C H A P T E R II

RING SOURCE PROBLEMS

Problem 1. Displacement produced in an elastic half-space by the impulsive torsional motion of a circular ring source.

Problem 2. SH-waves in an elastic half-space due to a ring source of increasing radius.

Problem 3. Torsional response of an elastic half-space to a nonuniformly expanding ring source.

Displacement Produced in an Elastic Half-Space by the Impulsive Torsional Motion of a Circular Ring Source

INTRODUCTION: At present much attention has been given to problems concerned with wave propagation in homogeneous as well as in inhomogeneous, isotropic, elastic media. Much of this work has been connected with problems of seismological interest, involving wave propagation. The normal loading problem of an elastic half-space was first investigated by Lamb (1904). This type of problem was then investigated by Eason (1964), Mitra (1964), Chakraborty and De (1971) and many others. In fact a class of elastic half-space problems involving an axisymmetric, normally applied, surface load was investigated by Gakenheimer (1971). He assumed that loads suddenly emanate from a point on the surface and expand radially at a constant rate. He used Cagniard's method to evaluate the inverse transforms. This paper has a particular reference to the work by Ghosh (1971) where techniques similar to those adopted here, are used. Many recent studies on elastic wave propagation are due to the work of Cagniard (1962), who developed a particular technique of finding the Laplace inversion, that has been found to be extremely useful in dealing with problems of this type.

The type of disturbing force considered in this paper is impulsive in time and acts over the circumference of a circular region of

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Published in Pure and Applied Geophysics, vol.119.P.102,(1980/81)

constant radius on the free surface of a semi-infinite, isotropic, elastic half-space. The effect of the inhomogeneity of the medium on the disturbance produced is determined in the integral form, whereas the displacement in the case of a homogeneous medium is determined exactly. The displacement at any point on the free surface is evaluated numerically and the graphs are drawn to show how the vibration of a point in the medium is affected due to the inhomogeneity of the medium, which enters into the expression for displacement through the factor ϵ.

Case I: Homogeneous medium

FORMULATION OF THE PROBLEM: Let (r,θ,z) be the cylindrical polar co-ordinates, z-axis being directed into the isotropic elastic medium, the plane boundary being $z=0$ with the origin at the centre of the ring source $r=a$, $z=0$.

The displacement is calculated at points inside and on the free surface of the medium, subject to the condition that the half-space is initially at rest and that the displacement remains bounded even for large values of z. For torsional motion of the ring all quantities depend on r, z and the time t, the only non-zero component of the displacement vector is the component v along the direction of θ increasing. The relevant non-vanishing stress components are

$$\tau_{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) \tag{1}$$

and

$$\tau_{\theta z} = \mu \frac{\partial v}{\partial z} \tag{2}$$

where μ is Lame's constant. The only non-zero equation of motion is

$$\frac{\partial}{\partial r}(\tau_{r\theta}) + \frac{\partial}{\partial z}(\tau_{\theta z}) + 2\,\frac{\tau_{r\theta}}{r} = \rho\,\frac{\partial^2 v}{\partial t^2} \tag{3}$$

where ρ is the density of the material, assumed constant. The boundary condition is

$$\tau_{\theta z} = P\delta(r - a)\delta(t) \text{ at } z = o, \tag{4}$$

where P is a constant, a is the radius of the ring source and $\delta(t)$ is Dirac's delta function.

Using (1) and (2) the equation (3) can be written in the form

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\beta^2}\;\frac{\partial^2 v}{\partial t^2} \tag{5}$$

where $\beta = \sqrt{(\mu/\rho)}$ is the shear wave velocity.

METHOD OF SOLUTION: We define for all positive real values of s the Laplace transform $f_1(r,z,s)$ of a function $f(r,z,t)$ by the relation

$$f_1(r,z,s) = \int_0^\infty f(r,z,t)e^{-st}\,dt. \tag{6}$$

Applying the Laplace transform (6) to the equation (5) we obtain

$$\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}\left(\frac{\partial v_1}{\partial r} - \frac{v_1}{r}\right) + \frac{\partial^2 v_1}{\partial z^2} = \frac{s^2 v_1}{\beta^2}\,. \tag{7}$$

Define the Hankel transform $v_2(\xi,z,s)$ of $v_1(r,z,s)$ by the equation

$$v_2(\xi,z,s) \int_0^\infty rJ_1(\xi r)v_1(r,z,s)dr, \tag{8}$$

where J_1 is a Bessel function.

Multiplying the equation (7) by $rJ_1(\xi r)$ and integrating with respect to r from o to ∞ we get,

$$\frac{d^2v_2}{dz^2} = \left(\xi^2 + \frac{s^2}{\beta^2}\right)v_2 \ . \tag{9}$$

The general solution of this equation which remains bounded as $z \to +\infty$ is

$$v_2 = A \exp\left[-z\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right]. \tag{10}$$

where A is to be determined from the boundary conditions,

$\tau_{\theta z_1} = P\delta(r - a)$ at $z = o$, where $\tau_{\theta z_1}$ is the Laplace transform of $\tau_{\theta z}$. From the Hankel transform $(\tau_{\theta z_1})_2$ of $\tau_{\theta z_1}$, we obtain by using (2)

$$(\tau_{\theta z_1})_2 = \mu\frac{dv_2}{dz} = PaJ_1(\xi a) \text{ at } z = o.$$

On $z = o$, $v_2 = A$ and $dv_2/dz = -A(\xi^2 + s^2/\beta^2)^{1/2}$.

Using these relations we get

$$A = -\frac{Pa}{\mu}\frac{J_1(\xi a)}{(\xi^2+s^2/\beta^2)^{1/2}} \ .$$

Substituting the value of A in (10) and inverting the Hankel transform (8), we obtain

$$v_1(r,z,s) = -\frac{Pa}{\mu}\int_0^\infty \frac{\xi J_1(\xi a)J_1(\xi r)}{(\xi^2+s^2/\beta^2)^{1/2}} \exp\left[-z\left(\xi^2+\frac{s^2}{\beta^2}\right)^{1/2}\right]d\xi. \qquad (11)$$

From a well-known result (Watson (1966),p.358)

$$J_1(\xi r)J_1(\xi a) = \frac{1}{\pi}\int_0^\pi J_0(\xi R)\cos\phi\, d\phi .$$

and

$$J_0(\xi R) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\xi R\sin\psi}\, d\psi \qquad \text{(Erdelyi(1953).p.14)}$$

where $R = \sqrt{(r^2+a^2-2ar\cos\phi)}$, we obtain

$$\frac{2\pi^2\mu v_1}{Pa} = -\int_0^\pi I_1\cos\phi\, d\phi \qquad (12)$$

where

$$I_1 = \int_0^{2\pi}\int_0^\infty \frac{\xi\exp\left[-z(\xi^2+s^2/\beta^2)^{1/2} + i\xi R\sin\psi\right]}{(\xi^2+s^2/\beta^2)^{1/2}}\, d\xi\, d\psi$$

If we put $p = \xi\sin\psi$ and $q = \xi\cos\psi$ in I_1, then

$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp\left[-z(p^2+q^2+s^2/\beta^2)^{1/2} + iRp\right]}{(p^2+q^2+s^2/\beta^2)^{1/2}}\, dp\, dq. \qquad (13)$$

To find the inversion of I_1, we adopt Cagniard's technique as modified by De Hoop(1959). Accordingly in (13), we put $p = ms$ and $q = ns$, then

$$I_1 = 2\int_0^\infty dn \int_{-\infty}^{\infty} \frac{s\ \exp\left\{-s\left[z(m^2+n^2+1/\beta^2)^{1/2} - iRm\right]\right\}}{(m^2+n^2+1/\beta^2)^{1/2}}\, dm \qquad (14)$$

In the above integral the path of integration with respect to m is the real axis (Fig.1) which is deformed in such a way that $-iRm + z(m^2+n^2+1/\beta^2)^{1/2} = t$, where t is real and positive. The deformed path of integration is the branch Γ of a hyperbola whose equation is

$$m = \frac{iRt \pm z\left[t^2 - (z^2+R^2)(n^2+1/\beta^2)\right]^{1/2}}{z^2+R^2}, \quad \left\{(z^2+R^2)(n^2+1/\beta^2)\right\}^{1/2} < t < \infty.$$

In the course of deformation of the path of integration it is essential to know the singularities of the function $s/(m^2+n^2+1/\beta^2)^{1/2}$ in the m-plane which are the branch points $\pm\, i(n^2+1/\beta^2)^{1/2}$.

Since the hyperbolic path Γ does not cross any of the singularities during its deformation, it is possible by virtue of Cauchy's theorem and Jordon's lemma, to replace the integration along the real m-axis by an integration along the hyperbolic path Γ.

We assume

$$m_+ = \frac{iRt + z\left[t^2-(z^2+R^2)(n^2+1/\beta^2)\right]^{1/2}}{z^2+R^2}$$

and

$$m_- = \frac{iRt - z\left[t^2-(z^2+R^2)(n^2+1/\beta^2)\right]^{1/2}}{z^2+R^2}$$

then

$$\frac{dm_\pm}{dt} = \frac{iR\left[t^2-(z^2+R^2)(n^2+1/\beta^2)\right]^{1/2} \pm zt}{(z^2+R^2)\left[t^2-(z^2+R^2)(n^2+1/\beta^2)\right]^{1/2}}$$

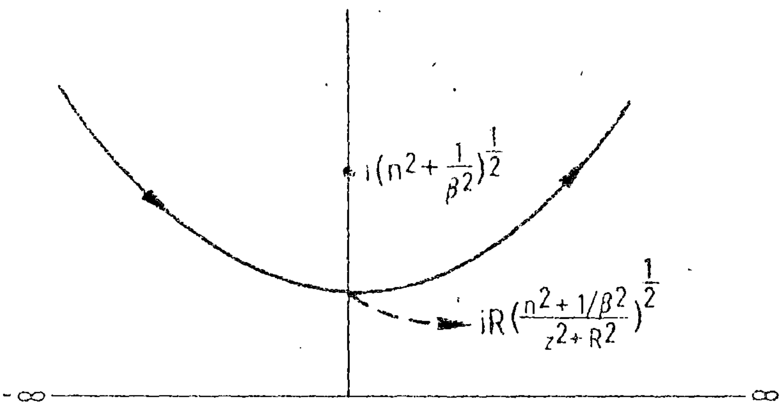

Figure 1
Paths of integration in the complex *m*-plane.

The point where Γ cuts the imaginary axis is given by

$$t = \left\{(z^2 + R^2)(n^2 + 1/\beta^2)\right\}^{1/2}$$

and the point is

$$m = \frac{iR(n^2+1/\beta^2)^{1/2}}{(z^2+R^2)^{1/2}}$$

which is below the branch point $i(n^2+1/\beta^2)^{1/2}$. Hence (14) can be written as

$$I_1 = 2\int_0^\infty dn \int_{\left\{(z^2+R^2)(n^2+1/\beta^2)\right\}^{1/2}}^\infty se^{-st}\left[\frac{1}{(m_+^2+n^2+1/\beta^2)^{1/2}}\frac{dm_+}{dt} - \frac{1}{(m_-^2+n^2+1/\beta^2)^{1/2}}\frac{dm_-}{dt}\right]dt \quad (15)$$

Now using the fact that $m_- = -\bar{m}_+$ and $dm_-/dt = -(d\bar{m}_+/dt)$ where $\bar{m}$ is the complex conjugate of m, (15) can be written as

$$I_1 = 4\int_0^\infty dn \int_{\left\{(z^2+R^2)(n^2+1/\beta^2)\right\}^{1/2}}^\infty se^{-st}\ \mathrm{Re}\left[\frac{(dm_+/dt)}{(m_+^2+n^2+1/\beta^2)^{1/2}}\right]dt$$

Changing the order of integration, we get,

$$I_1 = 4\int_{(z^2+R^2)^{1/2}/\beta}^\infty se^{-st}\, dt \int_0^{\left[t^2/(z^2+R^2)-1/\beta^2\right]^{1/2}} \mathrm{Re}\left[\frac{dm_+/dt}{(m_+^2+n^2+1/\beta^2)^{1/2}}\right]dn \quad (16)$$

Now,

$$\mathrm{Re}\left[\frac{(dm_+/dt)}{(m_+^2+n^2+1/\beta^2)^{1/2}}\right] = \frac{1}{\left\{t^2-(z^2+R^2)(n^2+1/\beta^2)\right\}^{1/2}}$$

Substituting this result in (16), we obtain

$$I_1 = 4\int_{\frac{1}{\beta}(z^2+R^2)^{1/2}}^{\infty} se^{-st}\,dt \int_0^{\left[t^2/(z^2+R^2)-1/\beta^2\right]^{1/2}} \frac{dn}{\left[t^2-(z^2+R^2)(n^2+1/\beta^2)\right]^{1/2}}$$

$$= \frac{2\pi}{(z^2+R^2)^{1/2}} \int_{\left[(z^2+R^2)^{1/2}\right]/\beta}^{\infty} se^{-st}\,dt.$$

Hence the Laplace inversion of I_1 is

$$I = \frac{2\pi}{(z^2+R^2)^{1/2}}\ \frac{d}{dt}\left[H\left\{t-\frac{(z^2+R^2)^{1/2}}{\beta}\right\}\right]$$

$$= \frac{2\pi}{(z^2+R^2)^{1/2}}\ \delta\left[t-\frac{(z^2+R^2)^{1/2}}{\beta}\right] \qquad (17)$$

Therefore the Laplace inversion of (12) by using the Laplace inversion of I_1 as given in (17) is

$$v(r,z,t) = -\frac{Pa}{\pi/\mu}\int_0^{\pi}\frac{\delta\left[t-\frac{(z^2+r^2+a^2-2ra\cos\phi)^{1/2}}{\beta}\right]}{(z^2+r^2+a^2-2ra\cos\phi)^{1/2}}\cos\phi\, d\phi. \qquad (18)$$

To evaluate the above integral we put

$$(z^2 + r^2 + a^2 - 2ra \cos\phi)^{1/2} = \beta\theta,$$

then $\dfrac{d\phi}{d\theta} = \dfrac{\beta^2\theta}{r\,a\,\sin\phi}$,

$$\sin\phi = \frac{1}{2ra}\left[2(\beta^2\theta^2 - z^2)(r^2+a^2) - (\beta^2\theta^2 - z^2)^2 - (r^2-a^2)^2\right]^{1/2}$$

and (18) can be written as

$$v(r,z,t) = -\frac{P\beta}{\pi\mu r}\int_{\frac{1}{\beta}\{z^2+(r-a)^2\}^{1/2}}^{\frac{1}{\beta}\{z^2+(r+a)^2\}^{1/2}} \frac{(z^2+r^2+a^2-\beta^2\theta^2)\,\delta(t-\theta)}{\left[2(r^2+a^2)(\beta^2\theta^2-z^2)-(r^2-a^2)^2-(\beta^2\theta^2-z^2)^2\right]^{1/2}}\,d\theta$$

$$= \frac{P\beta}{\pi\mu r}\;\frac{\beta^2 t^2 - z^2 - r^2 - a^2}{\left\{2(r^2+a^2)(\beta^2t^2-z^2)-(r^2-a^2)^2-(\beta^2t^2-z^2)^2\right\}^{1/2}}$$

for $\quad \frac{1}{\beta}\left\{z^2 + (r-a)^2\right\}^{1/2} < t < \frac{1}{\beta}\left\{z^2 + (r+a)^2\right\}^{1/2}$ (19)

<u>Case II: Inhomogeneous Medium</u>

FORMULATION OF THE PROBLEM: In this case the same problem of torsional motion of a semi-infinite elastic medium due to the presence of a ring source $r = a$, on the free surface $z = 0$ as in Case I is considered. The only difference is that the medium under consideration is in - homogeneous in nature, the coefficient of rigidity and the density of the medium are assumed to be

$$\mu = \mu_0(1 + \epsilon z)^2 \quad \text{and} \quad \rho = \rho_0(1 + \epsilon z)^2. \qquad (20)$$

Here also the non-vanishing stress components and the non-zero equations of motion are the same as in Case I, given by the equations (1),(2) and (3).

