_{m=0}^{\infty} C_{2m}\, Ce_{2m}(\xi, -q)\, ce_{2m}(\eta, -q) \tag{15}$$

in which $Ce_{2m}(\xi, -q)$ and $ce_{2m}(\eta, -q)$ are modified Mathieu function and ordinary Mathieu function of the first kind and of order $2m$ respectively, and

$$q = \frac{\beta_1{}^2 d^2}{4} \tag{16}$$

While solving a problem of bending of a plate with an elliptic hole, by taking a single Mathieu function of the second orcer instead of taking Mathieu functions of all orders, Naghdi [9] has shown that the results are satisfactory for larger elliptic holes. In this paper also similar approximation is made by taking Mathieu function of zero order and on this assumption Eq.15 reduces to

$$T = C_o\, Ce_o(\xi, -q)\, ce_o(\xi, -q) \tag{17}$$

The following boundary condition is imposed on T.

$$T = \text{Constant} = K_1 \text{ on } \xi = \xi_0$$

with the above boundary condition Eq. 17 yields

$$K_1 = C_o C_{eo}(\xi_0, -q)\, c_{eo}(\xi, -q) \tag{18}$$

Multiplying Eq. 18 by $c_{eo}(\eta, -q)$ and integrating with respect to η from 0 to 2π and using the orthogonality relation and normalisation (Mclachlan [10]) one gets

$$C_o = \frac{2A_0^{(0)} K_1}{C_{eo}(\xi_0, -q)} \tag{19}$$

in which $A_0^{(0)}$ is the first Fourier Coefficient in the expansion of $c_{eo}(\eta, -q)$ Therefore

$$T = \frac{2A_0^{(0)} K_1}{C_{eo}(\xi_0, -q)} C_{eo}(\xi, -q)\, c_{eo}(\eta, -q) \tag{20}$$

is determined.

Changing Eq. 9 to elliptic Co-ordinates and substituting the expresion fo $\nabla^2 T$ one gets

$$(\nabla^2 - P_1^2)(\nabla^2 - P_2^2)\, w = \lambda C_{eo}(\xi, -q)\, c_{eo}(\eta, -q) \tag{21}$$

in which

$$P_1^2 + P_2^2 = -\beta^2 \tag{22}$$

$$P_1^2\, P_2^2 = \frac{K}{D} \tag{23}$$

$$\lambda = -\frac{E\alpha f(h)}{D(1-\nu)} \frac{2\beta_1^2 A_0^{(0)} K_1}{C_{eo}(\xi_0, -q)} \tag{24}$$

$$\nabla^2 = \frac{2}{d^2(\cosh 2\xi - \cos 2\eta)} \left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right] \tag{25}$$

Complimentary function of Eq. 21 is given by

$$W = B_0 C_{eo}(\xi, -q_1)\, c_{eo}(\eta, -q_1) + D_0 C_{eo}(\xi, -q_2)\, c_{eo}(\eta, -q_2) \tag{26}$$

in which

$$q_1 = \frac{P_1^2 d^2}{4}\ ;\ q_2 = \frac{P_2^2 d^2}{4} \tag{27}$$

Clearly the particular integral of Eq. 21 is

$$\frac{\lambda}{(\beta_1^2 - P_2^2)(\beta_1^2 - P_1^2)} C_{eo}(\xi, -q)\, c_{eo}(\eta, -q) \tag{28}$$

Thus the complete solution of Eq. 21 is

$$W = B_0 Ce_0(\xi, -q_1)\, ce_0(\eta, -q_1) + D_0 Ce_0(\xi, -q_2)\, ce_0(\eta, +q_2)$$

$$+ \frac{\lambda}{(\beta_1^2 - P_2^2)(\beta_1^2 - P_1^2)} Ce_0(\xi, -q) ce_0(\eta, -q) \qquad (29)$$

If the outer boundary of the plate $\xi = \xi_0$ be clamped, the boundary conditions are

$$(w)_{\xi=\xi_0} = \left(\frac{\partial w}{\partial \xi}\right)_{\xi=\xi_0} = 0 \qquad (30)$$

