I N T R O D U C T I O N

The theory of propagation of waves in elastic solids was developed in the last century. The names of the academicians that may be associated for pioneer theoretical work on this line are, STOKES, POISSON, RAYLEIGH, KELVIN and others, who extended the theory of elasticity to the problem of vibrating bodies and propagation of waves in elastic material. During the first quarter of this century the subject lost much of its glamour and interest. This is perhaps partly because of the attractions of the new fields opened up by the discoveries in atomic physics and partly because of a gap between the advancement of theoretical and experimental work, as there were no practical methods available in laboratory for observing the passage of stress waves in elastic materials. But there has been a remarkable revival of interest in the subject as far back as thirties. Extensive study in seismic wave propagation, earthquake engineering and research on geophysical phenomena attract a number of theoretical and practical workers. Since then the interest in the subject has been gathering momentum. With the advent of sophisticated instruement, electronic tecbniques and high speed computers, the subject has become a very important field of research. A large number of original papers on both experimental and theoretical aspects of the subject have been appearing with various information.

Most of the experimental works carried out on the wave propagation are concerned with studying propagation in specimens of comparatively simple geometrical shape, the results of this experiment could be compared directly with exact or approximate theoretical predictions. The agreement, with experimental results and theoretical predictions, inspires confidence in taking up complicated problems and makes possible theoretical predictions and interpretations of observations.

The propagation of waves through homogeneous isotropic elastic material of unbounded extension is not a subject of very complexity. The waves are either dilatational or distortional or a combination there of. The picture changes radically as soon as there is a boundary. Interaction of two types of waves occurs, when boundary is present and this interaction presents an inherent difficulty in the solution of elasto-dynamic problems. More over the effect of a free surface on the generation and propagation of waves in elastic medium has been the subject of many investigations ever since the discovery in existence of surface waves by LORD RAYLEIGH.

In general, problems which mostly attract the researchers both theoretical and experimental, in relation to the generation and propagation of waves in an elastic medium may be classified as

i) diffraction of propagating waves through the medium due to any obstacle, cavity or a crack of any shape situated somewhere in the medium;

ii) reflection, refraction and diffraction of propagating waves due to mixed boundary conditions;

iii) wave motion generated due to a punch on some bounded region of the medium;

iv) radiation of waves i.e the wave motions generated due to some fixed external disturbance and propagating away from the source of disturbance;

v) wave motion generated in a medium when a source of disturbance moves along the medium.

Depending on the nature of the source of disturbance, shape of the punch or normal loading on the free surface and the presence of discontinuities in the medium, different complicated problems arise. The solution of these problems need an advance level of sophisticated mathematical techniques.

We present some of the mathematical techniques in short and give references to some of the problems along with their solutions. These may be important and interesting in engineering science, in earthquake engineering, in geophysics and in seismology, and definitely to the mathematicians because of the complicated mathematics involved in the formulation of the problems and in the determination of their solutions.

The dynamic response of an elastic half space due to an external load or a punch on the free surface and also the scattering of elastic waves by a finite crack or a strip inside an elastic medium may be investigated by the use of integral transform techniques.

The integral transform $\bar{f}(\zeta)$ of a function $f(x)$ defined in an interval (a, ∞) is an expression of the form

$$\bar{f}(\zeta) = \int_a^\infty f(x)K(x, \zeta)dx, \qquad (1)$$

where a is real number and ζ is a complex parameter varying over some region D of the complex plane. $K(x,\zeta)$ is called the kernal of the transformation. The transformation (1) becomes particularly useful if it possesses inverse mapping. In that case one can express $f(x)$ in terms of its integral transform by

$$f(x) = \frac{1}{2\pi i}\int_\Gamma \bar{f}(\zeta)M(x,\zeta)d\zeta \quad .$$

Here $M(x,\zeta)$ is a suitable function defined in $a < x < \infty$ and $\zeta \in D$ and is called the kernel of the inverse transform, which is defined for all x in the interval (a,∞). The complex parameter ζ is in the region D while Γ is a suitable path of integration in D.

