Chapter 1

Introduction

" Equipped with his five senses, man explores the universe around him and calls the adventure Science." —– Edwin Hubble.

1.1 Compact Stars

Stars are objects of great interest in astrophysics to unravel the mystery of the nature. The matters in the stars exist under extreme terrestrial conditions and their behaviour is a topic of research in theoretical and experimental physics in recent times. The pressure originates in the star from the burning of H_2 gas by thermonuclear fusion in a gravitationally bound hydrogen gas. The inward pull due to gravitation is balanced by the outward push of the radiation pressure leading to hydrostatic equilibrium. Hertzsprung and Russell took a large number of stars and plotted surface temperature against intensity which is known as HR diagram shown in Fig. (1.1). The maximum number of stars are found to lie in a region known as the Main-sequence stars, some stars lie below the main sequence region which are classified as White dwarf and those above are called Giant and Super Giant stars. The different phases of the stars namely, Normal star, White dwarf, Supernova, Neutron star, Pulsar and Black holes are found depending upon the equilibrium mechanism for hydrodynamical stability. The end of hydrogen burning phase in a low mass star leads to the fact that the outer layer of the star may drift away from the core as a gaseous shell. This is the red giant phase of the star. The remaining core of the star composed primarily of Helium, Carbon, Oxygen etc. is unable to maintain the thermonuclear fusion temperature when Fe^{56} nuclei results, afterwards the thermonuclear fusion process that absorbs energy from the surroundings leading to a decrease in the outward pressure. Thus the outward push due to the thermal radiation diminishes and the gravitational contraction wins leading to a contracting star and in this phase of contraction the electrons are torn out from the atoms and are freed. Thereby a repulsive electron degeneracy pressure dominates which in turn significantly resists the further collapse of the star. A stable configuration of the star is obtained when the degeneracy pressure balances the gravitational collapse resulting a White dwarf. A white dwarf of low mass ($M << M_{\odot}$) radiates the residual energy and eventually transforms to a fainter brown dwarf which thereafter disappears as a Black dwarf. However, a massive star evolves differently when it attains the main sequence stage. After millions of year a series of nuclear reactions namely, r-process leads to the p- process and the s -process are responsible for the formation of heavy nuclei which are heavier than iron in the star. Once iron (Fe^{56}) is produced at the core instead of generating energy, a phase is reached where the energy is absorbed form the surrounding. As a result, the iron core collapses because the outward push is less than the inward pull by the gravity causing an explosion called Supernova. A shock wave is formed that it blows up the outer layer of the star with mass shedding leaving a dense core made up of neutrons only, called Neutron star. Neutron stars with a core between (1.5 -3) $M_{\odot}$ forms a stable hydrodynamical equilibrium as the gravitational attraction is balanced by the repulsive neutron degeneracy pressure. If the mass of the core is greater than 3 $M_{\odot}$ (called TOV mass limit), the core further collapses to a new phase called Black hole. In a nutshell, the White dwarfs, Neutron stars and Black holes are collectively known as Compact star as their mass to radius ratio is greater than a normal star. For a slowly rotating star it can be described by a static geometry.

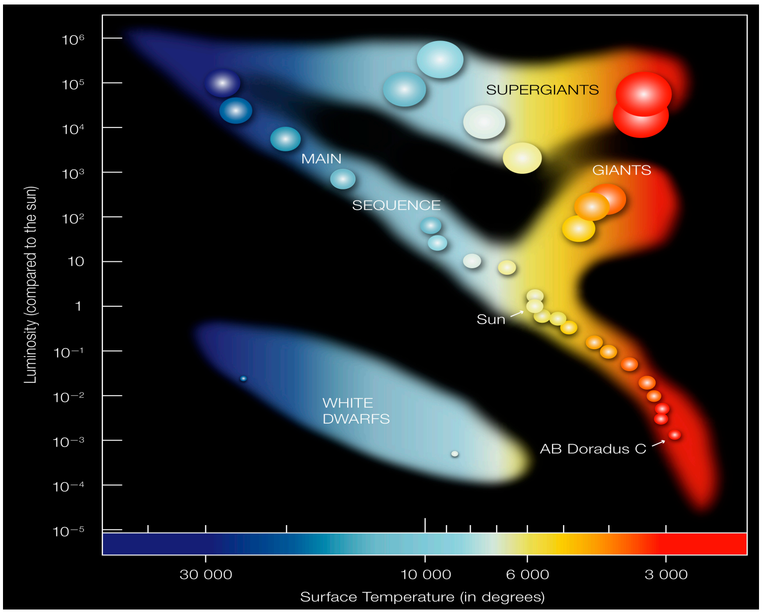

Fig. 1.1 Source: Hertzsprung–Russell diagram (HR Diagram) of Stars from "https://www.eso.org/public/images/eso0728c/"

1.2 Classification of Compact Objects

The compact objects are classified by defining a dimensionless parameter called compactification factor which is a ratio of the mass to radius: $u = \frac{M}{b}$, (where b is radius of a star and M is mass expressed in length unit). The stars are classified as : (i) Normal stars: $u \sim 10^{-5}$, (ii) White dwarfs: $u \sim 10^{-3}$, (iii) Neutron stars: $u \sim 0.1$ to 0.2, (iv) Strange stars: $u \sim 0.2$ to < 0.5, (v) Black holes: $u \sim 0.5$. In the next paragraph we describe in brief these astrophysical objects.

1.2.1 White Dwarf

White dwarfs are dense compact objects of radius about 10^3 km. and mean density $\sim 10^9$ kg/m^3. The hydrostatic equilibrium of a white dwarf is attained when the electron degeneracy pressure balances the inward gravitational pull. In 1785 Herschel [1] first discovered white dwarf which is named as *40 Eridani B*. In 1915 Adams [2] from the spectral measurement claimed that the nearest known white dwarf is *Sirius B*. In 1926 Fowler [3] shown using Fermi-Dirac statistics that the non-thermal electron degeneracy pressure balances the inward gravitational pull in such stars for its stability. In 1930 Chandrasekhar [4] theoretically modelled a white dwarf star considering the relativistic effects of degenerate electrons and obtained a limiting maximum mass of white dwarf $\sim 1.4/\mu^2$ $M_\odot$, which is known as the Chandrasekhar mass limit, where μ depends on the chemical composition of material content in a white dwarf.