Using Eq. 30 in Eq. 29 one gets the following two conditional equations

$$B_0 Ce_0(\xi_0, -q_1)\, ce_0(\eta, -q_1) + D_0 Ce_0(\xi_0, -q_2)\, ce_0(\eta, -q_2)$$

$$+ \frac{\lambda}{(\beta_1^2 - P_2^2)(\beta_1^2 - P_1^2)} Ce_0(\xi_0, -q)\, ce_0(\eta, -q) = 0 \qquad (31\,a)$$

$$B_0 C'e_0(\xi_0, -q_1)\, ce_0(\eta, -q_1) + D_0 C'e_0(\xi_0, -q_2)\, ce_0(\eta, -q_2)$$

$$+ \frac{\lambda}{(\beta_1^2 - P_2^2)(\beta_1^2 - P_1^2)} C'e_0(\xi_0, -q)\, ce_0(\eta, -q) = 0 \qquad (31\,b)$$

Multiplying Eqs. 31 *a* and 31 *b* by $ce_0(\eta, -q_1)$ and integrating with respect to η from 0 to 2π and using the orthogonality relation and normalisation one gets

$$B_0 = -\frac{\lambda\phi_2}{\pi\psi\phi_3}\{ Ce_0(\xi_0, -q_2)\, C'e_0(\xi_0, -q) - Ce_0(\xi_0, -q)$$

$$C'e_0(\xi_0, -q_2)\} \qquad (32)$$

$$D_0 = \frac{\lambda\phi_2}{\psi\phi_3\phi_1}\{ Ce_0(\xi_0, -q_1)\, C'e_0(\xi_0, -q)$$

$$- Ce_0(\xi_0, -q)\, C'e_0(\xi_0, -q_1)\} \qquad (33)$$

in which

$$\psi = (\beta_1^2 - P_2^2)(\beta_1^2 - P_1^2)$$

$$\phi_1 = 2\bar{A}_0^{(0)} \bar{\bar{A}}_0^{(0)} + \sum_{r=1}^{\infty} \bar{A}_{2r}^{(0)} \bar{\bar{A}}_{2r}^{(0)}$$

$$\phi_2 = 2A_0^{(0)} \bar{A}_0^{(0)} + \sum_{r=1}^{\infty} \bar{A}_{2r}^{(0)} A_{2r}^{(0)}$$

$$\phi_3 = Ce_0(\xi_0, -q_2)\, C'e_0(\xi_0, -q_1) - Ce_0(\xi_0, -q_1)\, C'e_0(\xi_0, -q_2)$$

$\bar{A}_{2r}^{(0)}$, $\bar{\bar{A}}_{2r}^{(0)}$ and $A_{2r}^{(0)}$ are the Fourier Coefficients in the expansion of $ce_0(\eta, q_1)$, $ce_0(\eta, -q_2)$ and $ce_0(\eta, -q)$ respectively.

To determine the constant β^2, Eq. 8 is transformed into elliptic Co-ordinates in the form

$$h_1h_2\left\{\frac{\partial}{\partial \xi}\left(\frac{u_\xi}{h_2}\right) + \frac{\partial}{\partial \eta}\left(\frac{u_\eta}{h_1}\right)\right\} + \frac{1}{2}h_1h_2\left\{\left(\frac{\partial w}{\partial \xi}\right)^2 + \left(\frac{\partial w}{\partial \eta}\right)^2\right\}$$

$$= \frac{\beta^2 h^2}{12} + (1 + \nu)\,\alpha\, T_0 \qquad (34)$$

in which

$$h_1 = h_2 = \frac{1}{d\sqrt{\sinh^2 \xi + \sin^2 \eta}}$$

The boundary conditions for u_ξ and u_η are

$$u_\xi = 0 = u_\eta \text{ at } \xi = \xi_0 \qquad (35)$$

Let

$$u_\xi = \sum_{n=0}^{\infty} P(\xi) \cos 2n\eta \qquad (36)$$

$$u_\eta = \sum_{n=1}^{\infty} G(\xi) \sin 2n\eta \qquad (37)$$

subject to the conditions

$$P(\xi_0) = G(\xi_0) = 0$$

Substituting Eqs. 29, 36, and 37 in Eq. 34 and integrating over the surface of the plate one gets

$$\int_0^{2\pi}\int_0^{\xi_0}\left\{\left(\frac{\partial w}{\partial \xi}\right)^2 + \left(\frac{\partial w}{\partial \eta}\right)^2\right\} d\xi d\eta$$

$$= d^2\left\{\frac{\beta^2 h^2}{6} + 2(1 + \nu)\,\alpha T_0\right\}\int_0^{2\pi}\int_0^{\xi_0}(\sinh^2 \xi + \text{Sin}^2 \eta)\, d\xi d\eta \qquad (38)$$

