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Scattering of antiplane shear wave by a propagating crack at the interface of two dissimilar elastic media

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Abstract. An analysis of the scattering of horizontally polarized shear wave by a semi-infinite crack running with uniform velocity along the interface of two dissimilar semi-infinite elastic media has been carried out. The mixed boundary value problem has been solved completely by the Wiener–Hopf technique. The effect of different values of the material parameter, the angle of incidence of incident wave and the crack propagation velocity on the stress intensity factor have been illustrated graphically.

Keywords. Diffraction of elastic waves; propagating crack; SH-wave; stress intensity factor.

1. Introduction

It is well known that the problems of diffraction of elastic wave by cracks or inclusions are of considerable importance in view of their application in seismology and geophysics. If the cracks or inclusions are located at the interface of layered media, the study becomes more relevant. The extensive use of composite materials in modern technology has also evoked interest in the wave propagation problems in layered media with interfacial discontinuities. Onder *et al* [5] studied the diffraction of monochromatic plane SH-waves obliquely incident on a rigid half plane between the two different semi-infinite media.

In this paper we have considered the problem of the diffraction of a plane harmonic SH-wave by a semi-infinite crack running uniformly along the interface of two dissimilar semi-infinite elastic media. The problem of scattering of plane harmonic polarized shear wave by a half plane crack in an infinite isotropic medium extending under antiplane strain was studied earlier by Jahanshahi [3]. Chen and Sih [1,2] also solved the in plane problem of the diffraction of stress waves by a running crack in an incident wave field in an infinite elastic medium. We have applied Fourier transform and Wiener–Hopf technique [4] to solve the mixed boundary value problem. The resulting integrals have been evaluated asymptotically to obtain the displacement and stress field near about the crack tip. It is found that the stress intensity factor depends sensitively upon the speed of crack propagation, the angle of incidence of the incoming wave and on the material properties of the elastic media. Quantitative assessment of the effect of the aforementioned parameters on the stress intensity factor has been made by displaying the numerical results graphically for two pairs of different materials.

2. Formulation of the problem and its solution

Let a plane crack move at a constant velocity V on the interface of two bonded dissimilar elastic semi-infinite medium due to the incidence of the plane harmonic SH-wave

$$v_1^0 = V_1 \exp[-i\{\Lambda_1(X\cos\Theta_1 + Y\sin\Theta_1) + \Omega T\}] \tag{1}$$

in the medium where the co-efficient of rigidity, density and shear-wave velocity respectively are given by μ_1, ρ_1 and C_1. The crack lies on the bimaterial interface along $Y=0$ with respect to the fixed rectangular co-ordinate system (X, Y, Z).

We assume that the displacement and stress due to the scattered field are

$$v_j = v_j(X, Y, T) \tag{2}$$

and

$$(\tau_{xz})_j = \mu_j \frac{\partial v_j}{\partial X}, \quad (\tau_{yz})_j = \mu_j \frac{\partial v_j}{\partial Y} \tag{3}$$

where the subscript $j=1,2$ refers to the upper and lower half-planes and T the time.

The equations of SH-wave motion in either elastic half-space are given by

$$\frac{\partial^2 v_j}{\partial X^2} + \frac{\partial^2 v_j}{\partial Y^2} = \frac{1}{C_j^2}\frac{\partial^2 v_j}{\partial T^2} \quad (j=1,2) \tag{4}$$

where $C_j = (\mu_j/\rho_j)^{1/2}$ is the shear-wave velocity. Without any loss of generality, we further assume that $C_1 > C_2$.

Due to the incident wave given in (1), the reflected and transmitted waves in the absence of the crack may be written in the form

$$v_1^r(X, Y, T) = V_1^r \exp[-i\{\Lambda_1(X\cos\Theta_1 - Y\sin\Theta_1) + \Omega T)\}]$$

and

$$v_2^T(X, Y, T) = V_2^T \exp[-i\{\Lambda_2(X\cos\Theta_2 + Y\sin\Theta_2) + \Omega T)\}] \tag{5}$$

where

$$V_1^r = \frac{\mu_1\Lambda_1\sin\Theta_1 - \mu_2\Lambda_2\sin\Theta_2}{\mu_1\Lambda_1\sin\Theta_1 + \mu_2\Lambda_2\sin\Theta_2} V_1 = A_1^R V_1 \quad \text{(say)}$$

and

$$V_2^T = \frac{2\mu_1\Lambda_1\sin\Theta_1}{\mu_1\Lambda_1\sin\Theta_1 + \mu_2\Lambda_2\sin\Theta_2} V_1 = A_2^T V_1 \quad \text{(say)} \tag{6}$$

with

$$\Lambda_1\cos\Theta_1 = \Lambda_2\cos\Theta_2.$$

V_1, V_1^r and V_2^T are the incident, reflected and transmitted wave amplitude respectively, Λ_j the wave number, $\Omega = \Lambda_j C_j$ the circular frequency and Θ_1, Θ_2 the angles of incidence and refraction respectively.

Assume that the crack has been moving in the horizontal direction along the interface for a sufficiently long time and that a steady state has been reached in the neighbourhood of the crack.

A set of moving co-ordinate systems (x, y, z, t) attached to the crack tip moving at

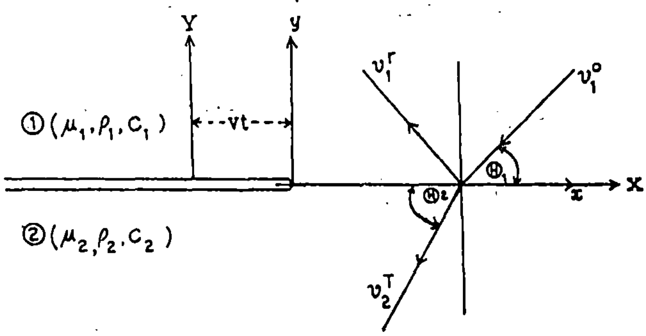

Figure 1. Geometry of the propagating crack.

a constant velocity V is introduced in accordance with

$$x = X - Vt, \quad y_j = s_j Y, \quad z = Z, \quad t = T \tag{7}$$

where $s_j = (1 - M_j^2)^{1/2}$ and $M_j = V/C_j$ is the Mach number.

In terms of the moving co-ordinate system (x, y, t), (4) becomes

$$\frac{\partial^2 v_j}{\partial x^2} + \frac{\partial^2 v_j}{\partial y_j^2} + \frac{1}{C_j^2 s_j^2}\frac{\partial}{\partial t}\left(2M_j C_j \frac{\partial v_j}{\partial x} - \frac{\partial v_j}{\partial t}\right) = 0. \tag{8}$$

It is convenient to define an apparent circular frequency $\omega = \alpha\Omega$ and the angles of reflection ϕ_1 and refraction ϕ_2 are given by

$$\cos\phi_j = M_j + (\Lambda_j/\lambda_j)\cos\Theta_j, \quad \sin\phi_j = (s_j/\alpha)\sin\Theta_j,$$

where

$$\alpha = (1 + M_j \cos\Theta_j) \quad \text{and} \quad \lambda_j = (\Lambda_j/s_j^2)\alpha. \tag{9}$$

Using these relations in a moving system, (1) and (5) take the form

$$\begin{bmatrix} v_1^0 \\ v_1^r \\ v_2^T \end{bmatrix} = \begin{bmatrix} w_1^0(x, y_1) \\ w_1^r(x, y_1) \\ w_2^T(x, y_2) \end{bmatrix} \exp\{i(M_1\lambda_1 x - \omega t)\} \tag{10}$$

where

$$\begin{aligned} w_1^0(x, y_1) &= V_1 \exp\{-i\lambda_1(x\cos\phi_1 + y_1 \sin\phi_1)\} \\ w_1^r(x, y_1) &= A_1^R V_1 \exp\{-i\lambda_1(x\cos\phi_1 - y_1\sin\phi_1)\} \\ w_2^T(x, y_2) &= A_2^T V_1 \exp[-i\{(\beta_2 + \lambda_2\cos\phi_2)x + \lambda_2 y_2 \sin\phi_2\}] \end{aligned} \tag{11}$$

and

$$\beta_2 = M_1\lambda_1\left(1 - \frac{\lambda_2 C_1}{\lambda_1 C_2}\right) < 0 \quad \text{since} \quad C_1 > C_2.$$

Using (10), we assume the solution of the governing equation (8) as

$$v_j(x, y_j, t) = w_j(x, y_j)\exp[i(M_j\lambda_j x - \omega t)]. \tag{12}$$

Substitution of (12) in (8) yields the Helmholtz equation

$$\frac{\partial^2 w_j}{\partial x^2} + \frac{\partial^2 w_j}{\partial y_j^2} + \lambda_j^2 w_j = 0 \quad (j = 1, 2). \tag{13}$$

Applying Fourier transform, (13) can be solved and the result is

$$w_1(x, y_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_1(\xi)\exp\{-i\xi x - (\xi^2 - \lambda_1^2)^{1/2} y_1\}\,d\xi, \quad (y_1 > 0)$$

and

$$w_2(x, y_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_2(\xi)\exp\{-i\xi x - (\xi^2 - \lambda_2^2)^{1/2} y_2\}\,d\xi, \quad (y_2 < 0) \tag{14}$$

where $A_1(\xi)$ and $A_2(\xi)$ are the unknown functions to be determined. From (12) and (14) we obtain the displacement components due to scattered field as

$$v_1 = \exp[i(M_1\lambda_1 x - \omega t]\frac{1}{2\pi}\int_{-\infty}^{\infty} A_1(u)\exp[-iux - \gamma_1 y_1]\,du, \quad (y_1 > 0)$$

and

$$v_2 = \exp[i(M_1\lambda_1 x - \omega t]\frac{1}{2\pi}\int_{-\infty}^{\infty} A_2(u)\exp[-iux + \gamma_2 y_2]\,du, \quad (y_2 < 0) \tag{15}$$

where

$$\gamma_1 = (u^2 - \lambda_1^2)^{1/2} \quad \text{and} \quad \gamma_2 = [(u - \beta_2)^2 - \lambda_2^2]^{1/2}. \tag{16}$$

Therefore, the expressions for the stresses are

$$(\tau_{xz})_1 = -i\mu_1 \exp[i(M_1\lambda_1 x - \omega t)] \times \frac{1}{2\pi}\int_{-\infty}^{\infty} (u - M_1\lambda_1)A_1(u)\exp[-iux - \gamma_1 y_1]\,du$$

$$(\tau_{xz})_2 = -i\mu_2 \exp[i(M_1\lambda_1 x - \omega t)] \times \frac{1}{2\pi}\int_{-\infty}^{\infty} (u - M_1\lambda_1)A_2(u)\exp[-iux + \gamma_2 y_2]\,du$$

and

$$(\tau_{y_1 z})_1 = -\mu_1 s_1 \exp[i(M_1\lambda_1 x - \omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty} \gamma_1 A_1(u)\exp[-iux - \gamma_1 y_1]\,du$$

$$(\tau_{y_2 z})_2 = \mu_2 s_2 \exp[i(M_1\lambda_1 x - \omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty} \gamma_2 A_2(u)\exp[-iux + \gamma_2 y_2]\,du. \tag{17}$$

The unknown functions $A_1(u)$ and $A_2(u)$ are to be determined from the following boundary conditions at the interface $y = 0$

(i) $v_1(x, 0) = v_2(x, 0), \quad x > 0$

(ii) $\mu_1 s_1 \frac{\partial v_1}{\partial y_1} = \mu_2 s_2 \frac{\partial v_2}{\partial y_2}, \quad -\infty < x < \infty$

and

(iii) $$\frac{\partial v_1^0}{\partial y_1}+\frac{\partial v_1^r}{\partial y_1}+\frac{\partial v_1}{\partial y_1}=0, \quad x<0, \quad y\to 0+.$$

From the boundary condition (ii) we obtain

$$\mu_1 s_1 \gamma_1 A_1(u)+\mu_2 s_2 \gamma_2 A_2(u)=0 \tag{18}$$

and from other two boundary conditions, we get

$$\int_{-\infty}^{\infty} B_1(u)\exp(-iux)du=0 \quad (x>0)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} M(u)B_1(u)\exp(-iux)du=N\exp[-i\lambda_1 x\cos\phi_1], \quad (x<0) \tag{19}$$

where

$$B_1(u)=\frac{\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2}{\mu_2 s_2\gamma_2}A_1(u)$$

$$M(u)=\gamma_1\frac{\mu_2 s_2\gamma_2}{(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)} \tag{20}$$

and

$$N=-\frac{i\Lambda_1 v_1\sin\Theta_1}{s_1}(1-A_1^R).$$

The solution of the dual integral equation may be obtained by a method based on the Wiener–Hopf technique. The first part of (19) can be satisfied if we choose

$$B_1(u)=L_-(u) \tag{21}$$

where $L_-(u)$ is a function of u, analytic in the lower half of the complex u-plane. The second part is satisfied if we take

$$M(u)B_1(u)=\frac{N}{i(u-\alpha_1)}\frac{U_+(u)}{U_+(\alpha_1)} \tag{22}$$

where $\alpha_1=\lambda_1\cos\phi_1$ and $U_+(u)$ is a function free from zeros and singularities in the upper half of the complex u-plane. Thus (22) is a solution of the second part of (19) can easily be shown by completing the path from $-\infty$ to ∞ by a semi-circle of infinite radius in the upper u-plane and then applying the residue theorem and Jordan's Lemma. The path of integration is chosen to avoid possible branch points and is indented below the pole $u=\alpha_1$.

Eliminating $B_1(u)$ from (21) and (22) we obtain

$$\frac{L_-(u)}{U_+(u)}=\frac{N}{i(u-\alpha_1)M(u)}\frac{1}{U_+(\alpha_1)} \tag{23}$$

and

$$M(u)=\frac{\mu_2 s_2}{\mu_1 s_1+\mu_2 s_2}(u^2-\lambda_1^2)^{1/2}F(u) \tag{24}$$

where

$$F(u)=\frac{(\mu_1 s_1+\mu_2 s_2)\gamma_2}{(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)}$$

and

$$F(u)\to 1 \quad \text{as} \quad |u|\to\infty.$$

The function $F(u)$ can be expressed as the product of two functions such that

$$F(u)=F_+(u)\cdot F_-(u) \tag{25}$$

where $F_+(u)$ and $F_-(u)$ are analytic in the upper and lower half of the complex u-plane respectively. The expressions for $F_+(u)$ and $F_-(u)$ have been derived in the appendix.

In view of (25), (24) assumes the form

$$\frac{U_+(u)}{(u+\lambda_1)^{1/2}F_+(u)}=\frac{L_-(u)}{\dfrac{N}{iU_+(\alpha_1)}\dfrac{\mu_1 s_1+\mu_2 s_2}{\mu_2 s_2(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)}} \tag{26}$$

where

$$U_+(u)=(u+\lambda_1)^{1/2}F_+(u). \tag{27}$$

So

$$L_-(u)=\frac{N}{i(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}\frac{\mu_1 s_1+\mu_2 s_2}{\mu_2 s_2(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)}. \tag{28}$$

Hence the functions $A_1(u)$ and $A_2(u)$ are

$$A_1(u)=\frac{N}{i(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}\frac{\gamma_2(\mu_1 s_1+\mu_2 s_2)}{(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)}$$

and

$$A_2(u)=\frac{-N}{i(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}\frac{\mu_1 s_1\gamma_1(\mu_1 s_1+\mu_2 s_2)}{\mu_2 s_2(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)}. \tag{29}$$

The singular behaviour of the stress components for the scattered waves at the crack-tip is due to the divergence of the integrals around $x=y_j=0$ in (17). Making use of (29) and asymptotic expressions of the integrands of (17) for large values of u, we obtain near about the crack-tip,

$$(\tau_{xz})_1=\frac{B(1+i)}{s_1}\int_0^\infty u^{-1/2}\exp[-s_1 uY](\cos ux-\sin ux)\,du$$

$$(\tau_{xz})_2=\frac{-B(1+i)}{s_2}\int_0^\infty u^{-1/2}\exp[-s_2 u|Y|](\cos ux-\sin ux)\,du$$

$$(\tau_{yz})_1=-B(1+i)\int_0^\infty u^{-1/2}\exp[-s_1 uY](\cos ux+\sin ux)\,du$$

$$(\tau_{yz})_2=-B(1+i)\int_0^\infty u^{-1/2}\exp[-s_2 u|Y|](\cos ux+\sin ux)\,du \tag{30}$$

where

$$B = -\frac{N\mu_1 s_1}{2\pi(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}; \quad y_j = s_j Y \quad (j=1,2).$$

Using the results

$$\int_0^\infty u^{-1/2}\exp[-s_1 uY]\cos ux\,dx = (\pi/2)^{1/2}\left[\frac{(s_1^2y^2+x^2)^{1/2}+s_1y}{s_1^2y^2+x^2}\right]^{1/2}$$
$$\int_0^\infty u^{-1/2}\exp[-s_1 uY]\sin ux\,dx = (\pi/2)^{1/2}\left[\frac{(s_1^2y^2+x^2)^{1/2}-s_1y}{s_1^2y^2+x^2}\right]^{1/2} \tag{31}$$

the stresses near about the crack tip given by (30) can be evaluated. The displacement near about the crack tip can be obtained from the crack tip stresses by integration.

Now introducing the factor $\exp[i(M_1\lambda_1 x-\omega t)]$ and taking the real part, the stresses and displacements near about the moving crack-tip are found to be equal to

$$\begin{bmatrix}(\tau_{yz})_j\\ (\tau_{xz})_j\\ v_j\end{bmatrix} = \mathrm{Re}\begin{bmatrix} K_1\left[\dfrac{(s_j^2Y^2+x^2)^{1/2}+x}{s_j^2Y^2+x^2}\right]^{1/2}\\ (-1)^j\dfrac{K_1}{s_1}\left[\dfrac{(s_j^2Y^2+x^2)^{1/2}-x}{s_j^2Y^2+x^2}\right]^{1/2}\\ (-1)^{j+1}\dfrac{2K_1}{\mu_j s_j}[(x^2+s_j^2Y^2)^{1/2}-x]^{1/2}\end{bmatrix}\exp\left[i\left(M_1\lambda_1 x-\omega t-\frac{\pi}{4}\right)\right] \tag{32}$$

where

$$K_1 = (2/\pi)^{1/2}\frac{\mu_1\mu_2\Lambda_1\Lambda_2 V_1\sin\Theta_1\sin\Theta_2}{(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)(\mu_1\Lambda_1\sin\Theta_1+\mu_2\Lambda_2\sin\Theta_2)}. \tag{33}$$

In the case of crack propagation in an isotropic elastic medium using the result $\mu_1=\mu_2$, $\rho_1=\rho_2$ and $F_+(\alpha_1)=1$, we obtain

$$K_1 = (1/\pi)^{1/2}\mu_1\Lambda_1^{1/2}V_1(1-M_1)^{1/2}\sin(\Theta_1/2). \tag{34}$$

Putting $r=(x^2+y^2)^{1/2}$, $\tan\phi = |Y|/x$, the expression of displacements and stresses given by (32) near about the moving crack-tip is found to be equal to

$$v_1 = \frac{2K_1}{\mu_1 s_1}r^{1/2}\{(1-M_1^2\sin^2\phi)^{1/2}-\cos\phi\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{3/2})$$

$$v_2 = -\frac{2K_1}{\mu_2 s_2}r^{1/2}\{(1-M_2^2\sin^2\phi)^{1/2}-\cos\phi\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{3/2})$$

$$(\tau_{yz})_1 = \frac{K_1}{r^{1/2}}\left\{\frac{(1-M_1^2\sin^2\phi)^{1/2}+\cos\phi}{1-M_1^2\sin^2\phi}\right\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{1/2})$$

$$(\tau_{yz})_2 = \frac{K_1}{r^{1/2}}\left\{\frac{(1-M_2^2\sin^2\phi)^{1/2}+\cos\phi}{1-M_2^2\sin^2\phi}\right\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{1/2})$$

$$(\tau_{xz})_1 = -\frac{K_1}{s_1}\frac{1}{r^{1/2}}\left\{\frac{(1-M_1^2\sin^2\phi)^{1/2}-\cos\phi}{1-M_1^2\sin^2\phi}\right\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{1/2})$$

$$(\tau_{xz})_2 = \frac{K_1}{s_2}\frac{1}{r^{1/2}}\left\{\frac{(1-M_2^2\sin^2\phi)^{1/2}-\cos\phi}{1-M_2^2\sin^2\phi}\right\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{1/2}). \tag{35}$$

Taking the value of K_1 given by (34), the results given by (35) agree with the results of the crack propagation in an isotropic elastic medium as given by Jahanshahi [3].

When the crack is stationary, the corresponding results of stresses and displacements near about the crack-tip can be derived by making M_1 and M_2 approach zero and are given by

$$\begin{aligned}(\tau_{yz})_1 &= K_1^*(2/r)^{1/2}\cos\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{1/2})\\ (\tau_{yz})_2 &= K_1^*(2/r)^{1/2}\cos\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{1/2})\\ (\tau_{xz})_1 &= -K_1^*(2/r)^{1/2}\sin\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{1/2})\\ (\tau_{xz})_2 &= K_1^*(2/r)^{1/2}\sin\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{1/2})\end{aligned} \tag{36}$$

and

$$v_1 = \frac{2\sqrt{2}K_1^*}{\mu_1}(r)^{1/2}\sin\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{3/2})$$

$$v_2 = \frac{-2\sqrt{2}K_1^*}{\mu_2}(r)^{1/2}\sin\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{3/2}) \tag{37}$$

where

$$K_1^* = \sqrt{2/\pi}\,\frac{\mu_1\mu_2\Lambda_1\Lambda_2 V_1\sin\Theta_1\sin\Theta_2}{(\Lambda_1\cos\phi_1+\Lambda_1)^{1/2}F_+^*(\Lambda_1\cos\phi_1)(\mu_1\Lambda_1\sin\phi_1+\mu_2\Lambda_2\sin\phi_2)} \tag{38}$$

and

$$F_+^*(\Lambda_1\cos\phi_1) = \exp\left[\frac{1}{\pi}\int_{\Lambda_1}^{\Lambda_2}\tan^{-1}\left\{\frac{\mu_1(s^2-\Lambda_1^2)^{1/2}}{\mu_2(\Lambda_2^2-s^2)^{1/2}}\right\}\frac{ds}{s+\Lambda_1\cos\phi_1}\right]. \tag{39}$$

If we put $\mu_1=\mu_2$, $\rho_1=\rho_2$, the corresponding results of the stationary crack in an isotropic elastic medium are found to be

$$(\tau_{yz})_{1,2} = V_1(\sin\tfrac{1}{2}\Theta_1)(\cos\tfrac{1}{2}\phi)\cos(\Omega t+\tfrac{1}{4}\pi)\left[\frac{2\Lambda_1\mu_1^2}{\pi r}\right]^{1/2}+0(r^{1/2})$$

$$(\tau_{xz})_{1,2} = \mp V_1(\sin\tfrac{1}{2}\Theta_1)(\cos\tfrac{1}{2}\phi)\cos(\Omega t+\tfrac{1}{4}\pi)\left[\frac{2\Lambda_1\mu_1^2}{\pi r}\right]^{1/2}+0(r^{1/2})$$

and

$$v_{1,2} = \pm V_1(\sin\tfrac{1}{2}\Theta_1)(\sin\tfrac{1}{2}\phi)\cos(\Omega t+\tfrac{1}{4}\pi)\left[\frac{8\Lambda_1 r}{\pi}\right]^{1/2}+0(r^{3/2}) \tag{40}$$

which are same as given by Jahanshahi [3].

3. Results and discussion

K_{I} given by (33) is the dynamic stress intensity factor at the moving crack-tip and K_{I}^{*} given by (38) is the value of the corresponding quantity when the crack is stationary. The variation of $K_{\mathrm{I}}/K_{\mathrm{I}}^{*}$ with the values of V/C_2 where V is the crack speed has been depicted graphically for the following two sets of materials.

First set:

Wrought iron	$\rho_1 = 7{\cdot}8\ \mathrm{g/cm^3}$,	$\mu_1 = 7{\cdot}7 \times 10^{11}\ \mathrm{dyn/cm^2}$
Copper	$\rho_2 = 8{\cdot}96\ \mathrm{g/cm^3}$,	$\mu_2 = 4{\cdot}5 \times 10^{11}\ \mathrm{dyn/cm^2}$

Second set:

Steel	$\rho_1 = 7{\cdot}6\ \mathrm{g/cm^3}$,	$\mu_1 = 8{\cdot}32 \times 10^{11}\ \mathrm{dyn/cm^2}$
Aluminium	$\rho_2 = 2{\cdot}7\ \mathrm{g/cm^3}$,	$\mu_2 = 2{\cdot}63 \times 10^{11}\ \mathrm{dyn/cm^2}$.

It is found that in both the cases the stress intensity factor gradually decreases with an increase in the value of V/C_2 and approaches zero as $V/C_2 \to 1$; the decrease in the value of $K_{\mathrm{I}}/K_{\mathrm{I}}^{*}$ for the second set being more rapid than for the first set. We also find that in both the cases for any fixed value of C_1/C_2, $K_{\mathrm{I}}/K_{\mathrm{I}}^{*}$ decreases with decrease in the value of Θ.

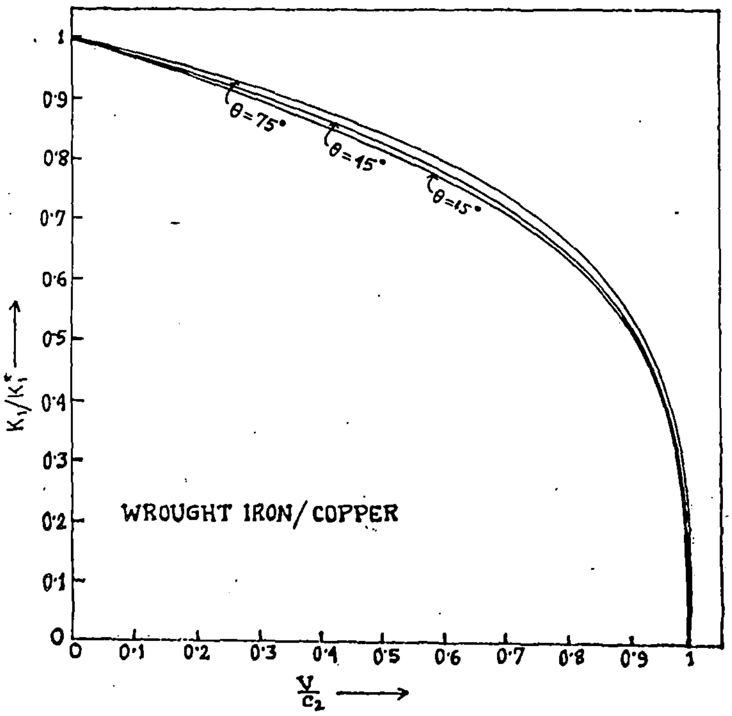

Figure 2. Stress intensity factor vs dimensionless crack speed (wrought iron/copper).

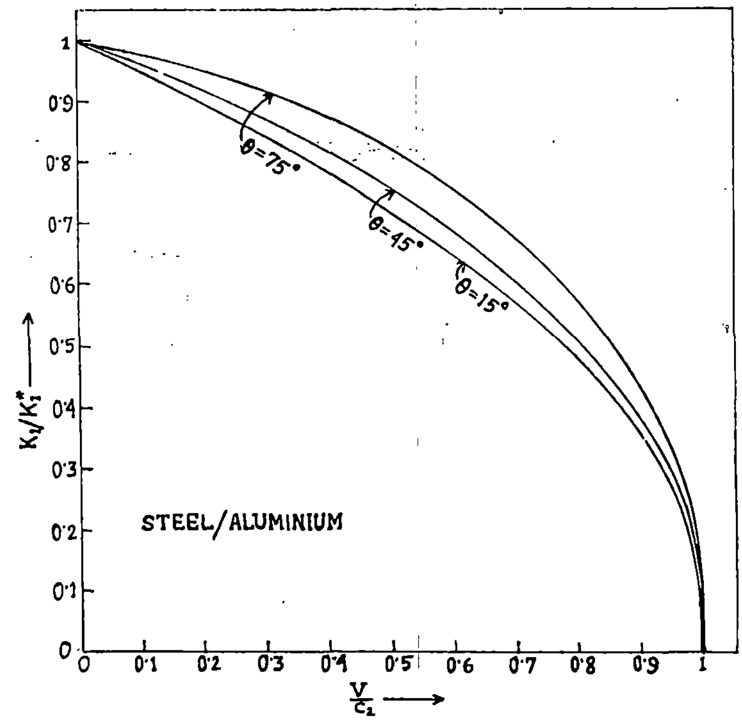

Figure 3. Stress intensity factor vs dimensionless crack speed (steel/aluminium).

Appendix

Factorization of $F(\xi)$ into $F_+(\xi)$ and $F_-(\xi)$

Consider

$$F(\xi)=\frac{(\mu_1 s_1+\mu_2 s_2)\gamma_2}{(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)} \tag{A1}$$

The branch points of $F(\xi)$ are at $\xi=\lambda_1,\ -\lambda_1,\ \lambda_2+\beta_2,\ -(\lambda_2-\beta_2)$ where

$$-(\lambda_2-\beta_2)<-\lambda_1<\lambda_1<\lambda_2+\beta_2 \text{ since } C_2<C_1.$$

Since $F(\xi)\to 1$ as $|\xi|\to\infty$, $F(\xi)$ possesses no singularity within the rectangular contour (shown in figure 4), by Cauchy's residue theorem we can write

$$\log F(\xi)=\frac{1}{2\pi i}\int_{c_+ + c_-}\frac{\log F(s)}{s-\xi}\mathrm{d}s \tag{A2}$$

$$=\frac{1}{2\pi i}\int_{c_+}\frac{\log F(s)}{s-\xi}\mathrm{d}s+\frac{1}{2\pi i}\int_{c_-}\frac{\log F(s)}{s-\xi}\mathrm{d}s$$

$$\log F(\xi)=\log F_+(\xi)+\log F_-(\xi), \tag{A3}$$

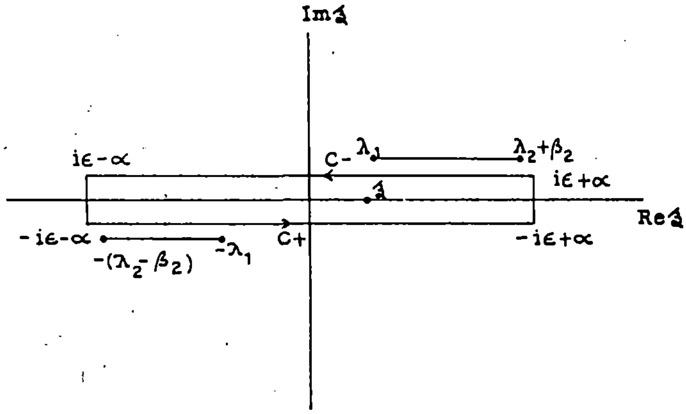

Figure 4. Rectangular contour in the complex ξ-plane.

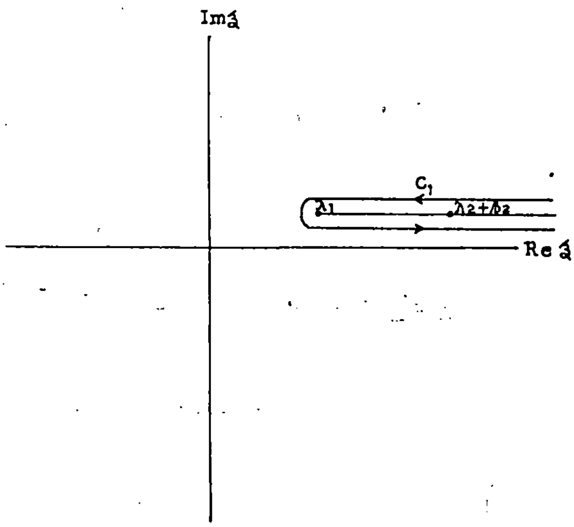

Figure 5. Path of integration C_1 round the branch cut.

where $F_+(\xi)$ and $F_-(\xi)$ are analytic in the upper and lower half of the complex ξ-plane respectively and can be expressed as

$$F_+(\xi) = \exp\left[\frac{1}{2\pi i}\int_{c_+} \frac{\log F(s)}{s-\xi} \mathrm{d}s\right]$$

and

$$F_-(\xi) = \exp\left[\frac{1}{2\pi i}\int_{c_-} \frac{\log F(s)}{s-\xi} \mathrm{d}s\right]. \tag{A4}$$

In order to evaluate $F_-(\xi)$ the path of integration C_- can be deformed to the path C_1 round the branch cut through λ_1 and $\lambda_2 + \beta_2$ as shown in figure 5.

After a little algebraic manipulation it can be shown that

$$F_-(\xi)=\exp\left[\frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2+\beta_2}\frac{1}{s-\xi}\log\left\{1+i\frac{m_1}{m_2}\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2-s)^2]^{1/2}}\right\}ds\right.$$
$$\left.-\frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2+\beta_2}\frac{1}{s-\xi}\log\left\{1-i\frac{m_1}{m_2}\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2-s)^2]^{1/2}}\right\}ds\right] \tag{A5}$$

which after simplification becomes

$$F_-(\xi)=\exp\left[\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2+\beta_2}\frac{1}{s-\xi}\tan^{-1}\left\{\frac{m_1}{m_2}\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2-s)^2]^{1/2}}\right\}ds\right] \tag{A6}$$

where

$$m_1=\frac{\mu_1 s_1}{\mu_1 s_1+\mu_2 s_2}\quad\text{and}\quad m=\frac{\mu_2 s_2}{\mu_1 s_1+\mu_2 s_2}. \tag{A7}$$

Similarly it can be shown that

$$F_+(\xi)=\exp\left[\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2-\beta_2}\frac{1}{s+\xi}\tan^{-1}\left\{\frac{m_1}{m_2}\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2+s)^2]^{1/2}}\right\}ds\right]. \tag{A8}$$

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Diffraction of SH-waves by a Griffith crack in nonhomogeneous elastic strip

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IN THIS PAPER the scattering of elastic SH-waves by a Griffith crack situated in an infinitely long inhomogeneous strip has been analyzed. Assuming that the shear modulus (μ) and density (ϱ) of the material vary in the vertical direction and applying Fourier transform, the mixed boundary value problem has been reduced to the solution of dual integral equations which finally has been reduced to the solution of a Fredholm integral equation of second kind. The numerical values of stress intensity factor and crack opening displacement have been illustrated graphically to show the effect of inhomogeneity of the material.

1. Introduction

THE NATURAL or artificial materials are usually inhomogeneous; so in recent years great attention has been given to the study of diffraction of elastic waves by cracks or obstacles in inhomogeneous media in view of their application in fracture mechanics. Many problems have been solved involving one or more cracks in an infinite homogeneous elastic medium. LOEBER and SIH [1] and MAL [2] have studied the problem of diffraction of elastic waves by a Griffith crack in an infinite medium. The problem of finite crack at the interface of two elastic half-spaces has been discussed by SRIVASTAVA *et al.* [3] and BOSTROM [4]. SINGH *et al.* [5, 6] considered the problem of scattering of a SH-wave by cracks or strips in a nonhomogeneous infinite elastic medium. Papers involving cracks located in an infinitely long elastic strip are very few. The problem of an infinite elastic strip containing an arbitrary number of unequal Griffith cracks, located parallel to its surfaces and opened by an arbitrary internal pressure, has been treated by ADAMS [7]. Finite crack perpendicular to the surface of the infinitely long elastic strip has been studied by CHEN [8] (for an impact load) and by SRIVASTAVA *et al.* [9] (for normally incident waves). Recently SHINDO *et al.* [10] considered the problem of impact response of a finite crack in an orthotropic strip. In our paper, the diffraction of normally incident SH-waves by a Griffith crack situated in an infinitely long inhomogeneous elastic strip has been discussed. The shear modulus (μ) and the density (ϱ) of the material have been assumed to vary in the vertical direction. Applying the Fourier transform, the mixed boundary value problem has been converted to the solution of dual integral equations. The dual integral equations have been finally reduced to a Fredholm integral equation of second kind by applying the Abel transform. Expressions for the stress intensity factor and crack opening displacement have been derived. The numerical values of stress intensity factor and crack opening displacement have been depicted by means of graphs to show the effect of material inhomogeneity.

2. Formulation of the problem

Consider the problem of diffraction of SH-waves by a Griffith crack in an inhomogeneous elastic strip of width $2h_1$. The crack is located in the region $-a \le x_1 \le a$, $-\infty < y_1 < \infty$, $z_1 = 0$ (Fig. 1). Normalizing all the lengths with respect to a and putting $x_1/a = x$, $y_1/a = y$, $z_1/a = z$, $h_1/a = h$ it is found that the location of the crack is $-1 \leq x \leq 1$, $-\infty < y < \infty$, $z = 0$ referred to a Cartesian coordinate system (x, y, z). Let a plane harmonic SH-wave originating at $z = -\infty$ impinge on the crack normally to the x-axis. The variation of the shear modulus μ and the density ϱ is taken in the vertical (z) direction in such a manner that the shear velocity $(\mu_0/\varrho_0)^{1/2}$ is constant. The only non-vanishing y-component of the displacement which is independent of y is $v = v(x, z, t)$.

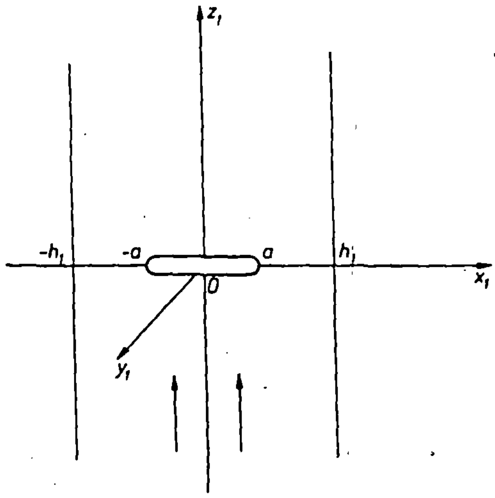

FIG. 1. Crack in the inhomogeneous strip.

The equation of motion is given by

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) = \varrho\frac{\partial^2 v}{\partial t^2}. \tag{2.1}$$

If we consider $v(x, z, t)$ in the form

$$v(x, z, t) = \frac{W(x, z, t)}{\sqrt{\mu(z)}}, \tag{2.2}$$

then

$$\mu\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial z^2}\right) + \frac{1}{2}\left[\frac{1}{2\mu}\left(\frac{\partial\mu}{\partial z}\right)^2 - \frac{\partial^2\mu}{\partial z^2}\right]W = \varrho\frac{\partial^2 W}{\partial t^2}. \tag{2.3}$$

Putting $W(x, z, t) = F(x)G(z)e^{-i\omega t}$ and $\mu(z) = \mu_0 f(z)$, $\varrho(z) = \varrho_0 f(z)$ in Eq. (2.3) where μ_0, ϱ_0 are constants, such that $(\mu_0/\varrho_0)^{1/2} = c_2$ is the shear wave velocity, it is found that $F(x)$ and $G(z)$ satisfy the following equations

$$\frac{\partial^2 F}{\partial x^2} + n^2 F = 0, \tag{2.4}$$

$$\frac{\partial^2 G}{\partial z^2} + \left(\frac{a^2\omega^2}{c_2^2} - b^2 - n^2\right)G = 0, \tag{2.5}$$

provided $f(z)$ is of the form

$$-\frac{1}{4}\left(\frac{\partial f}{\partial z}/f\right)^2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial z^2}/f\right) = b^2, \tag{2.6}$$

where a and b are constants.

Let us assume $f(z)$ in the form

$$f(z) = \cosh^2(bz) \tag{2.7}$$

so that Eq. (2.6) is automatically satisfied.

Now the shear modulus $\mu(z)$ and density of the medium $\varrho(z)$ are

$$\mu = \mu_0 \cosh^2(bz), \qquad \varrho = \varrho_0 \cosh^2(bz). \tag{2.8}$$

Using Eqs. (2.8), (2.2) and $W(x, z, t) = W(x, z)e^{-i\omega t}$, Eq. (2.1) takes the form

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial z^2} + k^2 W = 0, \quad k^2 = (k_2^2 - b^2), \quad k_2 = \frac{a\omega}{c_2}. \tag{2.9}$$

The displacement component $v^{(i)}(x, z, t)$ and stress $\tau^{(i)}(x, z, t)$ due to incident waves are given by

$$v^{(i)}(x, z, t) = \frac{A_0 e^{i(kz-\omega t)}}{\sqrt{\mu_0}\cosh(bz)} \tag{2.10}$$

and

$$\tau_{yz}^{(i)}(x, z, t) = A_0\sqrt{\mu_0}[ik\cosh(bz) - b\sinh(bz)]e^{i(kz-\omega t)}, \tag{2.11}$$

where A_0 is a constant.

Henceforth the time factor $e^{-i\omega t}$ will be suppressed in the sequel.

Solution of Eq. (2.9) is

$$W(x, z) = \int_0^\infty B_1(\xi)e^{-\beta z}\cos(\xi x)d\xi + \int_0^\infty C_1(\zeta)\cosh(\alpha x)\sin(\zeta z)d\zeta, \tag{2.12}$$

where

$$\alpha = (\zeta^2 - k^2)^{1/2}, \quad \zeta > k, \quad \beta = (\xi^2 - k^2)^{1/2}, \quad \xi > k,$$
$$= i(k^2 - \zeta^2)^{1/2}, \quad \zeta < k, \quad = -i(k^2 - \xi^2)^{1/2}, \quad \xi < k.$$

Now displacement $v(x, z)$ and stresses $\tau_{yz}(x, z), \tau_{xy}(x, z)$ due to the scattered field are

$$v(x, z) = \frac{1}{\cosh(bz)}\left[\int_0^\infty B(\xi)e^{-\beta z}\cos\xi x\, d\xi + \int_0^\infty C(\zeta)\cosh(\alpha x)\sin\zeta z\, d\zeta\right], \tag{2.13}$$

$$\begin{aligned}\tau_{yz}(x, z) = -\mu_0 b\sinh(bz)\Big[\int_0^\infty B(\xi)e^{-\beta z}\cos\xi x\, d\xi \\ + \int_0^\infty C(\zeta)\cosh(\alpha x)\sin\zeta z\, d\zeta\Big] + \mu_0\cosh(bz) \\ \Big[-\int_0^\infty \beta B(\xi)e^{-\beta z}\cos\xi x\, d\xi + \int_0^\infty \zeta C(\zeta)\cosh(\alpha x)\cos\zeta z\, d\zeta\Big],\end{aligned} \tag{2.14}$$

$$\tau_{xy}(x,z) = \mu_0 \cosh(bz)\Big[- \int_0^\infty \xi B(\xi) e^{-\beta z} \sin \xi x \, d\xi + \int_0^\infty \alpha C(\zeta) \sinh(\alpha x) \sin \zeta z \, d\zeta \Big], \tag{2.15}$$

where

$$B(\xi) = \frac{1}{\sqrt{\mu_0}} B_1(\xi), \quad C(\zeta) = \frac{1}{\sqrt{\mu_0}} C_1(\zeta).$$

The boundary conditions are

$$\tau_{yz}(x,0) = -\tau_0, \qquad |x| \le 1, \tag{2.16}$$

$$v(x,0) = 0, \qquad 1 \le |x| \le h, \tag{2.17}$$

$$\tau_{xy}(\pm h, z) = 0, \qquad |z| < \infty, \tag{2.18}$$

where $\tau_0 = ikA_0\sqrt{\mu_0}$.