With the aid of these transforms one can replace some of the independent variables, so that in many cases it is possible to reduce the governing partial differential equations to ordinary differential equation in the transformed space with one less independent variable. If the reduced equation for $\bar{f}$ can be solved, the solution f can be expressed in terms of the inversion integral, which may then be evaluated. The inversion from the transformed space to the space of actual variables usually involves very complicated integrations. In many cases even the numerical integration can not be performed successfully because of the highly oscillatory character of the integrands. [cf. ERINGEN and SUHUBI (1975), chap.7; ACHENBACH (1976), Chap.7]. It is mentioned in the preceeding para that the inversion from the transformed space to the original space of variables involves various complications. In particular, mixed boundary value problems like the dynamic response of a punch on an elas-tic half space and the problems involving the presence of a crack or a strip inside an elastic medium may be reduced to FREDHOLM's integral equation of first kind or to dual integral equations.

Different techniques have been applied by many authors to tackle these type of problems. From these stand point, these problems may be divided into two categories: one for low frequency oscillation of the source or long wave scattering or transmission and the other for high frequency oscillation or short wave scattering or transmisson in the medium. The terms long and short are used in comparison to the region of the source of distrubance or the size of the strip or crack etc. in side the medium to the wave length of distrubance. In case of low frequency oscillations NOBLE's (1963) method of solving dual integral equations, TRATER's (1962) technique for solving dual integral equations, Matched Asymptotic Expansion, and variational principle are found to be very useful whereas in case of high frequency oscillations WEINER-HOPF (NOBLE, 1958) technique, or Geometrical Ray Theory are found to be most suitable.

Suppose that a mixed boundary value problem is formulated by suitable integral transform so as to be governed by a set of dual integral equation of the form

$$\int_0^\infty G(p)f(p)J_\nu(rp)\,dp = g(r),\quad 0 < r < 1$$

$$\text{and}\quad \int_0^\infty f(p)J_\nu(rp)\,dp = 0,\quad r > 1 \tag{2}$$

where G(p), g(r) are the known functions of the variable indicated while J_ν is a Bessel function of the first kind of order ν. We are supposed to determine f(p).

According to NOBLE (1963) if G(p) can be expressed in the form

$G(p) = p^\gamma[1 + H(p)]$ and that if H(p) tends to zero as $p \to \infty$ then he proved that

$$f(p) = \frac{p^{1-\frac{\gamma}{2}}}{2^{-1+\frac{1}{2}\gamma}\Gamma(\frac{1}{2}\gamma)}\int_0^1 \xi^{1/2}\,\theta(\xi)\,J_{\nu+\frac{1}{2}\gamma}(\xi p)\,d\xi$$

where $\theta(\xi)$ satisfies the FREDHOLM integral equation of the second kind

$$\Theta(x)+\frac{1}{\pi}\int_{0}^{1}K(x,\xi)\Theta(\xi)\,d\xi = F(x) \qquad (3)$$

with kernal

$$K(x,\xi)\pi(x\xi)^{1/2}\int_{0}^{\infty}pH(p)J_{\nu+\frac{1}{2}\gamma}(xp)J_{\nu+\frac{1}{2}\gamma}(\xi p)dp$$

and if $o<\gamma<2$ then

$$F(x)=x^{-\nu-\frac{1}{2}\gamma+\frac{1}{2}}\int_{0}^{x}g(r)r^{\nu+1}(x^{2}-r^{2})^{-1+\frac{1}{2}\gamma}\,d\gamma .$$

The integral equation (3) can be solved for $\Theta(x)$ and consequently $f(p)$ can be determined.

ROBERTSON (1966) considered a circular rigid disc pressed on the free surface of an elastic half space and vibrating on the surface having a smooth contact. The displacement being specified under the disc and the surface outside the disc is assumed to be stress f free. This mixed boundary value problem is reduced to a set of dual integral equations. The solution of this dual integral equations is then reduced to the FREDHOLM integral equation of the second kind, an iterative solution of whi-ch is then obtained for low frequency oscillation of the disc.

In another paper ROBERTSON (1967,b) considered a longitudinal wave harmonic in time to be incident normal to a penny-shaped crack on a semi-infinite elastic solid. This problem is also formulated by a set of dual integral equations. An equivalent FREDHOLM integral equation of the second kind is determined by the use of NOBLE's (1963) method, which is also solved by iterative method assuming low frequency oscillation of the applied stress on the crack surface. GLADWELL (1968) assumed a circular indentor vibrating about one of its diameter. A displacement under the indentor is prescribed and stress out side the indentor is taken to be zero. This mixed boundary value problem is formulated so as to be governed by a set of dual integral equations which is again converted to an equivalent Fredholm integral equation of the second kind by the use of NOBLE's method and the solution is obtained for low frequency oscillation of the indentor.