After evaluating the integrals the following equation leading to β is obtained.

$$B_0^2[(2\{\bar{A}_0^{(0)}\}^2 + \sum_{r=1}^{\infty}\{\bar{A}_{2r}^{(0)}\}^2)\{\sum_{r=1}^{\infty} 4r^2\{A_{2r}'^{(0)}\}^2\psi_1$$

$$+ \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} 2rs(-1)^r(-1)^s A_{2r}^{\prime(0)} A_{2s}^{\prime(0)} \psi_2\}$$

$$+ (\sum_{r=1}^{\infty} 4r^2 \{\bar{A}_{2r}^{(0)}\}^2) \{(A_0^{\prime(0)})^2 \xi_0 + A_0^{\prime(0)} \sum_{r=1}^{\infty} A_{2r}^{\prime(0)} (-1)^r \psi_5$$

$$+ \sum_{r=1}^{\infty} A_{2r}^{\prime(0)} \psi_4 + \tfrac{1}{2} \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} (-1)^r (-1)^s A_{2r}^{\prime(0)} A_{2s}^{\prime(0)} \psi_3\}]$$

$$+ D_0^2 [(2\{\bar{\bar{A}}_0^{(0)}\}^2 + \sum_{r=1}^{\infty} \{\bar{\bar{A}}_{2r}^{(0)}\}^2) \{\sum_{r=1}^{\infty} 4r^2 \{A_{2r}^{\prime\prime(0)}\}^2 \psi_1$$

$$+ \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} 2rs(-1)^r(-1)^s A_{2r}^{\prime\prime(0)} A_{2s}^{\prime\prime(0)} \psi_2\}$$

$$+ (\sum_{r=0}^{\infty} 4r^2 \{\bar{\bar{A}}_{2r}^{(0)}\}^2) \{(A_0^{\prime\prime(0)})^2 \xi_0 + A_0^{\prime\prime(0)} \sum_{r=1}^{\infty} A_{2r}^{\prime\prime(0)} (-1)^r \psi_5$$

$$+ \sum_{r=1}^{\infty} A_{2r}^{\prime\prime(0)} \psi_4 + \tfrac{1}{2} \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} (-1)^r (-1)^s A_{2r}^{\prime\prime(0)} A_{2s}^{\prime\prime(0)} \psi_3\}]$$

$$+ 2B_0 D_0 [(2\bar{A}_0^{(0)} \bar{\bar{A}}_0^{(0)} + \sum_{r=1}^{\infty} \bar{A}_{2r}^{(0)} \bar{\bar{A}}_{2r}^{(0)}) \{\sum_{r=1}^{\infty} 4r^2 A_{2r}^{\prime(0)} A_{2r}^{\prime\prime(0)} \psi_1$$

$$+ \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} 2rs(-1)^r(-1)^s A_{2s}^{\prime(0)} A_{2r}^{\prime\prime(0)} \psi_2\}$$

$$+ (\sum_{r=1}^{\infty} 4r^2 \bar{A}_{2r}^{(0)} \bar{\bar{A}}_{2r}^{(0)}) \{A_0^{\prime\prime(0)} A_0^{\prime(0)} \xi_0 + A_0^{\prime\prime(0)} \sum_{r=1}^{\infty} (-1)^r A_{2r}^{\prime(0)} \psi_5$$

$$+ A_0^{\prime(0)} \sum_{r=1}^{\infty} (-1)^r A_{2r}^{\prime\prime(0)} \psi_5 + \sum_{r=1}^{\infty} A_{2r}^{\prime\prime(0)} A_{2r}^{\prime(0)} \psi_4$$

$$+ \tfrac{1}{2} \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} (-1)^r (-1)^s A_{2r}^{\prime\prime(0)} A_{2s}^{\prime(0)} \psi_3\}]$$

$$+ \frac{2B_0\lambda}{\psi} [(2A_0^{(0)} \bar{A}_0^{(0)} + \sum_{r=1}^{\infty} A_{2r}^{(0)} \bar{A}_{2r}^{(0)}) \{\sum_{r=1}^{\infty} 4r^2 A_{2r}^{\prime(0)} a_{2r}^{(0)} \psi_1$$