From the boundary condition (2.18) $C(\zeta)$ is found to be expressible in terms of $B(\xi)$ as follows:

$$C(\zeta) = \frac{2\zeta}{\pi \alpha \sinh(\alpha h)} \int_0^\infty \frac{\xi B(\xi) \sin(\xi h)}{\xi^2 + \alpha^2} \, d\xi. \tag{2.19}$$

Next, the use of Eq. (2.19) in the boundary condition (2.16) and (2.17) yields the following dual integral equations from which the unknown function $B(\xi)$ is to be determined:

$$\int_0^\infty \xi [1 + M(\xi)] B(\xi) \cos(\xi x) \, d\xi = p(x), \quad |x| \le 1 \tag{2.20}$$

and

$$\int_0^\infty B(\xi) \cos(\xi x) \, d\xi = 0, \quad 1 \le |x| \le h \tag{2.21}$$

where

$$M(\xi) = \left(\frac{\beta}{\xi} - 1 \right), \tag{2.22}$$

$$p(x) = \frac{\tau_0}{\mu_0} + \frac{2}{\pi} \int_0^\infty \frac{\zeta^2 \cosh(\alpha x)}{\alpha \sinh(\alpha h)} \, d\zeta \int_0^\infty \frac{\xi B(\xi) \sin(\xi h)}{\xi^2 + \alpha^2} \, d\xi. \tag{2.23}$$

3. Method of solution

In order to solve the dual integral equations (2.20) and (2.21), $B(\xi)$ is taken in the form

$$B(\xi) = \frac{\tau_0}{\mu_0} \int_0^1 t \phi(t) J_0(\xi t) \, dt, \tag{3.1}$$

so that Eq. (2.21) is automatically satisfied.

Substitution of the value of $B(\xi)$ from Eq. (3.1) in Eq. (2.20), yields a Fredholm integral equation of second kind

$$\phi(t) + \int_0^1 u[L_1(u,t) + L_2(u,t)]\phi(u)\,du = 1, \tag{3.2}$$

where

$$L_1(u,t) = \int_0^\infty \xi M(\xi) J_0(\xi u) J_0(\xi t)\,d\xi, \tag{3.3}$$

$$L_2(u,t) = -\int_0^\infty \frac{\zeta^2 I_0(\alpha t) I_0(\alpha u) e^{-\alpha h}}{\alpha \sinh(\alpha h)}\,d\zeta. \tag{3.4}$$

Using contour integration technique [3], the infinite integral arising in the kernel $L_1(u,t)$ can be converted to a finite integral and is given by

$$\begin{aligned} L_1(u,t) &= -ik^2 \int_0^1 (1-\eta^2)^{1/2} J_0(k\eta t) H_0^{(1)}(k\eta u)\,d\eta, \qquad u > t, \\ &= -ik^2 \int_0^1 (1-\eta^2)^{1/2} J_0(k\eta u) H_0^{(1)}(k\eta t)\,d\eta, \qquad u < t. \end{aligned} \tag{3.5}$$

Now

$$\begin{aligned} L_2(u,t) &= \int_0^k \frac{\zeta^2 J_0(\alpha_1 t) J_0(\alpha_1 u) e^{i\alpha_1 h}}{\alpha_1 \sin(\alpha_1 h)}\,d\zeta - \int_k^\infty \frac{\zeta^2 I_0(\alpha t) I_0(\alpha u) e^{-\alpha h}}{\alpha \sinh(\alpha h)}\,d\zeta \\ &= \int_0^k \frac{\zeta^2}{\alpha_1} J_0(\alpha_1 t) J_0(\alpha_1 u)\,\mathrm{ctg}(\alpha_1 h)\,d\zeta + i \int_0^k \frac{\zeta^2}{\alpha_1} J_0(\alpha_1 t) J_0(\alpha_1 u)\,d\zeta \\ &\qquad - \int_k^\infty \frac{\zeta^2 I_0(\alpha t) I_0(\alpha u) e^{-\alpha h}}{\alpha \sinh(\alpha h)}\,d\zeta, \end{aligned}$$

where

$$\alpha_1 = (k^2 - \zeta^2)^{1/2}.$$

Putting $\zeta^2 = k^2(1-y^2)$ in the first and second integrals and $\zeta^2 = k^2(1+y^2)$ in the third integral, it is found that

$$\begin{aligned} L_2(u,t) = k^2 \Big[&\int_0^1 (1-y^2)^{1/2} J_0(kyt) J_0(kyu)\,\mathrm{ctg}(kyh)\,dy \\ &+ i \int_0^1 (1-y^2)^{1/2} J_0(kyt) J_0(kyu)\,dy \\ &- \int_0^\infty (1+y^2)^{1/2} I_0(kyt) I_0(kyu) e^{-kyh}\,\mathrm{cosech}(kyh)\,dy \Big]. \end{aligned} \tag{3.6}$$

4. Stress intensity factor and crack opening displacement

From Eq. (2.14) the stress τ_{yz} on the plane $z = 0$ can be written as

$$\tau_{yz}(x,0) = \mu_0\Big[-\int_0^\infty \beta B(\xi)\cos\xi x\, d\xi + \int_0^\infty \zeta C(\zeta)\cosh(\alpha x)\, d\zeta\Big]. \tag{4.1}$$

Substituting the value of $C(\zeta)$ and $B(\xi)$ from Eqs. (2.19) and (3.1), the expression for the stress can finally be presented as

$$\tau_{yz}(x,0) = \frac{\tau_0 x}{(x^2-1)^{1/2}}\phi(1) + O(1), \qquad |x| > 1.$$

Defining the stress intensity factor N by

$$N = \operatorname*{Lt}_{x\to 1^+}\left|\frac{(x-1)^{1/2}\tau_{yz}(x,0)}{\tau_0}\right|,$$

we obtain

$$N = \frac{1}{\sqrt{2}}|\phi(1)|. \tag{4.2}$$

Now the crack opening displacement $\Delta v(x,0) = v(x,0^+) - v(x,0^-)$ can be obtained from Eq. (2.13) as

$$\Delta v(x,0) = 2\int_0^\infty B(\xi)\cos(\xi x)\, d\xi, \qquad |x| \le 1,$$

which, on substitution of the value of $B(\xi)$ from Eq. (3.1), takes the form

$$\Delta x(x,0) = \frac{2\tau_0}{\mu_0}\int_x^1 \frac{t\phi(t)}{(t^2-x^2)^{1/2}}\, dt, \qquad |x| \le 1. \tag{4.3}$$

5. Numerical results and discussion

Using the method of FOX and GOODWIN [11], the Fredholm integral equation given by Eq. (3.2) has been solved numerically for different values of the material inhomogeneity parameters. In this method the integral in Eq. (3.2) has been represented at first by a quadrature formula involving the values of the desired function $\phi(t)$ at the pivotal points inside the specified range of integration, and then converted to a set of simultaneous linear algebraic equations; their solutions yield the first approximations to the required pivotal values of $\phi(t)$. Applying the difference-correction technique, the first approximations have been improved. After solving the integral equation (3.2) numerically, the stress intensity factor N and the crack opening displacement $\mu_0\Delta v(x,0)/\tau_0$ have been calculated numerically and plotted separately against the dimensional frequency k_2 ($0.5 \le k_2 \le 1$) and dimensionless distance x ($0 \le x \le 1$), respectively, for different values of the material inhomogeneity parameter b and strip width $2h$.

In Fig. 2, the effect of the width of the strip on the stress intensity factor for a homogeneous material has been shown; the effect of inhomogeneity of the material on the stress intensity factor for different widths of the strip has been depicted in Figs. 3–5.

It is found that in both the homogeneous and nonhomogeneous cases, the effect of the strip width decreases with the increase of the frequency, and the graphs of the stress

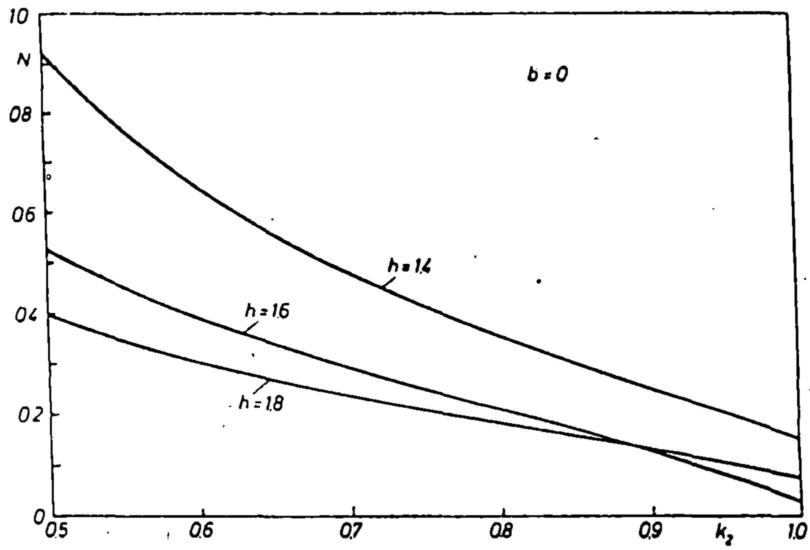

FIG. 2. Stress intensity factor N vs. dimensionless frequency k_2 for homogeneous medium ($b = 0$).

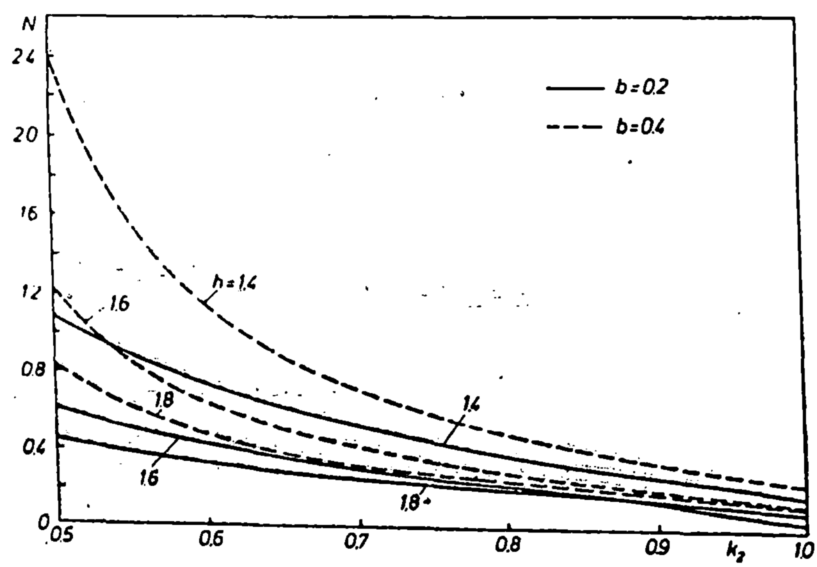

FIG. 3. Stress intensity factor N vs. dimensionless frequency k_2.

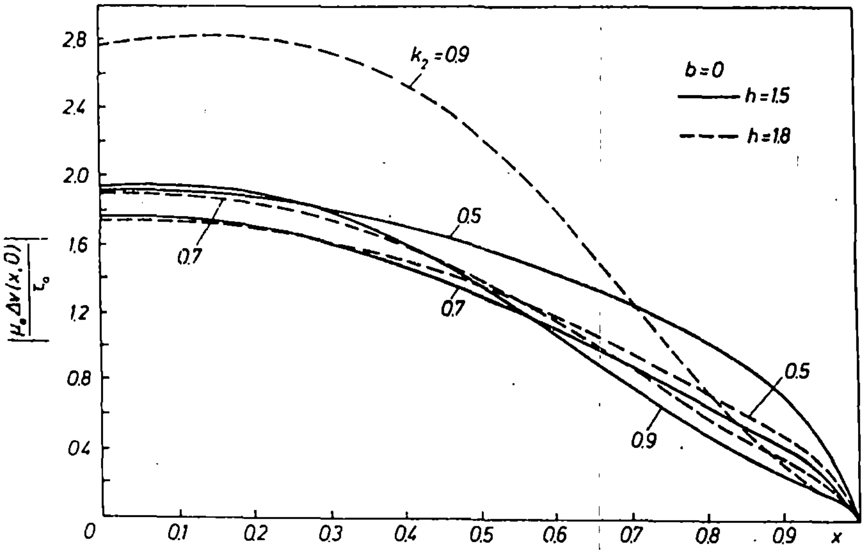

FIG. 4. Crack opening displacement vs. dimensionless distance $x(b = 0)$.

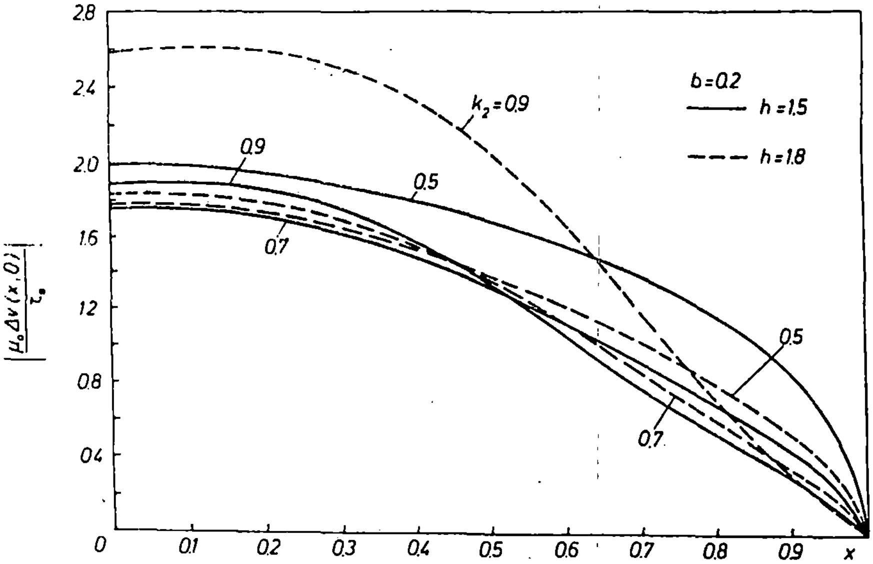

FIG. 5. Crack opening displacement vs. dimensionless distance $x(b = 0.2)$.

intensity factor N become flat with the increase of strip width $2h$. From Fig. 3 it is clear that the effect of inhomogeneity parameter b is prominent for low frequency k_2 and stress intensity factor is greater for higher values of the inhomogeneity parameter b.

In Figs. 4–8 the crack opening displacements against dimensionless distance x for different values of the material inhomogeneity parameter b and the strip width $2h$ have been illustrated by means of graphs. Case $b = 0$ corresponds to the homogeneous case (Fig. 4). From Figs. 4–6 it is seen that for a fixed value of inhomogeneity parameter b,

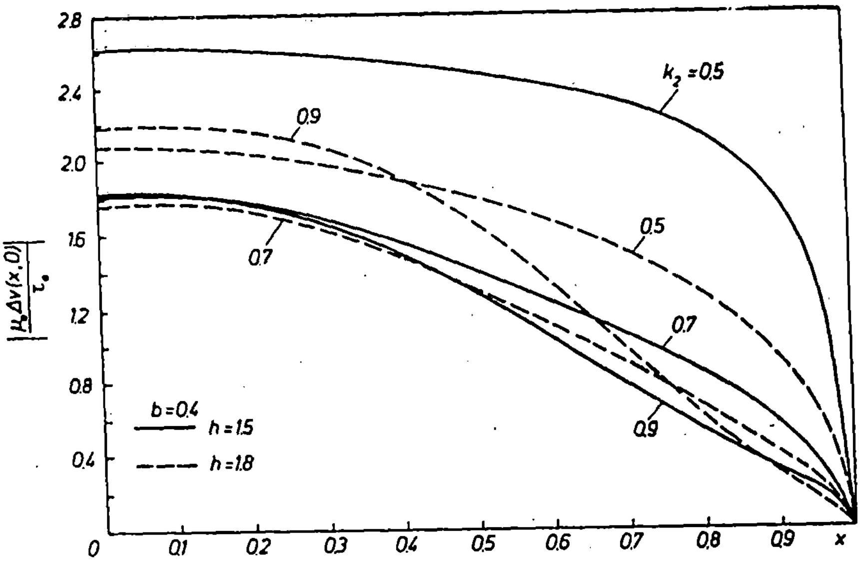

FIG. 6. Crack opening displacement vs. dimensionless distance $x (b = 0.4)$.

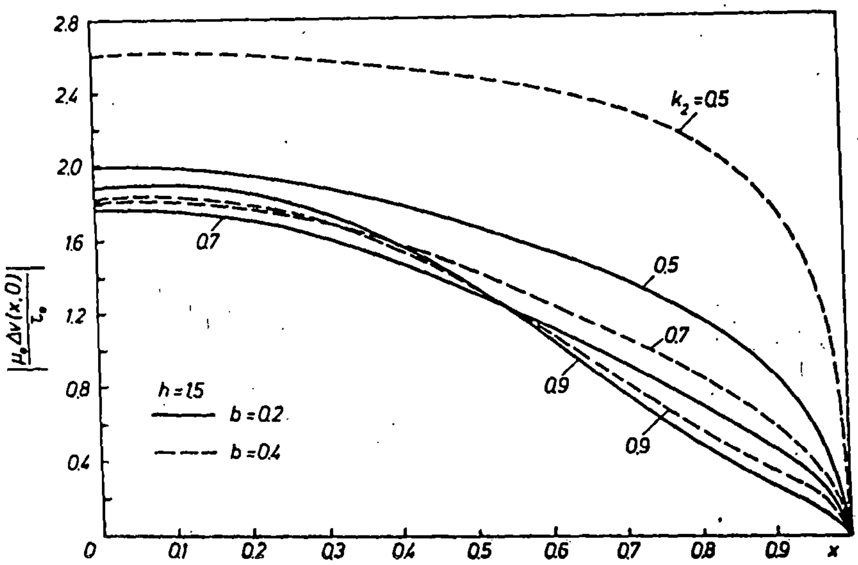

FIG. 7. Crack opening displacement vs. dimensionless distance $x (h = 1.5)$.

the crack opening displacement is greater for lower values of h when the frequencies are small, but the reverse effect is found for higher frequencies.

Next, in Figs. 7 and 8 we see that for a fixed value of h, the crack opening displacement is greater for higher values of the inhomogeneity parameter b when the frequencies are small, but for higher frequencies the effect is just reverse.

Finally it is found in all the cases that the crack opening displacement reaches its maximum at about $x = 0$, and then it gradually decreases and becomes zero at $x = 1$.

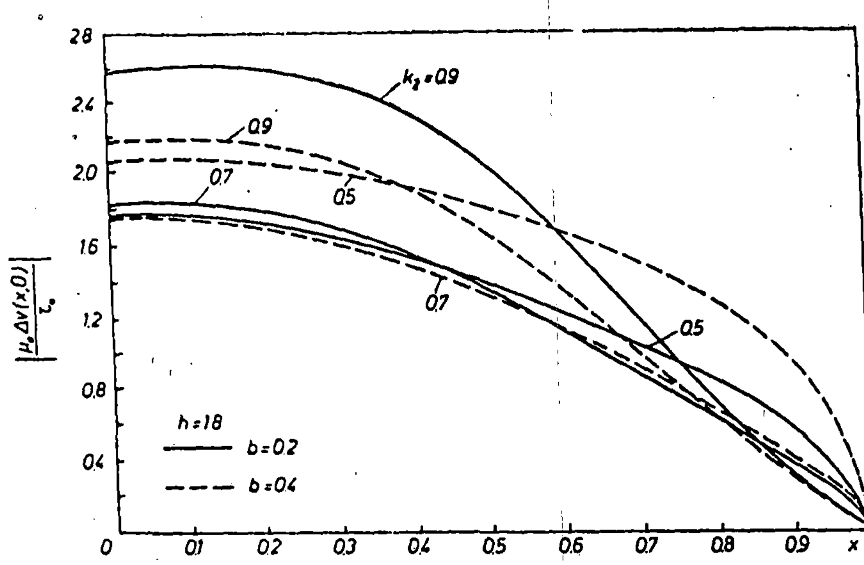

FIG. 8. Crack opening displacement vs. dimensionless distance x ($h = 1.8$).

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AN ELASTIC STRIP WITH THREE CO-PLANAR MOVING GRIFFITH CRACKS

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Abstract—The dynamic anti-plane problem of determining stress and displacement due to three co-planar Griffith cracks moving steadily at a subsonic speed in an infinite elastic strip has been considered. Employing Fourier integral transform, the problem when the rigidly clamped edges of the strip are pulled apart in opposite directions has been reduced to solving a set of four integral equations. These integral equations have been solved using the finite Hilbert transform technique and Cook's result [*Glas. Math. J.* 11, 9 (1970)] to obtain the exact form of crack opening displacement and stress intensity factors. Numerical results for stress intensity factors are presented in the form of graphs.

1. INTRODUCTION

IN FRACTURE MECHANICS, the problem of diffraction of elastic waves by cracks of finite dimension in a strip of elastic material has been examined by several investigators. Sih and Chen [1] investigated the problem of propagation of a crack of finite length in a strip under plane extension. Closed-form solutions for a finite length crack moving in a strip under anti-plane shear stress were obtained by Singh *et al.* [2]. Using a finite Hilbert transform technique developed by Srivastava and Lowengrub [3], Lowengrub and Srivastava [4] solved the static problem of distribution of stress and displacement in an infinitely long elastic strip containing two co-planar Griffith cracks. Recently, several dynamic problems of determining stress and displacement due to moving Griffith cracks have been solved by Das and Ghosh [5–8] and by Das [9, 10]. Dhawan and Dhaliwal [11] also solved the static problem of determining the stress distribution in an infinite transversely isotropic medium containing three co-planar Griffith cracks.

In this paper, the problem of propagation of three co-planar Griffith cracks in a fixed direction with constant velocity V in an infinitely long but finite width elastic strip is considered. Employing the Fourier integral transform, the problem when the lateral boundaries are assumed to be clamped and displaced by an equal amount has been reduced to solving a set of four integral equations which are solved using the finite Hilbert transform technique and Cook's result [12] to derive the exact form of stress intensity factors and crack opening displacement. Numerical results for stress intensity factors are presented graphically to show their variations with crack speed, crack length and the separating distance between the cracks.

2. STATEMENT OF THE PROBLEM

Consider an infinitely long elastic strip occupying the region $-h \leqslant y \leqslant h$, weakened by three co-planar Griffith cracks moving steadily at a constant velocity V in the X-direction, referred to a fixed coordinate system (X, Y, Z) as shown in Fig. 1.

In dynamic problems of anti-plane shear, the non-vanishing component of displacement W directed in the Z-direction satisfies the equation of motion:

$$W_{,XX} + W_{,YY} = \frac{1}{C_2^2} W_{,TT}, \tag{1}$$

where $C_2 = (\mu/\rho)^{1/2}$ is the shear wave velocity, ρ is the material density and $W_{,X}$ represents partial derivatives of W with respect to X.

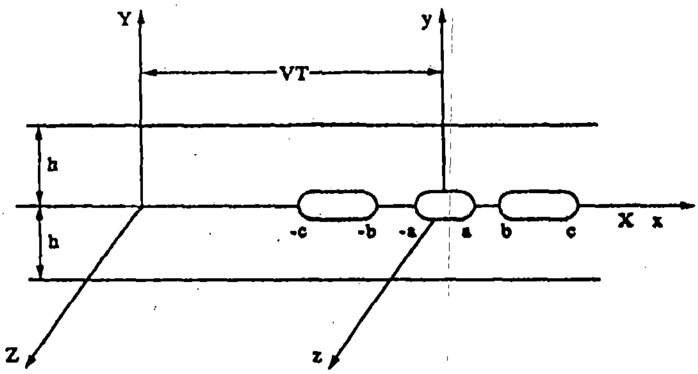

Fig. 1. Geometry and coordinate system.

For cracks moving at a constant velocity V in the X-direction, it is convenient to introduce the Galilean transformation:

$$x = X - VT, \quad y = Y, \quad z = Z, \quad t = T, \tag{2}$$

where (x, y, z) represents the translating coordinate system shown in Fig. 1.

Let three co-planar Griffith cracks of finite length located along the X-axis be moving steadily with velocity V in the direction of the X-axis so that their positions referred to translating coordinates (x, y, z) are $-c < x < -b$, $-a < x < a$ and $b < x < c$ on $y = 0$. The edges of the strip $y = \pm h$ are assumed to be clamped and displaced by an equal amount W_0, where W_0 is a constant. The boundary conditions of the proposed problem are

$$\sigma_{yz}(x, 0) = 0, \quad |x| < a, \quad b < |x| < c \tag{3}$$

$$W(x, \pm h) = \pm W_0, \quad -\infty < x < \infty \tag{4}$$

$$W(x, 0) = 0, \quad a < |x| < b, \quad |x| > c. \tag{5}$$

In order to apply the integral transform technique it is required to solve a different but equivalent problem which can be obtained from the clamped strip problem (without any cracks) while the uniform strain is applied. The equivalent stress conditions on the cracks are

$$\sigma_{yz}(x, 0) = \frac{\mu W_0}{h}, \quad |x| < a, \quad b < |x| < c \tag{6}$$

and the boundary conditions for the displacement are

$$W(x, \pm h) = 0, \quad -\infty < x < \infty \tag{7}$$

$$W(x, 0) = 0, \quad a < |x| < b, \quad |x| > c. \tag{8}$$

In the moving coordinate system, the equation of motion becomes independent of time and takes the form

$$s^2 W_{,xx} + W_{,yy} = 0, \tag{9}$$

with

$$s = \sqrt{(1 - V^2/C_2^2)}. \tag{10}$$

Introducing

$$\bar{W}_c(\xi, y) = \int_0^\infty W(x, y)\cos(\xi x)\, dx$$

$$W(x, y) = \frac{2}{\pi}\int_0^\infty \bar{W}_c(\xi, y)\cos(\xi x)\, d\xi \tag{11}$$

in eq. (3), the solution of eq. (3) is obtained as

$$W(x,y)=\frac{2}{\pi}\int_0^\infty [C_1(\xi)e^{-\xi ys}+C_3(\xi)e^{\xi ys}]\cos(\xi x)\,d\xi, \tag{12}$$

with

$$\sigma_{yz}(x,y)=-\frac{2\mu s}{\pi}\int_0^\infty \xi[C_1(\xi)e^{-\xi ys}-C_3(\xi)e^{\xi ys}]\cos(\xi x)\,d\xi. \tag{13}$$

Using the expression for $W(x,y)$ given in (6) in eq. (9), it has been found that

$$C_1(\xi)=\frac{C(\xi)}{1-e^{-2\xi hs}}$$

$$C_3(\xi)=-\frac{C(\xi)e^{-2\xi hs}}{1-e^{-2\xi hs}}, \tag{14}$$

where the unknown function $C(\xi)$ is to be determined.

From conditions (8) and (10) it is determined that $C(\xi)$ satisfies the following quadruple integral equations

$$\int_0^\infty \xi C(\xi)\coth(\xi hs)\cos(\xi x)\,d\xi=\frac{\pi W_0}{2hs},\quad x\in I_1,I_3 \tag{15a, b}$$

and

$$\int_0^\infty C(\xi)\cos(\xi x)\,d\xi=0,\quad x\in I_2,I_4, \tag{16}$$

where

$$I_1=(0,a),\quad I_2=(a,b),\quad I_3=(b,c),\quad I_4=(c,\infty).$$

3. METHOD OF SOLUTION

In order to solve the quadruple integral equations given by eqs (15) and (16), let us take

$$C(\xi)=\frac{1}{\xi}\int_0^a h(u)\sin(\xi u)\,du+\frac{1}{\xi}\int_b^c g(v^2)\operatorname{sech}^2(ev)\sin(\xi v)\,dv, \tag{17}$$

where $h(u)$ and $g(v^2)$ are the unknown functions to be determined from the boundary conditions of the proposed problem. Substituting the value of $C(\xi)$ given by (17) in (16) and using the following result:

$$\int_0^\infty \frac{\sin(\xi u)\cos(\xi x)}{\xi}\,d\xi=\begin{cases}\pi/2, & u>x>0\\ \pi/4, & u=x>0\\ 0, & x>u>0,\end{cases}$$

it is found that this choice of $C(\xi)$ leads to the condition

$$\int_b^c g(v^2)\operatorname{sech}^2(ev)\,dv=0. \tag{18}$$

Rewriting eq. (15a) as

$$\frac{d}{dx}\int_0^\infty C(\xi)\coth(\xi hs)\sin(\xi x)\,d\xi=\frac{\pi W_0}{2hs},\quad x\in I_1 \tag{19}$$

and inserting the value of $C(\xi)$ from eq. (17) in (19), it is found that $h(u)$ is the solution of the following singular integral equation:

$$\int_0^a h(u)\log\left|\frac{\tanh(ex)+\tanh(eu)}{\tanh(ex)-\tanh(eu)}\right|du=\pi f(x),\quad x\in I_1, \tag{20}$$

with

$$f(x)=\int_0^x\left[\frac{W_0}{hs}-\frac{2}{\pi}\int_b^c\frac{eg(v^2)\operatorname{sech}^2(ex')\operatorname{sech}^2(ev)\tanh(ev)}{\tanh^2(ev)-\tanh^2(ex')}\,dv\right]dx',$$

where the following result [13] has been used:

$$\int_0^\infty \coth(\xi hs)\frac{\sin(\xi x)\sin(\xi u)}{\xi}\,d\xi = \tfrac{1}{2}\log\left|\frac{\tan(ex)+\tanh(eu)}{\tanh(ex)-\tanh(eu)}\right|, \quad e=\frac{\pi}{2hs}. \tag{21}$$

Now using Cook's result [12], the solution of (20) has been obtained with the aid of the following result:

$$\int_0^u \frac{\sqrt{[\tanh^2(ea)-\tanh^2(ex)]}\,e\,\mathrm{sech}^2(ex)\,dx}{[\tanh^2(ex)-\tanh^2(eu)][\tanh^2(ev)-\tanh^2(ex)]}$$
$$= -\frac{\pi}{2\tanh(ev)}\frac{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}{\tanh^2(ev)-\tanh^2(eu)} \quad \text{for } u\in I_1 \text{ and } v\in I_3,$$

$$h(u)=\frac{-2e\tanh(eu)\mathrm{sech}^2(eu)}{\pi\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}}\left[\frac{W_0}{hs}\int_0^a\frac{\sqrt{[\tanh^2(ea)-\tanh^2(ex)]}}{\tanh^2(ex)-\tanh^2(eu)}dx\right.$$
$$\left.+\int_b^c\frac{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}{\tanh^2(ev)-\tanh^2(eu)}\times g(v^2)\mathrm{sech}^2(ev)\,dv\right]. \tag{22}$$

Substituting the resulting value of $C(\xi)$, obtained using eq. (22) in eq. (17), in condition (15b) and making use of the following results:

$$\int_0^a \frac{e\,\mathrm{sech}^2(eu)\tanh^2(eu)\,du}{[\tanh^2(eu)-\tanh^2(ex)][\tanh^2(ev)-\tanh^2(eu)]\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}}$$
$$=\frac{\pi}{2[\tanh^2(ev)-\tanh^2(ex)]}\left[\frac{\tanh(ev)}{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}-\frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}\right],$$

$$\int_0^a \frac{e\,\mathrm{sech}^2(eu)\tanh^2(eu)\,du}{[\tanh^2(eu)-\tanh^2(ex)][\tanh^2(ey')-\tanh^2(eu)]\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}}$$
$$=\frac{\pi}{2[\tanh^2(ex)-\tanh^2(ey')]}\frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}, \quad \text{for } x,v\in I_3 \text{ and } y'\in I_1,$$

it can be shown that $g(v^2)$ is the solution of the following singular integral equation:

$$\int_b^c\frac{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}{\tanh^2(ev)-\tanh^2(ex)}eg(v^2)\mathrm{sech}^2(ev)\,dv=\frac{\pi W_0}{2hs}\left[\frac{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}{\mathrm{sech}^2(ex)\tanh(ex)}\right.$$
$$\left.+\frac{e}{\pi}\int_0^a\frac{\sqrt{[\tanh^2(ea)-\tanh^2(ey')]}}{\tanh^2(ex)-\tanh^2(ey')}dy'\right], \quad \text{for } x\in I_3. \tag{23}$$

Using the finite Hilbert transform technique [3], and the following result:

$$\int_b^c\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ex)}{\tanh^2(ex)-\tanh^2(eb)}\right]}\frac{2\,\mathrm{sech}^2(ex)\tanh(ex)\,dx}{[\tanh^2(ex)-\tanh^2(ey')][\tanh^2(ex)-\tanh^2(ev)]}$$
$$=-\frac{\pi}{e[\tanh^2(ev)-\tanh^2(ey')]}\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ey')}{\tanh^2(eb)-\tanh^2(ey')}\right]},$$

the solution of eq. (23) is found as

$$g(v^2)=-\frac{2eW_0}{h\pi s}\frac{\tanh^2(ev)\sqrt{[\tanh^2(ev)-\tanh^2(eb)]}}{\sqrt{\{[\tanh^2(ev)-\tanh^2(ea)][\tanh^2(ec)-\tanh^2(ev)]\}}}$$
$$\times\left[\int_b^c\sqrt{\left[\frac{\tan^2(ec)-\tanh^2(ex)}{\tan^2(ex)-\tanh^2(eb)}\right]}\frac{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}{\tanh^2(ex)-\tanh^2(ev)}dx\right.$$
$$\left.-\int_0^a\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ey')}{\tanh^2(eb)-\tanh^2(ey')}\right]}\frac{\sqrt{[\tan^2(ea)-\tanh^2(ey')]}}{\tanh^2(ev)-\tanh^2(ey')}dy'\right]$$
$$+\frac{C_1\tanh(ev)}{\sqrt{\{[\tanh^2(ev)-\tanh^2(ea)][\tanh^2(ev)-\tanh^2(eb)][\tanh^2(ec)-\tanh^2(ev)]\}}}. \tag{24}$$

Next substituting the value of $g(v^2)$ from eq. (24) in eq. (22) and finally using the following result:

$$\int_b^c \sqrt{\left[\frac{\tanh^2(ev)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ev)}\right]}\frac{2\,\mathrm{sech}^2(ev)\tanh(ev)\,dv}{[\tanh^2(ev)-\tanh^2(eu)][\tanh^2(ex')-\tanh^2(ev)]}$$

$$=\frac{\pi}{e[\tan^2(eu)-\tanh^2(ex')]}\left[\sqrt{\left[\frac{\tanh^2(eb)-\tanh^2(eu)}{\tanh^2(ec)-\tanh^2(eu)}\right]}-\sqrt{\left[\frac{\tanh^2(eb)-\tanh^2(ex')}{\tanh^2(ec)-\tanh^2(ex')}\right]}\right],$$

for $u, x' \in I_1$,

$h(u)$ is derived in the form:

$$h(u)=-\frac{2eW_0}{\mu\pi s}\frac{\mathrm{sech}^2(eu)\tanh(eu)\sqrt{[\tanh^2(eb)-\tanh^2(eu)]}}{\sqrt{\{[\tanh^2(ea)-\tanh^2(eu)][\tanh^2(ec)-\tanh^2(eu)]\}}}$$

$$\times\left[\int_0^a\sqrt{\left[\frac{\tanh^2(ea)-\tanh^2(ey')}{\tanh^2(eb)-\tanh^2(ey')}\right]}\frac{\sqrt{[\tanh^2(ec)-\tanh^2(ey')]}}{\tanh^2(ey')-\tanh^2(eu)}\,dy'\right.$$

$$\left.+\int_b^c\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ex)}{\tanh^2(ex)-\tanh^2(eb)}\right]}\frac{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}{\tanh^2(ex)-\tanh^2(eu)}\,dx\right]$$

$$-\frac{C_1\tan(eu)\mathrm{sech}^2(eu)}{\sqrt{\{[\tanh^2(ea)-\tanh^2(eu)][\tanh^2(eb)-\tanh^2(eu)][\tanh^2(ec)-\tanh^2(eu)]\}}}. \tag{25}$$

Substitution of the value of $g(v^2)$ from eq. (24) in the condition (18) yields

$$C_1=-\frac{2eW_0}{\pi hs}\left[\int_b^c\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ex)}{\tanh^2(ex)-\tanh^2(eb)}\right]}\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}\right.$$

$$\times\left\{\frac{\tanh^2(ex)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ex)}\times\Pi\left\{\frac{\pi}{2},\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ex)},q\right\}\Big/F\left(\frac{\pi}{2},q\right)+1\right\}dx$$

$$+\int_0^a\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(es)}{\tanh^2(eb)-\tanh^2(es)}\right]}\sqrt{[\tanh^2(ea)-\tanh^2(es)]}$$

$$\left.\times\left\{1-\frac{\tanh^2(eb)-\tanh^2(es)}{\tanh^2(ec)-\tanh^2(es)}\Pi\left\{\frac{\pi}{2},\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(es)},q\right\}\Big/F\left(\frac{\pi}{2},q\right)\right\}ds,\right. \tag{26}$$

where $F(\phi, q)$ and $\Pi(\phi, n, q)$ are elliptic integrals of the first and third kinds respectively and

$$q=\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ea)}\right]}.$$

The relevant displacement and stress components in the plane of the crack can now be shown to be given by

$$W(x,0)=\int_x^a h(u)\,du, \qquad 0\leqslant x\leqslant a$$

$$=\int_x^c g(v^2)\cosh(ev)\,dv, \qquad b\leqslant x\leqslant c \tag{27}$$

and

$$[\sigma_{yz}(x,0)]_{a<x<b}=\frac{2\mu s}{\pi}\left[\int_0^a\frac{eh(u)\tanh(eu)\,du}{\tanh^2(ex)-\tanh^2(eu)}-\int_b^c\frac{eg(v^2)\tanh(ev)\mathrm{sech}^2(ev)}{\tanh^2(ev)-\tanh^2(ex)}\,dv\right]\mathrm{sech}^2(ex)$$

$$[\sigma_{yz}(x,0)]_{x>c}=\frac{2\mu s}{\pi}\left[\int_0^a\frac{eh(u)\tanh(eu)\,du}{\tanh^2(ex)-\tanh^2(eu)}+\int_b^c\frac{eg(v^2)\tanh(ev)\mathrm{sech}^2(ev)}{\tanh^2(ex)-\tanh^2(ev)}\,dv\right]\mathrm{sech}^2(ex). \tag{28}$$

