Abstract

This thesis presents some new results on the construction of an extended scale invariant analytic framework using the concept of asymptotic duality at the level of the real continuum $\mathbb{R}$ and its applications in some nonlinear systems.

Rather than working directly on a Cantor set like fractal set, we begin from the ordinary real number system (continuum) $\mathbb{R}$ instead and extend it to a non-standard (and non-Archimedean) like continuum $\mathbb{R}^*$ accommodating relative infinitesimals and infinitely large new elements relative to a preassigned scale, that are supposed to be accessed, in actual applications, in an asymptotic limiting sense, involving a nontrivial, finitely valued, ultrametric norm, respecting some scaling and duality (inversion) transformations. As the ultrametric valuation turns out to be discretely valued, the ordinary linear neighbourhood of 0 in $\mathbb{R}$ generaly gets extended into a Cantor set like structure in the extended set $\mathbb{R}^*$. Consequently, an arbitrarily small asymptotic real variable $x \to 0$ would get a relatively finite non-negative real value $x \mapsto v(x)$. Moreover, the nontrivial scale invariant function $v(x)$ has the structure of a Lebesgue-Cantor staircase function that has nontrivial variations only on that preassigned Cantor set like extended neighbourhood. The entire constructions leading to the above scenario is called *asymptotic duality* transformations.

Next, a measure theoretic realization of the asymptotic valuation $v(x)$

in an extended Cantor set like neighbourhood $\mathbf{O}^*$ of $0 \in \mathbb{R}$ is presented. It is shown that $v(x)$ can be interpreted as a regular measure that is absolutely continuous with respect to the associated Hausdorff measure in the sense of the Radon-Nikodym theorem.

Subsequently, the extended non-Archimedean space $\mathbb{R}^*$ equipped with discretely valued valuation $v(x)$ is interpreted as a connected deformed real continuum $\mathcal{R}$ in which an asymptotic neighbourhood is defined as $\mathcal{O} = v(\mathbf{O}^*)$. The duality transformations and corresponding valuations are classified as self dual, weakly self dual and strictly dual asymptotics. Next, a geometric characterization of the deformed extended set $\mathcal{O}$ is carried out for self dual and strictly dual valuations. For self dual valuations $\mathcal{O}$ can be realized either as a smooth or a piece-wise smooth broken curve (line), when a strictly dual valuation corresponds to a fractal like extension. Such distinct geometric features are shown to relate to cases when the original extended set $\mathbf{O}^*$ could be covered either by a finite or countable set of clopen balls, on each of which constant values of the valuation v are assigned, respecting continuity induced by the ultrametric norm.

An extension of asymptotic duality concepts in a function space is then considered. Introducing function space dependent valuation $v_F(f)$ for a given function $f(x)$, $x \in \mathbb{R}$, we next study asymptotic continuity and asymptotic differentiability in the associated extended space. Some simple examples are worked out to show how classical discontinuity and nondifferentiability are realized as asymptotically continuous and differtentiable in a point-wise manner.

Applications of above formulated ideas and results are then applied to the middle third Cantor set to formulated a differential calculus on such sets. A function having nontrivial variations only on a Cantor set can be described by a fractal differential equation, when a fractal

derivative is defined as an asymtptotic derivative.
Finally, an application of this asymptotic duality formulations is also studied in the context of the KdV type nonlinear evolutionary equation. It is shown explicitly how an intrinsically realized *seed deformation* in a neighbouhood of an initial solitary wave profile, aided by asymptotic duality principle, could subsequently induce a global deformation on the original solitary wave, so as to realize exotic wave forms such as rouge wave, breather wave, periodic singular wave and etc, even in absence of of any external excitation. Consequently, the realizations of such exotic wave patterns can be interpreted as manifestations of another non-physical level of excitation that generally remain nascent at ordinary scales, but could become activated and realizable here as asymptotic duality principle.

The thesis is based on the following three publications:
(1) *D. P. Datta, S. Sarkar*, "Duality structure, asymptotic analysis and emergent fractal sets." *Nonlinear Studies*, Vol. 25, No. 3, pp. 609-640, 2018. (See arXiv:1602.01486[math.CA]).
(2)*D. P. Datta, S. Sarkar and S. Raut.*, "Novel Excitation of local fractional dynamics" *Nonlinear Studies*, Vol. 27, No. 4, pp. 935-956, 2020.
(3)*S. Sarkar, S. Raut and D. P. Datta*, "Asymptotically Deformed KdV Systems and Spontaneous Emergence of Complex Structures" *Chaos, Soliton & Fractals*
Communicated, 2023.