CHAPTER – II

UNSTEADY FLOW OF ELASTICO-VISCOUS LIQUID

UNSTEADY CHANNEL FLOW OF A NON-NEWTONIAN FLUID *

2.1 Introduction

A number of workers studied non-Newtonian fluid flow between two parallel plates due to its applications in various fields of science and technology. Sharma [1] considered plane Poiseuille flow through tube and case of a parallel plate visco-meter. Agarwal and Jain [2] extended the paper of Sharma [1] by introducing magnetic field in the system. He considered the fluid to satisfy the Oldroyd model. Frater [3] discussed the flow of an elastico-viscous liquid between torsionally oscillating discs. The plane Poiseuille flow, plane Couette flow and the Couette flow of micropolar fulids was investigated by Rajagopalan [4]. Subsequently, Johri [5] discussed the problem of an elastico-viscous flow (Rivlin-Ericksen model) induced by circular oscillations of two infinite parallel discs. Later, Johri [6] studied the problem of unsteady slow flow of Oldroyd B-liquid between two infinite parallel discs. The flow was induced by the elliptic harmonic oscillations of the discs.

Here we consider the flow of visco-elastic fluid (Kuvshiniski model [7]) between two oscillating parallel flat plates as we are aware of very little work [8,9] in this model.

---

* Published in the 'Indian J. Maths.', Vol.31, p-137, 1989.

2.2 Basic equations and their solutions

We consider the flow of an incompressible elastico-viscous liquid bounded by two infinite parallel flat plates executing simple harmonic oscillations with a frequency ω in their own planes $y = \pm h$. We assume that the fulid was at rest for $t \leq 0$. Since the plates are infinite, the flow is a parallel flow in which the stream lines are along the x-axis and the velocities are functions of distance y and time t only.

Under these assumptions, the equations (1.12) – (1.15) reduce to

$$\left(1 + \lambda_o \frac{\partial}{\partial t} \right) \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad , \qquad (2.1)$$

where u is the velocity in the direction of x, λ_o is the relaxation time and ν is the kinematic coefficient of viscosity.

The flow is generated by the non-torsional oscillations of the plates given by

$$u = a \sin \omega t \text{ at } y = \pm h \quad , \qquad (2.2)$$

where a is the amplitude of oscillation and ω is the imposed oscillations. For a liquid at rest for all $t \leq 0$, it may be assumed that the initial state of stress is zero. The initial conditions are

$$u(y,t) = 0, \quad \frac{\partial u(y,t)}{\partial t} = 0 \text{ at } t = 0 \text{ for all } y. \qquad (2.3)$$

Introducing non-dimensional parameters $u' = \frac{u}{h\omega}$, $t' = t\omega$, $y' = \frac{y}{h}$, $\alpha = \lambda_0 \omega$, $R = \frac{\nu}{h^2 \omega}$, $a' = \frac{a}{h\omega}$ in equations (2.1) – (2.3) and dropping dashes, we have

$$\left(1 + \alpha \frac{\partial}{\partial t}\right) \frac{\partial u}{\partial t} = R \frac{\partial^2 u}{\partial y^2} \quad , \qquad (2.4)$$

$$u = \frac{\partial u}{\partial t} = 0 \text{ at } t = 0 \text{ for all } y \quad , \qquad (2.5)$$

$$u = a \sin t \text{ at } y = \pm 1 \ , \ t > 0. \qquad (2.6)$$

Using the Laplace transform in equation (2.4) subject to initial and boundary conditions in (2.5) and (2.6), it turns out taht the solution of the problem can be readily represented by the Laplace inversion integral in the form

$$u\,(y,t) = \frac{1}{2\pi i}\, a \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{1+p^2}\, \frac{\cosh My}{\cosh M}\, e^{pt}\, dp \ , \qquad (2.7)$$