Almost in a similar way MAL, ANG and KNOPOFF (1968) determined the solution of the problem of diffraction of axisymmetric harmonic elastic waves by a rigid circular disc, MAL (1969) studied the diffraction of elastic waves in presence of a penny-shaped crack inside a semi-infinite medium and SINGH, DHALIWAL and VRBIK (1983) considered the mixed boundary value problem arising out of the interaction of the anti-plane shear waves to an arbitray angle to the moving crack.

WICKHAM (1977) considered the time harmonic vibration of frequency w of a rigid infinite strip in a semi-infinite homogeneous isotropic elastic solid. The motion is forced by prescribed displacement distribution $v_o(x)e^{-iwt}$, normal to the infinite strip $|x|<1, y=o$, $-\infty<z<\infty$. It is assumed that the tangential stress at the plane $y=o$, is zero and the normal stress is also zero for $|x|>1$. Boundary conditions are formulated as

$$v(x,o) = v_o(x) \quad ; \quad |x|\leq 1$$

$$\tau_{yy}(x,o) = o \quad ; \quad |x|>1$$

$$\tau_{xy}(x,o) = o \quad ; \quad -\infty < x < \infty.$$

In this paper the approach is some what different from the others, which has already been discussed. The author assumed a function $f(x)$

(unknown) to be the normal stress below the strip. Thus $\tau_{yy}(x,o)$ is assumed to be known on $(-\infty, \infty)$. Assuming the normal stress to be known on $y = o$, the integral representation for the potentials ϕ and Ψ are determined by the use of Green's function. with the help of ϕ and Ψ (which involve the unknown function $f(x)$) an integral equation of the first kind for $f(x)$ is obtained, which is then converted to an integral equation of the second kind by NOBLE's method and the iterative solution is obtained for small wave number.

In this connection I could not resist the temptation of referring to a paper by FABRIKANT and SANKAR (1984), where a mixed boundary value problem in a non-homogeneous medium is considered. An asymmetric contact in the form of a circle $\rho = a$ is assumed to be present on the half space. An arbitrary normal displacement is prescribed inside the circle $\rho = a$, while the boundary $Z = 0$ of the half space out side the circle is x stress free and the tangential stress vanishes all over the plane $Z=0$,. Following ROSTOVTSEV (1964) the solution of the problem is reduced to the solution of a two dimensional integral equation in polar co-ordinates. The authors then proved that it is possible to reduce the integral equation to a sequence of two Abel type integral operator and another operator introduced by them. The inverse of the operators can be found easily and the exact solution of the two dimensional equation is obtained in the closed form.

In all the cases discussed above, dual integral equations are converted to Fredholm's integral equation of the second kind which is then solved iteratively for low frequency oscillation. But TRANTER's method (1962) of solving dual integral equation is different. If a mixed boundary value problem is formulated as in (2), then according to TRANTER f(p) (which is to be determined) is taken in the form

$$f(p) = p^{1-k} \sum_{m=0}^{\infty} a_m \, J_{\nu+2m+k}(p) \, , \qquad (4)$$

where m is a positive integer or zero, k is real and positive and the real part of ν is greater than -1. It may be seen that under the conditions stated above the integral

$$\int_0^\infty p^{1-k} \, J_{\nu+2m+k}(p) \, J_\nu(rp) dp \qquad (5)$$

converges for both $r > 1$ and $o < r < 1$ and its value is also given by WATSON (1944). For a choice of f(p) as given in (4) and using the value of the integral (5) it can be proved that the second of the equation (2) is automatically satisfied. The coefficients a_m have to be so chosen that the form of f(p) as given in (4) also satisfies the first equation of (2). Again from WATSON (1944) we have the m value of the integral in (5) to be equal to

$$\frac{r^\nu \Gamma(\nu+m+1)}{2^{k-1}\Gamma(\nu+1)\Gamma(m+k)} \, {}_2F_1(\nu+m+1, \, 1-k-m; \, \nu+1; \, r^2) \qquad (6)$$

when $o < r < 1$.