1.2.2 Neutron Star

Landau introduced the concept of degenerate neutron pressure immediately after the discovery of neutron by Chadwick in 1932. Baade and Zwicky (1934) [5] predicted the concept of ‘neutron stars’. Oppenheimer and Volkoff [6] in 1939 proposed the model of neutron star assuming the matter to be composed of an ideal gas of free neutrons at very high densities. Tolman - Oppenheimer - Volkoff further obtained a limiting value of the maximum mass of the neutron star to be $3M_\odot$, which is known as TOV mass limit [7]. A Pulsar is a rotating neutron star which is highly magnetized discovered in 1967 by Hewish and his student Bell [8]. The first pulsar named as *PSR 1919+ 21*. It is also known subsequently that two more pulsars, the *Crab* and the *Vela* are the remnants of the supernova explosions. It is known to us now from different astronomical missions that there are many such pulsars in the sky. The equation of state of the interior matter content inside the neutron star is not known exactly yet. These are the stars with properties that are not possible to realize in the terrestrial laboratory.

A pulsar with slow rotations can be investigated theoretically by a static star in relativistic theory to begin with in order to understand its physical features.

1.2.3 Strange Star

When the core density of a neutron star is higher than the nuclear density, the nuclear barrier might disappear and quarks became deconfined, such stars may evolve into a new phase called Strange star. In such a state of matter composition, nucleons loose their identities and a soup of quarks originates which are asymptotically free. Astrophysicist are curious to study stars with deconfined quark matter phase to unravel the mystery of the compact astrophysical objects. The ordinary hadronic matter in the core of the neutron star may transmute to quark phase of matter due to high pressure. Witten [9] proposed that the strange quark matter can be considered as a true ground state of strong interaction. It was Gerlach [10] who used astronomical data to deduce the equation of state of "cool" matter at supranuclear densities. In 1998 Schertler [11] proposed that in the case of deconfined quark state the equation of state (EoS) may be considered as the " effective mass bag model", making use of the MIT Bag model [12] for hadrons in nuclear physics. The model presumes that all the three flavours quarks are confined in a bag which behave as non-interacting, massless particles. Therefore, it is possible to begin with a simple EoS for massless strange quarks given by,

$$p_r = \beta(\rho - 4B_g) \tag{1.1}$$

where β is a constant and B_g is the Bag parameter. The Eq.(1.1) reduces to the MIT Bag model, when β is 0.28 for massive strange quarks with mass 250 MeV [13] and $\frac{1}{3}$ for massless strange quarks. According to the strange quark hypothesis it is noted that the confined state of individual hadron possibly be transformed into strange quark matter, leading to a dense core of neutron resulting a neutron star and further evolution of a neutron star results to a strange star phase [14, 15].

1.2.4 Black Hole

Black holes are compact astrophysical objects known from the vacuum solution of Einstein's General Theory of Relativity (GR). The black holes can be classified into four categories depending on their mass and size, namely primordial black hole (smallest in size), super massive black hole (mass is almost 10^3 - 10^9 $M_\odot$), stellar black hole (5-100 $M_\odot$) and intermediate black hole (10^2-10^6 $M_\odot$).

In relativity, the spacetime geometry exterior to any spherical collapsing object can be described by the Schwarzschild metric. The Schwarzschild metric [16] is given by,

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2. \tag{1.2}$$

Where M is the mass of a sphere having radius r, taking G = 1 and c = 1. At $r = 0$, the above metric is singular, which is called the curvature singularity. The singularity at $r = 2M$ is referred to as the coordinate singularity. Here, $r = r_s = 2M$ is termed as the Schwarzschild radius. The surface of the black hole at $r_s = 2M$ is known as the *event horizon*, which divides the spacetime into two regions for a Schwarzschild black hole. From the region outside the event horizon, *i.e.* $r > r_s$, light can escape, but for $r < r_s$ light cannot escape due to the strong gravity. In 1918, Reissner-Nordstr*ö*m described the charged static black hole, which is known as Reissner-Nordstr*ö*m (RN) black hole [17, 18]. The RN black hole metric is given by,

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2. \tag{1.3}$$

Where Q is the charged enclosed in the black hole. Subsequently rotating black hole solution was obtained by Kerr, which is called Kerr metric [19]. Thereafter, the rotating charged black hole is obtained which is known as Kerr-Newman metric [20, 21]. In this case the two parameters: the mass (M) and spin parameter (a) are important for characterizing the most generic astrophysical black hole.

One of the most difficult problems in astronomy is the measurement of the spin of a rotating black hole. However, the black hole shadow provides an estimation of the spin of a black hole. Black hole casts a shadow as it comes in front of a luminous background. The study of the shadow is a direct proof of the black hole horizon. The shadow of a black hole is a dark area with a dazzling photon ring. The border of the black hole shadow is defined by light beams that approaches the photon sphere asymptotically. It is known that outside the event horizon, photon sphere is formed (for Schwarzschild black holes, $r = 3M$). There are two international efforts now underway: the American-funded Event Horizon Telescope (EHT) [22] and the European-funded Black Hole Cam project. (Fig. 1.4). The collaboration of these two telescopes created first photograph of the supermassive black hole Sgr A* which exists in the centre of our galaxy. The shadow of a black hole with a greater mass is bigger, while the shadow of a black hole with a larger spin parameter is the cause of more distortion in the shape of the BH. Synge (1966) [23] first theoretically investigated the shadows of spherically symmetric and static black holes. Bardeen studied theoretically the shadows of rotating black holes. The first simulated image was created by Luminet [24] in 1979. Due to the lack of spin parameter, the shadows of non-rotating black holes are found to have circular in shape. The difference between capture radius of the co-rotating and counter-rotating photons causes a crater in the black hole shadow in the case of rotating black hole. It is an exciting field of research at present moment and astronomers are very much involved to study the shadows of rotating black holes both theoretically and experimentally [25–30].

1.3 Objective

The objective of the proposed research is to obtain relativistic solutions of compact objects in hydrostatic equilibrium in Einstein's general theory of relativity (GR) and in modified theories of gravity to construct stellar models. Considering different spacetime geometries relevant to compact stars, stellar models will be constructed for compact objects. As the equation of state (EoS) of the matters inside a compact object is not yet known, the EoS will be predicted considering model parameters for a given spacetime. The physical properties of a charged or uncharged static star will be determined satisfying all the criteria for a physically viable stellar scenario. The observational parameters of known star (such as mass and radius) will be used to construct realistic stellar models.