$p > 0$, γ is greater than the real part of the singularities of the integrand and

$$M = \sqrt{[p\,(1+\alpha\, p)\,/\,R\,]} \quad .$$

Using the theory of residues, we evaluate the integral (2.7) and the expression for the velocity field becomes

$$u\,(y,t) = \frac{a}{E_3^2 + E_4^2}\, [\,(E_1E_3 + E_2E_4)\sin t$$

$$+ (E_2E_3 - E_1E_4)\cos t\,]$$

$$+ a\pi R \sum_n \sum_{i=1}^{2} \frac{(-1)^n(2n+1)}{{p_n^i}^2 + 1}\, Q(p_n^i).\cos\left(\frac{2n+1}{2}\right)\pi y\, e^{p_n^i t} \ , \qquad (2.8)$$

where

$E_1 = \cosh Cy \cos Dy,$

$E_2 = \sinh Cy \sin Dy,$

$$E_3 = \cosh C \quad \cos D ,$$

$$E_4 = \sinh C \quad \sin D ,$$

$$C, D = \sqrt{[\sqrt{(1+\alpha^2)} \mp \alpha] / \sqrt{2R}} ,$$

p_n^i 's are the roots of the quadratic

$$\alpha {p_n^i}^2 + p_n^i + R \left(\frac{2n+1}{2}\right)^2 . \pi^2 = 0 \qquad (2.9)$$

and

$$Q(p_n^i) = \frac{1}{(1 + 2\alpha p_n^i)} .$$

The first term of Eqn. (2.8) (under bracket) represents the steady state velocity field and the last infinite sum corresponds to the transient component of the solution which decays exponentially as $t \to \infty$ and the final steady state oscillations is achieved. $u(y,t)$ may also be represented by

$$u(y,t) = A \sin(t + \theta) + Tr. , \qquad (2.10)$$

where A is amplitude of oscillations, θ is phase difference with the oscillations of the plate and Tr. is the transient part of the velocity profile.

The skin-friction on the upper plate wall is

$$\tau_{xy} = \left[\left(1 - \alpha \frac{\partial}{\partial t}\right) \frac{\partial u}{\partial y} \right]_{y=1}$$

$$= \frac{a}{E_3^2 + E_4^2} [(E_9 + \alpha E_{10}) \sin t + (E_{10} - \alpha E_9) \cos t]$$

$$+ a \pi R \sum_n \sum_{i=1}^{2} \frac{2n+1}{{p_n^i}^2 + 1} . Q(p_n^i) e^{p_n^i t} . \left(\frac{2n+1}{2}\right) \pi . (\alpha p_n^i - 1), \qquad (2.11)$$

where

$$E_9 = E_5 E_7 + E_6 E_8 ,$$

$$E_{10} = E_5 E_8 - E_6 E_7 ,$$

$$E_5 = C E_3 + D E_4 ,$$

$E_6 = C\,E_4 - D\,E_3$,

$E_7 = \sinh C \cos D$,

$E_8 = \cosh C \sin D$.

2.3 Discussion

To have a physical insight into the problem, velocity distribution is calculated for different values of α. Figure 2.1 shows that maximum velocity is shifted towards the axis of the channel in presence of elastic elements in the liquid but for Newtonian liquid the case is reversed. It can be remarked that flow of elastico-viscous liquid has a boundary layer character near the plate walls. Figure 2.1 also depicts that effect of elasticity is more prominent near the axis of the channel.

From table 2.1, it may be concluded that amplitude of steady state oscillations of the velocity profile increases with the increase in the elastic parameter. There will be a phase lag of oscillations on the axis of the channel. Phase lag increases with the increase in the value of elastic parameter.

Table 2.2 shows that the effect of elastic element is to increase the magnitude of shear stress on the plate wall.

Table 2.1: Amplitude (A) and phase difference (θ) of the velocity profile at $y = 0$.

