

CHAPTER 2

**MEROMORPHIC FUNCTIONS
AND THEIR
RELATIVE DEFECTS**

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2.1 Introduction, Definitions and Notations.

Let f be a non-constant meromorphic function defined in the open complex plane \mathbb{C} . The term $S(r, f)$ denotes any quantity satisfying $S(r, f) = o\{T(r, f)\}$ i.e. $\frac{S(r, f)}{T(r, f)} \rightarrow 0$ as $r \rightarrow \infty$ through all values of r if f is of finite order and except possibly for a set of r of finite linear measure otherwise. For $a \in \mathbb{C} \cup \{\infty\}$ let $n(t, a; f)$ denotes the number of roots of $f = a$ in $|z| \leq t$, the multiple roots being counted according to their multiplicities and $N(r, a; f)$ is defined in the usual way in terms of $n(t, a; f)$. Similarly $\bar{n}(t, a; f)$ denotes the number of distinct roots of $f = a$ in $|z| \leq t$ and $\bar{N}(r, a; f)$ is also defined in the usual way in terms of $\bar{n}(t, a; f)$.

The Nevanlinna defect $\delta(a; f)$ and the Valiron defect $\Delta(a; f)$ of a are respectively defined in the following manner:

$$\delta(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \rightarrow \infty} \frac{m(r, a; f)}{T(r, f)}$$

The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [11]; **International Journal of Pure and Applied Mathematics**, see [12] and **International Journal of Contemporary Mathematical Sciences**, see [17].

and

$$\Delta(a; f) = 1 - \liminf_{r \rightarrow \infty} \frac{N(r, a; f)}{T(r, f)} = \limsup_{r \rightarrow \infty} \frac{m(r, a; f)}{T(r, f)}.$$

Milloux [30] introduced the concept of absolute defect of a with respect to the derivative f' . Later Xiong [42] extended this definition to $f^{(k)}$. He [42] introduced the notion of

$$\delta_R^{(k)}(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f^{(k)})}{T(r, f)},$$

for $k = 1, 2, 3, \dots$, and called it the relative Nevanlinna defect of a with respect to $f^{(k)}$. In [42] he has shown various relations between the usual defects and the relative defects.

The following definition is well known.

Definition 2.1.1. The order ρ_f of a meromorphic function f is defined in the following way

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}.$$

A meromorphic function f is said to be of finite order if $\rho_f < \infty$.

For a meromorphic function f of finite order ρ_f , a function $\rho_f(r)$ is called a proximate order of f if the following conditions hold:

- (i) $\rho_f(r)$ is non-negative and continuous for $r > r_0$, say.
- (ii) $\rho_f(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\rho'_f(r+0)$ and $\rho'_f(r-0)$ exist.
- (iii) $\lim_{r \rightarrow \infty} \rho_f(r) = \rho_f$,
- (iv) $\lim_{r \rightarrow \infty} r \rho'_f(r) \log r = 0$ and,
- (v) $\limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f(r)}} = 1$.

The existence of such a proximate order is proved by Lahiri [28]. In the value distribution theory the proximate deficiency of the value a is defined as

$$\delta_{\rho_f}(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f)}{r^{\rho_f(r)}}.$$

Using property (v) one can easily verify that $\delta(a; f) \leq \delta_{\rho_f}(a; f)$ for every complex number a in the following way:

$$\begin{aligned} \delta_{\rho_f}(a; f) &= 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f)}{r^{\rho_f(r)}} \\ &= 1 - \limsup_{r \rightarrow \infty} \left\{ \frac{N(r, a; f)}{T(r, f)} \cdot \frac{T(r, f)}{r^{\rho_f(r)}} \right\} \\ &\geq 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f)}{T(r, f)} \cdot \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \\ &= 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f)}{T(r, f)} = \delta(a; f) \end{aligned}$$

i.e., $\delta(a; f) \leq \delta_{\rho_f}(a; f)$.

Likewise one may define

$$\Theta_{\rho_f}(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, a; f)}{r^{\rho_f(r)}}$$

and call this proximate deficiency of the value a for distinct a -points. Clearly for every complex number a , $\Theta(a; f) \leq \Theta_{\rho_f}(a; f)$ which can be proved as follows:

$$\begin{aligned} \Theta_{\rho_f}(a; f) &= 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, a; f)}{r^{\rho_f(r)}} \\ &= 1 - \limsup_{r \rightarrow \infty} \left\{ \frac{\bar{N}(r, a; f)}{T(r, f)} \cdot \frac{T(r, f)}{r^{\rho_f(r)}} \right\} \\ &\geq 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, a; f)}{T(r, f)} \cdot \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \\ &= 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, a; f)}{T(r, f)} = \Theta(a; f) \end{aligned}$$

i.e., $\Theta(a; f) \leq \Theta_{\rho_f}(a; f)$.

From the second fundamental theorem it is not difficult to prove that the set $\{a : \Theta_{\rho_f}(a; f) > 0\}$ is countable and $\sum_a \Theta_{\rho_f}(a; f) \leq 2$. Also for a complex number a , we put

$$\Delta_{\rho_f}(a; f) = 1 - \liminf_{r \rightarrow \infty} \frac{N(r, a; f)}{r^{\rho_f(r)}}.$$

In this chapter we use the term

$${}_R\delta_{\rho_f}^{(k)}(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f^{(k)})}{r^{\rho_f(r)}}.$$

for relative proximate deficiency of a with respect to $f^{(k)}$ and

$${}_R\Theta_{\rho_f}^{(k)}(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, a; f^{(k)})}{r^{\rho_f(r)}}.$$

for relative proximate deficiency of a with respect to $f^{(k)}$ for distinct a -points and establish various relations between $\delta_{\rho_f}(a; f)$, $\Theta_{\rho_f}(a; f)$, ${}_R\delta_{\rho_f}^{(k)}(a; f)$ and ${}_R\Theta_{\rho_f}^{(k)}(a; f)$, where $k = 1, 2, 3, \dots$.

Let f and g be two non constant meromorphic functions defined in the open complex plane \mathbb{C} . If for $a \in \mathbb{C} \cup \{\infty\}$ f and g have the same a -points with the same multiplicities, we say that f and g share the value a CM(counting multiplicities).

The following definitions are well known.

Definition 2.1.2. We denote by $N(r, a; f | = 1)$ the counting function of simple a -points for $a \in \mathbb{C} \cup \{\infty\}$.

Definition 2.1.3. Let k be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f | \geq k)$ the counting function of those a -points of f whose multiplicities are not less than k , where an a -point is counted according to its multiplicity. Also by $\bar{N}(r, a; f | \geq k)$ we denote the corresponding reduced counting function.

We denote by $N(r, a; f | \leq k)$ the counting function $N(r, a; f) - N(r, a; f | \geq k + 1)$.

Definition 2.1.4. The quantities $\delta_{1)}(a; f)$ and $\delta_{2)}(a; f)$ are defined as follows:

$$\delta_{1)}(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f | = 1)}{T(r, f)}$$

and

$$\delta_{2)}(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f | \leq 2)}{T(r, f)}$$

for $a \in \mathbb{C} \cup \{\infty\}$.

We now introduce the following definition.

Definition 2.1.5. The quantities $\delta_{1)\rho_f}(a; f)$ and $\delta_{2)\rho_f}(a; f)$ are defined in the following way

$$\delta_{1)\rho_f}(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f | = 1)}{r^{\rho_f(r)}}$$

and

$$\delta_{2)\rho_f}(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f | \leq 2)}{r^{\rho_f(r)}}$$

for $a \in \mathbb{C} \cup \{\infty\}$.

Definition 2.1.6. A function b is said to be a small function with respect to f if $T(r, b) = o\{T(r, f)\}$.

In this chapter we also intend to establish a few results on the proximate deficiency of meromorphic functions involving sharing of values of them.

2.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.2.1. Let f be a meromorphic function of finite order. Then

$$\lim_{r \rightarrow \infty} \frac{S(r, f)}{r^{\rho_f(r)}} = 0.$$

Proof. Since f is of finite order then

$$S(r, f) = o\{T(r, f)\}.$$

$$\text{i.e., } \lim_{r \rightarrow \infty} \frac{S(r, f)}{T(r, f)} = 0.$$

So

$$\limsup_{r \rightarrow \infty} \frac{S(r, f)}{r^{\rho_f(r)}} = \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \cdot \lim_{r \rightarrow \infty} \frac{S(r, f)}{T(r, f)} = 0.$$

$$\text{i.e., } \lim_{r \rightarrow \infty} \frac{S(r, f)}{r^{\rho_f(r)}} = 0.$$

This proves the lemma. ■

Lemma 2.2.2. [45] If f, g share $0, 1, \infty$ CM and $f \not\equiv g$ then

$$(i) N(r, 1; f | \geq 2) = N(r, 1; g | \geq 2) = S(r, f) = S(r, g),$$

$$(ii) N(r, 0; f | \geq 2) = N(r, 0; g | \geq 2) = S(r, f) = S(r, g),$$

$$\text{and (iii) } N(r, \infty; f | \geq 2) = N(r, \infty; g | \geq 2) = S(r, f) = S(r, g).$$

Lemma 2.2.3. [25] If f, g share $0, 1, \infty$ CM and $f \not\equiv g$ then for any complex number $a (\neq 0, 1, \infty)$

$$(i) N(r, a; f | \geq 3) = S(r, f) = S(r, g)$$

$$\text{and (ii) } N(r, a; g | \geq 3) = S(r, f) = S(r, g).$$

2.3 Theorems.

In this section we present the main results of this chapter.

