Chapter 1

Introduction

1.1 The equation of radiative transfer

The basic equation of radiative transfer for the flow of radiation through the outer layers of the star governing the radiation field in a medium which absorbs, emits and scatters radiation is given by

$$-\frac{dI_\nu}{\kappa_\nu \rho ds} = I_\nu - \Im_\nu \tag{1.1}$$

where I_ν the specific intensity,κ_ν the mass absorption coefficient of the medium, ρ the density of the medium, s the height of the medium and $\Im_\nu$ the source function which is the ratio of emission coefficient j_ν and absorption coefficient κ_ν for radiation of frequency ν. Due to the functionally dependency of source function on the intensity at a point, the equation of radiative transfer is generally an integro-differential equation.

1.1.1 Equation of transfer in different media and geometries:

(1)Plane parallel medium::
Here medium is considered to be stratified in planes perpendicular to oz-axis. The radiative properties in each plane are uniform. We define optical depth τ by

$$\tau = \int_s^\infty \kappa_\nu \rho ds \tag{1.2}$$

where s is the height of the medium. Below there are some cases in these geometries.

(a) Local thermodynamic equilibrium with no scattering:

In this case Kirchoff's law holds and the source function $\Im_\nu$ is given by

$$\Im_\nu = B_\nu(T) \tag{1.3}$$

where $B_\nu(T)$, the Planck function given by

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{exp(h\nu/kT) - 1} \tag{1.4}$$

where h is the Planck constant,k the Boltzmann constant and T is the characteristic temperature. The equation of transfer in this case is

$$-\mu\frac{dI(\tau_\nu,\mu)}{d\tau_\nu} = I(\tau_\nu,\mu) - B_\nu(T), \;\; where\ \tau_\nu = \int_s^\infty \kappa_\nu \rho ds \tag{1.5}$$

is the optical depth and $\mu = \cos\theta$,θ being the angle the pencil of incident radiation makes with the outward drawn normal from an element of area $d\sigma$.

(b) Medium where the scattering is isotropic:

Here the source function takes the form

$$\Im_\nu = \frac{1}{2}\int_{-1}^{+1} I(\tau,\mu')d\mu' \tag{1.6}$$

and the equation of transfer has the form

$$\mu\frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{1}{2}\int_{-1}^{+1} I(\tau,\mu')d\mu' \tag{1.7}$$

(c) In the case of scattering medium , source function is

$$\Im_\nu = \frac{\varpi}{2}\int_{-1}^{+1} p(\mu,\mu')I(\tau,\mu')d\mu' \tag{1.8}$$

where $p(\mu,\mu')$ is the phase function which governs the directional distribution of intensity ,ϖ is the albedo of single scattering which equals to 1 for single scattering (conservative case) and equals to zero for the case of true absorption. In nonconservative case $\varpi < 1$ and in the case of neutron transport $\varpi > 1$ .If the scattering is isotropic $p(\mu, \mu') = 1$ and in anisotropic case $p(\mu, \mu')$ has different well known form e.g

$$\begin{aligned}
p(\mu, \mu') &= (1 + x\mu\mu') \; Planetary\ scattering \\
&= 1 + \frac{1}{2} P_2(\mu) P_2(\mu') \; Rayleigh\ scattering \\
&= 1 + \frac{\alpha}{2} P_2(\mu) P_2(\mu') \; Pomraning\ phase\ function,\ \alpha = \frac{5\lambda}{5 - 3\lambda} \\
&= \sum_{k=0}^{\infty} \omega_k P_k(\mu) P_k(\mu') \; General\ phase\ function \\
&= 1 + b_0 P_4(\mu) \; Carlstedt\ \&\ Mullikin's\ phase\ function \\
&= 1 + 3g P_1(\mu) P_1(\mu') + 5g^2 P_2(\mu) P_2(\mu') + 7g^3 P_3(\mu) P_3(\mu') \\
&\quad Henyey - Greenstein\ phase\ function
\end{aligned}$$

(d) Coherent and noncoherent scattering:
In a same frequency when an atom absorbs and emits , it is called coherent scattering ,otherwise it is known as non coherent scattering. The equation of radiative transfer for coherent scattering is in the form of

$$\mu \frac{dI(\tau, \mu)}{d\tau} = (1 + \eta) I(\tau, \mu) + (1 - \epsilon)\eta J_\nu + (1 + \epsilon\eta) B_\nu(T) \tag{1.9}$$

where $\eta = \frac{l_\nu}{\kappa}$, l_ν = line absorption coefficient and κ = continuous absorption coefficient.
Noncoherent scattering again has two sections. Interlocking of lines and purely noncoherent scattering. When two, three, or more substates of lower energy states have a common substate in the upper energy state and an electron at the upper substate coming from any of the lower substates have equal probability to go down to any of the lower substates and thus differing in the absorption and emission frequencies, the lines so formed are called interlocked to each other. The case may be totally reversed i.e., two, three, or more substates in the upper state may have a common substate in the lower state. The equation of transfer for r-th line of interlocked multiplet is

$$\begin{aligned}
\mu \frac{dI_r(\tau, \mu)}{d\tau} &= (1 + \eta_r) I_r(\tau, \mu) - (1 + \epsilon\eta_r) B_\nu(T) \\
&\quad -(1 - \epsilon)\alpha_r \sum_{p=1}^{k} \frac{\eta_p}{2} \int_{-1}^{1} I_p(\tau, \mu') d\mu'
\end{aligned} \tag{1.10}$$

$$r = 1, 2, \dots k$$

where

$$\alpha_r = \frac{\eta_r}{\eta_1 + \eta_2 + \ldots + \eta_k} \tag{1.11}$$

so that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_k = 1 \tag{1.12}$$

and η_r , the ratio of line to the continuuam absorption coefficient for the r-th line is independent of depth but is function of frequency. ϵ the coefficient of thermal emission, is independent of both frequency and depth. The equation of transfer for noncoherent scattering is of the form

$$\mu\frac{dI_\nu(\tau,\mu)}{d\tau} = (1+\epsilon)I_\nu(\tau,\mu) - (1-\epsilon\eta)B_\nu(T) - (1-\epsilon)\eta\{aJ_\nu + bJ_{\nu_0} + c\overline{J}\} \tag{1.13}$$

(e) Grey medium:
If the medium possesses radiative properties independent of frequency, the medium is termed as a grey medium. In this case the equation of transfer is

$$-\mu\frac{dI(\tau,\mu)}{d\tau} = (1-\varpi)\frac{n^2\sigma T^4}{\pi} + \frac{\varpi}{2}\int_{-1}^{+1} p(\mu,\mu')I(\tau,\mu')d\mu' \tag{1.14}$$

where n is the refractive index of the medium and σ is the Stefan constant. In plane parallel system there are chiefly two types of problem. One is semi-infinite atmosphere bounded at $\tau = 0$ and extended to infinity $(\tau \to \infty)$ in the other direction and the other is finite atmosphere bounded by $\tau = 0$ and $\tau = \tau_1$

(2)Spherical geometry:
The equation of transfer in spherical geometry is of the form

$$\mu\frac{\partial I_\nu}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I_\nu}{\partial \mu} = j_\nu - \kappa_\nu I_\nu \tag{1.15}$$

In case of cylindrical geometry it takes the form

$$\sin\theta\cos\phi\frac{\partial I_\nu}{\partial r} - \frac{\sin\theta\sin\phi}{r}\frac{\partial I_\nu}{\partial \phi} = j_\nu - \kappa_\nu I_\nu \tag{1.16}$$

In three dimensional geometry, the equation is of the form
(a) Cartesian:

$$\mu\frac{\partial I_\nu}{\partial x} + \eta\frac{\partial I_\nu}{\partial y} + \xi\frac{\partial I_\nu}{\partial z} = j_\nu - \kappa_\nu I_\nu \tag{1.17}$$

(b) General Spherical:

$$\mu\frac{\partial I_\nu}{\partial r} + \frac{\eta}{r}\frac{\partial I_\nu}{\partial \theta} + \frac{\xi}{r\sin\theta}\frac{\partial I_\nu}{\partial \theta} + \frac{1-\mu^2}{r}\frac{\partial I_\nu}{\partial \mu} + \frac{\xi\cot\theta}{r}\frac{\partial I_\nu}{\partial \phi} = j_\nu - \kappa_\nu I_\nu \quad (1.18)$$

(c) General Cylindrical:

$$\mu\frac{\partial I_\nu}{\partial r} + \frac{\eta}{r}\frac{\partial I_\nu}{\partial \Theta} + \xi\frac{\partial I_\nu}{\partial z} - \frac{\eta}{r}\frac{\partial I_\nu}{\partial \phi} = j_\nu - \kappa_\nu I_\nu \quad (1.19)$$

where

$$\begin{aligned} I_r &= I_r(r,\Theta,z,\theta,\phi) \\ \xi &= \cos\theta \\ \mu &= \sin\theta\cos\phi \\ \eta &= \sin\theta\sin\phi \\ \textit{and} &\quad \xi^2+\mu^2+\eta^2 = 1 \end{aligned}$$

1.2 The double interval spherical harmonic method:

In the equation of transfer the intensity $I(\tau, \mu)$ is an unknown function of τ and μ. An approximate solution of this transfer equation can be obtained by the First Eddington approximation, often called the Eddington or the Milne-Eddington approximation and by spherical harmonic method by expanding intensity $I(\tau, \mu)$ in a series of Legendre polynomials. However these methods have some difficulties [38] due to discontinuity of intensity $I(\tau, \mu)$ at some boundaries. Here in below the description of spherical harmonic method and its necessary modifications [65] are made.