Now insertion of the values of $h(u)$ and $g(v^2)$ as given by eqs (25) and (24) in the expressions (28) yields, after some algebraic manipulations,

$$
\begin{aligned}
[\sigma_{yz}(x,0)]_{a<x<b} = \frac{2\mu e W_0}{\pi h s}\Bigg[& -\sqrt{\left[\frac{\tanh^2(eb)-\tanh^2(ea)}{\tanh^2(ec)-\tanh^2(ea)}\right]}\frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}} \\
& \times\left\{\int_0^a F_2(u,x)\,\mathrm{d}u+\int_b^c F_2(v,x)\,\mathrm{d}v\right\}-\frac{2e[\tanh^2(ec)-\tanh^2(eb)]}{\pi} \\
& \times\left\{\int_0^a F_2(u',x)\,\mathrm{d}u'\int_0^a F_4(c,u)\times F_3(0,x,u)\,\mathrm{d}u\right. \\
& \left.+\int_b^c F_2(v,x)\,\mathrm{d}v\int_0^a F_4(c,u)F_3(v,x,u)\,\mathrm{d}u\right\} \\
& +\frac{\mu s h}{e W_0}C_1\left\{\frac{\pi}{2}\frac{1-\tanh(ex)/\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}{\sqrt{\{[\tanh^2(eb)-\tanh^2(ea)][\tanh^2(ec)-\tanh^2(ea)]\}}}\right. \\
& \left.+e\int_0^a F_4(c,u)F_5(u,x)\,\mathrm{d}u\right\}+\frac{e[\tanh^2(eb)-\tanh^2(ea)]}{\pi} \\
& \times\left\{\int_b^c F_2(v',x)\,\mathrm{d}v'\int_b^c F_4(a,v)F_6(v',x,v)\,\mathrm{d}v+\int_0^a F_2(u,x)\,\mathrm{d}u\right. \\
& \times\int_b^c F_4(a,v)F_6(u,x,v)\,\mathrm{d}v-\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(eb)-\tanh^2(ea)} \\
& \left.\times\int_0^a F_1(u,x)\,\mathrm{d}u\int_0^a F_4(c,u')F_9(u,u')\,\mathrm{d}u'\right\}-\frac{\mu s h}{e W_0}\frac{C_1}{X_1} \\
& \times\left\{\frac{\pi}{2}\frac{\tanh(ec)}{\sqrt{[\tanh^2(ec)-\tanh^2(ea)]}}+e\tanh^2(ea)\int_b^c F_7(x,v)\,\mathrm{d}v\right\}\Bigg]\mathrm{sech}^2(ex)
\end{aligned}
$$

and

$$
\begin{aligned}
[\sigma_{yz}(x,0)]_{x>c} = \frac{2W_0 e\mu}{\pi h s}\Bigg[& -\sqrt{\left[\frac{\tanh^2(eb)-\tanh^2(ea)}{\tanh^2(ec)-\tanh^2(ea)}\right]}\frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}} \\
& \times\left\{\int_0^a F_2(u,x)\,\mathrm{d}u+\int_b^c F_2(v,x)\,\mathrm{d}v\right\}-\frac{2e[\tanh^2(ec)-\tanh^2(eb)]}{\pi} \\
& \times\left\{\int_0^a F_2(u',x)\,\mathrm{d}u'\int_0^a F_4(c,u)F_3(0,x,u)\,\mathrm{d}u\right. \\
& \left.+\int_b^c F_2(v,x)\,\mathrm{d}v\int_0^a F_4(c,u)F_3(v,x,u)\,\mathrm{d}u\right\}+\frac{\mu s h}{e W_0}C_1 \\
& \times\left\{\frac{\pi}{2}\frac{1-\tanh(ex)/\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}{\sqrt{\{[\tanh^2(ec)-\tanh^2(ea)][\tanh^2(eb)-\tanh^2(ea)]\}}}+e\right. \\
& \left.\times\int_0^a F_4(c,u)F_5(u,x)\,\mathrm{d}u\right\}-\frac{e[\tanh^2(eb)-\tanh^2(ea)]}{\pi} \\
& \times\left\{\int_b^c F_2(v',x)\,\mathrm{d}v'\int_b^c F_4(a,v)F_8(v',v,x)\,\mathrm{d}v+\int_0^a F_2(u,x)\,\mathrm{d}u\right. \\
& \times\int_b^c F_4(a,v)F_8(u,v,x)\,\mathrm{d}v+\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(eb)-\tanh^2(ea)} \\
& \left.\times\int_0^a F_1(u,x)\,\mathrm{d}u\int_0^a F_4(c,u')F_9(u,u')\,\mathrm{d}u'\right\}
\end{aligned}
$$

$$+\frac{\mu sh}{eW_0}\frac{C_1}{X_1}\left\{\frac{\pi}{2}\frac{\tanh(ec)}{\sqrt{[\tanh^2(ec)-\tanh^2(ea)]}}+e\tanh^2(ea)\int_b^c F_7(x,v)\,\mathrm{d}v\right\}$$

$$-\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ea)}\right]}\frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ec)]}}$$

$$\times\left\{\int_0^a F_2(u,x)\,\mathrm{d}u+\int_b^c F_2(v,x)\,\mathrm{d}v\right\}\Bigg]\mathrm{sech}^2(ex), \tag{29}$$

where

$$F_1(u,x)=\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eu)}{\tanh^2(eb)-\tanh^2(eu)}\right]}\frac{\tanh(eu)}{\tanh^2(ex)-\tanh^2(eu)}$$

$$F_2(v,x)=\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ev)}{\tanh^2(ev)-\tanh^2(eb)}\right]}\frac{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}{\tanh^2(ev)-\tanh^2(ex)}$$

$$F_3(v,x,u)=\frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}\tan^{-1}\left\{\frac{\tanh(eu)}{\tanh(ex)}\sqrt{\left[\frac{\tanh^2(ex)-\tanh^2(ea)}{\tanh^2(ea)-\tanh^2(eu)}\right]}\right\}$$

$$-\frac{\tanh(ev)}{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}\tan^{-1}\left\{\frac{\tanh(eu)}{\tanh(ev)}\sqrt{\left[\frac{\tanh^2(ev)-\tanh^2(ea)}{\tanh^2(ea)-\tanh^2(eu)}\right]}\right\}$$

$$F_4(\omega,u)=\frac{\mathrm{sech}^2(eu)\tanh(eu)}{\sqrt{\{[\tanh^2(e\omega)-\tanh^2(eu)]^3[\tanh^2(eb)-\tanh^2(eu)]\}}}$$

$$F_5(u,x)=[2\tanh^2(eu)-\tanh^2(ec)-\tanh^2(eb)]\left\{\sin^{-1}\left(\frac{\tanh(eu)}{\tanh(ea)}\right)-F_3(0,x,u)\right\}$$

$$F_6(u,x,v)=\frac{\tanh(ex)}{\sqrt{[\tanh^2(ec)-\tanh^2(ex)]}}$$

$$\times\log\left|\frac{\tanh(ex)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}+\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(ex)]}}{\tanh(ex)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}-\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(ex)]}}\right|$$

$$-\frac{\tanh(eu)}{\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}}$$

$$\times\log\left|\frac{\tanh(eu)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}+\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}}{\tanh(eu)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}-\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}}\right|$$

$$F_7(x,v)=\tan^{-1}\left(\frac{\sqrt{[\tanh^2(ec)-\tanh^2(ex)]}}{\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}}\frac{\sqrt{[\tanh^2(ev)-\tanh^2(eb)]}}{\sqrt{[\tanh^2(eb)-\tanh^2(ex)]}}\right)$$

$$\times\frac{\mathrm{sech}^2(ev)}{\sqrt{\{[\tanh^2(ev)-\tanh^2(ea)]^3\}}}$$

$$F_8(u,v,x)=-\frac{2\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ec)]}}\tan^{-1}\left\{\frac{\tanh(ev)}{\tanh(ex)}\sqrt{\left[\frac{\tanh^2(ex)-\tanh^2(ec)}{\tanh^2(ec)-\tanh^2(ev)}\right]}\right\}$$

$$+\frac{\tanh(eu)}{\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}}$$

$$\times\log\left|\frac{\tanh(eu)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}+\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}}{\tanh(eu)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}-\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}}\right|$$

$$F_9(u,u')=\log\left|\frac{\tanh(eu)\sqrt{[\tanh^2(ea)-\tanh^2(eu')]}+\tanh(eu')\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}}{\tanh(eu)\sqrt{[\tanh^2(ea)-\tanh^2(eu')]}-\tanh(eu')\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}}\right|$$

and

$$X_1=\sqrt{\{[\tanh^2(eb)-\tanh^2(ex)][\tanh^2(ec)-\tanh^2(ex)]\}}. \tag{30}$$

The dynamic stress intensity factors are defined by

$$N_a = \lim_{x\to a^+} \sqrt{[2(x-a)]}[\sigma_{yz}(x,0)]_{a<x<b}$$

$$N_b = \lim_{x\to b^-} \sqrt{[2(b-x)]}[\sigma_{yz}(x,0)]_{a<x<b}$$

$$N_c = \lim_{x\to c^+} \sqrt{[2(x-c)]}[\sigma_{yz}(x,0)]_{x>c}. \tag{31}$$

Substitution of the results given by eqs (29) in expressions (31) yields

$$N_a = \sqrt{\left[\frac{\tanh(ea)}{e}\right]}\left[-\sqrt{\left[\frac{\tanh^2(eb)-\tanh^2(ea)}{\tanh^2(ec)-\tanh^2(ea)}\right]}\frac{2W_0 e}{\pi h}\left\{\int_0^a F_2(u,a)\,du + \int_b^c F_2(v,a)\,dv\right\}\right.$$
$$\left. - \frac{\mu s C_1}{\sqrt{\{[\tanh^2(eb)-\tanh^2(ea)][\tanh^2(ec)-\tanh^2(ea)]\}}}\right]\operatorname{sech}(ea)$$

$$N_b = -\frac{\mu s C_1}{\sqrt{\{[\tanh^2(eb)-\tanh^2(ea)][\tanh^2(ec)-\tanh^2(eb)]\}}}\sqrt{\left[\frac{\tanh(eb)}{e}\right]}\operatorname{sech}(eb)$$

$$N_c = \sqrt{\left[\frac{\tanh(ec)}{e}\right]}\left[-\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ea)}\right]}\frac{2W_0 e}{\pi h}\left\{\int_0^a F_2(u,c)\,du + \int_b^c F_2(v,c)\,dv\right\}\right.$$
$$\left. + \frac{\mu s C_1}{\sqrt{\{[\tanh^2(ec)-\tanh^2(ea)][\tanh^2(ec)-\tanh^2(eb)]\}}}\right]\sec(ec). \tag{32a–c}$$

Again insertion of the values of $h(u)$ and $g(v^2)$, given by eqs (24) and (25), in the expressions for displacements given by eqs (27) yields

$$[W(x,0)]_{0<x<a} = -\frac{W_0}{h\mu\pi s}\left[\frac{2[\tanh^2(eb)-\tanh^2(ea)]}{\sqrt{[\tanh^2(ec)-\tanh^2(ea)]}}\left\{\int_b^c \Pi\left\{\lambda, \frac{\tanh^2(ev)-\tanh^2(eb)}{\tanh^2(ev)-\tanh^2(ea)}, q\right\}\right.\right.$$
$$\times\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ev)}{\tanh(ev)-\tanh^2(eb)}\right]}\frac{dv}{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}$$
$$-\int_0^a \Pi\left\{\lambda, \frac{\tanh^2(eb)-\tanh^2(eu)}{\tanh^2(ea)-\tanh^2(eu)}, q\right\}\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eu)}{\tanh^2(eb)-\tanh^2(eu)}\right]}$$
$$\left.\left.\times\frac{du}{\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}}\right\}\right] - \frac{C_1 F(\lambda,q)}{e\sqrt{[\tanh^2(ec)-\tanh^2(ea)]}}$$

and

$$[W(x,0)]_{b<x<c} = \left[\frac{2W_0}{h\mu\pi s}\left(\int_b^c \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ev)}{\tanh^2(ev)-\tanh^2(eb)}\right]}\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}\right.\right.$$
$$\times\left\{F(\lambda',q) + \frac{\tanh^2(ev)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ev)}\Pi\left\{\lambda', \frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ev)}, q\right\}\right\}dv$$
$$+\int_0^a \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eu)}{\tanh^2(eb)-\tanh^2(eu)}\right]}\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}$$
$$\left.\times\left\{F(\lambda,q) - \frac{\tanh^2(eb)-\tanh^2(eu)}{\tanh^2(ec)-\tanh^2(eu)}\Pi\left\{\lambda', \frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(eu)}, q\right\}\right\}du\right)$$
$$\left. + \frac{C_1}{e}F(\lambda',q)\right]\frac{1}{\sqrt{[\tanh^2(ec)-\tanh^2(ea)]}}, \tag{33a, b}$$

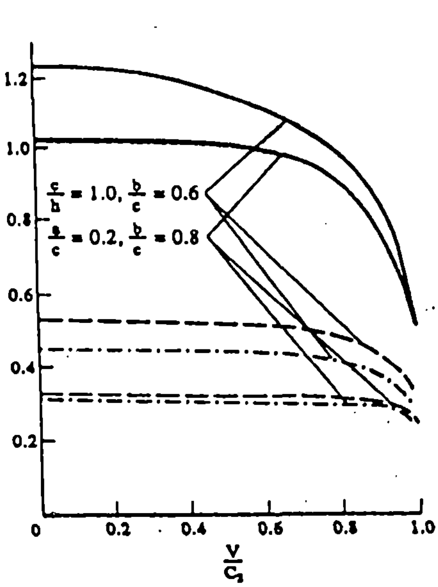

Fig. 2. Variations of stress intensity factors with V/C_2: (——) $hN_a/\mu W_0\sqrt{a}$, (- - -) $hN_b/\mu W_0\sqrt{b}$; (-·-·-) $hN_c/\mu W_0\sqrt{c}$.

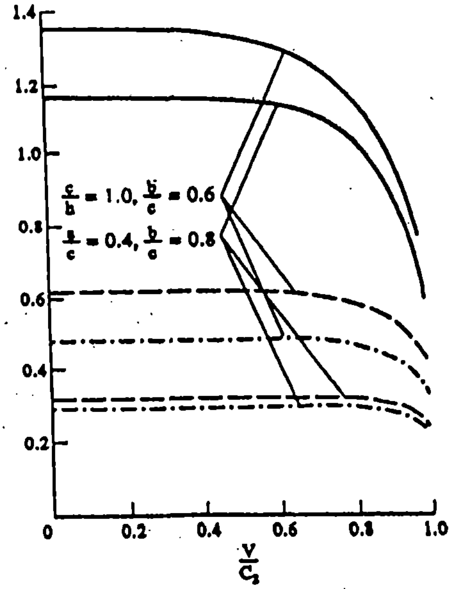

Fig. 3. Variations of stress intensity factors with V/C_2: (——) $hN_a/\mu W_0\sqrt{a}$; (- - -) $hN_b/\mu W_0\sqrt{b}$; (-·-·-) $hN_c/\mu W_0\sqrt{c}$.

where

$$\sin \lambda = \sqrt{\left[\frac{\tanh^2(ea) - \tanh^2(ex)}{\tanh^2(eb) - \tanh^2(ex)}\right]}, \quad \sin \lambda' = \sqrt{\left[\frac{\tanh^2(ec) - \tanh^2(ex)}{\tanh^2(ec) - \tanh^2(eb)}\right]}$$

$$\text{and } F(\phi, q), \Pi(\phi, n, q)$$

and q have been defined earlier.

On putting $b = c$ and simplifying, it may be noted that the results (33a) and (32a) become those given by eqs (3.18) and (3.21) of Singh *et al.* [2] and for $a = 0$ the results given by (32b), (32c) and (33b) coincide with those given by eqs (4.21), (4.22) and (4.17) of Das and Gosh [5].

4. NUMERICAL RESULTS AND DISCUSSION

Numerical results for stress intensity factors at the tips of the cracks for different values of crack speed, crack length and the separating distance between the cracks are presented in this

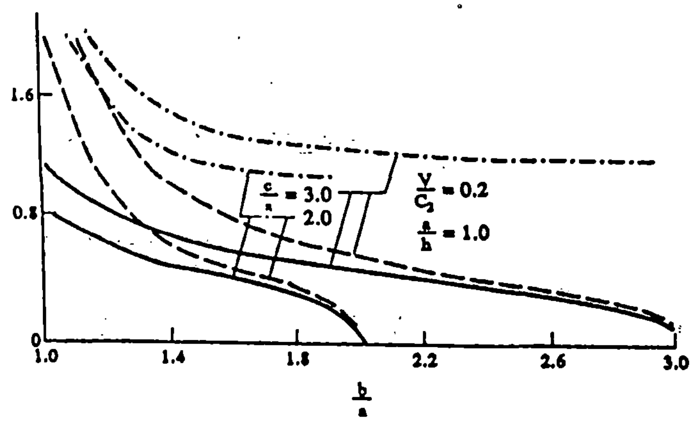

Fig. 4. Stress intensity factors vs b/a: (-·-·-) $hN_a/\mu W_0\sqrt{a}$; (- - -) $hN_b/\mu W_0\sqrt{b}$; (——) $hN_c/\mu W_0\sqrt{c}$.

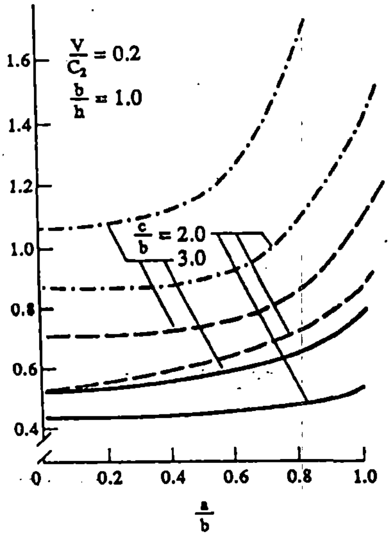

Fig. 5. Stress intensity factors vs a/b: (–·–·–) $hN_a/\mu W_0\sqrt{a}$; (– – –) $hN_b/\mu W_0\sqrt{b}$; (——) $hN_c/\mu W_0\sqrt{c}$.

section. The crack length dependence of the stress intensity factors and its variations with V/C_2 are shown in Figs 2–5. It is shown in Figs 2 and 3 that stress intensity factors at the edges of the cracks decrease with an increase in the values of V/C_2 and have a prominent variation when $V/C_2 \to 1$. Variations of stress intensity factors at the edge $x = a$ become more prominent than those at the tips $x = b$ and $x = c$ when the length of the inner crack increases.

Variations of stress intensity factors at the edges of the cracks with a/b for different values of c/b and those with b/a for different values of c/a are plotted in Figs 4 and 5, respectively. It is found that when the separating distance between the inner crack and outer pair of cracks decreases the stress intensity factors at the tips $x = a$ and $x = b$ become more prominent than that at the edge $x = c$.

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INTERACTION OF ELASTIC WAVES WITH TWO COPLANAR GRIFFITH CRACKS IN AN ORTHOTROPIC MEDIUM

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Abstract—The problem of diffraction of normally incident elastic waves by two coplanar Griffith cracks situated in an infinite orthotropic medium has been analyzed. Fourier and Hilbert transforms have been used to solve this mixed boundary value problem. Approximate analytical results for stress intensity factors and crack opening displacement have been derived when the wave lengths are large compared to the crack length. Numerical values of stress intensity factors and the crack opening displacement for several orthotropic materials have been calculated and plotted graphically to show the effect of material orthotropy.

INTRODUCTION

DYNAMIC fracture problems involving anisotropic materials weakened by crack-like imperfections have drawn much attention by investigators because of the increased usage of macroscopically anisotropic construction materials such as fibre reinforced composites. The different possible location of cracks with respect to the planes of material symmetry introduce great modifications in the strain and stress distribution. The problems are also of considerable interest in seismology and exploration geophysics. The problems involving single or two Griffith cracks in isotropic elastic medium have been studied by many authors [1–6]. Mathematical difficulties encountered in solving the governing equations of the anisotropic elasticity theory are responsible for the availability of few results only for special classes of materials. Kassir and Bandyopadhyay [7] have studied the elastodynamic response of an infinite orthotropic solid containing a crack under the action of impact loading and the elastodynamic problem of a finite Griffith crack in an orthotropic strip under normal impact was investigated by Shindo [8]. Problem involving a moving Griffith crack in an orthotropic strip has also been studied by De and Patra [9]. Recently, Kundu and Bostrom [10] solved the problem of scattering of elastic waves by a circular crack situated in a transversely isotropic solid.

In our paper, the diffraction of normally incident time harmonic elastic waves by two coplanar Griffith cracks in an infinite orthotropic medium has been investigated. The faces of each of the cracks are assumed to be separated by a small distance so that, during small deformations of the solid, the crack faces do not come into contact. The resulting mixed boundary value problem is reduced to the solution of a triple integral equation which has further been reduced to the solution of an integro-differential equation. Iterative solution valid for low frequency has been obtained. Analytical formulae for stress intensity factor and crack opening displacement have been derived. Making the distance between two crack zero the corresponding results for single crack have been presented. Finally, choosing the engineering elastic constants of the orthotropic material suitably the results for isotropic material have been deduced and compared with the results obtained by Jain and Kanwal [5]. To display the influence of the material orthotropy numerical values of stress intensity factors and crack opening displacement have been plotted for several orthotropic materials.

STATEMENT AND FORMULATION OF THE PROBLEM

Consider the plane problem of diffraction of normally incident longitudinal wave by two symmetrical co-planar Griffith cracks situated in an infinite orthotropic elastic medium. The cracks are assumed to occupy the region $b \leq |X| \leq a$, $Y = 0$, $|Z| < \infty$. It is convenient to normalize all lengths with respect to "a" and so setting $X/a = x$, $Y/a = y$, $Z/a = z$, $b/a = c$, the new position of the cracks are defined by $c \leq |x| \leq 1$, $y = 0$, $|z| < \infty$ (Fig. 1).

Let a plane time harmonic elastic wave originating at $y = -\infty$ be incident normally on the two cracks is defined by $v_0 = \exp[i(ky - \omega t)]$ where $k = a\omega/c_s\sqrt{c_{22}}$, $c_s = (\mu_{12}/\rho)^{1/2}$ with ρ being the density of the material. In the isotropic solid, c_s, represents the velocity of the shear wave.

The non-zero stress components τ_{yy} and τ_{xy} are given by

$$\begin{aligned} \tau_{yy}/\mu_{12} &= c_{12}u_{,x} + c_{22}v_{,y} \\ \tau_{xy}/\mu_{12} &= u_{,y} + v_{,x}, \end{aligned} \tag{1}$$

where u, v denote the component of the displacement in the x, y directions, respectively and comma denotes partial differentiation with respect to the co-ordinates or time; $c_{ij}(i, j = 1, 2)$ are nondimensional parameters related to the elastic constants by the relations

$$\begin{aligned} c_{11} &= E_1/\mu_{12}(1 - v_{12}^2 E_2/E_1) \\ c_{22} &= E_2/\mu_{12}(1 - v_{12}^2 E_2/E_1) = c_{11} E_2/E_1 \\ c_{12} &= v_{12}E_2/\mu_{12}(1 - v_{12}^2 E_2/E_1) = v_{12}c_{22} = v_{21}c_{11} \end{aligned} \tag{2}$$

for generalized plane stress, and by

$$\begin{aligned} c_{11} &= (E_1/\Delta\mu_{12})(1 - v_{23}v_{32}) \\ c_{22} &= (E_2/\Delta\mu_{12})(1 - v_{13}v_{31}) \\ c_{12} &= E_1(v_{21} + v_{13}v_{32}E_2/E_1)/\Delta\mu_{12} \\ &= E_2(v_{12} + v_{23}v_{31}E_1/E_2)/\Delta\mu_{12} \\ \Delta &= 1 - v_{12}v_{21} - v_{23}v_{32} - v_{31}v_{13} - v_{12}v_{23}v_{31} - v_{13}v_{21}v_{32} \end{aligned} \tag{3}$$

for plane strain. In the above equations E_i, μ_{ij} and v_{ij} $(i, j = 1, 2, 3)$ denote the engineering elastic constants of the material where the subscripts 1, 2, 3 correspond to the x, y, z directions which coincide with the axes of material orthotropy and the constants E_i and v_{ij} satisfy the Maxwell's relation

$$v_{ij}/E_i = v_{ji}/E_j. \tag{4}$$

The equations of motion for orthotropic material, in terms of displacements are

$$\begin{aligned} c_{11}u_{,xx} + u_{,yy} + (1 + c_{12})v_{,xy} &= \frac{a^2}{c_s^2}u_{,tt} \\ c_{22}v_{,yy} + v_{,xx} + (1 + c_{12})u_{,xy} &= \frac{a^2}{c_s^2}v_{,tt}. \end{aligned} \tag{5}$$

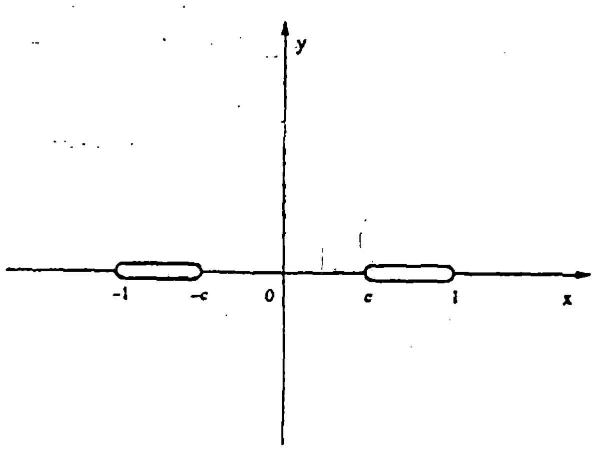

Fig. 1. Geometry of the cracks.

Therefore, substituting $u(x, y, t) = u(x, y)\exp(-i\omega t)$ and $v(x, y, t) = v(x, y)\exp(-i\omega t)$ in eq. (5) we obtain

$$c_{11}u_{,xx} + u_{,yy} + (1 + c_{12})v_{,xy} + k_s^2 u = 0 \tag{6}$$

and

$$c_{22}v_{,yy} + v_{,xx} + (1 + c_{12})u_{,xy} + k_s^2 v = 0$$

where $k_s^2 = a^2\omega^2/c_s^2$.

The boundary conditions of the problem are

$$\tau_{xy}(x, 0) = 0, \quad |x| < \infty \tag{7}$$

$$\tau_{yy}(x, 0) + \tau_{yy}^{(0)}(x, 0) = 0, \quad c \leqslant |x| \leqslant 1 \tag{8}$$

$$v(x, 0) = 0, \quad |x| < c, \quad |x| > 1. \tag{9}$$

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables would be omitted in the sequel.

The solutions of eqs (6) can be taken as

$$u(x, y) = \frac{2}{\pi}\int_0^\infty [A_1(\xi)\exp(-\gamma_1|y|) + A_2(\xi)\exp(-\gamma_2|y|)]\sin \xi x \, d\xi \tag{10}$$

$$v(x, y) = \pm\frac{2}{\pi}\int_0^\infty \frac{1}{\xi}[\alpha_1 A_1(\xi)\exp(-\gamma_1|y|) + \alpha_2 A_2(\xi)\exp(-\gamma_2|y|)]\cos \xi x \, d\xi, \quad y \gtrless 0 \tag{11}$$

where

$$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1 + c_{12})\gamma_i}, \quad i = 1, 2 \tag{12}$$

and $A_i(\xi)(i = 1, 2)$ are the unknown functions to be determined, γ_1^2, γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1 + c_{22})k_s^2\}\gamma^2 + (c_{11}\xi^2 - k_s^2)(\xi^2 - k_s^2) = 0. \tag{13}$$

From the boundary condition (7) it is found that

$$A_2(\xi) = -\beta A_1(\xi) \tag{14}$$

where

$$\beta = \frac{\gamma_1 + \alpha_1}{\gamma_2 + \alpha_2}. \tag{15}$$

Employing eq. (14) the expressions for displacements and stresses reduce to

$$u(x, y) = \frac{2}{\pi}\int_0^\infty [\exp(-\gamma_1|y|) - \beta\exp(-\gamma_2|y|)]A_1(\xi)\sin \xi x \, d\xi, \tag{16}$$

$$v(x, y) = \pm\frac{2}{\pi}\int_0^\infty \frac{1}{\xi}[\alpha_1\exp(-\gamma_1|y|) - \beta\alpha_2\exp(-\gamma_2|y|)]A_1(\xi)\cos \xi x \, d\xi, \quad y \gtrless 0 \tag{17}$$

$$\tau_{xy}/\mu_{12} = \mp\frac{2}{\pi}\int_0^\infty (\gamma_1 + \alpha_1)[\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)]A_1(\xi)\sin \xi x \, d\xi, \quad y \gtrless 0 \tag{18}$$

$$\begin{aligned}\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty \Bigg[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \\ - \beta\left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\Bigg]A_1(\xi)\cos \xi x \, d\xi.\end{aligned} \tag{19}$$

We further substitute

$$A(\xi) = \frac{\alpha_1 - \beta\alpha_2}{\xi}A_1(\xi)$$

so that the boundary conditions (9) and (8) yield the following integral equations in $A(\xi)$

$$\int_0^\infty A(\xi)\cos \xi x \, d\xi = 0, \quad |x| < c, \quad |x| > 1 \tag{20}$$

and

$$\int_0^\infty H(\xi)A(\xi)\cos\xi x\,d\xi = -\frac{\pi p_0}{2\mu_{12}}, \quad c \leqslant |x| \leqslant 1 \tag{21}$$

where $p_0 = ik\mu_{12}c_{22}$

and

$$H(\xi) = \frac{c_{12}\xi^2 - c_{22}\alpha_1\gamma_1 - \beta(c_{12}\xi^2 - c_{22}\alpha_2\gamma_2)}{(\alpha_1 - \beta\alpha_2)}. \tag{22}$$

METHOD OF SOLUTION

In order to solve the set of integral eqs (20) and (21), assume

$$A(\xi) = \frac{1}{\xi}\int_c^1 h(t^2)\sin(\xi t)\,dt \tag{23}$$

where $h(t^2)$ is an unknown function to be determined from the boundary conditions.

Inserting the value of $A(\xi)$ from eq. (23) in eq. (20) and using the following result [11]

$$\int_0^\infty \frac{\sin(\xi t)\cos(\xi x)}{\xi}\,d\xi = \begin{cases} \frac{\pi}{2}, & t > x \\ 0, & t < x \end{cases}$$

it is found that the choice of $A(\xi)$ leads to the equation

$$\int_c^1 h(t^2)\,dt = 0. \tag{24}$$

Further substitution of $A(\xi)$ from eq. (23) in eq. (21) leads to

$$\int_c^1 h(t^2)\,dt \int_0^\infty \sin(\xi t)\cos(\xi x)\,d\xi = q_0 -$$

$$-\frac{d}{dx}\int_c^1 h(t^2)\,dt\int_0^\infty \xi H_1(\xi)\frac{\sin(\xi t)\sin(\xi x)}{\xi^2}\,d\xi, \quad c \leqslant |x| \leqslant 1 \tag{25}$$

where

$$q_0 = -\frac{\pi p_0}{2\theta\mu_{12}} \tag{26}$$

$$H_1(\xi) = \frac{H(\xi)}{\xi\theta} - 1 \quad \to 0 \text{ as } \xi \to \infty \tag{27}$$

$$\theta = \frac{(c_{12}^2 + c_{12} - c_{11}c_{22})(c_{12}N_1N_2 - c_{11}) - c_{22}[c_{12}N_1^2N_2^2 + c_{11}(N_1^2 + N_1N_2 + N_2^2)]}{c_{11}(1 + c_{12})(N_1 + N_2)} \tag{28}$$

$$N_1^2 = \frac{1}{2c_{22}}[-(c_{12}^2 + 2c_{12} - c_{11}c_{22}) + \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}}]$$

$$N_2^2 = \frac{1}{2c_{22}}[-(c_{12}^2 + 2c_{12} - c_{11}c_{22}) - \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}}]. \tag{29}$$

Using the relation

$$\frac{\sin\xi x\sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_0(\xi w)J_0(\xi v)\,dv\,dw}{(x^2 - w^2)^{1/2}(t^2 - v^2)^{1/2}} \tag{30}$$

eq. (25) can be rewritten in the following form

$$\int_c^1 \frac{th(t^2)}{t^2 - x^2}\,dt = q_0 - \frac{d}{dx}\int_c^1 h(t^2)\,dt\int_0^x\int_0^t \frac{vwL(v,w)\,dw\,dv}{(x^2 - w^2)^{1/2}(t^2 - v^2)^{1/2}}, \quad c \leqslant |x| \leqslant 1 \tag{31}$$

where

$$L(v,w) = \int_0^\infty \xi H_1(\xi)J_0(\xi w)J_0(\xi v)\,d\xi. \tag{32}$$

Applying a contour integration technique, the infinite integral in $L(v, w)$ can be converted to the following finite integrals (details given in the appendix)

$$L(v,w) = -ik_s^2\Bigg[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - c_{22}\bar{\alpha}_1\bar{\gamma}_1 - \beta(c_{12}\eta^2 - c_{22}\bar{\alpha}_2\bar{\gamma}_2)}{\theta(\bar{\alpha}_1 - \beta\bar{\alpha}_2)} \times J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,\mathrm{d}\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{\beta(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta(\hat{\alpha}_1 - \beta\hat{\alpha}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,\mathrm{d}\eta\Bigg], \quad w > v \tag{33}$$

where

$$\begin{aligned}
\bar{\gamma}_1 &= [\tfrac{1}{2}\{R_1 - (R_1^2 - 4\bar{R}_2)^{1/2}\}]^{1/2}\\
\bar{\gamma}_2 &= [\tfrac{1}{2}\{R_1 + (R_1^2 - r\bar{R}_2)^{1/2}\}]^{1/2}\\
\hat{\gamma}_1 &= [\tfrac{1}{2}\{-R_1 + (R_1^2 + 4R_2')^{1/2}\}]^{1/2}\\
\hat{\gamma}_2 &= [\tfrac{1}{2}\{R_1 + (R_1^2 + 4R_2')^{1/2}\}]^{1/2}\\
R_1 &= \frac{1}{c_{22}}\{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\eta^2 + (1 + c_{22})\}\\
\bar{R}_2 &= \frac{c_{11}}{c_{22}}(1 - \eta^2)\left(\frac{1}{c_{11}} - \eta^2\right)\\
R_2' &= \frac{c_{11}}{c_{22}}(1 - \eta^2)\left(\eta^2 - \frac{1}{c_{11}}\right)\\
\bar{\alpha}_i &= \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_i^2}{(1 + c_{12})\bar{\gamma}_i}\;(i = 1, 2)\\
\hat{\alpha}_i &= \frac{c_{11}\eta^2 - 1 + (-1)^i\hat{\gamma}_i^2}{(1 + c_{12})\hat{\gamma}_i}\;(i = 1, 2)\\
\bar{\beta} &= \frac{\bar{\alpha}_1 - \bar{\gamma}_1}{\bar{\alpha}_2 - \bar{\gamma}_2}\\
\hat{\beta} &= \frac{\hat{\alpha}_1 + \hat{\gamma}_1}{\hat{\alpha}_2 - \hat{\gamma}_2}.
\end{aligned} \tag{34}$$

The corresponding expression of $L(v, w)$ for $w < v$ follows from (33) by interchanging w and v.

Employing the series expansions for the Bessel function J_0 and the Hankel function $H_0^{(1)}$ in eq. (33), it is found that

$$L(v, w) = \frac{2}{\pi} P k_s^2 \log k_s + O(k_s^2) \tag{35}$$

where

$$P = \frac{1}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - c_{22}\bar{\alpha}_1\bar{\gamma}_1 - \bar{\beta}(c_{12}\eta^2 - c_{22}\bar{\alpha}_2\bar{\gamma}_2)}{(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)}\,\mathrm{d}\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)}\,\mathrm{d}\eta\right].$$

Now, let us expand $h(t^2)$ in the form

$$h(t^2) = h_0(t^2) + k_s^2 \log k_s h_1(t^2) + O(k_s^2). \tag{36}$$

Inserting the above expansion of $h(t^2)$ and the value of $L(v, w)$ given by eq. (35) into eq. (31) and equating the coefficients of like powers of k_s, we obtain the equations

$$\int_c^1 \frac{t h_0(t^2)}{t^2 - x^2}\,\mathrm{d}t = q_0, \quad c \leq |x| \leq 1 \tag{37}$$

and

$$\int_c^1 \frac{t h_1(t^2)}{t^2 - x^2}\,\mathrm{d}t = -\frac{2P}{\pi}\int_c^1 t h_0(t^2)\,\mathrm{d}t, \quad c \leq |x| \leq 1. \tag{38}$$

Using the finite Hilbert transform technique [12], the solutions of the above integral equations can be obtained as

$$h_0(t^2) = \frac{2}{\pi} q_0 \left(\frac{t^2-c^2}{1-t^2}\right)^{1/2} + \frac{D_1}{\sqrt{(1-t^2)(t^2-c^2)}} \tag{39}$$

$$h_1(t^2) = -\frac{2}{\pi} P \left[\frac{q_0(1-c^2)}{\pi} + D_1\right]\left(\frac{t^2-c^2}{1-t^2}\right)^{1/2} + \frac{D_2}{\sqrt{(1-t^2)(t^2-c^2)}} \tag{40}$$

where D_1 and D_2 are constants to be determined using the condition given by eq. (24) so that

$$\int_c^1 h_0(t^2)\,dt = 0 \tag{41}$$

and

$$\int_c^1 h_1(t^2)\,dt = 0.$$

Substitution of the values of $h_0(t^2)$ and $h_1(t^2)$ given by eqs (39) and (40) in (41), yields

$$D_1 = \frac{2}{\pi} q_0 \left[c^2 - \frac{E}{F}\right] \tag{42}$$

$$D_2 = \frac{2}{\pi^2} q_0 \left[1 + c^2 - \frac{2E}{F}\right]\left[\frac{E}{F} - c^2\right], \tag{43}$$

where $F = F[\frac{\pi}{2}, \sqrt{1-c^2}]$ and $E = E[\pi/2, \sqrt{1-c^2}]$ are the elliptic integrals of first and second kind, respectively. Substituting the value of D_1 and D_2 given by eqs (42) and (43) into eqs (39) and (40), we obtain

$$h_0(t^2) = -\frac{p_0}{\mu_{12}\theta} \frac{\left[t^2 - \frac{E}{F}\right]}{\sqrt{(1-t^2)(t^2-c^2)}} \tag{44}$$

$$h_1(t^2) = -\frac{P p_0}{\pi \mu_{12}\theta} \frac{\left[1 + c^2 - \frac{2E}{F}\right]\left[t^2 - \frac{E}{F}\right]}{\sqrt{(1-t^2)(t^2-c^2)}} \tag{45}$$

CRACK OPENING DISPLACEMENT AND STRESS INTENSITY FACTORS

The crack opening displacement and the normal stress component in the plane of the crack can be written as

$$\Delta v(x,0) = v(x,0+) - v(x,0-) = 2\int_x^1 h(t^2)\,dt, \quad c \leq x \leq 1 \tag{46}$$

and

$$\tau_{yy}(x,0) = \frac{2\mu_{12}\theta}{\pi}\int_c^1 \frac{t h(t^2)}{t^2-x^2}\,dt, \quad 0 < x < c \tag{47}$$

$$= -\frac{2\mu_{12}\theta}{\pi}\int_c^1 \frac{t h(t^2)}{x^2-t^2}\,dt, \quad x > 1. \tag{48}$$

Expressions (47) and (48) with the aid of the eqs (36), (44) and (45) yield

$$\tau_{yy}(x,0) = -p_0\left[1 + \frac{\left[x^2 - \frac{E}{F}\right]}{\sqrt{(1-x^2)(c^2-x^2)}}\right]\left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\}k_s^2 \log k_s\right] + O(k_s^2), \quad 0 < x < c \tag{49}$$

$$\tau_{yy}(x,0) = -p_0\left[1 - \frac{\left[x^2 - \frac{E}{F}\right]}{\sqrt{(x^2-1)(x^2-c^2)}}\right]\left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\}k_s^2 \log k_s\right] + O(k_s^2), \quad x > 1, \tag{50}$$

The stress intensity factors are defined as (in physical units)

$$K_c = \lim_{x \to c-} \left[\frac{\sqrt{(c-x)}\,\tau_{yy}(x,0)}{p_0} \right]_{0<x<c} \tag{51}$$

$$K_1 = \lim_{x \to 1+} \left[\frac{\sqrt{(x-1)}\,\tau_{yy}(x,0)}{p_0} \right]_{x>1}. \tag{52}$$

Substituting eqs (49) and (50) into (51) and (52) it can be shown that

$$K_c = -\frac{\left[c^2 - \dfrac{E}{F}\right]}{\sqrt{2c(1-c^2)}} \left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\} k_s^2 \log k_s \right] + O(k_s^2) \tag{53}$$

$$K_1 = \frac{\left[1 - \dfrac{E}{F}\right]}{\sqrt{2(1-c^2)}} \left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\} k_s^2 \log k_s \right] + O(k_s^2) \tag{54}$$

Further substituting eqs (36), (44) and (45) in the expression given by eq. (46), the crack opening displacement is obtained as

$$\Delta v(x,0) = \frac{2p_0}{\mu_{12}\theta} \left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\} k_s^2 \log k_s \right] \left[\frac{E\left(\dfrac{\pi}{2}, q\right)}{F\left(\dfrac{\pi}{2}, q\right)} F(\lambda, q) - E(\lambda, q) \right] + O(k_s^2), \quad c \leqslant x \leqslant 1 \tag{55}$$

where

$$\sin\lambda = \sqrt{\frac{1-x^2}{1-c^2}} \text{ and } q = \sqrt{(1-c^2)}.$$

Letting $c \to 0$ in the expression for stress intensity factor and crack opening displacement, the results for a single crack occupying the region $|x| \leqslant 1, y = 0, |z| < \infty$ are found to be

$$K_1 = \frac{1}{\sqrt{2}} \left[1 - \frac{P}{\pi} k_s^2 \log k_s \right] + O(k_s^2) \tag{56}$$

$$\Delta v(x,0) = -\frac{2p_0}{\mu_{12}\theta} \sqrt{(1-x^2)} \left[1 - \frac{P}{\pi} k_s^2 \log k_s \right] + O(k_s^2), \quad 0 \leqslant x \leqslant 1. \tag{57}$$

For isotropic medium, putting

$$c_{11} = c_{22} = \frac{\lambda + 2\mu}{\mu}, \quad \mu_{12} = \mu, \quad c_{12} = c_{11} - 2 = \frac{\lambda}{\mu}$$

so that

$$\alpha_1 = \gamma_1, \quad \alpha_1 = \xi^2/\gamma_2, \quad k_s = m_2, \quad k_s/\sqrt{c_{11}} = m_1, \quad \tau^2 = \frac{1}{c_{11}}$$

$$N_1 = 1 = N_2, \quad \theta = -2(1-\tau^2) \text{ and } P = \frac{\pi}{2} c_1,$$

where

$$c_1 = \frac{3\tau^4 - 4\tau^2 - 3}{4(1-\tau^2)}, \quad \gamma_i = (\xi^2 - m_i^2)^{1/2} \text{ and } m_i = \frac{a\omega}{c_i} (i = 1, 2)$$

the expressions for displacement and stress are found to be

$$v(x, \pm 0) = \mp \frac{p_0}{2\mu(1-\tau^2)} \left[1 - \frac{c_1}{2}\left\{1 + c^2 - \frac{2E}{F}\right\} m_2^2 \log m_2 \right] \left[\frac{E\left(\dfrac{\pi}{2}, q\right)}{F\left(\dfrac{\pi}{2}, q\right)} F(\lambda, q) - E(\lambda, q) \right] + O(m_2^2), \quad c \leqslant x \leqslant 1$$

$= 0, \quad |x| < c, \quad |x| > 1$

$0 \quad |x| < c \quad |x| > 1$

and

$$\tau_{yy}(x,0) = -p_0\left[1+\frac{\left[x^2-\frac{E}{F}\right]}{\sqrt{(1-x^2)(c^2-x^2)}}\right]\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^{3/2}\log m_2\right]+O(m_2^{3/2})\quad 0<x<c$$

$$= -p_0,\quad c\leqslant|x|\leqslant 1$$

$$= -p_0\left[1-\frac{\left[x^2-\frac{E}{F}\right]}{\sqrt{(x^2-1)(x^2-c^2)}}\right]\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^{3/2}\log m_2\right]+O(m_2^{3/2}),\quad |x|>1.$$

Now, the crack opening displacement and stress intensity factors are found to be

$$\Delta v(x,0) = -\frac{p_0}{\mu(1-\tau^2)}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^{3/2}\log m_2\right]$$

$$\times\left[\frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)}F(\lambda,q)-E(\lambda,q)\right]+O(m_2^{3/2}),\quad c\leqslant x\leqslant 1$$

and

$$K_c = -\frac{\left[c^2-\frac{E}{F}\right]}{\sqrt{2c(1-c^2)}}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^{3/2}\log m_2\right]+O(m_2^{3/2})$$

$$K_1 = \frac{\left[1-\frac{E}{F}\right]}{\sqrt{2(1-c^2)}}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^{3/2}\log m_2\right]+O(m_2^{3/2})$$

which coincide with the results obtained by Jain and Kanwal [5] up to the order of $m_2^{3/2}\log m_2$ in the isotropic case.