Theorem 2.3.1. Let f be a transcendental meromorphic function of finite order ρ_f such that for any positive integer k , $f^{(k)}$ is non-constant. Also let $a \neq 0, \infty$ be any small function related to f . Then

$$\Theta_{\rho_f}(\infty; f) + \delta_{\rho_f}(0; f) + {}_R\delta_{\rho_f}^{(k)}(a; f) \leq 2.$$

Proof. Let us consider the following identity

$$\frac{1}{f} = \frac{1}{a} \left\{ \frac{f^{(k)}}{f} - \left(a \frac{f^{(k+1)}}{f} - a' \frac{f^{(k)}}{f} \right) \frac{f^{(k)} - a}{af^{(k+1)} - a'f^{(k)}} \right\}.$$

Now in view of Milloux's theorem {p.55, [24]}, the relation

$T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$ and $T(r, a) = S(r, f)$, we obtain that

$$m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{f^{(k)} - a}{af^{(k+1)} - a'f^{(k)}}\right) + S(r, f)$$

$$\begin{aligned} \text{i.e.,} \quad & m\left(r, \frac{1}{f}\right) \\ & \leq T\left(r, \frac{f^{(k)} - a}{af^{(k+1)} - a'f^{(k)}}\right) - N\left(r, \frac{f^{(k)} - a}{af^{(k+1)} - a'f^{(k)}}\right) + S(r, f) \end{aligned}$$

$$\begin{aligned} \text{i.e.,} \quad & m\left(r, \frac{1}{f}\right) \\ & \leq m\left(r, \frac{af^{(k+1)} - a'f^{(k)}}{f^{(k)} - a}\right) + N\left(r, \frac{af^{(k+1)} - a'f^{(k)}}{f^{(k)} - a}\right) \\ & \quad - N\left(r, \frac{f^{(k)} - a}{af^{(k+1)} - a'f^{(k)}}\right) + S(r, f). \end{aligned} \tag{2.1}$$

Now adding $N\left(r, \frac{1}{f}\right)$ to both sides of (2.1) and using the Jensen formula {p.34, [24]}, we get from (2.1) in view of

$$T(r, f) = T\left(r, \frac{1}{f}\right) + O(1),$$

$$T(r, f) \leq \bar{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - a}\right) + S(r, f). \quad (2.2)$$

Now on dividing both sides of (2.2) by $r^{\rho_f(r)}$ and taking limit superior it follows by Lemma 2.2.1 that

$$\begin{aligned} & \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \\ & \leq \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, f)}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f}\right)}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f^{(k)} - a}\right)}{r^{\rho_f(r)}} \end{aligned}$$

$$\text{i.e., } 1 \leq \{1 - \Theta_{\rho_f}(\infty; f)\} + \{1 - \delta_{\rho_f}(0; f)\} + \left\{1 - {}_R\delta_{\rho_f}^{(k)}(a; f)\right\}$$

$$\text{i.e., } \Theta_{\rho_f}(\infty; f) + \delta_{\rho_f}(0; f) + {}_R\delta_{\rho_f}^{(k)}(a; f) \leq 2.$$

This proves the theorem. ■

Remark 2.3.1. The condition $a \neq 0, \infty$ in Theorem 2.3.1 is essential which is evident from the following two examples.

Example 2.3.1. Let $f = \exp z$, $k = 2$ and $a = 0$.

Then $\rho_f = 1$.

Now $N(r, 0; f) = \bar{N}(r, \infty; f) = 0$.

Also $N(r, 0; f^{(2)}) = N(r, 0; \exp z) = 0$.

So

$$\delta_{\rho_f}(0; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, 0; f)}{r^{\rho_f(r)}} = 1,$$

$$\Theta_{\rho_f}(\infty; f) = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, \infty; f)}{r^{\rho_f(r)}} = 1$$

and

$${}_R\delta_{\rho_f}^{(k)}(a; f) = {}_R\delta_{\rho_f}^{(2)}(0; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, 0; f^{(2)})}{r^{\rho_f(r)}} = 1.$$

Therefore

$$\Theta_{\rho_f}(\infty; f) + \delta_{\rho_f}(0; f) + {}_R\delta_{\rho_f}^{(2)}(a; f) = 3,$$

which contradicts Theorem 2.3.1.

Example 2.3.2. Let $f = \exp z$, $k = 2$ and $a = \infty$.

Then $\rho_f = 1$ and $\delta_{\rho_f}(0; f) = \Theta_{\rho_f}(\infty; f) = 1$.

Also $N(r, \infty; f^{(2)}) = N(r, \infty; \exp z) = 0$.

Then

$${}_R\delta_{\rho_f}^{(k)}(a; f) = {}_R\delta_{\rho_f}^{(2)}(\infty; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, \infty; f^{(2)})}{r^{\rho_f(r)}} = 1.$$

Therefore

$$\Theta_{\rho_f}(\infty; f) + \delta_{\rho_f}(0; f) + {}_R\delta_{\rho_f}^{(2)}(a; f) = 3,$$

which is contrary to the conclusion of Theorem 2.3.1.

Theorem 2.3.2. Let f be a meromorphic function such that $\rho_f < \infty$. Then for every non-negative integer k :

$${}_R\Theta_{\rho_f}^{(k)}(a; f) + \Theta_{\rho_f}(\infty; f) + \delta_{\rho_f}(0; f) \leq 2,$$

where a is any non-zero finite complex number.

Proof. Since

$$\frac{a}{f} = \frac{f^{(k)}}{f} - \frac{f^{(k)} - a}{f^{(k+1)}} \cdot \frac{f^{(k+1)}}{f},$$

it follows by the relation $T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$ and by Milloux's theorem {p.55. [24]}

$$m\left(r, \frac{a}{f}\right) \leq m\left(r, \frac{f^{(k)} - a}{f^{(k+1)}}\right) + S(r, f)$$

$$\text{i.e., } m\left(r, \frac{a}{f}\right) \leq T\left(r, \frac{f^{(k)} - a}{f^{(k+1)}}\right) - N\left(r, \frac{f^{(k)} - a}{f^{(k+1)}}\right) + S(r, f)$$

$$\text{i.e., } m\left(r, \frac{a}{f}\right) \leq T\left(r, \frac{f^{(k+1)}}{f^{(k)} - a}\right) - N\left(r, \frac{f^{(k)} - a}{f^{(k+1)}}\right) + S(r, f)$$

$$\begin{aligned}
\text{i.e., } m\left(r, \frac{a}{f}\right) &\leq m\left(r, \frac{f^{(k+1)}}{f^{(k)} - a}\right) + N\left(r, \frac{f^{(k+1)}}{f^{(k)} - a}\right) \\
&\quad - N\left(r, \frac{f^{(k)} - a}{f^{(k+1)}}\right) + S(r, f) \\
\text{i.e., } m\left(r, \frac{a}{f}\right) &\leq N\left(r, \frac{f^{(k+1)}}{f^{(k)} - a}\right) \\
&\quad - N\left(r, \frac{f^{(k)} - a}{f^{(k+1)}}\right) + S(r, f). \tag{2.3}
\end{aligned}$$

Now adding $N\left(r, \frac{a}{f}\right)$ to both sides of (2.3) and using $T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$, we get in view of {p.34, [24]},

$$\begin{aligned}
&m\left(r, \frac{a}{f}\right) + N\left(r, \frac{a}{f}\right) \\
&\leq N\left(r, f^{(k+1)}\right) + N\left(r, \frac{1}{f^{(k)} - a}\right) - N\left(r, f^{(k)} - a\right) \\
&\quad - N\left(r, \frac{1}{f^{(k+1)}}\right) + N\left(r, \frac{a}{f}\right) + S(r, f) \\
\text{i.e., } &T\left(r, \frac{a}{f}\right) \\
&\leq N\left(r, \frac{1}{f^{(k)} - a}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) \\
&\quad + \left\{N\left(r, f^{(k+1)}\right) - N\left(r, f^{(k)}\right)\right\} + N\left(r, \frac{1}{f}\right) + S(r, f) \\
\text{i.e., } &T\left(r, \frac{1}{f}\right) \\
&\leq \left\{N\left(r, \frac{1}{f^{(k)} - a}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right)\right\} + \bar{N}(r, f) \\
&\quad + N\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.4}
\end{aligned}$$

Let $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ denotes the zeros of $f^{(k+1)}$ which are not the zeros of $f^{(k)} - a$. So

$$N\left(r, \frac{1}{f^{(k)} - a}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) = \bar{N}\left(r, \frac{1}{f^{(k)} - a}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right).$$

Therefore from (2.4) we get that

$$\begin{aligned} & T(r, f) \\ & \leq \bar{N}\left(r, \frac{1}{f^{(k)} - a}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + \bar{N}(r, f) \\ & \quad + N\left(r, \frac{1}{f}\right) + S(r, f) \end{aligned}$$

$$\begin{aligned} \text{i.e.,} \quad & T(r, f) \\ & \leq \bar{N}\left(r, \frac{1}{f^{(k)} - a}\right) + \bar{N}(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f). \quad (2.5) \end{aligned}$$

Dividing both sides of (2.5) by $r^{\rho_f(r)}$ and taking limit superior it follows by Lemma 2.2.1 that

$$\begin{aligned} & \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \\ & \leq \limsup_{r \rightarrow \infty} \frac{\bar{N}\left(r, \frac{1}{f^{(k)} - a}\right)}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, f)}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f}\right)}{r^{\rho_f(r)}} \end{aligned}$$

$$\text{i.e., } 1 \leq \left\{1 - {}_R\Theta_{\rho_f}^{(k)}(a; f)\right\} + \left\{1 - \Theta_{\rho_f}(\infty; f)\right\} + \left\{1 - \delta_{\rho_f}(0; f)\right\}$$

$$\text{i.e., } {}_R\Theta_{\rho_f}^{(k)}(a; f) + \Theta_{\rho_f}(\infty; f) + \delta_{\rho_f}(0; f) \leq 2.$$

Thus the theorem is established. ■

Remark 2.3.2. The condition that 'a is any non zero finite complex number' in Theorem 2.3.2 is necessary which can be shown in the following two examples.

Example 2.3.3. Let $f = \exp z$, $k = 2$ and $a = 0$.

Then $\rho_f = 1$ and $\Theta_{\rho_f}(\infty; f) = \delta_{\rho_f}(0; f) = 1$.

Also $\bar{N}(r, 0; f^{(2)}) = \bar{N}(r, 0; \exp z) = 0$.

So

$${}_R\Theta_{\rho_f}^{(k)}(a; f) = {}_R\Theta_{\rho_f}^{(2)}(0; f) = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, 0; f^{(2)})}{r^{\rho_f(r)}} = 1.$$

Therefore

$${}_R\Theta_{\rho_f}^{(2)}(a; f) + \Theta_{\rho_f}(\infty; f) + \delta_{\rho_f}(0; f) = 3,$$

which contradicts Theorem 2.3.2.

Example 2.3.4. Let $f = \exp z$, $k = 2$ and $a = \infty$.

Then $\rho_f = 1$ and $\Theta_{\rho_f}(\infty; f) = \delta_{\rho_f}(0; f) = 1$.

Also $\bar{N}(r, \infty; f^{(2)}) = \bar{N}(r, \infty; \exp z) = 0$ and

$${}_R\Theta_{\rho_f}^{(k)}(a; f) = {}_R\Theta_{\rho_f}^{(2)}(\infty; f) = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, \infty; f^{(2)})}{r^{\rho_f(r)}} = 1.$$

Therefore

$${}_R\Theta_{\rho_f}^{(2)}(a; f) + \Theta_{\rho_f}(\infty; f) + \delta_{\rho_f}(0; f) = 3,$$

which is contrary to Theorem 2.3.2.

