NOTE ON THE UNSTEADY FLOW OF VISCOUS INCOMPRESSIBLE FLUID OF FINITE DEPTH OVER A RIGID PLANE BASE DUE TO AN ARBITRARY TIME - VARYING PRESSURE GRADIENT *

1. Introduction :- The problem of unidirectional flow of viscous incompressible fluid acted upon by pressure gradient has received the attention of several investigators. Mithal[12] (1960) has discussed the non-steady flow of a viscous incompressible fluid through a straight tube of circular cross-section in the presence of time-varying pressure gradient. The non-steady flow of viscous incompressible fluid through a straight channel with two parallel flat walls has also been studied by Satya Prakash[20] (1967) in the presence of time varying pressure gradient. In the present note, we consider the laminar motion set up in a viscous incompressible fluid by pressure-gradient acting in the direction of the flow. In such a case the pressure gradient is a function of time but is independent of space co-ordinates. The general solution is obtained by using the method of Laplace Transform. Velocity distributions for some particular types of pressure gradients namely (i) finite (ii) impulsive and (iii) admitting a periodic pulse are calculated in finite forms.

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2. Formulation of problem :-

A co-ordinate system is chosen in which the origin lies on the free-surface of the fluid and X-axis is in the direction of the flow and Y-axis is normal to X and normal to the boundary and it is directed vertically upwards. Let the depth of the fluid be h. Let us consider the motion in the direction of X - axis, then the equation of continuity becomes,

$$\frac{\partial u}{\partial x} = 0 \qquad (1)$$

Therefore u is a function of y, t Say   u = u(y,t)

The Stokes - Navier's equation for slow motion becomes

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (2)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (3)$$

From (1), (2) and (3) it follows that $\frac{\partial p}{\partial x}$ is a function of time alone.

Let us suppose that

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = f(t)$$

Let us assume that there is no external force and ρ denotes the density of the fluid, and ν , u, t and p denote the Kinematic Coefficient of viscosity, the component of velocity along X-direction, time and the pressure at any point (x,y) respectively

The initial and boundary conditions are

$$\left.\begin{array}{llll} (a)\ u = 0 & \text{when } -h \leq y \leq 0 & \text{and } t = & 0 \\ (b)\ u = 0 & \text{when } y = -h & \text{and } t > & 0 \\ (c)\ \frac{\partial u}{\partial y} = 0 & \text{when } y = 0 & \text{and } t > & 0 \end{array}\right| \quad (4)$$

3. Method of Solution :-

Let us introduce the Laplace Transform[25] $\bar u(\lambda) = \int_0^\infty e^{-\lambda t} u \, dt$

The subsidiary equation corresponding to (2) is

$$\frac{d^2 \bar u}{dy^2} - \frac{\lambda}{\nu}\bar u = -\frac{\bar f(\lambda)}{\nu} \qquad (5)$$

Condition (4) become

$$\left.\begin{array}{lll} (a)\ \bar u = 0 & \text{when} & -h = y \\ (b)\ \frac{\partial \bar u}{\partial y} = 0 & \text{when} & y = 0 \end{array}\right\} \qquad (6)$$

Solution of equation (5) together with the condition (6) is

$$\bar u = \frac{\bar f(\lambda)}{\lambda}\left(1 - \frac{\cosh\sqrt{\frac{\lambda}{\nu}}\,y}{\cosh\sqrt{\frac{\lambda}{\nu}}\,h}\right) \qquad (7)$$

Now $\bar f(\lambda)$ is the Laplace Transform of $f(t)$

and $\left(\frac{1}{\lambda} - \frac{1}{\lambda}\,\frac{\cosh\sqrt{\frac{\lambda}{\nu}}\,y}{\cosh\sqrt{\frac{\lambda}{\nu}}\,h}\right)$ is the Laplace

Transform of $\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n \cos(2n+1)\frac{\pi y}{2h}}{(2n+1)} e^{-(2n+1)^2\pi^2\nu t/4h^2}$ $= g(y,t)$ say

Thus applying Convolution theorem[5], we have

$$u(y,t) = \int_0^t f(\tau)\, g(y, t-\tau)\, d\tau \tag{8}$$

$$= \int_0^t g(y,\tau)\, f(t-\tau)\, d\tau \tag{8a}$$

$$= \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n \cos(2n+1)\pi y/2h}{2n+1}\int_0^t e^{-(2n+1)^2\pi^2\nu\tau/4h^2} f(t-\tau)\, d\tau \tag{8b}$$

4. Particular types of Pressure gradients.

a) Constant pressure gradient :-

We take $f(t) = A\, H(t)$

where $H(t)$ is Heavisides unit function i.e. $H(t) = 1$ for $t > 0$

$H(t) = 0$ for $t \leq 0$

Substituting from (9) in (8) we get

$$u(y,t) = \frac{4A}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n \cos(2n+1)\pi y/2h}{2n+1}\int_0^t e^{-(2n+1)^2\pi^2\nu\tau/4h^2} H(t-\tau)\, d\tau \tag{10}$$

where $H(t-\tau) = 1$ when $t > \tau$

$= 0$ when $t \leq \tau$

Thus when $t \leq \tau$, we have $u(y,t) = 0$ (11)

and when $t > \tau$ we have

$$u(y,t) = \frac{4A}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \cos (2n+1)\pi y/2h}{2n+1} \int_0^t e^{-(2n+1)^2 \pi^2 \nu \tau/4h^2} d\tau$$

$$= \frac{16Ah^2}{\pi^3 \nu} \sum_{n=0}^{\infty} \frac{(-1)^n \cos (2n+1)\pi y/2h}{2n+1} \left(1 - e^{-(2n+1)^2 \pi^2 \nu t/4h^2}\right) \qquad (12)$$

(b) Impulsive pressure gradient

We take $f(t) = A\,\delta(t)$

The motion is set up by application of an impulse A at $t = 0$, and δ is the Dirac - delta function

Hence, $f(t-\tau) = A\,\delta(t-\tau)$ (13)

Making substitution from (13) in (8b) we get

$$u(y,t) = A \int_0^t g(y,\tau)\, \delta(t-\tau)\, d\tau$$

$$= A\, g(y,t)$$

$$= \frac{4A}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \cos (2n+1)\pi y/2h}{2n+1} e^{-(2n+1)^2 \pi^2 \nu t/4h^2} \qquad (14)$$

(c) Periodic pulse of pressure - gradient :-

We take $f(t) = B \sin \omega t$ when $0 \leq t \leq \pi/\omega$

$= 0$ when $t > \pi/\omega$

i.e. the motion is set up by a single periodic pulse of pressure - gradient.

Now $\bar{F}(\lambda) = B\int_0^\infty e^{-\lambda t}\sin\omega t\,dt = \frac{B\omega}{\lambda^2+\omega^2}\left(1+e^{-\frac{\pi\lambda}{\omega}}\right)$ (15)

Making substitution from (15) in equation (7) we get,

$$\bar{u}(y,\lambda) = \frac{B\omega}{\lambda(\lambda^2+\omega^2)}\left(1-\frac{\cosh\sqrt{\frac{\lambda}{\nu}}\,y}{\cosh\sqrt{\frac{\lambda}{\nu}}\,h}\right)\left(1+e^{-\frac{\pi\lambda}{\omega}}\right)$$

Now

$$\mathcal{L}^{-1}\left[\frac{B\omega}{\lambda(\lambda^2+\omega^2)}\left(1-\frac{\cosh\sqrt{\frac{\lambda}{\nu}}\,y}{\cosh\sqrt{\frac{\lambda}{\nu}}\,h}\right)\right] = -\frac{B}{\omega}\cos\omega t$$

$$+\frac{B}{2\omega}\left[e^{i\omega t}\frac{\cosh\sqrt{\frac{i\omega}{\nu}}\,y}{\cosh\sqrt{\frac{i\omega}{\nu}}\,h}+e^{-i\omega t}\frac{\cos\sqrt{\frac{i\omega}{\nu}}\,y}{\cos\sqrt{\frac{i\omega}{\nu}}\,h}\right]$$

$$+\frac{4B\omega}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\pi y/2h}{(2n+1)\left[\frac{(2n+1)^4\pi^4\nu^2}{16h^4}+\omega^2\right]}e^{-(2n+1)^2\pi^2\nu t/4h^2} = \phi(y,t)\text{ say}$$