$$+ \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} 2rs(-1)^r(-1)^s A_{2s}^{\prime(0)} a_{2r}^{(0)} \psi_2 + (\sum_{r=1} 4r^2 A_{2r}^{(0)} \bar{A}_{2r}^{(0)})$$

$$\times \{a_0^{(0)} A_0^{\prime(0)} \xi_0 + a_0^{(0)} \sum_{r=1}^{\infty} (-1)^r A_{2r}^{\prime(0)} \psi_5 + A_0^{\prime(0)} \sum_{r=1}^{\infty} (-1)^r a_{2r}^{(0)} \psi_5$$

$$+ \sum_{r=1}^{\infty} a_{2r}^{(0)} A_{2r}^{\prime(0)} \psi_4 + \tfrac{1}{2} \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} (-1)^r (-1)^s a_{2r}^{(0)} A_{2s}^{\prime(0)} \psi_3\}]$$

$$+\frac{2D_0\lambda}{\psi}[(2A_0^{(0)}\bar{\bar{A}}_0^{(0)}+\sum_{r=1}^{\infty}A_{2r}^{(0)}\bar{\bar{A}}_{2r}^{(0)})\{\sum_{r=1}^{\infty}4r^2A_{2r}^{\prime\prime(0)}a_{2r}^{(0)}\psi_1$$

$$+\sum_{\substack{r=1\\r\neq s}}^{\infty}\sum_{s=1}^{\infty}2rs(-1)^s(-1)^rA_{2p}^{\prime\prime(0)}a_{2r}^{(0)}\psi_2\}+(\sum_{r=1}^{\infty}4r^2A_{2r}^{(0)}\bar{\bar{A}}_{2r}^{(0)})$$

$$\times\{a_0^{(0)}A_0^{\prime\prime 0}\xi_0+a_0^{(0)}\sum_{r=1}^{\infty}(-1)^rA_{2r}^{\prime\prime(0)}\psi_5+A_0^{\prime\prime(0)}\sum_{r=1}^{\infty}(-1)^ra_{2r}^{(0)}\psi_5$$

$$+\sum_{r=1}^{\infty}a_{2r}^{(0)}A_{2r}^{\prime\prime(0)}\psi_4+\tfrac{1}{2}\sum_{\substack{r=1\\r\neq s}}^{\infty}\sum_{s=1}^{\infty}(-1)^r(-1)^sa_{2r}^{(0)}A_{2s}^{\prime\prime(0)}\psi_3\}]$$

$$+\frac{\lambda^2}{\psi^2}[(2\{A_0^{(0)}\}^2+\sum_{r=1}^{\infty}\{A_{2r}^{(0)}\}^2)\{\sum_{r=1}^{\infty}4r^2\{a_{2r}^{(0)}\}^2\psi_1$$

$$+\sum_{\substack{r=1\\r\neq s}}^{\infty}\sum_{s=1}^{\infty}2rs(-1)^r(-1)^sa_{2r}^{(0)}a_{2s}^{(0)}\psi_2\}+(\sum_{r=1}^{\infty}4r^2\{A_{2r}^{(0)}\}^2)$$

$$\times\{(a_0^{(0)})^2\xi_0+a_0^{(0)}\sum_{r=1}^{\infty}a_{2r}^{(0)}(-1)^r\psi_5+\sum_{r=1}^{\infty}a_{2r}^{(0)}\psi_4$$

$$+\tfrac{1}{2}\sum_{\substack{r=1\\r\neq s}}^{\infty}\sum_{s=1}^{\infty}(-1)^r(-1)^sa_{2r}^{(0)}a_{2s}^{(0)}\psi_3\}]$$

$$=\frac{d^2}{2}\left\{\frac{\beta^2h^2}{6}+2(1+\nu)\alpha T_0\right\}\sinh 2\xi_0 \tag{39}$$

where

$$\psi_1=\frac{\text{Sinh }4r\xi_0}{8r}-\frac{\xi_0}{2}$$

$$\psi_2=\frac{\text{Sinh }\overline{2r+2s}\,\xi_0}{2r+2s}-\frac{\text{Sinh }\overline{2r-2s}\,\xi_0}{2r-2s}$$

$$\psi_3=\frac{\text{Sinh }\overline{2r+2s}\,\xi_0}{2r+2s}+\frac{\text{Sinh }\overline{2r-2r}\,\xi_0}{2r-2s}$$

$$\psi_4=\frac{\xi_0}{2}+\frac{\text{Sinh }4r\xi_0}{8r}$$

$$\psi_5=\frac{\text{Sinh }2r\,\xi_0}{2r}$$

and

$a_{2r}^{(0)}$, $A_{2r}^{\prime(0)}$, and $A_{2r}^{\prime\prime(0)}$ are the Fourier Coefficients in the expansions of $Ce_0(\xi,-q)$, $Ce_0(\xi,-q_1)$, and $Ce_0(\xi,-q_2)$ respectively.