The origin of single interval spherical harmonic method is due to Eddington [20], Gratton [24]. Chandrasekhar [8, 10] also developed a systematic method and suggested a procedure for solving integro-differential equation of transfer by this method to any order of approximation. This method was extensively used to solve various radiative transfer problems in plane parallel medium in stellar atmosphere and in neutron transport.

According to Eddington [20, 38] , in the spherical harmonic method for the solution of transfer equation , the intensity $I(\tau, \mu)$ has an expansion in series of Legendre polynomials $P_j(\mu)$ and unknown functions $A_j(\tau)$. Legendre polynomials $P_j(\mu)$ forms a complete set of orthogonal functions in the interval $(-1, 1)$ in which μ varies and we express $I(\tau, \mu)$ as

$$I(\tau, \mu) = \sum_{j=0}^{m} A_j(\tau) P_j(\mu) \tag{1.20}$$

where the sum terminates after a finite number of terms and the solution of equation of transfer is thus reduced to the determination of unknown functions $A_j(\tau)$.
In the grey case (isotropic conservative case in which κ_ν is function of depth variable)the equation of transfer is

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2} \int_{-1}^{1} I(\tau, \mu) d\mu \tag{1.21}$$

and the fundamental quantities mean intensity $\overline{I}(\tau)$ and source function $\Im(\tau)$ are defined as

$$\overline{I}(\tau) = \frac{1}{2} \int_{-1}^{1} I(\tau, \mu) d\mu \tag{1.22}$$

$$\overline{\Im}(\tau) = 2\int_{-1}^{1} I(\tau,\mu)\mu d\mu = F \tag{1.23}$$

where F is the net flux.
With the help of Rodrigues formula

$$P_n(\mu) = \frac{1}{2^n n!}\frac{d^n}{d\mu^n}(\mu^2-1)^n\,,\ \ n = 0,1,2,\ldots \tag{1.24}$$

the above fundamental quantities can be written as

$$\overline{I}(\tau) = \frac{1}{2}\int_{-1}^{1} I(\tau,\mu)P_0(\mu)d\mu \tag{1.25}$$

$$\overline{\Im}(\tau) = 2\int_{-1}^{1} I(\tau,\mu)P_1(\mu)d\mu = F \tag{1.26}$$

Using orthogonality property of Legendre polynomial $P_j(\mu)$ given by

$$\int_{-1}^{1} P_m(\mu)P_n(\mu) = \begin{cases} 0, & if\ m \neq n \\ \frac{2}{2n+1}, & if\ m = n \end{cases} \tag{1.27}$$

we get mean intensity

$$\begin{aligned} \overline{I}(\tau) &= \frac{1}{2}\int_{-1}^{1} I(\tau,\mu)d\mu \\ &= \frac{1}{2}\left[A_0(\tau)\int_{1}^{-1} P_0^2(\mu)d\mu + \sum_{j=1}^{m}\left\{A_j(\tau)\int_{-1}^{1} P_j(\mu)P_0(\mu)d\mu\right\}\right] \\ &= A_0(\tau) \end{aligned} \tag{1.28}$$

and source function

$$\begin{aligned} \overline{\Im}(\tau) &= 2\int_{-1}^{1} I(\tau,\mu)P_1(\mu)d\mu \\ &= 2\Big[A_0(\tau)\int_{-1}^{1} P_0(\mu)P_1(\mu)d\mu + A_1(\tau)\int_{-1}^{1} P_1^2(\mu)d\mu \\ &\quad + \sum_{j=2}^{m}\left\{A_j(\tau)\int_{-1}^{1} P_j(\mu)P_1(\mu)d\mu\right\}\Big] \\ &= \frac{4}{3}A_1(\tau) \end{aligned} \tag{1.29}$$

We know that in Grey case the conservation of flux integral gives $B = \overline{I}(\tau)$, where B is Planck function and the equation of transfer (1.21) now takes the form

$$\mu\frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - A_0(\tau) \tag{1.30}$$

Now using intensity expression (1.20) in this above transfer equation (1.30) we obtain

$$\mu \sum_{j=0}^{m} \left\{ P_j(\mu) \frac{d}{d\tau} A_j(\tau) \right\} = \sum_{j=1}^{m} \{ A_j(\tau) P_j(\mu) \} \tag{1.31}$$

in which replacing each $\mu P_j(\mu)$ by the classical recurrence formula

$$\mu P_j(\mu) = \frac{j+1}{2j+1} P_{j+1}(\mu) + \frac{j}{2j+1} P_{j-1}(\mu) \ , j \in N \tag{1.32}$$

we get

$$\mu P_0(\mu) \frac{d}{d\tau} A_0(\tau) + \sum_{j=1}^{m} \left(\frac{j+1}{2j+1} P_{j+1}(\mu) + \frac{j}{2j+1} P_{j-1}(\mu) \right) \frac{dA_j(\tau)}{d\tau} = \sum_{j=1}^{m} A_j(\tau) P_j(\mu) \tag{1.33}$$

or

$$\frac{1}{3} P_0 \frac{dA_1(\tau)}{d\tau} + \sum_{j=1}^{m} \left\{ \frac{j}{2j-1} \frac{d}{d\tau} A_{j-1}(\tau) + \frac{j+1}{2j+3} \frac{d}{d\tau} A_{j+1}(\tau) \right\} P_j(\mu) = \sum_{j=1}^{m} A_j(\tau) P_j(\mu) \tag{1.34}$$

Now comparing the coefficients of $P_j(\mu)$ in the above equation (1.34) we find that

$$\frac{d}{d\tau} A_1(\tau) = 0 \tag{1.35}$$

$$\frac{d}{d\tau} A_0(\tau) + \frac{2}{5} \frac{d}{d\tau} A_2(\tau) = A_1(\tau) \tag{1.36}$$

$$\frac{2}{3} \frac{d}{d\tau} A_1(\tau) + \frac{3}{7} \frac{d}{d\tau} A_3(\tau) = A_2(\tau) \tag{1.37}$$

$$\ldots\ldots\ldots = \ldots$$

so in general we have for $j = 2, 3, \ldots$

$$\frac{j}{2j-1} \frac{d}{d\tau} A_{j-1}(\tau) + \frac{j+1}{2j+3} \frac{d}{d\tau} A_{j+1}(\tau) = A_j(\tau) \tag{1.38}$$

for the simple case (j=2)

$$A_1(\tau) = \frac{3}{4} F = C_1 \tag{1.39}$$

$$A_0(\tau) = \frac{3}{4} F\tau - \frac{2}{5} A_2(\tau) + C_0 \tag{1.40}$$

where C_0 and C_1 are constants of integration.
Now extending $m = 2n$ in equation (1.20)and ignoring the final equation $\left[\frac{dA_{2n}}{d\tau} = 0,\ j = 2,3,\dots 2n\right]$ we find that $\left[writing\ ,\ \frac{d}{d\tau} \equiv D, A_j(\tau) \equiv A_j\right]$

$$\left.\begin{array}{ll} DA_1 & = 0,\ j = 1 \\ -A_2 + \frac{3}{7}DA_3 & = 0,\ j = 2 \\ \frac{j}{2j-1}DA_{j-1} - A_j + \frac{j+1}{2j+3}DA_{j+1} & = 0,\ j = 3,4,\dots 2n-1 \\ \frac{2n}{4n-1}DA_{2n-1} - A_{2n} & = 0,\ j = 2n \end{array}\right\} \quad (1.41)$$

From the above set of equation (1.41) with the help of second and forth , together with those given by $j = 4,6,\dots 2n-2$; $DA_3, DA_5, \dots, DA_{2n-1}$ can be eliminated and another first integral is obtained.Therefore we get a linear relation between $A_2, A_4, \dots, A_{2n}$ with constant coefficients. Now for a particular $A_m(m = 2,3,\dots 2n)$, the system (1.41) can be written as

$$F(D)A_m = 0 \quad (1.42)$$

where

$$F(D) = \left|\begin{array}{ccccccc} -1 & \frac{3}{7}D & \dots & \dots & \dots & 0 & 0 \\ \frac{3}{5}D & -1 & \frac{4}{9}D & \dots & \dots & 0 & 0 \\ 0 & \frac{4}{7}D & -1 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{2n-1}{4n-3}D & -1 & \frac{2n}{4n+1}D \\ 0 & 0 & 0 & \dots & \dots & \frac{2n}{4n-1}D & -1 \end{array}\right| \quad (1.43)$$

It contains only derivatives of even order and its characteristic equation $F(\lambda) = 0$ has all its roots real and of modulus greater than unity.
Let the roots of equation $F(D) = 0$ are $\pm\alpha_j(j = 2,3,\dots,n)$ where each $\alpha_j > 1$, then the solution will be given by

$$A_m = \sum_{j=2}^{n} C_{\pm j} e^{\pm\alpha_j\tau} \quad (1.44)$$

where $C_{\pm j}(j = 2,3,\dots,n)$ are $(2n-2)$ constants of integration with C_0, C_1 are obtained from (1.39, 1.40) and the method reduces to the determination of constants.

Determination of constants:

The functions $A_2, A_3, \ldots, A_m$ are depending linearly on the same exponentials and same constants of integration. We have to determine the constants $C_{\pm j}$ by the following boundary conditions.