Then $u(y,t) = \phi(y,t)+\phi\left(y,t-\frac{\pi}{\omega}\right)H\left(t-\frac{\pi}{\omega}\right)$ (Ref - 4) (16)

where $H\left(t-\frac{\pi}{\omega}\right) = 0$ when $t \leq \frac{\pi}{\omega}$

$=1$ when $t > \frac{\pi}{\omega}$

Thus when $0 \leq t \leq \frac{\pi}{\omega}$

$$u(y,t) = -\frac{B}{\omega}\cos\omega t+\frac{B}{2\omega}\left[e^{i\omega t}\frac{\cosh\sqrt{\frac{i\omega}{\nu}}\,y}{\cosh\sqrt{\frac{i\omega}{\nu}}\,h}+e^{-i\omega t}\frac{\cos\sqrt{\frac{i\omega}{\nu}}\,y}{\cos\sqrt{\frac{i\omega}{\nu}}\,h}\right]$$

$$+\frac{4B\omega}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\pi y/2h}{(2n+1)\left[\frac{(2n+1)^4\pi^4\nu^2}{16h^4}+\omega^2\right]}e^{-(2n+1)^2\pi^2\nu t/4h^2}$$

(17)

When $t > \frac{\pi}{\omega}$

u(y,t) = $$-\frac{B}{\omega}\cos\omega t + \frac{B}{2\omega}\left[e^{i\omega t}\frac{\cosh\sqrt{\frac{i\omega}{\nu}}\,y}{\cosh\sqrt{\frac{i\omega}{\nu}}\,h} + e^{-i\omega t}\frac{\cos\sqrt{\frac{i\omega}{\nu}}\,y}{\cos\sqrt{\frac{i\omega}{\nu}}\,h}\right]$$

$$+\frac{4B\omega}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\pi y/2h}{(2n+1)\left[\frac{(2n+1)^4\pi^4\nu^2}{16h^4}+\omega^2\right]}\,e^{-(2n+1)^2\pi^2\nu t/4h^2}$$

$$-\frac{B}{\omega}\cdot\cos\omega\left(t-\frac{\pi}{\omega}\right) + \frac{B}{2\omega}\left[e^{i\omega(t-\pi/\omega)}\frac{\cosh\sqrt{\frac{i\omega}{\nu}}\,y}{\cosh\sqrt{\frac{i\omega}{\nu}}\,h} + e^{-i\omega(t-\pi/\omega)}\frac{\cos\sqrt{\frac{i\omega}{\nu}}\,y}{\cos\sqrt{\frac{i\omega}{\nu}}\,h}\right]$$

$$+\frac{4B\omega}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\pi y/2h}{(2n+1)\left[\frac{(2n+1)^4\pi^4\nu^2}{16h^4}+\omega^2\right]}\,e^{-(2n+1)^2\pi\nu\left(t-\frac{\pi}{\omega}\right)/4h^2}$$

or

u(y,t) =

$$\frac{4B\omega}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\pi y/2h}{(2n+1)\left[\frac{(2n+1)^4\pi^4\nu^2}{16h^4}+\omega^2\right]}\times$$

$$\times\left[e^{-(2n+1)^2\pi^2\nu t/4h^2} + e^{-(2n+1)^2\pi^2\nu\left(t-\frac{\pi}{\omega}\right)/4h^2}\right]$$

(18)

Equation (8b) gives the velocity distribution for the viscous incompressible fluid with an arbitrary pressure gradient which is a function of time. The expression (10), (14), and (16) are the velocity distribution of the same fluid with constant, impulsive and periodic pressure gradients respectively.

A graphical representations of the velocity distribution in the range $0 \leq \omega t \leq \pi$ and $\omega t > \pi$ are depicted in Fig. 1 and Fig. 2 respectively [Assuming the value of $kh = 1$ and $y = -\frac{h}{2}$, $\sqrt{\frac{\omega}{2\nu}} = k$]. Fig. 1 and Fig. 2 show that velocity reaches its maximum when $t \approx \frac{2.18}{\omega}$ and after that it exponentially decays as t increases. A variation in the distribution of velocity with the different height is also shown in Fig. 3, assuming the fixed value of $\omega t = \frac{\pi}{2}$