Theorem 2.3.3. If f be a meromorphic function of finite order ρ_f , then for any positive integer k ,

$$2{}_R\delta_{\rho_f}^{(k)}(\infty; f) + \delta_{\rho_f}(a; f) + \delta_{\rho_f}(0; f) \leq 3.$$

where a be any non-zero finite complex number.

Proof. From the identity

$$\frac{a}{f} = 1 - \frac{f-a}{f^{(k)}} \cdot \frac{f^{(k)}}{f}$$

and $T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$ it easily follows that

$$T\left(r, \frac{a}{f}\right) \leq T\left(r, \frac{f^{(k)}}{f-a}\right) + T\left(r, \frac{f^{(k)}}{f}\right) + S(r, f)$$

$$\begin{aligned} \text{i.e., } T\left(r, \frac{1}{f}\right) &\leq N\left(r, \frac{f^{(k)}}{f-a}\right) + m\left(r, \frac{f^{(k)}}{f-a}\right) + N\left(r, \frac{f^{(k)}}{f}\right) \\ &\quad + m\left(r, \frac{f^{(k)}}{f}\right) + S(r, f). \end{aligned} \quad (2.6)$$

Using Milloux's theorem $\{p.55, [24]\}$ we obtain from (2.6) that

$$T\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{f^{(k)}}{f-a}\right) + N\left(r, \frac{f^{(k)}}{f}\right) + S(r, f)$$

$$\begin{aligned} \text{i.e., } T(r, f) &\leq N(r, f^{(k)}) + N\left(r, \frac{1}{f-a}\right) + N(r, f^{(k)}) \\ &\quad + N\left(r, \frac{1}{f}\right) + S(r, f) \end{aligned}$$

$$\begin{aligned} \text{i.e., } T(r, f) &\leq 2N(r, f^{(k)}) + N\left(r, \frac{1}{f-a}\right) \\ &\quad + N\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned} \quad (2.7)$$

On dividing both sides of (2.7) by $r^{\rho_f(r)}$ and taking limit superior we get from (2.7) in view of Lemma 2.2.1,

$$\begin{aligned} &\limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \\ &\leq 2 \limsup_{r \rightarrow \infty} \frac{N(r, f^{(k)})}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f-a}\right)}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f}\right)}{r^{\rho_f(r)}} \end{aligned}$$

$$\text{i.e., } 1 \leq 2 \left\{ 1 - {}_R\delta_{\rho_f}^{(k)}(\infty; f) \right\} + \{1 - \delta_{\rho_f}(a; f)\} + \{1 - \delta_{\rho_f}(0; f)\}$$

$$\text{i.e., } 2{}_R\delta_{\rho_f}^{(k)}(\infty; f) + \delta_{\rho_f}(a; f) + \delta_{\rho_f}(0; f) \leq 3.$$

This proves the theorem. ■

Remark 2.3.3. The condition that ‘ a be any non zero finite complex number’ in Theorem 2.3.3 is essential which is evident from the following two examples.

Example 2.3.5. Let $f = \exp z$, $k = 2$ and $a = 0$.

$$\text{Then } \rho_f = 1 \text{ and } \delta_{\rho_f}(a; f) = \delta_{\rho_f}(0; f) = 1.$$

$$\text{Also } N(r, \infty; f^{(2)}) = N(r, \infty; \exp z) = 0.$$

$$\text{So } {}_R\delta_{\rho_f}^{(2)}(\infty; f) = 1.$$

Therefore

$$2{}_R\delta_{\rho_f}^{(2)}(\infty; f) + \delta_{\rho_f}(a; f) + \delta_{\rho_f}(0; f) = 4,$$

which is contrary to Theorem 2.3.3.

Example 2.3.6. Let $f = \exp z$, $k = 2$ and $a = \infty$.

$$\text{Then } \rho_f = 1, \delta_{\rho_f}(0; f) = 1 \text{ and } {}_R\delta_{\rho_f}^{(2)}(\infty; f) = 1.$$

$$\text{Also } N(r, \infty; f) = 0$$

$$\text{Then } \delta_{\rho_f}(a; f) = \delta_{\rho_f}(\infty; f) = 1.$$

Therefore

$$2{}_R\delta_{\rho_f}^{(2)}(\infty; f) + \delta_{\rho_f}(a; f) + \delta_{\rho_f}(0; f) = 4,$$

which contradicts Theorem 2.3.3.

Theorem 2.3.4. Let f be a meromorphic function such that $\rho_f < \infty$. Also let $a_i (i = 1, 2, \dots, p)$ and $b_j (j = 1, 2, \dots, q)$ be two sets of distinct finite non-zero complex numbers where p and q are both finite positive integers. Then for any positive integer k ,

$$\Theta_{\rho_f}(0; f) + q\delta_{\rho_f}(0; f) + 2\Theta_{\rho_f}(\infty; f) + \sum_{i=1}^p \Theta_{\rho_f}(a_i; f) + \sum_{j=1}^q R\Theta_{\rho_f}^{(k)}(b_j; f) \leq q+3.$$

Proof. From the relation $T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$ and by Milloux's theorem {p.55, [24]}, we get that

$$\begin{aligned} T(r, f) &= T\left(r, \frac{1}{f}\right) + O(1) \\ &\leq N\left(r, \frac{1}{f}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{1}{f^{(k)}}\right) + O(1). \end{aligned}$$

Thus,

$$T(r, f) \leq N\left(r, \frac{1}{f}\right) + T(r, f^{(k)}) - N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \quad (2.8)$$

Again by Nevanlinna's second fundamental theorem,

$$\begin{aligned} &qT(r, f^{(k)}) \\ &\leq \bar{N}(r, f^{(k)}) + \sum_{j=1}^q \bar{N}\left(r, \frac{1}{f^{(k)} - b_j}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \end{aligned} \quad (2.9)$$

Now from (2.8) and (2.9) it follows that

$$\begin{aligned} &qT(r, f) \\ &\leq qN\left(r, \frac{1}{f}\right) + \bar{N}(r, f^{(k)}) + \bar{N}\left(r, \frac{1}{f^{(k)}}\right) - qN\left(r, \frac{1}{f^{(k)}}\right) \\ &\quad + \sum_{j=1}^q \bar{N}\left(r, \frac{1}{f^{(k)} - b_j}\right) + S(r, f). \end{aligned} \quad (2.10)$$

Since $\bar{N}(r, f^{(k)}) = \bar{N}(r, f)$ and $\bar{N}\left(r, \frac{1}{f^{(k)}}\right) - qN\left(r, \frac{1}{f^{(k)}}\right) \leq 0$, it follows from (2.10) that

$$\begin{aligned} & qT(r, f) \\ & \leq \bar{N}(r, f) + \sum_{j=1}^q \bar{N}\left(r, \frac{1}{f^{(k)} - b_j}\right) \\ & \quad + qN\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned} \quad (2.11)$$

In view of Nevanlinna's second fundamental theorem,

$$\begin{aligned} & pT(r, f) \\ & \leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + \sum_{i=1}^p \bar{N}\left(r, \frac{1}{f - a_i}\right) + S(r, f). \end{aligned} \quad (2.12)$$

Adding (2.11) and (2.12) it follows that

$$\begin{aligned} & (p + q)T(r, f) \\ & \leq \bar{N}\left(r, \frac{1}{f}\right) + qN\left(r, \frac{1}{f}\right) + 2\bar{N}(r, f) + \sum_{i=1}^p \bar{N}\left(r, \frac{1}{f - a_i}\right) \\ & \quad + \sum_{j=1}^q \bar{N}\left(r, \frac{1}{f^{(k)} - b_j}\right) + S(r, f). \end{aligned} \quad (2.13)$$

On dividing both sides of (2.13) by $r^{\rho_f(r)}$ and taking limit superior we obtain by Lemma 2.2.1 that

$$\begin{aligned} & (p + q) \\ & \leq \limsup_{r \rightarrow \infty} \frac{\bar{N}\left(r, \frac{1}{f}\right)}{r^{\rho_f(r)}} + q \limsup_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f}\right)}{r^{\rho_f(r)}} + 2 \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, f)}{r^{\rho_f(r)}} \\ & \quad + \sum_{i=1}^p \limsup_{r \rightarrow \infty} \frac{\bar{N}\left(r, \frac{1}{f - a_i}\right)}{r^{\rho_f(r)}} + \sum_{j=1}^q \limsup_{r \rightarrow \infty} \frac{\bar{N}\left(r, \frac{1}{f^{(k)} - b_j}\right)}{r^{\rho_f(r)}} \end{aligned}$$

$$\begin{aligned} \text{i.e.,} \quad & (p + q) \\ & \leq \{1 - \Theta_{\rho_f}(0; f)\} + q \{1 - \delta_{\rho_f}(0; f)\} + 2 \{1 - \Theta_{\rho_f}(\infty; f)\} \\ & \quad + \left\{ p - \sum_{i=1}^p \Theta_{\rho_f}(a_i; f) \right\} + \left\{ q - \sum_{j=1}^q {}_R\Theta_{\rho_f}^{(k)}(b_j; f) \right\} \end{aligned}$$

$$\begin{aligned} \text{i.e.,} \quad & \Theta_{\rho_f}(0; f) + q\delta_{\rho_f}(0; f) + 2\Theta_{\rho_f}(\infty; f) \\ & \quad + \sum_{i=1}^p \Theta_{\rho_f}(a_i; f) + \sum_{j=1}^q {}_R\Theta_{\rho_f}^{(k)}(b_j; f) \leq q + 3. \end{aligned}$$

Thus the theorem follows. ■

Remark 2.3.4. The condition that ‘ $a_i (i = 1, 2, \dots, p)$ and $b_j (j = 1, 2, \dots, q)$ be two sets of distinct finite non-zero complex numbers’ in Theorem 2.3.4 is essential which is evident from the following two examples.

Example 2.3.7. Let $f = \exp z$, $k = 2$, $p = q = 1$, $a_1 = 0$ and $b_1 = \infty$.

Then $\rho_f = 1$.

Also $\Theta_{\rho_f}(0; f) = \delta_{\rho_f}(0; f) = \Theta_{\rho_f}(\infty; f) = 1$,

$\Theta_{\rho_f}(a_1; f) = \Theta_{\rho_f}(0; f) = 1$ and ${}_R\Theta_{\rho_f}^{(2)}(b_1; f) = {}_R\Theta_{\rho_f}^{(2)}(\infty; \exp z) = 1$.