METHOD OF SOLUTION: Firstly we put $\bar{v} = (1 + \epsilon z)v$ in the equations (1),(2) and (3). The transformed equations are

$$\tau_{r\theta} = \mu_o(1 + z)\left(\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}\right),$$

$$\tau_{\theta z} = \mu_o\left\{(1 + \epsilon z)\frac{\partial \bar{v}}{\partial z} - \epsilon \bar{v}\right\} \tag{21}$$

and

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\left(\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}\right) + \frac{\partial^2 \bar{v}}{\partial z^2} = \frac{1}{\beta^2}\frac{\partial^2 \bar{v}}{\partial t^2} \tag{22}$$

where $\beta = \sqrt{(\mu_o / \rho_o)}$.

Taking the Laplace transform of the equation with respect to t, we obtain

$$\frac{\partial^2 \bar{v}_1}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}_1}{\partial r} - \left(\frac{1}{r^2} + \frac{s^2}{\beta^2}\right)\bar{v}_1 + \frac{\partial^2 \bar{v}_1}{\partial z^2} = 0 \tag{23}$$

where s is the Laplace transform parameter which is real and positive. Taking the Hankel transform of the equation (23) we have

$$\frac{d^2 \bar{v}_2}{dz^2} = \left(\xi^2 + \frac{s^2}{\beta^2}\right)\bar{v}_2 \,. \tag{24}$$

The general solution of this equation which remains bounded for large values of z is

$$\bar{\bar{v}}_2 = B \exp\left[- z\left(\xi^2 + \frac{s^2}{\beta^2} \right)^{1/2} \right] . \qquad (25)$$

Applying the Hankel transform and the Laplace transform on the boundary condition

$$\tau_{\theta z} = \mu_o\left[(1 + \epsilon z) \frac{\partial \bar{\bar{v}}}{\partial z} - \epsilon \bar{\bar{v}} \right] = P\delta(r - a)\delta(t)$$

and using (25), the value of B is found to be

$$B = - \frac{PaJ_1(\xi a)}{\mu \left\{\epsilon + (\xi^2 + s^2/\beta^2)^{1/2}\right\}}$$

Substituting this value of B in (25), it follows that

$$\bar{\bar{v}}_2 = - \frac{PaJ_1(\xi a)}{\mu_o \left\{\epsilon + (\xi^2 + s^2/\beta^2)^{1/2}\right\}} \exp\left[- z\left(\xi^2 + \frac{s^2}{\beta^2} \right)^{1/2}\right]. \qquad (26)$$

Taking the Hankel inversion of (26) , we have

$$\bar{v}_1 = - \frac{Pa}{\mu_o} \int_0^\infty \frac{\xi J_1(\xi a)J_1(\xi r)}{\left\{\epsilon + (\xi^2 + s^2/\beta^2)^{1/2}\right\}} \exp\left[- z\left(\xi^2 + \frac{s^2}{\beta^2} \right)^{1/2}\right] d\xi , \qquad (27)$$

Now,

$$\int_0^\infty \exp\left[- k \left\{\epsilon + (\xi^2 + s^2/\beta^2)^{1/2} \right\}\right] dk = \frac{1}{\epsilon + (\xi^2 + s^2/\beta^2)^{1/2}}$$

Using the above result, (27) is written as

$$\bar{v}_1 = -\frac{Pa}{\mu_o}\int_0^\infty e^{-\epsilon k}\, dk \int_0^\infty \xi J_1(\xi a) J_1(\xi r) \exp\left[-(z+k)\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right] d\xi \qquad (28)$$

We now replace $J_1(\xi a)J_1(\xi r)$ of (28) by the integral, which was used to modify equation (11). Finally we get

$$\bar{v}_1 = -\frac{Pa}{2\pi^2\mu_o}\int_0^\infty e^{-\epsilon k}\, dk \int_0^\pi I_2 \cos\phi \, d\phi , \qquad (29)$$

where

$$I_2 = \int_0^{2\pi} d\psi \int_0^\infty \xi \exp\left[-(z+k)\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2} + i\xi R \sin\psi\right] d\xi.$$

Assuming $p = \xi \sin\psi$ and $q = \xi \cos\psi$, it follows that

$$I_2 = \int_{-\infty}^\infty \int_{-\infty}^\infty \exp\left[-(z+k)(p^2+q^2+s^2/\beta^2)^{1/2} + i\, Rp\right] dp.dq$$

$$= 2\int_0^\infty dn \int_{-\infty}^\infty s^2 \exp\left[-s\left\{(z+k)\left(m^2+n^2+\frac{1}{\beta^2}\right)^{1/2} - iRm\right\}\right] dm. \qquad (30)$$

where , $p = ms$ and $q = ns$.

As in Case I, here also the path of integration with respect to m which is the real axis is deformed such that
$-iRm + (z+k)(m^2 + n^2 + 1/\beta^2)^{1/2} = t$, whre t is real and positive. The deformed path is a branch Γ_1 of a hyperbola the equation of which is

$$m = \frac{iRt \pm (z+k)\left[t^2 - \{(z+k)^2 + R^2\}(n^2 + 1/\beta^2)\right]^{1/2}}{(z+k)^2 + R^2},$$

$$\{(z+k)^2 + R^2\}^{1/2} \left(n^2 + \frac{1}{\beta^2}\right)^{1/2} < t < \infty.$$

Noting that the point where Γ_1 cuts the imaginary axis is

$$m = \frac{iR(n^2 + 1/\beta^2)^{1/2}}{\{(z+k)^2 + R^2\}^{1/2}}$$

when

$$t = \{(z+k)^2 + R^2\}^{1/2} \left(n^2 + \frac{1}{\beta^2}\right)^{1/2},$$

one gets from the equation (30)

$$I_2 = 4 \int_0^\infty dn \int_{\{(z+k)^2+R^2\}^{1/2}(n^2+1/\beta^2)^{1/2}}^\infty s^2 e^{-st} \operatorname{Re}\left(\frac{dm_+}{dt}\right) dt$$

Changing the order of integration, we obtain

$$I_2 = 4(z+k)/\{(z+k)^2 + R^2\}^{3/2} \int_{\frac{1}{\beta}\{(z+k)^2+R^2\}^{1/2}}^\infty ts^2 e^{-st} dt \; X$$

$$X \int_0^{\left[t^2/\{(z+k^2)+R^2\}-(1/\beta^2)\right]^{1/2}} \frac{dn}{\left[t^2/\{(z+k)^2+R^2\}-1/\beta^2-n^2\right]^{1/2}}$$

$$= \frac{2\pi(z + k)}{\left\{(z + k)^2 + R^2\right\}^{3/2}} \int_{\left[\left\{(z+k)^2+R^2\right\}^{1/2}\right]/\beta}^{\infty} ts^2\, e^{-st}\, dt. \qquad (31)$$

Hence the Laplace inversion of (31) is

$$I=2\pi(z+k)/\left\{(z+k)^2+R^2\right\}^{3/2} \frac{d^2}{dt^2} \left[t\, H\left\{t- \frac{1}{\beta} \left[(z+k)^2+R^2\right]^{1/2}\right\}\right]$$

$$= \frac{2\pi(z+k)}{\left\{(z+k)^2+R^2\right\}^{3/2}} \left\{2\delta\left[t-\left(\frac{(z+k)^2+ R^2}{\beta^2} \right)^{1/2}\right]+t\delta'\left[t-\left(\frac{(z+k)^2 +R^2}{\beta^2} \right)^{1/2}\right]\right\}.$$

Taking the Laplace transform of (29) and using the value of I, it is found that

$$\bar{\bar{v}} = - \frac{Pa}{\pi\mu_o} \int_0^{\infty} (z+k)Je^{-\epsilon k}\, dk. \qquad (32)$$

where

$$J= \int_0^{\pi} \left\{2\delta\left[t-\left(\frac{(z+k)^2+R^2}{\beta^2} \right)^{1/2}\right]+t\delta'\left[t-\left(\frac{(z+k)^2 +R^2}{\beta^2} \right)^{1/2}\right]\right\} \times$$

$$\times \frac{\cos \phi}{\left\{(z+k)^2+R^2\right\}^{3/2}}\, d\phi.$$

To evaluate the above integral we put

$$l= \frac{1}{\beta} \left\{(z+k)^2 + R^2\right\}^{1/2},$$

then

$$J = \int_{(1/\beta)\left\{(z+k)^2+(r-a)^2\right\}^{1/2}}^{(1/\beta)\left\{(z+k)^2+(r+a)^2\right\}^{1/2}} \left\{2\delta(t-\ell)+t\delta'(t-\ell)\right\} \frac{\cos\phi}{\ell^3\beta^3} \frac{d\phi}{d\ell}\, d\ell ,$$

where

$$\frac{d\phi}{d\ell} = \frac{\beta^2\ell}{ra\sin\phi} \quad \text{and} \quad \cos\phi = \frac{(z+k)^2 + r^2 + a^2 - \beta^2\ell^2}{2ra}.$$

Substituting these values, we get

$$J = \frac{1}{ra\beta} \int_{(1/\beta)\left\{(z+k)^2+(r-a)^2\right\}^{1/2}}^{(1/\beta)\left\{(z+k)^2+(r+a)^2\right\}^{1/2}} f(\ell,k)\left[2\delta(t-l)+t\delta'(t-l)\right]d\ell, \qquad (33)$$

where

$$f(\ell,k)= \frac{(z+k)^2 + r^2 + a^2 - \beta^2\ell^2}{\ell^2\left[2(r^2+a^2)\left\{\beta^2\ell^2-(z+k)^2\right\}-(r^2-a^2)^2-\left\{\beta^2\ell^2-(z+k)^2\right\}^2\right]^{1/2}}$$

and it is to be remembered that δ' is the derivative of the Dirac's δ-function with respect to t. Integrating (33),we get

$$J = \frac{1}{ra\beta}\left[2f(t,k)-tf(\ell_1,k)\delta(t-\ell_1)+tf(\ell_2,k)\delta(t-\ell_2)+tf'(t,k)\right] \qquad (34)$$

where

$$\ell_1= \frac{1}{\beta}\left\{(z+k)^2+(r+a)^2\right\}^{1/2}, \quad \ell_2 = \frac{1}{\beta}\left\{(z+k)^2+(r-a)^2\right\}^{1/2} \quad \ell_2 \leqslant t \leqslant \ell_1.$$

It is to be noted that if t does not belong to $[\ell_2, \ell_1]$ then the integrand in (33) is zero, consequently $J = 0$.