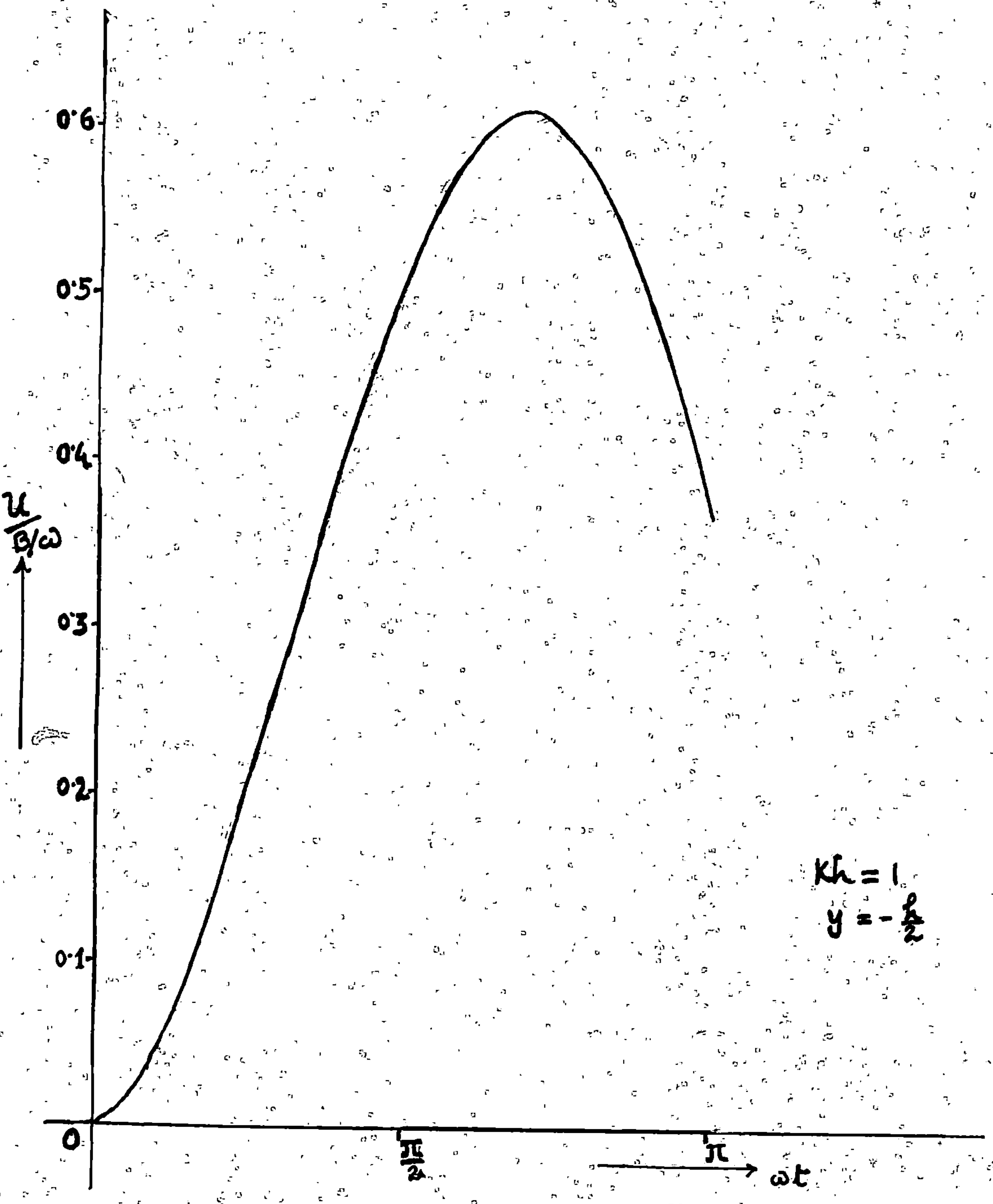

Fig-1

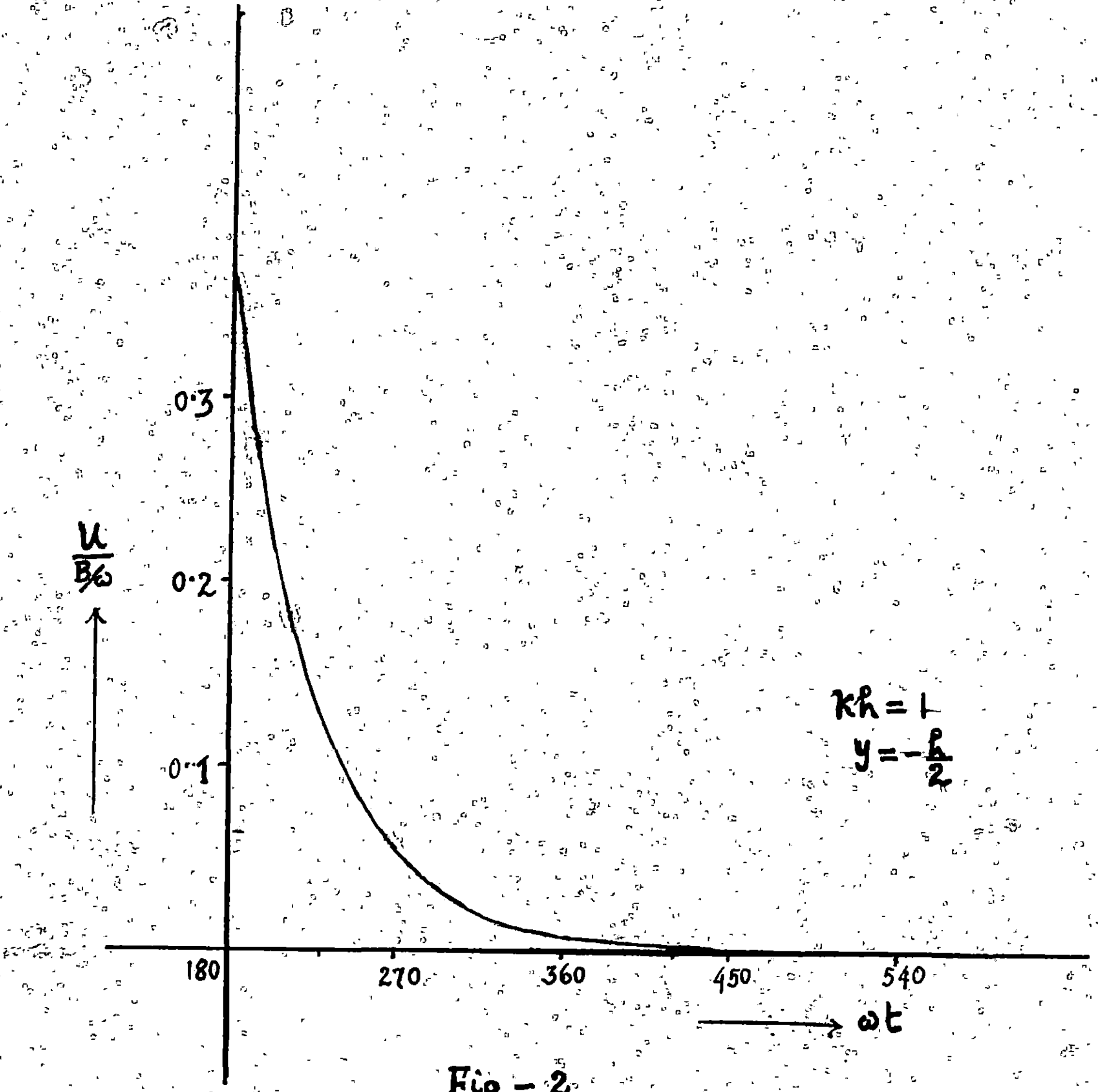

Fig - 2

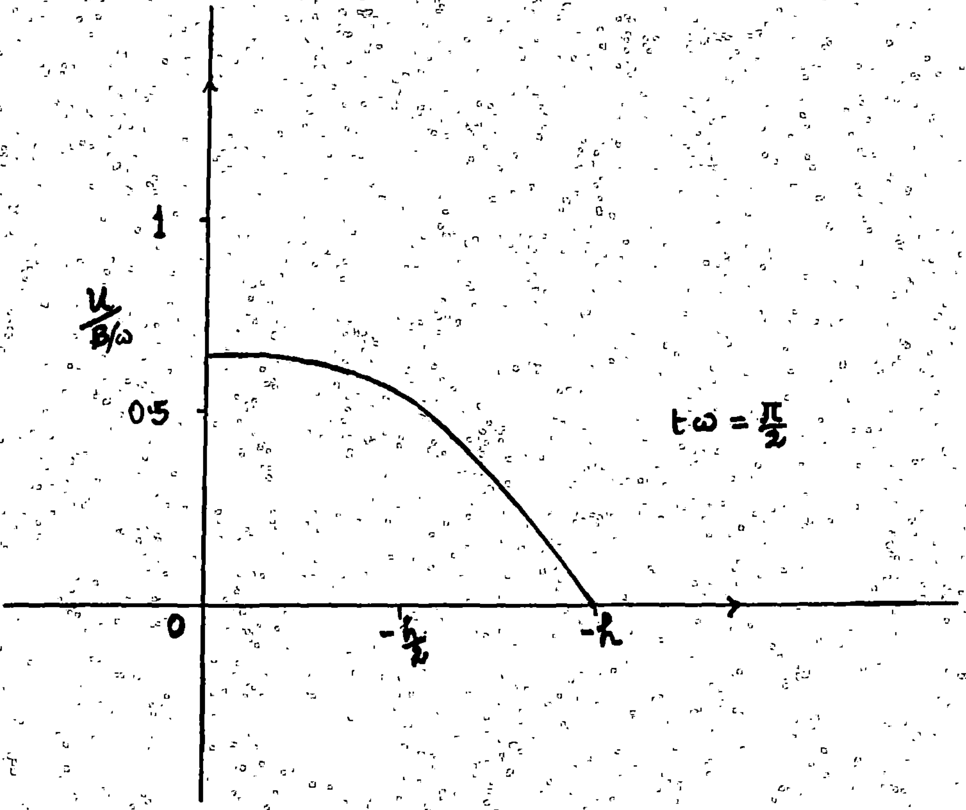

Fig - 3

NOTES ON THE PULSATING FLOW OF VISCOUS FLUID OF FINITE DEPTH DUE TO A SURFACE FORCE.

Introduction :- The problem of pulsating flow of viscous incompressible fluid under time-dependent pressure gradient superposed on a Constant pressure gradient has received the attention of several investigators.

Uchida[29](1956) has considered the pulsating viscous flow superposed on a steady laminar motion in a pipe of circular Cross-section, Verma[30](1960) considered the flow of viscous liquid under exponential pressure gradient superposed on steady laminar flow between Coaxial Cylinders. Following the method, Bhattacharyya[2](1966) solved the problem of flow of two incompressible immiscible fluids between two parallel plates. In the present paper, the method is applied to determine the unsteady flow of viscous liquid set up by a prescribed periodic tangential force superposed on a steady laminar flow over a rigid plane base. It is found from the general expression for velocity distribution that layers of the fluid are moving with different phases having their amplutude diminishing with increasing frequency. In case of simple periodic pulsation some physical features of the motion have been pointed out. Total flux and skin friction have also been calculated.

Formulation of the problem :- A co-ordinate system is chosen in which the origin lies on the free surface and X-axis is taken along the direction of the flow. Y-axis is taken along the normal to the axis of X and is directed vertically downwards.

Let h be the depth of the fluid below the surface, the length of the channel is very large in comparison to the width. Let u be the velocity component along the X-axis. Y-component of velocity is zero, since the flow is assumed to be unidirectional. Let μ, ρ denote the coefficient of viscosity and density of the fluid respectively. Neglecting the presence of extreneous forces, the equation governing the unsteady flow of viscous incompressible fluid are

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u \tag{1}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2}$$

where $\nu = \frac{\mu}{\rho}$

The equation of continuity is

$$\frac{\partial u}{\partial x} = 0 \tag{3}$$

Equation (3) indicates that the velocity to be constant in the direction parallel to the axis of X.