The value of f(p) from (4) is substituted in the first equation of (2). After some algebraic simplification with the help of Jacobi polynomial, and using the value of the integral (5) as given in (6), the Hankel inversion formula and also noting that

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b} \, {}_2F_1(c-a,c-b;c;z). $$

We finally obtain

$$ \sum_{m=0}^{\infty} a_m \int_0^{\infty} G(p) p^{1-2k} J_{\nu+2m+k}(p) J_{\nu+2n+k}(p)\, dp = E(\nu,n,k) \tag{7} $$

where $\dfrac{2^{k-1}\Gamma(\nu+1)\Gamma(n+k)}{\Gamma(\nu+n+1)} E(\nu, n, k)$

$$ = \int_0^1 g(r) r^{\nu+1} (1-r^2)^{k-1} \mathfrak{F}_n(k+\nu, \nu+1, r^2)\, dr. \tag{8} $$

The Jacobi polynomial being defined by

$$ \mathfrak{F}_n(\alpha,\gamma,x) = {}_2F_1(-n, \alpha+n; \gamma; x). $$

As a special case when $g(r)$ in (7) is replaced by Ar^{ν} (A being constant) and making use of the result [cf. WATSON (1944), p.404], the equation (7) can be written as

$$ a_n + \sum_{m=0}^{\infty} L_{m,n}\, a_m = (2\nu+4n+2k) E(\nu,n,k) \tag{9} $$

where $(2\nu+4n+2k)^{-1} L_{m,n}$

$$ = \int_0^{\infty} \left\{ p^{2-2k} G(p) - 1 \right\} p^{-1} J_{\nu+2m+k}(p) J_{\nu+2n+k}(p)\, dp. $$

Since Jacobi polynomials satisfy the relations

$\mathfrak{F}_0(\alpha, \gamma, x) = 1$ and

$$\int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma}\, \mathfrak{F}_m(\alpha,\gamma,x)\, \mathfrak{F}_n(\alpha,\gamma,x)\,dx = o$$

when $m \neq n$, we have from (8) , for $g(r) = Ar^{\nu}$

$$E(\nu,n,k) = A \frac{\Gamma(\nu+1)}{2^k\Gamma(k+\nu+1)} \quad \text{for } n = o$$

$$= 0 \qquad \text{for } n > o \tag{10}$$

Therefore by the use of (10), equation (9) takes the form

$$a_o + \sum_{m=o}^{\infty} L_{m,o}\, a_m = A \frac{2^{1-k}\Gamma(\nu+1)}{\Gamma(\nu+k)} \tag{11}$$

$$a_n + \sum_{m=o}^{\infty} L_{m,n}\, a_m = 0 \ , \ n > 0.$$

The iterative solution of the equation (11) is

$$a_n = A \frac{2^{1-k}\Gamma(\nu+1)}{\Gamma(\nu+k)}\left[\delta_n - c_n + c_n' - c_n'' + \dots\dots\dots\right],$$

where $\delta_n = o$ for $n > o$, $\delta_o = 1$ and

$$C_n = L_{o,n},\ C_n' = \sum_{m=o}^{\infty} L_{m,n}\, C_m,\ C_n'' = \sum_{m=o}^{\infty} L_{m,n}\, C_m' \text{ etc.}$$

With the determination of the constants, the asymptotic solution of the dual integral equations in (2) is determined.

BOSE (1968) considered a rigid circular disc indented in a semi infinite elastic solid which performs small oscillations normal to its plane without loosing contact with the surface of the solid. With the heop of Hankel transform, the problem has been reduced to the solution of dual integral equation and then following the method of TRANTER to the solution of an infinite set of algebraic equations. At present a new technique known as matched asymptotic expansion has been introduced and developed to solve the mixed boundary value problems involving wave propagation in elastic solids due to low frequency vibration of the scatterer.

This method has primarily been used in solving the celebrated Navier Stokes equations of Hydrodynamics. It was developed by PRANDTL (1905) to solve the problem of high speed viscous flow past of a body. Some times this method is found to be convenient to solve scattering and diffraction of elastic waves by cracks and bodies of finite length.

Diffraction and scattering of elastic waves due to the presence of a finite crack or an obstacle of finite dimension can be solved by the method of matched asymptotic expansion in case when the dimension of the scatterer is small compared to the wave length of the propagating waves.

In this method, two expansions are developed simultaneously firstly, an inner expansion valid close to the scatterer and secondly, an outer expansion valid far away from it. The inner expansion is constrained to obey the boundary condition at the surface of the scatterer while the outer expansion generated so as to satisfy the condition at infinity but not at the surface of the scatterer.

To render the problem determinate, it is necessary to use the fact that the inner and outer expansions are different forms of the same function. This leads to the matching of these two expansions in an intermediate region. This makes it possible to derive alternatively the successive terms in each expansion. In this way construction of a uniformly valid composite expansion is done.