α	A	θ
0	0.0999999	-4.9999×10^{-4}
1	0.1000499	-5.0016×10^{-4}
2	0.1000521	-5.0036×10^{-4}
3	0.1001500	-5.0049×10^{-4}
4	0.1002002	-5.0066×10^{-4}
5	0.1002504	-5.0082×10^{-4}

Table 2.2: Shear stress (τ_{xy}) at the upper plate.

t	$\alpha = 1$	$\alpha = 2$	$\alpha = 2.25$
1	-0.0897863	0.2881691	0.3091253
1.5	7.00048×10^{-3}	-0.1274758	-0.1462511
5	0.0275514	0.1739753	0.1973641

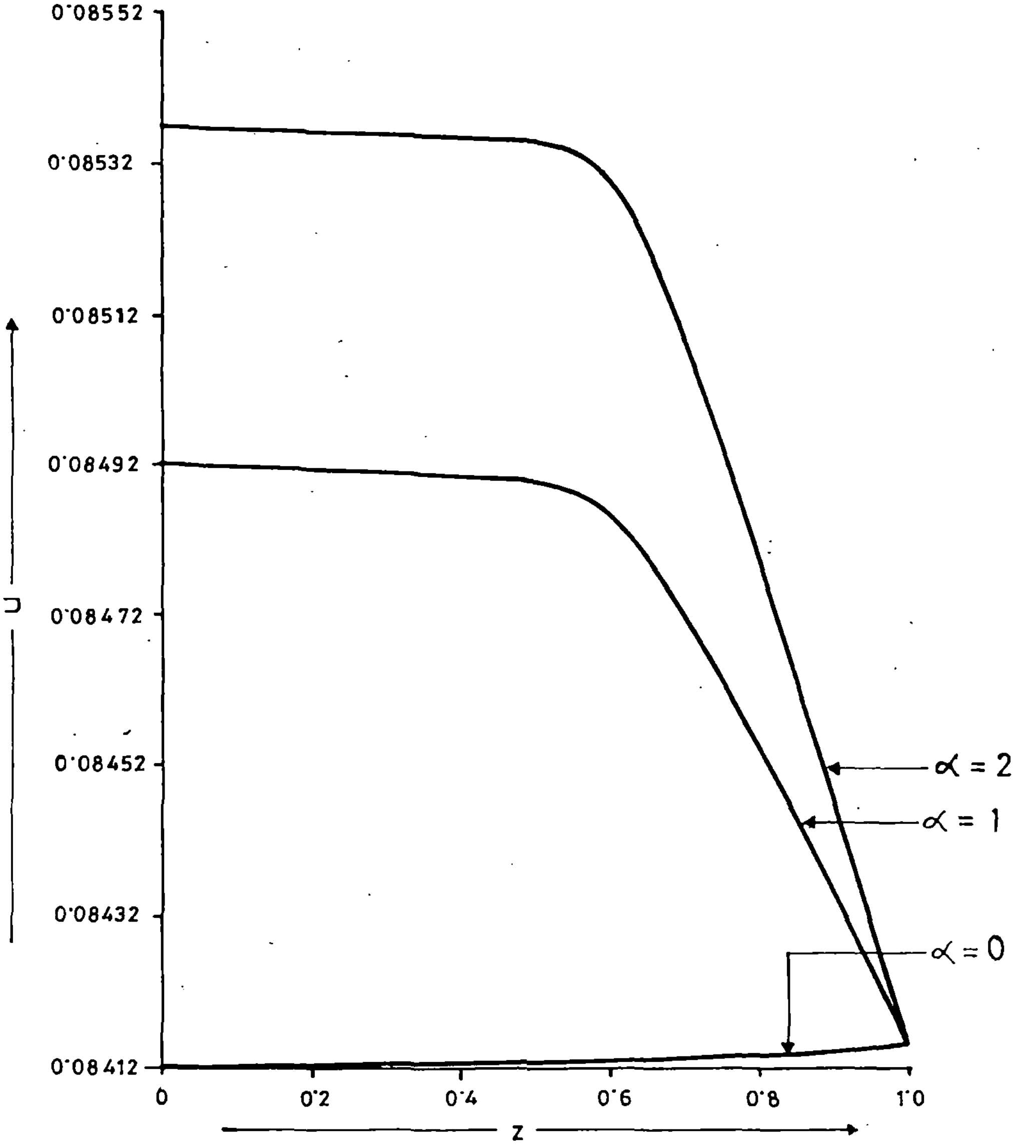

Fig. 2·1. Velocity profile of liquid particles at $t=1$ when $a=0{\cdot}1$, $R=1000$.