Boundary condition for the problem

$u = 0$ on $y = h$, $t \geqslant 0$ (4)

For pulsating flow, we assumed that shearing force to be given by

$$\mu \frac{\partial u}{\partial y} = A_0 + \sum_{n=1}^{\infty} A_{cn} \cos nt + \sum_{n=1}^{\infty} A_{sn} \sin nt$$

$$= A_0 + R_e \sum_{n=1}^{\infty} A_n e^{int} \tag{5}$$

on $y = 0$

where A_0 is real constant and

$A_n = A_{cn} - i A_{sn}$, A_{cn} and A_{sn} are real constants representing the amplutude of elemental variations.

Solution :- If we take the pressure to be constant, then the equation (1), (2) and (3) are satisfied and equation (1) becomes

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad (6)$$

Now for the expression of velocity, we assume the Fourier series

$$u = u_0 + \sum_{n=1}^{\infty} U_{cn} \cos nt + \sum_{n=1}^{\infty} U_{sn} \sin nt$$

$$= u_0 + R_e \sum_{n=1}^{\infty} U_n e^{int} \qquad (7)$$

where $U_n = U_{cn} - i U_{sn}$, U_{cn} and U_{sn} are function of y only.

Making substitution from (7) in equation (6) we get

$$i \sum_{n=1}^{\infty} n U_n e^{int} = \nu \left[\frac{d^2 u_0}{dy^2} + \sum_{n=1}^{\infty} \frac{d^2 U_n}{dy^2} e^{int} \right]$$

comparing terms of the same family, we get

$$\frac{d^2 u_0}{dy^2} = 0 \qquad (8)$$

$$\frac{d^2 U_n}{dy^2} = \frac{in}{\nu} U_n \qquad (9)$$

The boundary conditions for the problem are

$$\left.\begin{array}{lllr} \text{(a)} & u_n = 0 & \text{on } y = h\,,\ t \geqslant 0 \\ \text{(b)} & \mu \dfrac{du_n}{dy} = A_n & \text{on } y = 0\ ,\ t \geqslant 0 \end{array}\right\} \qquad (10)$$

where $n = 0, 1, 2, \ldots.$

The differential equation (8) and (9) admit of the following solutions satisfying the boundary condition (10)

$$u_0 = -\frac{A_0}{\mu}(h-y) \qquad (11)$$

and $$u_n = -\frac{A_n}{m\mu}\,\frac{e^{m(h-y)} - e^{-m(h-y)}}{e^{mh} + e^{-mh}}$$

where $m = K(1 + i)$, $K = \sqrt{\frac{n}{2\nu}}$

Thus the general solution is given by the real part of the following expression

$$u = -\frac{A_0}{\mu}(h-y) - \frac{Re}{\mu}\sum_{n=1}^{\infty}\frac{A_n e^{int}\left[e^{m(h-y)} - e^{-m(h-y)}\right]}{m\left(e^{mh} + e^{-mh}\right)}$$

$$= -\frac{A_0}{\mu}(h-y) + \frac{1}{\sqrt{2}\mu}\sum_{n=1}^{\infty}\frac{A_1}{KA_2}\left[A_{cn}\cos\left(nt + \frac{\pi}{4} + \theta_1 + \theta_2\right)\right.$$

$$\left. + A_{sn}\sin\left(nt + \frac{\pi}{4} + \theta_1 + \theta_2\right)\right] \qquad (13)$$

where

$$\left.\begin{aligned} A_1^2 &= \operatorname{Sinh}^2 k(h-y)\cos^2 k(h-y) + \operatorname{Cosh}^2 k(h-y)\operatorname{Sin}^2 k(h-y) \\ A_2^2 &= \operatorname{Sinh}^2 kh \sin^2 kh + \operatorname{Cosh}^2 kh \operatorname{Cos}^2 kh \end{aligned}\right\} \quad (14)$$

and

$$\left.\begin{aligned} \theta_1 &= \tan^{-1}\left[\tan k(h-y)\operatorname{Coth} k(h-y)\right] \\ \theta_2 &= \tan^{-1}\left[\operatorname{Cot} kh \operatorname{Coth} kh\right] \end{aligned}\right\} \quad (15)$$

Thus the fluid layers execute simple harmonic motion having their amplutudes proportional to the amplutude of the disturbing force and amplitude diminishes with increasing frequency.

Rapid Vibration

when a liquid of small viscosity pulsates rapidly, the parameter Kh becomes very large provided h is also large

then Sinh Kh $\rightarrow \infty$

and Cosh Kh $\rightarrow \infty$

when kh $\dot{>}> 1$

Then the motion on the free surface is given by

$$u = -\frac{A_0}{\mu}(h-y) + \frac{1}{\sqrt{2}\mu}\sum_{n=1}^{\infty}\frac{1}{k}\left[A_{cn}\operatorname{Cos}\left(nt+\frac{3\pi}{4}\right) + A_{sn}\operatorname{Sin}\left(nt+\frac{3\pi}{4}\right)\right] \quad (16)$$

Thus in the rapid pulsation, fluid flows on the free surface with phase ahead of 135° from the wave of disturbing force and its amplitude diminishes with increasing frequency but it increases when ν decreases.

Let us consider the case of simple periodic pulsation given by

$$\mu \frac{\partial u}{\partial y} = A_0 + A_{cn} \cos nt \qquad \text{on } y = 0$$

we write $\mu \frac{\partial u}{\partial y} = \mu \left(\frac{\partial u}{\partial y}\right)_0 + \mu \left(\frac{\partial u}{\partial y}\right)_n$

then $\mu \left(\frac{\partial u}{\partial y}\right)_0 = A_0$ (17)

and $\mu \left(\frac{\partial u}{\partial y}\right)_n = A_{cn} \cos nt$

The expression for velocity distribution becomes

$u = u_0 + u_n$

where $u_0 = -\frac{A_0}{\mu}(h - y)$ (18)

and $u_n = \frac{A_1}{\sqrt{2}\mu K A_2} A_{cn} \cos\left(nt + \frac{\pi}{4} + \theta_1 + \theta_2\right)$ (19)

where A_1, A_2 are given by (14)

and θ_1, θ_2 are given by (15)

or $u_n = \sigma_u A_{cn} \cos(nt + \delta_u)$ (20)