When $c\to 0$, we recover the stress intensity factor and crack opening displacement for a single crack

$$K_1 = \frac{1}{\sqrt{2}}\left[1-\frac{c_1}{2}m_2^{3/2}\log m_2\right]+O(m_2^{3/2})$$

$$\Delta v(x,0) = \frac{p_0}{\mu(1-\tau^2)}\sqrt{(1-x^2)}\left[1-\frac{c_1}{2}m_2^{3/2}\log m_2\right]+O(m_2^{3/2}),\quad 0\leqslant x\leqslant 1$$

which agrees with the result of Mal [2].

NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) K_c and K_1 given by (53) and (54) at the inner and outer tips of the cracks and crack opening displacement (COD) given by (55) have been plotted against dimensionless frequency k_t and distance, respectively for three different types of orthotropic materials whose engineering constants have been listed in Table 1.

From Fig. 2 it is found that SIF K_c at the inner tip of the crack increases at a slow rate with the increase in the value of frequency k_t $(0.1\leqslant k_t\leqslant 0.6)$. On the other hand the rate of increase of

Table 1. Engineering elastic constants

	E_1(Pa)	E_2(Pa)	μ_{12}(Pa)	ν_{12}
Type I	Modulite II graphite–epoxy composite:			
	15.3×10^9	158.0×10^9	5.52×10^9	0.033
Type II	E-Type glass–epoxy composite:			
	9.79×10^9	42.3×10^9	3.66×10^9	0.063
Type III	Stainless steel–aluminium composite:			
	79.76×10^9	85.91×10^9	30.02×10^9	0.31

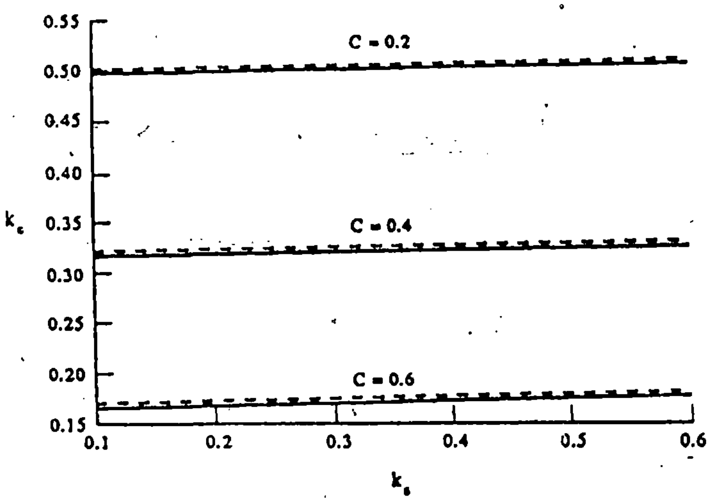

Fig. 2. Stress intensity factor K_c vs frequency k_s for generalized plane stress. (—, Type I; ----, Type II).

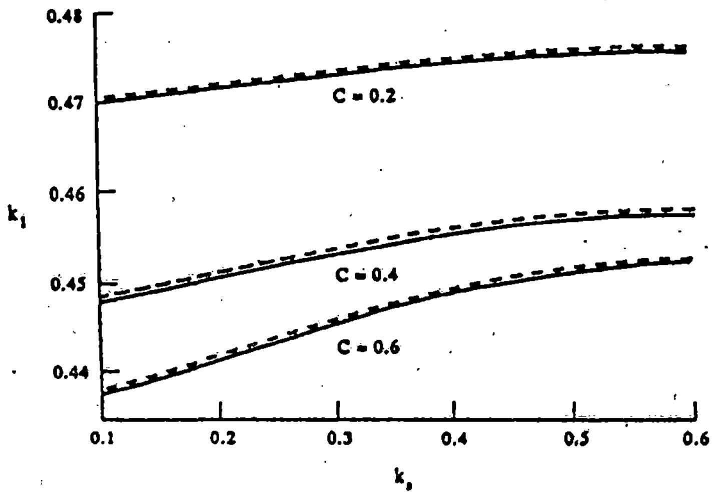

Fig. 3. Stress intensity factor K_1 vs frequency k_s for generalized plane stress. (—, Type I; ----, Type II).

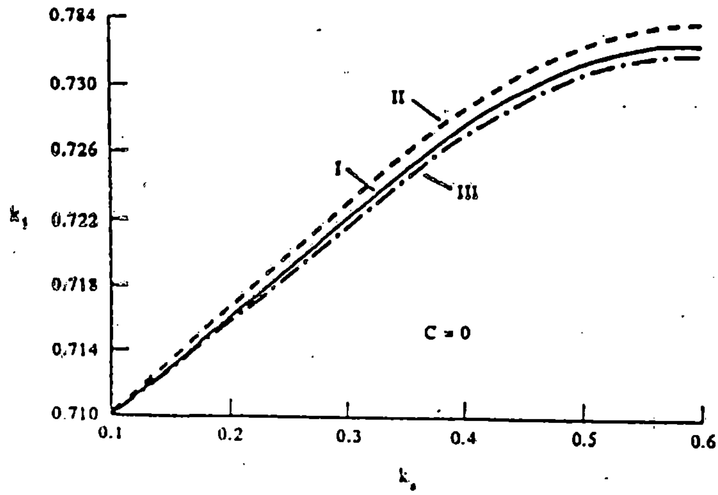

Fig. 4. Stress intensity factor K_1 vs frequency k_s for generalized plane stress. (Single crack, $c = 0$).

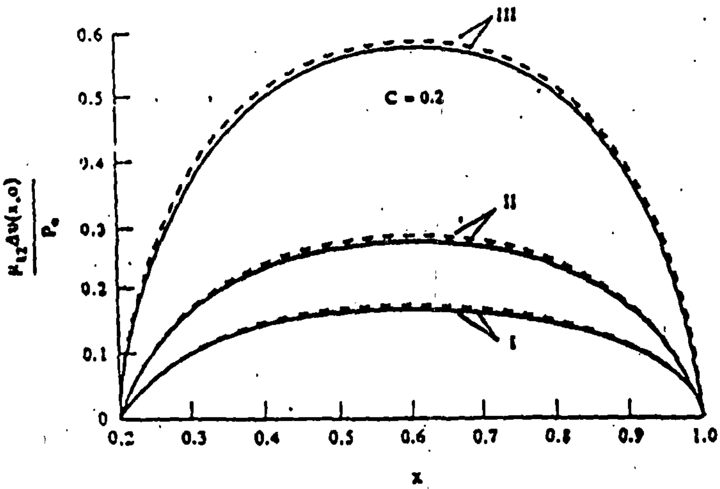

Fig. 5. Crack opening displacement (COD) vs distance ($c = 0.2$) for generalized plane stress. (—, $k_s = 0.2$; -----, $k_s = 0.6$).

the SIF K_I (Fig. 3) with frequency k_s at the outer tip of the crack is found to be higher than that of K_c.

In both the cases the value of SIF is higher for small values of c, i.e. for greater crack length SIF is higher. But it is interesting to note that for different materials the variation of SIFs in both the cases are not significant. In the case of single crack ($c = 0$) the variation of SIF with material properties has been shown in Fig. 4.

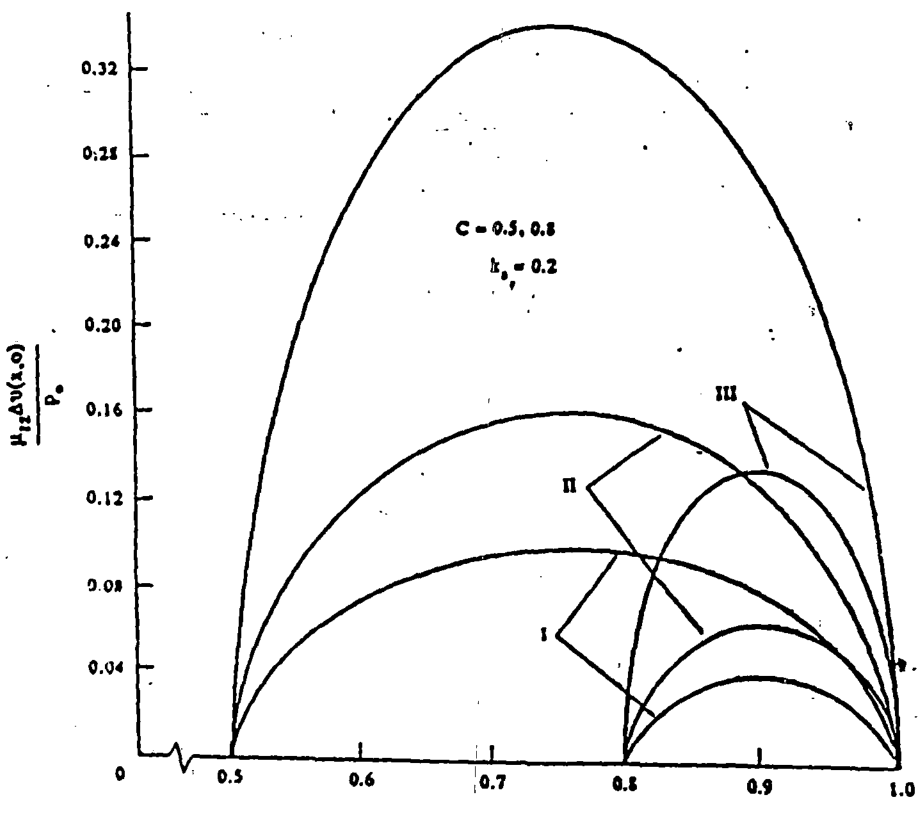

Fig. 6. Crack opening displacement (COD) vs distance ($c = 0.5$ and $c = 0.8$) for generalized plane stress.

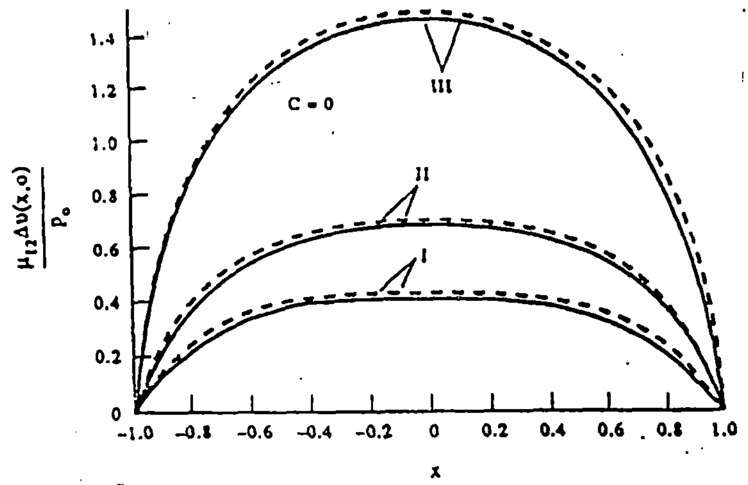

Fig. 7. Crack opening displacement (COD) vs distance (single crack, $c = 0$) for generalized plane stress. (—, $k_r = 0.2$; - - - -, $k_r = 0.6$).

The COD has been plotted for different crack length. In each case COD increases gradually from zero, attains maximum value and then decreases to zero. It is found that with the increase in the values of c (i.e. for small crack length) the values of COD decreases (Figs 5 and 6). For a fixed material the variation of COD with frequency is found to be insignificant, but it is noticed that for smaller values of c (Figs 5 and 7) the variation of COD with frequency is palpable. $c = 0$ (Fig. 7) correspond to the case of single crack.

In all the cases where different values of c has been considered the variation of COD is found to be prominent for different orthotropic materials.

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APPENDIX

Evaluation of L(v, w)

The integral $L(v, w)$, given by eq. (32) is

$$L(v, w) = \int_0^{\infty} M(\xi, \gamma_1, \gamma_2) J_0(\xi w) J_0(\xi v) \mathrm{d}\xi \tag{A1}$$

where

$$M(\xi, \gamma_1, \gamma_2) = \xi H_1(\xi) = \frac{c_{12}\xi^2 - c_{22}\alpha_1\gamma_1 - \beta(c_{12}\xi^2 - c_{22}\alpha_2\gamma_2)}{\eta(\alpha_1 - \beta\alpha_2)} - \xi \tag{A2}$$

$$\gamma_1 = [\tfrac{1}{2}\{-B_1 + (B_1^2 - 4B_2)^{1/2}\}]^{1/2}$$
$$\gamma_2 = [\tfrac{1}{2}\{-B_1 - (B_1^2 - 4B_2)^{1/2}\}]^{1/2}$$
$$B_1 = \frac{1}{c_{22}}\{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1 + c_{22})k_s^2\} \tag{A3}$$
$$B_2 = \frac{1}{c_{22}}(\xi^2 - k_s^2)(c_{11}\xi^2 - k_s^2).$$

To evaluate the integral (A1) we consider two contour integrals

$$I_1 = \int_{\Gamma_1} M(\xi, \gamma_1, \gamma_2)J_0(\xi v)H_0^{(1)}(\xi w)\,d\xi, \quad w > v$$
$$I_2 = \int_{\Gamma_2} M(\xi, \gamma_1, \gamma_2)J_0(\xi v)H_0^{(2)}(\xi w)\,d\xi, \quad w > v, \tag{A4}$$

where Γ_1 and Γ_2 are the closed contours defined in Fig. 8, and $H_0^{(1)}, H_0^{(2)}$ are the zero order Hankel functions of the first and second kind, respectively.

Assuming the relation

$$\left\{\frac{(c_{12}^2 + 2c_{12} - c_{11}c_{22})(1 + c_{22})}{c_{22}^2} + \frac{2(1 + c_{11})}{c_{22}}\right\}^2 - \left\{\frac{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2}{c_{22}^2} - \frac{4c_{11}}{c_{22}}\right\} \times \left\{\frac{(1 + c_{22})^2}{c_{22}^2} - \frac{4}{c_{22}}\right\} < 0 \tag{A5}$$

it is noted the branch points $\xi = \lambda_i (i = 1-4)$ corresponding to the roots of the equation $B_1^2 - 4B_2 = 0$ are always complex. Now, the branch points corresponding to the roots of the equations

$$-B_1 + (B_1^2 - 4B_2)^{1/2} = 0 \text{ and } -B_1 - (B_1^2 - 4B_2)^{1/2} = 0$$

are $\xi = \pm k_s$ and $\xi = \pm k_s/\sqrt{c_{11}}$, respectively where it has been assumed that

$$(c_{11}c_{22} - c_{12}^2 - 2c_{12}) > (1 + c_{22}) \tag{A6}$$

and

$$c_{12}^2 + 2c_{12} + c_{11} > 0.$$

Therefore under the above conditions, $\xi = \pm k_s/\sqrt{c_{11}}$ and $\xi = \pm k_s$ are the branch points of γ_1 and γ_2, respectively. Equations (A5) and (A6) are true for most of the orthotropic materials. The integrals in eq. (A4) can be shown to be zero

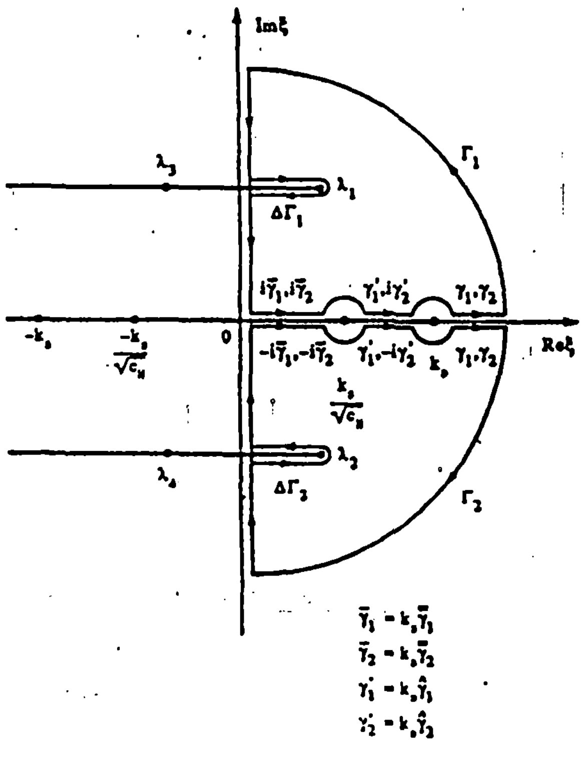

Fig. 8. Contours of integration for integral in eq. (A1).

on the contours $\Delta\Gamma_1$ and $\Delta\Gamma_2$ (Fig. 8) around the branch cuts from λ_1 and λ_2. Thus integrating along the contours Γ_1 and Γ_2 the integral $L(v, w)$ for $w > v$ can be finally written as

$$L(v, w) = -ik\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - c_{22}\bar{\alpha}_1\bar{\gamma}_1 - \beta(c_{12}\eta^2 - c_{22}\bar{\alpha}_2\bar{\gamma}_2)}{\theta(\bar{\alpha}_1 - \beta\bar{\alpha}_2)} \times J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\mathrm{d}\eta \right.$$

$$\left. - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta\,[illegible]\,(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\mathrm{d}\eta\right], \quad w > v$$

where $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\alpha}_1, \bar{\alpha}_2, \beta, \hat{\beta}, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\gamma}_2$ are given by eq. (34).

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DIFFRACTION OF ELASTIC WAVES BY THREE COPLANAR GRIFFITH CRACKS IN AN ORTHOTROPIC MEDIUM

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Abstract—The dynamic response of three co-planar Griffith cracks situated in an infinite orthotropic medium due to elastic waves incident normally on the cracks has been treated. The Fourier transform technique has been used to reduce the elastodynamic problem to the solution of a set of four integral equations. These integral equations have been solved by using the finite Hilbert transform technique and Cook's result. The analytical forms of crack opening displacement and stress intensity factors have been derived for low frequency vibration. Numerical results of crack opening displacement and stress intensity factors for several orthotropic materials have been calculated and plotted graphically to display the influence of the material orthotropy.

1. INTRODUCTION

Recently, with the increased usage of macroscopically anisotropic construction materials such as fibre-reinforced materials, the study of diffraction of elastic waves with cracks or inclusions has attracted the attention of scientists. The different possible location of cracks with respect to the planes of material symmetry is of great interest in Seismology and Exploration Geophysics. The problem of scattering of elastic waves by cracks of finite dimension in isotropic medium has been investigated by several investigators. Many investigators [1–6] have solved the diffraction problem involving single or two cracks in an isotropic medium. Dhawan and Dhaliwal [7] solved the statical problem involving three coplanar cracks in an infinite transversely isotropic medium. The dynamic problem of singular stresses around cracks in orthotropic medium are few in number. Kassir and Bandyopadhyay [8] solved the problem of elastodynamic response of an infinite orthotropic solid containing a crack under the action of impact loading. The problem of normal impact response of a finite Griffith crack in an orthotropic strip has been solved by Shindo [9]. De and Patra [10] have also solved the problem involving a moving Griffith crack in an orthotropic strip. Recently Kundu and Bostrom [11] treated the diffraction problem of a circular crack in orthotropic medium.

To the best knowledge of the authors, the problem of diffraction of elastic waves by three coplanar Griffith cracks in an orthotropic material has not been considered. In our paper, the interaction of normally incident time harmonic elastic waves with three coplanar Griffith cracks in an orthotropic medium has been investigated. It is assumed that the faces of each of the cracks do not come into contact during small deformation of the solid. The resulting mixed boundary value problem is reduced to the solution of a set of four integral equations which has been reduced to the solution of an integro-differential equation. Iteration method has been used to obtain the low frequency solution of the problem. This enables us to obtain approximate value of the crack opening displacements and stress intensity factors. Making the length of the central crack tend to zero, the corresponding results for two Griffith cracks have been obtained. Numerical results for stress intensity factors and crack opening displacements have been plotted against dimensionless frequency and distance respectively for different orthotropic materials which have been shown graphically.

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2. STATEMENT AND FORMULATION OF THE PROBLEM

Consider the interaction of normally incident longitudinal wave with three coplanar Griffith cracks situated in an infinite orthotropic elastic medium. The cracks are assumed to occupy the position $|X| \le d_1$, $d_2 \le |X| \le d$, $Y = 0$, $|Z| < \infty$. Let E_i, μ_{ij} and ν_{ij} $(i, j = 1, 2, 3)$ denote the engineering elastic constants of the material where the subscripts 1, 2, 3 correspond to the X, Y, Z directions chosen to coincide with the axes of material orthotropy. Normalizing all the lengths with respect to 'd' and setting $X/d = x$, $Y/d = y$, $Z/d = z$, $d_1/d = b$, $d_2/d = c$, the cracks are defined by $|x| \le b$, $c \le |x| \le 1$, $y = 0$, $|z| < \infty$ (Fig. 1).

Displacement components are also made dimensionless with respect to 'd' so that dimensionless components of displacement in x, y directions are assumed to be u, v respectively, where

$$u = u(x, y, t) \quad \text{and} \quad v = v(x, y, t).$$

Let a time harmonic plane elastic wave originating at $y = -\infty$ and incident normally on the three cracks be given by $v = v_0 \exp[i(ky - \omega t)]/d$ where $k = d\omega/c_s\sqrt{c_{22}}$, $c_s = (\mu_{12}/\rho)^{1/2}$, v_0 is a constant, ω and v_0/d are the frequency and dimensionless amplitude of the incident wave respectively, ρ being the density of the material. In the isotropic solid, c_s represents the velocity of the shear wave.

The non-zero stress components τ_{yy} and τ_{xy} are given by

$$\tau_{yy}/\mu_{12} = c_{12}u_{,x} + c_{22}v_{,y}$$

$$\tau_{xy}/\mu_{12} = u_{,y} + v_{,x} \tag{2.1}$$

where u, v denote the component of the displacement in the x, y directions respectively and comma denotes partial differentiation with respect to the coordinates or time ; $c_{ij}(i, j = 1, 2)$ are

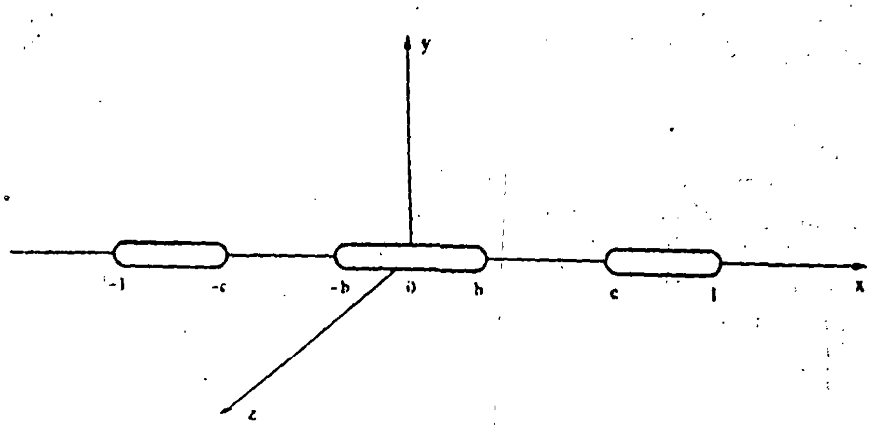

Fig. 1. Geometry of the cracks.

nondimensional parameters related to the elastic constant by the relations:

$$c_{11} = E_1/\mu_{12}(1 - \nu_{12}^2 E_2/E_1)$$
$$c_{22} = E_2/\mu_{12}(1 - \nu_{12}^2 E_2/E_1) = c_{11}E_2/E_1$$
$$c_{12} = \nu_{12}E_2/\mu_{12}(1 - \nu_{12}^2 E_2/E_1) = \nu_{12}c_{22} = \nu_{21}c_{11} \tag{2.2}$$

for generalized plane stress, and by

$$c_{11} = (E_1/\Delta\mu_{12})(1 - \nu_{23}\nu_{32})$$
$$c_{22} = (E_2/\Delta\mu_{12})(1 - \nu_{13}\nu_{31})$$
$$c_{12} = E_1(\nu_{21} + \nu_{13}\nu_{32}E_2/E_1)/\Delta\mu_{12} = E_2(\nu_{12} + \nu_{23}\nu_{31}E_1/E_2)/\Delta\mu_{12}$$
$$\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{21}\nu_{32} \tag{2.3}$$

for plane strain. The constants E_i and ν_{ij} satisfy Maxwell's relation:

$$\nu_{ij}/E_i = \nu_{ji}/E_j. \tag{2.4}$$

The displacement equations of motion for orthotropic material are

$$c_{11}u_{,xx} + u_{,yy} + (1 + c_{12})v_{,xy} = \frac{d^2}{c_s^2}u_{,tt}$$
$$c_{22}v_{,yy} + v_{,xx} + (1 + c_{12})u_{,xy} = \frac{d^2}{c_s^2}v_{,tt} \tag{2.5}$$

Substitution of $u(x, y, t) = u(x, y)\exp(-i\omega t)$ and $v(x, y, t) = v(x, y)\exp(-i\omega t)$ in equations (2.5) reduces them to

$$c_{11}u_{,xx} + u_{,yy} + (1 + c_{12})v_{,xy} + k_s^2 u = 0$$
$$c_{22}v_{,yy} + v_{,xx} + (1 + c_{12})u_{,xy} + k_s^2 v = 0 \tag{2.6}$$

with $k_s^2 = d^2\omega^2/c_s^2$, which are to be solved subject to the boundary conditions

$$v(x, 0) = 0, \quad b \le |x| \le c, \qquad |x| \le 1 \tag{2.7}$$
$$\tau_{xy}(x, 0) = 0, \qquad |x| < \infty \tag{2.8}$$
$$\tau_{yy}(x, 0) + \tau_{yy}^{(in)}(x, 0) = 0, \qquad |x| < b, \qquad c < |x| < 1. \tag{2.9}$$

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables would be omitted in the sequel.

Using the condition (2.8), the solutions of equations (2.6) may be written as

$$u(x, y) = \frac{2}{\pi}\int_0^\infty [\exp(-\gamma_1|y|) - \beta\exp(-\gamma_2|y|)]A_1(\xi)\sin(\xi x)\,d\xi \tag{2.10}$$

$$v(x, y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}[\alpha_1\exp(-\gamma_1|y|) - \beta\alpha_2\exp(-\gamma_2|y|)]A_1(\xi)\cos(\xi x)\,d\xi, \qquad y > 0 \tag{2.11}$$

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and the stress components are given by

$$\tau_{xy}/\mu_{12} = -\frac{2}{\pi}\int_0^\infty (\gamma_1+\alpha_1)[\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)]A_1(\xi)\sin(\xi x)\,d\xi, \qquad y>0 \tag{2.12}$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty \left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \beta\left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\right]A_1(\xi)\cos(\xi x) \tag{2.13}$$

where

$$\alpha_i = \frac{c_{11}\xi^2 - k_1^2 - \gamma_i^2}{(1+c_{12})\gamma_i}, \qquad i = 1, 2 \tag{2.14}$$

$$\beta = \frac{\gamma_1+\alpha_1}{\gamma_2+\alpha_2} \tag{2.15}$$

$A_1(\xi)$ is the unknown function to be determined, and γ_1^2, γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1+c_{22})k_1^2\}\gamma^2 + (c_{11}\xi^2 - k_1^2)(\xi^2 - k_1^2) = 0. \tag{2.16}$$

With the aid of the boundary conditions, (2.7) and (2.9) $A(\xi)$ is found to satisfy the integral equations

$$\int_0^\infty A(\xi)\cos(\xi x)\,d\xi = 0, \qquad x \in I_2, I_4 \tag{2.17}$$

and

$$\int_0^\infty H(\xi)A(\xi)\cos(\xi x)\,d\xi = -\frac{\pi p_0}{2\mu_{12}}, \qquad x \in I_1, I_3 \tag{2.18a, b}$$

where $I_1 = (0, b)$, $I_2 = (b, c)$, $I_3 = (c, 1)$, $I_4 = (1, \infty)$ and

$$p_0 = ik\mu_{12}c_{22}v_0/d \tag{2.19}$$

$$A(\xi) = \frac{\alpha_1 - \beta\alpha_2}{\xi}A_1(\xi) \tag{2.20}$$

$$H(\xi) = \frac{c_{12}\xi^2 - c_{22}\alpha_1\gamma_1 - \beta(c_{12}\xi^2 - c_{22}\alpha_2\gamma_2)}{(\alpha_1 - \beta\alpha_2)}. \tag{2.21}$$

3. METHOD OF SOLUTION

The solution of the integral equations (2.17) and (2.18) is taken in the form

$$A(\xi) = \frac{1}{\xi}\int_0^b h(t)\sin(\xi t)\,dt + \frac{1}{\xi}\int_c^1 g(u^2)\sin(\xi u)\,du \tag{3.1}$$

where $h(t)$ and $g(u^2)$ are the unknown functions to be determined. Substituting the value of $A(\xi)$ from (3.1) in (2.17) and using the following result [12]

$$\int_0^\infty \frac{\sin(\xi t)\cos(\xi x)}{\xi}\,d\xi = \begin{cases} \frac{\pi}{2}, & t > x \\ 0, & t < x \end{cases}$$

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it is found that the choice of $A(\xi)$ leads to the equation

$$\int_c^1 g(u^2)\,\mathrm{d}u = 0. \tag{3.2}$$

Further substituting $A(\xi)$ from (3.1) in (2.18a) and using the result [13]

$$\int_0^\infty \xi^{-1}\sin(\xi u)\sin(\xi x)\,\mathrm{d}\xi = \frac{1}{2}\log\left|\frac{u+x}{u-x}\right|$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h(t)\log\left|\frac{t+x}{t-x}\right|\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(u^2)\log\left|\frac{u+x}{u-x}\right|\mathrm{d}u \\ = 2\Big[q_0 - \frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h(t)\,\mathrm{d}t\int_0^\infty H_1(\xi)\xi^{-1}\sin(\xi t)\sin(\xi x)\,\mathrm{d}\xi \\ - \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(u^2)\,\mathrm{d}u\int_0^\infty H_1(\xi)\xi^{-1}\sin(\xi u)\sin(\xi x)\,\mathrm{d}\xi\Big], \qquad x \in I_1\end{aligned} \tag{3.3}$$

where

$$q_0 = -\frac{\pi p_0}{2\theta\mu_{12}} \tag{3.4}$$

$$H_1(\xi) = \frac{H(\xi)}{\xi\theta} - 1 \to 0 \quad \text{as} \quad \xi \to \infty \tag{3.5}$$

$$\theta = \frac{(c_{12}^2 + c_{12} - c_{11}c_{22})(c_{12}N_1N_2 - c_{11}) - c_{22}[c_{12}N_1^2N_2^2 + c_{11}(N_1^2 + N_1N_2 + N_2^2)]}{c_{11}(1 + c_{12})(N_1 + N_2)} \tag{3.6}$$

$$\begin{aligned}N_1^2 &= \frac{1}{2c_{22}}\{c_{11}c_{22} - c_{12}^2 - 2c_{12} + [(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}]^{1/2}\} \\ N_2^2 &= \frac{1}{2c_{22}}\{c_{11}c_{22} - c_{12}^2 - 2c_{12} - [(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}]^{1/2}\}.\end{aligned} \tag{3.7}$$

Using the relation

$$\frac{\sin\xi x\sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_0(\xi w)J_0(\xi v)\,\mathrm{d}v\,\mathrm{d}w}{(x^2 - w^2)^{1/2}(t^2 - v^2)^{1/2}}$$

equation (3.3) can now be rewritten in the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h(t)\log\left|\frac{t+x}{t-x}\right|\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(u^2)\log\left|\frac{u+x}{u-x}\right|\mathrm{d}u \\ = 2\Big[q_0 - \frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h(t)\,\mathrm{d}t\int_0^x\int_0^t \frac{vwL(v,w)\,\mathrm{d}w\,\mathrm{d}v}{(x^2 - w^2)^{1/2}(t^2 - v^2)^{1/2}} \\ - \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(u^2)\,\mathrm{d}u\int_0^x\int_0^u \frac{vwL(v,w)\,\mathrm{d}w\,\mathrm{d}v}{(x^2 - w^2)^{1/2}(u^2 - v^2)^{1/2}}, \qquad x \in I_1\end{aligned} \tag{3.8}$$

where

$$L(v, w) = \int_0^\infty \xi H_1(\xi)J_0(\xi w)J_0(\xi v)\,\mathrm{d}\xi \tag{3.9}$$

and $J_0(\ )$ is the Bessel function of order zero.

Applying a contour integration technique [14], the infinite integral in $L(v, w)$ can be converted to the following finite integrals

$$L(v,w) = -ik_s^2\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - \bar{\alpha}_1\bar{\gamma}_1 c_{22} - \bar{\beta}(c_{12}\eta^2 - \bar{\alpha}_2\bar{\gamma}_2 c_{22})}{\theta(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,d\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,d\eta\right], \qquad w > v \quad (3.10)$$

where

$$\bar{\gamma}_1 = \left[\frac{1}{2}\{R_1 - (R_1^2 - 4\bar{R}_2)^{1/2}\}\right]^{1/2}$$

$$\bar{\gamma}_2 = \left[\frac{1}{2}\{R_1 + (R_1^2 - 4\bar{R}_2)^{1/2}\}\right]^{1/2}$$

$$\hat{\gamma}_1 = \left[\frac{1}{2}\{-R_1 + (R_1^2 + 4R_2')^{1/2}\}\right]^{1/2}$$

$$\hat{\gamma}_2 = \left[\frac{1}{2}\{R_1 + (R_1^2 + 4R_2')^{1/2}\}\right]^{1/2}$$

$$R_1 = \frac{1}{c_{22}}\{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\eta^2 + (1 + c_{22})\}$$

$$\bar{R}_2 = \frac{c_{11}}{c_{22}}(1 - \eta^2)\left(\frac{1}{c_{11}} - \eta^2\right)$$

$$R_2' = \frac{c_{11}}{c_{22}}(1 - \eta^2)\left(\eta^2 - \frac{1}{c_{11}}\right)$$

$$\bar{\alpha}_i = \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_i^2}{(1 + c_{12})\bar{\gamma}_i}, \qquad i = 1, 2$$

$$\hat{\alpha}_i = \frac{c_{11}\eta^2 - 1 + (-1)^i\hat{\gamma}_i^2}{(1 + c_{12})\hat{\gamma}_i}, \qquad i = 1, 2$$

$$\bar{\beta} = \frac{\bar{\alpha}_1 - \bar{\gamma}_1}{\bar{\alpha}_2 - \bar{\gamma}_2}$$

$$\hat{\beta} = \frac{\hat{\alpha}_1 + \hat{\gamma}_1}{\hat{\alpha}_2 - \hat{\gamma}_2}. \qquad (3.11)$$

The corresponding expression of $L(v, w)$ for $w < v$ is obtained by interchanging v and w in (3.10).

Employing the series expansions for the Bessel function J_0 and the Hankel function $H_0^{(1)}$ in equation (3.10), it is found that

$$L(v, w) = \frac{2}{\pi}Pk_s^2 \log k_s + O(k_s^2) \qquad (3.12)$$

where

$$P = \frac{1}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - \bar{\alpha}_1\bar{\gamma}_1 c_{22} - \bar{\beta}(c_{12}\eta^2 - \bar{\alpha}_2\bar{\gamma}_2 c_{22})}{(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)}\,d\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)}\,d\eta\right]. \qquad (3.13)$$

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Let us now expand $h(t)$ and $g(u^2)$ in the form

$$h(t) = h_0(t) + k_s^2 \log k_s\, h_1(t) + O(k_s^2)$$

and

$$g(u^2) = g_0(u^2) + k_s^2 \log k_s\, g_1(u^2) + O(k_s^2). \tag{3.14}$$

Substituting the above equations (3.14) and the value of $L(v, w)$ given by (3.10) in equations (3.8) and (3.2) and equating the coefficients of like powers of k_s, the following equations are derived.

$$\frac{d}{dx}\int_0^b h_0(t)\log\left|\frac{t+x}{t-x}\right| dt + 2\int_c^1 \frac{u g_0(u^2)}{u^2 - x^2}\, du = 2q_0, \qquad x \in I_1, I_3 \tag{3.15a, b}$$

$$\frac{d}{dx}\int_0^b h_1(t)\log\left|\frac{t+x}{t-x}\right| dt + 2\int_c^1 \frac{u g_1(u^2)}{u^2 - x^2}\, du = -\frac{4P}{\pi}\left[\int_0^b t h_0(t)\, dt + \int_c^1 u g_0(u^2)\, du\right], \qquad x \in I_1, I_3 \tag{3.16a, b}$$

and

$$\int_c^1 g_i(u^2)\, du = 0 \qquad (i = 0, 1). \tag{3.17a, b}$$

Rewriting equation (3.15a) as

$$\int_0^b h_0(t)\log\left|\frac{t+x}{t-x}\right| dt = \pi F_1(x), \qquad x \in I_1 \tag{3.18}$$

where

$$F_1(x) = -\int_0^x \left[\frac{p_0}{\mu_{12}\theta} + \frac{2}{\pi}\int_c^1 \frac{u g_0(u^2)}{u^2 - y^2}\, du\right] dy.$$

The solution of the integral equation (3.18) with the help of Cook's result [15] is found to be

$$h_0(t) = -\frac{p_0}{\mu_{12}\theta}\frac{t}{(b^2 - t^2)^{1/2}} - \frac{2}{\pi}\frac{t}{(b^2 - t^2)^{1/2}}\int_c^1 \frac{\sqrt{u^2 - b^2}\, g_0(u^2)}{u^2 - t^2}\, du. \tag{3.19}$$

Substitution of the value of $h_0(t)$ from (3.19) in (3.15b) with the aid of the result

$$\int_0^b \frac{1}{(b^2 - t^2)^{1/2}}\frac{t^2\, dt}{(x^2 - t^2)(u^2 - t^2)} = \frac{\pi}{2}\left[\frac{x}{(x^2 - b^2)^{1/2}} - \frac{u}{(u^2 - b^2)^{1/2}}\right], \qquad x \in I_3$$

yields the singular integral equation

$$\int_c^1 \frac{\sqrt{u^2 - b^2}\, g_0(u^2)}{u^2 - x^2}\, du = -\frac{\pi}{2}\frac{p_0}{\mu_{12}\theta}, \qquad x \in I_3. \tag{3.20}$$

Next using the finite Hilbert transform technique [13] the solution of the integral equation is found to be

$$g_0(u^2) = -\frac{p_0}{\mu_{12}\theta}\sqrt{\frac{u^2(u^2 - c^2)}{(u^2 - b^2)(1 - u^2)}} + \frac{u D_1}{\sqrt{(u^2 - b^2)(u^2 - c^2)(1 - u^2)}} \tag{3.21}$$

where D_1 is unknown constant to be determined from equation (3.17a).

Now substituting the value of $g_0(u^2)$ from (3.21) in (3.19) and performing the integrations, $h_0(t)$ is obtained in the following form

$$h_0(t) = -\frac{p_0}{\mu_{12}\theta}\sqrt{\frac{t^2(c^2 - t^2)}{(b^2 - t^2)(1 - t^2)}} + \frac{t D_1}{(b^2 - t^2)(c^2 - t^2)(1 - t^2)}. \tag{3.22}$$

By the procedure similar to one which led to the derivations of the solutions of (3.15) as given

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by (3.21) and (3.22), the solutions of equation (3.16a, b) can also be obtained and they are found to be

$$h_1(t) = -\frac{4PR}{\pi^2}\sqrt{\frac{t^2(c^2-t^2)}{(b^2-t^2)(1-t^2)}} - \frac{tD_2}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}} \tag{3.23}$$

$$g_1(u^2) = -\frac{4PR}{\pi^2}\sqrt{\frac{u^2(u^2-c^2)}{(u^2-b^2)(1-u^2)}} + \frac{uD_2}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}} \tag{3.24}$$

where

$$R = -\frac{p_0}{\mu_{12}\theta}[I_0^b + I_c^1] - D_1[J_0^b - J_c^1]$$

$$I_m^n = \int_m^n \frac{t^2\sqrt{(c^2-t^2)}}{\sqrt{(b^2-t^2)(1-t^2)}}\,dt$$

$$J_m^n = \int_m^n \frac{t^2\,dt}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}}. \tag{3.25}$$

The constant D_2 is to be determined from equation (3.17b).