Substituting the value of J in (32), we get

$$\bar{\bar{v}} = -\frac{P}{\pi/\mu\, c\beta r}\int_0^\infty (z+k)e^{-\epsilon k}\Big[2f(t,k) - tf(\ell_1,k)\delta(t-\ell_1) + tf(\ell_2,k)\delta(t-\ell_2) + tf'(t,k)\Big]dk. \qquad (35)$$

Now, $\ell_2 \leqslant t \leqslant \ell_1$ implies that

$$\left\{\beta^2 t^2 - (r+a)^2\right\}^{1/2} - z \leqslant k \leqslant \left\{\beta^2 t^2 - (r-a)^2\right\}^{1/2} - z. \qquad (36)$$

In evaluating the integral (35), the following sub-cases are to be considered, keeping in mind that k satisfies (36) and that $k \geqslant 0$

i) If $\left\{\beta^2 t^2 - (r-a)^2\right\}^{1/2} - z < 0$, that is if $\beta t < \left\{z^2 + (r-a)^2\right\}^{1/2}$ then, t does not belong to $[\ell_2, \ell_1]$, so $J = 0$. Consequently $\bar{\bar{v}} = 0$. This is in accordance with the physical condition of the problem because a disturbance cannot reach a point Q (Fig.2) before the time $(1/\beta)\left\{z^2 + (r-a)^2\right\}^{1/2}$, which is the time of arrival of the disturbance at the point Q from the nearest point of the ring source.

ii) $\left\{\beta^2 t^2 - (r+a)^2\right\}^{1/2} - z < 0 < \left\{\beta^2 t^2 - (r-a)^2\right\}^{1/2} - z$, that is,

$$\left\{z^2 + (r-a)^2\right\}^{1/2} < \beta t < \left\{z^2 + (r+a)^2\right\}^{1/2}.$$

In this case (35) takes the form

$$\bar{v} = \frac{-p}{\pi/^{\mu}o\beta r} \int_0^{\{\beta^2t^2-(r-a)^2\}^{1/2}-z} (z+k)e^{-\epsilon k}\Big[2f(t,k)-tf(l_1,k)\delta(t-l_1) + tf(l_2,k)\delta(t-l_2)+tf'(t,k)\Big]dk. \tag{37}$$

The integrand of (37) is considered as a generalized function, so the finite part of the integral (37) is retained (JONES(1966),p.89)and we get

$$\bar{v} = \frac{p\beta}{\pi r/^{\mu}o} \frac{\beta^2t^2 - z^2 - r^2 - a^2}{[2(r^2+a^2)(\beta^2t^2-z^2)-(r^2-a^2)^2-(\beta^2t^2-z^2)^2]^{1/2}}$$

$$+ \frac{p\beta\epsilon}{\pi r/^{\mu}o} \int_0^{\{\beta^2t^2-(r-a)^2\}^{1/2}-z}$$

$$\times \frac{\{(z+k)^2+r^2+a^2-\beta^2t^2\}e^{-\epsilon k}\, dk}{[2(r^2+a^2)\{\beta^2t^2-(z+k)^2\}-(r^2-a^2)^2-\{\beta^2t^2-(z+k)^2\}^2]^{1/2}}$$

Hence

$$v = \frac{p\beta}{\pi r/^{\mu}o(1+\epsilon z)} \frac{\beta^2t^2 - z^2 - r^2 - a^2}{[2(r^2+a^2)(\beta^2t^2-z^2)-(r^2-a^2)^2-(\beta^2t^2-z^2)^2]^{1/2}}$$

$$+ \frac{p\beta\epsilon}{\pi r/^{\mu}o(1+ z)} \int_0^{\{\beta^2t^2-(r-a)^2\}^{1/2}-z}$$

$$\times \frac{\{(z+k)^2 + r^2 + a^2 - \beta^2t^2\}e^{-\epsilon k}\, dk}{[2(r^2+a^2)\{\beta^2t^2-(z+k)^2\}-(r^2-a^2)^2-\{\beta^2t^2-(z+k)^2\}^2]^{1/2}} \tag{38}$$

In (38) if we put $\epsilon = o$, we get the same result that we have determined in (19) of Case I.

iii) If $\left\{\beta^2 t^2 - (r+a)^2\right\}^{1/2} - z > o$, that is if $\beta t > \left\{z^2 + (r+a)^2\right\}^{1/2}$ then

$$v = \frac{B\beta\epsilon}{\pi r/\mu_o(1+\epsilon z)} \int_{\left\{\beta^2 t^2 - (r+a)^2\right\}^{1/2} - z}^{\left\{\beta^2 t^2 - (r-a)^2\right\}^{1/2} - z}$$

$$\times \frac{\left\{(z+k)^2 + r^2 + a^2 - \beta^2 t^2\right\} e^{-\epsilon k}\, dk}{\left[2(r^2+a^2)\left\{\beta^2 t^2 - (z+k)^2\right\} - (r^2-a^2)^2 - \left\{\beta^2 t^2 - (z+k)^2\right\}^2\right]^{1/2}} \qquad (39)$$

It is interesting to note that in the case of a homogeneous medium there is no displacement at a point Q (Fig.2) after the time $t = (1/\beta)\left\{z^2 + (r+a)^2\right\}^{1/2}$, which is the time required by the disturbance to reach the point Q directly from the farthest point on the ring source from the point Q. But in the case of an inhomogeneous medium the disturbance reaches a point Q even after the time $t = (1/\beta)\left\{z^2 + (r+a)^2\right\}^{1/2}$ which is the maximum time required by a direct wave to reach the point Q from the farthest point on the source for the point Q. This is due to the fact that in the case of an inhomogeneous medium the region $z > o$ may be considered as an assembly of an infinite number of thin layers of material of infinitesimal thickness of continuously varying density and coefficient of rigidity. That is why the disturbance, which reaches the point Q after successive reflection and refraction in different layers of the medium, arrives at Q after the time $\beta t = \left\{z^2 + (r+a)^2\right\}^{1/2}$. The disturbance comes continuously after the time $\beta t = \left\{z^2 + (r+a)^2\right\}^{1/2}$ with decreasing intensity.

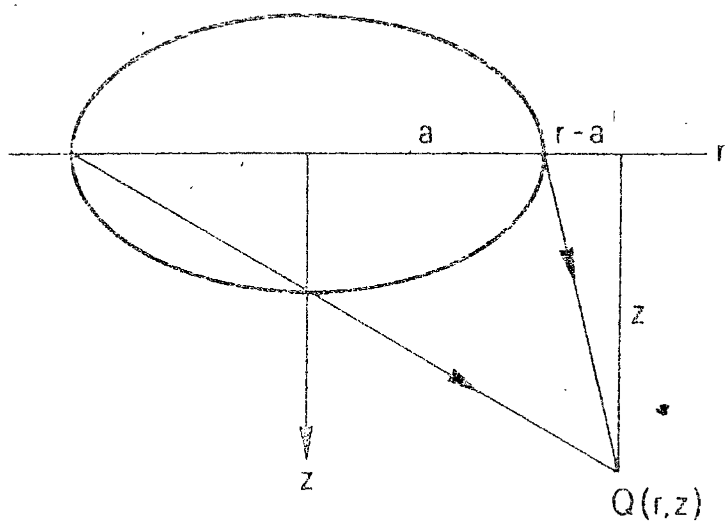

Figure 2
Arrival of the direct wave to Q from the nearest and the farthest point of the source.

<u>Numerical solution on the free surface z = o</u>

In order to obtain the displacement on the free surface we make the substitution

$$\left[2(r^2+a^2)(\beta^2t^2-k^2)-(r^2-a^2)^2-(\beta^2t^2-k^2)^2\right]^{1/2} = 2ra\sin\theta$$

which transforms the equations (38) and (39) to the forms given by

$$\frac{v\pi\mu_0 a}{P\beta} = d = d_1 + d_2$$

where

$$d_1 = \frac{a}{r}\frac{\dfrac{\beta^2t^2}{a^2}-\dfrac{r^2}{a^2}-1}{\left[2\left(\dfrac{r^2}{a^2}+1\right)\dfrac{\beta^2t^2}{a^2}-\left(\dfrac{r^2}{a^2}-1\right)^2-\dfrac{\beta^4t^4}{a^4}\right]^{1/2}}$$

$$d_2 = \epsilon a\int_0^{\cos^{-1}\Lambda}\frac{\cos\theta\left\{\exp -\epsilon a\left[2\frac{r}{a}\cos\theta+\dfrac{\beta^2t^2}{a^2}-\dfrac{r^2}{a^2}-1\right]^{1/2}\right\}}{\left[2\frac{r}{a}\cos\theta+\dfrac{\beta^2t^2}{a^2}-\dfrac{r^2}{a^2}-1\right]^{1/2}}\,d\theta.$$

(40)

$$\Lambda = \frac{r^2+a^2-\beta^2t^2}{2ra},\quad r-a<\beta t<r+a,$$

$$\frac{v\pi\mu_0 a}{P\beta} = d' = \epsilon a\int_0^{\pi}\frac{\cos\theta\exp\left\{-\epsilon a\left[2\frac{r}{a}\cos\theta+\dfrac{\beta^2t^2}{a^2}-\dfrac{r^2}{a^2}-1\right]^{1/2}\right\}}{\left[2\frac{r}{a}\cos\theta+\dfrac{\beta^2t^2}{a^2}-\dfrac{r^2}{a^2}-1\right]}\,d\theta,$$

(41)

for $\beta t > r + a$ respectively.

If $\epsilon a = o$, then from (40) it follows that $d = d_1$, which corresponds to the displacement inhomogeneous medium. The integrals in (40) and (41) giving the displacements d and d' have been numerically evaluated for different values of ϵa at different points on the free surface and are presented in Tables 1-4 for different values of $\beta t/a$.

Concluding remarks

From Tables 1-4 it is found that the difference in the values of the displacement at any point corresponding to $\epsilon a = o$ and $\epsilon a = 10$ gradually diminishes with the

Table 1

$r/a = 2.\ (r/a) - 1 < (\beta t/a) < (r/a) + 1$

$\beta t/a$	d when ϵa=o	d when ϵa=1	d when ϵa = 10
1.2	-0.97596	-0.32841	-0.50851
1.4	-0.58468	-0.08456	-0.41435
1.6	-0.38490	0.00497	-0.31149
1.8	-0.24498	0.05256	-0.21268
2.0	-0.12909	0.08585	-0.11623
2.2	-0.02001	0.11644	-0.01716
2.4	0.09676	0.15276	0.09355
2.6	0.24498	0.20795	0.23612
2.8	0.50411	0.32902	0.48230

Table 2

$r/a = 10, \; (r/a) - 1 < (\beta t/a) < (r/a) + 1$

$\beta t/a$	d when ϵa = o	d when ϵa = 1	d when ϵa = 10
9.2	-0.14221	-0.00782	-0.13509
9.4	-0.08155	-0.00314	-0.08070
9.6	-0.04927	-0.00063	-0.04911
9.8	-0.02559	0.00324	-0.02556
10.0	-0.00500	0.00868	-0.00500
10.2	0.01537	0.01588	0.01537
10.4	0.03834	0.02557	0.03833
10.6	0.06901	0.03990	0.06900
10.8	0.12546	0.06799	0.12542

Table 3

$r/a = 50, \; (r/a) - 1 < (\beta t/a) < (r/a) + 1$

$\beta t/a$	d when ϵa = o	d when ϵa = 1	d when ϵa = 10
49.2	-0.02700	-0.00822	-0.02699
49.4	-0.01525	-0.00693	-0.01525
49.6	-0.00894	-0.00474	-0.00894
50.0	-0.00020	0.00028	-0.00020
50.2	0.00387	0.00318	0.00387
50.4	0.00851	0.00665	0.00851
50.6	0.01475	0.01130	0.01475
50.8	0.02633	0.01944	0.02633

Table 4

$r/a = 2, (\beta t/a) > (r/a) + 1$

$\beta t/a$	d' when $\epsilon a = 1$	d' when $\epsilon a = 10$
3.2	-0.17211	
3.4	-0.07793	
3.6	-0.04250	
3.8	-0.02533	d' is of the order of 10^{-7}
4.0	-0.01593	
4.2	-0.01040	
4.4	-0.00697	
4.6	-0.00477	
4.8	-0.00332	

When $r = 10a$ or $\epsilon a = 10$, d' is very small.

increase in the value of r/a. This is also apparent from the expression for d_2 in (40) because the exponential term

$$\exp\left\{-\epsilon a\left[2\frac{r}{a}\cos\theta + \frac{\beta^2 t^2}{a^2} - \frac{r^2}{a^2} - 1\right]^{1/2}\right\}$$

in the integrand for large values of r/a decreases rapidly with the increase in value of ϵa.

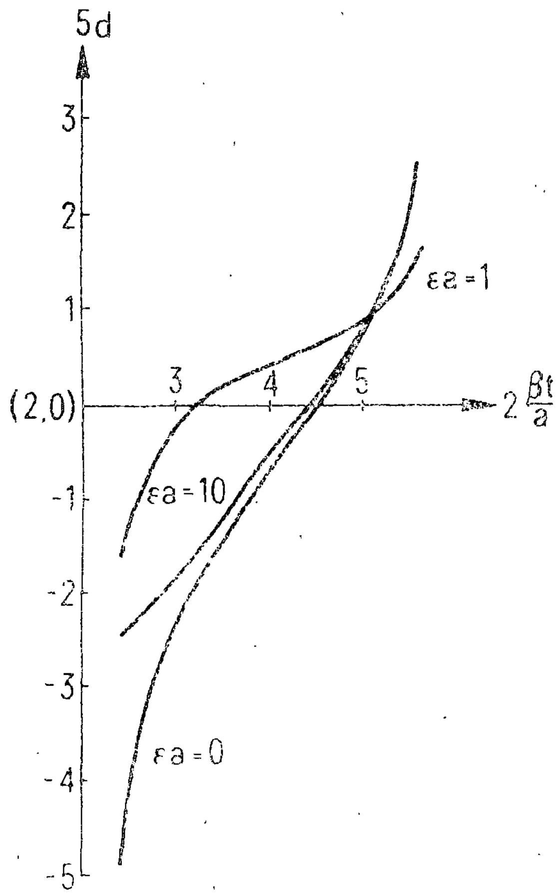

Figure 3

$r = 2a$, variation in displacement near the source for $\varepsilon a = 0, 1, 10$.

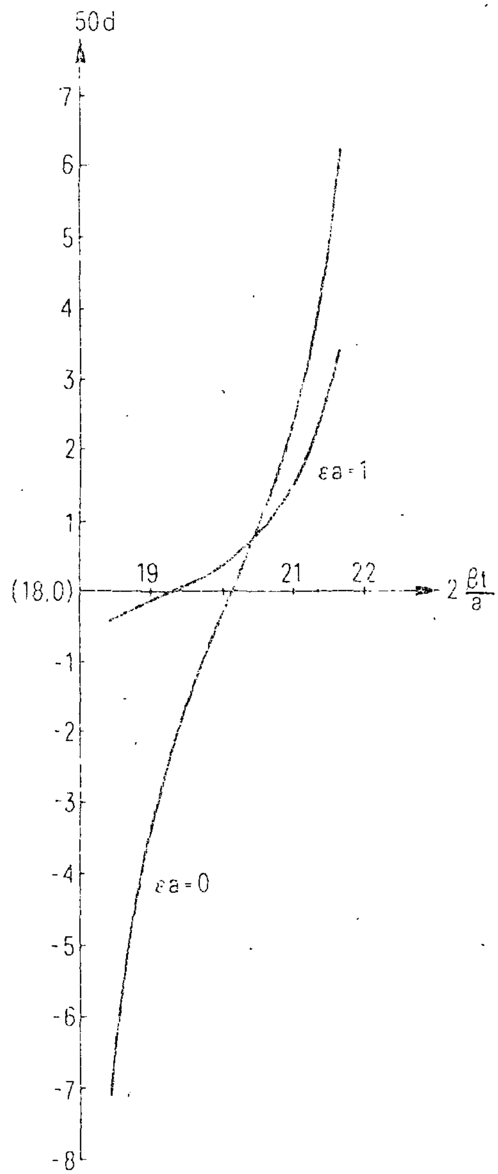

Figure 4
$r = 10a$, variation in displacement at a moderate distance from the source for $\varepsilon a = 0, 1$.