Boundary conditions

$$\Im(\tau) = F = constant \text{ (Net flux is constant)} \tag{1.45}$$

$$I(\tau,\mu)e^{-\tau} \to 0 \text{ as } \tau \to \infty \text{ (convergence of intensity)} \tag{1.46}$$

$$I(0,\mu) \equiv 0 \text{ for } \mu < 0 \quad \text{(absence of incident radiation from outside on the free surface)} \tag{1.47}$$

Now from (1.45) the constancy of net flux gives $C_1 = \frac{3}{4}F$. Therefore the remaining constants are C_0 and other $2(n-1)$ constants. From the second boundary condition (1.46) we have

$$C_{-j} = 0 \;\; (j = 2, 3, \ldots, n) \tag{1.48}$$

Thus the remaining n constants are $C_0, C_2, C_3, C_4, \ldots, C_n$ and remaining boundary condition(1.47) is $I(0,\mu) \equiv 0$ *for* $\mu < 0$.
Now according to (1.47)

$$I(0,\mu) = \sum_{j=0}^{2n} A_j(0)P_j(\mu) \tag{1.49}$$

must be satisfied $\forall$ values of $\mu \in (0,-1)$. Thus we arrive at the system containing infinite number of linear equation with n unknowns. Therefore we conclude that the system (1.49) is incompatible. We have no alternatives but to choose arbitrary n equations corresponding to n arbitrary values of $\mu \in (0,-1)$ to determine n constants of integration $C_0, C_2, C_3, C_4, \ldots, C_n$. The equation (1.49) is satisfied $\forall$ values of $\mu \in (0,-1)$. This means that we are trying to determine n constants from an infinite set of linear equation. Hence arbitrariness in the determination of constants cannot be avoided. Use of various boundary conditions is an attempt to bypass arbitrariness. For example Mark [41] met it by choosing some strategic values of μ for which the condition (1.49) hold good.

Kourganoff [38] analysed to reduce this arbitrariness by using the least square method but even then this arbitrariness cannot be removed completely. According to his suggestion $I(0, \mu) \equiv 0 \; for \;\; \mu \in (0, -1)$ is equivalent to

$$\sigma = \int_0^1 [I(0, \mu)]^2 \, d\mu = \;\; minimum \tag{1.50}$$

which on using, (1.49) reduces to

$$\sigma = \int_0^1 \left[\sum_{j=0}^{2n} A_j(0) P_j(\mu) \right]^2 d\mu = \;\; minimum \tag{1.51}$$

Differentiating partially (1.51) w.r.t $A_j(0)$ and using the orthogonality (1.27)of $P_j(\mu)$ we find that

$$\frac{2}{2i+1} A_j(0) = \sum_{j=0}^{2n} A_j(0) \int_0^1 P_i(\mu) P_j(\mu) d\mu \, , \; i = 0, 1, 2, \ldots, 2n. \tag{1.52}$$

In which there are now $2n + 1$ relations involving n unknowns. Therefore we still have arbitrariness and inconsistency which are minimized but not completely removed. Kourganoff [38] pointed out the defect to the above fact that the emergent intensity $I(\tau, \mu)$ which is discontinuous at the free surface at $\mu = 0$ is represented by a finite number of continuous Legendre polynomials. He suggested that the situation would improve if double interval representation of emergent intensity $I(\tau, \mu)$ is tried. This suggestion, was infact made by Yvon earlier[footnote Kourganoff [38],p 301](Henceforth we shall call it double interval spherical harmonic method or DISHM) and elaborately demonstrated by Mertens [43].

Mertens [43] represented $I(\tau, \mu)$ as $I_+(\tau, \mu)$ and $I_-(\tau, \mu)$ at the two separate ranges $(0, 1)$ and $(-1, 0)$ for μ and expressed

$$I_+(\tau, \mu) = \sum_{l=o}^{L} (2l + 1) I_l^+(\tau) \mu P_l(2\mu - 1), \mu \in (0, 1) \tag{1.53}$$

$$I_-(\tau, \mu) = \sum_{l=o}^{L} (2l + 1) I_l^-(\tau) \mu P_l(2\mu + 1), \mu \in (-1, 0) \tag{1.54}$$

with boundary condition

$$I_l^-(\tau) \equiv 0, \;\; l = 0, 1, 2, \ldots, n \tag{1.55}$$

$$\left.\begin{array}{l} I_l^+(\tau)e^{-\tau} \to 0, \\ I_l^-(\tau)e^{-\tau} \to 0, \end{array}\right\} \quad as \ \tau \to \infty \tag{1.56}$$

Double interval spherical harmonic method has been used with some modifications by Sykes [57], Gross and Zeiring [25], Max Krook [39, 40] and others. These, however, are equivalent to Merten's method and share it's limitations.

It was found that, on close approximation, Merten's [43] representation of the specific intensity was still discontinuous at $\mu = 0$ to the interior. At this stage Wilson and Sen [65] introduced a double interval spherical harmonic method which preserves the advantages of the representation of Mertens [43]. They introduced intensity in plane parallel medium as

$$I_+(\tau,\mu) = A(\tau) + \sum_{l=o}^{L}(2l+1)I_l^+(\tau)\mu P_l(2\mu-1), \mu \in (0,1) \tag{1.57}$$

$$I_-(\tau,\mu) = A(\tau) + \sum_{l=o}^{L}(2l+1)I_l^-(\tau)\mu P_l(2\mu+1), \mu \in (-1,0) \tag{1.58}$$

in which inclusion of the same term $A(\tau)$ in both (1.57) and (1.58) ensures the continuity of the intensities at $\mu = 0$.
and in Spherical geometry

$$I_+(r,\mu) = A(r) + \sum_{l=o}^{l_0}(2l+1)I_l^+(r)\mu P_l(2\mu-1), \mu \in (0,1) \tag{1.59}$$

$$I_-(r,\mu) = A(r) + \sum_{l=o}^{l_0}(2l+1)I_l^-(r)\mu P_l(2\mu+1), \mu \in (-1,0) \tag{1.60}$$

with A(r) is a function of r only, r being the distance measured outward from the centre of the sphere.
Here in below, some modifications of DISHM after Wilson and Sen [65] are noted.
(1) Wan, Wilson and Sen's [60] form

$$i_+(\tau,\mu) = \phi(\tau) + \sum_{l=o}^{l_0}(2l+1)i_l^+(\tau)\mu P_l(2\mu-1), \mu \in (0,1) \tag{1.61}$$

$$i_-(\tau,\mu) = \phi(\tau) + \sum_{l=o}^{l_0}(2l+1)i_l^-(\tau)\mu P_l(2\mu+1), \mu \in (-1,0) \tag{1.62}$$

(2) Karanjai and Talukdar's form[32]

$$\begin{aligned} I_+(\tau,\mu) &= I(0,0)\left[A\tau+\phi(\mu)+\sum_{l=o}^{l_0}(2l+1)I_l^+(\tau)\mu P_l(2\mu-1)\right], \\ &\quad \mu\in(0,1) && (1.63) \\ I_-(\tau,\mu) &= I(0,0)\left[A\tau+\phi(\mu)+\sum_{l=o}^{l_0}(2l+1)I_l^-(\tau)\mu P_l(2\mu+1)\right], \\ &\quad \mu\in(-1,0) && (1.64) \end{aligned}$$

where I(0,0) is the specific intensity in the direction normal to the surface which is constant and A is an arbitrary constant.
(3) Raychaudhuri and Karanjai's form[49, 50, 51]

$$\begin{aligned} I_+(\tau,\mu) &= I(0,0)\left[\phi(\tau)+\Psi(\mu)+\sum_{l=o}^{l_0}(2l+1)I_l^+(\tau)\mu P_l(2\mu-1)\right], \\ &\quad \mu\in(0,1) && (1.65) \\ I_-(\tau,\mu) &= I(0,0)\left[\phi(\tau)+\Psi(\mu)+\sum_{l=o}^{l_0}(2l+1)I_l^-(\tau)\mu P_l(2\mu+1)\right], \\ &\quad \mu\in(-1,0) && (1.66) \end{aligned}$$

1.2.1 Application of SHM in solving Radiative transfer problem

Wilson and Sen [65] introduced a double interval spherical harmonic method for solving the equation of transfer in plane parallel medium

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau} = I(\tau,\mu) - \frac{1}{2}\int_{-1}^{+1} I(\tau,\mu')d\mu' \quad (1.67)$$

where $I(\tau,\mu)$ is the specific intensity, optical depth $\tau = \int_z^\infty \kappa\rho dz$; κ, ρ are scattering coefficient and density of the medium, $\mu = \cos\theta$, θ being the angle of incident radiation.
They considered the boundary conditions
(1) Absence of incident radiation from outside at the free surface

$$I(0,\mu) \equiv 0 \ \forall \ \mu \in (-1,0) \quad (1.68)$$

(2) The convergence of intensity

$$I(\tau,\mu)e^{-\tau} \to 0 \ as \ \tau \to \infty \quad (1.69)$$

They evaluated $q(\tau)$ and the ratio $\frac{I(0,\mu)}{F}$ for first(P_1) and second(P_2) approximation and compared the results with those of Chandrasekhar [11] and Mertens [43] and conclude that P_2 approximation is better than P_1 approximation.
Wilson and Sen [66] extended their previous work [65] in plane geometry with anisotropic scattering using general phase function. They considered the transfer equation

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau} = I(\tau,\mu) - \frac{1}{2}\int_{-1}^{+1} p(\mu,\mu')I(\tau,\mu')d\mu' \quad (1.70)$$

where $p(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} p(\mu,\Phi;\mu',\Phi')d\mu'$ is the phase function giving the measure of the probability of a ray in the direction (μ',Φ') being scattered into the direction (μ,Φ), Φ is the azimuthal angle.
They used the boundary condition (1.68) and (1.69) and intensities as given in (1.57),(1.58) taking first term $A(\tau) = A\tau$. The form of general phase function considered by them was

$$p(\mu,\mu') = \sum_{k=0}^{\infty} w_k P_k(\mu)P_k(\mu') \quad (1.71)$$

where w_k's are constant and P_k are Legendre polynomials. For a particular case they used Rayleigh phase function $p(\mu,\mu') = 1 + \frac{1}{2}P_2(\mu)P_2(\mu')$ and obtained the results of $\frac{I(0,\mu)}{F}$ in P_1-approximation. They compared these results with those of Chandrasekhar [9].