In order to determine the values of the unknown constants D_1 and D_2, $g_0(u^2)$ and $g_1(u^2)$ as given by (3.21) and (3.24) respectively are substituted in (3.17a, b) and it is found that

$$D_j = A_j\left[(1-b^2)\frac{E}{F} - (c^2-b^2)\right], \qquad (j=1,2) \tag{3.26}$$

and

$$A_1 = \frac{p_0}{\mu_{12}\theta}, \qquad A_2 = \frac{4PR}{\pi^2} \tag{3.27}$$

where $F = F\left(\frac{\pi}{2}, q\right)$ and $E = E\left(\frac{\pi}{2}, q\right)$ are the elliptic integrals of first and second kind respectively and $q = \sqrt{\frac{1-c^2}{1-b^2}}$. Substitution of the values of D_j $(j=1,2)$ given by equations (3.26) in equations (3.21)–(3.24) yields

$$h_{j-1}(t) = -A_j\left[(1-b^2)\frac{E}{F} + (b^2-t^2)\right]\frac{t}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}} \qquad (j=1,2) \tag{3.28}$$

$$g_{j-1}(u^2) = -A_j\left[(1-b^2)\frac{E}{F} - (u^2-b^2)\right]\frac{u}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}} \qquad (j=1,2). \tag{3.29}$$

4. STRESS INTENSITY FACTORS AND CRACK OPENING DISPLACEMENTS

The stress intensity factors are defined as (in physical units)

$$N_b = \lim_{x\to b+}\left[\frac{\sqrt{(x-b)}\,\tau_{yy}(x,0)}{p_0}\right]_{b<x<c} \tag{4.1}$$

$$N_c = \lim_{x\to c-}\left[\frac{\sqrt{(c-x)}\,\tau_{yy}(x,0)}{p_0}\right]_{b<x<c} \tag{4.2}$$

$$N_1 = \lim_{x\to 1+}\left[\frac{\sqrt{(x-1)}\,\tau_{yy}(x,0)}{p_0}\right]_{x>1} \tag{4.3}$$

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and the crack opening displacement can now be shown to be given by

$$\Delta v(x,0) = v(x,0+) - v(x,0-) = 2\int_x^b h(t)\,\mathrm{d}t, \qquad 0 \le x \le b \tag{4.4}$$

$$= 2\int_x^1 g(u^2)\,\mathrm{d}u, \qquad c \le x \le 1. \tag{4.5}$$

Substituting the values of the function $h(t)$ and $g(u^2)$, the stress component τ_{yy} can be evaluated from the expressions (2.13), (2.21) and (3.1). After evaluation of the value of τ_{xy}' and putting it in relations (4.1)–(4.3) it is found that

$$N_b = \sqrt{\frac{b(1-b^2)}{2(c^2-b^2)}}\,\frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)}\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2) \tag{4.6}$$

$$N_c = \sqrt{\frac{c}{2(c^2-b^2)(1-c^2)}}\left[(1-b^2)\frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)} - (c^2-b^2)\right]\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2) \tag{4.7}$$

$$N_1 = \sqrt{\frac{(1-b^2)}{2(1-c^2)}}\left[1 - \frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)}\right]\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2) \tag{4.8}$$

where

$$M_2 = \left[I_0^b + I_c^1 + \left\{(1-b^2)\frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)} - (c^2-b^2)\right\}(J_0^b - J_c^1)\right].$$

Expressions (4.4–4.5) with the aid of the equations (3.28)–(3.29) yield

$$\Delta v(x,0) = \frac{2p_0}{\mu_{12}\theta}\left[\sqrt{(1-b^2)}\,F(\beta,q)\left\{\frac{E(\beta,q)}{F(\beta,q)} - \frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)}\right\} - \sqrt{\frac{(1-x^2)(b^2-x^2)}{(c^2-x^2)}}\right]$$
$$\times\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2), \qquad 0 \le x \le b \tag{4.9}$$

and

$$\Delta v(x,0) = \frac{2p_0}{\mu_{12}\theta}\left[\sqrt{(1-b^2)}\,F(\lambda,q)\left\{\frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)} - \frac{E(\lambda,q)}{F(\lambda,q)}\right\}\right]$$
$$\times\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2), \qquad c \le x \le 1 \tag{4.10}$$

where

$$\sin\beta = \sqrt{\frac{b^2-x^2}{c^2-x^2}} \quad\text{and}\quad \sin\lambda = \sqrt{\frac{1-x^2}{1-b^2}}.$$

000 62371 adj = +28.5pts

When $b \to 0$, we recover the stress intensity factor and the crack opening displacement for two Griffith cracks occupying the region $c \le |x| \le 1$, $y = 0$, $|z| < \infty$:

$$N_c = -\frac{\left[c^2 - \frac{E}{F}\right]}{\sqrt{2c(1-c^2)}}\left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\}k_s^2 \log k_s\right] + O(k_s^2)$$

$$N_1 = -\frac{\left[1 - \frac{E}{F}\right]}{\sqrt{2(1-c^2)}}\left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\}k_s^2 \log k_s\right] + O(k_s^2) \tag{4.11}$$

and

$$\Delta v(x, 0) = \frac{2p_0}{\mu_{12}\theta}\left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\}k_s^2 \log k_s\right] \times \left[\frac{E\left(\frac{\pi}{2}, q\right)}{F\left(\frac{\pi}{2}, q\right)}F(\lambda, q) - E(\lambda, q)\right] + O(k_s^2), \qquad c \le x \le 1 \tag{4.12}$$

where $M_2 = \frac{\pi}{4}(1 + c^2 - 2E/F)$ has been used.

It is noted that if further $c \to 0$, the crack merge into a single crack of width two units. In this case $F \to \infty$ and $M_2 \to \pi/4$; so the results for stress intensity factor and crack opening displacements corresponding to the single crack are found to be

$$N_1 = \frac{1}{\sqrt{2}}\left[1 - \frac{P}{\pi}k_s^2 \log k_s\right] + O(k_s^2) \tag{4.13}$$

and

$$\Delta v(x, 0) = -\frac{2p_0}{\mu_{12}\theta}\sqrt{(1-x^2)}\left[1 - \frac{P}{\pi}k_s^2 \log k_s\right] + O(k_s^2), \qquad 0 \le x \le 1. \tag{4.14}$$

The results given by (4.11)–(4.14) are found to be in agreement with the results of Sarkar *et al.* [16].

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) N_b, N_c and N_1 given by (4.6), (4.7) and (4.8) at the tips of the cracks and crack opening displacements (COD) given by (4.9) and (4.10) have been plotted against dimensionless frequency k_s and distance respectively for three different types of orthotropic materials whose engineering constants have been listed in Table 1.

Table 1. Engineering elastic constants

	E_1 (Pa)	E_2 (Pa)	μ_{12} (Pa)	ν_{12}
Type I	Modulite II graphite–epoxy composite:			
	15.3×10^9	158.0×10^9	5.52×10^9	0.033
Type II	E-Type glass–epoxy composite:			
	9.79×10^9	42.3×10^9	3.66×10^9	0.063
Type III	Stainless steel–aluminium composite:			
	79.76×10^9	85.91×10^9	30.02×10^9	0.31

000 62371 adj = + 21pts

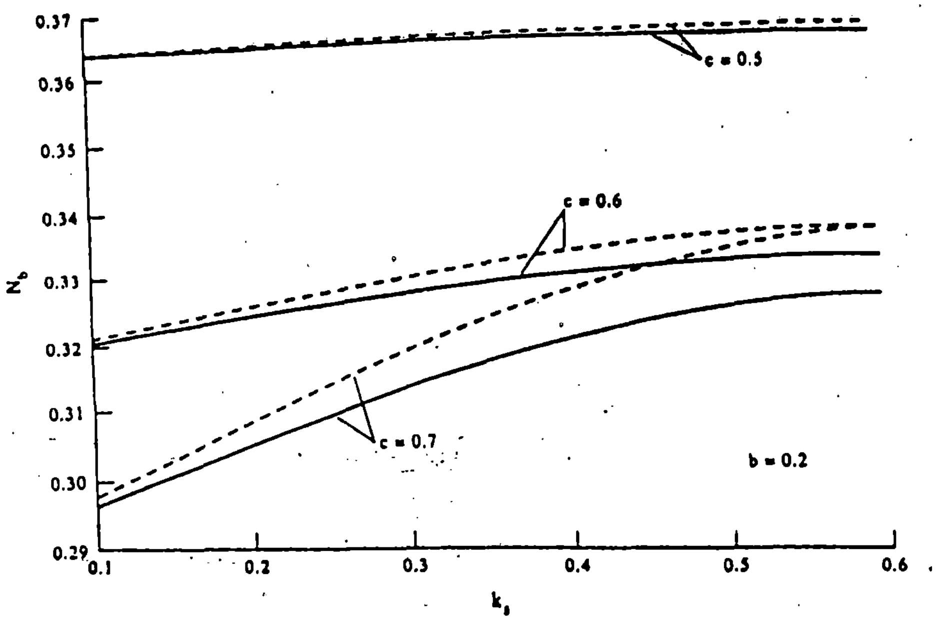

Fig. 2. Stress intensity factor N_b vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

Keeping the length of the central crack fixed ($b = 0.2$) SIFs at the tips of the central and outer cracks have been plotted against frequency k_s ($0.1 \leq k_s \leq 0.6$) for different lengths ($c = 0.5, 0.6, 0.7$) of the outer crack (Figs 2–4). It is noted from the graphs (Figs 2–4) that with the decrease in the value of outer crack length, i.e. with the increase in the value of the distance between inner and outer cracks the rate of increase in the SIF is higher with the increase in the value of the frequency k_s.

The same nature of SIFs are seen (Figs 5–7) in the case when the length of the outer cracks

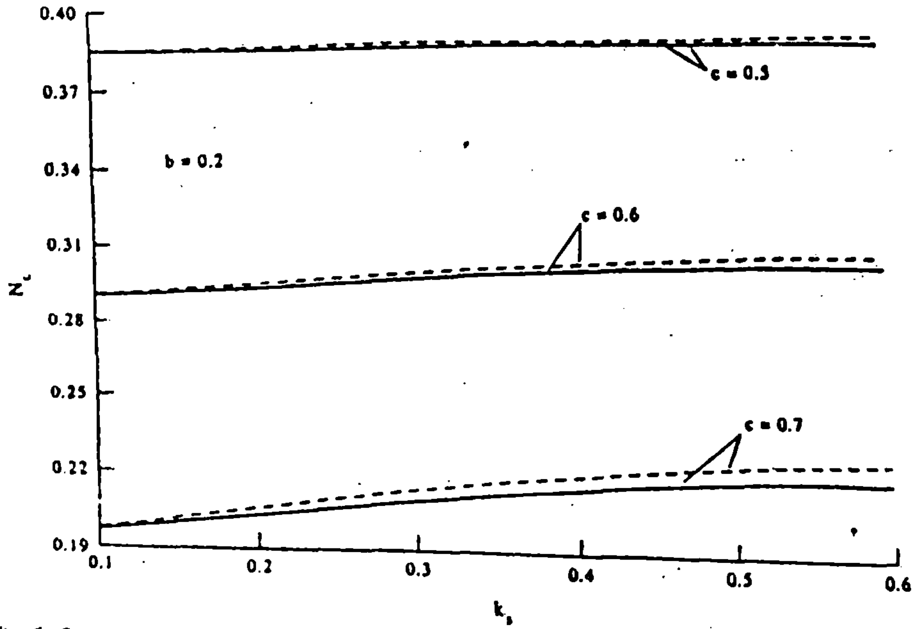

Fig. 3. Stress intensity factor N_c vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

ooo 62371 adj = + 11pts

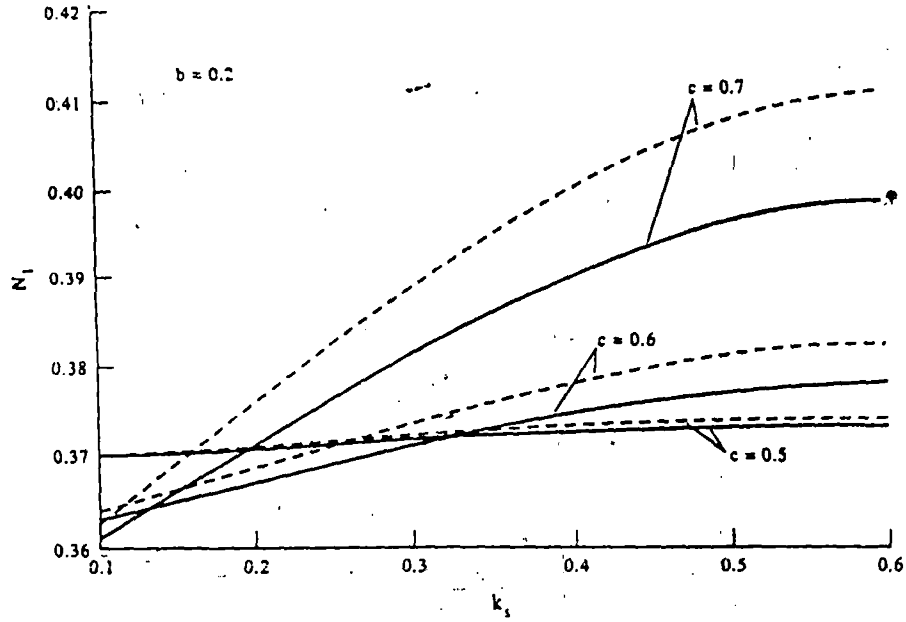

Fig. 4. Stress intensity factor N_1 vs frequency k_s for generalized plane stress. (——) type I; (- - - - -) type III.

are fixed ($c = 0.7$) and the length of the central crack increases ($b = 0.3$, 0.4, 0.5). It is interesting to note that for fixed c ($= 0.7$) the SIFs N_b and N_c increase with the increase in the value of b, but the effect is just reverse in case of N_1.

The COD $\mu_{12}\Delta v(x, 0)/p_0$ has been plotted for different crack lengths. It is found from Figs 8 and 9 that with the increase in the value of crack length the value of COD increases. For a fixed material the variation of COD with frequency is found to be insignificant.

In all the cases where different values of crack length have been considered the variation of COD is found to be prominant for different orthotropic materials.

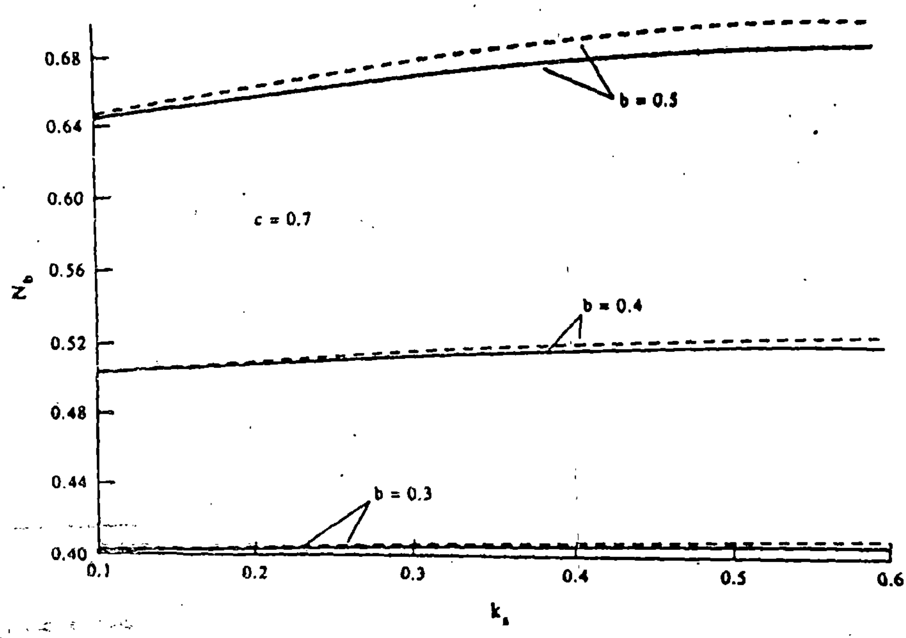

Fig. 5. Stress intensity factor N_b vs frequency k_s for generalized plane stress. (——) type I; (- - - - -) type III.

ooo 62371 adj = + 6pts

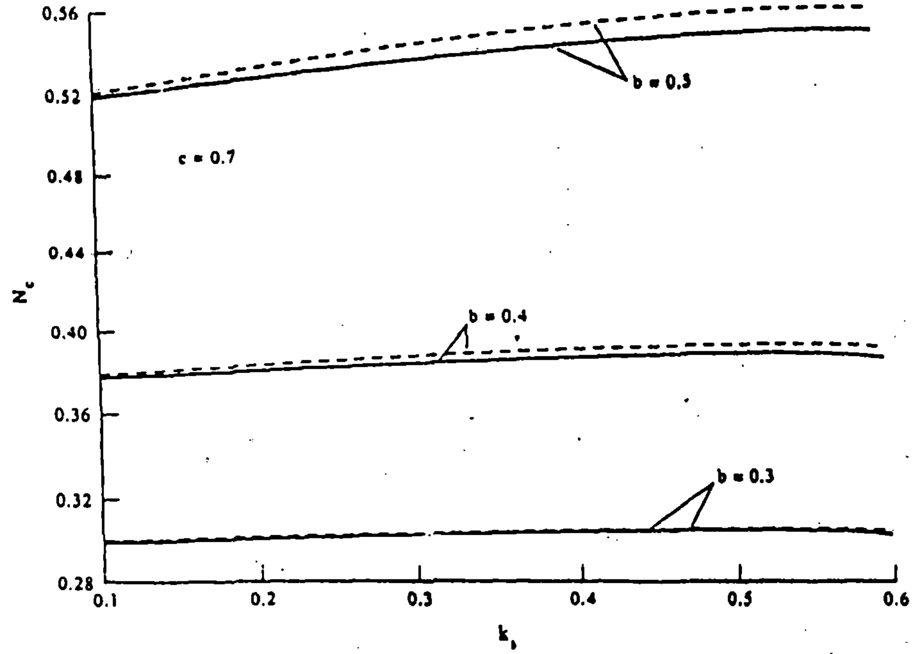

Fig. 6. Stress intensity factor N_c vs frequency k_s for generalized plane stress. (——) type I, (-----) type III.

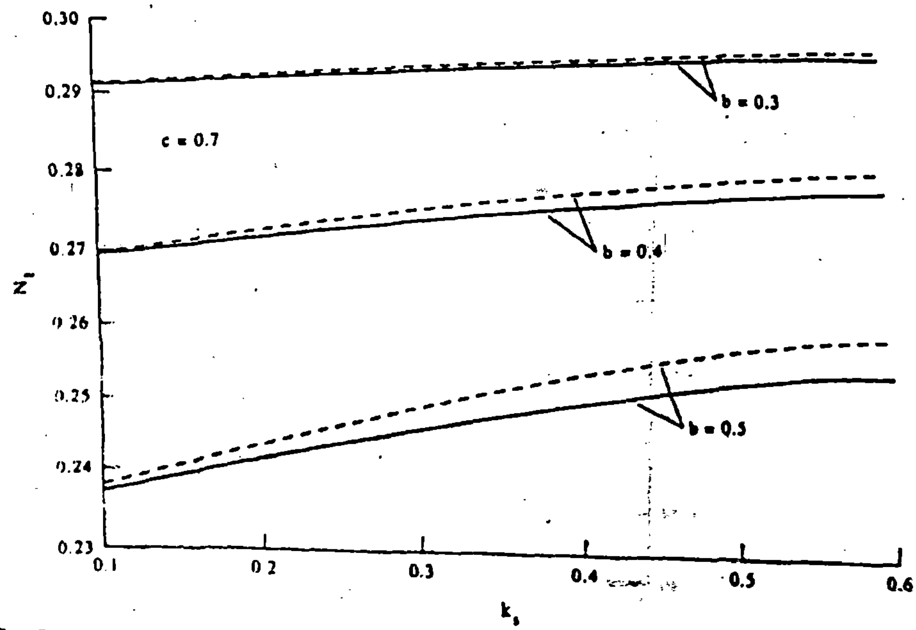

Fig. 7. Stress intensity factor N_1 vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

000 62371 adj = + 4pts

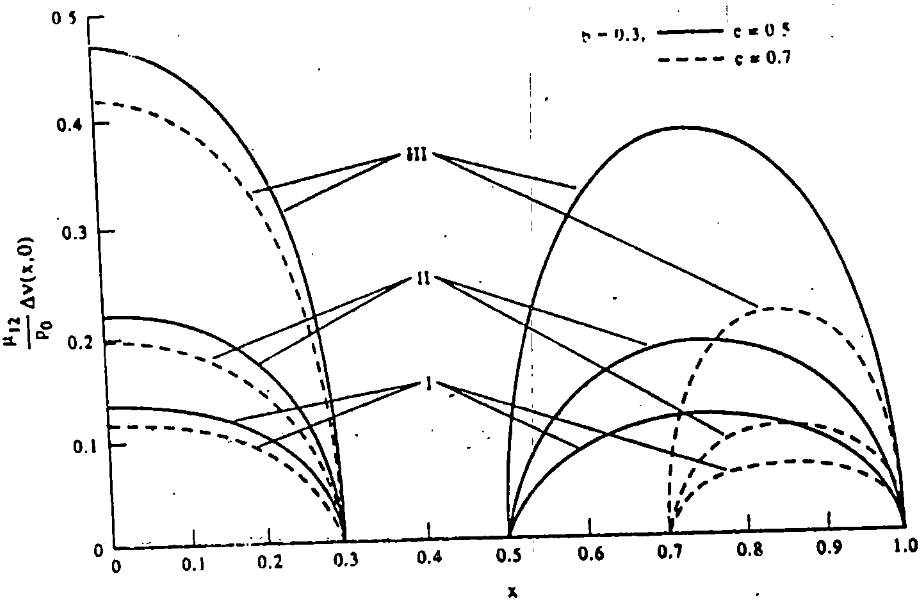

Fig. 8. Crack opening displacement vs distance for generalized plane stress ($k_s = 0.5$, b = 0.3, c = 0.5, 0.7).

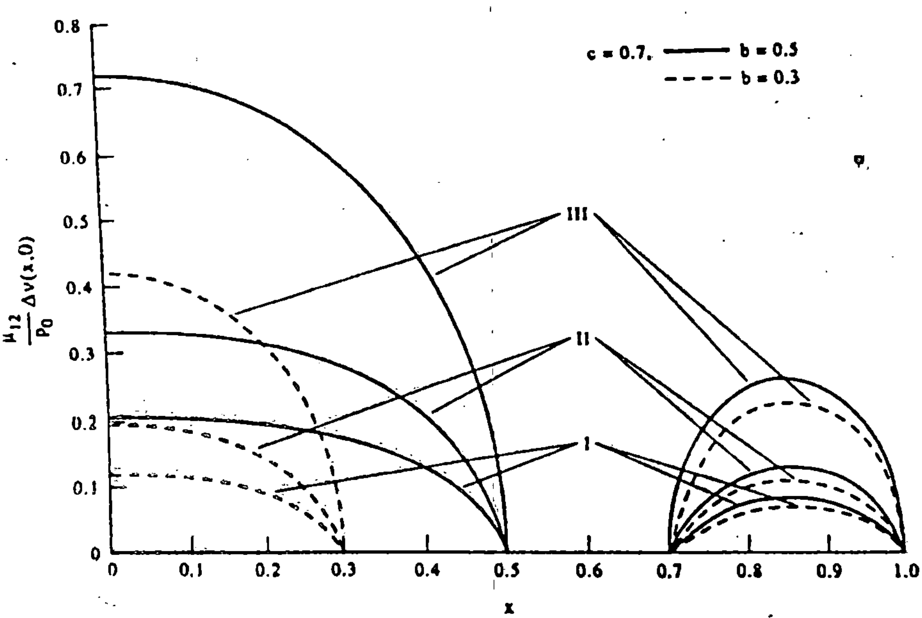

Fig. 9. Crack opening displacement vs distance for generalized plane stress ($k_s = 0.5$, b = 0.3, 0.5, c = 0.7).

000 62371 adj = + 633pts change = + 633pts

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INTERACTION OF ELASTIC WAVES WITH TWO COPLANAR GRIFFITH CRACKS IN AN ORTHOTROPIC MEDIUM

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Abstract—The problem of diffraction of normally incident elastic waves by two coplanar Griffith cracks situated in an infinite orthotropic medium has been analyzed. Fourier and Hilbert transforms have been used to solve this mixed boundary value problem. Approximate analytical results for stress intensity factors and crack opening displacement have been derived when the wave lengths are large compared to the crack length. Numerical values of stress intensity factors and the crack opening displacement for several orthotropic materials have been calculated and plotted graphically to show the effect of material orthotropy.

INTRODUCTION

DYNAMIC fracture problems involving anisotropic materials weakened by crack-like imperfections have drawn much attention by investigators because of the increased usage of macroscopically anisotropic construction materials such as fibre reinforced composites. The different possible location of cracks with respect to the planes of material symmetry introduce great modifications in the strain and stress distribution. The problems are also of considerable interest in seismology and exploration geophysics. The problems involving single or two Griffith cracks in isotropic elastic medium have been studied by many authors [1–6]. Mathematical difficulties encountered in solving the governing equations of the anisotropic elasticity theory are responsible for the availability of few results only for special classes of materials. Kassir and Bandyopadhyay [7] have studied the elastodynamic response of an infinite orthotropic solid containing a crack under the action of impact loading and the elastodynamic problem of a finite Griffith crack in an orthotropic strip under normal impact was investigated by Shindo [8]. The problem involving a moving Griffith crack in an orthotropic strip has also been studied by De and Patra [9]. Recently, Kundu and Bostrom [10] solved the problem of scattering of elastic waves by a circular crack situated in a transversely isotropic solid.

In our paper, the diffraction of normally incident time harmonic elastic waves by two coplanar Griffith cracks in an infinite orthotropic medium has been investigated. The faces of each of the cracks are assumed to be separated by a small distance so that, during small deformations of the solid, the crack faces do not come into contact. The resulting mixed boundary value problem is reduced to the solution of a triple integral equation which has further been reduced to the solution of an integro-differential equation. Iterative solution valid for low frequency has been obtained. Analytical formulae for stress intensity factor and crack opening displacement have been derived. Making the distance between two crack zero, the corresponding results for single crack have been presented. Finally, choosing the engineering elastic constants of the orthotropic material suitably the results for isotropic material have been deduced and compared with the results obtained by Jain and Kanwal [5]. To display the influence of the material orthotropy numerical values of stress intensity factors and crack opening displacement have been plotted for several orthotropic materials.

STATEMENT AND FORMULATION OF THE PROBLEM

Consider the plane problem of diffraction of normally incident longitudinal wave by two symmetrical co-planar Griffith cracks situated in an infinite orthotropic elastic medium. The cracks are assumed to occupy the region $b \leqslant |X| \leqslant a$, $Y = 0$, $|Z| < \infty$. It is convenient to normalize all lengths with respect to "a" and so setting $X/a = x$, $Y/a = y$, $Z/a = z$, $b/a = c$, the new positions of the cracks are defined by $c \leqslant |x| \leqslant 1$, $y = 0$, $|z| < \infty$ (Fig. 1).

Let a plane time harmonic elastic wave originating at $y = -\infty$ be incident normally on the two cracks, and is defined by $v_0 = \exp[i(ky - \omega t)]$ where $k = a\hat{\omega}/c_s\sqrt{c_{22}}$, $c_s = (\mu_{12}/\rho)^{1/2}$ with ρ being the density of the material. In the isotropic solid, c_s represents the velocity of the shear wave.

The non-zero stress components τ_{yy} and τ_{xy} are given by

$$\tau_{yy}/\mu_{12} = c_{12}u_{,x} + c_{22}v_{,y}$$
$$\tau_{xy}/\mu_{12} = u_{,y} + v_{,x}, \tag{1}$$

where u, v denote the component of the displacement in the x, y directions, respectively and comma denotes partial differentiation with respect to the co-ordinates or time; $c_{ij}(i, j = 1, 2)$ are non-dimensional parameters related to the elastic constants by the relations

$$c_{11} = E_1/\mu_{12}(1 - \nu_{12}^2 E_2/E_1)$$
$$c_{22} = E_2/\mu_{12}(1 - \nu_{12}^2 E_2/E_1) = c_{11}E_2/E_1 \tag{2}$$
$$c_{12} = \nu_{12}E_2/\mu_{12}(1 - \nu_{12}^2 E_2/E_1) = \nu_{12}c_{22} = \nu_{21}c_{11}$$

for generalized plane stress, and by

$$c_{11} = (E_1/\Delta\mu_{12})(1 - \nu_{23}\nu_{32})$$
$$c_{22} = (E_2/\Delta\mu_{12})(1 - \nu_{13}\nu_{31})$$
$$c_{12} = E_1(\nu_{21} + \nu_{13}\nu_{32}E_2/E_1)/\Delta\mu_{12} \tag{3}$$
$$= E_2(\nu_{12} + \nu_{23}\nu_{31}E_1/E_2)/\Delta\mu_{12}$$
$$\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{21}\nu_{32}$$

for plane strain. In the above equations E_i, μ_{ij} and ν_{ij} $(i, j = 1, 2, 3)$ denote the engineering elastic constants of the material where the subscripts 1, 2, 3 correspond to the x, y, z directions which coincide with the axes of material orthotropy and the constants E_i and ν_{ij} satisfy the Maxwell's relation

$$\nu_{ij}/E_i = \nu_{ji}/E_j. \tag{4}$$

The equations of motion for orthotropic material, in terms of displacements are

$$c_{11}u_{,xx} + u_{,yy} + (1 + c_{12})v_{,xy} = \frac{a^2}{c_s^2}u_{,tt}$$
$$c_{22}v_{,yy} + v_{,xx} + (1 + c_{12})u_{,xy} = \frac{a^2}{c_s^2}v_{,tt}. \tag{5}$$

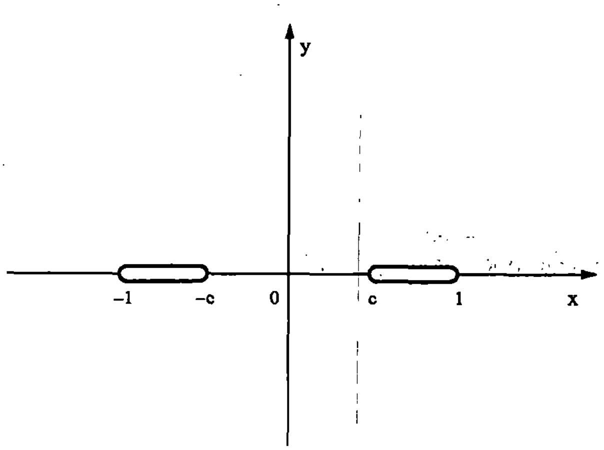

Fig. 1. Geometry of the cracks.

Therefore, substituting $u(x, y, t) = u(x, y)\exp(-i\omega t)$ and $v(x, y, t) = v(x, y)\exp(-i\omega t)$ in eq. (5) we obtain

$$c_{11}u_{,xx} + u_{,yy} + (1 + c_{12})v_{,xy} + k_s^2 u = 0 \tag{6}$$

and

$$c_{22}v_{,yy} + v_{,xx} + (1 + c_{12})u_{,xy} + k_s^2 v = 0$$

where $k_s^2 = a^2\omega^2/c_s^2$.

The boundary conditions of the problem are

$$\tau_{xy}(x, 0) = 0, \quad |x| < \infty \tag{7}$$

$$\tau_{yy}(x, 0) + \tau_{yy}^{(0)}(x, 0) = 0, \quad c \leqslant |x| \leqslant 1 \tag{8}$$

$$v(x, 0) = 0, \quad |x| < c, \quad |x| > 1. \tag{9}$$

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables would be omitted in the sequel.

The solutions of eqs (6) can be taken as

$$u(x, y) = \frac{2}{\pi}\int_0^\infty [A_1(\xi)\exp(-\gamma_1|y|) + A_2(\xi)\exp(-\gamma_2|y|)]\sin\xi x\, d\xi \tag{10}$$

$$v(x, y) = \pm\frac{2}{\pi}\int_0^\infty \frac{1}{\xi}[\alpha_1 A_1(\xi)\exp(-\gamma_1|y|) + \alpha_2 A_2(\xi)\exp(-\gamma_2|y|)]\cos\xi x\, d\xi, \quad y \gtrless 0 \tag{11}$$

where

$$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1 + c_{12})\gamma_i}; \quad i = 1, 2 \tag{12}$$

and $A_i(\xi)(i = 1, 2)$ are the unknown functions to be determined, γ_1^2, γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1 + c_{22})k_s^2\}\gamma^2 + (c_{11}\xi^2 - k_s^2)(\xi^2 - k_s^2) = 0. \tag{13}$$

From the boundary condition (7) it is found that

$$A_2(\xi) = -\beta A_1(\xi) \tag{14}$$

where

$$\beta = \frac{\gamma_1 + \alpha_1}{\gamma_2 + \alpha_2}. \tag{15}$$

Employing eq. (14) the expressions for displacements and stresses reduce to

$$u(x, y) = \frac{2}{\pi}\int_0^\infty [\exp(-\gamma_1|y|) - \beta\exp(-\gamma_2|y|)]A_1(\xi)\sin\xi x\, d\xi, \tag{16}$$

$$v(x, y) = \pm\frac{2}{\pi}\int_0^\infty \frac{1}{\xi}[\alpha_1\exp(-\gamma_1|y|) - \beta\alpha_2\exp(-\gamma_2|y|)]A_1(\xi)\cos\xi x\, d\xi, \quad y \gtrless 0 \tag{17}$$

$$\tau_{xy}/\mu_{12} = \mp\frac{2}{\pi}\int_0^\infty (\gamma_1 + \alpha_1)[\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)]A_1(\xi)\sin\xi x\, d\xi, \quad y \gtrless 0 \tag{18}$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty \left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \beta\left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\right]A_1(\xi)\cos\xi x\, d\xi. \tag{19}$$

We further substitute

$$A(\xi) = \frac{\alpha_1 - \beta\alpha_2}{\xi}A_1(\xi)$$

so that the boundary conditions (8) and (9) yield the following integral equations in $A(\xi)$

$$\int_0^\infty A(\xi)\cos\xi x\, d\xi = 0, \quad |x| < c, \quad |x| > 1 \tag{20}$$

and

$$\int_0^\infty H(\xi)A(\xi)\cos\xi x\,\mathrm{d}\xi = -\frac{\pi p_0}{2\mu_{12}}, \quad c \leqslant |x| \leqslant 1 \tag{21}$$

where $p_0 = ik\mu_{12}c_{22}$
and

$$H(\xi) = \frac{c_{12}\xi^2 - c_{22}\alpha_1\gamma_1 - \beta(c_{12}\xi^2 - c_{22}\alpha_2\gamma_2)}{(\alpha_1 - \beta\alpha_2)}. \tag{22}$$

METHOD OF SOLUTION

In order to solve the set of integral eqs (20) and (21), assume

$$A(\xi) = \frac{1}{\xi}\int_c^1 h(t^2)\sin(\xi t)\mathrm{d}t \tag{23}$$

where $h(t^2)$ is an unknown function to be determined from the boundary conditions.

Inserting the value of $A(\xi)$ from eq. (23) in eq. (20) and using the following result [11]

$$\int_0^\infty \frac{\sin(\xi t)\cos(\xi x)}{\xi}\mathrm{d}\xi = \begin{cases} \frac{\pi}{2}, & t > x \\ 0, & t < x \end{cases}$$

it is found that the choice of $A(\xi)$ leads to the equation

$$\int_c^1 h(t^2)\mathrm{d}t = 0. \tag{24}$$

Further substitution of $A(\xi)$ from eq. (23) in eq. (21) leads to

$$\int_c^1 h(t^2)\mathrm{d}t\int_0^\infty \sin(\xi t)\cos(\xi x)\mathrm{d}\xi = q_0$$
$$-\frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 h(t^2)\mathrm{d}t\int_0^\infty \xi H_1(\xi)\frac{\sin(\xi t)\sin(\xi x)}{\xi^2}\mathrm{d}\xi, \quad c \leqslant |x| \leqslant 1 \tag{25}$$

where

$$q_0 = -\frac{\pi p_0}{2\theta\mu_{12}} \tag{26}$$

$$H_1(\xi) = \frac{H(\xi)}{\xi\theta} - 1 \tag{27}$$

$$\theta = \frac{(c_{12}^2 + c_{12} - c_{11}c_{22})(c_{12}N_1N_2 - c_{11}) - c_{22}[c_{12}N_1^2N_2^2 + c_{11}(N_1^2 + N_1N_2 + N_2^2)]}{c_{11}(1 + c_{12})(N_1 + N_2)} \tag{28}$$

$$N_1^2 = \frac{1}{2c_{22}}[-(c_{12}^2 + 2c_{12} - c_{11}c_{22}) + \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}}]$$

$$N_2^2 = \frac{1}{2c_{22}}[-(c_{12}^2 + 2c_{12} - c_{11}c_{22}) - \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}}]. \tag{29}$$

Using the relation

$$\frac{\sin\xi x\sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_0(\xi w)J_0(\xi v)\mathrm{d}v\,\mathrm{d}w}{(x^2 - w^2)^{1/2}(t^2 - v)^{1/2}} \tag{30}$$

eq. (25) can be rewritten in the following form

$$\int_c^1 \frac{th(t^2)}{t^2 - x^2}\mathrm{d}t = q_0 - \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 h(t^2)\mathrm{d}t\int_0^x\int_0^t \frac{vwL(v,w)\mathrm{d}w\,\mathrm{d}v}{(x^2 - w^2)^{1/2}(t^2 - v^2)^{1/2}}, \quad c \leqslant |x| \leqslant 1 \tag{31}$$

where

$$L(v,w) = \int_0^\infty \xi H_1(\xi)J_0(\xi w)J_0(\xi v)\mathrm{d}\xi. \tag{32}$$

Applying a contour integration technique, the infinite integral in $L(v, w)$ can be converted to the following finite integrals (details given in the Appendix)

$$L(v, w) = -ik_s^2\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - c_{22}\bar{\alpha}_1\bar{\gamma}_1 - \bar{\beta}(c_{12}\eta^2 - c_{22}\bar{\alpha}_2\bar{\gamma}_2)}{\theta(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)} \times J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\mathrm{d}\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\mathrm{d}\eta\right], \quad w > v \tag{33}$$

where

$$\begin{aligned}
\bar{\gamma}_1 &= [\tfrac{1}{2}\{R_1 - (R_1^2 - 4\bar{R}_2)^{1/2}\}]^{1/2}\\
\bar{\gamma}_2 &= [\tfrac{1}{2}\{R_1 + (R_1^2 - 4\bar{R}_2)^{1/2}\}]^{1/2}\\
\hat{\gamma}_1 &= [\tfrac{1}{2}\{-R_1 + (R_1^2 + 4R_2')^{1/2}\}]^{1/2}\\
\hat{\gamma}_2 &= [\tfrac{1}{2}\{R_1 + (R_1^2 + 4R_2')^{1/2}\}]^{1/2}\\
R_1 &= \frac{1}{c_{22}}\{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\eta^2 + (1 + c_{22})\}\\
\bar{R}_2 &= \frac{c_{11}}{c_{22}}(1-\eta^2)\left(\frac{1}{c_{11}} - \eta^2\right)\\
R_2' &= \frac{c_{11}}{c_{22}}(1-\eta^2)\left(\eta^2 - \frac{1}{c_{11}}\right)\\
\bar{\alpha}_i &= \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_i^2}{(1+c_{12})\bar{\gamma}_i}\,(i = 1, 2)\\
\hat{\alpha}_i &= \frac{c_{11}\eta^2 - 1 + (-1)^i\hat{\gamma}_i^2}{(1+c_{12})\hat{\gamma}_i}\,(i = 1, 2)\\
\bar{\beta} &= \frac{\bar{\alpha}_1 - \bar{\gamma}_1}{\bar{\alpha}_2 - \bar{\gamma}_2}\\
\hat{\beta} &= \frac{\hat{\alpha}_1 + \hat{\gamma}_1}{\hat{\alpha}_2 - \hat{\gamma}_2}.
\end{aligned} \tag{34}$$

The corresponding expression of $L(v, w)$ for $w < v$ follows from eq. (33) by interchanging w and v.