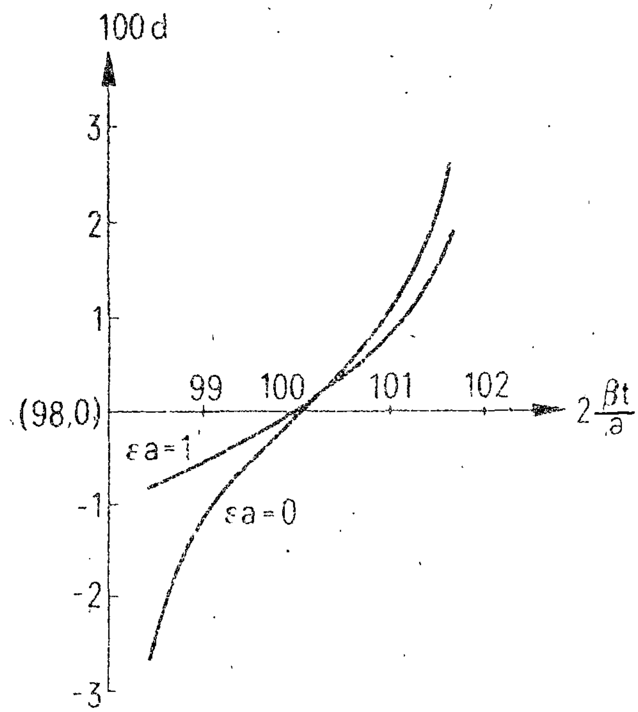

Figure 5

$r = 50a$, variation in displacement at a large distance from the source for $\varepsilon a = 0, 1$.

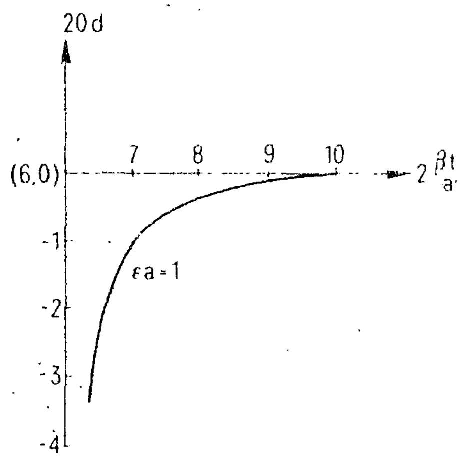

Figure 6

$r = 2a$, variation in displacement after the maximum time required by a direct wave to arrive from the farthest point of the source when $\varepsilon a = 1$

SH-WAVES IN AN ELASTIC HALF SPACE DUE TO A RING SOURCE OF INCREASING RADIUS.

INTRODUCTION: The torsional vibration of an elastic half space due to a surface force which is periodic in time was first considered by Reissner (1937). Reissner and Sagoci (1944) determined the distribution of the stresses in the interior of a semi-infinite, homogenous isotropic elastic material due to a periodic shear stresses applied in an axially symmetric manner to a circular area of the plane surface by means of a rigid disk, the torsional displacement being prescribed under the disk. Verma (1957) discussed the static distribution of stresses and displacement when shearing stress is prescribed on the circumfeience of a circle on the plane boundary. Datta (1961) discussed the corresponding problem when shearing stress decreases exponentially with time. Ghosh (1964) exactly evaluated the displacement at any point of the medium when a twisting moment in the form $M\delta(t)$ is applied to the disk by following Cagniard (1939) and Dix (1954). Ghosh (1971) also discussed the axisymmetric problem of propagation of a stress discontinuity over a circular region by using Cagniard's (1939) method as modified by De-Hoop (1959). In the present paper the author determines the displacement in the integral form due to a ring source which increases steadily when the twisting impulse is prescribed by $P\delta(r-ct)H(t)$, where δ,H are two dimensional delta function and Heaviside function respectively, and then the exact evaluation of the displacement is determined after the first arrival of the shear wave and, the displacement at any point for large values of the time t.

Published in Ind. J. Mech. Math., Vol.XVIII, No.,2 P.74 (1980)

FORMULATION OF THE PROBLEM:

The isotropic, elastic, semi infinite medium is supposed to occupy the region $z > o$. We choose cylindrical polar co-ordinates (r, θ, z) with the z-axis directed into the medium, the plane boundary being $z = o$ with origin at the centre of the source. The displacement is calculated at points inside the medium assuming that the half space is, initially, at rest and that the displacement remains bounded even as $z \to +\infty$. Since the motion is symmetrical about z-axis for torsional motion of the ring source, all quantities depend on r, z and the time t. The only non-vanishing component of the displacement vector is the component v along the direction of θ increasing. Hence the non-vanishing stress components are

$$\tau_{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \text{ and } \quad \tau_{\theta z} = \mu \frac{\partial v}{\partial z} \qquad (1)$$

where μ is the coefficient of rigidity. The only non-zero equation of motion is

$$\frac{\partial}{\partial r}(\tau_{r\theta}) + \frac{\partial}{\partial z}(\tau_{\theta z}) + 2\frac{\tau_{r\theta}}{r} = \rho \frac{\partial^2 v}{\partial t^2} \qquad (2)$$

where ρ is the density of the medium, assumed constant. The boundary condition

$$\tau_{\theta z} = P\,\delta(r - ct)\,H(t) \text{ at } z = 0 \qquad (3)$$

c, P being constant H is the Heaviside function and δ is the two dimensional delta function given by

$$2\pi \int_0^\infty \delta(r)\, r\,dr = 1.$$

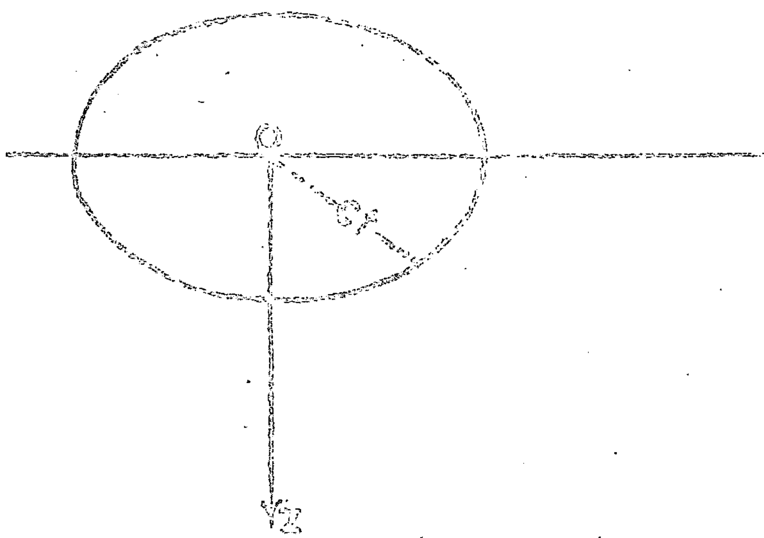

Fig.1 Co-ordinates system in the medium.

SOLUTION: We define for all positive real values of s, the Laplace transform $f_1(r, z, s)$ of a function $f(r, z, t)$ by

$$f_1(r,z,s) \int_0^\infty e^{-st} f(r,z,t)dt \quad (4)$$

Substituting the value of $\tau_{r\theta}$ and $\tau_{\theta z}$ in equation (2) and then applying the Laplace transform (4), we obtain

$$\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}\left(\frac{\partial v_1}{\partial r} - \frac{v_1}{r}\right) + \frac{\partial^2 v_1}{\partial z^2} = \frac{1}{\beta^2} s^2 v_1 \quad (5)$$

where $\beta = \sqrt{(\mu/\rho)}$ is the shear wave velocity.

Defining v_2 by the equation.

$$v_2(\xi,z,s) = \int_0^\infty r\, J_1(\xi r) v_1(r,z,s)dr \quad (6)$$

and then multiplying the equation (5) by $rJ_1(\xi r)$ and integrating with respect to r from o to ∞, we get

$$\frac{d^2 v_2}{dz^2} = (\xi^2 + s^2/\beta^2) v_2 \quad (7)$$

Taking ξ real, the general solution of the equation (7) which remains bounded for large values of z, is

$$v_2 = A \exp\left[-z(\xi^2 + s^2/\beta^2)^{1/2}\right]. \quad (8)$$

The Laplace transform of $\tau_{\theta z}$ is

$$(\tau_{\theta z})_1 = P\int_0^\infty e^{-st}\,\delta(r-ct)H(t)dt$$

$$= \frac{P}{2\pi cr}\,e^{-sr/c}$$

It's Hankel transform is

$$(\tau_{\theta z})_2 = \frac{P}{2\pi c}\int_0^\infty e^{-sr/c}\,J_1(\xi r)dr$$

$$= \frac{P}{2\pi c}\left[1 - \frac{s}{c}\left(\xi^2 + \frac{s^2}{c^2}\right)^{-1/2}\right]$$

[See Erdelyi, et al 1964, p. 19].

Noting that on $z = o$,

$\frac{dv_2}{dz} = -A(\xi^2 + s^2/\beta^2)^{1/2}$ and using the boundary condition,

we get

$$A = -\frac{P}{2\pi\mu\xi c}\;\frac{\left[1-\frac{s}{c}\left(\xi^2+\frac{s^2}{c^2}\right)^{-1/2}\right]}{(\xi^2 + s^2/\beta^2)^{1/2}}.$$

Substituting this value of A in (8) and inverting the Hankel transform (6), we obtain

$$v_1 = -\frac{P}{2\pi\mu c}\int_0^\infty \frac{1-\frac{s}{c}\left(\xi^2+\frac{s^2}{c^2}\right)^{-1/2}}{(\xi^2+s^2/\beta^2)^{1/2}}\,J_1(\xi r)e^{-z(\xi^2+s^2/\beta^2)^{1/2}}\,d\xi \qquad (9)$$

Now,

$$J_1(\xi r)= \frac{1}{2\pi}\int_0^{2\pi} e^{i\xi r \sin\psi}(\cos\psi - i\sin\psi)\,d\psi.$$ (See Erdelyi, A. etal 1953 P.14)

Substituting this value of $J_1(\xi r)$ in (9) and putting $p=\xi\sin\psi$ and $q=\xi\cos\psi$, we get

$$v_1= \frac{-P}{4\pi^2\mu c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(q-ip)\left\{(p^2+q^2+\frac{s^2}{c^2})^{1/2}-\frac{s}{c}\right\}}{(p^2+q^2+s^2/c^2)^{1/2}(p^2+q^2+s^2/\beta^2)^{1/2}}$$

$$\times\ \frac{e^{-z(p^2+q^2+s^2/\beta^2)^{1/2}+irp}}{(p^2+q^2)}\,dp\,dq.$$

To find the inversion of v_1, we put

$p=ms$ and $q=ns$ in the above integral, then we have

$$v_1= \frac{-P}{4\pi^2\mu c}\int_{-\infty}^{\infty}dn\int_{-\infty}^{\infty}\frac{(m-in)\left\{(m^2+n^2+1/c^2)^{1/2}-1/c\right\}e^{-s\left[z(m^2+n^2+1/\beta^2)^{-1/2}-irm\right]}}{(m^2+n^2+1/c^2)^{1/2}(m^2+n^2+1/\beta^2)^{1/2}(m^2+n^2)}\,dm$$

$$= \frac{iP}{2\pi^2\mu c}\int_{0}^{\infty}dn\int_{-\infty}^{\infty}\frac{m\left\{(m^2+n^2+1/c^2)^{1/2}-1/c\right\}e^{-s\left[z(m^2+n^2+1/\beta^2)^{1/2}-irm\right]}}{(m^2+n^2+1/c^2)^{1/2}(m^2+n^2+1/\beta^2)^{1/2}(m^2+n^2)}\,dm \qquad (10)$$

In the integral of the equation (10), the path of integration with respect to m is the real axis which is deformed in such a way that

$$-irm+z(m^2+n^2+1/\beta^2)^{1/2}=t \qquad (11)$$

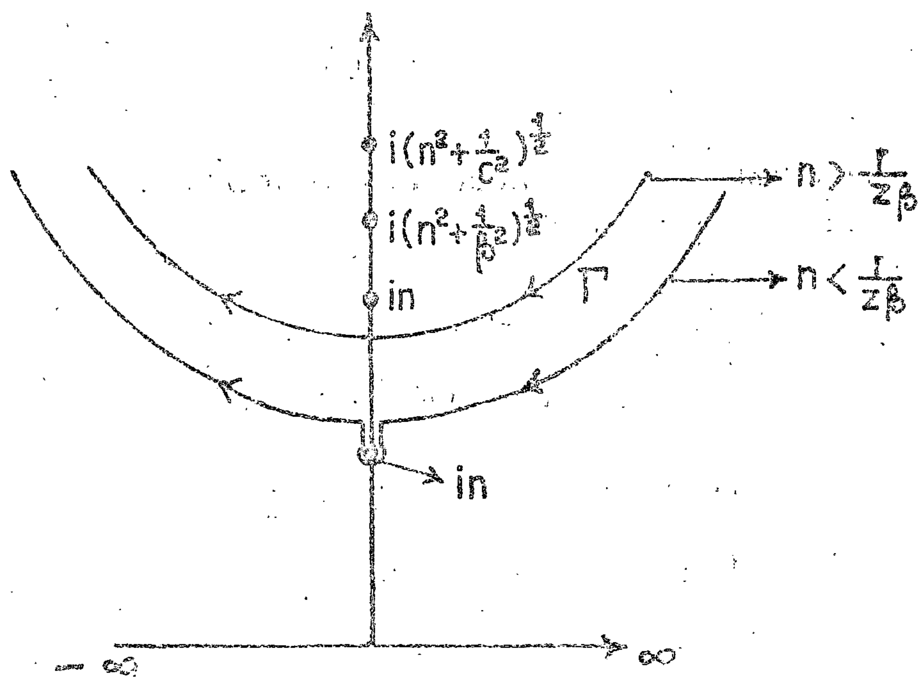

Fig. 2 Path of integration in the complex m-plane.

where t is real and positive. The deformed path of integration is the branch Γ of a hyperbola whose equation is

$$m = \frac{irt \pm z\left[t^2-(z^2+r^2)(n^2+1/\beta^2)\right]^{1/2}}{z^2+r^2}, \quad \left\{(z^2+r^2)\left(n^2+\frac{1}{\beta^2}\right)\right\}^{1/2} < t < \infty$$

We write

$$m_{\pm} = \frac{irt \pm z\left[t^2-(z^2+r^2)(n^2+1/\beta^2)\right]^{1/2}}{z^2+r^2},$$

$$K(m,n) = \frac{m\left\{(m^2+n^2+1/c^2)^{1/2} - 1/c\right\}}{(m^2+n^2+1/c^2)^{1/2}(m^2+n^2+1/\beta^2)^{1/2}(m^2+n^2)}$$

and we obtain finally the expression of v_1 in the form

$$v_1 = \frac{-P}{\pi^2 \mu c} \int_{1/\beta(z^2+r^2)^{1/2}}^{\infty} e^{-st}\, dt \int_0^{\sqrt{\gamma}} \operatorname{Im}\left[K(m_+, n)\frac{dm_+}{dt}\right] dn.$$

where $\gamma = \left\{t^2/(z^2+r^2)\right\} - \beta^{-2}$.