Therefore

$$\Theta_{\rho_f}(0; f) + q\delta_{\rho_f}(0; f) + 2\Theta_{\rho_f}(\infty; f) + \Theta_{\rho_f}(a_1; f) + \Theta_{\rho_f}^{(k)}(b_1; f) = 6.$$

But $q + 3 = 4$.

Example 2.3.8. Let $f = \exp z$, $k = 2$, $p = q = 1$, $a_1 = \infty$ and $b_1 = 0$.

So $\rho_f = 1$.

Also $\Theta_{\rho_f}(0; f) = \delta_{\rho_f}(0; f) = \Theta_{\rho_f}(\infty; f) = 1$,

$\Theta_{\rho_f}(a_1; f) = \Theta_{\rho_f}(\infty; f) = 1$ and ${}_R\Theta_{\rho_f}^{(2)}(b_1; f) = {}_R\Theta_{\rho_f}^{(2)}(0; \exp z) = 1$.

Therefore

$$\Theta_{\rho_f}(0; f) + q\delta_{\rho_f}(0; f) + 2\Theta_{\rho_f}(\infty; f) + \Theta_{\rho_f}(a_1; f) + \Theta_{\rho_f}^{(k)}(b_1; f) = 6.$$

But $q + 3 = 4$.

Theorem 2.3.5. For any meromorphic function f of finite order ρ_f ,

$$\Theta_{\rho_f}(\infty; f) + {}_R\Theta_{\rho_f}^{(k)}(0; f) + q \sum_{i=1}^p \delta_{\rho_f}(a_i; f) + \sum_{j=1}^q {}_R\Theta_{\rho_f}^{(k)}(b_j; f) \leq q + 2,$$

where $a_i (i = 1, 2, \dots, p)$ and $b_j (j = 1, 2, \dots, q)$ are any two sets of distinct finite non-zero complex numbers; p, q are finite positive integers and k is any positive integer.

Proof. Let us consider

$$F = \sum_{i=1}^p \frac{1}{f - a_i}.$$

Then by {p.32-33, [24]} we obtain that

$$\sum_{i=1}^p m \left(r, \frac{1}{f - a_i} \right) \leq m(r, F) + O(1)$$

$$\begin{aligned} \text{i.e.,} \quad & \sum_{i=1}^p m \left(r, \frac{1}{f - a_i} \right) \\ & \leq m(r, F f^{(k)}) + m \left(r, \frac{1}{f^{(k)}} \right) + O(1) \\ & \leq \sum_{i=1}^p m \left(r, \frac{f^{(k)}}{f - a_i} \right) + m \left(r, \frac{1}{f^{(k)}} \right) + O(1) \\ & \leq m \left(r, \frac{1}{f^{(k)}} \right) + S(r, f). \end{aligned} \tag{2.14}$$

Adding $\sum_{i=1}^p N \left(r, \frac{1}{f - a_i} \right)$ to both sides of (2.14) we get that

$$\sum_{i=1}^p T \left(r, \frac{1}{f - a_i} \right) \leq T \left(r, \frac{1}{f^{(k)}} \right) + \sum_{i=1}^p N \left(r, \frac{1}{f - a_i} \right) + S(r, f)$$

$$\text{i.e., } pqT(r, f) \leq qT(r, f^{(k)}) + q \sum_{i=1}^p N \left(r, \frac{1}{f - a_i} \right) + S(r, f). \quad (2.15)$$

Again by Nevanlinna's second fundamental theorem and in view of $\bar{N}(r, f^{(k)}) = \bar{N}(r, f)$ we obtain that

$$\begin{aligned} & qT(r, f^{(k)}) \\ & \leq \bar{N}(r, f) + \bar{N} \left(r, \frac{1}{f^{(k)}} \right) \\ & \quad + \sum_{j=1}^q \bar{N} \left(r, \frac{1}{f^{(k)} - b_j} \right) + S(r, f). \end{aligned} \quad (2.16)$$

Now from (2.15) and (2.16) it follows that

$$\begin{aligned} & pqT(r, f) \\ & \leq \bar{N}(r, f) + \bar{N} \left(r, \frac{1}{f^{(k)}} \right) + \sum_{j=1}^q \bar{N} \left(r, \frac{1}{f^{(k)} - b_j} \right) \\ & \quad + q \sum_{i=1}^p N \left(r, \frac{1}{f - a_i} \right) + S(r, f). \end{aligned} \quad (2.17)$$

Dividing both sides of (2.17) by $r^{\rho_f(r)}$ and taking limit superior we get in view of Lemma 2.2.1 that

$$\begin{aligned} & pq \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \\ & \leq \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, f)}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{\bar{N} \left(r, \frac{1}{f^{(k)}} \right)}{r^{\rho_f(r)}} + \sum_{j=1}^q \limsup_{r \rightarrow \infty} \frac{\bar{N} \left(r, \frac{1}{f^{(k)} - b_j} \right)}{r^{\rho_f(r)}} \\ & \quad + q \sum_{i=1}^p \limsup_{r \rightarrow \infty} \frac{N \left(r, \frac{1}{f - a_i} \right)}{r^{\rho_f(r)}} \end{aligned}$$

$$\begin{aligned} \text{i.e.,} \quad & pq \\ & \leq \{1 - \Theta_{\rho_f}(\infty; f)\} + \{1 - {}_R\Theta_{\rho_f}^{(k)}(0; f)\} \\ & + \sum_{j=1}^q \{1 - {}_R\Theta_{\rho_f}^{(k)}(b_j; f)\} + q \left\{ p - \sum_{i=1}^p \delta_{\rho_f}(a_i; f) \right\} \end{aligned}$$

$$\text{i.e., } \Theta_{\rho_f}(\infty; f) + {}_R\Theta_{\rho_f}^{(k)}(0; f) + \sum_{j=1}^q {}_R\Theta_{\rho_f}^{(k)}(b_j; f) + q \sum_{i=1}^p \delta_{\rho_f}(a_i; f) \leq q + 2.$$

This proves the theorem. ■

Remark 2.3.5. The condition that ' $a_i (i = 1, 2, \dots, p)$ and $b_j (j = 1, 2, \dots, q)$ are any two sets of distinct finite non-zero complex numbers' in Theorem 2.3.5 is necessary which can be shown by the following two examples.

Example 2.3.9. Let $f = \exp z$, $k = 2$, $p = q = 1$, $a_1 = 0$ and $b_1 = \infty$.

Then $\rho_f = 1$.

$$\text{Also } \Theta_{\rho_f}(\infty; f) = {}_R\Theta_{\rho_f}^{(2)}(0; f) = 1, \delta_{\rho_f}(a_1; f) = \delta_{\rho_f}(0; f) = 1$$

$$\text{and } {}_R\Theta_{\rho_f}^{(2)}(b_1; f) = {}_R\Theta_{\rho_f}^{(2)}(\infty; \exp z) = 1.$$

Therefore

$$\Theta_{\rho_f}(\infty; f) + {}_R\Theta_{\rho_f}^{(2)}(0; f) + q\delta_{\rho_f}(a_1; f) + {}_R\Theta_{\rho_f}^{(2)}(b_1; f) = 4.$$

But $q + 2 = 3$.

Example 2.3.10. Let $f = \exp z$, $k = 2$, $p = q = 1$, $a_1 = \infty$ and $b_1 = 0$.

So $\rho_f = 1$.

$$\text{Also } \Theta_{\rho_f}(\infty; f) = {}_R\Theta_{\rho_f}^{(2)}(0; f) = 1, \delta_{\rho_f}(a_1; f) = \delta_{\rho_f}(\infty; f) = 1$$

$$\text{and } {}_R\Theta_{\rho_f}^{(2)}(b_1; f) = {}_R\Theta_{\rho_f}^{(2)}(0; \exp z) = 1.$$

Therefore

$$\Theta_{\rho_f}(\infty; f) + {}_R\Theta_{\rho_f}^{(2)}(0; f) + q\delta_{\rho_f}(a_1; f) + {}_R\Theta_{\rho_f}^{(2)}(b_1; f) = 4.$$

But $q + 2 = 3$.

Theorem 2.3.6. If f be a meromorphic function of finite order and $a_j (j = 1, 2, \dots, p), b_j (j = 1, 2, \dots, q)$ are finite distinct complex numbers such that $a_j \neq 0, b_j \neq 0$, then for any positive integer k

$$\begin{aligned} & 3\Theta_{\rho_f}(\infty; f) + \rho_f \Theta_{\rho_f}^{(k)}(0; f) + \Theta_{\rho_f}(0; f) + q\delta_{\rho_f}(0; f) \\ & + (1+q) \sum_{j=1}^p \delta_{\rho_f}(a_j; f) + 2 \sum_{j=1}^q \rho_f \Theta_{\rho_f}^{(k)}(b_j; f) \leq 2\sigma_f + 5, \end{aligned}$$

where p, q are finite positive integers.

Proof. Adding (2.13) and (2.17) we get that

$$\begin{aligned} & (p+q+pq)T(r; f) \\ & \leq 3N(r; f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) \\ & + qN\left(r, \frac{1}{f}\right) + 2 \rho_f \sum_{j=1}^q N\left(r, \frac{1}{f-b_j}\right) \\ & + 2 \sum_{j=1}^p N\left(r, \frac{1}{f-a_j}\right) + S(r; f) \end{aligned} \quad (2.18)$$

Now dividing both sides of (2.18) by $r^{\rho_f(r)}$ and taking limit superior it follows by Lemma 2.2.1 that

$$\begin{aligned} & (p+q+pq) \limsup_{r \rightarrow \infty} \frac{T(r; f)}{r^{\rho_f(r)}} \\ & \leq 3 \limsup_{r \rightarrow \infty} \frac{\tilde{N}(r; f)}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f}\right)}{r^{\rho_f(r)}} \\ & + \limsup_{r \rightarrow \infty} \frac{\tilde{N}\left(r, \frac{1}{f}\right)}{r^{\rho_f(r)}} + q \limsup_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f}\right)}{r^{\rho_f(r)}} \\ & + (1+q) \sum_{j=1}^p \limsup_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f-a_j}\right)}{r^{\rho_f(r)}} + 2 \sum_{j=1}^q \limsup_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f-b_j}\right)}{r^{\rho_f(r)}} \end{aligned}$$

$$\begin{aligned}
& \text{i.e.,} \quad (p + q + pq) \\
& \leq 3 \{1 - \Theta_{\rho_f}(\infty; f)\} + \left\{1 - {}_R\Theta_{\rho_f}^{(k)}(0; f)\right\} \\
& \quad + \{1 - \Theta_{\rho_f}(0; f)\} + q \{1 - \delta_{\rho_f}(0; f)\} \\
& \quad + (1 + q) \left\{p - \sum_{i=1}^p \delta_{\rho_f}(a_i; f)\right\} + 2 \left\{q - \sum_{j=1}^q {}_R\Theta_{\rho_f}^{(k)}(b_j; f)\right\} \\
& \text{i.e.,} \quad 3\Theta_{\rho_f}(\infty; f) + {}_R\Theta_{\rho_f}^{(k)}(0; f) + \Theta_{\rho_f}(0; f) + q\delta_{\rho_f}(0; f) \\
& \quad + (1 + q) \sum_{i=1}^p \delta_{\rho_f}(a_i; f) + 2 \sum_{j=1}^q {}_R\Theta_{\rho_f}^{(k)}(b_j; f) \leq 2q + 5. \quad (2.19)
\end{aligned}$$

Thus the theorem is proved. ■

Remark 2.3.6. The condition that ' $a_i (i = 1, 2, \dots, p)$ and $b_j (j = 1, 2, \dots, q)$ are finite distinct complex numbers such that $a_i \neq 0, b_j \neq 0$ ' in Theorem 2.3.6 is essential which is evident from the following two examples.

