CHAPTER 1

Introduction

1.1 Overview

This dissertation is concerned with the optimization and complexity of a certain class of facility location problems. We investigate ways in which plane analytical coordinate geometry and spherical trigonometry can be used as tools in the solutions of various facility location problems involving geodesic and rectilinear norms.

Owing to its potential application in many practical situations, these problems have been treated extensively in the literature. The distance functions most commonly encountered are euclidean, rectilinear and geodesic distances (see, for example [2], [4], [6], [14], [23], [26], [28], [39], [53], [57] and [58]).

Facility location problems have attracted the attention of the ancient Greeks. Fermat [11] in early seventeenth century formulated a version of the distance location problem as a geometrical problem which may be enunciated as follows:

Let there be given three points in a plane. To find a fourth point such that the sum of the distances from it to the three points is a minimum.

This problem was solved by Torricelli [25] in 1640. A German economist named Alfred Weber [56] generalised this problem by introducing weights in the analysis. The Weber problem consists in locating a warehouse so as to render the total weighted distance between it and the demand points a minimum. Such problems are referred to as the minisum problem in the literature and Harold Kuhn [33] must be credited with trying a purely mathematical solution procedure to solve the problem. Very often instead of minimising the total distance travelled the maximum distance is required to be minimised. Being naturally called the minimax criterion, this is most suitable for locating an emergency facility where maximum delay in rendering a service is more important than average or total delay (see, e.g., [26] and [39]). There is still another criterion which involves maximising the minimum distance. Known as the maximin criterion, this is applicable in case there is an obnoxious or undesirable facility (see, e.g., [6], [12], [19] and [41]), such as a nuclear plant or for that matter any polluting source, and the facility is to be so located that it is as far away as possible from the points it actually serves.

The study and development of methodologies to determine the location of new facilities is carried out in such a manner that the potential users of the facilities are benefited most. The investigation is conducted by constructing suitable models involving one or more new facilities, and solving them.

The above criteria may be used to solve a single facility or a multifacility location problem (see, e.g., [2], [38], [42] and [49]). The former seeks to locate a facility amongst several demand points whereas the latter concerns locating any given number of variable points representing facilities with respect to any given number of fixed points representing potential users. The m-centre problem which is a generalisation of the 1-centre problem forms the most important class of problems in location analysis and has been extensively studied (see, e.g., [21], [25], [32], [39] and [46]).

Lee [34], and Shamos and Hoey [52] presented algorithms to construct the Voronoi diagram and proved that the smallest circle (in the L_p-metric) enclosing the set could be solved very efficiently.

Spherical location problems typically involve finding the optimal location of a service facility among a number of demand points situated on the surface of a sphere (see, e.g., [1], [2], [14]-[19], [31], [35]-[37], [39], [45], [50] and [58]). Quite a good number of military, civil and commercial logistics and location problems being concerned with globally distributed demand points (see, e.g., [35], [45] and [58]), the planar distance approximation becomes irrelevant and the choice naturally falls on a metric that spans over a sphere.

This research considers the minimax spherical problem that finds the location of a service facility for which the largest distance to a demand point is minimized. The minimax spherical problem differs from its planar counterpart (see, e.g., [19], [45] and [60]) in that its objective function is nonconvex and nondifferentiable. In a special case where all demand points lie on a hemisphere, geometrical algorithms (see, e.g., [5], [9], [23], [26], [29], [40], [43], [44] and [52]) for the two-dimensional minimax problem using the euclidean norm can be applied. In addition, Sarkar and Chaudhuri [50] present an efficient geometrical algorithm based on geodesic distance and Litwhiler's method [35] stereographically projects spherical surfaces bounded by planes on to plane circles so that techniques for two and three-dimensional spaces can be used. Given a set of demand points on a sphere, algorithms in [48] and [55] determine whether the points lie on a hemisphere.

When the demand points do not lie on a hemisphere, Drezner and Wesolowsky [19] proposed a steepest descent technique to solve the minimax problem. Recently, Patel [45] *proposed* another algorithm based on a factored secant update technique. Both algorithms only produce local optimal solutions. In theory, one can obtain a global optimal solution by generating all local solutions. However, the approach is difficult, if not impossible, to implement in practice.

As regards the scope of application of the minimax criterion to the weighted rectilinear distance location problem (see, e.g., [20] and [25]) we might consider locating a new facility, say a polyclinic or a fire station in a large metropolitan area where the objective is to minimize rectilinear travel distance of a potential user plus a nonnegative constant, the weight being any positive number quantifying the nature of interaction between the facility and the category of user. On receiving a call for any emergency service, the person at the reception requires some time to pass the information to the appropriate location where the service is to be rendered from. Thus there is a time lag from the instant a demand for being serviced is requisitioned to the instant the particular facility is made available. This explains why a response parameter has been considered in the ongoing analysis.

The time complexity of various types of location problem may be found (see, e.g., [40] and [53]). A method-oriented selective survey of representative problems in location research occurs extensively in the works of [4], [13] and [28].

1.2. Chapter Summaries

In chapter 2 we have presented three different algorithms for spherical minimax location problems.