The following problems are investigated:

- To construct relativistic models of compact star in GR with different geometries in usual four as well as in higher dimensions.
- To study the anisotropic behaviour of compact objects in different modified theories of gravity.
- To study isotropic and anisotropic compact objects with electromagnetic field in different geometries.
- To study collapsing stars, new solutions of Black Holes (BH) and its shadow.

1.4 Methodology

1.4.1 General theory of relativity (GR) to study the physical features of Compact Objects

The study of compact stellar objects is a significant area of research in modern astrophysics. A strong gravitational field originates due to a highly condensed state of matter present inside the compact stellar objects. At the extreme condition in the vicinity of these objects, a number of various unusual phenomena such as high-energy X-ray and gamma-ray radiation, high-frequency oscillations, and relativistic jets might have occurred. In recent time a huge number of reliable data obtainable from the different missions and the observational data are useful to support the theoretical model in the presence of different model parameters. Simple Newtonian mechanics is not enough to investigate the properties of the high density objects, which cannot be created in the terrestrial laboratories. The general theory of relativity (GR) plays an important role to probe the compact objects which is an active field of research since it can be used also as test bed for relativity as well as particle behavior in the extremely high-density regime. We still do not have a proper understanding of the interior composition of different classes of compact stars. Due to their extreme properties in the high gravity regime, investigations of compact objects are of fundamental importance for our basic understanding of physics.

The Einstein's GR is a fundamental theory for understanding compact objects. GR has a strong connection between geometry of spacetime and matter, which implies that the gravity is proportional to the curvature of spacetime, *i.e.*, more the curvature of spacetime more stronger the gravity. On the other hand, Einstein's GR reduces to the Newton's gravitational theory in the weak field limit. Einstein proposed a relation between the intrinsic parameters of a non-Euclidean spacetime geometry to the distribution of gravitating matter and energy [31] as:

$$R_{ab} - \frac{1}{2} g_{ab}\, R = k^2\, T_{ab}. \tag{1.4}$$

where, $k^2 = \frac{8\pi G}{c^4}$ is the coupling constant, G is the gravitational constant, R_{ab} is Ricci tensor, R is the Ricci Scalar, g_{ab} is the metric tensor and T_{ab} is the energy-momentum tensor, where a, b are the indices for a given dimensions.

German Mathematician David Hilbert (1915) showed that the Einstein field equation can be derived using the action principle. The Einstein-Hilbert action is given by,

$$S = \frac{1}{2k^2} \int \sqrt{-g}\, L_{\mathscr{G}}\, d^4x + \int \sqrt{-g}\, L_m\, d^4x \tag{1.5}$$

where, the gravitational Lagrangian density $L_{\mathscr{G}}$ = R (Ricci Scalar), L_{matter} denotes the Lagrangian matter density. The Einstein's field equations is also obtained by varying the action given by Eq.(1.5) with respect to $g^{\mu\nu}$, as $\frac{\delta S}{\delta g^{\mu\nu}}$ = 0.
The spacetime in the interior of a spherically symmetric compact star in equilibrium is described by the static metric in 4-dimensions, which is given by,

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\mu(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.6}$$

in (t, r, θ, ϕ) coordinate, $\nu(r)$ and $\mu(r)$ are the two unknown metric potentials for a static configuration.
Energy Momentum tensor:- The general form of the energy momentum tensor is given by,

$$T_{ab} = (\rho + p_\perp)u_a u_b + p_\perp g_{ab} + (p_r - p_\perp)X_a X_b + q_a u_b + q_b u_a - \zeta\Theta(g_{ab} + u_a u_b) \tag{1.7}$$

where, ρ is the energy-density of the fluid, p_r is the radial pressure, $p_\perp$ is the tangential pressure, X_a is an unit- four vector along the radial direction and u_a is the 4- velocity of the fluid such that $X_a X^a = 1$, $X_a u^a = 0$, ${q_b}^a = {\delta_b}^a$ is the radial heat flux vector and $u^a q_b = 0$.
The fluids are classified as follows:
• Perfect fluid when $p_r = p_\perp$ and $X^a = 0$.
• Anisotropic fluid when $p_r \neq p_\perp$ and $X^a \neq 0$.
• Viscous fluid with anisotropic conditions $p_r \neq p_\perp$, $q_{ab} = 0$, $X^a \neq 0$ and $\zeta \neq 0$ with a coefficient of viscosity, *i.e.*, $\zeta > 0$ can be considered.
For perfect fluid, Eq. (1.7) reduces to,

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} \tag{1.8}$$

1.4.2 Anisotropic Star

During the last couple of decades it has been realized that a deviation from local isotropy in the interior pressure is important in compact star with extreme internal dense matter. By anisotropic pressure we mean the radial component of pressure (p_r) which is different from that of the tangential pressure ($p_\perp$). Bowers and Liang extensively [32] studied the anisotropic relativistic matter distributions in general relativity and obtained solutions for a static spherically symmetric configuration in order to determine the changes in the surface redshift and gravitational mass generalizing the hydrostatic equation. The theoretical investigations by Ruderman [33] pointed out that at very high densities of order 10^{15} g/cm^3, nuclear matter tends to become anisotropic in nature. Kippenhahn [34] proposed that for a relativistic star anisotropy might occur due to the existence of a solid core or type 3A super fluid. Weber [14] showed that a strong magnetic field in a compact star may generate an anisotropic pressure. It is also known that anisotropy may occur in the astrophysical objects for various reasons namely, viscosity, phase transition [35], pion condensation [36] etc. The shear of the fluid is also considered as another source for the origin of anisotropy in a self gravitating body [37]. Small anisotropy, according to Herrera and Santos [38], can substantially alter the system's stability. The role of local anisotropy is investigated in this work. Thus, by incorporating anisotropic pressure in the stress-energy tensor of the material composition, different classes of static as well as non-static models of relativistic stars can be constructed.

For an anisotropic fluid distribution the energy momentum tensor without heat flow and viscosity given by Eq.(1.7) reduces to,

$$T_{ab} = (\rho + p_{\perp})u_a u_b + p_{\perp} g_{ab} + (p_r - p_{\perp})X_a X_b \tag{1.9}$$

For a static star we generally consider two types of stars with (i) isotropic and (ii) anisotropic fluid distribution. When $p_{\perp} = p_r$, it corresponds to isotropic star and the Einstein field equation given by Eq.(1.4) are obtained using the metric Eq.(1.6).