Taking the Laplace inversion, we get

$$v = \frac{-P}{\pi^2 \mu c} H\left[t - \beta^{-1}(z^2+r^2)^{1/2}\right] \int_0^{\sqrt{\gamma}} \operatorname{Im}\left[K(m_+, n)\frac{dm_+}{dt}\right] dn \tag{12}$$

APPROXIMATE EVALUATION OF THE DISPLACEMENT:

Case1. <u>Displacement after the first arrival.</u>

To integrate $\int_0^{\sqrt{\gamma}} \operatorname{Im}\left[K(m_+, n)\frac{dm_+}{dt}\right] dn$ (13)

we put $n = \sqrt{\gamma}\sin\alpha$ and $t_\beta = \beta^{-1}(z^2+r^2)^{1/2}$,

which is the time taken by the shear wave to reach the point(r,θ,z).

The integral (13) after the substitution takes the form

$$\int_0^{\pi/2} \mathrm{Im}\left[K(m_+,n)\frac{dm_+}{dt}\frac{dn}{d\alpha}\right]d\alpha. \tag{14}$$

Now, $\mathrm{Im}\left[K(m_+,n)\frac{dm_+}{dt}\frac{dn}{d\alpha}\right] = \frac{\beta}{r}\left[\frac{\beta(z^2+r^2)^{1/2}}{\beta^2(z^2+r^2)-c^2r^2\;1/2} - 1\right]$

as $t \to t_\beta$.

Hence from (12) , we obtain

$$v = \frac{-P\delta}{2\pi/^{\mu}cr}\left\{\frac{\beta(z^2+r^2)^{1/2}}{\beta^2(z^2+r^2)-c^2r^2\;1/2} - 1\right\} H(t-t_\beta),$$

which is the displacement at any point (r,z) just after the arrival of the disturbance. It is interesting to note that the displacement due to the first arrival of the disturbance at any point of the z-axis is zero which is also expected from the physical stand point. It is to be noted that the displacement at any point on the free surface z=o varies inversely as r.

Case 2. <u>Displacement after sufficiently large time when $z \neq o$.</u>

In this case, $\mathrm{Im}\left[K(m_+,n)\frac{dm_+}{dt}\frac{dn}{d\alpha}\right]$

$$= \frac{(z^2+r^2)^{1/2}}{t}\,\frac{r\{z^2\sin^2\alpha - (r^2+z^2)\cos^2\alpha\}}{(z^2+r^2\cos^2\alpha)^2} -$$

$$- \frac{(z^2+r^2)^{3/2}}{ct^2}\cdot\frac{rz\{z^2-3(r^2+z^2)\cos^2\alpha+r^2\cos^4\alpha\}}{(z^2+r^2\cos^2\alpha)^3}.$$

The terms containing $1/t^3$ and higher orders are neglected. After the above substitution (14) takes the following form

$$\frac{r(z^2+r_1^2)^{1/2}}{t}\int_0^{\pi/2}\frac{z^2\sin^2\alpha-(r^2+z^2)\cos^2\alpha}{(z^2+r^2\cos^2\alpha)^2}\,d\alpha -$$

$$-\frac{rz(z^2+r^2)^{3/2}}{ct^2}\int_0^{\pi/2}\frac{z^2-3(r^2+z^2)\cos^2\alpha+r^2\cos^4\alpha}{(z^2+r^2\cos^2\alpha)^3}\,d\alpha \tag{15}$$

The first integral of (15) is zero, hence for the large value of the time t compared to t_β, the displacement is given by

$$v=-Pr(4z^2+5r^2)/(4\pi\mu c^2t^2z^2).$$

In this case the displacement at any point varies inversely as t^2. Also this is to be noted that the displacement increase with the increase of r when t is very large, which is in conformity with the physical condition because the radius of the ring source after large time t is infinitely large.

Case 3. <u>Displacement at the free surface.</u>

In this case taking z=o, we obtain from Eq.(10)

$$v_1=\frac{iP}{2\pi^2\mu c_s}\Bigg[\int_0^\infty dn\int_{-\infty}^\infty\frac{me^{isrm}}{(m^2+n^2+1/\beta^2)^{1/2}(m^2+n^2)} -$$

$$-\frac{1}{c}\int_0^\infty dn\int_{-\infty}^\infty\frac{me^{isrm}}{(m^2+n^2+1/c^2)^{1/2}(m^2+n^2+1/\beta^2)^{1/2}(m^2+n^2)}\Bigg]dm.$$

(16)

The path of integration in the complex m-plane is the real axis, which is deformed in such a way that

$-irm=t$, where t is real and positive. Taking the integral over the deformed path we get

$$v_1 = \frac{Pr}{\pi^2 \mu c_s}\left[\int_0^\infty dn \int_{r(n^2+1/\beta^2)^{1/2}}^{\infty} \frac{te^{-st}}{\left\{t^2-r^2(n^2+1/\beta^2)\right\}^{1/2}(t^2-r^2n^2)}\,dt -\right.$$

$$\left. - \frac{r}{c}\left\{\int_0^\infty dn \int_{r(n^2+1/\beta^2)^{1/2}}^{r(n^2+1/c^2)^{1/2}} \frac{te^{-st}}{\left\{r^2(n^2+1/c^2)-t^2\right\}^{1/2}\left\{t^2-r^2(n^2+1/\beta^2)\right\}^{1/2}(t^2-r^2n^2)}\,dt\right.\right]$$

Changing the order of integration. We obtain

$$v_1 = \frac{Pr}{\pi^2 \mu c_s}\left[\int_{r/\beta}^\infty te^{-st}\,dt \int_0^{(t^2/r^2-1/\beta^2)^{1/2}} \frac{dn}{\left\{t^2-r^2(n^2+1/\beta^2)\right\}^{1/2}(t^2-r^2n^2)} -\right.$$

$$- \frac{r}{c}\left\{\int_{r/\beta}^{r/c} te^{-st}\,dt \int^{(t^2/r^2-1/\beta^2)^{1/2}} \frac{dn}{\left\{r^2(n^2+1/c^2)-t^2\right\}^{1/2}\left\{t^2-r^2(n^2+1/\beta^2)\right\}^{1/2}(t^2-r^2n^2)} +\right.$$

$$\left.\left. + \int_{r/c}^\infty te^{-st}\,dt \int_{(t^2/r^2-1/c^2)^{1/2}}^{(t^2/r^2-1/\beta^2)^{1/2}} \frac{dn}{\left\{r^2(n^2+1/c^2)-t^2\right\}^{1/2}\left\{t^2-r^2(n^2+1/\beta^2)\right\}^{1/2}(t^2-r^2n^2)}\right\}\right]$$

Taking Laplace inversion of the above integral, we finally obtain.

$$v = \frac{P\beta}{2\pi\mu cr} H(t-r/\beta) - \frac{Pt\beta^3}{\pi^2\mu cr^2(\beta^2-c^2)^{1/2}}$$

$$X\left\{[H(t-r/\beta)-H(t-r/c)]\Pi(n,R)+H(t-r/c).\Pi(n,R,\bar{\phi})\right\}.$$

where

$\Pi(R,n)$ is the complete elliptic integral of 3rd kind,

$\Pi(R,n,\bar{\phi})$ is the elliptic integral of 3rd kind,

$R^2 = \dfrac{c^2(t^2\beta^2-r^2)}{r^2(\beta^2-c^2)}$, $n^2 = (t^2\beta^2 - r^2)/r^2$ and

$$\bar{\phi} = \cos^{-1}\left[\beta/c\ (c^2t^2-r^2)/(\beta^2t^2-r^2)^{1/2}\right].$$

TORSIONAL RESPONSE OF AN ELASTIC HALF SPACE TO A NONUNIFORMLY EXPANDING RING SOURCE.

INTRODUCTION: The study of the dynamic behaviour of an elastic solid under various forms of moving loads and torsional pressure has been gaining importance day by day. This is because of their importance in seismology, structural design and under ground exploration.

Gakenheimer (1971) in one of his papers presented in details the problem of a load emanating from a point on the surface and then expanding radially at a constant rate. He considered the cases when the loads are disk-shaped or ring-shaped and the expanding rates are supper seismic, transseismic and sub seismic. Almost at the same time Ghosh (1971) also considered the problem of propagation of a stress discontinuity over an expanding circular region with a constant velocity which is less than the shear wave velocity of the medium. Freund (1973) considered the non uniformly moving line load as well as point load. Stronge (1970) discussed the problem of an accelerating line load in an acoustic half space. The nonuniform pressure distribution problem applied to an elastic half space over a circular zone are discussed by Brock (1980) and by Roy (1979). Almost a same type of problem has been considered by Aggarwal and Ablow (1965). There it was assumed that circularly symmetric load spreads out from a point

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Published in ZAMM, vol. 63, p. 7346 (1983)

on an acoustic half space with decelerating speed. Ghosh (1980/81) determined exactly the displacement produced by SH-type of waves when a torsional force is prescribed over a circular region on the free surface of a homogeneous isotropic medium and that in the integral form in case of a non homogeneous medium.

In the present paper, the displacement at any point (r,z) in the semi infinite medium is determined in the integral form by prescribing a time dependent torsional force over the rim of a circular zone. The ring is assumed to expand in an arbitrary manner with time. It is found that the displacement field contains besides the usual SH-waves, contribution from conical waves which arise due to the motion of the source. The region of conical waves which depend on the nature of the motion of the source and the initial speed of expansion of the source are investigated in details. Different wave front surfaces are located and first motion responses near different wave arrivals have been obtained.

Finally numerical evaluation of the displacement on the free surface has been made for a decelerating ring source whose radius at time t is of the form $h(t)=At^{1/2}$. Displacements at points on the free surface for different position of the source have been shown by means of graphs.

FORMULATION OF THE PROBLEM: Consider a homogeneous isotropic elastic half space on the free surface of which a ring source producing SH-type of waves is expanding with non uniform velocity. (r,θ,z) are the cylindrical polar co-ordinates, z-axis being directed into the medium and the plane boundary being $z = 0$.

The origin of co-ordinates is at the centre of the ring $r = h\ (t),\ z = o$. The ring is assumed to expand with uniform acceleration or with deceleration and an impulsive torque applied to the ring is prescribed.

The displacement is determined in the integral form at any point inside and on the free surface of the medium, subject to the condition that the half-space is initially at rest and that the displacements remain bounded for large values of z. For torsional motion of the ring all quantities depend on r, z and the time t. We assume that h(t) is non negative and monotone increasing function. The only non-zero component of the displacement vector is the component v along the direction of θ increasing. The relevant non vanishing stress components are

$$\tau_{r\theta} = \mu \left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \text{ and } \quad \tau_{\theta z} = \mu \frac{\partial v}{\partial z} \qquad (1\ a,b)$$

where μ is the Lame's constant. The non zero equation of the displacement field is

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\beta^2}\ \frac{\partial^2 v}{\partial t^2} \qquad (2)$$

where β is the shear wave velocity. The boundary condition of the motion is

$$\tau_{\theta z} = -P\ \delta\left[h(t) - r\right]H(t),\ z = o \qquad (3)$$

Where P is a constant, $\delta(\ )$ is Dirac's delta function, $H(\ )$ is the Heaviside step function and h(t) is the radius of the ring at time t. Initial conditions of the motion are given by

$$h(t) = o, \ t = o \text{ and } \dot{h}(t) > o, \ t > o \tag{4}$$

where dot denotes the time derivative.

METHOD OF SOLUTION: We define Laplace transform $f_1(r,z,p)$ of the function $f(r,z,t)$ by

$$f_1(r,z,p) = \int_0^\infty \exp(-pt)\, f(r,z,t)dt \tag{5}$$

where p is real and positive and Hankel transform $f_2(\xi,z,p)$ of $f_1(r,z,p)$ by

$$f_2(\xi,z,p) = \int_0^\infty r\, J_1\ (\xi r)\ f_1(r,z,p)dr \tag{6}$$

where J_n is the Bessel function of the first kind of order n.

Applying Laplace and Hankel transforms, to the equation (2) successively we obtain

$$\frac{d^2v_2}{dz^2} - k^2\, v_2 = o \tag{7}$$

where $k^2 = \xi^2 + (p^2 / \beta^2)$.

The solution of the equation (7) which remains bounded as $z \to +\infty$ is

$$v_2 = K \exp\ (-kz) \tag{8}$$

The value of the constant K is determined, by using the condition (4), the equation (8) and the Hankel transform of the Laplace transform of the equation (3) It is found to be

$$K = \frac{p}{\mu} \int_0^\infty \frac{1}{k}\, h(\tau) J_1(\xi h(\tau))\exp(-p\tau)\,d\tau . \qquad (9)$$

Substituting the value of K in (8) and then taking Hankel's inversion one gets

$$v_1 = \frac{p}{\mu} \int_0^\infty h(\tau)\exp(-p\tau) \int_0^\infty \frac{\xi}{k} J_1(\xi r) J_1(\xi h(\tau))\exp(-kz)\,d\xi\,d\tau . \qquad (10)$$

LAPLACE INVERSION: In this section the Laplace inverse transform is evaluated by Cagniard's technique.