Employing the series expansions for the Bessel function J_0 and the Hankel function $H_0^{(1)}$ in eq. (33), it is found that

$$L(v, w) = \frac{2}{\pi}Pk_s^2\log k_s + O(k_s^2) \tag{35}$$

where

$$P = \frac{1}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - c_{22}\bar{\alpha}_1\bar{\gamma}_1 - \bar{\beta}(c_{12}\eta^2 - c_{22}\bar{\alpha}_2\bar{\gamma}_2)}{(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)}\mathrm{d}\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)}\mathrm{d}\eta\right].$$

Now, let us expand $h(t^2)$ in the form

$$h(t^2) = h_0(t^2) + k_s^2\log k_s h_1(t^2) + O(k_s^2). \tag{36}$$

Inserting the above expansion of $h(t^2)$ and the value of $L(v, w)$ given by eq. (35) into eq. (31) and equating the coefficients of like powers of k_s, we obtain the equations

$$\int_c^1 \frac{th_0(t^2)}{t^2 - x^2}\mathrm{d}t = q_0, \quad c \leqslant |x| \leqslant 1 \tag{37}$$

and

$$\int_c^1 \frac{th_1(t^2)}{t^2 - x^2}\mathrm{d}t = -\frac{2P}{\pi}\int_c^1 th_0(t^2)\mathrm{d}t, \quad c \leqslant |x| \leqslant 1. \tag{38}$$

Using the finite Hilbert transform technique [12], the solutions of the above integral equations can be obtained as

$$h_0(t^2) = \frac{2}{\pi} q_0 \left(\frac{t^2 - c^2}{1 - t^2} \right)^{1/2} + \frac{D_1}{\sqrt{(1 - t^2)(t^2 - c^2)}} \tag{39}$$

$$h_1(t^2) = -\frac{2}{\pi} P \left[\frac{q_0(1 - c^2)}{\pi} + D_1 \right] \left(\frac{t^2 - c^2}{1 - t^2} \right)^{1/2} + \frac{D_2}{\sqrt{(1 - t^2)(t^2 - c^2)}}, \tag{40}$$

where D_1 and D_2 are constants to be determined using the condition given by eq. (24) so that

$$\int_c^1 h_0(t^2)\mathrm{d}t = 0 \tag{41}$$

and

$$\int_c^1 h_1(t^2)\mathrm{d}t = 0.$$

Substitution of the values of $h_0(t^2)$ and $h_1(t^2)$ given by eqs (39) and (40) in (41), yields

$$D_1 = \frac{2}{\pi} q_0 \left[c^2 - \frac{E}{F} \right] \tag{42}$$

$$D_2 = \frac{2}{\pi^2} q_0 \left[1 + c^2 - \frac{2E}{F} \right] \left[\frac{E}{F} - c^2 \right], \tag{43}$$

where $F = F[\pi/2, \sqrt{1 - c^2}]$ and $E = E[\pi/2, \sqrt{1 - c^2}]$ are the elliptic integrals of first and second kind, respectively. Substituting the value of D_1 and D_2 given by eqs (42) and (43) into eqs (39) and (40), we obtain

$$h_0(t^2) = -\frac{p_0}{\mu_{12}\theta} \frac{\left[t^2 - \frac{E}{F} \right]}{\sqrt{(1 - t^2)(t^2 - c^2)}} \tag{44}$$

$$h_1(t^2) = -\frac{P p_0}{\pi \mu_{12}\theta} \frac{\left[1 + c^2 - \frac{2E}{F} \right] \left[t^2 - \frac{E}{F} \right]}{\sqrt{(1 - t^2)(t^2 - c^2)}}. \tag{45}$$

CRACK OPENING DISPLACEMENT AND STRESS INTENSITY FACTORS

The crack opening displacement and the normal stress component in the plane of the crack can be written as

$$\Delta v(x, 0) = v(x, 0+) - v(x, 0-) = 2 \int_x^1 h(t^2)\mathrm{d}t, \quad c \leqslant x \leqslant 1 \tag{46}$$

and

$$\tau_{yy}(x, 0) = \frac{2\mu_{12}\theta}{\pi} \int_c^1 \frac{t h(t^2)}{t^2 - x^2} \mathrm{d}t, \quad 0 < x < c \tag{47}$$

$$= -\frac{2\mu_{12}\theta}{\pi} \int_c^1 \frac{t h(t^2)}{x^2 - t^2} \mathrm{d}t, \quad x > 1. \tag{48}$$

Expressions (47) and (48) with the aid of the eqs (36), (44) and (45) yield

$$\tau_{yy}(x, 0) = -p_0 \left[1 + \frac{\left[x^2 - \frac{E}{F} \right]}{\sqrt{(1 - x^2)(c^2 - x^2)}} \right] \left[1 - \frac{P}{\pi} \left\{ 1 + c^2 - \frac{2E}{F} \right\} k_s^2 \log k_s \right] + O(k_s^2), \quad 0 < x < c \tag{49}$$

$$\tau_{yy}(x, 0) = -p_0 \left[1 - \frac{\left[x^2 - \frac{E}{F} \right]}{\sqrt{(x^2 - 1)(x^2 - c^2)}} \right] \left[1 - \frac{P}{\pi} \left\{ 1 + c^2 - \frac{2E}{F} \right\} k_s^2 \log k_s \right] + O(k_s^2), \quad x > 1. \tag{50}$$

The stress intensity factors are defined as (in physical units)

$$K_c = \operatorname*{Lim}_{x \to c-} \left[\frac{\sqrt{(c-x)}\tau_{yy}(x,0)}{p_0} \right]_{0<x<c} \tag{51}$$

$$K_1 = \operatorname*{Lim}_{x \to 1+} \left[\frac{\sqrt{(x-1)}\tau_{yy}(x,0)}{p_0} \right]_{x>1}. \tag{52}$$

Substituting eqs (49) and (50) into (51) and (52) it can be shown that

$$K_c = -\frac{\left[c^2 - \frac{E}{F}\right]}{\sqrt{2c(1-c^2)}} \left[1 - \frac{P}{\pi} \left\{ 1 + c^2 - \frac{2E}{F} \right\} k_s^2 \log k_s \right] + O(k_s^2) \tag{53}$$

$$K_1 = \frac{\left[1 - \frac{E}{F}\right]}{\sqrt{2(1-c^2)}} \left[1 - \frac{P}{\pi} \left\{ 1 + c^2 - \frac{2E}{F} \right\} k_s^2 \log k_s \right] + O(k_s^2). \tag{54}$$

Further substituting eqs (36), (44) and (45) in the expression given by eq. (46), the crack opening displacement is obtained as

$$\Delta v(x,0) = \frac{2p_0}{\mu_{12}\theta} \left[1 - \frac{P}{\pi} \left\{ 1 + c^2 - \frac{2E}{F} \right\} k_s^2 \log k_s \right] \left[\frac{E\left(\frac{\pi}{2}, q\right)}{F\left(\frac{\pi}{2}, q\right)} F(\lambda, q) - E(\lambda, q) \right] + O(k_s^2), \quad c \leqslant x \leqslant 1 \tag{55}$$

where

$$\sin \lambda = \sqrt{\frac{1-x^2}{1-c^2}} \text{ and } q = \sqrt{(1-c^2)}.$$

Letting $c \to 0$ in the expression for stress intensity factor and crack opening displacement, the results for a single crack occupying the region $|x| \leqslant 1, y = 0, |z| < \infty$ are found to be

$$K_1 = \frac{1}{\sqrt{2}} \left[1 - \frac{P}{\pi} k_s^2 \log k_s \right] + O(k_s^2) \tag{56}$$

$$\Delta v(x,0) = -\frac{2p_0}{\mu_{12}\theta} \sqrt{(1-x^2)} \left[1 - \frac{P}{\pi} k_s^2 \log k_s \right] + O(k_s^2), \quad 0 \leqslant x \leqslant 1. \tag{57}$$

For isotropic medium, putting

$$c_{11} = c_{22} = \frac{\lambda + 2\mu}{\mu}, \quad \mu_{12} = \mu, \quad c_{12} = c_{11} - 2 = \frac{\lambda}{\mu}$$

so that

$$\alpha_1 = \gamma_1, \quad \alpha_1 = \xi^2/\gamma_2, \quad k_s = m_2, \quad k_s/\sqrt{c_{11}} = m_1, \quad \tau^2 = \frac{1}{c_{11}}$$

$$N_1 = 1 = N_2, \quad \theta = -2(1-\tau^2) \text{ and } P = \frac{\pi}{2} c_1,$$

where

$$c_1 = \frac{3\tau^4 - 4\tau^2 - 3}{4(1-\tau^2)}, \quad \gamma_i = (\xi^2 - m_i^2)^{1/2} \text{ and } m_i = \frac{a\omega}{c_i} (i = 1, 2)$$

the expressions for displacement and stress are found to be

$$v(x, \pm 0) = \mp \frac{p_0}{2\mu(1-\tau^2)} \left[1 - \frac{c_1}{2} \left\{ 1 + c^2 - \frac{2E}{F} \right\} m_2^2 \log m_2 \right]$$

$$\times \left[\frac{E\left(\frac{\pi}{2}, q\right)}{F\left(\frac{\pi}{2}, q\right)} F(\lambda, q) - E(\lambda, q) \right] + O(m_2^2), \quad c \leqslant x \leqslant 1$$

$$= 0, \quad |x| < c, \quad |x| > 1$$

and

$$\tau_{yy}(x,0) = -p_0\left[1+\frac{\left[x^2-\frac{E}{F}\right]}{\sqrt{(1-x^2)(c^2-x^2)}}\right]\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]+O(m_2^2)\ 0<x<c$$

$$= -p_0,\quad c \leqslant |x| \leqslant 1$$

$$= -p_0\left[1-\frac{\left[x^2-\frac{E}{F}\right]}{\sqrt{(x^2-1)(x^2-c^2)}}\right]\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]+O(m_2^2),\quad |x|>1.$$

Now, the crack opening displacement and stress intensity factors are found to be

$$\Delta v(x,0) = -\frac{p_0}{\mu(1-\tau^2)}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right] \times\left[\frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)}F(\lambda,q)-E(\lambda,q)\right]+O(m_2^2),\quad c\leqslant x\leqslant 1$$

and

$$K_c = -\frac{\left[c^2-\frac{E}{F}\right]}{\sqrt{2c(1-c^2)}}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]+O(m_2^2)$$

$$K_1 = \frac{\left[1-\frac{E}{F}\right]}{\sqrt{2(1-c^2)}}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]+O(m_2^2)$$

which coincide with the results obtained by Jain and Kanwal [5] up to the order of $m_2^2 \log m_2$ in the isotropic case.

When $c \to 0$, we recover the stress intensity factor and crack opening displacement for a single crack

$$K_1 = \frac{1}{\sqrt{2}}\left[1-\frac{c_1}{2}m_2^2\log m_2\right]+O(m_2^2)$$

$$\Delta v(x,0) = \frac{p_0}{\mu(1-\tau^2)}\sqrt{(1-x^2)}\left[1-\frac{c_1}{2}m_2^2\log m_2\right]+O(m_2^2),\quad 0\leqslant x\leqslant 1$$

which agrees with the result of Mal [2].

NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) K_c and K_1 given by eqs (53) and (54) at the inner and outer tips of the cracks, and crack opening displacement (COD) given by eq. (55) have been plotted against dimensionless frequency k_s and distance, respectively for three different types of orthotropic materials whose engineering constants have been listed in Table 1.

From Fig. 2 it is found that SIF K_c at the inner tip of the crack increases at a slow rate with the increase in the value of frequency $k_s (0.1 \leqslant k_s \leqslant 0.6)$. On the other hand the rate of increase of

Table 1. Engineering elastic constants

	E_1 (Pa)	E_2 (Pa)	μ_{12} (Pa)	ν_{12}
Type I	Modulite II graphite–epoxy composite:			
	15.3×10^9	158.0×10^9	5.52×10^9	0.033
Type II	E-Type glass–epoxy composite:			
	9.79×10^9	42.3×10^9	3.66×10^9	0.063
Type III	Stainless steel–aluminium composite:			
	79.76×10^9	85.91×10^9	30.02×10^9	0.31

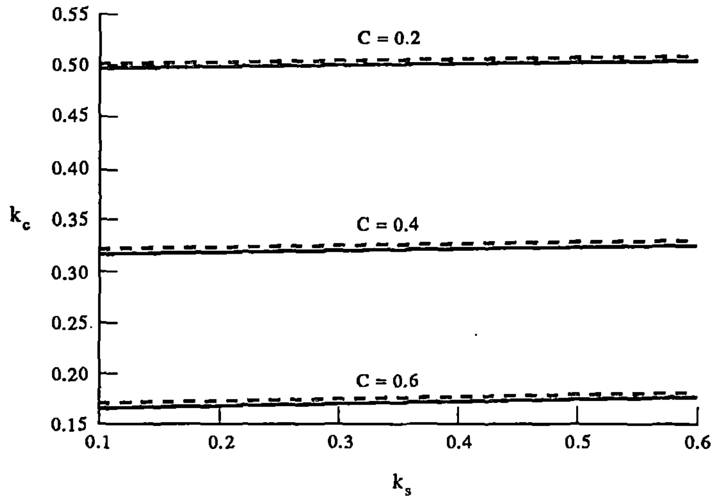

Fig. 2. Stress intensity factor K_c vs frequency k_s for generalized plane stress. (—, Type I; - - - -, Type II).

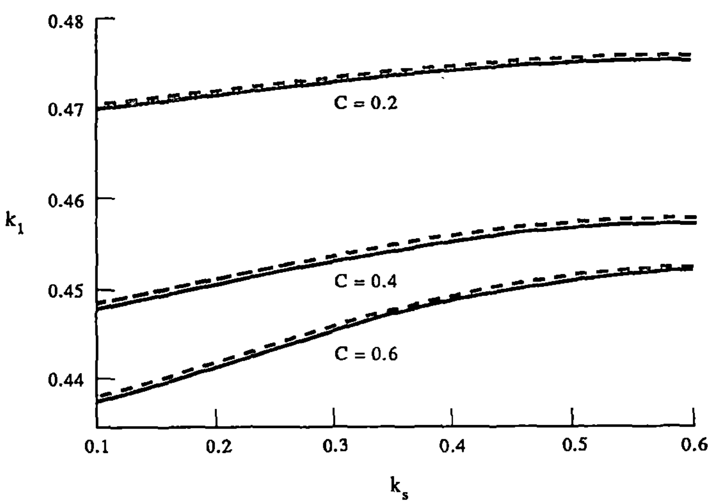

Fig. 3. Stress intensity factor K_I vs frequency k_s for generalized plane stress. (—, Type I; - - - -, Type II).

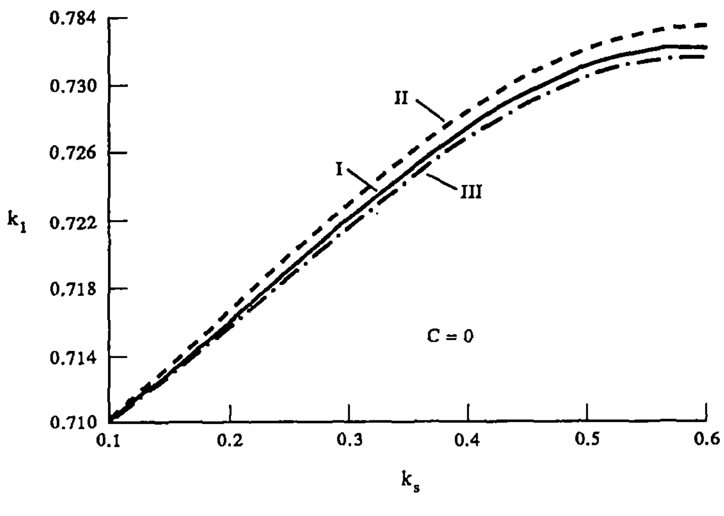

Fig. 4. Stress intensity factor K_I vs frequency k_s for generalized plane stress. (Single crack, $c = 0$).

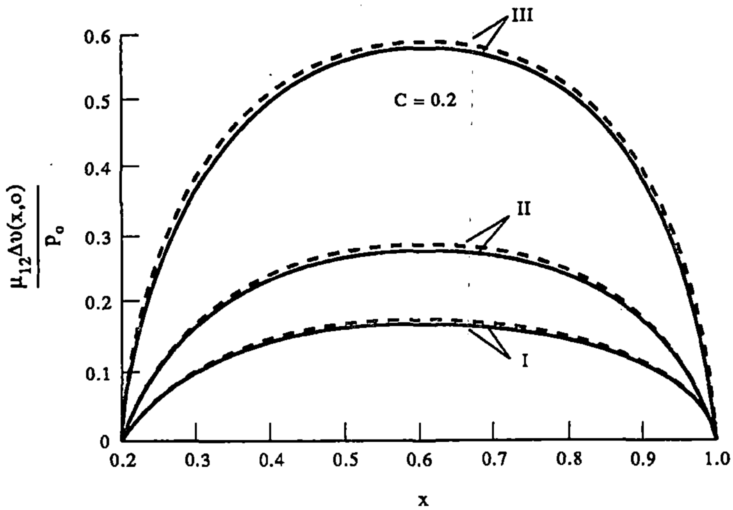

Fig. 5. Crack opening displacement (COD) vs distance ($c = 0.2$) for generalized plane stress. (—, $k_s = 0.2$; ----, $k_s = 0.6$).

the SIF K_I (Fig. 3) with frequency k_s at the outer tip of the crack is found to be higher than that of K_c.

In both the cases the value of SIF is higher for small values of c, i.e. for greater crack length SIF is higher. But it is interesting to note that for different materials the variation of SIFs in both the cases are not significant. In the case of single crack ($c = 0$) the variation of SIF with material properties has been shown in Fig. 4.

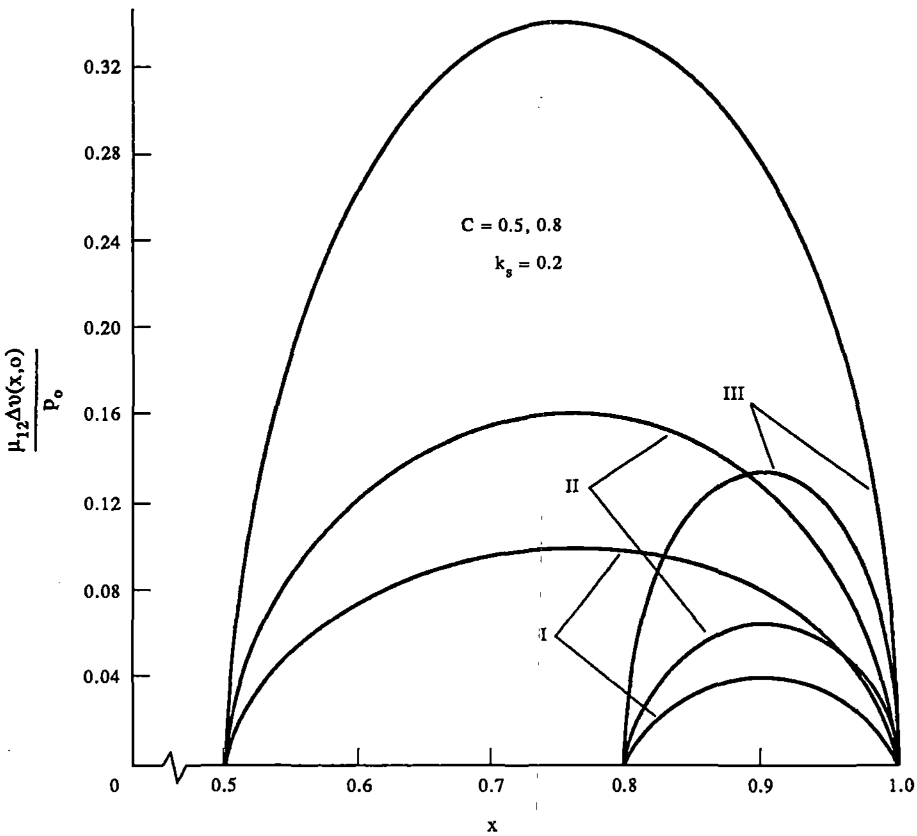

Fig. 6. Crack opening displacement (COD) vs distance ($c = 0.5$ and $c = 0.8$) for generalized plane stress.

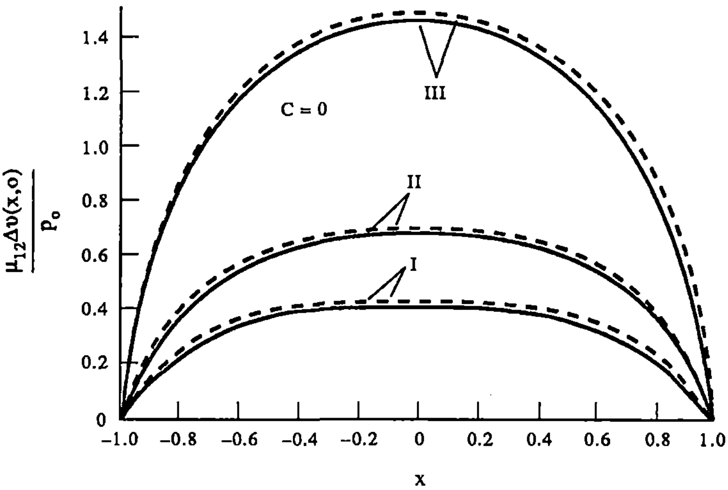

Fig. 7. Crack opening displacement (COD) vs distance (single crack, $c = 0$) for generalized plane stress. (—, $k_s = 0.2$; - - - -, $k_s = 0.6$).

The COD has been plotted for different crack lengths. In each case COD increases gradually from zero, attains maximum value and then decreases to zero. It is found that with the increase in the values of c (i.e. for small crack length) the values of COD decrease (Figs 5 and 6). For a fixed material the variation of COD with frequency is found to be insignificant, but it is noticed that for smaller values of c (Figs 5 and 7) the variation of COD with frequency is palpable; $c = 0$ (Fig. 7) corresponds to the case of single crack.

In all the cases where different values of c have been considered the variation of COD is found to be prominent for different orthotropic materials.

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APPENDIX

Evaluation of L(v, w)

The integral $L(v, w)$ given by eq. (32) is

$$L(v, w) = \int_0^\infty M(\xi, \gamma_1, \gamma_2) J_0(\xi w) J_0(\xi v) d\xi \tag{A1}$$

where

$$M(\xi,\gamma_1,\gamma_2)=\xi H_1(\xi)=\frac{c_{12}\xi^2-c_{22}\alpha_1\gamma_1-\beta(c_{12}\xi^2-c_{22}\alpha_2\gamma_2)}{\theta(\alpha_1+\beta\alpha_2)}-\xi \quad \text{(A2)}$$

$$\gamma_1=[\tfrac{1}{2}\{-B_1+(B_1^2-4B_2)^{1/2}\}]^{1/2}$$

$$\gamma_2=[\tfrac{1}{2}\{-B_1-(B_1^2-4B_2)^{1/2}\}]^{1/2}$$

$$B_1=\frac{1}{c_{22}}\{(c_{12}^2+2c_{12}-c_{11}c_{22})\xi^2+(1+c_{22})k_s^2\} \quad \text{(A3)}$$

$$B_2=\frac{1}{c_{22}}(\xi^2-k_s^2)(c_{11}\xi^2-k_s^2).$$

To evaluate the integral (A1) we consider two contour integrals

$$I_1=\int_{\Gamma_1}M(\xi,\gamma_1,\gamma_2)J_0(\xi v)H_0^{(1)}(\xi w)\mathrm{d}\xi,\quad w>v$$

$$I_2=\int_{\Gamma_2}M(\xi,\gamma_1,\gamma_2)J_0(\xi v)H_0^{(2)}(\xi w)\mathrm{d}\xi,\quad w>v, \quad \text{(A4)}$$

where Γ_1 and Γ_2 are the closed contours defined in Fig. 8, and $H_0^{(1)}$, $H_0^{(2)}$ are the zero order Hankel functions of the first and second kind, respectively.

Assuming the relation

$$\left\{\frac{(c_{12}^2+2c_{12}-c_{11}c_{22})(1+c_{22})}{c_{22}^2}+\frac{2(1+c_{11})}{c_{22}}\right\}^2-\left\{\frac{(c_{12}^2+2c_{12}-c_{11}c_{22})^2}{c_{22}^2}-\frac{4c_{11}}{c_{22}}\right\}\times\left\{\frac{(1+c_{22})^2}{c_{22}^2}-\frac{4}{c_{22}}\right\}<0 \quad \text{(A5)}$$

it is noted that the branch points $\xi=\lambda_i(i=1-4)$ corresponding to the roots of the equation $B_1^2-4B_2=0$ are always complex. Now, the branch points corresponding to the roots of the equations

$$-B_1+(B_1^2-4B_2)^{1/2}=0 \text{ and } -B_1-(B_1^2-4B_2)^{1/2}=0$$

are $\xi=\pm k_s$ and $\xi=\pm k_s/\sqrt{c_{11}}$, respectively where it has been assumed that

$$(c_{11}c_{22}-c_{12}^2-2c_{12})>(1+c_{22}) \quad \text{(A6)}$$

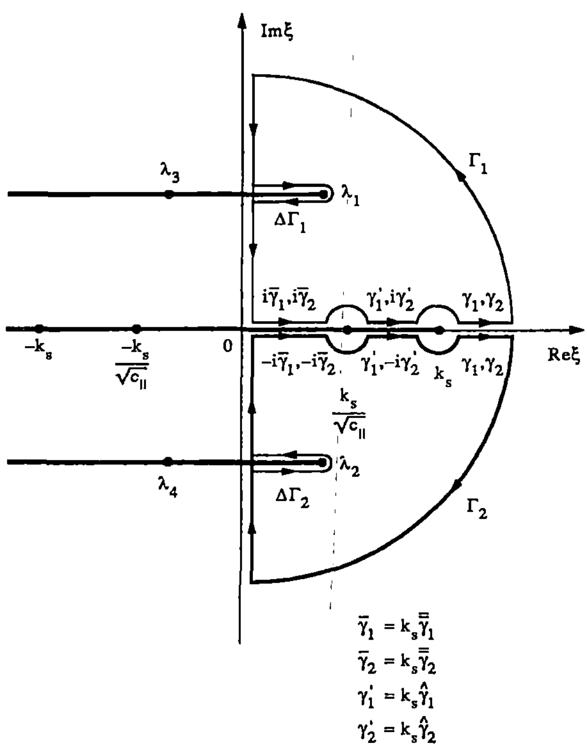

Fig. 8. Contours of integration for integral in eq. (A1).

and

$$c_{12}^2 + 2c_{12} + c_{11} > 0.$$

Therefore under the above conditions, $\xi = \pm k_s/\sqrt{c_{11}}$ and $\xi = \pm k_s$ are the branch points of γ_1 and γ_2, respectively. Equations (A5) and (A6) are true for most of the orthotropic materials. The integrals in eq. (A4) can be shown to be zero on the contours $\Delta\Gamma_1$ and $\Delta\Gamma_2$ (Fig. 8) around the branch cuts from λ_1 and λ_2. Thus integrating along the contours Γ_1 and Γ_2 the integral $L(v, w)$ for $w > v$ can be finally written as

$$L(v, w) = -ik_s^2\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - c_{22}\bar{\alpha}_1\bar{\gamma}_1 - \bar{\beta}(c_{12}\eta^2 - c_{22}\bar{\alpha}_2\bar{\gamma}_2)}{\theta(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)} \times J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\mathrm{d}\eta \right.$$
$$\left. - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\mathrm{d}\eta\right], \quad w > v$$

where $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\beta}, \hat{\beta}, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\gamma}_2$ are given by eq. (34).

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Steady State Propagation of a Series of Parallel Cracks in Anti-Plane State of Strain in an Inhomogeneous Elastic Medium

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Abstract

The problem of a series of semi-infinite, parallel and equally spaced cracks subjected to identical loads satisfying the conditions of anti-plane state of strain and steadily propagating in an infinite inhomogeneous medium has been solved by the application of Wiener-Hopf technique. Elastic moduli and density are assumed to vary exponentially in the direction of propagation of the cracks. The problem of crack propagation in the case of constant strain on the crack edges has been treated. Expressions of the stress and crack opening displacement have been derived in closed form and the effect of the inhomogeneity of the medium has been shown by means of graphs.

1. Introduction

Many authors have studied the dynamic crack propagation in a homogeneous elastic medium. The problem presents an interest for better understanding of the brittle behaviour of materials. Scattering of elastic wave by a single crack has been studied in great detail. But the literature involving the scattering of elastic waves by a series of cracks is very few. It is only recently that Angel and Achenbach [1] studied the reflection and transmission of elastic waves by a periodic array of cracks. Matczynski [2] also considered the quasi static problem of an infinite homogeneous elastic medium weakened by an infinite number of semi-infinite equally spaced parallel cracks. However, natural or artificial materials are

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generally inhomogeneous and propagation of cracks in an inhomogeneous medium has not been studied. Recently, steady state solutions have been derived by Atkinson [3] for crack propagation in media with spatially varying elastic moduli when the crack propagates in a plane where the elastic moduli are constant. Atkinson and List [4] also considered the steady state crack propagation in variable moduli media when the crack moves in the direction of the modulus variation. Steady state crack propagation due to shear waves in a medium of monoclinic type has recently been studied by Chattopadhyay and Bandyopadhyay [5].

In our paper, we have considered the steady state propagation of a series of semi-infinite, rectilinear parallel and uniformly spaced cracks in an infinite inhomogeneous medium. Cracks are assumed to move steadily in the direction of modulus variation, it being assumed that the moduli vary exponentially. We further assume that the medium possesses constant elastic wave speeds. These assumptions are necessary for the steady state solution to exist. We assume that the loading is such that Mode III conditions prevail. Mode III is the simplest mode to analyze mathematically. Nevertheless, it can be expected that the results for the stress intensity factor obtained here will be qualitatively similar to other modes, even though the specific structure of the stress variation near the crack tip will differ in each case. Following Atkinson and List [4], we have also assumed in our paper that the edges of the cracks are loaded on their entire length by constant strain.

2. Formulation of the Problem

Consider an infinite elastic medium with spatially varying density and elastic moduli divided partially by an infinite number of semi-infinite, rectilinear, parallel and uniformly spaced cracks.

The semi-infinite cracks are situated parallel to the negative x_1-axis at $2h$ distance apart and move along positive x_1-direction at a constant velocity $c < c_2$.

The cracks are assumed to propagate steadily in the direction of modulous variation. We assume that the elastic moduli and density both vary exponentially in the same manner; so that the medium may have constant elastic wave speeds.

Owing to symmetry of the problem, it is reduced to the problem of an infinite elastic strip of thickness $2h$ weakened in the middle plane $x_2 = 0$ by a semi-infinite crack $x_1 < 0$, the surfaces $x_2 = \pm h$ of the strip being rigidly clamped.

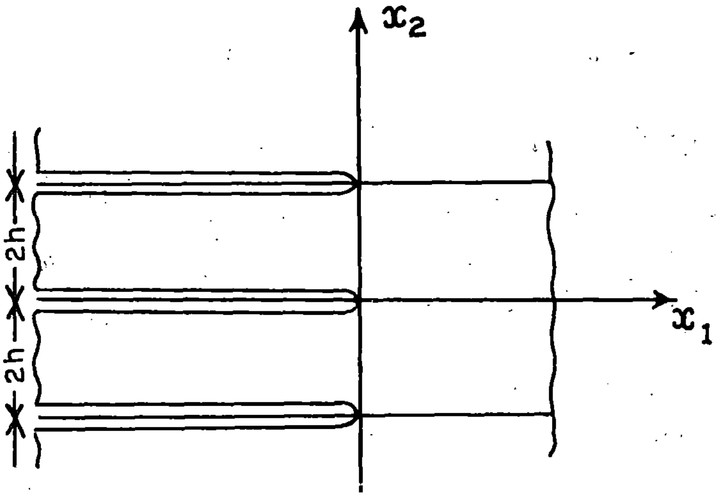


Fig.1

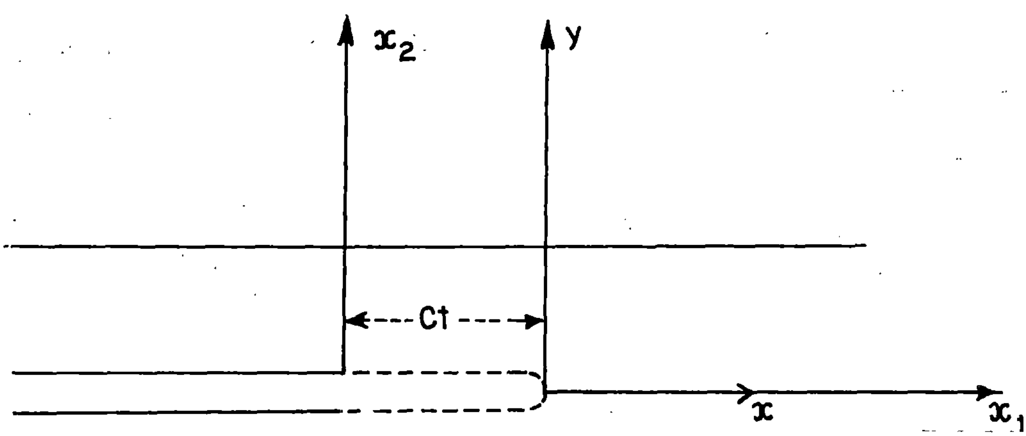


Fig.2

The displacement $\vec{U}$ in the anti-plane state of strain in a rectangular co-ordinate system (x_1, x_2, x_3) is in the form

$$\vec{U} = [0, 0, w(x_1, x_2, t)] \tag{1}$$

The non-vanishing components of this state of strain are given by the following relations:-

$$e_{13} = \frac{\partial w}{\partial x_1}, \qquad e_{23} = \frac{\partial w}{\partial x_2}$$

$$\sigma_{13} = \mu \frac{\partial w}{\partial x_1} = \mu_0 e^{2\alpha x_1} \frac{\partial w}{\partial x_1}, \quad \sigma_{23} = \mu \frac{\partial w}{\partial x_2} = \mu_0 e^{2\alpha x_1} \frac{\partial w}{\partial x_2} \tag{2}$$

where the shear modulous $\mu(x_1) == \mu_0 e^{2\alpha x_1}$, μ_0 and α are constants.

Using relation (2), the equation of motion of SH-waves is

$$\frac{\partial}{\partial x_1}\left[\mu(x_1)\frac{\partial w}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[\mu(x_1)\frac{\partial w}{\partial x_2}\right] = \rho(x_1)\frac{\partial^2 w}{\partial t^2}$$

or,
$$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + 2\alpha \frac{\partial w}{\partial x_1} = c_2^{-2}\frac{\partial^2 w}{\partial t^2} \tag{3}$$

where $\rho(x_1) = \rho_0 e^{2\alpha x_1}$; so $c_2 = \sqrt{\mu(x_1)/\rho(x_1)} = \sqrt{\mu_0/\rho_0}$ is the shear wave velocity.

The fixed coordinate system may be replaced by the conventional system (x, y, z) moving with the crack tip.

$$x_1 = x + ct, \quad x_2 = y, \quad x_3 = z \tag{4}$$

Using relation (4), equation (3) becomes

$$\left(1 - \frac{c^2}{c_2^2}\right)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 2\alpha\frac{\partial w}{\partial x} = 0 \tag{5}$$

Applying complex Fourier transform in x, equation (5) becomes

$$\frac{d^2\overline{W}}{dy^2} - \beta^2 \overline{W} = 0 \tag{6}$$

where $\beta^2 = \left(1 - \frac{c^2}{c_2^2}\right) \zeta^2 + 2 i \alpha \zeta$ (7.1)

and $\overline{W}(\zeta, y) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} w(x,y)\, e^{i\zeta x}\, dx$ (7.2)

The solution of equation (6) becomes

$$\overline{W}(\zeta, y) = A \sinh(\beta y) + B \cosh(\beta y) \tag{8}$$

where the constants A and B are to be determined.

3. Solution of the problem for constant strain $\frac{\partial w}{\partial y} = P$ of the crack edges $x < 0$

We now consider the problem when the constant strain given by

$$\frac{\partial w}{\partial y} = P \tag{9}$$

is applied to the crack face $y = 0$, $x < 0$.

We shall therefore consider the steady state crack propagation under the boundary conditions.

$$\frac{\partial w}{\partial y} = P, \qquad \text{for } x < 0, y = 0 \tag{10.1}$$

$$w(x,y) = 0, \qquad \text{for } x > 0, y = 0 \tag{10.2}$$

$$w(x,y) = 0, \qquad \text{for } |x| < \infty, y = h \tag{10.3}$$

Now we can write

$$\frac{\partial w}{\partial y} = P, \qquad \text{for } x < 0, y = 0$$

$$= e(x), \qquad \text{for } x > 0, y = 0$$

where $e(x)$ is the unknown function which is to be detemined.

In our case

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{\partial w}{\partial y} e^{i\zeta x} dx = (2\pi)^{-1/2} \int_{-\infty}^{0} \frac{\partial w}{\partial y} e^{i\zeta x} dx + (2\pi)^{-1/2} \int_{0}^{\infty} \frac{\partial w}{\partial y} e^{i\zeta x} dx$$

$$\frac{\partial \overline{W}}{\partial y}(\zeta, 0) = (2\pi)^{-1/2} \int_{-\infty}^{0} P\, e^{i\zeta x} dx + (2\pi)^{-1/2} \int_{0}^{\infty} e(x)\, e^{i\zeta x} dx \tag{11}$$

Therefore using (8) and writing $(2\pi)^{-1/2} \int_{0}^{\infty} e(x)\, e^{i\zeta x} dx = E_{+}(\zeta)$,

$$\beta A = (2\pi)^{-1/2} \frac{P}{i\zeta} + E_{+}(\zeta) \qquad \text{for } -k < \operatorname{Im} \zeta < 0 \tag{12}$$

$$\text{if } e(x) \sim O(e^{-kx}) \quad \text{as} \, x \to \infty$$

Using the conditions (10.2) and (10.3), it can be easily shown that

$$A = -\frac{\overline{W}_{-}(\zeta, 0)}{\tanh(\beta h)} \tag{13}$$

where $\overline{W}_{-}(\zeta, 0) = (2\pi)^{-1/2} \int_{-\infty}^{0} w(x, 0)\, e^{i\zeta x} dx$ is analytic in the lower half-plane $\operatorname{Im} \zeta < k_1$, if we assume $w(x, 0) \sim O(e^{k_1 x})$ as $x \to -\infty$.

Eliminating A by equations (12) and (13)

$$-\beta \frac{\overline{W}_{-}(\zeta, 0)}{\tanh(\beta h)} = \frac{-iP}{\sqrt{2\pi}} \frac{1}{\zeta} + E_{+}(\zeta) \tag{14}$$

Let $$K(\zeta) = \beta \coth(\beta h) = \frac{1}{h} \beta h \frac{\cosh(\beta h)}{\sinh(\beta h)} = \frac{1}{h} \prod_{n=1}^{\infty} \left\{ \frac{1 - \left(\frac{i\beta h}{\pi(n - 1/2)} \right)^2}{1 - \left(\frac{i\beta h}{n\pi} \right)^2} \right\} \tag{15}$$

[cf. Noble [8], eqns. (3.96a) and (3.96b), p.123]

Now consider

$$1-\left(\frac{i\beta h}{n\pi}\right)^2 = 1+\left(\frac{\beta h}{n\pi}\right)^2 = \left(\frac{h}{n\pi}\right)^2\left[\nu^2\zeta^2+2i\alpha\zeta+\left(\frac{n\pi}{h}\right)^2\right]$$

$$= \left(\frac{\nu h}{n\pi}\right)^2\left[\zeta^2+\frac{2i\alpha\zeta}{\nu^2}+\left(\frac{n\pi}{\nu h}\right)^2\right] \tag{16}$$

where $\nu^2 = 1 - c^2/c_2^2$.

So equation (16) can be written as

$$1-\left(\frac{i\beta h}{n\pi}\right)^2 = \left(\frac{\nu h}{n\pi}\right)^2 (\zeta + i\eta_n^+)(\zeta + i\eta_n^-)$$

where $\quad \eta_n^{\pm} = \frac{\alpha}{\nu^2} \pm \left[\frac{\alpha^2}{\nu^4}+\left(\frac{n\pi}{\nu h}\right)^2\right]^{1/2}$

Similarly, $1-\left(\frac{i\beta h}{(n-1/2)\pi}\right)^2 = \left(\frac{\nu h}{n\pi}\right)^2 (\zeta+i\eta_{n-1/2}^+)(\zeta+i\eta_{n-1/2}^-)$

It may be noted that η_n^- and $\eta_{n-1/2}^-$ are negative real quantities.

So equation (15) becomes

$$K(\zeta) = \frac{1}{h}\prod_{n=1}^{\infty}\frac{(\zeta+i\eta_{n-1/2}^-)(\zeta+i\eta_{n-1/2}^+)\,n^2}{(\zeta+i\eta_n^-)(\zeta+i\eta_n^+)(n-1/2)^2}$$

$$= \frac{1}{h}\prod_{n=1}^{\infty}\frac{(\zeta+i\eta_{n-1/2}^-)}{(\zeta+i\eta_n^-)}\frac{n}{(n-1/2)}\cdot\prod_{n=1}^{\infty}\frac{(\zeta+i\eta_{n-1/2}^+)}{(\zeta+i\eta_n^+)}\frac{n}{(n-1/2)}$$

$$= K^-(\zeta)\,.K^+(\zeta) \qquad \text{(say)} \tag{17}$$

where $K^-(\zeta)$ is analytic in the lower half-plane given by $\operatorname{Im}\zeta < -\eta_{1/2}^-$ where as $K^+(\zeta)$ is analytic in the upper half plane given by $\operatorname{Im}\zeta > -\eta_{1/2}^+$.

Now $$K^{+}(\zeta)=\prod_{n=1}^{\infty}\frac{(\zeta+i\eta^{+}_{n-1/2})}{(\zeta+i\eta^{+}_{n})}\frac{(n-0)}{(n-1/2)}$$

$$=\prod_{n=1}^{\infty}\frac{\left[\zeta+i\left(\frac{\alpha}{\nu^2}+\left(\frac{\alpha^2}{\nu^4}+\frac{(n-1/2)^2\pi^2}{\nu^2h^2}\right)^{1/2}\right)\right](n-0)}{\left[\zeta+i\left(\frac{\alpha}{\nu^2}+\left(\frac{\alpha^2}{\nu^4}+\frac{n^2\pi^2}{\nu^2h^2}\right)^{1/2}\right)\right](n-1/2)}$$

$$=\prod_{n=1}^{\infty}\frac{\left[\frac{\zeta\nu h}{\pi}+i\left(\frac{\alpha h}{\nu\pi}+\left\{\frac{\alpha^2h^2}{\nu^2\pi^2}+(n-1/2)^2\right\}^{1/2}\right)\right](n-0)}{\left[\frac{\zeta\nu h}{\pi}+i\left(\frac{\alpha h}{\nu\pi}+\left\{\frac{\alpha^2h^2}{\nu^2\pi^2}+n^2\right\}^{1/2}\right)\right](n-1/2)}$$

Now elastic moduli and density are assumed to be varying slowly with x_1 so that αh may be assumed to be small.

So neglecting α^2h^2 we get

$$K^{+}(\zeta)=\prod_{n=1}^{\infty}\frac{\left[\frac{\zeta\nu h}{\pi}+i\left(\frac{\alpha h}{\nu\pi}+(n-1/2)\right\}\right](n-0)}{\left[\frac{\zeta\nu h}{\pi}+i\left\{\frac{\alpha h}{\nu\pi}+n\right\}\right](n-1/2)}$$

$$=\prod_{n=1}^{\infty}\frac{\left[n-\left(\frac{1}{2}+\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right)\right](n-0)}{\left[n-\left(\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right)\right](n-1/2)} \tag{18}$$

Next using the formula

$$\prod_{n=1}^{\infty}\left\{\frac{(n-a_1)\,\ldots\ldots\ldots\ldots\,(n-a_k)}{(n-b_1)\,\ldots\ldots\ldots\ldots\,(n-b_k)}\right\}=\prod_{m=1}^{k}\frac{\Gamma(1-b_m)}{\Gamma(1-a_m)}$$

which expresses the general infinite product in terms of the Gamma functions (cf. Whittaker and Watson [6], p.239)) we obtain from (18)

$$K^{+}(\zeta)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left[1-\left(\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right)\right]}{\Gamma(1)\,\Gamma\left[\frac{1}{2}-\left(\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right)\right]} \tag{19}$$

Similarly, for small values of αh, neglecting $\alpha^2 h^2$, it can be easily shown that

$$K^{-}(\zeta)=\frac{\sqrt{\pi}}{h}\frac{\Gamma\left[1+\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[\frac{1}{2}+\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right]} \tag{20}$$

Now writing $\beta\coth(\beta h)=K(\zeta)=K^{+}(\zeta)K^{-}(\zeta)$, equation (14) becomes

$$-K^{+}(\zeta)\;K^{-}(\zeta)\;\overline{W}_{-}(\zeta,0)=-\frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}+E_{+}(\zeta)$$

so, $$-K^{-}(\zeta)\,\overline{W}_{-}(\zeta,0)=-\frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\frac{1}{K^{+}(\zeta)}+\frac{E_{+}(\zeta)}{K^{+}(\zeta)}$$

$$=-\frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\left[\frac{1}{K^{+}(\zeta)}-\frac{1}{K^{+}(0)}\right]-\frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\frac{1}{K^{+}(0)}+\frac{E_{+}(\zeta)}{K^{+}(\zeta)}$$

Therefore,

$$-K^{-}(\zeta)\,\overline{W}_{-}(\zeta,0)+\frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\frac{1}{K^{+}(0)}=\frac{E_{+}(\zeta)}{K^{+}(\zeta)}-\frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\left[\frac{1}{K^{+}(\zeta)}-\frac{1}{K^{+}(0)}\right] \tag{21}$$

The expression on the left hand side of equation (21) is regular in the half-plane $\operatorname{Im}\zeta<0$ whereas R.H.S. is regular in $\operatorname{Im}\zeta>-K_1$ where $K_1=\min(k,\eta^{+}_{1/2})$. The equation (21) holds in the strip $-K_1<\operatorname{Im}\zeta<0$ and therefore using analytic continuation and Liouville's theorem we can write

$$\overline{W}_{-}(\zeta,0)=\frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\frac{1}{K^{+}(0)K^{-}(\zeta)} \tag{22}$$

and $$E_{+}(\zeta)=\frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\left[1-\frac{K^{+}(\zeta)}{K^{+}(0)}\right] \tag{23}$$

Therefore, by help of (11) and (23), we obtain

$$\frac{\partial\overline{W}}{\partial y}(\zeta,0)=-\frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\frac{K^{+}(\zeta)}{K^{+}(0)}$$

So, $$\frac{\partial w}{\partial y}=-\frac{iP}{\sqrt{2\pi}}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon}\frac{1}{\zeta}\frac{K^{+}(\zeta)}{K^{+}(0)}e^{-i\zeta x}\,d\zeta \quad \text{where } -K_1<\varepsilon<0 \tag{24}$$

For $x<0$, considering a semi-circular contour in the upper half ζ-plane it can easily be verified that

$$\frac{\partial w}{\partial y}=P$$

Now for $x>0$, substituting the values of $K^{+}(\zeta)$ and $K^{+}(0)$ from (19) and (24) we obtain

$$\frac{\partial w}{\partial y}=-\frac{iP}{2\pi}\frac{\Gamma\left[\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{\alpha h}{\nu\pi}\right]}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon}\frac{1}{\zeta}\frac{\Gamma\left[1-\frac{i\zeta\nu h}{\pi}+\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[\frac{1}{2}-\frac{i\zeta\nu h}{\pi}+\frac{\alpha h}{\nu\pi}\right]}e^{-i\zeta x}\,d\zeta \quad (x>0)$$

$$=-\frac{iP}{2\pi}\frac{\Gamma\left[\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{\alpha h}{\nu\pi}\right]}e^{-\frac{\pi x}{\nu h}\left(\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right)}\int_{-i\infty+s}^{i\infty+s}\frac{1}{\left(\frac{1}{2}-p+\frac{\alpha h}{\nu\pi}\right)}\times\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p)}e^{\frac{\pi x}{\nu h}p}\,dp$$

where $s=\frac{1}{2}+\frac{\alpha h}{\nu\pi}-\frac{\nu h\varepsilon}{\pi}$

$$=\frac{iP}{2\pi}\frac{\Gamma\left[\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{\alpha h}{\nu\pi}\right]}e^{-\frac{\pi x}{\nu h}\left(\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right)}\int_{-i\infty+s}^{i\infty+s}\frac{\Gamma\left(p-\frac{1}{2}-\frac{\alpha h}{\nu\pi}\right)}{\Gamma\left(p+\frac{1}{2}-\frac{\alpha h}{\nu\pi}\right)}\times\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)}e^{\frac{\pi x}{\nu h}p}\,dp$$

$$= -P \frac{\Gamma\left[\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{\alpha h}{\nu\pi}\right]} e^{-\frac{\pi x}{\nu h}\left(\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right)} \frac{e^{\frac{\pi x}{\nu h}\left(\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right)}\left(1-e^{-\frac{\pi x}{\nu h}}\right)^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}$$

$$\times {}_2F_1\left(-\frac{1}{2}; -\frac{\alpha h}{\nu\pi}; \frac{1}{2}; 1-e^{-\frac{\pi x}{\nu h}}\right)$$

[cf. Erdelyi et. al. [7], formula no. 7. p.262]

$$= -\frac{P}{\sqrt{\pi}} \frac{\Gamma\left[\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{\alpha h}{\nu\pi}\right]} \frac{1}{\sqrt{1-\exp(-\pi x/\nu h)}} {}_2F_1\left(-\frac{1}{2}; -\frac{\alpha h}{\nu\pi}; \frac{1}{2}; 1-e^{-\frac{\pi x}{\nu h}}\right)$$

where ${}_2F_1\left(-\frac{1}{2}; -\frac{\alpha h}{\nu\pi}; \frac{1}{2}; 1-e^{-\frac{\pi x}{\nu h}}\right)$ is the hypergemotric function.