Thus the field equations for an anisotropic fluid distribution are given by,

$$\rho = \frac{1}{k^2}\left[\frac{\left(1 - e^{-2\mu}\right)}{r^2} + \frac{2\mu' e^{-2\mu}}{r}\right] \tag{1.10}$$

$$p_r = \frac{1}{k^2}\left[\frac{2\nu' e^{-2\mu}}{r} - \frac{\left(1 - e^{-2\mu}\right)}{r^2}\right] \tag{1.11}$$

$$p_{\perp} = \frac{e^{-2\mu}}{k^2}\left[\nu'' + \nu'^2 - \nu'\mu' + \frac{\nu'}{r} - \frac{\mu'}{r}\right] \tag{1.12}$$

where $k^2 = \frac{8\pi G}{c^4}$ and $()'$ implies derivative w.r.t r. Here we have used the gravitational unit which corresponds to $k^2 = 1$. We define anisotropy, $\Delta = p_{\perp} - p_r$. The stellar models for compact objects are obtained knowing the metric functions $\mu(r)$ and $\nu(r)$. Using Eqs. (1.11) and (1.12), the anisotropy pressure parameter is given by,

$$\nu'' + \nu'^2 - \nu'\mu' - \frac{\nu'}{r} - \frac{\mu'}{r} - \frac{\left(1 - e^{2\mu}\right)}{r^2} = \Delta e^{2\mu}. \tag{1.13}$$

However, in the case of isotropic pressure ($\Delta = 0$), the Eq.(1.13) reduces to,

$$\nu'' + \nu'^2 - \nu'\mu' - \frac{\nu'}{r} - \frac{\mu'}{r} - \frac{\left(1 - e^{2\mu}\right)}{r^2} = 0. \tag{1.14}$$

It is a second order differential equation which can be solved knowing either μ or ν as a function of r to construct the stellar models.

1.4.3 Electromagnetic Field

The effect of change in a compact star can be studied by coupling the Einstein equations with the Maxwell equations. It is interesting to note that gravitational collapse of a charged spherical object to a point singularity may be halted by incorporating charge. The study of self-gravitating spherically symmetric systems in the presence of an electromagnetic field is therefore, a topic of interest in compact objects. Majumdar [39], De and Raychaudhuri [40] and Papapetrou [41] studied relativistic charged dust models. Whitman and Burch [42] showed that a homogeneous fluid sphere with a considerable amount of net surface charge is more stable than the same system without charge. Bonnor and Wickramasuriya [43] also studied systems where the electrostatic repulsion is counter-balanced by the gravitational pull. In a high-density regime of relativistic compact objects with a strong gravitational field, Ghezzi [44] and later Ghezzi and Letelier [45] showed that it is possible to construct a stable equilibrium configuration which can support a huge amount of charge. The exterior gravitational field of a static charged fluid distribution is therefore can be matched with the Reissner-Nordstr*ö*m metric for a realistic stellar model.

• Maxwell's Electromagnetic Fluid
The tensorial form of inhomogenous pair of Maxwell equation is given by,

$$\nabla_a F^{ab} = 4\pi J^b \tag{1.15}$$

where, F^{ab} is the electromagnetic field tensor, J^b is the current density in S.I. system. The homogeneous pair of Maxwell's equation is given by,

$$F_{ab;\,c} + F_{bc;\,a} + F_{ca;\,b} = 0 \tag{1.16}$$

where, $(;)$ represents the co-variant derivative. The four potential and the current density are given by,

$$A_a(t,r) = A(t,r)\delta_a^0, \quad J^a = \rho_{EM}(t,r)u^a. \tag{1.17}$$

where, δ_a^0 is the Kronecker delta and u^a is the velocity vector and ρ_{EM} is the electromagnetic charge density. The covariant form of the electric field and magnetic field are,

$$E_a = F_{ab}u^a, \tag{1.18}$$

$$B_a = \frac{1}{2}\varepsilon_{abcd}u^b F^{cd}. \tag{1.19}$$

where, ε_{abcd} represents antisymmetric Levi - Civita tensor.

1.4.4 Tolman-Oppenheimer- Volkoff (TOV) method

In Tolman-Oppenheimer- Volkoff (TOV) method, a known equation of state is considered to obtain the stellar models. There are two different approaches to obtain relativistic stellar equilibrium configurations. In one case knowing the matter compositions, the geometry can be determined and in other case the geometry determines the matter present in the compact objects. In the first method, if the EOS of a spherically symmetric relativistic star is known, physical features of the star can be investigated by solving the relevant Einstein field equations. In the other case Oppenheimer-Volkoff investigated the gravitational equilibrium of neutron stars making use of Tolman's equations for the static fluid spheres [6, 7]. The Tolman, Oppenheimer and Volkoff (TOV) equations is given by,

$$\frac{dp}{d\rho} = -(\rho + p)\frac{2M(r) + pr^3}{r^2(1 - \frac{2M(r)}{r})} \tag{1.20}$$

The variation of mass in such star is given by,

$$\frac{dM}{dr} = \frac{1}{2}\rho r^2 \tag{1.21}$$

where, ρ is the density. These above equations completely determine the structure and features of a spherically symmetric system in equilibrium however, it can be used significantly for a prescribed EoS. Here, $M(r)$ denotes the mass contained within a spherical region of radius r. The TOV equation can be derived by solving the Einstein field equations, for the spherically symmetric spacetime metric of the form :

$$ds^2 = -\,e^{2\nu(r)}dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \tag{1.22}$$

in the (t, r, θ, ϕ) coordinate system. Now using eqs.(1.20) and (1.21), the derivative of $\nu(r)$ can be obtained as,

$$\frac{d\nu(r)}{dr} = -\frac{1}{\rho(r)+p(r)}\frac{dp(r)}{dr} \tag{1.23}$$

where, $\frac{dp(r)}{dr}$ is the pressure gradient in the isotropic fluid case. Using an equation of state ($p = f(\rho)$) connecting density (ρ) to pressure (p), the Tolman-Oppenheimer-Volkoff equation entirely specifies the structure of a spherically symmetric sphere of isotropic mater in equilibrium. When one considers the following: $2M(r) << r$, $M(r) << 4\pi r^3 p(r)$ and $\rho >> p$, the TOV equation reduces to the hydrostatic equation in Newtonian mechanics.