Example 2.3.11. Let $f = \exp z, k = 2, p = q = 1, a_1 = 0$ and $b_1 = \infty$.

Then $\rho_f = 1$.

Also $\Theta_{\rho_f}(\infty; f) = {}_R\Theta_{\rho_f}^{(2)}(0; f) = \Theta_{\rho_f}(0; f) = \delta_{\rho_f}(0; f) = 1$,

$\delta_{\rho_f}(a_1; f) = \delta_{\rho_f}(0; f) = 1$ and ${}_R\Theta_{\rho_f}^{(2)}(b_1; f) = {}_R\Theta_{\rho_f}^{(2)}(\infty; \exp z) = 1$.

Therefore

$$\begin{aligned}
& 3\Theta_{\rho_f}(\infty; f) + {}_R\Theta_{\rho_f}^{(k)}(0; f) + \Theta_{\rho_f}(0; f) + q\delta_{\rho_f}(0; f) \\
& \quad + (1 + q)\delta_{\rho_f}(a_i; f) + 2 {}_R\Theta_{\rho_f}^{(k)}(b_j; f) = 10.
\end{aligned}$$

But $2q + 5 = 7$.

Example 2.3.12. Let $f = \exp z, k = 2, p = q = 1, a_1 = \infty$ and $b_1 = 0$.

Then $\rho_f = 1$.

Also $\Theta_{\rho_f}(\infty; f) = {}_R\Theta_{\rho_f}^{(2)}(0; f) = \Theta_{\rho_f}(0; f) = \delta_{\rho_f}(0; f) = 1$,

$$\delta_{\rho_f}(a_1; f) = \delta_{\rho_f}(\infty; f) = 1 \text{ and } {}_R\Theta_{\rho_f}^{(2)}(b_1; f) = {}_R\Theta_{\rho_f}^{(2)}(0; \exp z) = 1.$$

Therefore

$$\begin{aligned} 3\Theta_{\rho_f}(\infty; f) + {}_R\Theta_{\rho_f}^{(k)}(0; f) + \Theta_{\rho_f}(0; f) + q\delta_{\rho_f}(0; f) \\ + (1+q)\delta_{\rho_f}(a_i; f) + 2{}_R\Theta_{\rho_f}^{(k)}(b_j; f) = 10. \end{aligned}$$

But $2q + 5 = 7$.

Corollary 2.3.1. If f is an entire function such that $\rho_f < \infty$ and $\sum_{i=1}^p \delta_{\rho_f}(a_i; f) = 1$, then for any positive integer k

$${}_R\Theta_{\rho_f}^{(k)}(0; f) + \Theta_{\rho_f}(0; f) + q\delta_{\rho_f}(0; f) + 2\sum_{j=1}^q {}_R\Theta_{\rho_f}^{(k)}(b_j; f) \leq q + 1.$$

Proof. Since f is an entire function.

$$\Theta_{\rho_f}(\infty; f) = 1.$$

Now in view of $\sum_{i=1}^p \delta_{\rho_f}(a_i; f) = 1$ and $\Theta_{\rho_f}(\infty; f) = 1$ it follows from (2.19) that

$$3 + {}_R\Theta_{\rho_f}^{(k)}(0; f) + \Theta_{\rho_f}(0; f) + q\delta_{\rho_f}(0; f) + (1+q) + 2\sum_{j=1}^q {}_R\Theta_{\rho_f}^{(k)}(b_j; f) \leq 2q + 5$$

$$\text{i.e., } {}_R\Theta_{\rho_f}^{(k)}(0; f) + \Theta_{\rho_f}(0; f) + q\delta_{\rho_f}(0; f) + 2\sum_{j=1}^q {}_R\Theta_{\rho_f}^{(k)}(b_j; f) \leq q + 1.$$

This proves the corollary. ■

Remark 2.3.7. The condition $\sum_{i=1}^p \delta_{\rho_f}(a_i; f) = 1$ in Corollary 2.3.1 is necessary which can be verified from the following example.

Example 2.3.13. Let $f = \exp z$, $k = 2$, $p = q = 2$, $a_1 = 0$, $a_2 = \infty$, $b_1 = 0$ and $b_2 = \infty$.

So $\rho_f = 1$.

Also $\delta_{\rho_f}(0; f) = \delta_{\rho_f}(\infty; f) = \Theta_{\rho_f}(0; f) = {}_R\Theta_{\rho_f}^{(k)}(0; f) = 1$.

$$\sum_{i=1}^2 \delta_{\rho_f}(a_i; f) = \delta_{\rho_f}(a_1; f) + \delta_{\rho_f}(a_2; f) = 2$$

and ${}_R\Theta_{\rho_f}^{(k)}(0; f) = {}_R\Theta_{\rho_f}^{(k)}(\infty; f) = 1$.

Therefore

$${}_R\Theta_{\rho_f}^{(k)}(0; f) + \Theta_{\rho_f}(0; f) + q\delta_{\rho_f}(0; f) + 2\sum_{j=1}^2 {}_R\Theta_{\rho_f}^{(k)}(b_j; f) = 8.$$

But $q + 1 = 3$.

Theorem 2.3.7. Let f be a meromorphic function of finite order ρ_f . Also let $a_i (i = 1, 2, \dots, p)$ be finite, distinct, non-zero complex numbers. Then for $p \geq 1$,

$$p\delta_{\rho_f}(0; f) + \Theta_{\rho_f}(\infty; f) + \sum_{i=1}^p {}_R\Theta_{\rho_f}^{(k)}(a_i; f) \leq p + 1.$$

Proof. From the relation $T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$ and in view of Milloux's theorem {p.55, [24]}, it follows that

$$T(r, f) \leq N\left(r, \frac{1}{f}\right) + m\left(r, \frac{f^{k_j}}{f}\right) + m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f)$$

$$\text{i.e., } T(r, f) \leq N\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f)$$

$$\text{i.e., } T(r, f) \leq N\left(r, \frac{1}{f}\right) + T\left(r, \frac{1}{f^{(k)}}\right) - N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f)$$

$$\begin{aligned} \text{i.e., } pT(r, f) &\leq pN\left(r, \frac{1}{f}\right) + pT\left(r, \frac{1}{f^{(k)}}\right) \\ &\quad - pN\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \end{aligned} \quad (2.20)$$

Since $S(r, f^{(k)}) = S(r, f)$, by Nevanlinna's second fundamental theorem we obtain that

$$\begin{aligned} &pT(r, f^{(k)}) \\ &\leq \bar{N}(r, f^{(k)}) + \bar{N}\left(r, \frac{1}{f^{(k)}}\right) \\ &\quad + \sum_{i=1}^p \bar{N}\left(r, \frac{1}{f^{(k)} - a_i}\right) + S(r, f) \end{aligned} \quad (2.21)$$

As $\bar{N}(r, f^{(k)}) = \bar{N}(r, f)$, we get from (2.20) and (2.21) that

$$\begin{aligned} &pT(r, f) \\ &\leq pN\left(r, \frac{1}{f}\right) + \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f^{(k)}}\right) + \sum_{i=1}^p \bar{N}\left(r, \frac{1}{f^{(k)} - a_i}\right) \\ &\quad - pN\left(r, \frac{1}{f^{(k)}}\right) + S(r, f) \end{aligned} \quad (2.22)$$

Since $\bar{N}\left(r, \frac{1}{f^{(k)}}\right) - pN\left(r, \frac{1}{f^{(k)}}\right) \leq 0$, it follows from (2.22) that

$$\begin{aligned} &pT(r, f) \\ &\leq pN\left(r, \frac{1}{f}\right) + \bar{N}(r, f) \\ &\quad + \sum_{i=1}^p \bar{N}\left(r, \frac{1}{f^{(k)} - a_i}\right) + S(r, f). \end{aligned} \quad (2.23)$$

Dividing both sides of (2.23) by $r^{\rho_f(r)}$ and taking limit superior we get

in view of Lemma 2.2.1 that

$$\begin{aligned} & p \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \\ & \leq p \limsup_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f}\right)}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, f)}{r^{\rho_f(r)}} + \sum_{i=1}^p \limsup_{r \rightarrow \infty} \frac{\bar{N}\left(r, \frac{1}{f^{(k)} - a_i}\right)}{r^{\rho_f(r)}} \end{aligned}$$

$$\text{i.e., } p \leq p \{1 - \delta_{\rho_f}(0; f)\} + \{1 - \Theta_{\rho_f}(\infty; f)\} + \left\{ p - \sum_{i=1}^p R\Theta_{\rho_f}^{(k)}(a_i; f) \right\}$$

$$\text{i.e., } p\delta_{\rho_f}(0; f) + \Theta_{\rho_f}(\infty; f) + \sum_{i=1}^p R\Theta_{\rho_f}^{(k)}(a_i; f) \leq p+1.$$

This proves the theorem. ■

Remark 2.3.8. The condition that ' $a_i (i = 1, 2, \dots, p)$ be finite, distinct, non-zero complex numbers' in Theorem 2.3.7 is essential which is evident from the following two examples.

Example 2.3.14. Let $f = \exp z$, $k = 2$, $p = 1$ and $a_1 = 0$.

Then $\rho_f = 1$.