In order to explain the mathematical procedure, consider the dynamical equations of elasticity

$$(\lambda+\mu)\,\mathrm{grad}\,\Delta+\mu\,\nabla^2\vec{u}-\rho\frac{\partial^2\vec{u}}{\partial t^2}=0 \qquad (12)$$

Putting $\vec{u}=\vec{u}_o\,e^{iwt}$ and dropping the zero subscript one derives

$$(\lambda+\mu)\,\mathrm{grad}\,\Delta+\mu\,\nabla^2\vec{u}+\rho w^2\vec{u}=0 \qquad (13)$$

Introducing a characteristic geometric length l in the problem and putting

$\vec{u}'=\vec{u}/a$, $\vec{r}'=\vec{r}/a$ and finally dropping the primes, the above equation in the dimensionless form can be written as

$$\frac{(\lambda+\mu)}{\rho w^2 a^2}\,\mathrm{grad}\,\Delta+\frac{\mu}{\rho w^2 a^2}\nabla^2\vec{u}+\vec{u}=0 \qquad (14)$$

Setting $\vec{u}=\mathrm{grad}\,\Phi+\mathrm{rot}\,\vec{F}$, it is found that Φ and $\vec{F}$ satisfy the equation

$$\nabla^2\Phi+M^2\Phi=0 \qquad (15)$$

$$\nabla^2\vec{F}+m^2\vec{F}=0 \qquad (16)$$

where M and m are two dimensionless numbers defined by $M=\eta m$, and

$$m^2=\frac{\rho w^2 a^2}{\mu} \quad \text{and} \quad \eta=\sqrt{\mu/(\lambda+2\mu)}\ .$$

The analysis is based on the asumption that M and m are small. Let an axially symmetric body be depressed by an amount $d_o e^{iwt}$ along its axis of symmetry into the elastic space by an exciting periodic force. If $\vec{I}$ stands for the unit vector along the axis of symmetry taken as x-axis, then the boundary conditions become

$\vec{u} = d_o \vec{I} e^{iwt}$ at the surface of the body while

$\vec{u} = o$ as $r \to \infty$.

Applying the principle of superposition and using nondimensional unit this is equivalent to a problem with boundary condition

$\vec{u} = o$ at the surface of the body

$\vec{u} = -\frac{d_o}{a} \vec{I}$ as $r \to \infty$.

In order to obtain appropriate inner solution for $\vec{u}$, we assume an expansion of the form

$$\vec{u} = \vec{u}_o + \vec{u}_1 + \ldots\ldots\ldots + \vec{u}_n + \ldots\ldots \tag{17}$$

such that

$$u_n = m^n \operatorname{grad} \varphi_n + M^n \operatorname{rot} \vec{f}_n \tag{18}$$

and

$$\Phi = \varphi_o + M\varphi_1 + M^2\varphi_2 + \ldots\ldots\ldots + M^n\varphi_n \ldots\ldots \tag{19}$$

$$\vec{F} = \vec{f}_o + m\vec{f}_1 + m^2\vec{f}_2 + \ldots\ldots\ldots m^n \vec{f}_n + \ldots$$

where φ_n and $\vec{f}_n$ satisfy the equations

$$\nabla^2 \varphi_o = o; \quad \nabla^2\varphi_1 = o; \quad \nabla^2\varphi_2 + \varphi_o = o \tag{20}$$

and

$$\nabla^2\vec{f}_o = o; \quad \nabla^2\vec{f}_1 = o; \quad \nabla^2\vec{f}_2 + \vec{f}_o = o$$

The expansion given by (17) is assumed to satisfy the condition at the surface of the body only.

As such these expansions are valid only in the vicinity of the body. Next consider the matched outer expansion. To find the required outer expansion for Φ, we set

$$\bar{x} = Mx \ , \ \bar{y} = My, \ \bar{z} = Mz.$$

As $M \to o$, the point (x,y,z) will move to infinity and it is in this neighbourhood that we are interested in finding the appropriate expansion. We Write

$$\Phi = Mh_1(\bar{x},\bar{y},\bar{z}) + M^2 h_2(\bar{x},\bar{y},\bar{z}) + \ldots\ldots$$

where $\bar{\nabla}^2 h_j + h_j = o \ ; \ j = 1,2, \ldots\ldots$

In a similar fashion to find the outer expansion for $\vec{F}$, we set

$$x^* = mx \ , \ y^* = my, \ \ z^* = mz \text{ and}$$

$$\vec{F} = m\vec{g}_1(x^*,y^*,z^*) + m^2\vec{g}_2(x^*,y^*,z^*) + \ldots\ldots$$

where $\nabla^{*2}\vec{g}_j + \vec{g}_j = o \ ; \ j = 1,2, \ldots\ldots$.