Wilson and Sen [67] extended their work [65, 66] in spherical geometry where they solved the classical problem of diffusion of radiation through a homogeneous sphere. They considered the equation of transfer for that problem

$$\mu\frac{\partial I(r,\mu)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I(r,\mu)}{\partial \mu} + I(r,\mu) = \frac{1}{2}\int_{-1}^{+1} I(r,\mu')d\mu' \tag{1.72}$$

They used two different expression for intensities given by (1.59) and (1.60) and boundary conditions

$$\left.\begin{array}{l} A(R) = 0 \\ I_0^-(R) = 0 \\ I_1^-(R) = 0 \end{array}\right\} \tag{1.73}$$

They obtained the results for mean intensity $J(r)$ in P_1 approximation and compared that with those of Chandrasekhar [9].

Wilson and Sen [69] solved the transfer problem by their modified SHM in a spherically symmetric , finite stellar atmosphere where $\kappa\rho \propto r^{-2}$ with transfer equation

$$\mu\frac{\partial I(r,\mu)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I(r,\mu)}{\partial \mu} + \kappa\rho I(r,\mu) = \frac{\kappa\rho}{2}\int_{-1}^{+1} I(r,\mu')d\mu' \tag{1.74}$$

κ, ρ are mass absorption coefficient and density of the material and all other symbols with their usual meanings.
They considered the same form of intensity given by (1.59) and (1.60) noting that the function $A(r)$ that appears in the expression of intensity depends on the nature of the physical problem. In P_1 - approximation, they evaluated the mean intensity $J(x)$ at $x = 2$ where $x = \frac{k_0 C}{r}$ within the boundaries $R \in (2k_0C, k_0C)$ comparing the results with the results of Chandrasekhar [11].

Wilson and Sen [70] solved the problem of radiative transfer in spherically symmetric, finite planetary nebular shell with $\kappa\rho \propto r^{-2}$. They considered the transfer equation

$$\mu\frac{\partial I}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu} = -\kappa\rho\left[I - \frac{1}{2}\int_{-1}^{+1} I d\mu - \frac{Sr_1^2}{4r^2}e^{-(\tau_1-\tau)}\right] \tag{1.75}$$

where τ_1 is the radial optical thickness of the nebular shell and πS is the net flux of the radiant energy. Here also they considered the same form of intensity given by (1.59) and (1.60). The boundary conditions assumed by them was

(1) no incident radiation on the outer boundary $r = R$

$$I(R,\mu) = 0 \ for \ \mu \in (-1,0) \tag{1.76}$$

(2) the diffuse flux across the inner surface vanishes

$$F_{r=r_1} = 0 \tag{1.77}$$

They calculated mean intensity in P_1-approximation and compared the results with Sen [52].

Canosa & Penafiel [5] solved the *average intensity* form of the equation of radiative transfer given by

$$\begin{aligned}\mu\frac{\partial I(\tau,\mu)}{\partial \tau} + I(\tau,\mu) &= \frac{1}{2}\sum_{l=0}^{L}\frac{2l+1}{2}a_l(\tau)P_l(\tau)\int_{-1}^{+1}P_l(\mu')I(\tau,\mu',\Phi')d\mu' d\Phi' \\ &+\frac{1}{4}Fe^{-\frac{\tau}{\mu_0}}\sum_{l=0}^{L}\frac{2l+1}{2}a_l(\tau)P_l(\mu_0)P_l(\mu)\end{aligned} \tag{1.78}$$

They have taken the form of intensity as

$$I(\tau,\mu) = \sum_{l=0}^{L}\frac{2l+1}{2}f_l(\tau)P_l(\mu) \tag{1.79}$$

and assumed the normalized phase function

$$P(\cos\theta) = \sum_{l=0}^{L}\omega_l P_l(\cos\theta) \tag{1.80}$$

They performed test computations on Rayleigh and Mie phase function.

Devaux et al. [19] made a critical study of four methods of solution of the equation of transfer and compared both the accuracy of the results and required computation time. The SHM seems to have significant advantages over the others.

Wan, Wilson & Sen [60] applied the modified SHM for solving the radiative transfer problem in an isothermal slab with Rayleigh phase function. They considered the model consisting of an isothermal slab of optical depth τ_0 confined between grey and diffuse walls that absorbs and anisotropically scatter radiant energy. They considered the transfer equation for such a model as

$$\mu\frac{\partial i}{\partial \tau} + i = (1-\omega_0)i_0 + \frac{\omega_0}{2}\int_{-1}^{+1} p(\mu,\mu')i(\mu')d\mu' - S(\tau,\mu) \tag{1.81}$$

where i is the intensity , i_b, the black body intensity, ω_0, the albedo for single scattering. They expressed the intensity as

$$i_+(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0}(2l+1)i_l^+(\tau)\mu P_l(2\mu-1), 0 \le \mu \le 1 \tag{1.82}$$

$$i_-(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0}(2l+1)i_l^-(\tau)\mu P_l(2\mu+1), -1 \le \mu \le 0 \tag{1.83}$$

Using the boundary conditions

$$\left.\begin{array}{l} i(0,\mu) = B_1 \ \ for \ \ 0 \le \mu \le 0 \\ i(\tau_0,\mu) = B_2 \ \ for \ \ -1 \le \mu \le 0 \end{array}\right\} \tag{1.84}$$

they solved the problem taking both B_1, B_2 equal to zero and applied the first approximation to find the zeroth , first and second moments of intensity.

Peraiah [46] obtained a solution of radiative transfer equation in spherically symmetric media using spherical harmonic method. Here he approximated the angle derivative by an orthonormal polynomial which is represented by curvature matrix, for a given beam of rays. He considered the RT equation in spherical symmetry as

$$\frac{\mu}{r^2}\frac{\partial}{\partial r}\left[r^2 I(\mu,r)\right] + \frac{1}{r}\frac{\partial}{\partial \mu}\left[(1-\mu^2)I(\mu,r)\right] + k(r)I(r,\mu)$$
$$= k(r)\left[(1-w(r))B(r) + \frac{w(r)}{2}\int_{-1}^{+1} P(r,\mu,\mu')I(r,\mu')d\mu'\right] \tag{1.85}$$

where $k(r)$ is the absorption coefficient, $k(r) \ge 0$ and $w(r)$ is the albedo for single scattering, $0 \le w(r) \le 1$. $I(r,\mu)$ is the monochromatic specific intensity of the ray making an angle $\cos^{-1}\mu$ with the radius vector at the radial point r and B(r) is the Planck's function. $P(r,\mu,\mu')$ is the phase function which is assumed to be isotropic. The angles are discretized such that $0 < \mu_1 < \mu_2 < \ldots < \mu_m \leq 1$.

He took the form of intensity as

$$I(\mu) = \sum_{m=0}^{M} \alpha_m P_m(\mu) \tag{1.86}$$

and calculated the emergent intensities.

Karp, Greenstadt and Fillmore[36] solved the equation of RT for a plane parallel planetary atmosphere using SHM. They assumed that all the inhomogeneities were confined to the vertical direction and each layer of the atmosphere was taken to be homogeneous but with arbitrary optical thickness. They considered the equation of transfer with monochromatic radiation as

$$\mu \frac{dI(\tau, \mu, \Phi)}{d\tau} = I(\tau, \mu, \Phi) - J(\tau, \mu, \Phi) \tag{1.87}$$

where $I(\tau, \mu, \Phi)$ is the specific intensity and $J(\tau, \mu, \Phi)$ is the source function defined by

$$\begin{aligned} J(\tau, \mu, \Phi) &= \frac{1}{4} P(\tau; \mu, \Phi; -\mu_0, \Phi_0) F_0 e^{-\frac{\tau}{\mu_0}} \\ &+ \frac{1}{4\pi} \int_{-1}^{+1} \int_{0}^{2\pi} P(\tau; \mu, \Phi; \mu', \Phi') I(\tau, \mu', \phi') d\mu' d\phi' \end{aligned} \tag{1.88}$$

where the external illumination πF_0 was assumed to be unidirectional. They used the boundary conditions as

$$I(0; \mu < 0, \Phi) = I(\tau_b; \mu > 0, \Phi) = 0 \tag{1.89}$$

Using SHM they developed an algorithm for the method and computed the angle dependent intensity.