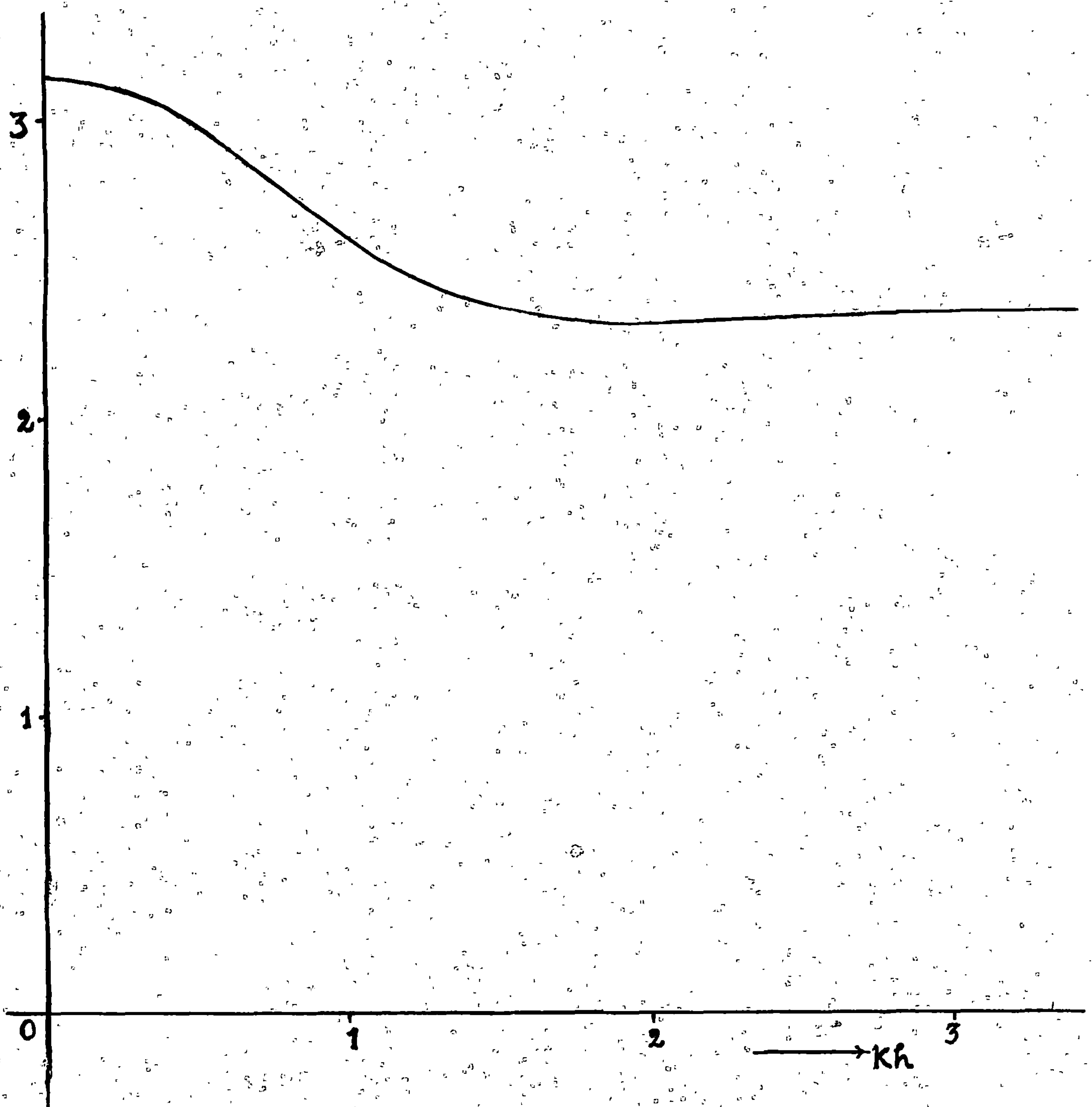

Fig-1

Thus the distribution of velocity is simple harmonic in nature having the time period $\frac{2\pi}{n}$ equal to that of the disturbing force. Different layers of the fluid are moving with different phases having different amplitudes. Here $\sigma_u = \frac{A_1}{\sqrt{2}\mu K A_2}$ represents the coefficient of amplitude of the velocity distribution which is a function of y. Variation of σ_u on the free surface for different kh have been shown in table I, and

$\delta_u = \frac{\pi}{4} + \theta_1 + \theta_2$ represents the phase ahead of the wave of disturbing force.

At the free surface, the coefficient of the amplitude attains its steady value $\frac{1}{\sqrt{2}\mu K}$ when δ_n becomes $\frac{3\pi}{4}$ for $kh \geqslant 2s\pi$, $s = 1,2,3\ldots$. Corresponding graph has been depicted in figure I.

Total Flux:-

The total flux Q is given by

$$Q = \int_0^h u \, dy$$

$$= -\frac{A_0 h^2}{2\mu} - \frac{1}{\mu}\sum_{n=1}^{\infty}\int_0^h \frac{A_n e^{int}\left[e^{m(h-y)} - e^{-m(h-y)}\right]}{m\left[e^{mh} + e^{-mh}\right]}\,dy$$

$$= -\frac{A_0}{2\mu}h^2 - \frac{\nu}{\mu}\sum_{n=1}^{\infty}\left\{\frac{A_{cn}}{n}\cos\left(nt-\frac{\pi}{2}\right) + \frac{A_{sn}}{n}\sin\left(nt-\frac{\pi}{2}\right)\right\}$$

$$-\frac{\nu}{\mu}\sum_{n=1}^{\infty}\left\{\frac{A_{cn}}{A_2 n}\cos\left(nt+\theta_2\right) + \frac{A_{sn}}{n A_2}\sin\left(nt+\theta_2\right)\right\} \qquad (21)$$

where A_2 and θ_2 are given by (14) and (15)

In a simple periodic pulsation produced by (17), (21) is reduced to

$$\psi = -\frac{A_0}{2\mu}h^2 - \frac{\nu}{\mu}\frac{A_{cn}}{n}\cos\left(nt-\frac{\pi}{2}\right)$$

$$-\frac{\nu}{\mu}\frac{A_{cn}}{n A_2}\cos\left(nt+\theta_2\right) \qquad (22)$$

From equation (22) it follows that total flux consists of two types of simple periodic pulse. Due to the 1st part $\frac{\nu}{\mu}\frac{A_{cn}}{n}\cos\left(nt-\frac{\pi}{2}\right)$, the flux Q has a phase lag $\pi/2$ to the wave of the shearing applied force, having the coefficient of amplitude $\sigma_{Q_1} = \frac{1}{\rho n}$ thus the amplitude due to this part gradually diminishes as the frequency of oscillation increases. Due to the 2nd part, flux Q has a phase ahead of θ_2 to the waves

kh	$\sqrt{n}\,\sigma_u$	kh	$\sqrt{n}\,\sigma_u$	kh	$\sqrt{n}\,\sigma_u$
0·1	0·1401	0·8	1·0138	1·5	1·1045
0·2	0·2820	1·0	1·1142	$\pi/2$	1·0876
0·3	0·4304	1·1	1·1320	2	1·0220
0·4	0·5600	1·2	1·1445	π	0·9925
0·5	0·6964	1·3	1·136	2π	1·0000
0·6	0·8162	1·4	1·121		

Table - I ($\rho=1$)

kh	$n\,\sigma_{a_2}$	kh	$n\,\sigma_{a_2}$	kh	$n\,\sigma_{a_2}$
0·1	1·0001	0·8	0·8859	1·5	0·4690
0·2	0·9996	1·0	0·7745	$\pi/2$	0·4340
0·3	0·9935	1·1	0·7095	2	0·2731
0·4	0·9925	1·2	0·6441	π	0·0860
0·5	0·9800	1·3	0·5818	2π	0·0037
0·6	0·9594	1·4	0·5145		

Table - II ($\rho=1$)

of the shearing force having the coefficient of amplitude $\sigma_{a_2} = \frac{1}{\rho n A_2}$. The values of σ_{a_2} for different kh have been tabulated in table II

Skin friction :-

Instantaneous frictional force acting on the wall is given by

$$\tau = -\mu \left(\frac{\partial u}{\partial y}\right)_{y=h}$$

$$= - A_0 - \sum_{n=1}^{\infty} \frac{A_{cn}}{A_2} \sin(nt+\theta_2) + \sum_{n=1}^{\infty} \frac{A_{sn}}{A_2} \cos(nt+\theta_2) \qquad (23)$$

where A_2 and θ_2 are given by (14) and (15) respectively. In a simple periodic pulse given by (17), equation (23) is reduced to

$$\tau = - A_0 + \frac{A_{cn}}{A_2} \cos\left(nt+\theta_2+\frac{\pi}{2}\right)$$

$$= - A_0 + A_{cn}\,\sigma_\tau \cos(nt+\delta_\tau) \quad (24)$$

where

$\sigma_\tau = \frac{1}{A_2}$ represents the coefficient of amplitude of shearing stress at the wall and δ_τ represents the phase ahead of the wave of the shearing force applied on the surface. It is found from (24) that τ is