Let us denote the outer expansion for $\vec{u}$ as

$$\vec{u} = \vec{U}_o + \vec{U}_1 + \ldots\ldots + \vec{U}_n + \ldots\ldots$$

with $\vec{U}_o = (-d_o/l)\,\vec{I}$ and

$$\vec{U}_n = M^n \text{ grad } h_n + m^n \text{ rot } \vec{g}_n; \ n = 1,2 \ldots\ldots \ .$$

We are now in a position to obtain the inner and outer solutions and match them appropriately in order to determine the unknown constants which arise in each of them. This is done using VAN DYKE'S (1964) asymptotic matching principle which amounts to the following:
the p-term inner expansion of (the q-term outer expansion) = the q-term outer expansion of (the p-term inner expansion), where p and q may be taken as any two integers equal or unequal.
Using the method of matched asymptotic expansion the disturbance due

to the action of a periodic symmetric force acting on a rigid circular disc attached to the free surface of an elastic half space has been studied by KANWAL (1965). Scattering of SH-waves by a rough half space of arbitrary slope and scattering of Rayleigh waves by a ridge have been studied by SABINA and WILLIS (1975,1977) by the method of matched asymptotic expansion. DUTTA and AKILY (1978) applied M.A.E. to obtain the scattered field when the wave length is large compared with the linear dimension of the inclusion in a half space.

KRIEGSMANN and REISS (1983) assumed that a localized inhomogeneity in the medium acts as a scatterer. An asymptotic expansion which is uniformly valid in space is obtained for low frequency scattering of a plane wave incident on the scatterer. It is assumed that the characteristic length of the scattering region is small compared to the wave length of the incident wave. The method of M.A.E is used in the analysis.

Another method that may be applied to solve the mixed boundary value problem is the variational principle.

Variational techniques have been applied with much success for several years in attacking diffraction and scattering problems in electromagnetic theory. While the power of variational techniques for obtaining approximate solution to problems of elastostatics is well known, similar methods do not appear to have been much in use to solve mixed boundary value problems in elastodynamics. The method of obtaining approximate solution of dual integral equations by variational method was developed by NOBLE (1958-59). Being guided by this method STALLYBRASS IN 1962 developed a variation procedure of solving the socalled punch or contact problems in which a rigid punch or die of arbitrary cross section and with a flat base is forced to oscillate in contact with an elastic medium occupying either a half space, or the infinite region bounded by parallel planes. He has shown that a function can be constructed whose stationary value is proportional to the amplitude of oscillations of the punch, if the boundary conditions of the problem are suitably restricted. It is known that the displacement field U_α generated in an elastic medium occupying a region D, by traction T_i applied to the bounding surface B with a harmonic time dependence e^{-iwt}, can be expressed in the form

$$U_\alpha(P) = \int_B U_\alpha^i(P,Q)\ T_i(Q)\ dA_Q,\quad P\ D+B, \tag{21}$$

where $U_\alpha^i(PQ) = U_i^\alpha(Q,P)$ and $B = B_u + B_T$.

dA_Q is an elemental area at Q , a point on B and $T_i \equiv \tau_{ij}\, n_j$ where τ_{ij} are the components of stress tensor and n_j stands for the components of the outer unit normal to B. The singular functions $U_\alpha^i(P,Q)$ is the Green's function which may be interpreted as the components of displacement in the rectangular cartesion direction x_α at P due to an oscillating unit concentrated surface force in the x_i direction at Q.

The boundary B of the region D is divided into two regions B_u and B_T where (i) on B_u, the boundary conditions are of mixed type (e.g. normal component of displacement vector and tangential component of the surface traction are prescribed).