It is known that the series

$$ {}_2F_1(a,b,c,z) = 1 + \frac{a.b}{1.c} z + \frac{a(a+1)\,b(b+1)}{1.2.c(c+1)} z^2 + $$

$$ + \frac{a(a+1)\,(a+2)\,b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)} z^3 + \ldots\ldots\ldots\ldots $$

therefore neglecting $\left(\frac{\alpha h}{\nu\pi}\right)^2$ and higher power of $\frac{\alpha h}{\nu\pi}$,

$$ {}_2F_1\left(-\frac{1}{2}; -\frac{\alpha h}{\nu\pi}; \frac{1}{2}; 1-e^{-\frac{\pi x}{\nu h}}\right) = 1 + \frac{\alpha h}{\nu\pi}\left(\frac{z}{1} + \frac{z^2}{2.3} + \frac{z^3}{3.4} + \frac{z^4}{4.7} + \ldots\ldots\right) $$

where $z = 1 - e^{-\frac{\pi x}{\nu h}}$;

After a little algebraic simplification it can be shown that for small $\frac{\alpha h}{\nu\pi}$

$$ {}_2F_1\left(-\frac{1}{2}; -\frac{\alpha h}{\nu\pi}; \frac{1}{2}; z\right) = 1 + \frac{\alpha h}{\nu\pi}\left[(1+\sqrt{z})\log(1+\sqrt{z}) + (1-\sqrt{z})\log(1-\sqrt{z})\right] $$

Therefore

$$\frac{\partial w}{\partial y} = -\frac{P}{\sqrt{\pi}} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \frac{1}{\sqrt{1 - \exp(-\pi x / \nu h)}}$$

$$\times \left\{1 + \frac{\alpha h}{\nu h}\left[(1 + \sqrt{z}) \log (1 + \sqrt{z}) + (1 - \sqrt{z}) \log (1 - \sqrt{z})\right]\right\} \quad (x > 0) \qquad (25)$$

Next in order to determine the crack opening displacement consider equation (22) viz.

$$\overline{W}_-(\zeta, 0) = \frac{iP}{\sqrt{2\pi}} \frac{1}{\zeta} \frac{1}{K^+(0) K^-(\zeta)}$$

which by help of equations (19) and (20) becomes

$$\overline{W}_-(\zeta, 0) = \frac{iP}{\sqrt{2\pi}} \frac{h}{\pi} \frac{\Gamma\left[\frac{1}{2} + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \frac{1}{\zeta}$$

Therefore

$$w(x, 0) = \frac{ihP}{\pi} \frac{1}{2\pi} \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} \frac{\Gamma\left[\frac{1}{2} + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \frac{1}{\zeta} e^{i\zeta x} \, d\zeta$$

Obviously for $x > 0, w(x, 0) \equiv 0$. In order to find $w(x, 0)$ for $x < 0$, we firstly evaluate $\frac{dw\,(x, 0)}{dx}$ which is given by

$$\frac{dw}{dx} = \frac{hP}{\pi} \frac{1}{2\pi} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} \frac{\Gamma\left[\frac{1}{2} + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]} e^{-i\zeta x} \, d\zeta$$

so,
$$\frac{dw}{dx} = \frac{P\, e^{\frac{\pi x}{\nu h}\left(\frac{1}{2} - \frac{\alpha h}{\nu\pi}\right)}}{\nu\sqrt{\pi}} \frac{1}{2\pi i} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \int_{s - i\infty}^{s + i\infty} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(p)}{\Gamma(p + \frac{1}{2})} e^{-\frac{\pi x}{\nu h} p} \, dp$$

where $p = \frac{1}{2} + \frac{i\nu\zeta h}{\pi} - \frac{\alpha h}{\nu\pi}$ and $s = \frac{1}{2} - \frac{\alpha h}{\nu\pi} + \frac{\nu h\varepsilon}{\pi}$

Using the table of inverse Laplace transform [7], we find

$$\frac{dw}{dx} = \frac{P\, e^{\frac{\pi x}{\nu h}\left(\frac{1}{2} - \frac{\alpha h}{\nu\pi}\right)}}{\nu\sqrt{\pi}} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \frac{1}{\sqrt{1 - \exp(\pi x/\nu h)}}$$

Integrating w.r.t. x we obtain

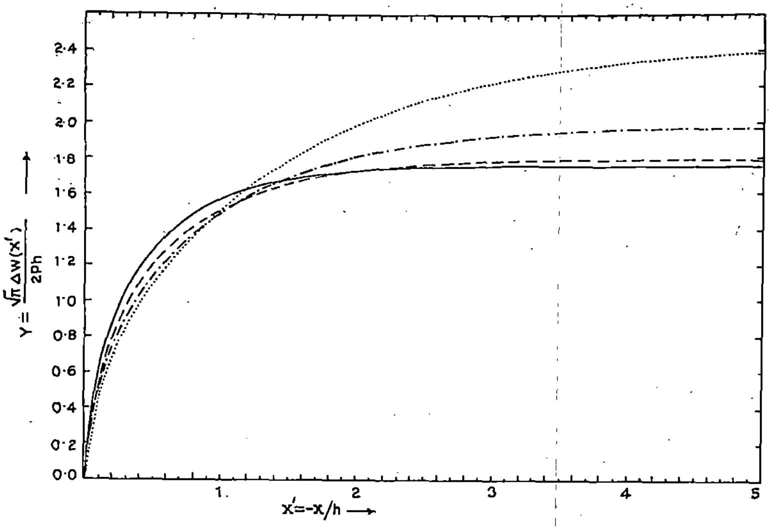
$$w(x, 0) = \frac{P}{\nu\sqrt{\pi}} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \int_0^x e^{-\frac{\alpha x}{\nu^2}} \frac{e^{\frac{\pi x}{2\nu h}}}{\sqrt{1 - \exp(\pi x/\nu h)}}\, dx \qquad (\text{for } x<0) \qquad (26)$$

Making $x \to -\infty$, it can easily be shown that

$$w(x, 0) \to -\frac{Ph}{\pi} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \frac{\Gamma\left[\frac{1}{2} - \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 - \frac{\alpha h}{\nu\pi}\right]} \qquad (27)$$

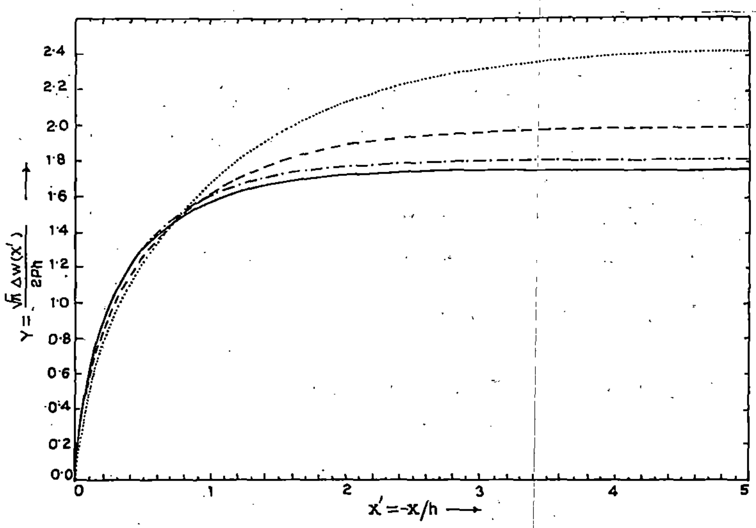
Putting $\alpha = 0$ in (25) and (26) expressions for $\frac{\partial w(x, 0)}{\partial y}$ and $w(x, 0)$ for homogeneous medium can be derived and they are found to be identical with the results given by Matczynski [2].

Crack opening displacement is obviously $\Delta w = 2w(x, 0)$ where $w(x, 0)$ is given by (26). In figs. 3-5 dimensionless values of the crack opening displacement given by $Y = \frac{\pi\Delta w}{2ph}$ have been plotted against the dimensionless distance $x' = -\frac{x}{h}$ along the length of the crack for different values of $\alpha_1 = \frac{\alpha h}{\nu\pi}$ and $c_1 = c/c_2$.



— c_1=.0, α_1=.0 --- c_1=.0, α_1=.1 -·-·- c_1=.0, α_1=.2 ······· c_1=.0, α_1=.3

Fig. 3



— c_1=.6, α_1=.0 -·-·- c_1=.6, α_1=.1 ---- c_1=.6, α_1=.2 ······ c_1=.6, α_1=.3

Fig. 4

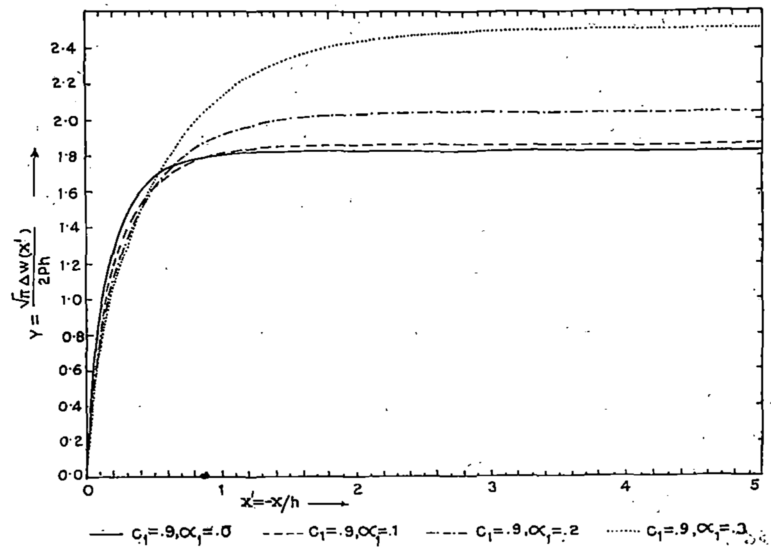


Fig. 5

It is interesting to note that for a fixed value of c_1, crack opening displacement increases with the increase in the values of the inhomogeneity parameter α_1 for large values of x' whereas for small values of x' ($x' \neq 0$), the result is just the opposite. Further it may be noted that for any given value of the inhomogeneity parameter α_1, crack opening displacement Y at any point x' increases with the increase in the crack propagation velocity.

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INPLANE PROBLEM OF DIFFRACTION OF ELASTIC WAVES BY A PERIODIC ARRAY OF COPLANAR GRIFFITH CRACKS

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Abstract—This paper represents the analysis of the problem of diffraction of longitudinal waves by a series of periodically spaced coplanar Griffith cracks in an infinite, isotropic elastic medium. Due to the periodicity of the geometry, the problem with a single crack in a strip with boundaries such that shear stress and normal displacement are zero on them. On use of Fourier transform the mixed boundary value problem for a typical strip has been reduced firstly to the solution of dual integral equations and finally to that of a Fredholm integral equations of the second kind. Numerical values of stress intensity factor and the crack opening displacement have been plotted graphically.

1. INTRODUCTION

THE PROBLEMS involving cracks or inclusions in elastodynamics are of much importance in view of their application in geophysics and earthquake engineering. Uptil now many problems have been solved involving one or two cracks in an infinite homogeneous elastic medium. Loeber and Sih [1] and Mal [2] have studied the problem of diffraction of elastic waves by a Griffith crack in an infinite medium. The problem of a finite crack at the interface of two elastic half spaces has been discussed by Srivastava *et al.* [3] and Bostrom [4]. Finite crack perpendicular to the surface of the infinitely long elastic strip has been studied by Chen [5] for impact load and by Srivastava *et al.* [6] for normally incident waves. But elastodynamic problems involving two or more Griffith cracks have not yet received much attention. Jain and Kanwal [7] have studied the problem of scattering of elastic waves by two Griffith cracks for normally incident waves and the same problem has been considered by Itou [8] for impact load. Angel and Achenbach [9] have studied the problem of reflection and transmission of elastic waves by a periodic array of cracks in an infinite isotropic medium. The problem of diffraction of SH-waves by a series of cuts in nonhomogeneous solid was investigated by De Sarkar [10]. The steady state vibration of an infinite isotropic medium with a periodic system of coplanar cracks has been discussed by Parton and Morozov [11] using the method of the finite Fourier transforms to reduce the relevant mixed relations.

In our paper, the diffraction of normally incident time harmonic elastic waves by a periodic array of coplanar Griffith cracks in infinite elastic medium has been analyzed. Due to geometrical symmetry the problem has been reduced to the solution of the problem of a single crack in a strip whose boundaries are shear free and constrained in a way not to permit normal displacement. Applying Fourier transform the problem has been converted to the solution of dual integral equations. The dual integral equations finally have been reduced to a Fredholm integral equation of second kind by applying Abel's transform. Expressions for stress intensity factor and crack opening displacement have been derived in closed form. The numerical values of stress intensity factor and crack opening displacement have been presented graphically to bring out the salient features of the problem.

2. FORMULATION OF THE PROBLEM

We consider a homogeneous, isotropic, linearly elastic, unbounded solid weakened by a infinite number of collinear cracks of equal length which are equally spaced on a line taken as the x_1-axis. The length of each crack is $2a$ and the period of the crack-array is $2h_1$ as shown in Fig. 1. The

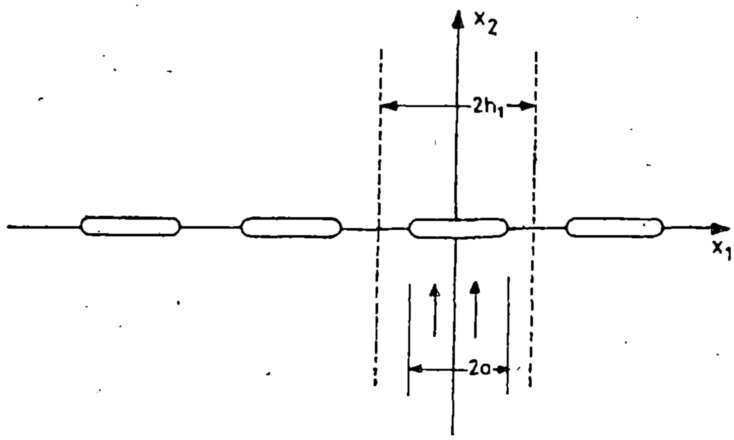

Fig. 1. Incidence of plane time-harmonic wave on a periodic array of cracks.

cracks lie in the plane $x_2 = 0$ and extend to infinity in the x_3-direction which is perpendicular to the plane of the figure. For convenience we make all the lengths dimensionless by writing

$$x_1/a = x, \quad x_2/a = y, \quad x_3/a = z, \quad h_1/a = h.$$

Let an incident time-harmonic body wave travel in the direction of the positive y-axis. The steady state term $e^{-i\omega t}$, which is common to all field variables, has been omitted in the sequel.

By simple symmetry considerations, the displacement and stress distribution due to the scattered field in the entire xy-plane can be derived by considering only the isotropic elastic strip $|x| \leqslant h$ with a central crack $|x| \leqslant 1$, $y = 0$; the boundaries of the strip $x = \pm h$ being shear free and constrained in a way not to permit normal displacement.

The displacement components are

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y} \tag{1}$$

and

$$v = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}$$

where ϕ and ψ are scalar and vector potentials satisfying the following equations.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{a^2}{c_1^2}\frac{\partial^2 \phi}{\partial t^2}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{a^2}{c_2^2}\frac{\partial^2 \psi}{\partial t^2} \tag{2}$$

where $c_1 = (\lambda + 2\mu/\rho)^{1/2}$ and $c_2 = (\mu/\rho)^{1/2}$ are the dilatational and shear wave velocities, λ, μ are the Lame's constant, ρ is the density of the material.

Therefore, substituting $\phi(x, y, t) = \phi(x, y)e^{-i\omega t}$ and $\psi(x, y, t) = \psi(x, y)e^{-i\omega t}$, our problem reduces to the solution of the equations

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + k_1^2 \phi = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + k_2^2 \psi = 0 \tag{3}$$

subject to the boundary conditions

$$\tau_{yy}(x, 0) = -p(x), \quad |x| < 1 \tag{4}$$

$$\tau_{xy}(x, 0) = 0, \quad |x| \leqslant h \tag{5}$$

$$v(x, 0) = 0, \quad 1 \leqslant |x| \leqslant h \tag{6}$$

$$\tau_{xy}(\pm h, y) = 0, \quad |y| < \infty \tag{7}$$

$$u(\pm h, y) = 0, \quad |y| < \infty \tag{8}$$

where $k_i = a\omega/c_i$ $(i = 1, 2)$.

Solutions of eq (3) are

$$\phi(x, y) = \sqrt{\frac{2}{\pi}}\left[\int_0^\infty A_1(\zeta)e^{-\alpha y}\cos\zeta\, x\, d\zeta + \int_0^\infty A_2(\xi)\cosh(\alpha_1 x)\cos\xi y\, d\xi\right]$$

and

$$\psi(x, y) = \sqrt{\frac{2}{\pi}}\left[\int_0^\infty B_1(\zeta)e^{-\beta y}\sin\zeta\, x\, d\zeta + \int_0^\infty B_2(\xi)\sinh(\beta_1 x)\sin\xi y\, d\xi\right] \tag{9}$$

where $A_1(\zeta)$, $A_2(\xi)$, $B_1(\zeta)$, $B_2(\xi)$ are constants and

$$\begin{aligned} \alpha &= (\zeta^2 - k_1^2)^{1/2}, & \zeta > k_1 \quad \beta &= (\zeta^2 - k_2 2)^{1/2}, & \zeta > k_2 \\ &= -i(k_1^2 - \zeta^2)^{1/2}, & \zeta < k_2 \quad &= -i(k_2^2 - \zeta^2)^{1/2}, & \zeta < k_2 \\ \alpha_1 &= (\xi^2 - k_1^2)^{1/2}, & \xi > k_1 \quad \beta_1 &= (\xi^2 - k_2^2)^{1/2}, & \xi > k_2 \\ &= -i(k_1^2 - \xi^2)^{1/2}, & \xi < k_1 \quad &= -i(k_2^2 - \xi^2)^{1/2}, & \xi < k_2. \end{aligned}$$

Now the stress τ_{xy} can be expressed as

$$\begin{aligned} a\tau_{xy}(x, y) = \sqrt{\frac{2}{\pi}}\Bigg[&-\mu\int_0^\infty(-2\zeta\alpha A_1(\zeta)e^{-\alpha y} + (\zeta^2 + \beta^2)B_1(\zeta)e^{-\beta y})\sin\zeta\, x\, d\zeta \\ &+ \mu\int_0^\infty(-2\xi\alpha_1 A_2(\xi)\sinh(\alpha_1 x) + (\xi^2 + \beta_1^2)B_2(\xi)\sinh(\beta_1 x))\sin\xi y\, d\xi\Bigg]. \end{aligned} \tag{10}$$

The boundary condition (5) yields

$$B_1(\zeta) = \frac{2\zeta\alpha}{\zeta^2 + \beta^2}A_1(\zeta). \tag{11}$$

Assuming $-\zeta A_1(\zeta) = A(\zeta)$, $\alpha_1 A_2(\xi) = C(\xi)$, $-\xi B_2(\xi) = D(\xi)$ and using the relation (11), expressions for displacements and stresses finally can be written as

$$\begin{aligned} u = &\sqrt{\frac{2}{\pi}}\int_0^\infty\left[e^{-\alpha y} - \frac{2\alpha\beta}{2\zeta^2 - k_2^2}e^{-\beta y}\right]A(\zeta)\sin\zeta\, x\, d\zeta \\ &+ \sqrt{\frac{2}{\pi}}\int_0^\infty[C(\xi)\sinh(\alpha_1 x) + D(\xi)\sinh(\beta_1 x)]\cos\xi y\, d\xi \end{aligned} \tag{12}$$

$$\begin{aligned} v = &\sqrt{\frac{2}{\pi}}\int_0^\infty\left[e^{-\alpha y} - \frac{2\zeta^2}{2\zeta^2 - k_2^2}e^{-\beta y}\right]\alpha\zeta^{-1}A(\zeta)\cos\zeta\, x\, d\zeta \\ &- \sqrt{\frac{2}{\pi}}\int_0^\infty[\xi\alpha_1^{-1}C(\xi)\cosh(\alpha_1 x) + \beta_1\xi^{-1}D(\xi)\cosh(\beta_1 x)]\sin\xi y\, d\xi \end{aligned} \tag{13}$$

$$\begin{aligned} a\tau_{yy} = &-\mu\sqrt{\frac{2}{\pi}}\int_0^\infty\left[(2\zeta^2 - k_2^2)e^{-\alpha y} - \frac{4\alpha\beta\zeta^2}{2\zeta^2 - k_2^2}e^{-\beta y}\right]\zeta^{-4}A(\zeta)\cos\zeta\, x\, d\zeta \\ &-\mu\sqrt{\frac{2}{\pi}}\int_0^\infty\left[\frac{(2\alpha_1^2 + k_2^2)}{\alpha_1}C(\xi)\cosh(\alpha_1 x) + 2\beta_1 D(\xi)\cosh(\beta_1 x)\right]\cos\xi y\, d\xi \end{aligned} \tag{14}$$

$$\sigma\tau_{xy} = -\mu\sqrt{\frac{2}{\pi}}\int_0^\infty [e^{-\alpha y} - e^{-\beta y}]2\alpha A(\zeta)\sin\zeta \times d\zeta$$

$$-\mu\sqrt{\frac{2}{\pi}}\int_0^\infty [2\xi C(\xi)\sinh(\alpha_1 x) + \xi^{-1}(2\xi^2 - k_2^2)D(\xi)\sinh(\beta_1 x)]\sin\xi y\, d\xi. \quad (15)$$

3. SOLUTION OF THE PROBLEM

The boundary conditions (4) and (6) yield the following two integral equations:

$$\int_0^\infty \frac{1}{\zeta}[1 + H(\zeta)]B(\zeta)\sin\zeta \times d\zeta = R(X), \quad 0 \leqslant |x| \leqslant 1 \quad (16)$$

$$\int_0^\infty \frac{1}{\zeta}B(\zeta)\cos\zeta \times d\zeta = 0, \quad 1 \leqslant |x| \leqslant h \quad (17)$$

where

$$B(\zeta) = \frac{2\alpha(k_1^2 - k_2^2)A(\zeta)}{2\zeta^2 - k_2^2} \quad (18)$$

$$H(\zeta) = \frac{(2\zeta^2 - k_2^2) - 4\alpha\beta\zeta^2}{2\alpha\zeta(k_1^2 - k_2^2)} - 1 \quad (19)$$

$$H(\zeta) \to 0 \quad \text{as } \zeta \to \infty.$$

$$R(x) = \sqrt{\frac{2}{\pi}}\mu^{-1}a\int_0^x p(x)dx - \int_0^\infty \left[\frac{(2\alpha_1^2 + k_2^2)}{\alpha_1^2}C(\xi)\sinh(\alpha_1 x) + 2D(\xi)\sinh(\beta_1 x)\right]d\xi. \quad (20)$$

Let us consider the solution of integral eqs (16) and (17) in the form

$$B(\zeta) = \sqrt{\frac{\pi}{2}}\zeta\int_0^1 tf(t)J_0(\zeta t)\, dt \quad (21)$$

so that the integral eq. (17) is automatically satisfied.

Now, substituting the value of $B(\zeta)$ from (21) in (16) and using Abel's transform we obtain the following Fredholm integral equation of second kind:

$$f(t) + \int_0^1 uf(u)L_1(t, u)\, du = Q(t) \quad (22)$$

where,

$$Q(t) = \frac{2a}{\mu\pi t}\frac{d}{dt}\int_0^t (t^2 - z^2)^{1/2}p(z)\, dz - \sqrt{\frac{2}{\pi}}\int_0^\infty)\alpha_1^{-1}(2\alpha_1^2 + k_2^2)I_0(\alpha_1 t)C(\xi) + 2\beta_1 I_0(\beta_1 t)D(\xi)]\, d\xi \quad (23)$$

and

$$L_1(t, u) = \int_0^\infty \zeta H(\zeta)J_0(\zeta u)J_0(\zeta t)\, d\zeta. \quad (24)$$

From the boundary conditions (7) and (8), the unknown functions $C(\xi)$ and $D(\xi)$ can be found to be related to $B(\zeta)$ as:

$$C(\xi) = \frac{2}{\pi k_2^2(k_1^2 - k_2^2)\sinh(\alpha_1 h)}\left[-\xi^2\int_0^\infty g_1(\xi, \zeta)B(\zeta)\, d\zeta + \frac{(2\xi^2 - k_2^2)}{2}\int_0^\infty g_2(\xi, \zeta)B(\zeta)\, d\zeta\right] \quad (25)$$

$$D(\xi) = \frac{2}{\pi k_2^2(k_1^2 - k_2^2)\sinh(\beta_1 h)}\left[\xi^2\int_0^\infty g_1(\xi, \zeta)B(\zeta)\, d\zeta - \xi^2\int_0^\infty g_2(\xi, \zeta)B(\zeta)\, d\zeta\right] \quad (26)$$

where,

$$g_1(\xi, \zeta) = \left\{\frac{2\beta_1^2 + k_2^2}{\zeta^2 + \beta_1^2} - \frac{2\alpha_1^2 + k_2^2}{\zeta^2 + \alpha_1^2}\right\}\sin(\zeta h)$$

$$g_2(\xi, \zeta) = \left\{\frac{2(\beta_1^2 + k_2^2)}{\zeta^2 + \beta_1^2} - \frac{2\alpha_1^2 + k_2^2}{\zeta^2 + \alpha_1^2}\right\}\sin(\zeta h). \quad (27)$$

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Next, substituting the value of $B(\zeta)$ from (21) in the expressions of $C(\xi)$ and $D(\xi)$ given by (25) and (26) and using the result (Gradsteyn [12])

$$\int_0^\infty \frac{\zeta \sin(\zeta h) J_0(\zeta u)}{\zeta^2+\alpha_1^2}\,d\zeta = \frac{\pi}{2} I_0(\alpha_1 u) e^{-\alpha_1 h}$$

$C(\xi)$ and $D(\xi)$ can be written in terms of $f(t)$ as

$$C(\xi) = \sqrt{\frac{\pi}{2}} \frac{1}{2(k_1^2-k_2^2)} \int_0^1 [(2\alpha_1^2+k_2^2) I_0(\alpha_1 u) e^{-\alpha_1 h}] \frac{u f(u)\,du}{\sinh(\alpha_1 h)}$$

$$D(\xi) = -\sqrt{\frac{\pi}{2}} \frac{\xi^2}{(k_1^2-k_2^2)} \int_0^1 [I_0(\beta_1 u) e^{-\beta_1 h}] \frac{u f(u)\,du}{\sinh(\beta_1 h)}. \tag{28}$$

Using the above relations (28) in (23) we obtain

$$Q(t) = \frac{2a}{\mu\pi t}\frac{d}{dt}\int_0^t \sqrt{t^2-z^2}\,p(z)\,dz + \int_0^1 u[L_2(t,u)+L_9(t,u)] f(u)\,du \tag{29}$$

where,

$$L_2(t,u) = -\frac{1}{2(k_1^2-k_2^2)} \int_0^\infty [\alpha_1^{-1}(2\alpha_1^2+k_2^2)^2 I_0(\alpha_1 t) L_0(\alpha_1 u) e^{-\alpha_1 h}] \frac{d\xi}{\sinh(\alpha_1 h)} \tag{30}$$

$$L_9(t,u) = \frac{2}{(k_1^2-k_2^2)} \int_0^\infty [\beta_1(\beta_1^2+k_2^2) I_0(\beta_1 t) I_0(\beta_1 u) e^{-\beta_1 h}] \frac{d\xi}{\sinh(\beta_1 h)}. \tag{31}$$

Next substituting $Q(t)$ from (29) in (22) and assuming $p(x)=p_0$ and $f(t)=ap_0 g(t)/\mu$ we finally obtain the following Fredholm integral equation of second kind for the determination of $g(t)$:

$$g(t) + \int_0^1 u g(u) L(t,u)\,du = 1 \tag{32}$$

where

$$L(t,u) = L_1(t,u) - L_2(t,u) - L_9(t,u) \tag{33}$$

and $L_1(t,u)$, $L_2(t,u)$ and $L_9(t,u)$ are given by (24), (30) and (31) respectively.

It is to be noted that the kernal $L_1(t,u)$ represented by the semi-infinite integral given by eq. (24) has a slow rate of convergence. In order to make the numerical analysis easier, the semi-infinite integral has therefore been converted to finite integrals by using simple contour integration technique (Srivastava *et al.* [3]) and is given by

$$L_1(t,u) = -\frac{ik_2^4}{2(k_2-k_1)}\left[\int_0^\gamma \frac{(2\eta^2-1)^2}{(\gamma^2-\eta^2)^{1/2}} J_0(k_2\eta u) H_0^{(1)}(k_2\eta t)\,d\eta + \int_0^1 4\eta^2(1-\eta^2)^{1/2} J_0(k_2\eta u) H_0^{(1)}(k_2\eta t)\,d\eta\right], \quad t>u \tag{34}$$

where $\gamma = k_1/k_2$. The corresponding expression of $L_1(t,u)$ for $t<u$ can be obtained by interchanging t and u in (34).

4. STRESS INTENSITY FACTOR AND DISPLACEMENT

The normal stress $\tau_{yy}(x,y)$ in the plane $y=0$ in the vicinity of the crack tip can be found from eq. (14) and is given by

$$\tau_{yy}(x,0) = -\mu\sqrt{\frac{2}{\pi}}\int_0^\infty B(\zeta)\cos\zeta x\,d\zeta + 0(1), \quad x>1$$

$$= -\frac{p_0 x}{\sqrt{x^2-1}} g(1) + 0(1), \quad x>1.$$

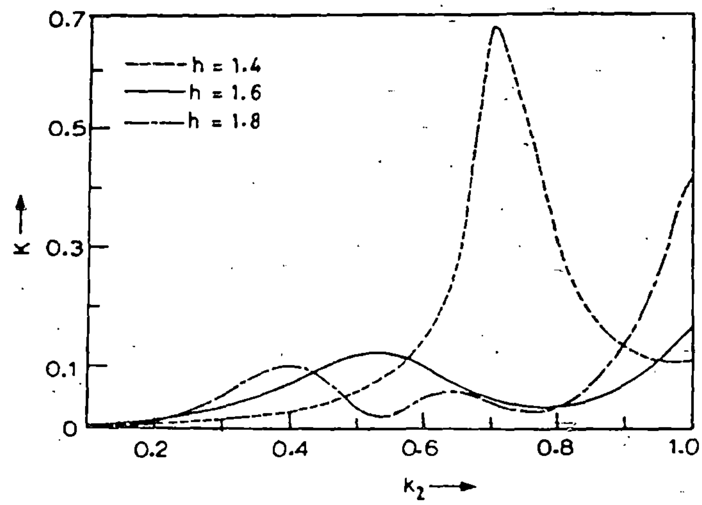

Fig. 2. Stress intensity factor K vs dimensionless frequency k_2.

Defining the stress intensity factor by

$$K = \underset{x\to 1+}{Lt} \left| \frac{\tau_{yy}(x, 0)\sqrt{x-1}}{p_0} \right|$$

it is found that

$$K = \frac{|g(1)|}{\sqrt{2}}. \tag{35}$$

Now the crack opening displacement $\Delta v(x, 0) = v(x, 0+) - v(x, 0-)$ can be obtained from (13) as

$$\Delta v(x, 0) = -\frac{k^2}{\sqrt{2\pi}(k_1^2 - k_2^2)} \int_0^\infty \frac{1}{\zeta} B(\zeta)\cos(\zeta x)\, d\zeta, \quad |x| \leqslant 1$$

which on substitution of the value of $B(\zeta)$ from (21) takes the form

$$\Delta v(x, 0) = \frac{ap_0}{\mu(1-\gamma^2)} \int_x^1 \frac{tg(t)\, dt}{(t^2 - x^2)^{1/2}}, \quad |x| \leqslant 1. \tag{36}$$

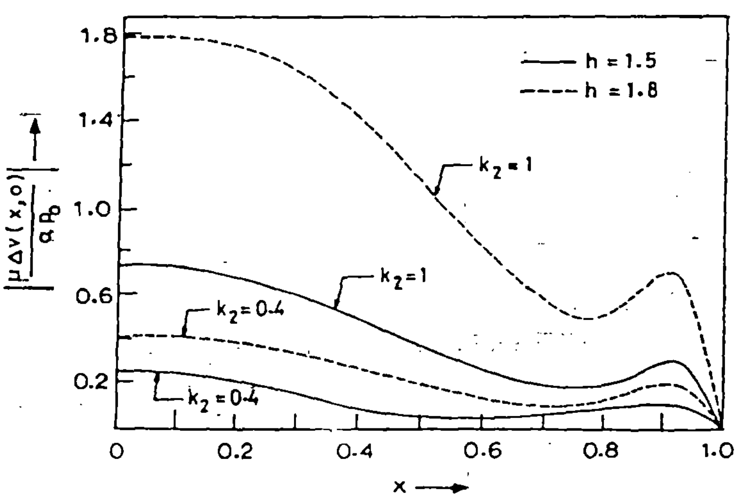

Fig. 3. Crack opening displacement vs distance.

5. NUMERICAL RESULTS AND DISCUSSION

Using the method of Fox and Goodwin [13] the Fredholm integral equation given by eq. (32) has been solved numerically for different values of dimensionless frequency k_2 and h, the separating distance of the cracks. At first the integral in (32) has been presented by a quadrature formula involving values of the desired function $g(t)$ at pivotal points inside the specified range of integration and then converted to a set of linear algebraic simultaneous equations, solving which the first approximation to the required pivotal values of $g(t)$ has been obtained. Applying difference–correction technique the first approximations has been improved. The standard numerical integration technique has been used to evaluate the kernals $L_1(t, u)$, $L_2(t, u)$ and $L_9(t, u)$ given by (34), (30) and (31). After solving the integral eq. (32) numerically, the stress intensity factor K and the crack opening displacement $\mu\Delta v(x, 0)/ap_0$ have been calculated numerically and plotted separately against dimensionless frequency $k_2(0 < k_2 \leqslant 1)$ and dimensionless distance $x(0 \leqslant x \leqslant 1)$ respectively for different values of h. The value of γ is taken to be $1/\sqrt{3}$. From Fig. 2 it is interesting to note that the number of oscillations in stress intensity factor K increases with the increase in the values of h. The crack opening displacement (Fig. 3) is greater for higher values of h and also for higher values of dimensionless frequency k_2.

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DIFFRACTION OF ELASTIC WAVES BY THREE COPLANAR GRIFFITH CRACKS IN AN ORTHOTROPIC MEDIUM

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Abstract—The dynamic response of three co-planar Griffith cracks situated in an infinite orthotropic medium due to elastic waves incident normally on the cracks has been treated. The Fourier transform technique has been used to reduce the elastodynamic problem to the solution of a set of four integral equations. These integral equations have been solved by using the finite Hilbert transform technique and Cook's result. The analytical forms of crack opening displacement and stress intensity factors have been derived for low frequency vibration. Numerical results of crack opening displacement and stress intensity factors for several orthotropic materials have been calculated and plotted graphically to display the influence of the material orthotropy.

1. INTRODUCTION

Recently, with the increased usage of macroscopically anisotropic construction materials such as fibre-reinforced materials, the study of diffraction of elastic waves with cracks or inclusions has attracted the attention of scientists. The different possible location of cracks with respect to the planes of material symmetry is of great interest in Seismology and Exploration Geophysics. The problem of scattering of elastic waves by cracks of finite dimension in isotropic medium has been investigated by several investigators. Many investigators [1–6] have solved the diffraction problem involving single or two cracks in an isotropic medium. Dhawan and Dhaliwal [7] solved the statical problem involving three coplanar cracks in an infinite transversely isotropic medium. The dynamic problem of singular stresses around cracks in orthotropic medium are few in number. Kassir and Bandyopadhyay [8] solved the problem of elastodynamic response of an infinite orthotropic solid containing a crack under the action of impact loading. The problem of normal impact response of a finite Griffith crack in an orthotropic strip has been solved by Shindo [9]. De and Patra [10] have also solved the problem involving a moving Griffith crack in an orthotropic strip. Recently Kundu and Bostrom [11] treated the diffraction problem of a circular crack in orthotropic medium.

To the best knowledge of the authors, the problem of diffraction of elastic waves by three coplanar Griffith cracks in an orthotropic material has not been considered. In our paper, the interaction of normally incident time harmonic elastic waves with three coplanar Griffith cracks in an orthotropic medium has been investigated. It is assumed that the faces of each of the cracks do not come into contact during small deformation of the solid. The resulting mixed boundary value problem is reduced to the solution of a set of four integral equations which has been reduced to the solution of an integro-differential equation. Iteration method has been used to obtain the low frequency solution of the problem. This enables us to obtain approximate value of the crack opening displacements and stress intensity factors. Making the length of the central crack tend to zero, the corresponding results for two Griffith cracks have been obtained. Numerical results for stress intensity factors and crack opening displacements have been plotted against dimensionless frequency and distance respectively for different orthotropic materials which have been shown graphically.

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2. STATEMENT AND FORMULATION OF THE PROBLEM

Consider the interaction of normally incident longitudinal wave with three coplanar Griffith cracks situated in an infinite orthotropic elastic medium. The cracks are assumed to occupy the position $|X| \leq d_1$, $d_2 \leq |X| \leq d$, $Y = 0$, $|Z| < \infty$. Let E_i, μ_{ij} and ν_{ij} $(i, j = 1, 2, 3)$ denote the engineering elastic constants of the material where the subscripts 1, 2, 3 correspond to the X, Y, Z directions chosen to coincide with the axes of material orthotropy. Normalizing all the lengths with respect to 'd' and setting $X/d = x$, $Y/d = y$, $Z/d = z$, $d_1/d = b$, $d_2/d = c$, the cracks are defined by $|x| \leq b$, $c \leq |x| \leq 1$, $y = 0$, $|z| < \infty$ (Fig. 1).

Displacement components are also made dimensionless with respect to 'd' so that dimensionless components of displacement in x, y directions are assumed to be u, v respectively, where

$$u = u(x, y, t) \quad \text{and} \quad v = v(x, y, t).$$

Let a time harmonic plane elastic wave originating at $y = -\infty$ and incident normally on the three cracks be given by $v = v_0 \exp[i(ky - \omega t)]/d$ where $k = d\omega/c_s\sqrt{c_{22}}$, $c_s = (\mu_{12}/\rho)^{1/2}$, v_0 is a constant, ω and v_0/d are the frequency and dimensionless amplitude of the incident wave respectively, ρ being the density of the material. In the isotropic solid, c_s represents the velocity of the shear wave.