A realistic stellar model can be obtained with the relativistic solution satisfying the following constraints :

- The energy density (ρ) and radial pressure (p_r) are finite and positive everywhere inside the star.
- At the boundary of a star ($r = b$), the radial pressure vanishes, *i.e.*, $p(r) = 0$ at $r = b$.
- The exterior metric of a charged star and an uncharged star are given by Reissner-Nordstr*ö*m metric and Schwarzschild metric respectively. Therefore the interior solution is to be matched with the exterior solution at the boundary.
- For a realistic star, the gradients of the energy density $\left(\frac{d\rho}{dr}\right)$ and the radial pressure $\left(\frac{dp_r}{dr}\right)$ are negative.
- Inside the star, Herrera's cracking condition, *i.e.*, $v^2 = \frac{dp}{d\rho} \leq 1$ is required to be satisfied to maintain causality condition.
- For a charged star the electric field intensity E is positive and nonsingular everywhere inside the star.
- The condition on adiabatic index, viz., $\Gamma > \frac{4}{3}$ ensures the stability of the stellar configuration [46] in the presence of radial perturbation.
- The energy conditions namely, (a) Null Energy condition (NEC), (b) Weak Energy condition (WEC) and (c) Strong Energy condition (SEC) are obeyed if the star is made up of normal fluid.

Even it is known unique vacuum solutions for the exterior spacetime with or without charge, the relativistic solutions describing the interior of a spherically symmetric matter distribution is not unique as huge number of solutions are possible. The above criteria are imposed to obtain a realistic solution in a gravitational theory. Delgaty and Lake [47] showed that only a few of the reported class of solutions of the Einstein field equations qualify the above tests for viable models of observed compact stars. The prescription of Delgaty and Lake is taken into consideration in the thesis for developing the stellar models.

1.4.5 An Alternative Approach : Given Spacetime Geometry

The TOV equation gives a relativistic solution for compact objects, such as neutron stars, when the EoS of matter inside the star is known apriori. However for a superdense stars, the EoS inside the compact object is not yet realized by any terrestrial experiment. In the absence of sufficient knowledge of matter composition, we can adopt an alternative method where for a given geometry of the spacetime it is possible to predict the EoS for the matter composition in a compact object. Such alternative methods have been employed in the past for the construction of stellar models by Vaidya and Tikeker and others. Earlier Duorah and Ray (1987) [48] and subsequently by other authors [49–54] introduced analytical solution for constructing stellar models of spherically symmetric static stars, out of which Finch-Skea showed that the analytic solution given by Duorah and Ray [55] did not permit a compact object. It has been shown by Finch and Skea (1989) that a modified version of the solution permitted a new regular and physically realistic solution for accommodating relativistic stellar objects, which is known as Finch-Skea metric. In the later approach, the stellar models were studied making use of the macroscopic properties of stars such as observational values of mass and radius of compact objects as input known parameters. The solutions obtained by adopting the alternative approach can be used to determine the thermodynamic relationship between density and pressure as well as to obtain the relation connecting them. The metric potentials considered in this case must be regular, well behaved, and capable of properly modelling a compact astrophysical star object and all the above features are present in FS metric.

The Finch-Skea metric can be used to study a large variety of stellar objects with or without charge. The Finch-Skea metric is given by,

$$e^{2\lambda(r)} = (1 + Cr^2), \tag{1.24}$$

$$e^{2\nu(r)} = D^2[\,(B - A\sqrt{1+Cr^2})\,cos\sqrt{1+Cr^2}$$

$$+(A + B\sqrt{1+Cr^2})\,sin\sqrt{1+Cr^2}]^2 \tag{1.25}$$

where A, B, C, D are metric parameters, which are to be determined from the boundary conditions for a physical viable stellar model, and other criterion as mentioned as above.
Within the framework of Einstein theory of relativity, various efforts have been made during the past decades to understand the nature and dynamics of collapse of stellar objects and possible outcome of massive stars that undergoes collapse. It has been observed that many physical conditions such as density inhomogeneity, viscosity, anisotropy, electromagnetic field *etc*. play a crucial role on the overall dynamics of the star in addition to predict the ultimate stage of a gravitationally collapsing system which is also an important area in research in the present decade.

1.5 Compact stars in Higher dimensions

In the last couple of decades there has been a considerable research activities to generalize the results of the well-known standard four-dimensional Einstein theory of gravity in the framework of higher dimensions because of success in superstring theories which can be formulated consistently in higher dimensions [9]. The history of higher dimensions goes back to the independent work done by Kaluza [56] and Klein [57], who introduced the concept of one extra dimension with the usual four dimensions in order to unify gravity with the electromagnetic interaction. The Kaluza - Klein approach however did not work well. But in the Brane world and in superstring theory it is revived again. Although existence of higher dimensions not yet recorded observationally and experimentally, the theoretical work is undertaken worldwide assuming the existence of extra dimensions to explore the different unearthed result which remains an academics interest. In the literature a number of research work to generalize spherically symmetric Schwarzschild and Reisner- Nordstrom black holes [16–18], Kerr black holes [19], Vaidya solution [49] have been implemented. Paul [58] studied the mass to radius ratio in higher dimensions for a uniform density star and found existence of new results which are not present only in 4-dimensions. Emparan and Real [59] provided a solution for a black ring in 5 dimensions, which indicates existence of different kinds of topologies in higher dimensions. Cassisi *et.al*. [60] pointed out the effects of higher dimensions in the case of stellar evolution showing the impact of extra dimensions. At present, higher dimensional studies become an active field of theoretical research to explore the features in astrophysics and cosmology not known in the usual 4-dimensional framework.