(ii) on B_T all the components of surface traction T_i will vanish.

$$\text{Let } U_\alpha \Big|_{B_u} = f_\alpha \text{ and } T_i \Big|_{B_u} = g_i \ ,$$

then using $T_i \,\big|_{B_T} = 0$, we obtain

$$f_\alpha(Q') = \int_{B_u} u_\alpha^i (Q' , Q) g_i(Q) dA_Q \qquad (22)$$

which provides a vector integral equation for the determination of the unknown components of the surface truction g_i on B_u. Instead of trying to find out an exact solution of this integral equation a functional is constructed which is stationary relative to small variations of the unknown components of g_i about their exact values. It can be shown that the functional

$$F_1(g_i^*) = 2\int_{B_u} g_i^*(Q) f_i(Q)dA_Q - \int_{B_u} g_i^*(Q)u_i^*(Q)dA_Q \qquad (23)$$

whbre $u_i^*(Q)= \int_{B_u} u_\alpha^i(Q,Q')g_i^*(Q')dA_Q$,

is stationary with respect to first variations of g_i^* about their correct values, as determined by the integral equation. The essence of the above refermulation is that the errors made by using $F_1[g_i^*]$ in place of $F_1[g_i]$ are of the order of magnitude of the squares of the errors in g_i^* relative to g_i. If therefore, we can arrange that $F_1[g_i]$ is proportional to a quantity of interest, and admissible functions g_i^*, in close proximity to the unknown functions g_i, can be obtained, then $F_1[g_i^*]$ will provide us with a good approximation to $F_1[g_i]$.

For purpose of calculation, it is more convenient to have a scalar invariant functional which may be obtained from (23) by replacing g_i^* by cg_i^* and using the stationary property of $F_1[g_i^*]$ to obtain a value for c, the functional obtained is

$$F_2[g_i^*] = \frac{\left[\int_{B_u} g_i^*(Q)f_i(Q)dA_Q\right]^2}{\int_{B_u} g_i^*(Q)U_i^*(Q)dA_Q} \qquad (24)$$

We find that

$$F_2[g_1] \equiv F_1[g_1].$$

In order to illustrate the power of the above variational method, the classical Reissner-Sagoci problem of the forced torsional oscillations of a rigid circular disk attached to an elastic half-space was reconsidered by STALLYBRASS (1962).

Approximation was obtained for sufficiently low values of a certain frequency parameter. The boundary value problem is to determine the solution of the differential equation (omitting the time factor e^{iwt})

$$\nabla^2 u_\varphi - \frac{u_\varphi}{\rho^2} + k^2 u_\varphi = 0 \qquad (25)$$

$$u_\varphi = u_\varphi(\rho, Z), \; k^2 = w^2/c^2$$

subject to the boundary conditions

$$u_\varphi = \beta\rho \; , \; Z = 0 \; , \; \rho \leq a \qquad (26,a)$$

$$T_{\varphi z} = \mu \frac{\partial u_\varphi}{\partial z} = 0, \; Z = 0, \; \rho > a \qquad (26,b)$$

where u_φ is the azimuthal component of displacement and β the amplitude of oscillation of the disk.

Considering general solution of (25) in the form

$$u_\varphi(s,z') = \beta \int_0^\infty \frac{A(\lambda)}{\sqrt{\lambda^2-\alpha^2}} e^{-z'\sqrt{\lambda^2-\alpha^2}} \lambda J_1(s\lambda) d\lambda$$

where $s = \rho/a$, $Z' = Z/a$, $\alpha = ka$, a non-dimensional frequency parameter, it can be shown that the displacement field $u_\varphi^*(s)$ corresponding to an arbitrary admissible stress $\tau_{\varphi z}^*(s)$, $s \leq 1$ can be obtained in the form

$$u_{\varphi}^{*}(s) = -\frac{a}{\mu}\int_{0}^{1}\left[\int_{0}^{\infty}\frac{\lambda}{\sqrt{\lambda^{2}-\alpha^{2}}}J_{1}(\lambda s')J_{1}(\lambda s)d\lambda\right]T_{\varphi z}^{*}(s')s'd\,s' \qquad (27)$$

which is the required relation between displacement and stress. Substituting this relation into the expression for the scalar invariant functional (24) and using the boundary condition (26,a) we get

$$F_{2}[T_{z\varphi}^{*}] = 2\pi\,\mu a^{3}\,\beta^{2}\,\frac{\left[\int_{0}^{1}T_{\varphi z}^{*}(s)s^{2}\,ds\right]^{2}}{\int_{0}^{\infty}\frac{\lambda}{\sqrt{(\lambda^{2}-\alpha^{2})}}\left[\int_{0}^{1}T_{\varphi z}^{*}(s)J_{1}(\lambda s)s\,ds\right]^{2}d\lambda}\,. \qquad (28)$$