PART TWO

OSCILLATING FLOW OF VISCO-ELASTIC OLDROYD FLUID IN A LONG CIRCULAR TUBE IN PRESENCE OF MAGNETIC FIELD *

2.4 Introduction

Oscillating flow of viscous fluid in a long circular tube under the influence of a pressure gradient was investigated theoretically and experimentally by Richardson and Tyler [10] and theoretically by Sexl [11].

Many common liquids such as oils, certain paints, blood, polymer solutions, some organic liquids and many new materials of industrial importance exhibit both viscous and elastic properties. Based on the macroscopic rheological behaviour of real materials, Oldroyd [12,13] formulated rheological equations of state for a class of incompressible visco-elastic liquids and initiated the study of visco-elastic flows. Based upon Oldroyd's equation of state, the flow phenomena have received a considerable attention for various configurations. Johri and Singhal [14] investigated the problem of oscillating flow of visco-elastic liquid of Maxwell type in a long circular tube under the influence of periodic pressure gradient.

---

* Published in the 'Proc. ISTAM', Vol.32, p-127, 1987.

The object of the present investigation is to discuss the oscillating flow of an incompressible electrically conducting visco-elastic (Oldroyd) liquid in a long circular tube in the presence of magnetic field of uniform strength. The effects of magnetic field and elastic elements in the liquid are shown in tabular form. In the presence of magnetic field, the velocity of Newtonian liquid is more than that of visco-elastic Oldroyd type liquid and decreases with the increase in elastic elements in the liquid. The effect of elasticity on the shear stress at the wall of tube is to increase its magnitude. Both the volume flow rate and shear stress increase with the increasing magnetic field. It is noted that if the oscillation be of high frequency, a boundary layer is formed close to the wall of tube. Such type of flow occurs e.g. under the influence of a reciprocating piston.

2.5 Mathematical formulation and solution of the problem

We consider axisymmetric flow of an electrically conducting (of conductivity σ) elastico-viscous liquid obeying (1.3) – (1.5) in presence of magnetic field of uniform strength (B_o) within a circular cylinder of radius 'a'. The time-dependent excitation is caused by the prescribed time-dependent pressure gradient.

Referring the problem to cylindrical polar coordinates (r,θ,x), the non-dimensional form (dropping dashes) of the governing equation of motion in the direction of x is

$$(1 + \alpha_1 \frac{\partial}{\partial t}) \frac{\partial u}{\partial t} = - (1 + \alpha_1 \frac{\partial}{\partial t}) \frac{\partial p}{\partial x}$$

$$+ (1 + \alpha_2 \frac{\partial}{\partial t}) (\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r})$$

$$- M^2 (1 + \alpha_1 \frac{\partial}{\partial t}) u , \tag{2.12}$$

where

$$x' = \frac{x}{a} , \quad t' = \frac{t\nu}{a^2} , \quad u' = \frac{ua}{\nu} , \quad p' = \frac{pa^2}{\rho\nu^2} , \quad r' = \frac{r}{a},$$

and $\alpha_1 = \lambda_1 \frac{\nu}{a^2}$, $\alpha_2 = \lambda_2 \frac{\nu}{a^2}$ are elastic parameters and

$M = \frac{B_0^2 \sigma a^2}{\rho \nu}$ is the magnetic parameter (ν is the kinematic coefficient of viscosity).