Karp and Petrack [37] made a comparison between spherical harmonic method and discrete ordinate method . There they showed that for higher terms of the Fourier expansions of the intensity the results were exact at the zeros of the Legendre polynomials. They considered the equation of transfer for plane parallel, scattering and absorbing atmosphere as

$$\mu \frac{dI(\mu, \Phi, \tau)}{d\tau} = I(\mu, \Phi, \tau) + \frac{1}{4\pi} \int_{0}^{2\pi} \int_{-1}^{+1} I(\mu', \Phi', \tau) P(\tau, \mu, \mu'; \Phi, \Phi') d\mu' d\Phi' \tag{1.90}$$

Introducing Fourier series they transfered the above equation (1.90) in the form

$$\mu\frac{dI^m(\mu,\tau)}{d\tau} = I^m(\mu,\tau) - \int_{-1}^{+1} I^m(\mu',\tau)P^m(\tau,\mu,\mu')d\mu' \tag{1.91}$$

and considered the phase function as

$$P^m(\tau,\mu,\mu') = \sum_{l=m}^{L+m} \beta_l(\tau)Y_l^m(\mu)Y_l^m(\mu') \tag{1.92}$$

where $Y_l^m(\mu)$ are associated Legendre polynomials and $\beta_l(\tau)$ are the coefficients of scattering phase function. They considered the form of intensity as

$$I^m(\mu,\tau) = \sum_{l=m}^{L+m} \frac{2l+1}{2} f_l^m(\tau)Y_l^m(\mu) \tag{1.93}$$

where$f_l^m(\tau)$ are the moments of the intensity. they conclude that the azimuth-dependent intensities computed from SHM was exact at the zeros of the Legendre polynomials.

Wells and Sidorowich [62] solved the RT problems by SHM with extreme forward scattering and demonstrated the method in slab geometry and developed the computational techniques with test results.

Aronson [2] compared P_N approximations with D_N approximations (double P_N approximation) for highly anisotropic scattering where he examined the results of both methods by taking different terms of the phase function for haze model and cloud model and considered the following form of phase function as

$$f(\theta) = \omega \sum b_l P_l(\cos\theta), \;\; b_0 = 1 \tag{1.94}$$

For haze model and for cloud model he took 82 terms and 300 terms in the phase function respectively and concluded that D_N approximations were better than P_N approximations.

Garcia and Siewert[22] developed a generalized spherical harmonics solution and considered the equation of transfer

$$\mu\frac{\partial}{\partial\tau}I(\tau,\mu,\Phi) + I(\tau,\mu,\Phi) = \frac{\omega}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} P(\mu,\mu';\Phi,\Phi')I(\tau,\mu,\mu')d\mu'd\Phi' \tag{1.95}$$

and solved (1.95) subject to the boundary conditions

$$I(0,\mu,\Phi) = \pi\delta(\mu-\mu_0)\delta(\Phi-\Phi_0)F \tag{1.96}$$

$$I(\tau_0,-\mu,\Phi) = \frac{\lambda_0}{\pi}L\int_0^{2\pi}\int_0^1 I(\tau_0,\mu',\Phi')\mu' d\mu' d\Phi' \tag{1.97}$$

where λ_0 is the coefficient for Lambert reflection and F is the flux vector.

Takeuchi[58] applied a new formulation of SHM to solve the equation of transfer for plane parallel media considering that all inhomogeneities confined in the vertical direction and with axially symmetric phase functions. He showed that the spherical harmonics approximation reduces the m-th Fourier component of the equation of radiative transfer to a system of infinite homogeneous ordinary differential equations where he expanded each term of intensity by normalized associated Legendre polynomials $Q_l^m(\mu)$ which are related to the associated Legendre polynomials $P_l^m(\mu)$ by

$$Q_l^m(\mu) = \frac{[(2l+1)(l-m)!]^{\frac{1}{2}}}{[2(l+m)!]^{\frac{1}{2}}}P_l^m(\mu) \tag{1.98}$$

Takeuchi[58] showed that the use of normalized associated Legendre polynomials made spherical harmonic method very easy to handle.

Tine, Aiello, Belleni and Pestellini[59] solved the transfer equation by SHM in spherically symmetrical media with radially varying parameters and anisotropic scattering where they considered the radiative transfer equation

$$\mu\frac{\partial I(r,\mu)}{\partial r}+\frac{1-r^2}{r}\frac{\partial I(r,\mu)}{\partial\mu} = -\alpha(r)I(r,\mu)+\frac{\alpha^{sca}(r)}{2}\int_{-1}^{+1} I(r,\mu')P(\mu,\mu')d\mu' \tag{1.99}$$

with boundary condition

$$I(r,\mu) = I^{est}(\mu), \quad \mu \le 0 \tag{1.100}$$

where $\alpha(r)$, $\alpha^{sca}(r)$ and $I^{est}(\mu)$ are respectively the extinction and scattering coefficients and externally incident radiation. They considered the form of phase function as

$$P(\mu,\mu') = \sum_{l=0}^{N}\sigma_l P_l(\mu)P_l(\mu') \tag{1.101}$$

and the form of intensity as

$$I(r,\mu) = \sum_{l=0}^{L}(2l+1)F_l(r)P_l(\mu) \tag{1.102}$$

They applied the technique to Henyey-Greenstein phase function.

Biswas and Karanjai[31] applied modified double interval SHM to solve the equation of radiative transfer taking Rayleigh phase function with thin atmosphere. They considered the equation of transfer

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau} + I(\tau,\mu) = \frac{1}{2}\int_{-1}^{+1} p(\mu,\mu')I(\tau,\mu')d\mu' \tag{1.103}$$

with boundary condition

$$I(0,\mu) = I_0, 0 \leq \mu \leq 1 \tag{1.104}$$

$$I(\tau_0,\mu) = 0, -1 \leq \mu \leq 0 \tag{1.105}$$

They considered the form of intensity as

$$I_+(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0}(2l+1)I_l^+(\tau)\mu P_l(2\mu-1), 0 \leq \mu \leq 1 \tag{1.106}$$

$$I_-(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0}(2l+1)I_l^-(\tau)\mu P_l(2\mu+1), -1 \leq \mu \leq 0 \tag{1.107}$$

Muldashev, Lyapustin and Sultangazin[45] designed correction function method for calculating the angular dependence of radiance, which corrects the solution of the spherical harmonics method so that the single scattering contribution is calculated exactly, regardless of the order of approximation P_N. They considered the problem of radiative transfer in the plane-parallel atmosphere bounded by a laterally uniform surface with Lambertian reflectance and the diffuse radiation field in the atmosphere, governed by the equation of radiative transfer

$$\begin{aligned}\mu\frac{\partial I(\tau,\mu,\phi)}{\partial\tau} + I(\tau,\mu,\phi) &= \frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} \chi(\tau,\gamma)I(\tau,\mu',\phi')d\mu' \\ &+\frac{1}{4}\chi(\tau,\gamma_0)S_0e^{-\frac{\tau}{\xi}}\end{aligned} \tag{1.108}$$

with boundary conditions

$$I(0,\mu,\phi) = 0, \ \ \mu > 0 \tag{1.109}$$

$$\begin{aligned}I(\tau_0,\mu,\phi) &= \rho S_0\xi e^{-\frac{\tau_0}{\xi}} + \frac{\rho}{\pi}\int_0^{2\pi} d\phi' \int_0^1 I(\tau_0,\mu',\phi')\mu' d\mu', \\ &\mu < 0\end{aligned} \tag{1.110}$$

where $I(\tau,\mu,\phi)$ is the diffuse radiance at optical depth τ, ρ is the surface albedo, γ is the angle of scattering, $\chi(\gamma,\tau)$ is the scattering function.

Finally they obtained the solution of the problem as follows:

$$\mathbf{u}^m(\tau,\mu) = u^m(\tau,\mu) - w_1^m(\tau,\mu) \tag{1.111}$$

where the part $w_1^m(\tau,\mu)$ allows to obtain an exact contribution of the single scattered radiation regardless of the order of approximation of the spherical harmonics method.

Recently, Ghosh and Karanjai[23] solved the equation of radiative transfer for coherent anisotropic scattering with Pomraning phase function by DISHM. They considered the equation of transfer given by Woolley and Stibbs[71] as

$$\begin{aligned}\mu\frac{dI(\tau,\mu)}{d\tau} &= (1+\eta)I(\tau,\mu) - \frac{(1-\epsilon)\eta}{2}\int_{-1}^{+1} p(\mu,\mu')I(\tau,\mu')d\mu' \\ &\quad -(1-\epsilon\eta)(a+b\tau) \end{aligned}\tag{1.112}$$

subject to the boundary conditions

$$I(0,\mu) \equiv 0 \ for \ -1 \le \mu \le 0 \tag{1.113}$$

$$I(\tau,\mu)e^{-\tau} \rightarrow 0 \ as \ \tau \rightarrow \infty \tag{1.114}$$

where η is the ratio of the line to the continuous absorption coefficient, ϵ is the coefficient of thermal emission, $I(\tau,\mu)$ is the specific intensity of radiation at an optical depth τ given by $\tau = \int_z^\infty \kappa\rho dz$, κ being the coefficient of scattering and ρ, the density of the medium. $\mu = \cos\theta$, where θ is the angle made by radiation with the outward drawn normal and $p(\mu,\mu')$ is the phase function.

They expressed $I(\tau,\mu)$ using two different expansions $I_+(\tau,\mu)$ and $I_-(\tau,\mu)$ for $\mu \in (0,1)$ as

$$I_+(\tau,\mu) = A\tau + \sum_{l=0}^{l_0}(2l+1)I_l^+(\tau)\mu P_l(2\mu-1) \quad for \ 0 \le \mu \le 1 \tag{1.115}$$

$$I_-(\tau,\mu) = A\tau + \sum_{l=0}^{l_0}(2l+1)I_l^+(\tau)\mu P_l(2\mu+1) \quad for \ -1 \le \mu \le 0 \tag{1.116}$$

and finally obtained the results for first and second approximation and showed that $A = \frac{3}{4}F$. They also calculated the values of a/A, source function and law of darkening.