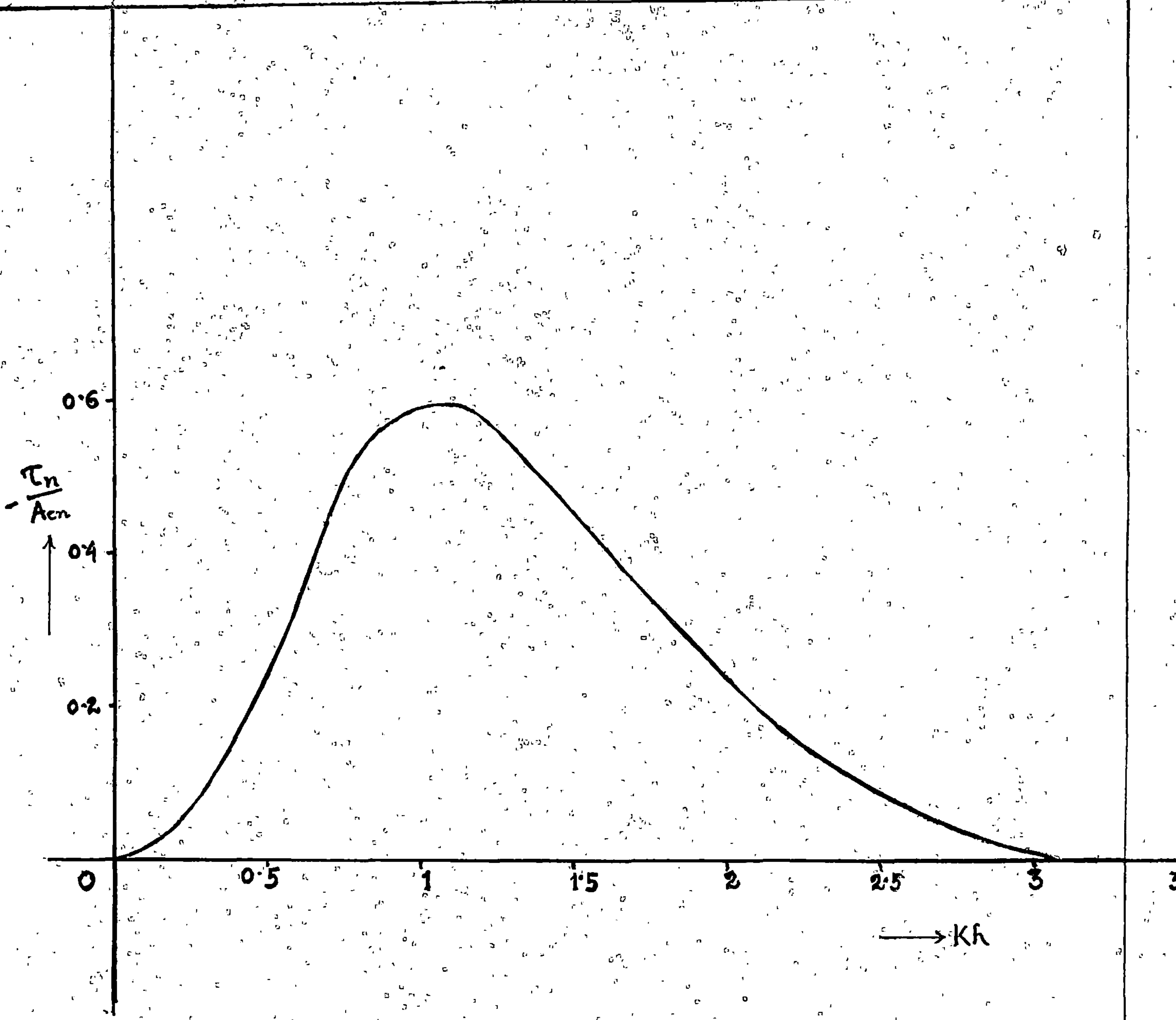

Fig – 2

independent of frequency of oscillation.

writing $\tau_n = -\frac{A_{cn}}{A_2} \sin(nt + \theta_2)$

τ_n has been depicted for different values of kh (taking $nt = \frac{\pi}{2}$) in figure 2.

UNSTEADY FLOW OF VISCOUS INCOMPRESSIBLE FLUID INDUCED BY A MOVING BOUNDARY *

Introduction :- The flow of an Ordinary viscous incompressible fluid past an oscillating flat plate has been discussed in detail by Schlichting.[21] The problem of motion of liquid due to oscillating boundary was investigated by Lord Rayleigh.[18]

In this paper, we have assumed that the fluid is at rest on a rigid boundary. The boundary is made to move by applying different types of force viz (i) impulsive (ii) force defined by sinosoidal function of time acting for a finite period (iii) a constant force acting for a short interval (iv) a force exponentially decaying with time upto a finite period respectively. The respective velocity distributions have been calculated by applying Laplace transform.

Formulation of the problem :-

We take the origin on the boundary $Z = 0$, the axis of X being taken in the direction of flow. Z axis is taken along the normal to the boundary. The unsteady motion of the viscous incompressible fluid induced by the motion of the boundary in its own plane can be regarded as

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a two dimensional motion in which the only Component of velocity different from zero is parallel to the plate. That means the component of velocity in the direction of Z is Zero. Assuming the pressure to be constant, the equation of motion can be written as

$$\rho \frac{\partial u}{\partial t} = \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right) \qquad (1)$$

(Neglecting the presence of any extreneous forces)

The equation of continuity is

$$\frac{\partial u}{\partial x} = 0 \qquad (2)$$

Hence equation (1) becomes

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2} \qquad (3)$$

ν stands for kinematic Coefficient of viscosity.

We suppose that the fluid on the infinite plane lamina $Z = 0$ is initially at rest and the lamina is suddently set into motion with a velocity $U\omega(t)$, where U is constant.

Thus the boundary conditions for the Problem become

$$\left. \begin{aligned} u(z,t) &= U\,\omega(t) \quad \text{when } z = 0,\ t > 0 \\ &= 0 \quad \text{when } z \to \infty,\ t > 0 \end{aligned} \right\} \qquad (4)$$

Initial condition is

$$u(z,t) = 0 \qquad \text{when } t = 0 \qquad (5)$$

3. Method of Solution :-

We introduce Laplace Transform[25]

$$\bar{u} = \int_0^\infty e^{-pt} u \, dt \quad , \quad Re(p) > 0$$

The Subsidiary Equation Corresponding to (3) is

$$\frac{d^2\bar{u}}{dz^2} = \frac{p}{\nu}\bar{u} \qquad (6)$$

Taking Laplace transform of equation (4) we get

$$\left.\begin{aligned} \bar{u}(z,p) &= U\bar{\omega}(p) && \text{when } z = 0 \\ &= 0 && \text{when } z \to \infty \end{aligned}\right\} \qquad (7)$$

The solution of equation (6) is

$$\bar{u} = A e^{\sqrt{p/\nu}\, z} + B e^{-\sqrt{p/\nu}\, z} \qquad (8)$$

Where A and B are functions independent of Z

Since u is finite for every value of Z and so is $\bar{u}$, Therefore we have A = 0

Thus $\bar{u} = B e^{-\sqrt{p/\nu}\, z}$ (9)

Condition (7) gives B =

$$U\bar{\omega}(p) \tag{10}$$

Making substitution from (10) in equation (9), we get

$$\bar{u} = U\bar{\omega}(p)\, e^{-z\sqrt{p/\nu}} \tag{11}$$

$$\text{Therefore } u = U\mathcal{L}^{-1}\left[\bar{\omega}(p)\, e^{-z\sqrt{p/\nu}}\right] \tag{12}$$

The inversion of equation (11) can be accomplished by using the Convolution theorem[5], giving

$$u(z,t) = \frac{Uz}{2\sqrt{\pi\nu}} \int_0^t \frac{\omega(t-\tau)\, e^{-z^2/4\nu\tau}}{\tau^{3/2}}\, d\tau . \tag{13}$$

4. Case I :- Impulsive motion :-

We choose $\omega(t) = \omega_0 \delta(t)$ where δ is the Dirac-Delta function.