1.6 Modified Theories of Gravity and Compact Stars

Although, GR is the fundamental and most fruitful theory to reveal the unsolved mysteries of the universe. It is known that some serious issues occur in GR at ultraviolet and infrared limits. Again the recent observational evidences of Galactic, extra Galactic and cosmic dynamics, e.g., the accelerating expansion phase of the present universe can not be understood in GR unless one considers exotic forms of matter energy namely dark matter and dark energy [61, 62]. The present observation indicates that dark matter-energy is expected up to 95% of total content of the universe. In order to accommodate all these features it is essential to modify the gravitational sector to fit the missing matter-energy of the observed universe if modification of matter sector is not considered. Thus the strong field regime can be understand by a proper modification of GR. As a result a number of modified theories of gravity have been proposed time to time namely $f(R)$ gravity, $f(R,T)$ gravity, $f(\mathbb{T})$ gravity, Einstein Gauss Bonnet (EGB) gravity etc., where R, T, $\mathbb{T}$ are Ricci scalar, energy-momentum tensor and torsion respectively. In all these theories instead of changing the matter sector of the Einstein field equations, the gravity sector has been modified by taking a generalized functional form of R, T, $\mathbb{T}$.
The $f(R)$ gravity [63] is the most elementary modification of GR which is extensively used to study the existence and stability of neutron stars and compact stars. Subsequently, Harko *et al*. (2011) [64] generalized $f(R)$ gravity by introducing an arbitrary function of the Ricci scalar R and the trace of the energy–momentum tensor T, which is $f(R,T)$ modified gravity in cosmology for admitting the observational universe. This is an extension of $f(R)$ gravity. It is known that the torsion based Teleparallel equivalent of general relativity (TEGR) [65] also plays an important role and can be employed as a basis to build a modified gravitational theory. The simplest class of torsion-based modifications is the $f(\mathbb{T})$ gravity, where $\mathbb{T}$ is the torsion scalar, where the Lagrangian is taken to be a non-linear function of $\mathbb{T}$. $f(\mathbb{T})$ gravity may be simple compared to $f(R)$ gravity as the $f(\mathbb{T})$ gravity give rise to second order differential equations whereas in $f(R)$ theories it is fourth order. In the next section the modified theories are discussed briefly.

1.6.1 $f(R)$ Theory

In recent times a number of theories of gravity proposed to accommodate the observed late universe as well as to solve the issues of non-renormalizability [66, 67] that appears in GR. It is known that GR cannot be quantized in a conventional manner. Now to find the effect of interaction between a set of quantized matter fields with a classical gravitational field the one-loop approximation played an important role. In 1962, Utiyama and DeWitt [68] showed

that for renormalization of a classical gravitational field interacting with quantized matter fields at one-loop approximation requires inclusion of higher order curvature terms in the corresponding Einstein-Hilbert action. Subsequently it is shown that at the low energy limit, the effective gravitational action contains the higher order curvature terms [69–71]. The incorporation of higher order curvature invariants with respect to Ricci scalar [11] such as R^2, $R^{ab}R_{ab}$, $R^{abcd}R_{abcd}$ etc. to the gravitational action raised a strong interest to the scientific community for accommodating various cosmological issues such as dark energy problem, late-time acceleration of the Universe and the observed astrophysical phenomena.
One of the most significant modified theory is $f(R)$ theory, where the Einstein-Hilbert action [72–74, 63] can be expressed as,

$$S = \frac{1}{2k^2} \int \sqrt{-g}\, f(R)\, d^4x + \int \sqrt{-g}\, L_m\, d^4x \tag{1.26}$$

Here, $f(R)$= $\mathscr{L}_{\mathscr{G}}$, which is expressed as analytic function of the Ricci scalar R. One can retrieved the standard Einstein-Hilbert action just by putting $f(R)$ = R in Eq.(1.26). To investigate some of the basic characteristics of higher-order gravity, the gravitational Lagrangian $\mathscr{L}_{\mathscr{G}}$ can be taken as $f(R)$ for its simplicity. In 2010, Sotiriou and Faraoni [75] expanded $f(R)$ as a power series expansion, containing both positive and negative power of Ricci scalar R,

$$f(R) = ... + \frac{\alpha_2}{R^2} + \frac{\alpha_1}{R} - 2\Lambda + R + \frac{R^2}{\beta_2} + \frac{R^3}{\beta_3} + ... \tag{1.27}$$

where the coefficients α_i and β_i (where i= 1,2, 3, ...) have the appropriate dimensions. One can obtain the field equation by varying the corresponding action $w.r.t$ g_{ab} which yields,

$$f_R\, R_{ab} - \frac{1}{2}\, f(R)\, g_{ab} - [\, \nabla_a \nabla_b - g_{ab}\, \Box\,]\, f_R = k^2\, T_{ab} \tag{1.28}$$

The above equation is a fourth order equation, f_R = $\frac{\partial f}{\partial R}$, ∇_a is the covariant derivative $w.r.t$ the symmetric connection associated with g_{ab}, T_{ab} is the the energy-momentum tensor and $\Box \equiv \frac{\partial_a(\sqrt{-g} g^{ab} \partial_b)}{\sqrt{-g}}$.

1.6.2 $f(R,T)$ theory

An extension of $f(R)$ gravity with the energy momentum tensor (T) $f(R,T)$ theory, where a non-minimal coupling between the matter and the geometry may exist and the idea introduced by Harko $et.al.$ (2011) [64]. In the basic $f(R,T)$ theory the Ricci Scalar R is coupled with the trace of the energy-momentum tensor T. This modified theory also follows from the corresponding models of curvature based gravity. In curvature based formulation, the

Einstein-Hilbert action can be modified by changing the geometric parts only, *i.e*, here the Lagrangian can be obtained by the minimal coupling of geometry with matter. In this theory, the matter is treated equally to geometry in order to investigate a variety of fascinating and innovative aspects in the observational universe, such to justify the presence of exotic forms of matter, such as dark matter [76]. The T dependency might result from the existence of an imperfect fluid or from the consideration of quantum effects. The modified gravitational action is given by

$$S = \frac{1}{2k^2}\int f(R,T)\sqrt{-g}\, d^4x + \int L_m\sqrt{-g}\, d^4x, \tag{1.29}$$

where $k^2 = \frac{8\pi G}{c^4}$, $f(R,T)$ is an arbitrary function of the Ricci scalar (R) and T is the trace of the energy-momentum tensor T_{ab}. The determinant of the metric tensor g_{ab} is given by g and L_m is the Lagrangian density of the matter part. The field equations for the modified gravity they can be obtained by varying the action S with respect to the metric tensor g_{ab} which yields,

$$\begin{aligned}(R_{ab} - \nabla_a\nabla_b)f_R(R,T) + g_{ab}\,\Box f_R(R,T)\\ -\frac{1}{2}g_{ab}\, f(R,T) = 8\pi\, T_{ab} - f_T(R,T)\,(T_{ab} + \Theta_{ab}),\end{aligned} \tag{1.30}$$

where $f_R(R,T)$ denotes the partial derivative of $f(R,T)$ with respect to R, and $f_T(R,T)$ denotes the partial derivative of $f(R,T)$ with respect to T. R_{ab} is the Ricci tensor, $\Box \equiv \frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}\, g^{ab}\partial_b)$ is the D'Alembert operator and ∇_a represents the co-variant derivative, which is associated with the Levi-Civita connection of the metric tensor g_{ab}. The energy momentum tensor T_{ab} for perfect fluid changes the role in the $f(R,T)$-modified gravity because of the presence of $\nabla_a\nabla_b R$ and $(\nabla_a R)(\nabla_b R)$ and terms which originate from trace of the energy momentum tensor T in the field equation.