1.3 The interlocking problem:

1.3.1 Formation of absorption lines : Coherent Scattering

When an atom absorbs and emits in the same frequency, it is called Coherent scattering. Otherwise Non Coherent scattering. In the plane parallel atmosphere, the basic equation of transfer for the flow of radiation in a direction making an angle θ with x-axis for a frequency ν within an absorption line be taken as

$$\cos\theta \frac{dI_\nu(\theta)}{\rho dx} = -\kappa_\nu I_\nu(\theta) + j_\nu \tag{1.117}$$

where absorption coefficient κ_ν and emission coefficient j_ν are rapidly varying functions of frequency ν and ρ is the density of the medium.
Now absorption coefficient κ_ν is divided into two parts.
(1) continuous absorption coefficient κ , where variation with frequency within the absorption line be neglected and
(2) line absorption coefficient l_ν , a rapidly changing part.
Thus (1.117) becomes

$$\cos\theta \frac{dI_\nu(\theta)}{\rho dx} = -(\kappa + l_\nu)I_\nu(\theta) + j_\nu \tag{1.118}$$

Also the emission in frequency ν per unit mass of atmosphere is composed of two parts
(1) re-emission from the selectively absorbing atoms which is the coherent scattering in the frequency ν is happening due to spontaneous emission of quanta by atoms excited by radiation. Assuming that the direction of emission from an excited atom is independent of the direction of the absorbed quantum, the re-emission by an assembly of atoms may be taken as isotropic. Thus,if all the energy absorbed by the atoms is emitted as isotropic radiation, this re-emission will be the absorption averaged over all directions of incidence of the radiation which is given by

$$\frac{1}{4\pi}\int l_\nu I_\nu(\theta) d\omega = l_\nu J_\nu \tag{1.119}$$

where ω is the solid angle.
(2) re-emission from the atmospheric constituents which give rise to the

continuous absorption. Here the emission takes place as in local thermodynamic equilibrium and by Kirchhoff's Law equals to $\kappa B(\nu, T)$.
Thus, the equation of transfer (1.118) becomes

$$\cos\theta \frac{dI_\nu(\theta)}{\rho dx} = -(\kappa + l_\nu)I_\nu(\theta) + l_\nu J_\nu + \kappa B(\nu, T) \tag{1.120}$$

Let us now consider the effect of collision on the emission of radiation by excited atoms. If the fraction ϵ of atoms excited by radiation of frequency ν is prevented from contributing to the coherent re-emission by super-elastic collisions, the line emission given by (1.119) will be reduced to $(1-\epsilon)l_\nu J_\nu$. And in thermodynamic equilibrium the conversion of radiation into kinetic energy by super-elastic collisions is exactly balanced by the reverse process of conversion of K.E into radiation by way of inelastic collisions. Thus the lost amount $\epsilon l_\nu J_\nu$ will be balanced by $\epsilon l_\nu B(\nu, T)$ since J_ν has its equilibrium value $B(\nu, T)$ and therefore j_ν becomes

$$j_\nu = (1-\epsilon)l_\nu J_\nu + \epsilon l_\nu B(\nu, T) + \kappa B(\nu, T) \tag{1.121}$$

and the transfer equation (1.120) becomes

$$\cos\theta \frac{dI_\nu(\theta)}{\rho dx} = -(\kappa + l_\nu)I_\nu(\theta) + (1-\epsilon)l_\nu J_\nu + (\kappa + \epsilon l_\nu)B(\nu, T) \tag{1.122}$$

Taking $\eta_\nu = l_\nu/\kappa, \mu = \cos\theta$ and introducing optical depth $d\tau = \kappa\rho dx$ the above (1.122) takes the form

$$\mu \frac{dI(\tau, \mu)}{d\tau} = (1+\eta_\nu)I(\tau, \nu) - (1+\epsilon\eta_\nu)B(\nu, T) - (1-\epsilon)\eta_\nu \int_{-1}^{+1} I(\tau, \mu')d\mu' \tag{1.123}$$

This result is due to Eddington [21].

1.3.2 Formation of absorption lines:Interlocking without redistribution

In the case of non-coherent scattering , if two or more lines from a lower substate have a common upper substate, the atom once exited to that upper substate by absorption in either line but the re-emission will take place according to the transition probabilities corresponding to the various lines and the atom may return to a substate different from which the

excitation was made. This phenomenon is called formation of interlocked lines.
Examples [71, 4, 64] of such interlocked principal lines

$$Al\begin{cases}\lambda 3962A\ ^2P_{\frac{3}{2}}-{}^2S_{\frac{1}{2}}\\ \lambda 3944A\ ^2P_{\frac{1}{2}}-{}^2S_{\frac{1}{2}}\end{cases}\quad and\ Mg\begin{cases}\lambda 5184A\ ^3P_2-{}^3S_1\\ \lambda 5173A\ ^3P_1-{}^3S_1\\ \lambda 5167A\ ^3P_0-{}^3S_1\end{cases}\qquad (1.124)$$

In the principal lines of Al, $^2P_{\frac{1}{2}}$ is the ground state and $^2P_{\frac{3}{2}}$ metastable. In the second case $^3P_2$ and $^3P_0$ are metastable and $^3P_1$ is linked by an intercombination line to the ground state $^1S_0$.
According to Woolley and Stibbs [71], we consider a stellar atmosphere consisting of plane parallel layers bounded at the plane $x = 0$ and extending to infinity in the negative direction of x-axis which is normal to the layers. Let us consider the case where there are k number of such lines. Let $\nu_1, \nu_2, \ldots, \nu_k$ be the central frequencies of the lines and suppose that when a quantum of frequency $\nu_1 + \Delta\nu$ is absorbed, the energy of the upper state is $E_0 + h\Delta\nu$ and the subsequent re-emission is with any of the frequencies $\nu_1 + \Delta\nu, \nu_2 + \Delta\nu, \nu_3 + \Delta\nu, \ldots, \nu_k + \Delta\nu$. These k frequencies are interlocked with each other. Now for each values of $\Delta\nu$ there are k transfer equation

$$-\mu\frac{dI_r(\theta)}{\rho dx} = \underbrace{-(\kappa_r + l_r)I_r(\theta)}_{absorption}$$

$$\underbrace{+(\kappa_r + \epsilon_r l_r)B(\nu_r, T) + (1-\epsilon_r)\sum_{j=1}^{k}\{p(r,j)l_j J_j\}}_{emission\ \&\ re-emission} \qquad (1.125)$$

$$r = 1, 2, \ldots, k.$$

where

$$absorption\ coefficient = \kappa_r + l_r \qquad (1.126)$$

κ_r and l_r are continuous absorption and line absorption coefficient for the r^{th} line , ρ is the density of the medium , ϵ_r is the thermal emissivity of the r^{th} line. J_r , the mean intensity of the r^{th} line , p(r , j) is the probability that the average atom in an assembly of lines absorbing atoms will emit in the state j after the absorption has taken place in the state r.

Now the number of transitions per c.c per sec from the k lower sub states to a band of sub-states to the upper state lying within $E_0 + h\Delta\nu$ to $E_0 + h(\Delta\nu + \delta\nu)$ is

$$\rho dx\delta\nu \sum_{j=1}^{k} \left\{ \frac{l(\nu_j + \Delta\nu)J(\nu_j + \Delta\nu)}{h(\nu_j + \Delta\nu)} \right\} \tag{1.127}$$

This must be equal to the number of transitions per c.c per sec leaving the upper substate into the k number of lines. Let N_i be the population of atom in state i . Then the population of the upper state with energies between $E_0 + h\Delta\nu$ and $E_0 + h(\Delta\nu + \delta\nu)$ is $N_u\delta\nu$. Therefore the number of transitions is

$$N_u\delta\nu \sum_{j=1}^{k} A_{uj} \tag{1.128}$$

where A_{uj} is the probability that the atom will undergo a spontaneous transition from upper state to a state j with the emission of quantum of energy $h\nu$. The secular equilibrium of the sub-state gives

$$N_u \sum_{j=1}^{k} A_{uj} = \rho dx \sum_{j=1}^{k} \left\{ \frac{l(\nu_j + \Delta\nu)J(\nu_j + \Delta\nu)}{h(\nu_j + \Delta\nu)} \right\} \tag{1.129}$$

Now the energy emitted in the r^{th} line is $N_u A_{ur} h(\nu_r + \Delta\nu)$ and similarly for the other lines.
Accordingly

$$p(r,j) = \frac{(\nu_r + \Delta\nu)}{(\nu_j + \Delta\nu)} \frac{A_{ur}}{A_{u1} + A_{u2} + \ldots + A_{uk}} \tag{1.130}$$

Now suppose that the k lines are so close that we can ignore differences in the frequencies. Then setting each of $\kappa_r = \kappa$ and each of $B(\nu_r, T) = B$ the above quantity (1.130) is written as

$$p(r,j) = \frac{A_{ur}}{A_{u1} + A_{u2} + \ldots + A_{uk}} \tag{1.131}$$

since p (r , j) does not involve j, we can set $p(r,j) = \alpha_r$ with $\sum_{i=1}^{k} \alpha_i = 1$. Then with each of $\epsilon_i = \epsilon$ and using the relations $d\tau = -\kappa\rho dx$ and $\eta_r = \frac{l_r}{\kappa}$, the equation (1.125) becomes

$$\mu\frac{dI_r(\tau,\mu)}{d\tau} = (1 + \eta_r)I_r(\tau,\mu) - (1 + \epsilon\eta_r)B - (1 - \epsilon)\alpha_r \sum_{p=1}^{k} \eta_p J_p \tag{1.132}$$