Velocity distribution becomes

$$u(z,t) = \frac{Uz\omega_0}{2\sqrt{\pi\nu}} \int_0^t \delta(t-\tau)\, \frac{e^{-z^2/4\nu\tau}}{\tau^{3/2}}\, d\tau$$

$$= \frac{Uz\omega_0}{2\sqrt{\pi\nu t^3}}\, e^{-z^2/4\nu t} \tag{14}$$

The retarding force on the boundary per unit area is

$$-2\mu \frac{\partial u}{\partial z}\Big]_{z=0} = -\frac{\mu U \omega_0}{\sqrt{\pi \nu t^3}}$$

Case II Periodic motion of the boundary

Let us assume $\omega(t)$ to be a sinosoidal function of time acting for a finite period

Let $\omega(t) = H(\tau - t) \sin \omega t$ (15)

where H(t) is Heavisides step function with the property

$$H(\tau - t) = 1 \quad \text{for} \quad \tau > t$$
$$= 0 \quad \text{for} \quad \tau \leq t$$

applying Laplace transform to the equation (15), we get

$$\bar{\omega}(p) = \int_0^\infty e^{-pt} H(\tau - t) \sin \omega t \, dt$$

$$= \frac{-p}{p^2+\omega^2} e^{-p\tau} \sin \omega \tau - \frac{\omega}{p^2+\omega^2} e^{-p\tau} \cos \omega \tau \qquad (16)$$

$$+ \frac{\omega}{p^2+\omega^2}.$$

Hence making substitution from equation (16) to the equation (12) we get

$$\bar{u} = U e^{-z\sqrt{p/\nu}} \left[\frac{-p}{p^2+\omega^2} e^{-pT} \sin\omega T - \frac{\omega}{p^2+\omega^2} e^{-pT} \cos\omega T + \frac{\omega}{p^2+\omega^2} \right] \qquad (17)$$

Inversion theorem gives

when $t > T$

$$u(z,t) = -\frac{U \sin\omega T}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{p(t-T)} \frac{p}{p^2+\omega^2} e^{-z\sqrt{p/\nu}} \, dp$$

$$- \frac{U\omega \cos\omega T}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{p(t-T)} e^{-z\sqrt{p/\nu}}}{p^2+\omega^2} \, dp + \frac{U\omega}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{pt} e^{-z\sqrt{p/\nu}}}{p^2+\omega^2} \, dp \qquad (18)$$

and when $t \leqslant T$

$$u(z,t) = \frac{U\omega}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{pt} e^{-z\sqrt{p/\nu}}}{p^2+\omega^2} \, dp \qquad (19)$$

where γ is greater than the real part of all the

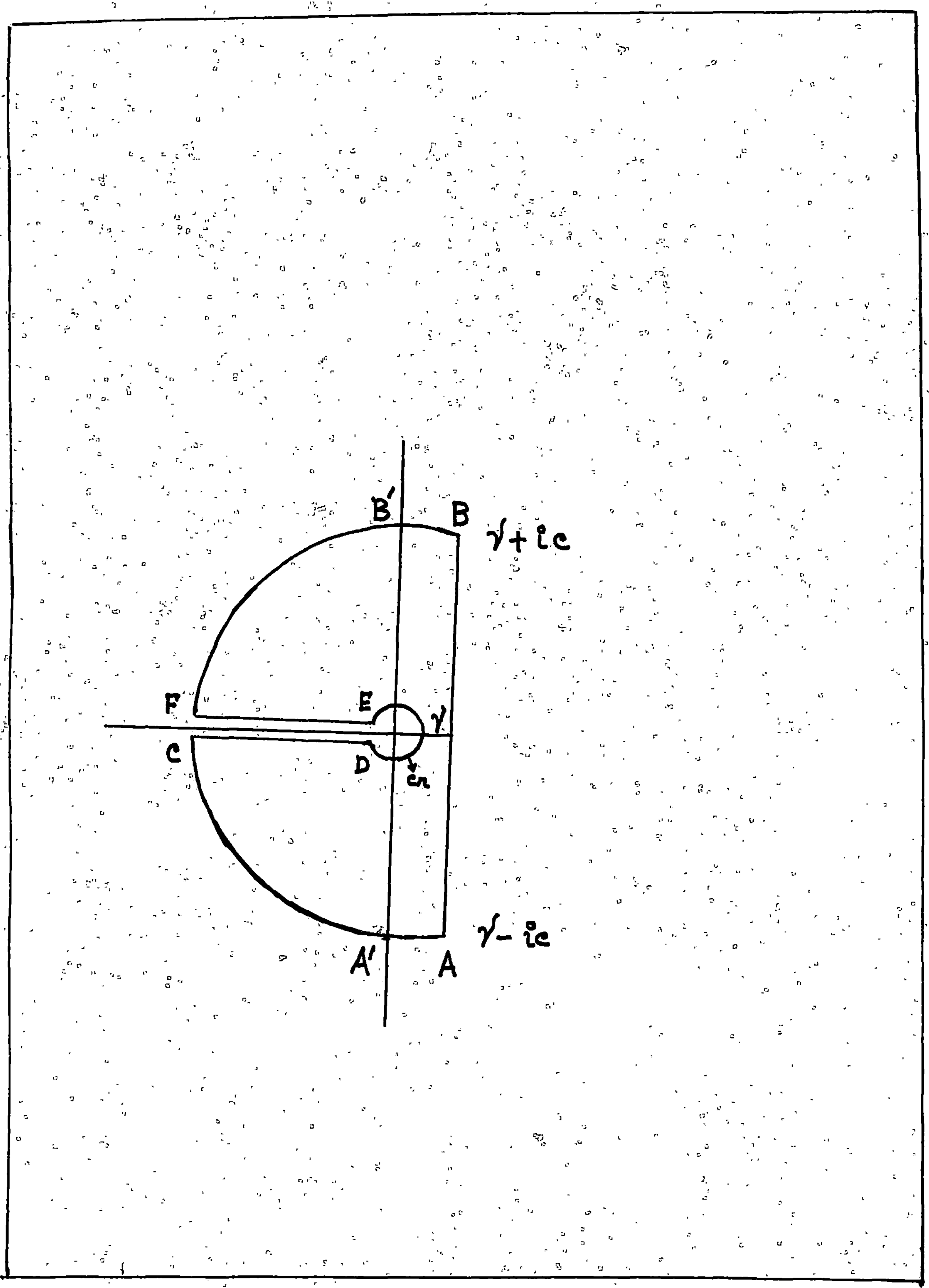

Figure 1.

singularities of the integrand in (18)

Now we consider the integral

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{pt}e^{-z\sqrt{p/\nu}}}{p^2+\omega^2}\,dp$$

there is a branch point at $p = 0$; so we use the contour shown in the annexed figure. The integral is single-valued in the region bounded by the closed circuit and has poles at $p = \pm i\omega$

Then by Cauchy's residue theorem

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{pt}e^{-z\sqrt{p/\nu}}}{p^2+\omega^2}\,dp = \frac{1}{2\pi i}\Big[\text{Sum of the integrals over } CD,\ \text{small circle } C_r \text{ and } EF, \text{ when } R\to\infty \text{ and } \epsilon\to 0 + \text{Sum of the residues at the poles } p=\pm i\omega\Big]$$

$$= \frac{1}{\pi}\int_0^\infty\frac{e^{-\rho t}\sin\sqrt{\rho/\nu}\,z}{\rho^2+\omega^2}\,d\rho + \frac{1}{\omega}e^{-z\sqrt{\frac{\omega}{2\nu}}}\sin\left(\omega t-\sqrt{\frac{\omega}{2\nu}}\,z\right)$$

where $p = \rho e^{-i\pi}$ over CD and

$p = \rho e^{i\pi}$ over EF.