Also $\delta_{\rho_f}(0; f) = \Theta_{\rho_f}(\infty; f) = 1$ and ${}_R\Theta_{\rho_f}^{(2)}(a_1; f) = {}_R\Theta_{\rho_f}^{(2)}(0; f) = 1$.

Therefore

$$p\delta_{\rho_f}(0; f) + \Theta_{\rho_f}(\infty; f) + {}_R\Theta_{\rho_f}^{(2)}(a_1; f) = 3.$$

But $p + 1 = 2$.

Example 2.3.15. Let $f = \exp z$, $k = 2$, $p = 1$ and $a_1 = \infty$.

Then $\rho_f = 1$.

Again $\delta_{\rho_f}(0; f) = \Theta_{\rho_f}(\infty; f) = 1$

and ${}_R\Theta_{\rho_f}^{(2)}(a_1; f) = {}_R\Theta_{\rho_f}^{(2)}(\infty; f) = 1$.

Therefore

$$p\delta_{\rho_f}(0; f) + \Theta_{\rho_f}(\infty; f) + {}_R\Theta_{\rho_f}^{(2)}(a_1; f) = 3.$$

But $p + 1 = 2$.

Theorem 2.3.8. If $a \neq 0, \infty$ and $b \neq 0, \infty$ be any two complex numbers, then for any meromorphic function f of finite order ρ_f ,

$${}_R\Theta_{\rho_f}^{(k)}(a; f) + {}_R\Theta_{\rho_f}^{(k)}(b; f) + \delta_{\rho_f}(0; f) \leq 2$$

Proof. In view of Milloux's theorem [p.55, [24]], we get that

$$\begin{aligned} m\left(r, \frac{1}{f}\right) &\leq m\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{1}{f^{(k)}}\right) \\ \text{i.e., } m\left(r, \frac{1}{f}\right) &\leq m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f) \\ \text{i.e., } m\left(r, \frac{1}{f}\right) &\leq T\left(r, \frac{1}{f^{(k)}}\right) - N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \end{aligned} \quad (2.24)$$

Now by the relation $T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$ and by Nevanlinna's second fundamental theorem we obtain from (2.24), on using

$$\bar{N}\left(r, \frac{1}{f^{(k)}}\right) - N\left(r, \frac{1}{f^{(k)}}\right) \leq 0.$$

$$\begin{aligned} &m\left(r, \frac{1}{f}\right) \\ &\leq \bar{N}\left(r, \frac{1}{f^{(k)}}\right) + \bar{N}\left(r, \frac{1}{f^{(k)} - a}\right) + \bar{N}\left(r, \frac{1}{f^{(k)} - b}\right) \\ &\quad - N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{f^{(k)} - a}\right) + \bar{N}\left(r, \frac{1}{f^{(k)} - b}\right) + S(r, f) \end{aligned}$$

$$\begin{aligned} \text{i.e., } T(r, f) &\leq \bar{N}\left(r, \frac{1}{f^{(k)} - a}\right) + \bar{N}\left(r, \frac{1}{f^{(k)} - b}\right) \\ &\quad + N\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned} \quad (2.25)$$

Now dividing both sides of (2.25) by $r^{\rho_f(r)}$ and taking limit superior it

follows from Lemma 2.2.1 that

$$\begin{aligned} & \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \\ & \leq \limsup_{r \rightarrow \infty} \frac{\bar{N}\left(r, \frac{1}{f^{(k)}-a}\right)}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{\bar{N}\left(r, \frac{1}{f^{(k)}-b}\right)}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f}\right)}{r^{\rho_f(r)}} \end{aligned}$$

$$\text{i.e., } 1 \leq \{1 - {}_R\Theta_{\rho_f}^{(k)}(a; f)\} + \{1 - {}_R\Theta_{\rho_f}^{(k)}(b; f)\} + \{1 - \delta_{\rho_f}(0; f)\}$$

$$\text{i.e., } {}_R\Theta_{\rho_f}^{(k)}(a; f) + {}_R\Theta_{\rho_f}^{(k)}(b; f) + \delta_{\rho_f}(0; f) \leq 2.$$

Thus the theorem is established. ■

Remark 2.3.9. The condition that ' $a \neq 0, \infty$ and $b \neq 0, \infty$ ' in Theorem 2.3.8 is essential which can be verified from the following examples.

Example 2.3.16. Let $f = \exp z$, $k = 2$ and $a = b = 0$

Then $\rho_f = 1$.

Also ${}_R\Theta_{\rho_f}^{(2)}(a; f) = {}_R\Theta_{\rho_f}^{(2)}(0; f) = {}_R\Theta_{\rho_f}^{(2)}(b; f) = 1$ and $\delta_{\rho_f}(0; f) = 1$.

Therefore

$${}_R\Theta_{\rho_f}^{(2)}(a; f) + {}_R\Theta_{\rho_f}^{(2)}(b; f) + \delta_{\rho_f}(0; f) = 3.$$

which is contrary to Theorem 2.3.8

Example 2.3.17. Let $f = \exp z$, $k = 2$, $a = 0$ and $b = \infty$.

Then $\rho_f = 1$.

Also ${}_R\Theta_{\rho_f}^{(2)}(a; f) = {}_R\Theta_{\rho_f}^{(2)}(0; f) = 1$, ${}_R\Theta_{\rho_f}^{(2)}(b; f) = {}_R\Theta_{\rho_f}^{(2)}(\infty; f) = 1$

and $\delta_{\rho_f}(0; f) = 1$.

Therefore

$${}_R\Theta_{\rho_f}^{(2)}(a; f) + {}_R\Theta_{\rho_f}^{(2)}(b; f) + \delta_{\rho_f}(0; f) = 3$$

which contradicts Theorem 2.3.8.

Example 2.3.18. Let $f = \exp z$, $k = 2$, $a = \infty$ and $b = 0$.

Then $\rho_f = 1$.

Also ${}_R\Theta_{\rho_f}^{(2)}(a; f) = {}_R\Theta_{\rho_f}^{(2)}(\infty; f) = 1$, ${}_R\Theta_{\rho_f}^{(2)}(b; f) = {}_R\Theta_{\rho_f}^{(2)}(0; f) = 1$
and $\delta_{\rho_f}(0; f) = 1$

Therefore

$${}_R\Theta_{\rho_f}^{(2)}(a; f) + {}_R\Theta_{\rho_f}^{(2)}(b; f) + \delta_{\rho_f}(0; f) = 3.$$

which is contrary to Theorem 2.3.8

Example 2.3.19. Let $f = \exp z$, $k = 2$ and $a = b = \infty$.

Then $\rho_f = 1$.

Again ${}_R\Theta_{\rho_f}^{(2)}(a; f) = {}_R\Theta_{\rho_f}^{(2)}(\infty; f) = {}_R\Theta_{\rho_f}^{(2)}(b; f) = 1$
and $\delta_{\rho_f}(0; f) = 1$.

Therefore

$${}_R\Theta_{\rho_f}^{(2)}(a; f) + {}_R\Theta_{\rho_f}^{(2)}(b; f) + \delta_{\rho_f}(0; f) = 3.$$

which is contrary to Theorem 2.3.8.

Theorem 2.3.9. Let f be a meromorphic function of finite order ρ_f and $T(r, f^{(k)}) \sim aT(r, f)$ i.e., $T(r, f^{(k)}) = aT(r, f) + S(r, f)$ where $a \geq 1$. Then

$$k\Theta_{\rho_f}(\infty; f) + a \leq k + 1.$$

Proof. Since f is of finite order it follows that

$$m(r, f^{(k)}) \leq m(r, f) + S(r, f)$$

and therefore

$$T(r, f^{(k)}) = N(r, f^{(k)}) + m(r, f^{(k)})$$

$$\text{i.e., } T(r, f^{(k)}) \leq N(r, f) + k\bar{N}(r, f) + m(r, f) + S(r, f)$$

$$\text{i.e., } T(r, f^{(k)}) \leq T(r, f) + k\bar{N}(r, f) + S(r, f). \quad (2.26)$$

But by hypothesis $T(r, f^{(k)}) \sim aT(r, f)$ and we get from (2.26) that

$$\begin{aligned} aT(r, f) &\leq T(r, f) + k\bar{N}(r, f) + S(r, f) \\ \text{i.e., } (a-1)T(r, f) &\leq k\bar{N}(r, f) + S(r, f). \end{aligned} \quad (2.27)$$

Now dividing both sides of (2.27) by $r^{\rho_f(r)}$ and taking limit superior it follows that

$$\begin{aligned} (a-1)\limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f(r)}} &\leq k\limsup_{r \rightarrow \infty} \frac{\bar{N}(r, f)}{r^{\rho_f(r)}} \\ \text{i.e., } (a-1) &\leq k\{1 - \Theta_{\rho_f}(\infty; f)\} \\ \text{i.e., } k\Theta_{\rho_f}(\infty; f) + a &\leq k + 1. \end{aligned}$$

This proves the theorem. ■

Remark 2.3.10. The sign ' \leq ' in Theorem 2.3.9 cannot be replaced by '<' only as we see in the following example.

Example 2.3.20. Let $f = \exp z$, $k = 2$ and $a = 1$.

Then $\rho_f = 1$.

Also $T(r, f^{(2)}) \sim aT(r, f)$ and $\Theta_{\rho_f}(\infty; f) = 1$.

Therefore $k\Theta_{\rho_f}(\infty; f) + a = 3 = k + 1$.

Theorem 2.3.10. Let $a \neq 0$, ∞ be any complex number. Then for any meromorphic function f with finite order ρ_f ,

$$\Theta_{\rho_f}(0; f) + \Theta_{\rho_f}(a; f) + {}_R\delta_{\rho_f}^{(k)}(\infty; f) \leq 2$$

Proof. The following identity is well known {cf.p.43, [24]}

$$T(r, f) \leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f-a}\right) + S(r, f). \quad (2.28)$$

Now from (2.28) we have for all non-negative integer k ,

$$\begin{aligned} T(r, f) &\leq k\bar{N}(r, f) + \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{f-a}\right) + S(r, f) \end{aligned}$$

$$\begin{aligned} \text{i.e., } T(r, f) &\leq N(r, f^{(k)}) + N\left(r, \frac{1}{f}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{f-a}\right) + S(r, f). \end{aligned} \quad (2.29)$$

Dividing both sides of (2.29) by $r^{\rho_f(r)}$ and taking limit superior it follows in view of Lemma 2.2.1 that

$$\begin{aligned} &\limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \\ &\leq \limsup_{r \rightarrow \infty} \frac{N(r, f^{(k)})}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{\bar{N}\left(r, \frac{1}{f}\right)}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{\bar{N}\left(r, \frac{1}{f-a}\right)}{r^{\rho_f(r)}} \end{aligned}$$

$$\text{i.e., } 1 \leq \{1 - {}_R\delta_{\rho_f}^{(k)}(\infty; f)\} + \{1 - \Theta_{\rho_f}(0; f)\} + \{1 - \Theta_{\rho_f}(a; f)\}$$

$$\text{i.e., } {}_R\delta_{\rho_f}^{(k)}(\infty; f) + \Theta_{\rho_f}(0; f) + \Theta_{\rho_f}(a; f) \leq 2.$$

Thus the theorem is proved. ■

Remark 2.3.11. The condition ' $a \neq 0, \infty$ ' in Theorem 2.3.10 is essential which is evident from the following two examples.