Since the lower states are nearly equal in excitation potential and they have the same classical damping constant , then η_i's are proportional to α_i's and we have

$$\alpha_r = \frac{\eta_r}{\eta_1 + \eta_2 + \ldots + \eta_k}, \quad (r = 1, 2, \ldots, k) \tag{1.133}$$

also the mean intensity is

$$J_p = \frac{1}{2} \int_{-1}^{1} I_p(\tau, \mu') d\mu' \tag{1.134}$$

Thus the equation of transfer for the r^{th} line of multiplets in case of interlocking without redistribution is

$$\begin{aligned} \mu \frac{dI_r(\tau,\mu)}{d\tau} &= (1+\eta_r) I_r(\tau,\mu) - (1+\epsilon\eta_r) B \\ &\quad -(1-\epsilon)\alpha_r \sum_{p=1}^{k} \frac{1}{2} \eta_p \int_{-1}^{1} I_p(\tau,\mu') d\mu' \end{aligned} \tag{1.135}$$

$r = 1, 2, \ldots, k.$

where

$$\alpha_r = \frac{\eta_r}{\eta_1 + \eta_2 + \ldots + \eta_k} \tag{1.136}$$

and

$$\alpha_1 + \alpha_2 + \ldots + \alpha_k = 1 \tag{1.137}$$

1.3.3 Works on Interlocking problem

Woollley and Stibbs [71] applied the theory of the formation of absorption lines by non-coherent scattering to the case of interlocking without redistribution and deduced the equation of transfer in the Milne-Eddington model. They have also yielded a solution for the case of a triplet using Eddington's approximation taking the Planck function linear.

Busbridge and Stibbs [4] applied the *principle of invariance* [11] governing the law of diffuse reflection with a slight modification to solve the equation of interlocked multiplet lines in the Milne-Eddington model (1.135) . They assumed that no redistribution in frequency takes place, other than that due to interlocking , that the ratios of the line absorption coefficients to the continuous absorption coefficients are independent of depth, and that the lower states of the lines are sharp. They used a linear approximation of the Planck function of the form

$$B(\nu, T) = a + b\tau \tag{1.138}$$

and obtained the final form of emergent intensity $I_r(0,\mu)$ in the r^{th} line as

$$I_r(0,\mu) = (a + bn_r\mu)H(n_r\mu)\left\{\left(\sum_{p=1}^{k}\alpha_p\lambda_p\right)^{\frac{1}{2}} + \frac{1}{2}\sum_{p=1}^{k}(\alpha_p n_p - \alpha_r n_r)(1-\lambda_p)\right.$$
$$\left.\times\int_0^1 \frac{\mu' H(n_p\mu')}{n_r\mu + n_p\mu'}d\mu'\right\} + \frac{1}{2}b\alpha_r n_r H(n_r\mu)\sum_{p=1}^{k}(1-\lambda_p)\int_0^1 \mu' H(n_p\mu')d\mu' \tag{1.139}$$

and the integral equation for H-function

$$\frac{1}{H(n_r\mu)} = \left(\sum_{p=1}^{k}\alpha_p\lambda_p\right)^{1/2} + \frac{1}{2}\sum_{p=1}^{k}\alpha_p(1-\lambda_p)\int_0^1 \frac{n_p\mu' H(n_p\mu')}{n_r\mu + n_p\mu'}d\mu' \tag{1.140}$$

where they assumed

$$\lambda_r = \frac{1+\epsilon\eta_r}{1+\eta_r}, \quad n_r = \frac{1}{1+\eta_r} = \frac{\lambda_r}{1+\epsilon\eta_r}, \quad r = 1, 2, \ldots, k. \tag{1.141}$$

When $k = 1$ the solution (1.139) becomes Chandrasekhar's solution for coherent scattering(§84) [11].
They considered the case of doublet in detail and computed the values of H-function. They also evaluated the residual intensities of the two components as a function of $\mu(=\cos\theta)$ for three distances from the centre of the lines and compared with the residual intensities for doublet lines when there is no interlocking. They shown that interlocking causes a small reduction in the difference between the residual intensities of the two lines, the reduction being greater towards the limb.
An exact solution of the transfer equation for interlocked lines has been given in a short note by DasGupta [12] by his modified form of *Wiener-Hopf technique* [63] in terms of H-function and finite and infinite integrals admitting of approximate evaluations in closed form.

Case [7] introduced the method of *Singular eigenfunction expansions* and applied to problems in neutron transport theory. This method has found favour mostly in the realm of neutron transport theory and used by Siewert and McCormick [54] in a study of line formation with anisotropic scattering taking phase function $p(\mu,\mu') = 1 + \omega_1\mu\mu'$ and obtained the outgoing intensity $I(0,-\mu)$ as

$$I(0,-\mu) = b(1-c_0)^{1/2}H(\mu)\left[\mu + \frac{c_0\alpha_1}{2(1-c_0)^{1/2}} + \frac{a}{b}\right] \tag{1.142}$$

which is the same result given by Chandrasekhar(§84, eq 66) [11].

Siewert and Ozisik [55] applied the method of *Singular eigenfunction expansions* to solve rigorously the equation of radiative transfer describing the interlocking-doublet model in the theory of line scattering . They represented Planck function as a linear function of optical variable and taken η_j's to be constant.
They introduced the new optical variable $x = (1+\eta_N)\tau$ and assumed the parameters η_j to be ordered such that $\eta_1 > \eta_2 > \eta_3 > \ldots > \eta_N$, and defined

$$\sigma_i = \frac{1+\eta_i}{1+\eta_N}, \quad i = 1,2,\ldots,N \tag{1.143}$$

$$\lambda_i = \frac{1+\epsilon\eta_i}{1+\eta_i}, \quad i = 1,2,\ldots,N \tag{1.144}$$

$$\alpha = a, \quad \beta = \frac{b}{1+\eta_N} \tag{1.145}$$

$$c_{ij} = \frac{\frac{1}{2}(1-\epsilon)\alpha_i\eta_j}{1+\eta_N}, \quad i,j = 1,2,\ldots,N \tag{1.146}$$

and formulated the equation of transfer for multiplet in matrix form as

$$\mu\frac{\partial}{\partial x}\mathbf{I}(\mathbf{x},\mu) + \sum\mathbf{I}(\mathbf{x},\mu) = \mathbf{C}\int_{-1}^{1}\mathbf{I}(\mathbf{x},\mu')\mathbf{d}\mu' + \sum\mathbf{\Lambda}(\alpha+\beta\mathbf{x}) \tag{1.147}$$

where the elements of the $\mathbf{\Lambda}$-vector are λ_i, the elements of the $\mathbf{C}$-matrix are c_{ij}, Σ is a diagonal matrix with $(\Sigma)_{\mathrm{ii}} = \sigma_{\mathrm{i}}$, and $\mathbf{I}(\mathbf{x},\mu)$ is a column vector containing $I_i(x,\mu), i = 1,2,\ldots,N$ as elements.
Taking only the exit distribution, they developed a rigorous solution to the equation for the interlocking doublet and obtained

$$I_1(0,-\mu) \;=\; (\alpha+\beta\mu/\sigma)H(\mu/\sigma)\left[(1-\frac{2c_{11}}{\sigma}-2c_{22})^{1/2} - (\frac{c_{12}}{\sigma}-c_{22})\times\right.$$

$$\int_0^1 \frac{\mu' H(\mu')}{\mu' + \mu/\sigma} d\mu' \Bigg] + \beta H(\mu/\sigma) \left[c_{11} \int_0^{1/\sigma} \mu' H(\mu') d\mu' + \frac{c_{12}}{\sigma} \int_0^1 \mu' H(\mu') d\mu' \right], \mu \in (0,1) \tag{1.148}$$

and

$$\begin{aligned} I_2(0,-\mu) &= (\alpha + \beta\mu) H(\mu) \Bigg[(1 - \frac{2c_{11}}{\sigma} - 2c_{22})^{1/2} \\ &\quad -(c_{11} - \sigma c_{21}) \int_0^{1/\sigma} \frac{\mu' H(\mu')}{\mu' + \mu/\sigma} d\mu' \Bigg] \\ &+ \beta H(\mu) \left[\sigma c_{21} \int_0^{1/\sigma} \mu' H(\mu') d\mu' + c_{22} \int_0^1 \mu' H(\mu') d\mu' \right] \end{aligned} \tag{1.149}$$

$\mu \in (0,1)$

Karanjai [26] calculated the residual intensities for interlocked doublet and triplet using his approximate form of H-function

$$\begin{aligned} H(\tilde{\omega_0}, \mu) &= 1 + \frac{\alpha\mu}{1 - \tilde{\omega_0} + 2\mu\omega}, \\ & A = (1 - \tilde{\omega_0})^{\frac{1}{2}} \end{aligned} \tag{1.150}$$

where α is a function of $\tilde{\omega_0}$ given by

$$\alpha = \tilde{\omega_0}(1 - \tilde{\omega_0}) exp \left[\tilde{\omega_0} \left(1 + 2 \left| \tilde{\omega_0} - \frac{1}{2} \right| \right) \right] \tag{1.151}$$

and compared the results with those of Busbridge and Stibbs [4]. He pointed that the residual intensities of the weaker triplet line considered, have no drop near $\mu = 0$. The weaker doublet considered by Busbridge and Stibbs(case III) [4] possesses the same property. He noted that his approximate form of H-function gives a good result in the range $0 < \tilde{\omega_0} < 0.4$. The results in the range $0.4 < \tilde{\omega_0} < 0.9$ are correct up to 3 or 4 significant figures only, and in the range $0.9 < \tilde{\omega_0} \leq 1$, the approximate form is not at all a good one. Thus, it can be used in the calculation of weak multiplets.