Since β is determined, w is determined completely.

NUMERICAL CALCULATION

To find the deflection at a given point, one has to start from Eq. 39 with an assumed value of β leading to the corresponding value of λ. With this value of λ and considering Eqs. 32 and 33 the deflection will be obtained from Eq. (29).

For numerical calculation the following values have been assumed:

$$\xi = 0,\ \eta = \frac{\pi}{2},\ \xi_0 = 3,\ d^2 = 2\cdot 5,\ h = 1,\ f(h) = h,$$

$$K_F = \frac{K}{D}\ \xi_0^4 = 100,\ \ \epsilon = 0\cdot 03,\ \ \nu = 0\cdot 3,\ \ \alpha T_0 = 2\cdot 5 \times 10^{-3}.$$

The interfocal distance $2d$ being assumed and the values of β^2, P_1^2 and P_2^2 being known, the values of q, q_1 and q_2 are determined. q, q_1 and q_2 being

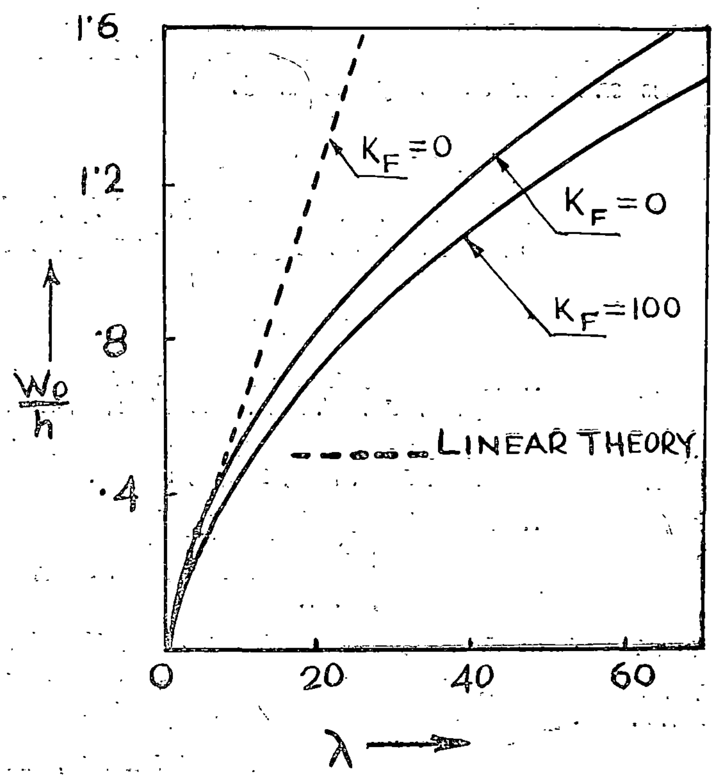

FIG. 1. Load-Deflection Curve

known the corresponding values of the Fourier Coefficients as well as those of Mathieu functions are determined. The maximum deflection W_0 is obtained at the centre of the plate. These deflections are graphically presented in Fig. 1 in which W_0/h for $K_F = 0$ and $K_F = 100$ are plotted against the non-dimensional load function λ. By setting $\beta \to 0$ the deflections according to the linear theory is obtained. For comparison Fig. 1 also includes a straightline which represents small deflections for $K_F = 0$. The results obtained in this study could not be compared in absence of any known results.

CONCLUSIONS

From Fig. 1 it is observed that the error according to the linear theory increases progressively with the increase in load function. Te solution proposed in this study is rapidly convergent and no computational difficulty other than computational effect is involved. The parameter q for the series $c_{eo}(\xi, q)$ may be real or imaginary and the corresponding coefficients can be computed with accuracy. The numerical results presented in this study are obtained by taking the first two terms of the series and sufficient for practical purposes. Since the deflection at any point is known the corresponding stresses can bew be easily estimated.

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