The algorithm presented in section 2.2 of this chapter solves a global single facility minimax location problem exactly in polynomial time. As far as we know there exists no algorithm which solves a global minimax location problem exactly in polynomial time. The algorithm we have developed is of complexity $O(n^3)$. It solves global as well as hemispherical minimax location problem. It also determines whether all demand points lie on a hemisphere or not. The procedure presented here is based on an enumeration technique and determines global optimal solutions in a finite number of steps. When a local minimum is obtained, one uses this information to obtain the next better solution(s). Thus the possibility of occurrence of inferior solution(s) in subsequent iterations can totally be eliminated. The Pascal Code of the problem has been developed. Using this code we have solved the problem given in [45]. From the solution of the problem it is clear that choosing the minimum from all local minima may sometimes fail. Because it is difficult to generate all local minima although theoretically it is sound. In fact the result given in [45] is not correct.

The algorithm given in section 3.2 of chapter 2 solves a hemispherical minimax location problem, involving a geodesic norm, exactly in polynomial time. The time complexity of this algorithm is $O(n^2)$. No geometrical algorithm for solving the spherical problem when the demand points are assumed to lie on a hemisphere exists, except the one which relies on primal feasibility [50]. Although the present approach is based on dual feasibility [23] adapted from the planar case [29], the two differ significantly in their implementations. Both select initially a pair of points. But the former proceeds in such a way that if a demand point happens to be within a spherical disc (see [58]) at any iteration it continues to remain in a larger spherical disc at subsequent iterations until optimality is reached. The complexity of the algorithm discussed in [50] is $O(n^2)$, but from a computational point of view it takes much more time to solve a problem than the present method. We have developed the Pascal codes of the present algorithm and the algorithm given in [50] to make a comparison of the running time of the algorithms.

In section 4.2, chapter 2, we have developed an alternative algorithm, based on geometry, for solving a hemispherical minimax location problem. The method yields exact solution having a time complexity $O(n^2)$. It has been shown that the solution of the minimax problem when the norm under consideration is geodesic is equivalent to solving a maximization problem using the euclidean norm. We have proved that a hemispherical minimax problem reduces to finding a small circle of maximum radius on the surface of the sphere which contains either two demand points at the ends of a diameter or three demand points forming an acute triangle such that all demand points lie on one side of the plane of the small circle and the centre of the sphere on the other side. The basic difference between this algorithm and the algorithm given in section 3.2 of this chapter is that while the former depends on the maximization of the euclidean distance the latter uses the properties of spherical triangles. Pascal code of the problem has been developed to compare the performance of the present and the existing algorithms. The algorithm presented in this section is significantly faster than all existing algorithms. From a computational point of view the algorithm given in section 3.2 takes much more time than the present algorithm, despite the former being $O(n^2)$. This is so because the algorithm presented in section 3.2 depends on computation of the trigonometric functions whereas the present algorithm uses only euclidean distances.

In Chapter 3 we have included solutions of three minimax location problems involving a rectilinear norm.

The purpose of sections two and three of this chapter are to deal with the effect of a response parameter (see, e.g., [20] and [26]) when it is added to the weighted rectilinear distance measured from a new facility point to each of n existing demand points so that the maximum value of the cost function attains its minimum value. The methods of solution are based on the notion of equipolygon, i.e., the locus of points whose generalized weighted rectilinear distances from two given points are equal.

The algorithm developed in section 2.2, of Chapter 3, depends on the concept of dual feasibility. Elzinga and Hearn [23] used this technique to solve an equiweighted euclidean minimax location problem. Hearn and Vijoy [29] extended this approach to the weighted case. The linear programming model (see, e.g., [25], [42] and [59]) of this problem is also possible. The dual simplex method is most appropriate in the present context. To solve this problem in a computer, the present algorithm requires approximately one fifth less computer memory storage than is necessary for the dual simplex method. Consequently the present algorithm can solve problems, approximately five times larger, which can be solved by the dual simplex method. When both methods can solve a problem, the present method requires less computer CPU time. T-transformation (see, Francis and White [25]) and the present algorithm can solve problems of the same size. But from a computational point of view T-transform algorithm is not very efficient.

In section 3.2 of Chapter 3, we have developed an alternative algorithm for the weighted rectilinear minimax location problem. This algorithm uses the concept of primal feasibility. Chakraborti and Chaudhuri [9] applied this idea to solve the equiweighted euclidean minimax location problem. Hearn and Vijoy [29] extended this principle to the weighted case. Although the algorithms given in 3.2 and 2.2 have the same complexity the primal feasible algorithm developed in section 3.2 requires less computer CPU time than the dual feasible algorithm given in section 2.2 . This is so because the former algorithm depends on a unidirectional search whereas the latter on a bidirectional search.

In section 4 we have solved the minimax location problem for an arbitrary shaped constrained region (see, e.g., [3], [5], [7], [8], [22], [27] and [51]) using the rectilinear norm. The method of solution is based on the concept of dominating sides. For a point P in the x-y plane a side of the rectangle ABCD, whose sides are inclined at an angle of 45° and 135° with the positive direction of the x-axis, is said to be dominating if the rectilinear distance of P from any point on this side is greater than that from any point on the remaining sides. In the constrained case, we find the minimum corresponding to each dominating side and the minimum among these will give the global minimum.