1.6.3 $f(\mathbb{T})$ Theory

It is known that the General theory of Relativity (GR) is a geometric theory of gravitation based on the assumption that gravity manifests itself as the curvature of spacetime. In GR the spacetime structure may be determined using either one or both of Lorentzian metric (g) and a linear Levi-Civita connection and spacetime connections considered by Einstein are torsionless or free from any kind of vorticity. However it may be noted that apart from simplicity there is no reason to consider spacetime to be torsionless. Torsion in the theory may arise from the consideration of spin or from a gradient of scalar field without possessing any spin. The idea to modify GR with torsion [77–79] arises whether the spacetime connection is a symmetric or not. Essentially GR is a classical theory which does not accounted quantum effects, a situation which deals with gravity at a fundamental level. The paradigm is that the mass-energy is the source of curvature and in general, spin is the source of torsion. The consideration of a spacetime with torsion is an alternate approach equivalent to GR, was first introduced by Einstein himself known as Teleparallel Equivalent of General Relativity (TEGR) [80]. The most significant difference between GR and TEGR is that in the later case tetrad fields are present [81, 82]. In this case the tetrad fields are used to define a linear anti-symmetric Weitzenb*ö*ck connection related to torsion without a curvature [81, 80]. Although both GR and TEGR provides similar results there are some fundamental differences between the two theories. According to GR, curvature represents the geometrical picture of spacetime. In contrast TEGR represents the same gravitational interactions in terms of torsion which acts as a force. This implies that in TEGR the geodesic equations are analogous to the Lorentz force equations of electrodynamics [82]. There are so many theories of gravity which is actually based on the curvature-based formulation $e.g.$ in $f(R)$ theory, the Lagrangian is considered to be a non-linear function of the curvature scalar, R. However, we can reasonably think to start from TEGR, and use it as a basis to build a gravitational modification. The simplest class of these torsion-based modifications is the paradigm of $f(\mathbb{T})$ gravity [83–86] in which the Lagrangian is considered to be a non-linear function of the TEGR Lagrangian $\mathbb{T}$. TEGR coincides completely with general relativity at the level of equations, $f(\mathbb{T})$ gravity may be simple compared to $f(R)$ gravity as they give rise to second order differential equations whereas it is fourth order in case of $f(R)$ theories. In recent times, TEGR formalism and $f(\mathbb{T})$ gravity are found to be successful to accommodate cosmological and astrophysical phenomena. Assuming torsion in four as well as in higher dimensions, researchers tried to investigate the unification of gravity with electromagnetism. Also, by introducing the concept of torsion, one can obtain the repulsive nature of the energy-momentum tensor, which leads to the singular free cosmological model [87–90] where the spin alignment of primordial particles may be source of torsion.

In TEGR formalism, a linear anti-symmetric Weitzenb*ö*ck connection can be prescribed by the tetrad fields (θ^i_a), which form an orthonormal basis for the tangent space at each point of the manifold with spacetime coordinate x_a.

In $f(\mathbb{T})$ gravity, the line element of a manifold is given by,

$$ds^2 = g_{ab}\, dx^a dx^b = \eta_{ij}\theta^i_a\theta^j_b\, dx^a dx^b, \tag{1.31}$$

$$dx^a = e^a_i\theta^i; \;\; \theta^i = e^i_a dx^a, \tag{1.32}$$

where $\eta_{ij} = diag[1,-1,-1,-1]$, $e^a_i e^i_b = \delta^a_b$, the indices i, j are related to the tetrad field θ^i_a and the greek letters a, b corresponds to the space time coordinates. The root of the metric determinant is $\sqrt{-g} = det[e^i_a] = e$. The Weitzenbŏck's anti-symmetric connections for the vanishing Riemann tensor with non vanishing torsion is defined as

$$\widetilde{\Gamma}^{\alpha}_{ab} = e^{\alpha}_{i}\partial_b e^{i}_{a} = -e^{a}_{i}\partial_b e^{\alpha}_{i}. \tag{1.33}$$

The torsion and con-torsion tensors are:

$$\mathbb{T}^{\alpha}_{ab} = \widetilde{\Gamma}^{\alpha}_{ba} - \widetilde{\Gamma}^{\alpha}_{ab} = e^{\alpha}_{i}(\partial_a e^{i}_{b} - \partial_b e^{i}_{a}). \tag{1.34}$$

$$K^{ab}_{\alpha} = -\frac{1}{2}(\mathbb{T}^{ab}_{\alpha} - \mathbb{T}^{ba}_{\alpha} - \mathbb{T}^{ab}_{\alpha}). \tag{1.35}$$

The above two tensors given by Eqs.(1.34) and (1.35) can be used to construct a new tensor given by,

$$S^{ab}_{\alpha} = \frac{1}{2}(K^{ab}_{\alpha} + \delta^{a}_{\alpha}\mathbb{T}^{\beta b}_{\beta} - \delta^{b}_{\alpha}\mathbb{T}^{\beta a}_{\beta}). \tag{1.36}$$

The torsion scalar now can be expressed using the above tensor as,

$$\mathbb{T} = \mathbb{T}^{\alpha}_{ab}S^{ab}_{\alpha}. \tag{1.37}$$

Analogous to modified gravitational action in $f(R)$ theory one can obtain another modified gravity $f(\mathbb{T})$ replacing R by $\mathbb{T}$ as,

$$S = \int d^4x \left[\frac{1}{2k^2}f(\mathbb{T}) + L_m(\phi_A)\right] e \tag{1.38}$$

where, L_m is the matter Lagrangian and ϕ_A indicates the matter field. Variation of Eq.(1.38) with respect to tetrad field yield the following set of equation of motion:

$$e^{\alpha}_{i}S^{ab}_{\alpha}\partial_a\mathbb{T}f_{\mathbb{T}\mathbb{T}} + e^{-1}\partial_a(ee^{\alpha}_{i}S^{ab}_{\alpha})f_{\mathbb{T}} + e^{a}_{i}\mathbb{T}^{\lambda}_{ak}S^{bk}_{\lambda}f_{\mathbb{T}}$$

$$-\frac{1}{4}e^{b}_{i}f = -4\pi e^{\lambda}_{i}T^{b}_{\lambda}, \tag{1.39}$$

where T^b_λ is the energy-momentum tensor of a particular matter, whereas $f_{\mathbb{T}}$ and $f_{\mathbb{T}\mathbb{T}}$ represent first and second derivatives of $f(\mathbb{T})$ with respect to the torsion scalar $\mathbb{T}$ respectively.