The non-zero stress components τ_{yy} and τ_{xy} are given by

$$\tau_{yy}/\mu_{12} = c_{12}u_{,x} + c_{22}v_{,y}$$

$$\tau_{xy}/\mu_{12} = u_{,y} + v_{,x} \tag{2.1}$$

where u, v denote the component of the displacement in the x, y directions respectively and comma denotes partial differentiation with respect to the coordinates or time ; $c_{ij}(i, j = 1, 2)$ are

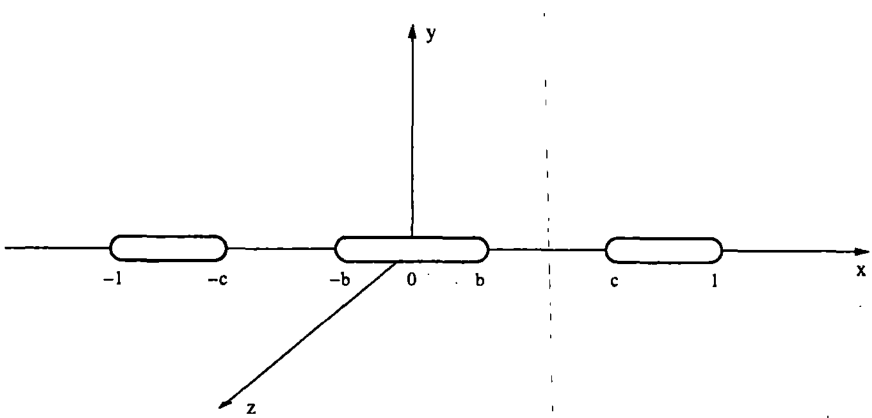

Fig. 1. Geometry of the cracks.

nondimensional parameters related to the elastic constant by the relations:

$$c_{11} = E_1/\mu_{12}(1 - \nu_{12}^2 E_2/E_1)$$

$$c_{22} = E_2/\mu_{12}(1 - \nu_{12}^2 E_2/E_1) = c_{11} E_2/E_1$$

$$c_{12} = \nu_{12} E_2/\mu_{12}(1 - \nu_{12}^2 E_2/E_1) = \nu_{12} c_{22} = \nu_{21} c_{11} \tag{2.2}$$

for generalized plane stress, and by

$$c_{11} = (E_1/\Delta\mu_{12})(1 - \nu_{23}\nu_{32})$$

$$c_{22} = (E_2/\Delta\mu_{12})(1 - \nu_{13}\nu_{31})$$

$$c_{12} = E_1(\nu_{21} + \nu_{13}\nu_{32}E_2/E_1)/\Delta\mu_{12} = E_2(\nu_{12} + \nu_{23}\nu_{31}E_1/E_2)/\Delta\mu_{12}$$

$$\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{21}\nu_{32} \tag{2.3}$$

for plane strain. The constants E_i and ν_{ij} satisfy Maxwell's relation:

$$\nu_{ij}/E_i = \nu_{ji}/E_j. \tag{2.4}$$

The displacement equations of motion for orthotropic material are

$$c_{11}u_{,xx} + u_{,yy} + (1 + c_{12})v_{,xy} = \frac{d^2}{c_s^2} u_{,tt}$$

$$c_{22}v_{,yy} + v_{,xx} + (1 + c_{12})u_{,xy} = \frac{d^2}{c_s^2} v_{,tt}. \tag{2.5}$$

Substitution of $u(x, y, t) = u(x, y)\exp(-i\omega t)$ and $v(x, y, t) = v(x, y)\exp(-i\omega t)$ in equations (2.5) reduces them to

$$c_{11}u_{,xx} + u_{,yy} + (1 + c_{12})v_{,xy} + k_s^2 u = 0$$

$$c_{22}v_{,yy} + v_{,xx} + (1 + c_{12})u_{,xy} + k_s^2 v = 0 \tag{2.6}$$

with $k_s^2 = d^2\omega^2/c_s^2$, which are to be solved subject to the boundary conditions

$$v(x, 0) = 0, \quad b \le |x| \le c, \qquad |x| \geqslant 1 \tag{2.7}$$

$$\tau_{xy}(x, 0) = 0, \qquad |x| < \infty \tag{2.8}$$

$$\tau_{yy}(x, 0) + \tau_{yy}^{(0)}(x, 0) = 0, \qquad |x| < b, \qquad c < |x| < 1. \tag{2.9}$$

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables would be omitted in the sequel.

Using the condition (2.8), the solutions of equations (2.6) may be written as

$$u(x, y) = \frac{2}{\pi}\int_0^\infty [\exp(-\gamma_1|y|) - \beta\exp(-\gamma_2|y|)]A_1(\xi)\sin(\xi x)\,d\xi \tag{2.10}$$

$$v(x, y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}[\alpha_1\exp(-\gamma_1|y|) - \beta\alpha_2\exp(-\gamma_2|y|)]A_1(\xi)\cos(\xi x)\,d\xi, \qquad y > 0 \tag{2.11}$$

and the stress components are given by

$$\tau_{xy}/\mu_{12} = -\frac{2}{\pi}\int_0^\infty (\gamma_1+\alpha_1)[\exp(-\gamma_1|y|)-\exp(-\gamma_2|y|)]A_1(\xi)\sin(\xi x)\,d\xi, \qquad y>0 \quad (2.12)$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty \left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \beta\left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\right]A_1(\xi)\cos(\xi x) \quad (2.13)$$

where

$$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1+c_{12})\gamma_i}, \qquad i=1,2 \quad (2.14)$$

$$\beta = \frac{\gamma_1+\alpha_1}{\gamma_2+\alpha_2} \quad (2.15)$$

$A_1(\xi)$ is the unknown function to be determined, and γ_1^2, γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1+c_{22})k_s^2\}\gamma^2 + (c_{11}\xi^2 - k_s^2)(\xi^2 - k_s^2) = 0. \quad (2.16)$$

With the aid of the boundary conditions, (2.7) and (2.9) $A(\xi)$ is found to satisfy the integral equations

$$\int_0^\infty A(\xi)\cos(\xi x)\,d\xi = 0, \qquad x \in I_2, I_4 \quad (2.17)$$

and

$$\int_0^\infty H(\xi)A(\xi)\cos(\xi x)\,d\xi = -\frac{\pi p_0}{2\mu_{12}}, \qquad x \in I_1, I_3 \quad (2.18a, b)$$

where $I_1 = (0, b)$, $I_2 = (b, c)$, $I_3 = (c, 1)$, $I_4 = (1, \infty)$ and

$$p_0 = ik\mu_{12}c_{22}v_0/d \quad (2.19)$$

$$A(\xi) = \frac{\alpha_1 - \beta\alpha_2}{\xi}A_1(\xi) \quad (2.20)$$

$$H(\xi) = \frac{c_{12}\xi^2 - c_{22}\alpha_1\gamma_1 - \beta(c_{12}\xi^2 - c_{22}\alpha_2\gamma_2)}{(\alpha_1 - \beta\alpha_2)}. \quad (2.21)$$

3. METHOD OF SOLUTION

The solution of the integral equations (2.17) and (2.18) is taken in the form

$$A(\xi) = \frac{1}{\xi}\int_0^b h(t)\sin(\xi t)\,dt + \frac{1}{\xi}\int_c^1 g(u^2)\sin(\xi u)\,du \quad (3.1)$$

where $h(t)$ and $g(u^2)$ are the unknown functions to be determined. Substituting the value of $A(\xi)$ from (3.1) in (2.17) and using the following result [12]

$$\int_0^\infty \frac{\sin(\xi t)\cos(\xi x)}{\xi}\,d\xi = \begin{cases} \frac{\pi}{2}, & t>x \\ 0, & t<x \end{cases}$$

it is found that the choice of $A(\xi)$ leads to the equation

$$\int_c^1 g(u^2)\,\mathrm{d}u = 0. \tag{3.2}$$

Further substituting $A(\xi)$ from (3.1) in (2.18a) and using the result [13]

$$\int_0^\infty \xi^{-1}\sin(\xi u)\sin(\xi x)\,\mathrm{d}\xi = \frac{1}{2}\log\left|\frac{u+x}{u-x}\right|$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h(t)\log\left|\frac{t+x}{t-x}\right|\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(u^2)\log\left|\frac{u+x}{u-x}\right|\mathrm{d}u \\ = 2\Big[q_0 - \frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h(t)\,\mathrm{d}t\int_0^\infty H_1(\xi)\xi^{-1}\sin(\xi t)\sin(\xi x)\,\mathrm{d}\xi \\ - \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(u^2)\,\mathrm{d}u\int_0^\infty H_1(\xi)\xi^{-1}\sin(\xi u)\sin(\xi x)\,\mathrm{d}\xi\Big], \qquad x\in I_1\end{aligned} \tag{3.3}$$

where

$$q_0 = -\frac{\pi p_0}{2\theta\mu_{12}} \tag{3.4}$$

$$H_1(\xi) = \frac{H(\xi)}{\xi\theta} - 1 \to 0 \quad \text{as} \quad \xi\to\infty \tag{3.5}$$

$$\theta = \frac{(c_{12}^2 + c_{12} - c_{11}c_{22})(c_{12}N_1N_2 - c_{11}) - c_{22}[c_{12}N_1^2N_2^2 + c_{11}(N_1^2 + N_1N_2 + N_2^2)]}{c_{11}(1+c_{12})(N_1+N_2)} \tag{3.6}$$

$$N_1^2 = \frac{1}{2c_{22}}\{c_{11}c_{22} - c_{12}^2 - 2c_{12} + [(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}]^{1/2}\}$$

$$N_2^2 = \frac{1}{2c_{22}}\{c_{11}c_{22} - c_{12}^2 - 2c_{12} - [(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}]^{1/2}\}. \tag{3.7}$$

Using the relation

$$\frac{\sin\xi x\sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_0(\xi w)J_0(\xi v)\,\mathrm{d}v\,\mathrm{d}w}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}$$

equation (3.3) can now be rewritten in the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h(t)\log\left|\frac{t+x}{t-x}\right|\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(u^2)\log\left|\frac{u+x}{u-x}\right|\mathrm{d}u \\ = 2\Big[q_0 - \frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h(t)\,\mathrm{d}t\int_0^x\int_0^t \frac{vwL(v,w)\,\mathrm{d}w\,\mathrm{d}v}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} \\ - \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(u^2)\,\mathrm{d}u\int_0^x\int_0^u \frac{vwL(v,w)\,\mathrm{d}w\,\mathrm{d}v}{(x^2-w^2)^{1/2}(u^2-v^2)^{1/2}}, \qquad x\in I_1\end{aligned} \tag{3.8}$$

where

$$L(v,w) = \int_0^\infty \xi H_1(\xi)J_0(\xi w)J_0(\xi v)\,\mathrm{d}\xi \tag{3.9}$$

and $J_0(\ )$ is the Bessel function of order zero.

Applying a contour integration technique [14], the infinite integral in $L(v, w)$ can be converted to the following finite integrals

$$L(v, w) = -ik_s^2\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - \bar{\alpha}_1\bar{\gamma}_1 c_{22} - \bar{\beta}(c_{12}\eta^2 - \bar{\alpha}_2\bar{\gamma}_2 c_{22})}{\theta(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,\mathrm{d}\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,\mathrm{d}\eta\right], \qquad w > v \quad (3.10)$$

where

$$\bar{\gamma}_1 = \left[\frac{1}{2}\{R_1 - (R_1^2 - 4\bar{R}_2)^{1/2}\}\right]^{1/2}$$

$$\bar{\gamma}_2 = \left[\frac{1}{2}\{R_1 + (R_1^2 - 4\bar{R}_2)^{1/2}\}\right]^{1/2}$$

$$\hat{\gamma}_1 = \left[\frac{1}{2}\{-R_1 + (R_1^2 + 4R_2')^{1/2}\}\right]^{1/2}$$

$$\hat{\gamma}_2 = \left[\frac{1}{2}\{R_1 + (R_1^2 + 4R_2')^{1/2}\}\right]^{1/2}$$

$$R_1 = \frac{1}{c_{22}}\{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\eta^2 + (1 + c_{22})\}$$

$$\bar{R}_2 = \frac{c_{11}}{c_{22}}(1 - \eta^2)\left(\frac{1}{c_{11}} - \eta^2\right)$$

$$R_2' = \frac{c_{11}}{c_{22}}(1 - \eta^2)\left(\eta^2 - \frac{1}{c_{11}}\right)$$

$$\bar{\alpha}_i = \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_i^2}{(1 + c_{12})\bar{\gamma}_i}, \qquad i = 1, 2$$

$$\hat{\alpha}_i = \frac{c_{11}\eta^2 - 1 + (-1)^i\hat{\gamma}_i^2}{(1 + c_{12})\hat{\gamma}_i}, \qquad i = 1, 2$$

$$\bar{\beta} = \frac{\bar{\alpha}_1 - \bar{\gamma}_1}{\bar{\alpha}_2 - \bar{\gamma}_2}$$

$$\hat{\beta} = \frac{\hat{\alpha}_1 + \hat{\gamma}_1}{\hat{\alpha}_2 - \hat{\gamma}_2}. \quad (3.11)$$

The corresponding expression of $L(v, w)$ for $w < v$ is obtained by interchanging v and w in (3.10).

Employing the series expansions for the Bessel function J_0 and the Hankel function $H_0^{(1)}$ in equation (3.10), it is found that

$$L(v, w) = \frac{2}{\pi}Pk_s^2 \log k_s + O(k_s^2) \quad (3.12)$$

where

$$P = \frac{1}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - \bar{\alpha}_1\bar{\gamma}_1 c_{22} - \bar{\beta}(c_{12}\eta^2 - \bar{\alpha}_2\bar{\gamma}_2 c_{22})}{(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)}\,\mathrm{d}\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)}\,\mathrm{d}\eta\right]. \quad (3.13)$$

Let us now expand $h(t)$ and $g(u^2)$ in the form

$$h(t) = h_0(t) + k_s^2 \log k_s \, h_1(t) + O(k_s^2)$$

and

$$g(u^2) = g_0(u^2) + k_s^2 \log k_s \, g_1(u^2) + O(k_s^2). \tag{3.14}$$

Substituting the above equations (3.14) and the value of $L(v, w)$ given by (3.10) in equations (3.8) and (3.2) and equating the coefficients of like powers of k_s, the following equations are derived.

$$\frac{d}{dx}\int_0^b h_0(t) \log \left|\frac{t+x}{t-x}\right| dt + 2\int_c^1 \frac{u g_0(u^2)}{u^2 - x^2} du = 2q_0, \qquad x \in I_1, I_3 \tag{3.15a, b}$$

$$\frac{d}{dx}\int_0^b h_1(t)\log \left|\frac{t+x}{t-x}\right| dt + 2\int_c^1 \frac{u g_1(u^2)}{u^2 - x^2} du = -\frac{4P}{\pi}\left[\int_0^b t h_0(t)\, dt + \int_c^1 u g_0(u^2)\, du\right],$$

$$x \in I_1, I_3 \tag{3.16a, b}$$

and

$$\int_c^1 g_i(u^2)\, du = 0 \qquad (i = 0, 1). \tag{3.17a, b}$$

Rewriting equation (3.15a) as

$$\int_0^b h_0(t)\log \left|\frac{t+x}{t-x}\right| dt = \pi F_1(x), \qquad x \in I_1 \tag{3.18}$$

where

$$F_1(x) = -\int_0^x \left[\frac{p_0}{\mu_{12}\theta} + \frac{2}{\pi}\int_c^1 \frac{u g_0(u^2)}{u^2 - y^2} du\right] dy.$$

The solution of the integral equation (3.18) with the help of Cook's result [15] is found to be

$$h_0(t) = -\frac{p_0}{\mu_{12}\theta}\frac{t}{(b^2 - t^2)^{1/2}} - \frac{2}{\pi}\frac{t}{(b^2 - t^2)^{1/2}}\int_c^1 \frac{\sqrt{u^2 - b^2}\, g_0(u^2)}{u^2 - t^2} du. \tag{3.19}$$

Substitution of the value of $h_0(t)$ from (3.19) in (3.15b) with the aid of the result

$$\int_0^b \frac{1}{(b^2 - t^2)^{1/2}} \frac{t^2\, dt}{(x^2 - t^2)(u^2 - t^2)} = \frac{\pi}{2}\left[\frac{x}{(x^2 - b^2)^{1/2}} - \frac{u}{(u^2 - b^2)^{1/2}}\right], \qquad x \in I_3$$

yields the singular integral equation

$$\int_c^1 \frac{\sqrt{u^2 - b^2}\, g_0(u^2)}{u^2 - x^2} du = -\frac{\pi}{2}\frac{p_0}{\mu_{12}\theta}, \qquad x \in I_3. \tag{3.20}$$

Next using the finite Hilbert transform technique [13] the solution of the integral equation is found to be

$$g_0(u^2) = -\frac{p_0}{\mu_{12}\theta}\sqrt{\frac{u^2(u^2 - c^2)}{(u^2 - b^2)(1 - u^2)}} + \frac{u D_1}{\sqrt{(u^2 - b^2)(u^2 - c^2)(1 - u^2)}} \tag{3.21}$$

where D_1 is unknown constant to be determined from equation (3.17a).

Now substituting the value of $g_0(u^2)$ from (3.21) in (3.19) and performing the integrations, $h_0(t)$ is obtained in the following form

$$h_0(t) = -\frac{p_0}{\mu_{12}\theta}\sqrt{\frac{t^2(c^2 - t^2)}{(b^2 - t^2)(1 - t^2)}} + \frac{t D_1}{(b^2 - t^2)(c^2 - t^2)(1 - t^2)}. \tag{3.22}$$

By the procedure similar to one which led to the derivations of the solutions of (3.15) as given by (3.21) and (3.22), the solutions of equation (3.16a, b) can also be obtained and they are found to be

$$h_1(t) = -\frac{4PR}{\pi^2}\sqrt{\frac{t^2(c^2-t^2)}{(b^2-t^2)(1-t^2)}} - \frac{tD_2}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}} \tag{3.23}$$

$$g_1(u^2) = -\frac{4PR}{\pi^2}\sqrt{\frac{u^2(u^2-c^2)}{(u^2-b^2)(1-u^2)}} + \frac{uD_2}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}} \tag{3.24}$$

where

$$R = -\frac{p_0}{\mu_{12}\theta}[I_0^b + I_c^1] - D_1[J_0^b - J_c^1]$$

$$I_m^n = \int_m^n \frac{t^2\sqrt{(c^2-t^2)}}{\sqrt{(b^2-t^2)(1-t^2)}}\,\mathrm{d}t$$

$$J_m^n = \int_m^n \frac{t^2\,\mathrm{d}t}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}}. \tag{3.25}$$

The constant D_2 is to be determined from equation (3.17b).

In order to determine the values of the unknown constants D_1 and D_2, $g_0(u^2)$ and $g_1(u^2)$ as given by (3.21) and (3.24) respectively are substituted in (3.17a, b) and it is found that

$$D_j = A_j\left[(1-b^2)\frac{E}{F} - (c^2-b^2)\right], \qquad (j=1,2) \tag{3.26}$$

and

$$A_1 = \frac{p_0}{\mu_{12}\theta}, \qquad A_2 = \frac{4PR}{\pi^2} \tag{3.27}$$

where $F = F\left(\frac{\pi}{2}, q\right)$ and $E = E\left(\frac{\pi}{2}, q\right)$ are the elliptic integrals of first and second kind respectively and $q = \sqrt{\frac{1-c^2}{1-b^2}}$. Substitution of the values of D_j $(j=1,2)$ given by equations (3.26) in equations (3.21)–(3.24) yields

$$h_{j-1}(t) = -A_j\left[(1-b^2)\frac{E}{F} + (b^2-t^2)\right]\frac{t}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}} \qquad (j=1,2) \tag{3.28}$$

$$g_{j-1}(u^2) = -A_j\left[(1-b^2)\frac{E}{F} - (u^2-b^2)\right]\frac{u}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}} \qquad (j=1,2). \tag{3.29}$$

4. STRESS INTENSITY FACTORS AND CRACK OPENING DISPLACEMENTS

The stress intensity factors are defined as (in physical units)

$$N_b = \lim_{x\to b+}\left[\frac{\sqrt{(x-b)}\,\tau_{yy}(x,0)}{p_0}\right]_{b<x<c} \tag{4.1}$$

$$N_c = \lim_{x\to c-}\left[\frac{\sqrt{(c-x)}\,\tau_{yy}(x,0)}{p_0}\right]_{b<x<c} \tag{4.2}$$

$$N_1 = \lim_{x\to 1+}\left[\frac{\sqrt{(x-1)}\,\tau_{yy}(x,0)}{p_0}\right]_{x>1} \tag{4.3}$$

and the crack opening displacement can now be shown to be given by

$$\Delta v(x,0)=v(x,0+)-v(x,0-)=2\int_x^b h(t)\,\mathrm{d}t,\qquad 0\leq x\leq b \tag{4.4}$$

$$=2\int_x^1 g(u^2)\,\mathrm{d}u,\qquad c\leq x\leq 1. \tag{4.5}$$

Substituting the values of the function $h(t)$ and $g(u^2)$, the stress component τ_{yy} can be evaluated from the expressions (2.13), (2.21) and (3.1). After evaluation of the value of τ_{yy} and putting it in relations (4.1)–(4.3) it is found that

$$N_b=\sqrt{\frac{b(1-b^2)}{2(c^2-b^2)}}\frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)}\left[1-\frac{4P}{\pi^2}M_2k_s^2\log k_s\right]+O(k_s^2) \tag{4.6}$$

$$N_c=\sqrt{\frac{c}{2(c^2-b^2)(1-c^2)}}\left[(1-b^2)\frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)}-(c^2-b^2)\right]\left[1-\frac{4P}{\pi^2}M_2k_s^2\log k_s\right]+O(k_s^2) \tag{4.7}$$

$$N_1=\sqrt{\frac{(1-b^2)}{2(1-c^2)}}\left[1-\frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)}\right]\left[1-\frac{4P}{\pi^2}M_2k_s^2\log k_s\right]+O(k_s^2) \tag{4.8}$$

where

$$M_2=\left[I_0^b+I_c^1+\left\{(1-b^2)\frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)}-(c^2-b^2)\right\}(J_0^b-J_c^1)\right].$$

Expressions (4.4)–(4.5) with the aid of the equations (3.28)–(3.29) yield

$$\Delta v(x,0)=\frac{2p_0}{\mu_{12}\theta}\left[\sqrt{(1-b^2)}\,F(\beta,q)\left\{\frac{E(\beta,q)}{F(\beta,q)}-\frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)}\right\}-\sqrt{\frac{(1-x^2)(b^2-x^2)}{(c^2-x^2)}}\right]$$
$$\times\left[1-\frac{4P}{\pi^2}M_2k_s^2\log k_s\right]+O(k_s^2),\qquad 0\leq x\leq b \tag{4.9}$$

and

$$\Delta v(x,0)=\frac{2p_0}{\mu_{12}\theta}\left[\sqrt{(1-b^2)}\,F(\lambda,q)\left\{\frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)}-\frac{E(\lambda,q)}{F(\lambda,q)}\right\}\right]$$
$$\times\left[1-\frac{4P}{\pi^2}M_2k_s^2\log k_s\right]+O(k_s^2),\qquad c\leq x\leq 1 \tag{4.10}$$

where

$$\sin\beta=\sqrt{\frac{b^2-x^2}{c^2-x^2}}\quad\text{and}\quad\sin\lambda=\sqrt{\frac{1-x^2}{1-b^2}}.$$

When $b \to 0$, we recover the stress intensity factor and the crack opening displacement for two Griffith cracks occupying the region $c \le |x| \le 1$, $y = 0$, $|z| < \infty$:

$$N_c = -\frac{\left[c^2 - \frac{E}{F}\right]}{\sqrt{2c(1-c^2)}}\left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\}k_s^2 \log k_s\right] + O(k_s^2)$$

$$N_1 = -\frac{\left[1 - \frac{E}{F}\right]}{\sqrt{2(1-c^2)}}\left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\}k_s^2 \log k_s\right] + O(k_s^2) \tag{4.11}$$

and

$$\Delta v(x, 0) = \frac{2p_0}{\mu_{12}\theta}\left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\}k_s^2 \log k_s\right] \times \left[\frac{E\left(\frac{\pi}{2}, q\right)}{F\left(\frac{\pi}{2}, q\right)}F(\lambda, q) - E(\lambda, q)\right] + O(k_s^2), \qquad c \le x \le 1 \tag{4.12}$$

where $M_2 = \frac{\pi}{4}(1 + c^2 - 2E/F)$ has been used.

It is noted that if further $c \to 0$, the crack merge into a single crack of width two units. In this case $F \to \infty$ and $M_2 \to \pi/4$; so the results for stress intensity factor and crack opening displacements corresponding to the single crack are found to be

$$N_1 = \frac{1}{\sqrt{2}}\left[1 - \frac{P}{\pi}k_s^2 \log k_s\right] + O(k_s^2) \tag{4.13}$$

and

$$\Delta v(x, 0) = -\frac{2p_0}{\mu_{12}\theta}\sqrt{(1-x^2)}\left[1 - \frac{P}{\pi}k_s^2 \log k_s\right] + O(k_s^2), \qquad 0 \le x \le 1. \tag{4.14}$$

The results given by (4.11)–(4.14) are found to be in agreement with the results of Sarkar *et al.* [16].

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) N_b, N_c and N_1 given by (4.6), (4.7) and (4.8) at the tips of the cracks and crack opening displacements (COD) given by (4.9) and (4.10) have been plotted against dimensionless frequency k_s and distance respectively for three different types of orthotropic materials whose engineering constants have been listed in Table 1.

Table 1. Engineering elastic constants

	E_1 (Pa)	E_2 (Pa)	μ_{12} (Pa)	ν_{12}
Type I	Modulite II graphite–epoxy composite:			
	15.3×10^9	158.0×10^9	5.52×10^9	0.033
Type II	E-Type glass–epoxy composite:			
	9.79×10^9	42.3×10^9	3.66×10^9	0.063
Type III	Stainless steel–aluminium composite:			
	79.76×10^9	85.91×10^9	30.02×10^9	0.31

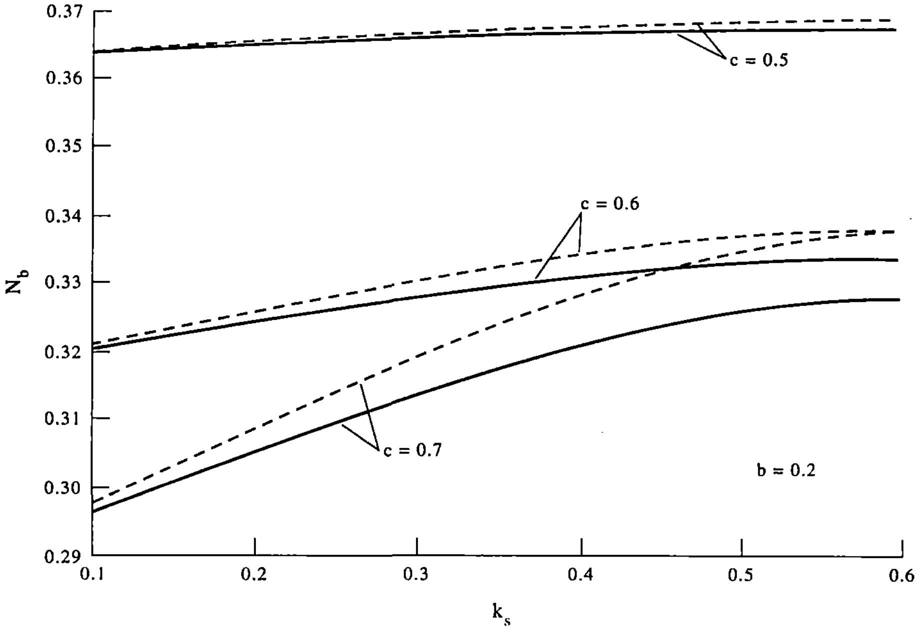

Fig. 2. Stress intensity factor N_b vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

Keeping the length of the central crack fixed ($b = 0.2$) SIFs at the tips of the central and outer cracks have been plotted against frequency k_s ($0.1 \leq k_s \leq 0.6$) for different lengths ($c = 0.5$, 0.6, 0.7) of the outer crack (Figs 2–4). It is noted from the graphs (Figs 2–4) that with the decrease in the value of outer crack length, i.e. with the increase in the value of the distance between inner and outer cracks the rate of increase in the SIF is higher with the increase in the value of the frequency k_s.

The same nature of SIFs are seen (Figs 5–7) in the case when the length of the outer cracks

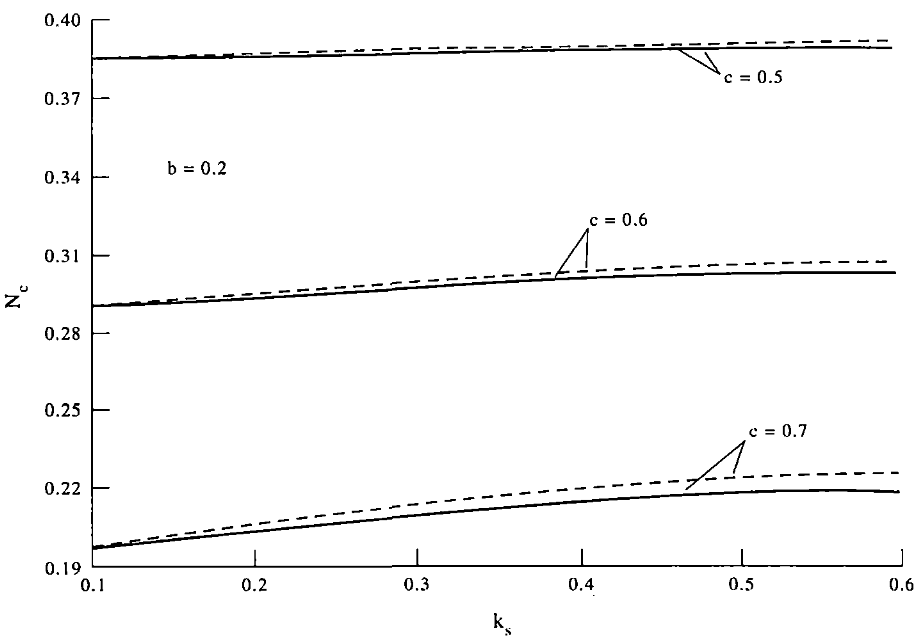

Fig. 3. Stress intensity factor N_c vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

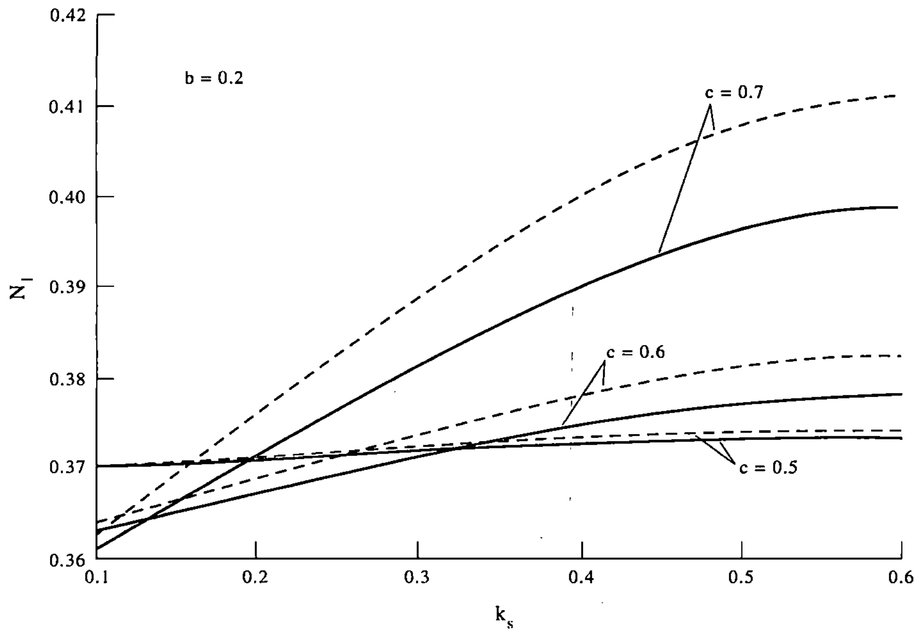

Fig. 4. Stress intensity factor N_{I} vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

are fixed ($c = 0.7$) and the length of the central crack increases ($b = 0.3$, 0.4, 0.5). It is interesting to note that for fixed c ($= 0.7$) the SIFs N_b and N_c increase with the increase in the value of b, but the effect is just reverse in case of N_{I}.

The COD $\mu_{12}\Delta v(x, 0)/p_0$ has been plotted for different crack lengths. It is found from Figs 8 and 9 that with the increase in the value of crack length the value of COD increases. For a fixed material the variation of COD with frequency is found to be insignificant.

In all the cases where different values of crack length have been considered the variation of COD is found to be prominant for different orthotropic materials.

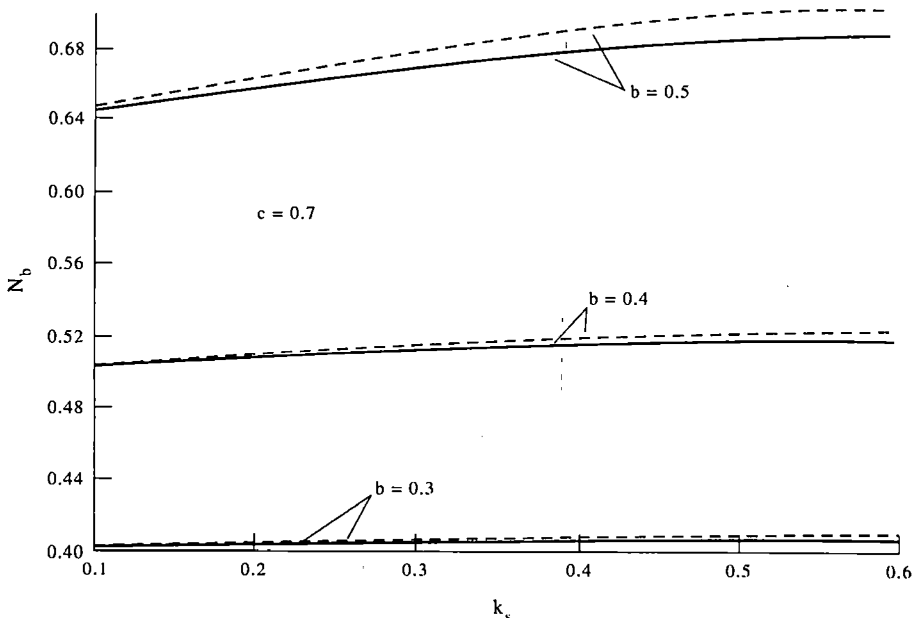

Fig. 5. Stress intensity factor N_b vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

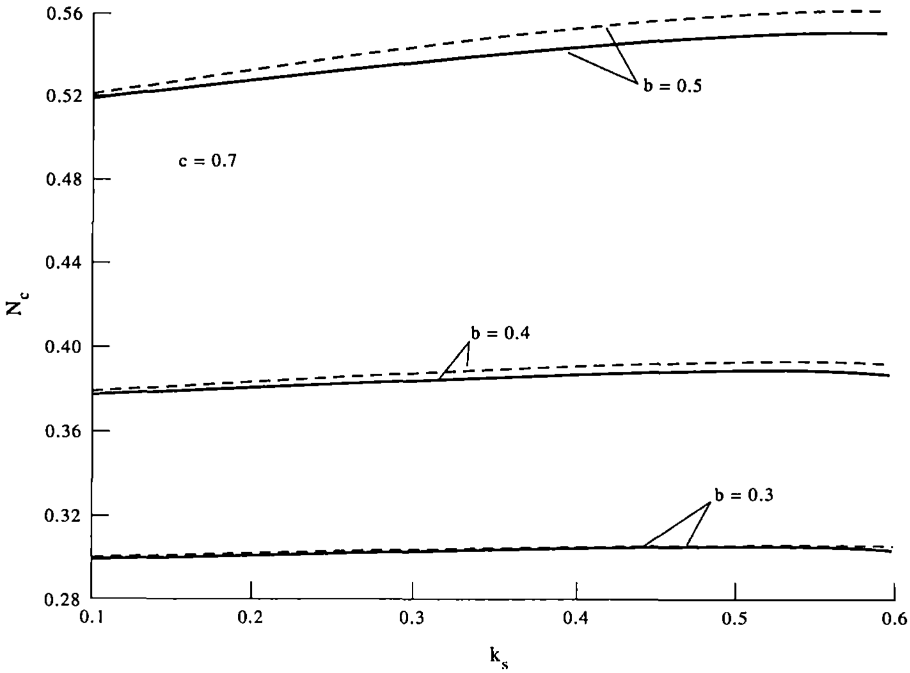

Fig. 6. Stress intensity factor N_c vs frequency k_s for generalized plane stress. (——) type I; (- - - - -) type III.

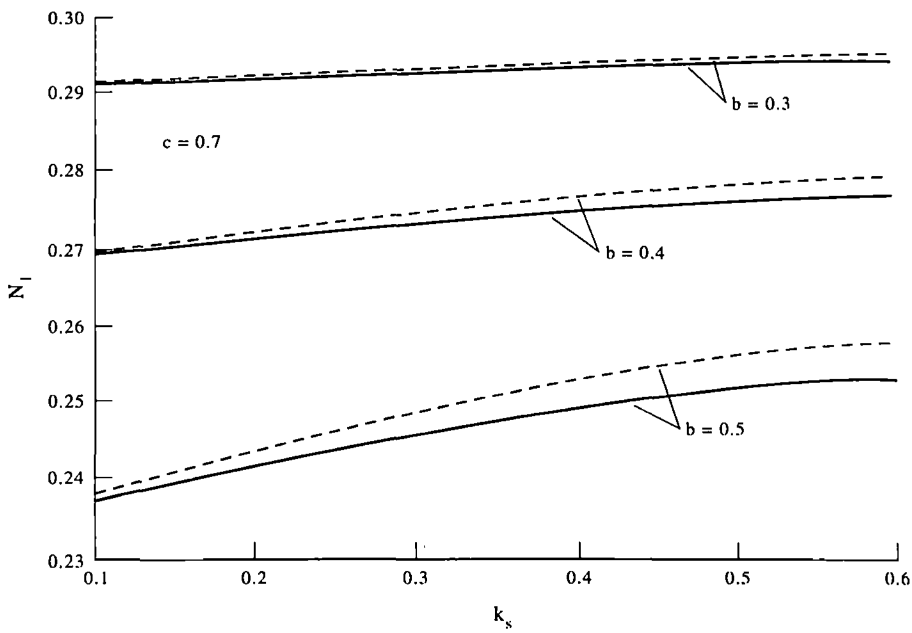

Fig. 7. Stress intensity factor N_I vs frequency k_s for generalized plane stress. (——) type I; (- - - - -) type III.

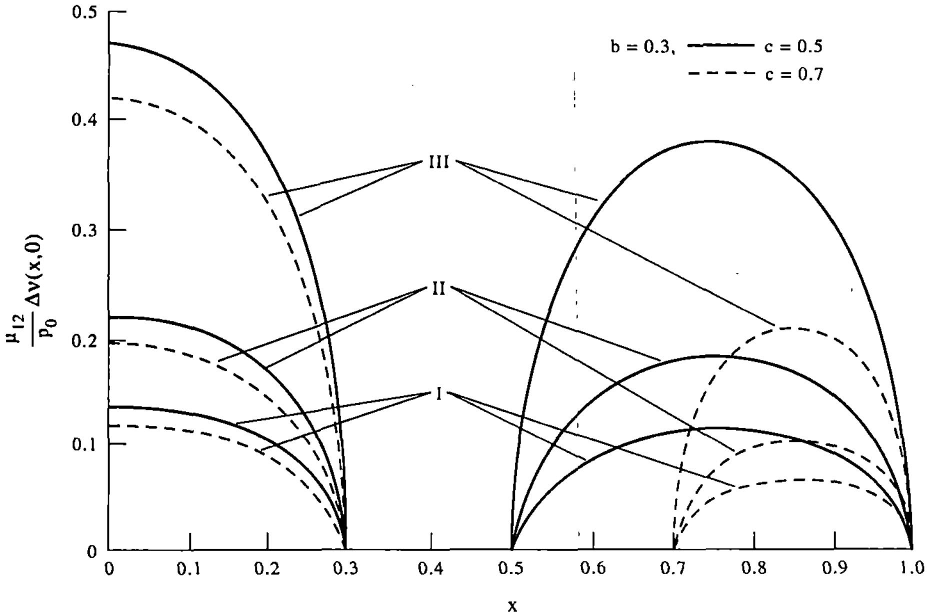

Fig. 8. Crack opening displacement vs distance for generalized plane stress ($k_s = 0.5$, $b = 0.3$, $c = 0.5$, 0.7).

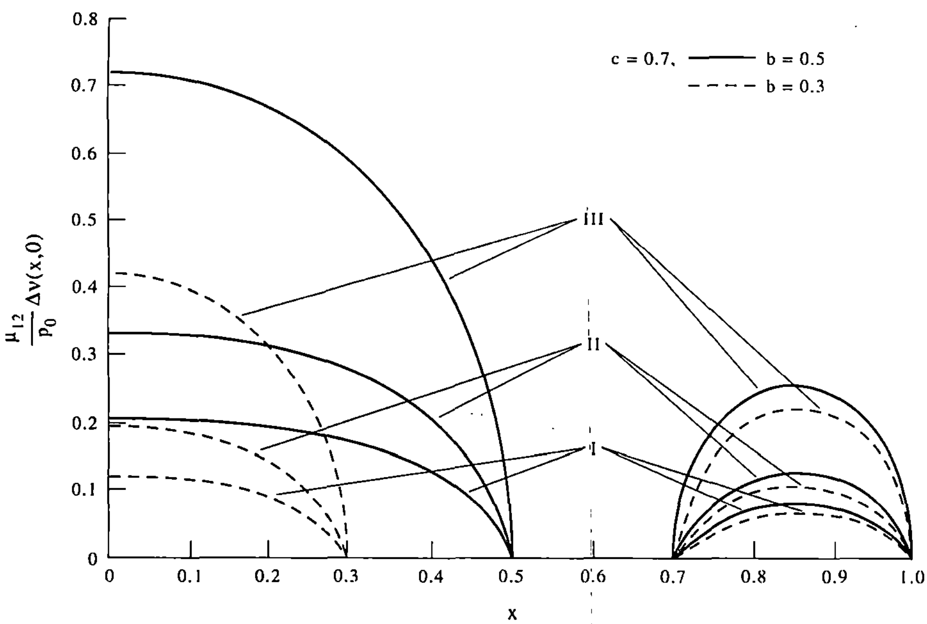

Fig. 9. Crack opening displacement vs distance for generalized plane stress ($k_s = 0.5$, $b = 0.3, 0.5$, $c = 0.7$).

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