We make use of the following results

$$J_1(\xi h(\tau)) J_1(\xi r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} J_0(\xi S) \cos\phi\, d\phi \quad \text{and}$$

$$J_0(\xi S) = \frac{1}{2\pi} \int_0^{2\pi} \exp(i\xi S \cos u)\, du, \text{ where}$$

$S = (r^2 + h^2(\tau) - 2r\, h(\tau) \cos\phi)^{1/2}$, to obtain the equation (10) as

$$v_1 = \frac{p}{4\pi^2 \mu} \int_0^\infty h(\tau)\exp(-p\tau) \int_{-\pi}^{\pi} I \cos\phi\, d\phi\, d\tau, \qquad (11)$$

where

$$I = \int_0^\infty \int_0^{2\pi} \frac{\xi}{k} \exp(i\xi S \cos(\psi - u) - kz)\,d\xi\,du \qquad (12)$$

and ψ is any constant angle.

In (12), we put

$\alpha' = \xi \cos u$ and $\beta' = \xi \sin u$, then substitute

$\alpha' = w\cos\psi - q\sin\psi$ and $\beta' = w\sin\psi + q\cos\psi$ and finally replace w by wp and q by qp to obtain (12) in the form

$$I = p \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\left[-p\left\{-iwS + (w^2+q^2+1/\beta^2)^{1/2}\ z\right\}\right]}{(w^2+q^2+1/\beta^2)^{1/2}}\, dw\, dq \qquad (13)$$

Equation (13) is well known form for determining the Laplace inversion of a function by applying Cagniard's technique as modified by De-Hoop. Substituting $t = -iwS + z(w^2+q^2+1/\beta^2)^{1/2}$ in (13) where t is real and positive, the Laplace inversion of (13) is found to be equal to

$$G(t) = 4\frac{d}{dt}\left\{H\left[t - \ /\beta\right] \int_{0}^{(t^2/\rho^2 - 1/\beta^2)^{1/2}} Rl\left[\frac{1}{(w_+^2 + q^2 + 1/\beta^2)^{1/2}} \frac{dw_+}{dt}\right] dq\right\} \qquad (14)$$

where $\rho^2 = z^2 + S^2$ and

$$w_+ = \frac{iSt + z\left\{t^2 - \ ^2(q^2 + 1/\beta^2)\right\}^{1/2}}{\rho^2}$$

Applying convolution theorem on (11), the Laplace inversion of v_1 is obtained in the form

$$v = \frac{p}{4\pi^2/\mu} \int_0^{\infty} h(\tau)\, d\tau \int_{-\pi}^{\pi} \cos\phi\, d\phi \int_0^{t} \delta(u - \tau) G(t-u)\, du,$$

which when simplified takes the form

$$v = \frac{p}{\pi/\mu} \int_0^{t} h(\tau)\, d\tau \int_0^{\pi} \frac{\cos\phi}{\rho}\ \delta\,(t - \tau - \rho/\beta)\, d\phi. \qquad (15)$$

Integrating over ϕ , we obtain

$$v = \frac{P\beta}{\pi r/\mu} \int_0^t \left\{ H\left[t-\mathcal{T}- \frac{\sqrt{z^2+(r-h(\mathcal{T})\ )^2}}{\beta} \right] - H\left[t-\mathcal{T}- \frac{\sqrt{z^2+(r+h(\mathcal{T})\ )^2}}{\beta} \right] \right\} Q(\mathcal{T})\, d\mathcal{T}$$

(16)

where

$$Q(\mathcal{T}) = \frac{z^2+r^2+h^2(\mathcal{T})-\beta^2(t-\mathcal{T})^2}{\left[\left\{ z^2+(r+h(\mathcal{T})\ )^2-\beta^2(t-\mathcal{T})^2 \right\} \left\{ \beta^2(t-\mathcal{T})^2-z^2-(r-h(\mathcal{T}))^2 \right\} \right]^{1/2}} .$$

To facilitate our discussion, equation (16) is written in an alternative form,

$$v = \frac{P\beta}{\pi r/\mu} \int_0^{t-z/\beta} \left\{ H\left[r-h(\mathcal{T})+ \sqrt{\beta^2(t-\mathcal{T})^2-z^2} \right] H\left[h(\mathcal{T})- \sqrt{\beta^2(t-\mathcal{T})^2-z^2} \right] + \right.$$

$$+ H\left[r+h(\mathcal{T})- \sqrt{\beta^2(t-\mathcal{T})^2-z^2} \right] H\left[-h(\mathcal{T})+ \sqrt{\beta^2(t-\mathcal{T})^2-z^2} \right] -$$

$$\left. - H\left[r-h(\mathcal{T})- \sqrt{\beta^2(t-\mathcal{T})^2-z^2} \right] \right\} Q(\mathcal{T})\, d\mathcal{T} . \qquad (17)$$

The region of support for $\mathcal{T}$ - integration is bounded by the curves:

I : $r = h(\mathcal{T}) + \sqrt{\beta^2(t-\mathcal{T})^2-z^2}$; $o < \mathcal{T} < t-z/\beta$. (18)

II : $r = h(\mathcal{T}) - \sqrt{\beta^2(t-\mathcal{T})^2-z^2}$; $\mathcal{T}_o < \mathcal{T} < t-z/\beta$. (19)

III: $r = -h(\mathcal{T}) + \sqrt{\beta^2(t-\mathcal{T})^2-z^2}$; $o < \mathcal{T} < \mathcal{T}_o$. (20)

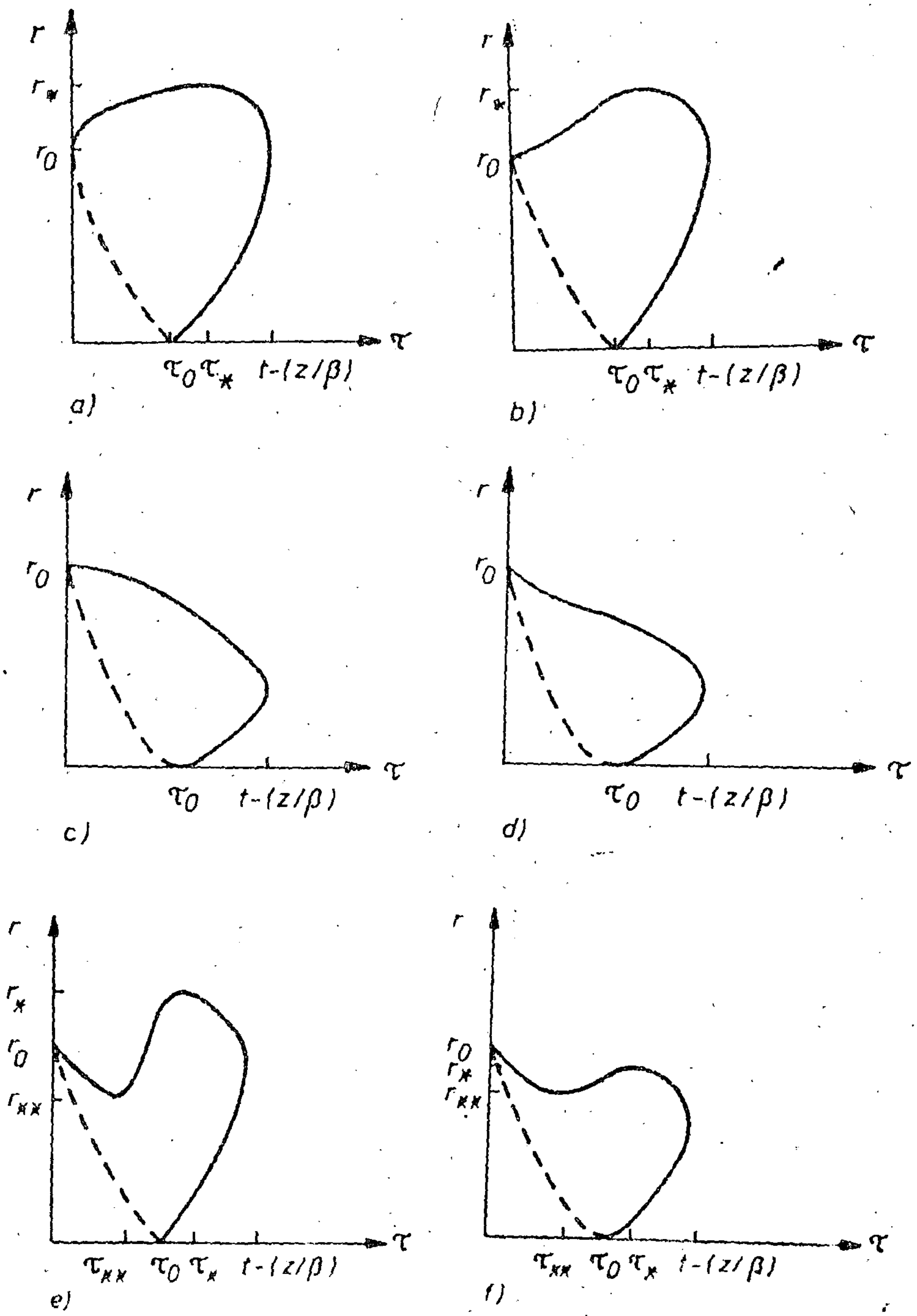

Fig. 1. Region of support for τ-integration for fixed z and t. ($a - b$) when a single maximum exists i.e. the starting velocity $h(0)$ is such that $(\partial r/\partial \tau) > 0$; ($c - d$) the case when no extremum exists; ($e - f$) when a maximum(r_*) and a minimum (r_{**}) both exist

The region of τ integration for $Q(\tau)$ bounded by the curves I, II and III are shown in the figs.1(a - f) and the following remarks can be made about them.

It is to be noted that the curves II and III are monotone increasing and decreasing in their respective region of existence viz. $(\tau_o, t - z/\beta)$ and (o, τ_o) where

$$h(\tau_o) = \left\{\beta^2(t - \tau_o)^2 - z^2\right\}^{1/2}$$

The curve I has extremum where

$$\frac{\partial r}{\partial \tau} = \dot{h}(\tau) - \frac{\beta^2(t - \tau)}{\left\{\beta^2(t - \tau)^2 - z^2\right\}^{1/2}} \tag{21}$$

vanishes and

$$\frac{\partial^2 r}{\partial \tau^2} = \ddot{h}(\tau) - \frac{\beta^2 z^2}{\left\{\beta^2(t - \tau)^2 - z^2\right\}^{3/2}} \tag{22}$$

does not vanish.

We consider the different cases that arise due to non uniform increase of the ring s-ource. Let the source increase with uniform acceleration $\ddot{h}(\tau) > o$. In this case, if the initial velocity $\dot{h}(o)$ of the source be such that $\left(\frac{\partial r}{\partial \tau}\right)_o > o$, then since $\left(\frac{\partial r}{\partial \tau}\right)_o > o$ and

$\left(\frac{\partial r}{\partial \tau}\right)_{t-z/\beta} < 0$, the curve I has only one maximum at $r = r_*$ because $\frac{\partial^2 r}{\partial \tau^2}$ either changes sign once from positive to negative or remains negative throughout in $(0, t-z/\beta)$. The corresponding cases are shown in Fig.1(a-b). Next let the initial velocity $\dot{h}(0)$ of the source be such that $\left(\frac{\partial r}{\partial \tau}\right)_0 < 0$. In this case, if $\left(\frac{\partial^2 r}{\partial \tau^2}\right)_0 < 0$, $\left(\frac{\partial^2 r}{\partial \tau^2}\right)$ will be negative throughout the interval $(0, t-z/\beta)$; the curve I then corresponds to Fig.1(c), since both $\left(\frac{\partial r}{\partial \tau}\right)_0$ and $\left(\frac{\partial r}{\partial \tau}\right)_{t-z/\beta}$ are negative. But if $\left(\frac{\partial^2 r}{\partial \tau^2}\right)_0$ be positive then $\frac{\partial^2 r}{\partial \tau^2}$ changes sign once from positive to negative in the interval $(0, t-z/\beta)$. Hence in this case the curve I has either no extremum which corresponds to Fig.1(d) or there is a maximum proceded by a minimum which is shown in Fig.1 (e,f). Finally, in case of a decelerating motion of the source i.e when $\ddot{h}(\tau)$(not necessarily a constant) < 0, throughout the interval, the curve has either only one maximum if $\left(\frac{\partial r}{\partial \tau}\right)_0 > 0$ as in Fig.1(a) or no extremum as in Fig.1(c) when $\left(\frac{\partial r}{\partial \tau}\right)_0 < 0$.

We consider the curves I and II together. Their combined equation is

$$(r-h(\tau))^2 = \beta^2(t-\tau)^2 - z^2 \tag{23}$$

For figures 1(c,d), τ is a single valued function of r. For the figs.1(a,b) τ may be a double valued function whereas for figs, 1(e,f), τ may be a triple valued function of r. Taking the equations (18) and (19) together, the values of τ are designated as $\tau = \tau_1$, $\tau = (\tau_1, \tau_2)$ and $\tau = (\tau_1, \tau_2, \tau_3)$ where $\tau_1 > \tau_2 > \tau_3$ depending on whether τ is single, double or triple valued function of r. In (20) r is a monotone decreasing function of τ, so the corresponding value of τ is designated as $\tau = \tau_4$.