We assume $- \frac{\partial p}{\partial x} = Re(Ke^{i\omega t})$. $\quad$ (2.13)

The boundary condition for the problem is

$u = 0$ on $r = 1$. $\quad$ (2.14)

We assume the velocity component in the form

$u = Re f(r) e^{i\omega t}$, $\quad$ (2.15)

where f(r) is a function of r to be determined.

On substituting (2.15) in (2.12), we get

$$\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \frac{(i\omega + M^2)(1 + i \omega \alpha_1)}{(1 + i \omega \alpha_2)} f + \frac{K(1 + i \omega \alpha_1)}{(1 + i \omega \alpha_2)} = 0, \tag{2.16}$$

whence the solution satisfying the condition (2.14) will be given by

$$u (r,t) = - \frac{K}{(i\omega + M^2)} \left\{ \frac{J_0 (r K_1)}{J_0(K_1)} - 1 \right\} e^{i\omega t} , \tag{2.17}$$

where

$$K_1^{\,2} = -\frac{(i\omega + M^2)(1 + i\omega\alpha_1)}{(1 + i\omega\alpha_2)},$$

$$= K_2^{\,2}\,(\cos 2\alpha - i\sin 2\alpha),$$

$$K_2^{\,2} = \frac{1}{(1+\omega^2\alpha_2^2)}\left[\left\{\omega^2\alpha_1 - M^2 - \omega^2\alpha_2(1 + M^2\alpha_1)^2\right\}^2\right.$$

$$\left. + \,\omega^2\left\{1 + M^2\alpha_1 - \alpha_2(M^2 - \omega^2\alpha_1)\right\}^2\right]^{1/2},$$

$$\tan 2\alpha = \frac{\omega\{1 + M^2\alpha_1 - \alpha_2(M^2 - \omega^2\alpha_1)\}}{\{\omega^2\alpha_1 - M^2 - \omega^2\alpha_2(1 + M^2\alpha_1)\}}.$$

The volume flow rate is given by

$$Q = -\frac{\pi K}{(i\omega + M^2)}\left[\frac{2J_1(K_1)}{K_1 J_0(K_1)} - 1\right]e^{i\omega t}$$

$$= -\frac{\pi K}{(i\omega + M^2)}\left[\frac{K_1^2}{8} - \frac{K_1^4}{32.3} + \,.........\right]e^{i\omega t}. \qquad (2.18)$$

For small values of K_2, i.e., for slow vibrations,

$$u = \frac{K_2^2(1 - r^2)\cos\{\omega t - (2\alpha + \zeta)\}}{4(M^4 + \omega^2)^{3/2}} \qquad (2.19)$$

where $\tan\zeta = \frac{\omega}{M^2}$.

So, in this case, there is a phase lag $(2\alpha + \zeta)$ from the phase of prescribed pressure gradient.

Now for large values of K_2, we use asymptotic expression

$$J_0(z) \sim \sqrt{\frac{2}{\pi z}} \quad e^{\,i\,(z - \frac{\pi}{4})}$$

whence

$$u = -\frac{K}{M^4 + \omega^2}\Big[R \left\{ M^2 \cos(\omega t + R_1) + \omega \sin(\omega t + R_1) \right\}$$

$$- \left\{ M^2 \cos \omega t + \omega \sin \omega t \right\} \Big] \quad , \qquad (2.20)$$

where

$$R_1 = K_2(r - 1)\cos \alpha,$$

$$R = \frac{1}{(r)^{1/2}} e^{-K_2(1 - r)\sin \alpha} \quad .$$

It is evident from (2.20) that for large values of K_2, the first term within third brackets in (2.20) decreases rapidly and for greater distances only the second term, which does not depend on the distance (1−r) from the wall of the cylinder, remains, since at large distance from the wall fluid moves as if it were frictionless. The flow has a distinct boundary layer character.