Therefore the solution of equation (18) and (19) become

$$u(z,t) = \frac{U\omega}{\pi}\int_0^\infty \frac{e^{-pt}\sin z\sqrt{p/\nu}}{p^2+\omega^2}\,dp + Ue^{-z\sqrt{\frac{\omega}{2\nu}}}\sin\left(\omega t - z\sqrt{\frac{\omega}{2\nu}}\right)$$

$$-\frac{U\omega\cos\omega\tau}{\pi}\int_0^\infty \frac{e^{-p(t-\tau)}\sin z\sqrt{\frac{p}{\nu}}}{p^2+\omega^2}\,dp - U\cos\omega\tau\, e^{-z\sqrt{\frac{\omega}{2\nu}}} \times$$

$$\times \sin\left[\omega(t-\tau) - z\sqrt{\frac{\omega}{2\nu}}\right] + \frac{U\sin\omega\tau}{\pi}\int_0^\infty \frac{e^{-p(t-\tau)}p\sin z\sqrt{\frac{p}{\nu}}}{p^2+\omega^2}\,dp$$

$$-U\sin\omega\tau\, e^{-z\sqrt{\frac{\omega}{2\nu}}}\cos\left[\omega(t-\tau) - z\sqrt{\frac{\omega}{2\nu}}\right]$$

$$\text{or}\quad U(z,t) = \frac{U\omega}{\pi}\int_0^\infty \frac{e^{-pt}\sin z\sqrt{\frac{p}{\nu}}}{p^2+\omega^2}\,dp - \frac{U\omega\cos\omega\tau}{\pi}\times$$

$$\times\int_0^\infty \frac{e^{-p(t-\tau)}\sin z\sqrt{\frac{p}{\nu}}}{p^2+\omega^2}\,dp + \frac{U}{\pi}\sin\omega\tau\int_0^\infty \frac{e^{-p(t-\tau)}p\sin z\sqrt{\frac{p}{\nu}}}{p^2+\omega^2}\,dp$$

$$\text{for } t > \tau \qquad (20)$$

and for t $\leqslant \tau$.

$$u(z,t) = \frac{U\omega}{\pi}\int_0^\infty \frac{e^{-pt}\sin z\sqrt{\frac{p}{\nu}}}{p^2+\omega^2}\,dp + Ue^{-z\sqrt{\frac{\omega}{2\nu}}}\sin\left(\omega t - z\sqrt{\frac{\omega}{2\nu}}\right) \qquad (21)$$

Case iii. Consider a constant force acting on the boundary for a finite period.

Let us choose

$$\omega(t) = \omega_0 \left[H(t) - H(t-T) \right] \tag{22}$$

where H(t) is Heaviside unit function.
The relation (22) shows that $\omega(t) = 0$ when $t > T$ and $\omega(t) = \omega_0$ when $t \leqslant T$. We are considering therefore the case when the applied force acts for a short interval.

The Laplace transform of equation (22) gives

$$\bar{\omega}(p) = \omega_0 \int_0^\infty e^{-pt} \left[H(t) - H(t-T) \right] dt$$

$$= \omega_0 \left[\frac{1 - e^{-pT}}{p} \right] \tag{23}$$

Making substitution from equation (23) to the equation(12) we have

$$\bar{u}(z, p) = \frac{U \omega_0}{p} \left(1 - e^{-pT}\right) e^{-z\sqrt{p/\nu}} \tag{24}$$

Inversion theorem gives

$$u(z,t) = \frac{U\omega_0}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{pt}\, e^{-z\sqrt{p/\nu}}}{p}\, dp \tag{25}$$

for $0 < t \leq T$.

and for $t > T$

$$u(z,t) = \frac{U\omega_0}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \left[e^{pt} - e^{p(t-T)}\right]\frac{e^{-z\sqrt{p/\nu}}}{p}\, dp \tag{26}$$

Where γ is greater than the real parts of all the singularities of the integraned in (26)

Solution of equation (25) and (26) are(Cf :- Carslaw and Jeager[4])

$$u(z,t) = U\omega_0\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2}\, d\eta\right]$$

$$= U\omega_0 \operatorname{erfc}\eta \qquad \text{for } 0 < t \leq T, \tag{27}$$

and

$$u(z,t) = U\omega_0\left\{\operatorname{erfc}\eta - \operatorname{erfc}\eta_1\right\}$$

$$\text{for } t > T \tag{28}$$

where $\eta = \frac{z}{2\sqrt{t\nu}}$ and $\eta_1 = \frac{z}{2\sqrt{\nu(t-\tau)}}$.

when $t \leq \tau$, the thickness of the layer of the fluid set in motion is of the order of $2\sqrt{\nu t}$.

The result in (27) is identical to the solution of the problem of flow of viscous fluid on both sides of an infinite plane lamina ($z = 0$) is initially at rest and the lamina is suddenly set in motion parallel to OX with a velocity $U\omega_0$ which is then maintained constant.

(Ref. H.Lamb :- Hydrodynamics - P - 590)

The Retarding forces on the boundary, per unit area are given by

$$-2\mu \left.\frac{\partial u}{\partial z}\right]_{z=0} = \frac{2\mu U\omega_0}{\sqrt{\pi\nu t}} \quad \text{when } 0<t\leq\tau. \qquad (29)$$

and

$$-2\mu \left.\frac{\partial u}{\partial z}\right]_{z=0} = \frac{2\mu U\omega_0}{\sqrt{\pi\nu}}\left[\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t-\tau}}\right] \qquad (30)$$

$$\text{for } t > \tau .$$

Case IV

In this case we suppose that a finite force is initially applied on the boundary and that this force decreases exponentially with time and after time τ , the force does not exist.

Let us choose

$$\omega(t) = \omega_0 \; H(\tau - t)\, e^{-\omega t} \tag{31}$$

Laplace transform of equation (31) is

$$\bar{\omega}(p) = \frac{\omega_0}{p+\omega}\left[1 - e^{-(p+\omega)\tau}\right] \tag{32}$$

Solution equation (12) with the use of equation (32) gives

$$\bar{u}(z,p) = \frac{U\omega_0}{p+\omega}\left[1 - e^{-(p+\omega)\tau}\right] e^{-z\sqrt{p/\nu}} \tag{33}$$

Inversion theorem gives

$$u(z,t) = \frac{U\omega_0}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{pt}\, e^{-z\sqrt{p/\nu}}}{p+\omega}\, dp$$

$$\text{for } t \leq T \qquad (34)$$

and for $t > T$

$$u(z,t) = \frac{U\omega_0}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{pt}\, e^{-z\sqrt{\frac{p}{\nu}}}}{p+\omega}\, dp$$

$$- \frac{U\omega_0\, e^{-\omega T}}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{p(t-T)}\, e^{-z\sqrt{p/\nu}}}{p+\omega}\, dp \qquad (35)$$

where γ is greater than the real part of all the singularities of the integrand in (34) and (35)

Thus for $t > T$

$$u(z,\, t) = \frac{U\omega_0}{2}\, e^{-\omega t}\left[e^{-iz\sqrt{\frac{\omega}{\nu}}}\, \mathrm{erfc}\left(\frac{z}{2\sqrt{t\nu}} - i\sqrt{\omega t}\right) + e^{iz\sqrt{\frac{\omega}{\nu}}}\, \mathrm{erfc}\left(\frac{z}{2\sqrt{t\nu}} + i\sqrt{\omega t}\right)\right]$$

$$-\frac{U\omega_0}{2} e^{-\omega\tau} e^{-\omega(t-\tau)} \left[e^{-iz\sqrt{\frac{\omega}{\nu}}} \operatorname{erfc}\left(\frac{z}{2\sqrt{(t-\tau)\nu}} - i\sqrt{\omega(t-\tau)}\right)\right.$$

$$\left. + e^{iz\sqrt{\frac{\omega}{\nu}}} \operatorname{erfc}\left(\frac{z}{2\sqrt{(t-\tau)\nu}} + i\sqrt{\omega(t-\tau)}\right)\right]$$

$$= \frac{U\omega_0 e^{-\omega t}}{2} \left[e^{-iz\sqrt{\frac{\omega}{\nu}}} \left\{ -\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu(t-\tau)}} - i\sqrt{\omega(t-\tau)}\right)\right.\right.$$

$$\left. + \operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} - i\sqrt{\omega t}\right)\right\} + e^{iz\sqrt{\frac{\omega}{\nu}}} \left\{ \operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} + i\sqrt{\omega t}\right)\right.$$

$$\left.\left. - \operatorname{erfc}\left(\frac{z}{2\sqrt{\nu(t-\tau)}} + i\sqrt{\omega(t-\tau)}\right)\right\}\right] \tag{36}$$

and for $t \leq \tau$

$u(z,t) =$

$$\frac{U\omega_0}{2} e^{-\omega t} \left[e^{-iz\sqrt{\frac{\omega}{\nu}}} \operatorname{erfc}\left(\frac{z}{2\sqrt{t\nu}} - i\sqrt{\omega t}\right)\right.$$

$$\left. + e^{iz\sqrt{\frac{\omega}{\nu}}} \operatorname{erfc}\left(\frac{z}{2\sqrt{t\nu}} + i\sqrt{\omega t}\right)\right] \tag{37}$$