1.6.4 Higher order curvature theory

To understand gravitational theories in higher dimensions there has been extensive work on various generalizations of GR making use of the higher dimensional spacetime. With the development of string theory there has been a spurt in activities in the study of black

holes and other compact objects in the framework of higher dimensions. Although we do not have direct evidence of observational or experimental proof to support the existence of higher dimensions but the success of superstring theory made it clear that the dimensions of spacetime may be more than the usual four dimensions. The higher dimensional theories must be consistent with the following features: (i) general covariance which states that the Lagrangian must be a scalar density constructed from the Riemann curvature yielding a non-trivial equation of motion, (ii) the equivalence principle and (iii) the equation of motion which are second order quasi-linear. The above mentioned properties uniquely correspond to the Lanczos–Lovelock Lagrangian (LL-gravity), which is a homogeneous polynomial in the Riemann curvature. In 1971, Lovelock [91] proposed a theory of gravity (also known as Lovelock gravity) as an extension of Einstein's theory of general relativity. It is the most general metric theory of gravity, providing conserved second order equations in any number of spacetime dimensions D. Lovelock's theory is a natural extension of Einstein's General Relativity in higher dimensions. The gravitational Lagrangian is a polynomial which has specific coefficients where the zeroth, linear and quadratic order terms correspond to the cosmological constant and Gauss–Bonnet (GB) terms [91, 92]. The simplest of the higher curvature gravity is the Gauss-Bonnet gravity (henceforth GB) with a coupling parameter in the action measuring the effect of higher curvature. The GB terms also arise in the one loop correction of the low-energy effective action in string theory. In four dimensions, the GB term in the Lagrangian is topologically invariant. The GB terms is a constant in $D = 4$ and but its dynamical effect is important in more than four *i.e.*, $D \geq 4$ dimensions.
Recently, there has been a spurt in research activities in the modified theories of gravity with GB terms in 4-D classical general relativity which is the contribution of lower energy limit in string theory. Gravitational propagation in higher dimensions has revealed a number of interesting aspects on kinematical and dynamical features of gravitational interactions. The Einstein-Gauss-Bonnet (EGB) gravity occurs naturally in heterotic string theory as the limit of the low-energy effective action. The GB combination with quadratic terms in curvature is considered in the Einstein-Hilbert action for a proper modification of GR. Zweibach [93] suggested that the string corrections due to Einstein action up to first order in the slope parameter α and fourth power of momenta is given by α (GB). Thereafter it was realized that the field redefinition theorem of 't Hooft and Veltman [94] can be applied in this case. It is known that on the Einstein shell $R_{ab} = 0$, an action of the form $R + a' \, R_{ab}^2 + b' \, R^2$, (where a', b' are constants) can be transformed into Ricci scalar: R itself (neglecting the higher order terms) by field redefinition :

$$g'_{ab} = g_{ab} + a' \, R_{ab} + g_{ab} \frac{a' + 2b'}{2 - D} \, R \tag{1.40}$$

where D represents number of spacetime dimensions. Later Deser and co-workers [95, 96] shown that the actions $R+\alpha\,(GB)$ and $R+\alpha R^2_{\mu\nu\alpha\beta}$ are not different. This result generalizes to all higher-order ghost terms. In cosmology an action in which the coefficients of R and $\alpha\,(GB)$ terms are considered arbitrary to begin with, which however can be determined when the relativistic cosmological solutions can be accommodated to fit the expected cosmological scenario. It is reported that the signature of the coefficient of GB for a realistic cosmology is however found opposite to that in a superstring theory [50]. The higher order theories of gravity have attracted much attention, as an alternative theory of gravity beyond GR, which shows quite different features from that in four dimensions. The approach opened up a new window for several novel aspects in cosmology, astrophysics and in particle physics. In standard 4D, EGB and Einstein gravity are indistinguishable. The departure from the standard 4D Einstein gravity is found in more than the usual 4D. There have been many interesting results in the 5-D EGB theory ranging from the vacuum exterior solution due to Boulware and Deser [96], which is a generalization of the Kerr-Schild vacuum solution. Thus Gauss-Bonnet terms have a rich structure in the theory which are relevant both in higher dimensional astrophysics and cosmology. The gravitational action with Gauss-Bonnet terms in higher dimensions is given by,

$$S=\int\sqrt{-g}\left[\frac{1}{2k^2}(R+\alpha\,L_{GB})\right]d^Dx+S_m \tag{1.41}$$

where, R is the Ricci scalar, L_{GB} is the Gauss-Bonnet terms, S_m is the matter action, g is the metric term in higher dimensions and the dimensional coupling parameter α. The Gauss-Bonnet Lagrangian is given by, $L_{GB}=R^2-4R_{ab}R^{ab}+R_{abcd}R^{abcd}$, where the indices a, b, c and d are indices (0,...., $D-1$). The field equation in the presence of matter field is derived from Eq.(1.41) which is given by,

$$G_{ab}+\alpha H_{ab}=k^2T_{ab} \tag{1.42}$$

where G_{ab} denotes the Einstein tensor, T_{ab} is the Total energy-momentum tensor and the Lanczos tensor is given by,

$$H_{ab}=2(RR_{ab}-2R_{ac}R^c_b-2R^{cd}R_{acbd}+R^{cde}_aR_{bcde})-\frac{1}{2}g_{ab}L_{GB}. \tag{1.43}$$

The above field equations are to be taken up to study astrophysical or cosmological issues.