The non-dimensional shear stress on the wall of the tube is given by

$$\tau\big|_{r=1} = \left[1 - (\alpha_1 - \alpha_2)\frac{\partial}{\partial t}\right]\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)$$

$$= \frac{KK_1}{i\omega + M^2} \quad \frac{J_1\,(K_1)}{J_0\,(K_1)} \left[1 - (\alpha_1 - \alpha_2)\,i\omega\right] . e^{i\omega t} . \qquad (2.21)$$

2.6 Numerical results and discussion

The effects of magnetic field and elastic elements over flow profile are shown in tabular form (tables 2.3 − 2.5). It is seen from table 2.3 that in presence of magnetic field, the velocity of Newtonian liquid is more than that of visco-elastic Oldroyd type liquid and it decreases with the increase in elastic elements. The effect of elasticity is to increase the shear stress on wall of the cylinder (table 2.4). It is clear from the table 2.5 that both the volume flow rate and shear stress increase with the increasing magnetic field.

Table 2.3: Effect of elastic parameters on the velocity profile when $M^2 = 1$, $\omega = 1$.

u	r=0	r=0.2	r=0.4	r=0.6	r=0.8	r=1.0
$\alpha_1=2, \alpha_2=0.2$	0.433673	0.412273	0.357052	0.265013	0.143392	0
$\alpha_1=1, \alpha_2=0.05$	0.434042	0.413166	0.358905	0.271357	0.151360	0
$\alpha_1=0, \alpha_2=0$	- 0.687967	-0.655684	-0.558585	-0.474683	-0.282026	0

Table 2.4: Effect of elasticity on the shear stress when $M^2 = 1$, $\omega = 1$.

	$\alpha_1 = 0$, $\alpha_2 = 0$	$\alpha_1 = 1$, $\alpha_2 = 0.05$	$\alpha_1 = 2$, $\alpha_2 = 0.2$
$\tau\vert_{r=1}$	0.5705354	0.7308212	- 0.7327728

Table 2.5: Effect of magnetic parameter on shear stress and volume flow rate when $\alpha_1 = 1$, $\alpha_2 = 0.05$, $\omega = 1$.

	$M^2 = 1$	$M^2 = 4$	$M^2 = 9$
$\tau\vert_{r=1}$	0.7308212	2.2408276	3.2824201
Q_λ	-8.3337×10^{-4}	-0.5246432	- 3.3630692

REFERENCES

[1] Sharma, S. K. (1959) ZAMM, Vol.39, 313.

[2] Agarwal, J. P. (1962) ZAMP, Vol.13, 152.
and
Jain, M. K.

[3] Frater, K. R. (1964) J. Fluid Mech., Vol. 19, 175.

[4] Rajagopalan, (1968) J. Indian Inst. Sci., Vol.50, 57.
Renuka.

[5] Johri, A. K. (1974) Agra Univ. Res.Sci.J.,Vol.23, 15.

[6] Johri, A. K. (1978) Indian J.Pure Appl.Math.,Vol.9,481.

[7] Kuvshiniski, E. V. (1951) J. Exptl. Theor. Phys. (USSR), Vol.21, 88.

[8] Soundalgekar, V. M. (1981) Indian J.Theoret.Phys.,Vol.29, 23.

[9] Gupta, Prem Chand., (1982) Indian J.Theoret.Phys.,Vol.30, 79.
Nair, B. K.
and
Sharma, R. G.

[10] Richardson, E. C. (1929) Proc. Phys. Soc., Lond., Vol.42, 1.
and
Tyler, E.

[11] Sexl, T. (1930) Z. Phys., Vol.61, 346.

[12] Oldroyd, J. G. (1950) Proc. Roy.Soc., Lond.,Vol.A200,523.

[13] Oldroyd, J. G. (1951) Quart. J. Mech. Appl. Math., Vol.4, 271.

[14] Johri, A. K. (1980) Acta Ciencia Indica, Vol.VIm, 63.
and
Singhal, R. K.