Example 2.3.21. Let $f = \exp z$, $k = 2$ and $a = 0$.

Then $\rho_f = 1$.

Also $\Theta_{\rho_f}(0; f) = \Theta_{\rho_f}(a; f) = 1$ and ${}_R\delta_{\rho_f}^{(k)}(\infty; f) = 1$.

Therefore

$$\Theta_{\rho_f}(0; f) + \Theta_{\rho_f}(a; f) + {}_R\delta_{\rho_f}^{(k)}(\infty; f) = 3,$$

which contradicts Theorem 2.3.10.

Example 2.3.22. Let $f = \exp z$, $k = 2$ and $a = \infty$.

Then $\rho_f = 1$.

Also $\Theta_{\rho_f}(0; f) = 1$, $\Theta_{\rho_f}(a; f) = \Theta_{\rho_f}(\infty; f) = 1$

and ${}_R\delta_{\rho_f}^{(k)}(\infty; f) = 1$.

Therefore

$$\Theta_{\rho_f}(0; f) + \Theta_{\rho_f}(a; f) + {}_R\delta_{\rho_f}^{(k)}(\infty; f) = 3,$$

which is contrary to Theorem 2.3.10

Theorem 2.3.11. If f be a meromorphic function of finite order and a, b, c be three distinct finite complex numbers with $b \neq 0, c \neq 0$, then

$$\delta_{\rho_f}(a; f) + {}_R\Theta_{\rho_f}^{(k)}(b; f) + {}_R\Theta_{\rho_f}^{(k)}(c; f) \leq 2.$$

Proof. By Milloux's theorem {p.55, [24]}, we get that

$$m\left(r, \frac{1}{f-a}\right) \leq m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \quad (2.30)$$

So by the relation $T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$ it follows from (2.30) that

$$m\left(r, \frac{1}{f-a}\right) \leq T(r, f^{(k)}) - N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \quad (2.31)$$

Again in view of Nevanlinna's second fundamental theorem and $S(r, f^{(k)}) = S(r, f)$ we get that

$$\begin{aligned} T(r, f^{(k)}) &\leq \bar{N}\left(r, \frac{1}{f^{(k)}}\right) + \bar{N}\left(r, \frac{1}{f^{(k)}-b}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{f^{(k)}-c}\right) + S(r, f) \end{aligned} \quad (2.32)$$

Combining (2.31) and (2.32) we obtain that

$$\begin{aligned} &m\left(r, \frac{1}{f-a}\right) \\ &\leq \bar{N}\left(r, \frac{1}{f^{(k)}}\right) + \bar{N}\left(r, \frac{1}{f^{(k)}-b}\right) + \bar{N}\left(r, \frac{1}{f^{(k)}-c}\right) \\ &\quad - N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \end{aligned}$$

Since $\bar{N}\left(r, \frac{1}{f^{(k)}}\right) - N\left(r, \frac{1}{f^{(k)}}\right) \leq 0$, it follows from above in view of $T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1)$,

$$\begin{aligned} T(r, f) &\leq \bar{N}\left(r, \frac{1}{f^{(k)}-b}\right) + \bar{N}\left(r, \frac{1}{f^{(k)}-c}\right) \\ &\quad + N\left(r, \frac{1}{f-a}\right) + S(r, f). \end{aligned} \quad (2.33)$$

Dividing both sides of (2.33) by $r^{\rho_f(r)}$ and taking limit superior we get by Lemma 2.2.1 that

$$\begin{aligned} &\limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \\ &\leq \limsup_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f-a}\right)}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{\bar{N}\left(r, \frac{1}{f^{(k)}-b}\right)}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{\bar{N}\left(r, \frac{1}{f^{(k)}-c}\right)}{r^{\rho_f(r)}} \end{aligned}$$

$$\text{i.e., } 1 \leq \{1 - \delta_{\rho_f}(a; f)\} + \{1 - {}_R\Theta_{\rho_f}^{(k)}(b; f)\} + \{1 - {}_R\Theta_{\rho_f}^{(k)}(c; f)\}$$

$$\text{i.e., } \delta_{\rho_f}(a; f) + {}_R\Theta_{\rho_f}^{(k)}(b; f) + {}_R\Theta_{\rho_f}^{(k)}(c; f) \leq 2.$$

This proves the theorem. ■

Theorem 2.3.12. Let f be a meromorphic function of finite order ρ_f . Also let almost all zeros of f has multiplicity $\geq n$. Then for all non-negative integer k ,

$$(k+1)\delta_{\rho_f}(0; f) + n {}_R\Theta_{\rho_f}^{(k)}(1; f) + n \Theta_{\rho_f}(\infty; f) \leq n + k + 1.$$

Proof. Let us consider the identity

$$\frac{1}{f} = \left(\frac{f^{(k)}}{f} - \frac{f^{(k)}-1}{f^{(k+1)}} \cdot \frac{f^{(k+1)}}{f} \right).$$

Then by Milloux's theorem {p.55, [24]}, we get that

$$m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{f^{(k)}-1}{f^{(k+1)}}\right) + S(r, f). \quad (2.34)$$

Using the relation $T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$ and in view of {p.34, [24]} we obtain that

$$\begin{aligned}
m\left(r, \frac{1}{f}\right) &\leq T\left(r, \frac{f^{(k)} - 1}{f^{(k+1)}}\right) - N\left(r, \frac{f^{(k)} - 1}{f^{(k+1)}}\right) - S(r, f) \\
\text{i.e., } m\left(r, \frac{1}{f}\right) &\leq T\left(r, \frac{f^{(k+1)}}{f^{(k)} - 1}\right) - N\left(r, \frac{f^{(k)} - 1}{f^{(k+1)}}\right) + S(r, f) \\
\text{i.e., } m\left(r, \frac{1}{f}\right) &\leq N\left(r, \frac{f^{(k+1)}}{f^{(k)} - 1}\right) - N\left(r, \frac{f^{(k)} - 1}{f^{(k+1)}}\right) + S(r, f) \\
\text{i.e., } m\left(r, \frac{1}{f}\right) &\leq N\left(r, f^{(k+1)}\right) + N\left(r, \frac{1}{f^{(k)} - 1}\right) - N\left(r, f^{(k)} - 1\right) \\
&\quad - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f) \\
\text{i.e., } T(r, f) &\leq N\left(r, f^{(k+1)}\right) + N\left(r, \frac{1}{f^{(k)} - 1}\right) - N\left(r, f^{(k)} - 1\right) \\
&\quad - N\left(r, \frac{1}{f^{(k+1)}}\right) + N\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.35}
\end{aligned}$$

Let $N_a\left(r, \frac{1}{f^{(k+1)}}\right)$ denotes the zeros of $f^{(k+1)}$ which are not the zeros of $(f^{(k)} - 1)$. So from (2.35) we get that

$$\begin{aligned}
&T(r, f) \\
&\leq N\left(r, \frac{1}{f}\right) + \left\{N\left(r, f^{(k+1)}\right) - N\left(r, f^{(k)}\right)\right\} \\
&\quad + \left\{\bar{N}\left(r, \frac{1}{f^{(k)}}\right) - N_a\left(r, \frac{1}{f^{(k+1)}}\right)\right\} + S(r, f)
\end{aligned}$$

$$\begin{aligned}
\text{i.e., } & T(r, f) \\
& \leq N\left(r, \frac{1}{f}\right) + \bar{N}\left(r, f^{(k)}\right) + \bar{N}\left(r, \frac{1}{f^{(k)} - 1}\right) \\
& \quad - N_a\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f).
\end{aligned} \tag{2.36}$$

Let $N_0\left(r, \frac{1}{f}\right)$ denotes all zeros of f taken with proper multiplicities if the multiplicity is $\leq k + 1$ and each zero of multiplicity $\geq k + 2$ being counted $(k + 1)$ times only. Also

$$\bar{N}\left(r, f^{(k)}\right) = N(r, f).$$

Therefore from (2.36) we obtain that

$$\begin{aligned}
& T(r, f) \\
& \leq N\left(r, \frac{1}{f}\right) + \bar{N}(r, f) - N_a\left(r, \frac{1}{f^{(k+1)}}\right) + \bar{N}\left(r, \frac{1}{f^{(k)} - 1}\right) + S(r, f)
\end{aligned}$$

$$\begin{aligned}
& \text{i.e., } T(r, f) \\
& \leq N_0\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f^{(k)} - 1}\right) + \bar{N}(r, f) + S(r, f).
\end{aligned} \tag{2.37}$$

Again we may write

$$nN_0\left(r, \frac{1}{f}\right) \leq (k + 1)N\left(r, \frac{1}{f}\right). \tag{2.38}$$

Since on the left hand side of (2.38) each zero is counted at least $n(k + 1)$ times as it is given that each zero of f is of order $\geq n$. So from (2.37) and (2.38) it follows that

$$\begin{aligned}
& T(r, f) \\
& \leq \left(\frac{k + 1}{n}\right) N\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f^{(k)} - 1}\right) \\
& \quad + \bar{N}(r, f) + S(r, f).
\end{aligned} \tag{2.39}$$

Dividing both sides of (2.39) by $r^{\rho_f(r)}$ and taking limit superior we get in view of Lemma 2.2.1 that

$$\begin{aligned} & \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \\ & \leq \left(\frac{k+1}{n} \right) \limsup_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f}\right)}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{\bar{N}\left(r, \frac{1}{f^{(k)}-1}\right)}{r^{\rho_f(r)}} \\ & \quad + \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, f)}{r^{\rho_f(r)}} + \limsup_{r \rightarrow \infty} \frac{S(r, f)}{r^{\rho_f(r)}} \end{aligned}$$

$$\text{i.e., } 1 \leq \left(\frac{k+1}{n} \right) \{1 - \delta_{\rho_f}(0; f)\} + \{1 - {}_R\Theta_{\rho_f}^{(k)}(1; f)\} + \{1 - \Theta_{\rho_f}(\infty; f)\}$$

$$\text{i.e., } (k+1)\delta_{\rho_f}(0; f) + n {}_R\Theta_{\rho_f}^{(k)}(1; f) + n \Theta_{\rho_f}(\infty; f) \leq n+k+1.$$

Thus the theorem follows. ■

We now give a lower bound for relative proximate defect of any complex number a with respect to the derivative $f^{(k)}$ in terms of k and $\Theta_{\rho_f}(\infty; f)$ where $k = 1, 2, 3, \dots$.