With the above values of the roots of the equations (18)-(20) and from a close examination of the different figs. 1(a-f), the displacement produced by the SH-type of waves is given by $v = v^1 + v^2 + v^3$, where

$$v^1 = BH(r_o - r)\ I(Q(\tau);\ \tau_4,\ \tau_1)$$

$$v^2 = B\left[H(r - r_o) - G(r - \max(r_*,\ r_o))\right] I(Q(\tau);\ \tau_2, \tau_1) \qquad (24)$$

$$v^3 = B\left[G(r - \min(r_*,\ r_o) - G(r - \min(r_{**}, r_o))\right] I(Q(\tau);\ \tau_3,\ \tau_2)$$

and

$$G(r - \max(r_*,\ r_o)) = \begin{array}{l} H(r - r_*) \text{ if } r_* = \max(r_*,\ r_o) \\ H(r - r_o) \text{ if } r_o = \max(r_*,\ r_o) \\ \qquad \text{or } r_* \text{ does not exist.} \end{array}$$

$$G(r - \min(r_*,\ r_o)) = \begin{array}{l} H(r - r_*) \text{ if } r_* = \min(r_*,\ r_o) \\ H(r - r_o) \text{ if } r_o = \min(r_*,\ r_o) \\ \qquad \text{or } r_* \text{ does not exist.} \end{array}$$

Similar meaning is attached to the symbol $G(r - \min(r_{**}, r_o))$. 'B' has been written for $\frac{P\beta}{\pi r \mu}$.

$r_o = \sqrt{\beta^2 t^2 - z^2}$ is the value of r at $\tau = o$ and

$$I\ (F(\tau);\ a,\ b) = \int_a^b F(\tau)\ d\tau\ .$$

WAVE FRONT ANALYSIS: In this section we locate and analyse the nature of the wave fronts.

The nature of the wave front changes due to non uniform expansion of the source and also it depends on the inital velocity $\dot{h}(o)$ ($=u_o$) of expansion of the source. We consider decelerating and accelerating expansion of the source for different initial velocities.

Case of decleration:

i) let $\dot{h}(o) = u_o < \beta$.

From (21) and (22) , $(\frac{\partial r}{\partial \tau})_o$ is negative for all z and $\frac{\partial^2 r}{\partial \tau^2}$ is also negative as $\ddot{h}(\tau)$ is negative. So the curve I in ($o, t-z/\beta$) is such that r decreases with the increase of τ . This corresponds to the region of integration as depicted in fig.1(c) and consequently the wave front is of the form as shown in fig.2(a).

ii) u_o($> \beta$) is finite.

It follows from (21), $(\frac{\partial r}{\partial \tau})_o$ is positive for $o < z < z_2$ and negative for $z > z_2$ where z_2 is obtained from

$u_o = \beta^2 t/(\beta^2 t^2 - z_2^2)^{1/2}$ i.e $z_2 = \beta t\ (1 - \beta^2/u_o^2)^{1/2}$ and further $\frac{\partial^2 r}{\partial \tau^2}$ is negative as $\ddot{h}(\tau)$ is negative. Therefore the region of

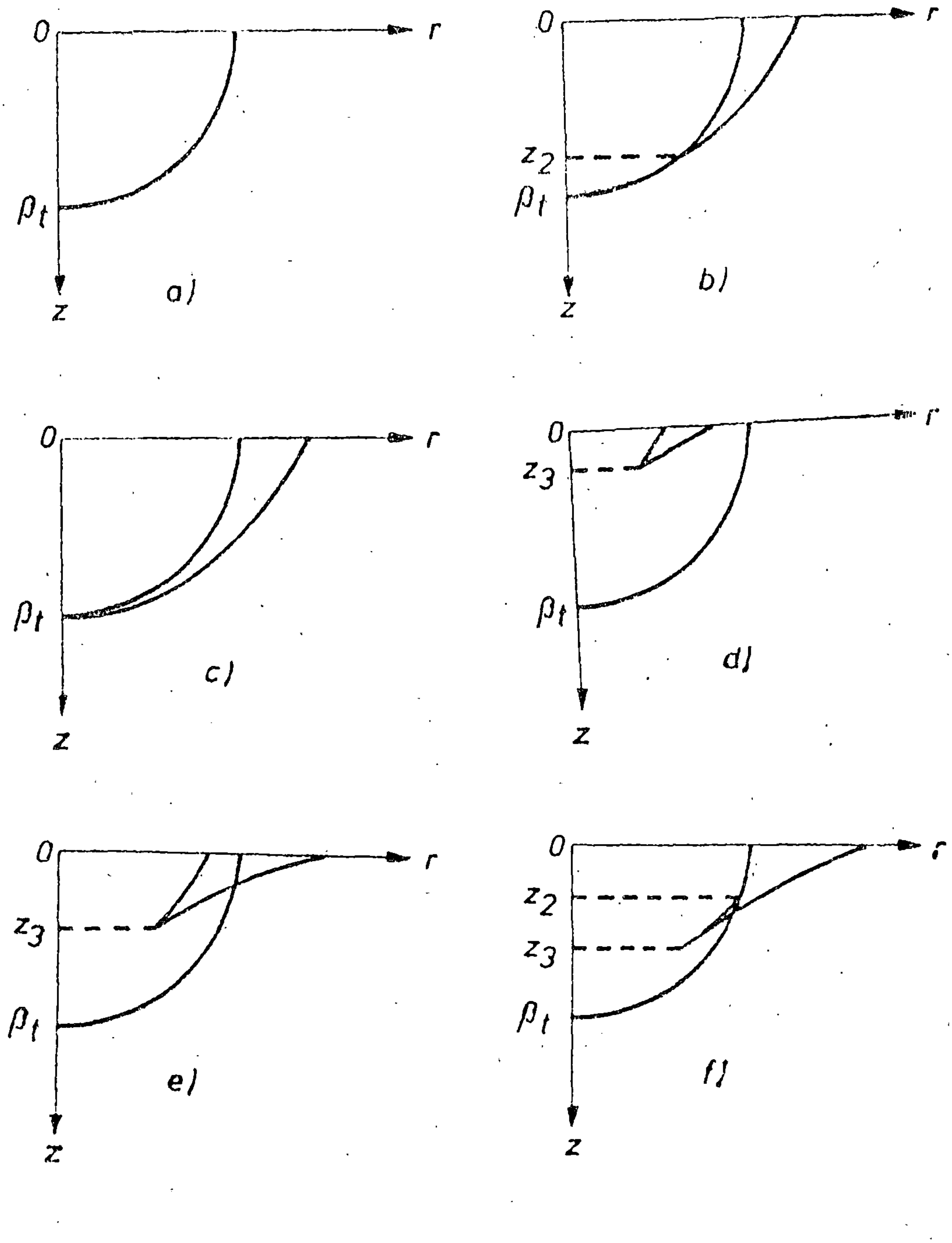

Fig. 2. Different types of wave fronts, at particular admissible values of time and position, which arise due to non uniformity and initial velocity of expansion of the ring source

integrations for $o < z < z_2$ and for $z > z_2$ correspond to the regions shown in the figs.1(a) and 1(c) respectively and consequently the wave front is given by the fig.2(b).

iii) u_o is infinitely large.

From (21) and (22), it follows that $(\frac{\partial r}{\partial \tau})_o$ is positive for all z and $(\frac{\partial^2 r}{\partial \tau^2})$ is negative for all z and for all τ. Hence the region of integration is fig.1(a) and the corresponding wave front is as shown in fig.2(c).

Case of acceleration:

i) We assume that the ring source expands with uniform acceleration f and starts with the velocity $u_o(=\dot{h}(o))$. First let $u_o < \beta$; then $(\partial r/\partial \tau)_o$ is negative for all z and $(\partial^2 r/\partial \tau^2)_o$ is positive for $o < z < z_1$ and negative for $z > z_1$, where z_1 is to be determined from the condition $(\partial^2 r/\partial \tau^2)_o = o$. For $z > z_1$, $(\partial^2 r/\partial \tau^2)$ is negative for τ in $(o, t-z/\beta)$. Consequently the region of integration is the fig.1(c). On the other hand if z lies in (o, z_1) then $(\partial^2 r/\partial \tau^2)$ is first positive and then negative as τ increases in $(o, t-z/\beta)$, so in this case the region of integration is either fig.1(d) or fig.1(e or f).

By using (22), z_1 is determined from the equation

$$f = \beta^2 z_1^2 / \beta^2 t^2 - z_1^2)^{3/2} \tag{25}$$

It is to be noted that $z_1 = o$ when $f = o$ and z_1 is a monotone increasing function of f. Further, in (o, z_1), $(\partial r/\partial \tau)$ may have two zeroes or there is no zero in the region $o < < t - z/\beta)$, depending on the value of z. The condition that $(\partial r/\partial \tau)$ may have two zeroes is, $o < z < z_3$, where

$$z_3 = \beta \left(\frac{u_o + ft}{f} \right) \left\{ 1 - \frac{\beta^{2/3}}{(u_o+ft)^{2/3}} \right\}^{3/2} \qquad \text{for } u_o + ft > \beta$$

$$= o \qquad \text{for } u_o + ft < \beta .$$

It can be shown further that $z_3 < z_1$. Hence for $o<z<z_3$ the region of integration is fig.1(e or f) and for $z_3 < z < z_1$, the region of integration is fig.1(d). Therefore for accelerating source with initial velocity $u_o < \beta$, the wave front is of the form as shown in fig.2(a) if the observation time be such that $(u_o + ft) \leq \beta$ and for $(u_o + ft) > \beta$ the wave front is like the figures as in 2(d) or 2(e) according as the position of the source at the observation time is inside or out side the characteristic surface $r^2 + z^2 + \beta^2 t^2$.

ii) Next let $u_o > \beta$ from (21) we have $(\partial r/\partial \mathrm{T})_o$ is positive for $o < z < z_2$ and is negative for $z > z_2$ where z_2 is given by

$$z_2 = \beta t (1 - \beta^2/u_o^2)^{1/2} \qquad (26)$$

Also $(\partial^2 r/\partial \mathrm{T}^2)_o$ is positive for $o < z < z_1$ and is negative for $z > z_1$, where z_1 is given by (25). So for $o < z < z_1$, $(\partial^2 r/\partial \mathrm{T}^2)$ is first positive and then negative in $o \leqslant \mathrm{T} < t - z/\beta$. We consider the case for $z_1 < z_2$ first. In this case

$$\beta^2 z_1^2/(\beta^2 t^2 - z_1^2)^{3/2} < \beta^2 z_2^2 / (\beta^2 t^2 - z_2^2)^{3/2} ,$$

since $\beta^2 z^2/(\beta^2 t^2 - z^2)^{3/2}$ is a monotone increasing function of z.

Using (25) and (26), we obtain

$$\beta^2 (u_o + ft) / u_o^3 < 1 \qquad (27)$$

Under the condition obtained in (27) the region of integration is like that of the fig.1(b) in the range $o < z < z_1$ and for $z_1 < z < z_2$, the region of integration is of the type as shown in fig.1(a). For $z_2 < z < \beta t$, the region of integration, is shown in fig.1(c). Therefore for $u_o > \beta$ and for the relation given in (27), it follows that the nature of the wave front is of the type as shown in fig.2(b).

Finally, we study the case when $z_1 > z_2$ i.e when $\beta^2(u_o+ft)/u_o^3 > 1$. Here for $o < z < z_2$ the region of integration is as in fig.1(b). Since z_3 is always less than z_1, so for $z_2 < z < z_3$ the region of integration is like fig.1(e or f) and for $z_3 < z < z_1$ the region of integration is like that as shown in fig.1(d). Fig.1(c) represents the region of integration for $z_1 < z < \beta t$. Accordingly the wave front takes the shape of the fig.2(f).

FIRST MOTION RESPONSES: The expression for the displacement as given in (24) is in the form of integrals over finite ranges. As such, computation of displacement for a given model can be done with the high power computer. However some idea about the nature of displacement at the time of the first arrival of wave fronts can be obtained by limiting process following Stronge(1970). The displacement field just after arrival time of the characteristic surface $r = r_*$ is from (24),

$$v = B\, I(Q\,(T);\ T_2,\ T_1) \qquad (28)$$

where as just before the arrival time the displacement is given by $v = o$.

To evaluate (28) near $r = r_*$, we put $r = r_* - \Delta r$ and $\tau = \tau_* + \theta$ in equations (18) and (19) . Using Taylor's expansion in the neighbourhood of (τ_*, r_*) and with the help of equations (21) and (22) we find the limits of integration of equation (28) in the new variable θ as

$$\theta_{1,2} = \pm \frac{\sqrt{2\Delta r}\left\{\beta^2(t - \tau_*)^2 - z^2\right\}^{3/4}}{\left[\beta^2 z^2 - \ddot{h}(\tau_*)\left\{\beta^2(t - \tau_*)^2 - z^2\right\}^{3/2}\right]^{1/2}} \tag{29}$$

The same procedure is followed to determine in the neighbourhood of (τ_* , r_*) , the value of Q which is found to be

$$Q(\tau_* + \theta) = \frac{\left[r_* h(\tau_*)\left\{\beta^2(t - \tau_*)^2 - z^2\right\}\right]^{1/2}}{\left[\theta^2\left\{\ddot{h}(\tau_*)(\beta^2(t - {}_*)^2 - z^2)^{3/2} - \beta^2 z^2\right\} + 2\Delta r\left\{\beta^2(t - \tau_*)^2 - z^2\right\}^{3/2}\right]^{1/2}} \tag{30}$$

where the lowest term in θ and Δr are retained. The value of the integral (28) after substituting the value of Q from (30) and the limits of integration for the new variable θ as obtained in (29) is found to be

$$\frac{p\,\beta}{r\,\mu}\left[\frac{r_*\,h(\tau_*)\left\{\beta^2\,(t - \tau_*)^2 - z^2\right\}}{\beta^2 z^2 - \ddot{h}(\tau_*)\left\{\beta^2(t - \tau_*)^2 - z^2\right\}^{3/2}}\right]^{1/2},$$ which is the

displacement at the first arrival of the wave front given by $r = r_*$.