DasGupta and Karanjai [13] applied Sobolev's method of *probability of quantum exit from the medium* [56] to solve the equation of transfer for interlocked multiplet lines to get an exact alternative form of emergent intensities in different lines. With slight modification they have written the equation of transfer for the r^{th} line of multiplets as

$$\mu\frac{dI_r(z,\mu)}{dz} = (k+\sigma_r)I_r(z,\mu) - \delta\sum_s P(rs)\sigma_s J_s - (k+\epsilon\sigma_r)B(\nu_r,T) \quad (1.152)$$

and obtained the emergent intensity $I_r(0,\mu)$ as

$$I_r(0,\mu) = M(x_r) + \delta b_0(1-\lambda_r)\sum_i \gamma_i(1-\lambda_i)\left[F(x_r) + \beta k Q(x_r)\right]$$
$$-\delta b_0(1-\lambda_r)\frac{\phi(x_r)}{2}\sum_i \xi_i\gamma_i \int_0^{\lambda_i} \frac{(1-\beta x)x\phi(x)}{x+x_r}dx \quad (1.153)$$

also they have obtained the integral equation for $\phi(\lambda_r\mu)$

$$\frac{1}{\phi(\lambda_s\mu)} = \left(\sum_i \sigma_i\lambda_i^*\right)^{1/2} + \frac{1}{2}\sum_i \sigma_i(1-\lambda_i^*)\int_0^1 \frac{\lambda_i\mu'\phi(\lambda_i\mu')}{\lambda_i\mu' + \lambda_s\mu}d\mu' \quad (1.154)$$

which is the same integral equation obtained by Busbridge and Stibbs [4], giving the H-function $H(n_r\mu)$(1.140).

DasGupta [14] obtained an exact solution of transfer equation for interlocked multiplet with linear Planck function by Laplace transform and Wiener-Hopf technique [63] using his new representation of H-function [15]. He has taken the equation of transfer for the r^{th} interlocked line (following Woolley and Stibbs [71]) in the form

$$\mu\frac{dI_r(t,\mu)}{dt} = n_r I_r(t,\mu) - \delta_r B(t) - a_r\sum_{p=1}^{N}\eta_p J_p(t) \quad (1.155)$$

and obtained the emergent intensity $I_\nu(t,\mu)$ in the case of coherent scattering

$$I_\nu(t,\mu) = \frac{b_1 H(z,\omega)(1-\omega)^{3/2}}{1+\epsilon\eta_\nu}\left(z + (1+\eta_\nu)\frac{b_0}{b_1} + \alpha_1\frac{\omega}{2}(1-\omega)^{-1/2}\right) \quad (1.156)$$

which is the same as Chandrasekhar's solution(eq.66 ,§84.4) [11].

Karanjai and Barman [27] solved the equation of transfer(1.135) for interlocked multiplets by *Discrete ordinate method* [11] with the basic assumption made by Woolley and Stibbs [71] and obtained the solution in the closed form

$$I_r^*(0,\mu) = (1-\omega_r)^{1/2}\zeta_r b H_r(\mu)\left[\mu + C_r\right] - (\zeta_r b\mu + a) \quad (1.157)$$

They also obtained the diffusely reflected intensity by the limiting process and the law of darkening.

Karanjai and Karanjai [28] solved the equation of radiative transfer(1.155) for interlocked multiplets by Laplace transform and Wiener-Hopf technique developed by Dasgupta [14] considering two nonlinear forms of Planck function $B_\nu(T)$, viz;
(1) an atmosphere [18] in which

$$B_\nu(T) = B(t) = b_0 + b_1 e^{-\alpha t} \tag{1.158}$$

and
(2) an atmosphere [3] in which

$$B_\nu(T) = B(t) = b_0 + b_1 t + E_2(t) \tag{1.159}$$

where $E_2(t)$ is the exponential integral given by

$$E_2(t) = \int_1^\infty \frac{e^{-tx}}{x^2} dx \tag{1.160}$$

In case (1), they obtained the form of $I_\nu(0, z)$ as

$$I_\nu(0, z) = H(z, w) b_0 (1 - w)^{1/2} \tag{1.161}$$

which is similar to the Chandrasekhar's result (with $b_1 = 0$,§84.4) [11]. In case (2), they have shown that the form of $I_\nu(0, z)$ is similar to the result obtained by DasGupta [14].

Deb *et al.* [16] solved the equation of transfer for interlocked multiplet(1.135) by *discrete ordinate method* [11] in an exponential atmosphere in which $B_\nu(T) = b_0 + b_1 e^{-\beta t}$ and obtained the emergent intensity in the nth approximation as

$$I_r(0, \mu) = \frac{b_0 \xi_r \beta (1 - w)^{1/2} H_r(\mu)}{1 + \beta \xi_r \mu} \left[\mu + \frac{1}{\xi_r \beta} + \frac{T_r b_1}{b_0 \xi_r \beta (1 - w)^{1/2} H_r(-\frac{1}{\beta \xi_r})} \right] \tag{1.162}$$

and compared this result with the results obtained by Karanjai and Barman [27]. Moreover, for large values of β the solution takes the form

$$I_r(0, \mu) = b_0 (1 - w_r)^{1/2} H_r(\mu) \tag{1.163}$$

i.e, Planck function $B_\nu(T)$ behaves like a constant or independent of τ. Karanjai and Deb [17] solved the same problem (1.135) by the method used by Busbridge and Stibbs [4] taking exponential Planck function [18] and obtained the final form of emergent intensity in the r^{th} line as

$$\begin{aligned} I_r(0,\mu) &= b_0 H(\xi_r\mu)\left\{\left(\sum_{p=1}^{k}\alpha_p\lambda_p\right)^{1/2} + \sum_{p=1}^{k}(\alpha_p\xi_p - \alpha_r\xi_r)(1-\lambda_p)\times\right. \\ &\left.\int_0^1 \frac{\mu' H(\xi_p\mu')}{\xi_r\mu + \xi_p\mu'}d\mu'\right\} + \frac{b_1 H(\xi_r\mu)}{1+\xi_r\beta\mu}\left\{T_r\left(\sum_{p=1}^{k}\alpha_p\lambda_p\right)^{1/2}\right. \\ &\left.+\frac{1}{2}\sum_{p=1}^{k}(\alpha_p\xi_p T_r - \alpha_r\xi_r T_p)(1-\lambda_p)\int_0^1 \frac{\mu' H(\xi_p\mu')}{\xi_r\mu + \xi_p\mu'}d\mu'\right\} \\ &+\frac{1}{2}\frac{b_1\alpha_r\xi_r\beta H(\xi_r\mu)}{1+\xi_r\beta\mu}\sum_{p=1}^{k}(1-\lambda_p)T_p\int_0^1 \frac{\mu' H(\xi_p\mu')}{\xi_p\beta\mu' - 1}d\mu' \end{aligned} \quad (1.164)$$

Rons [48] described a method for calculation of radiative transfer in overlapping resonance line doublets. His proposed method is an extension for overlapping doublets of the singlet method introduced by Mihalas et al. [44] for the treatment of the radiative transfer in the comoving frame in a monotonic outward increasing stellar wind, under the assumption that the transitions between the doublet levels can be neglected. He conclude that the doublet source function can be approximated by the source function of the 'contacted' line, i.e, the line constituted by the two components together at zero separation.

Basak & Karanjai [33] solved the equation of interlocked multiplet(1.135) by discrete ordinate method in anisotropically scattering media taking Rayleigh phase function. They used the boundary conditions

$$I_r(0,-\mu) = 0, (0 < \mu \leq 1) \quad (1.165)$$
$$I_r(\tau,\mu)e^{-\tau/\mu} \rightarrow 0 \;\; as \;\; \tau \rightarrow \infty \quad (1.166)$$

and obtained the diffusely reflected intensity in the form

$$I_r^*(0,\mu) = G_r(\mu + \lambda_r)H_r(\mu) - b_1\zeta_r\mu - b_0 \quad (1.167)$$
$$I_r(0,\mu) = G_r(\mu + \lambda_r)H_r(\mu) \quad (1.168)$$

Basak & Karanjai [34] applied the discrete ordinate method to solve the equation of radiative transfer(1.135) in anisotropically scattering media taking linear Planck function with three term scattering indicatrix given by

$$P(\mu,\mu') = 1 + \omega_1\mu\mu' + \frac{1}{4}\omega_2(3\mu^2 - 1)(3\mu'^2 - 1) \tag{1.169}$$

Here also they obtained the diffusely reflected intensity and emergent intensity in the form

$$I_r^*(0,\mu) = G_r(\mu + \lambda_r)H_r(\mu) - b_1\zeta_r\mu - b_0 \tag{1.170}$$

$$I_r(0,\mu) = G_r(\mu + \lambda_r)H_r(\mu) \tag{1.171}$$

Bask & Karanjai [35] also solved the same problem(1.135) by discrete ordinate method with linear form of Planck function and planetary phase function. Here also they used the boundary condition given by (1.165) & (1.166) and obtained the diffusely reflected intensity and emergent intensity in the form

$$I_r^*(0,\mu) = G_r(\mu + \lambda_r)H_r(\mu) - b_1\zeta_r\mu - b_0 \tag{1.172}$$

$$I_r(0,\mu) = G_r(\mu + \lambda_r)H_r(\mu) \tag{1.173}$$