Theorem 2.3.13. If f be a transcendental meromorphic function of finite order, then for any integer $k \geq 1$,

$$(i) \quad {}_R\delta_{\rho_f}^{(k)}(a; f) + k \geq k\Theta_{\rho_f}(\infty; f) \quad \text{for every complex number } a \neq \infty$$

$$\text{and (ii) } {}_R\delta_{\rho_f}^{(k)}(a; f) + k \geq k\Theta_{\rho_f}(\infty; f) + \delta_{\rho_f}(\infty; f), \text{ if } a = \infty.$$

Proof. (i) By the relation $T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$ we get that

$$m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) = m(r, f) + N(r, f) + S(r, f)$$

$$\text{i.e., } N\left(r, \frac{1}{f}\right) - N(r, f) = m(r, f) - m\left(r, \frac{1}{f}\right) + S(r, f)$$

$$\begin{aligned}
\text{i.e., } & N\left(r, \frac{1}{f}\right) - N(r, f) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f| d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{f} \right| d\theta + S(r, f) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log |f| d\theta + S(r, f). \tag{2.40}
\end{aligned}$$

Similarly using (2.40) we may write

$$\begin{aligned}
& N\left(r, f^{(k)} - a\right) - N\left(r, \frac{1}{f^{(k)} - a}\right) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{1}{f^{(k)} - a} \right| d\theta + S(r, f) \tag{2.41}
\end{aligned}$$

and

$$\begin{aligned}
& N\left(r, \frac{f^{(k)} - a}{f}\right) - N\left(r, \frac{f}{f^{(k)} - a}\right) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f}{f^{(k)} - a} \right| d\theta + S(r, f) \tag{2.42}
\end{aligned}$$

Combining (2.40), (2.41) and (2.42) it follows in view of the relation

$T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$ that

$$\begin{aligned}
& \left\{ N\left(r, \frac{1}{f}\right) - N(r, f) \right\} + \left\{ N\left(r, f^{(k)} - a\right) - N\left(r, \frac{1}{f^{(k)} - a}\right) \right\} \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f}{f^{(k)} - a} \right| d\theta + S(r, f) \\
&= N\left(r, \frac{f^{(k)} - a}{f}\right) - N\left(r, \frac{f}{f^{(k)} - a}\right)
\end{aligned}$$

$$\begin{aligned}
\text{i.e., } & N\left(r, \frac{1}{f^{(k)} - a}\right) \\
&= N\left(r, \frac{1}{f}\right) - N(r, f) + N(r, f^{(k)} - a) - N\left(r, \frac{f^{(k)} - a}{f}\right) \\
&\quad + N\left(r, \frac{f}{f^{(k)} - a}\right) + S(r, f) \\
&= N\left(r, \frac{1}{f}\right) - N(r, f) + N(r, f) + k\bar{N}(r, f) \\
&\quad - \left\{T\left(r, \frac{f^{(k)} - a}{f}\right) - m\left(r, \frac{f^{(k)} - a}{f}\right)\right\} \\
&\quad + \left\{T\left(r, \frac{f}{f^{(k)} - a}\right) - m\left(r, \frac{f}{f^{(k)} - a}\right)\right\} + S(r, f) \\
&= N\left(r, \frac{1}{f}\right) + k\bar{N}(r, f) + m\left(r, \frac{f^{(k)} - a}{f}\right) \\
&\quad - m\left(r, \frac{f}{f^{(k)} - a}\right) + S(r, f) \tag{2.43}
\end{aligned}$$

We immediately obtain from (2.43) that

$$\begin{aligned}
N\left(r, \frac{1}{f^{(k)} - a}\right) &\leq N\left(r, \frac{1}{f}\right) + k\bar{N}(r, f) - m\left(r, \frac{1}{f}\right) + S(r, f) \\
\text{i.e., } N\left(r, \frac{1}{f^{(k)} - a}\right) &\leq T(r, f) + k\bar{N}(r, f) + S(r, f). \tag{2.44}
\end{aligned}$$

Dividing both sides of (2.44) by $r^{\rho_f(r)}$ and taking limit superior we obtain that

$$\begin{aligned}
\limsup_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f^{(k)} - a}\right)}{r^{\rho_f(r)}} &\leq \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f(r)}} + k \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, f)}{r^{\rho_f(r)}} \\
\text{i.e., } 1 - {}_R\delta_{\rho_f}^{(k)}(a; f) &\leq 1 + k \{1 - \Theta_{\rho_f}(\infty; f)\}
\end{aligned}$$

$$\text{i.e., } {}_R\delta_{\rho_f}^{(k)}(a; f) + k \geq k\Theta_{\rho_f}(\infty; f).$$

This proves the first part of Theorem 2.3.13.

(ii) Since $N(r, f^{(k)}) = N(r, f) + k\bar{N}(r, f)$, dividing both sides by $r^{\rho_f(r)}$ and taking limit superior it follows that

$$\limsup_{r \rightarrow \infty} \frac{N(r, f^{(k)})}{r^{\rho_f(r)}} \leq \limsup_{r \rightarrow \infty} \frac{N(r, f)}{r^{\rho_f(r)}} + k \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, f)}{r^{\rho_f(r)}}$$

$$\text{i.e., } 1 - {}_R\delta_{\rho_f}^{(k)}(\infty; f) \leq \{1 - \delta_{\rho_f}(\infty; f)\} + k\{1 - \Theta_{\rho_f}(\infty; f)\}$$

$$\text{i.e., } {}_R\delta_{\rho_f}^{(k)}(\infty; f) + k \geq k\Theta_{\rho_f}(\infty; f) + \delta_{\rho_f}(\infty; f).$$

Thus the second part of Theorem 2.3.13 follows. ■

Theorem 2.3.14. If f, g share $0, 1, \infty$ CM and $f \neq g$ then

$$(i) \delta_{1|\rho_f}(0; f) + \delta_{1|\rho_f}(1; f) + \delta_{1|\rho_f}(\infty; f) + \sum_{a \neq 0, 1, \infty} \delta_{2|\rho_f}(a; f) \leq 2$$

$$\text{and (ii) } \delta_{1|\rho_g}(0; g) + \delta_{1|\rho_g}(1; g) + \delta_{1|\rho_g}(\infty; g) + \sum_{a \neq 0, 1, \infty} \delta_{2|\rho_g}(a; g) \leq 2.$$

Proof. By Nevanlinna's second fundamental theorem, Lemma 2.2.2 and Lemma 2.2.3 we get for pairwise distinct complex numbers $a_1, a_2, \dots, a_n (a_i \neq 0, 1, \infty; i = 1, 2, \dots, n)$

$$\begin{aligned} & (n+1)T(r, f) \\ & \leq N(r, 0; f | \neq 1) + N(r, 1; f | \neq 1) + N(r, \infty; f | \neq 1) \\ & \quad + \sum_{i=1}^n N(r, a_i; f | \leq 2) + S(r, f). \end{aligned} \tag{2.45}$$

On dividing both sides of (2.45) by $r^{\rho_f(r)}$ and taking limit superior we obtain that

$$\begin{aligned} (n+1) & \leq \{1 - \delta_{1|\rho_f}(0; f)\} + \{1 - \delta_{1|\rho_f}(1; f)\} \\ & \quad + \{1 - \delta_{1|\rho_f}(\infty; f)\} + n - \sum_{a \neq 0, 1, \infty} \delta_{2|\rho_f}(a; f), \end{aligned}$$

from which (i) follows.

Similarly by Nevanlinna's second fundamental theorem, Lemma 2.2.2 and Lemma 2.2.3 we obtain for pairwise distinct complex numbers $a_1, a_2, \dots, a_n (a_i \neq 0, 1, \infty; i = 1, 2, \dots, n)$

$$\begin{aligned} & (n+1)T(r, g) \\ & \leq N(r, 0; g | \neq 1) + N(r, 1; g | \neq 1) + N(r, \infty; g | \neq 1) \\ & \quad + \sum_{i=1}^n N(r, a_i; g | \leq 2) + S(r, g). \end{aligned} \quad (2.46)$$

On dividing both sides of (2.46) by $r^{\rho_g(r)}$ and taking limit superior we get that

$$\begin{aligned} (n+1) & \leq \{1 - \delta_{1|\rho_g}(0; g)\} + \{1 - \delta_{1|\rho_g}(1; g)\} \\ & \quad + \{1 - \delta_{1|\rho_g}(\infty; g)\} + n - \sum_{a \neq 0, 1, \infty} \delta_{2|\rho_g}(a; g), \end{aligned}$$

from which (ii) follows. ■

Theorem 2.3.15. If f, g share $0, 1, \infty$ CM and $f \not\equiv g$ then

$$(i) \delta_{1|}(0; f) + \delta_{1|}(1; f) + \delta_{1|}(\infty; f) + \sum_{a \neq 0, 1, \infty} \delta_{2|}(a; f) \leq 2$$

$$\text{and (ii) } \delta_{1|}(0; g) + \delta_{1|}(1; g) + \delta_{1|}(\infty; g) + \sum_{a \neq 0, 1, \infty} \delta_{2|}(a; g) \leq 2.$$

Proof. Dividing both sides of (2.45) by $T(r, f)$ and taking limit superior we get that

$$\begin{aligned} (n+1) & \leq \{1 - \delta_{1|}(0; f)\} + \{1 - \delta_{1|}(1; f)\} \\ & \quad + \{1 - \delta_{1|}(\infty; f)\} + n - \sum_{a \neq 0, 1, \infty} \delta_{2|}(a; f), \end{aligned}$$

from which (i) follows.

Again dividing both sides of (2.46) by $T(r, g)$ and taking limit superior we obtain that

$$(n+1) \leq \{1 - \delta_1(0; g)\} + \{1 - \delta_1(1; g)\} \\ + \{1 - \delta_1(\infty; g)\} + n - \sum_{a \neq 0, 1, \infty} \delta_2(a; g),$$

from which (ii) follows. ■