To find the displacement at the first arrival of the wave front given by $r = r_{**}$, we define $Q(\tau)$ in the neighbourhood of (τ_{**}, r_{**}) and out side the region of integration by

$$Q(\tau) = \frac{z^2 + r^2 + h^2(\tau) - \beta^2 (t-\tau)^2}{\left\{z^2+(r+h(\tau))^2 - \beta^2(t-\tau)^2\right\}^{1/2} \left|\left\{\beta^2(t-\tau)^2 - z^2 - (r-h(\tau))^2\right\}^{1/2}\right|} \tag{31}$$

and put $r = r_{**} + \Delta r$ and $\tau = \tau_{**} + \theta$. Following the same procedure as done in case of $r = r_*$, the displacement at the first arrival of the wave surface $r = r_{**}$ is found to be

$$\frac{p\beta}{r\mu}\left[\frac{r_{**}\, h(\tau_{**})\left\{\beta^2 (t-\tau_{**})^2 - z^2\right\}}{\ddot{h}(\tau_{**})\left\{\beta^2(t-\tau_{**})^2 - z^2\right\}^{3/2} - \beta^2 z^2}\right]^{1/2}.$$

The displacement at a point due to the first arrival of the wave fronts $r = r_*$ and $r = r_{**}$ simultaneously is also determined. At this point the wave fronts $r = r_*$ and $r = r_{**}$ form a cusp (cf.fig.2(d,e,f). In this case this is to be noted that at the cusp $r = r_* = r_{**} = \bar{r}$ (say) and $(\partial r/\partial\tau) = (\partial^2 r/\partial\tau^2) = 0$ where as $(\partial^3 r/\partial\tau^3) \neq 0$. Hence it follows from equation (24) that the displacement due to first arrival of this wave front at $r = \bar{r}$ is

$$\frac{p\beta}{\pi r\mu}\left[I(Q(\tau);\ \bar{\tau}, \tau_1) + I(Q(\tau); \tau_2, \bar{\tau})\right] \tag{32}$$

where $\bar{T} = T_* = T_{**}$ and T_1, T_2 are the two values of T close to $\bar{T}$ and correspond to the points lying on either side of $(\bar{T}, \bar{r})$ on the curve I and II together.

To evaluate the integrals in (32), $T = \bar{T} + \theta$ and $r = \bar{r} - \Delta r$ are put in the first integral where as $T = \bar{T} - \theta$ and $r = \bar{r} + \Delta r$ are put into the second integral of (32). Also this is to be remembered that outside the region of integration in the neighbourhood of $(\bar{T}, \bar{r})$, $Q(T)$ is defined as in (31).

After the above mentioned substitution in (28) and retaining the lowest order term of θ and Δr one gets the displacement due to first arrival at $r = \bar{r}$ as

$$\frac{2\,P\,\beta}{\pi\,\bar{r}\,\mu}\left[\frac{3\bar{r}h(\bar{T})\{\beta^2(t-\bar{T})^2 - z^2\}^2}{3\beta^4 z^2(t-\bar{T}) - \ddot{h}(\bar{T})\{\beta^2(t-\bar{T})^2 - z^2\}^{5/2}}\right]^{1/2}\int_0^a \frac{d\theta}{\sqrt{(a^3-\theta^3)}} \tag{33}$$

where

$$a^3 = \frac{6\Delta r(\bar{r} - h(\bar{T}))^5}{3\beta^4 z^2(t-\bar{T}) - \ddot{h}(\bar{T})(\bar{r} - h(\bar{T}))^5}.$$

By substituting $\theta^3 = a^3\sin^2\alpha$, the integral in (33) is evaluated. The displacement due to first arrival of the wave front $r=\bar{r}$ is found to be

$$\frac{2^{5/6}\,P\,\beta}{3^{2/3}\pi\,\bar{r}\,\mu(\Delta r)^{1/6}}\;\frac{\bar{r}^{1/2}\,h^{1/2}(\bar{T})\{\beta^2(t-\bar{T}) - z^2\}^{7/12}}{\left[3\beta^4 z^2(t-\bar{T}) - \ddot{h}(\bar{T})\{\beta^2(t-\bar{T})^2 - z^2\}^{5/2}\right]^{1/3}}\,B\left(\tfrac{1}{3}, \tfrac{1}{2}\right)$$

where $B(m,n)$ is the Beta function.

It is interesting to note that in this case the displacement due to first arrival at this point is infinitely large due to the presence of the factor $(\Delta r)^{1/6}$ in the denominator.

Finally we consider the characteristic surface $r^2 + z^2 = \beta^2 t^2$ which corresponds to a disturbance initiated at the origin when the torque is first applied at $\tau = o$. This disturbance spreads out from the origin with a velocity equal to β. To find the displacement due to the first arrival of this surface, following Aggarwal and Ablow (1965) let us consider the curve

$$\Gamma \quad : r = \left\{\beta^2 (t - \tau)^2 - z^2\right\}^{1/2} \tag{34}$$

and the lines

$$l_1 \quad : r = \sqrt{\beta^2 t^2 - z^2} - \epsilon_2 \tag{35}$$

$$l_2 \quad : r = \sqrt{\beta^2 t^2 - z^2} + \epsilon_1 \tag{36}$$

where ϵ_1 and ϵ_2 are very small positive quantities. Then to the first order of ϵ_1 and ϵ_2

$$l_1 \times \Gamma \equiv \frac{\epsilon_2 \sqrt{\beta^2 t^2 - z^2}}{\beta^2 t} = \tau' \text{(say)}$$

$$l_1 \times \text{III} \equiv \frac{\epsilon_2 \sqrt{\beta^2 t^2 - z^2}}{\dot{h}(o) \sqrt{\beta^2 t^2 - z^2} + \beta^2 t} = \tau_4 \text{ (say)}$$

and $$l_2 \times \text{I} \equiv \frac{\epsilon_1 \sqrt{\beta^2 t^2 - z^2}}{\dot{h}(o) \sqrt{\beta^2 t^2 - z^2} - \beta^2 t} = \tau_2 \text{ (say)};$$

ϵ_1, ϵ_2 are such that $\tau_2 < \tau'$ and tends to zero as $t \to \rho_0/\beta$,

where $\rho_o = \sqrt{r^2 + z^2}$. Then it follows immediately that $I(Q(\tau);\ \tau_4, \tau') \to o$ and $I(Q(\tau);\ \tau_2, \tau') \to o$ as $t \to \rho_o/\beta$.
Also $\{I(Q(\tau);\ \tau', \tau_1') - I(Q(\tau);\ \tau', \tau_1)\} \to o$ as $t \to \rho_o/\beta$,
where $\tau_1' \equiv l_1 \times I$ and $\tau_1 \equiv l_2 \times I$ are the values of τ which correspond to the points on the right of τ'. From this it follows that the displacement is continuous across the characteristic surface $\rho_o = \beta t$, showing that the displacement due to the first arrival of the characteristic surface $r^2 + z^2 = \beta^2 t^2$ is zero.

SURFACE DISPLACEMENT: In this section surface displacement has been determined numerically for a particular type of nonuniformly moving surface. We consider a decelerating ring source whose radius $h(\tau)$ at any time τ is assumed to be $h(\tau) = A\tau^{1/2}$. The displacement at any point (r,o) at the time of observation t is determined.

According to the position of the source the following three possible cases are considered.

i) Radius $h(\tau)$ of the ring coinciding with the rim of the conical wave front and moving with it so that $\beta t < h(t)$.

ii) $\beta t < h(t) < r_*$

iii) $h(t) < \beta t$.

To determine the displacement on the free surface, we put $z = o$ in the function $Q(\tau)$ of equation (24) and the variable of integration τ is changed to T, by substituting $T = \tau/t$.

$Q(\tau)$ is then obtained in the form

$$Q(Tt)=R(T)=\frac{\frac{r^2}{\beta^2 t^2}+\frac{A^2}{\beta^2 t}T-(1-T)^2}{\left[\left\{\left(\frac{r}{\beta t}+\frac{A}{\beta}\sqrt{\frac{T}{t}}\right)^2-(1-T)^2\right\}\left\{(1-T)^2-\left(\frac{r}{\beta t}-\frac{A}{\beta}\sqrt{\frac{T}{t}}\right)^2\right\}\right]^{1/2}}$$

on a close examination of the regions of integration as shown in Fig.1, the displacement v, in case of (i) is given by

$$\frac{\mu v}{P}=\frac{\beta t}{\pi r}\, I(R(T)\ ;\ T_1,\ T_2) \text{ for } o < r < \beta t$$

$$\frac{\mu v}{P}=\frac{\beta t}{\pi r}\, I(R(T)\ ;\ T_1,\ T_2) \text{ for } \beta t < r < h\ (t)$$

The displacement in case of (ii) is given by

$$\frac{\mu v}{P}=\frac{\beta t}{\pi r}\, I(R(T)\ ;\ T_1,\ T_2) \text{ for } r < \beta\ t.$$

$$\frac{\mu v}{P}=\frac{\beta t}{\pi r}\, I(R(T)\ ;\ T_1,\ T_2) \text{ for } \beta t < r < h\ (t)$$

$$\frac{\mu v}{P}=\frac{\beta t}{\pi r}\, I(R(T)\ ;\ T_1,\ T_3) \text{ for } h\ (t) < r < r_*$$

and the displacement in case (iii) is

$$\frac{\mu}{P}v=\frac{\beta t}{\pi r}\, I(R(T);\ T_1,\ T_2) \text{ for } o < r < h\ (t)$$

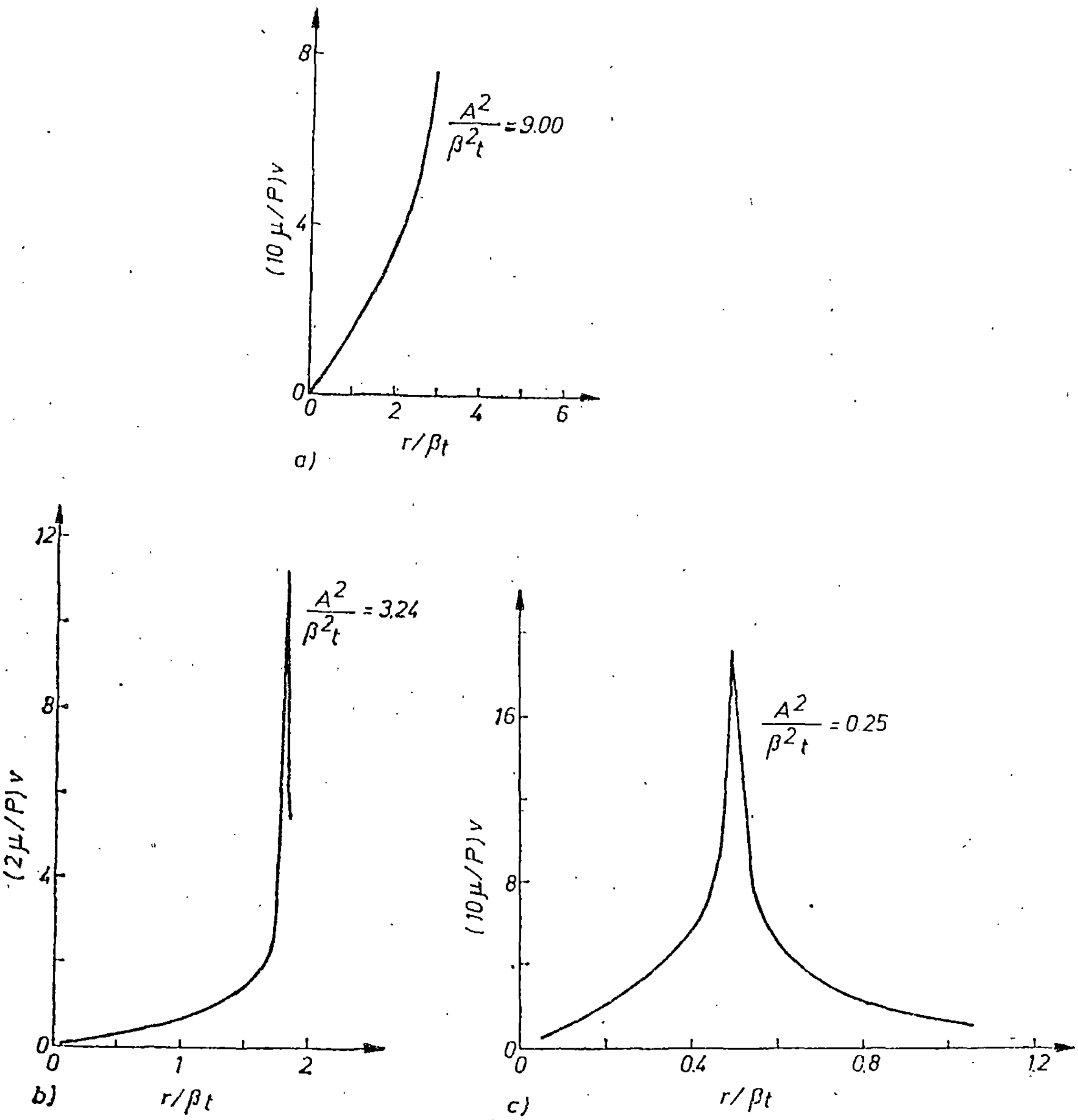

Fig. 3. Graphs showing $(\mu/P)\, v$ versus $(r/\beta t)$ when $z = 0$.

(a), (b), (c) correspond to the cases (i), (ii) and (iii) respectively

$$\frac{\mu}{P} v = \frac{\beta\ t}{\pi\ r} I\ (R\ (T)\ ;\ T_1, T_3) \text{ for } h\ (t) < r < \beta t$$

$$\frac{\mu}{P} v = \frac{\beta\ t}{\pi\ r} I\ (R\ (T);\ \ T_1, T_3) \text{ for } \beta t < r < r_*$$

where

$$T_1 = 1 - \frac{r}{\beta\ t} + \frac{1}{2}\ \frac{A^2}{\beta^2 t} - \frac{1}{2}\ \frac{A}{\beta\sqrt{t}} \sqrt{\frac{A^2}{\beta^2 t} + 4\ (\ 1 - \frac{r}{\beta t}\ )}$$

$$T_2 = 1 + \frac{r}{\beta\ t} + \frac{1}{2}\ \frac{A^2}{\beta^2 t} - \frac{1}{2}\ \frac{A}{\beta\sqrt{t}} \sqrt{\frac{A^2}{\beta^2 t} + 4\ (\ 1 + \frac{r}{\beta t}\ )}$$

$$T_3 = 1 - \frac{r}{\beta\ t} + \frac{1}{2}\ \frac{A^2}{\beta^2 t} + \frac{1}{2}\ \frac{A}{\beta\sqrt{t}} \sqrt{\frac{A^2}{\beta^2 t} + 4\ (\ 1 - \frac{r}{\beta t}\ )}$$

All the above integrals are numerically evaluated and the graphs are plotted by specifying admissible values of $(r/\beta t)$ against $(\mu/P)v$.