Now $F_{2}[T_{z\varphi}] = -2\pi\beta\int_{0}^{a}T_{\varphi z}(\rho)\,\rho^{2}\,d\rho = M_{o}\beta$, where M_{o} is the moment of the forces applied to the disk. Replacing $F_{2}[T_{\varphi z}^{*}]$ by $F_{2}[T_{\varphi z}]$ in (28), we obtain

$$\beta \approx \frac{M_{o}}{2\pi\,\mu a^{3}}\;\frac{\int_{0}^{\infty}\frac{\lambda}{\sqrt{(\lambda^{2}-\alpha^{2})}}\left[\int_{0}^{1}T_{\varphi z}^{*}(s)J_{1}(\lambda s)\,s\,ds\right]^{2}d\lambda}{\left[\int_{0}^{1}T_{\varphi z}^{*}(s)s^{2}\,ds\right]^{2}} \qquad (29)$$

the approximation is in the variational sense.

A natural first approximation of $T_{\varphi z}^{*}(s)$ for small values of frequency is to use the exact static stress distribution.

We therefore take

$$T_{\varphi z}^{*}(s) = \frac{s}{\sqrt{(1-s^{2})}}$$

which when substituted in (29) gives

$$\beta = \frac{9\,M_{o}}{16\,\mu a^{3}}\,I_{1} \qquad (30)$$

where $I_{1} = \int_{0}^{\infty}\frac{1}{\sqrt{(\lambda^{2}-\alpha^{2})}}J_{3/2}(\lambda)\,J_{3/2}(\lambda)\,d\lambda$.

Numerical values of M_o calculated from (30) can be found to be in good agreement with the exact value for low frequency.

The same variational principle has been applied by STALLYBRASS and SCHERER (1976) to solve the problem of forced vertical vibration of a rigid elliptical disk on an elastic half space. The methods discussed above are not applicable in case high frequency oscillations or short wave propagation. In these cases WIENER-HOPF technique finds extensive application in various mixed boundary value problems by means of integral transformations.

As a general case the three part mixed boundary value problems [cf. NOBLE (1958), p.196] may be formulated in complex ς -plane as

$$e^{i\varsigma q} F_+(\varsigma) + K(\varsigma) F_1(\varsigma) + e^{i\varsigma p} F_-(\varsigma)$$

$$= \frac{A}{\sqrt{2\pi}} \frac{e^{i(\varsigma - k\cos\theta)q} - e^{i(\varsigma - k\cos\theta)p}}{\varsigma - k\cos\theta} \qquad (31)$$

where A is a constant and $\varsigma = \sigma + i\,t$ and $k = k_1 + ik_2$. The equation (31) holds in the strip $-k_2 < t < k_2$, $F_+(\varsigma)$, $F_-(\varsigma)$ and $F_1(\varsigma)$ are the unknown functions:

$$F_+(\varsigma) = \frac{1}{\sqrt{2\pi}} \int_q^\infty f(x)\, e^{i\varsigma(x-q)}\, dx\,,$$

$$F_-(\varsigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^p f(x) e^{i\varsigma(x-p)}\, dx\,, \qquad (32)$$

$$F_1(\varsigma) = \frac{1}{\sqrt{2\pi}} \int_p^q f(x) e^{i\varsigma x}\, dx\,.$$

$F_+(\zeta)$ is assumed regular in $t > -k_2$, $F_-(\zeta)$ in $t < k_2$ and $F_1(\zeta)$ is an entire function. It is assumed that $K(\zeta)$ is regular in $-k_2 < t < k_2$, and has branch points at $\zeta = \pm k$, on the supposition that we can write $K(\zeta) = K_+(\zeta)\, K_-(\zeta)$; $K_+(-\zeta) = K_-(+\zeta)$ and taking $0 < \theta < \frac{1}{2}\pi$, we get $k_2 \cos\theta > 0$. It will then prove convenient to rearrange (31) so as to apply the Wiener-Hopf technique in a strip $-k_2 < t < k_2 \cos\theta$. Multiply (31) by $\exp(-i\zeta q)\{K_+(\zeta)\}^{-1}$ and rearrange in the form

$$\frac{F_+(\zeta)}{K_+(\zeta)} - \frac{A}{\sqrt{2\pi}}\,\frac{\exp(-ik\cos\theta\, q)}{\zeta - k\cos\theta}\left[\frac{1}{K_+(\zeta)} - \frac{1}{K_+(k\cos\theta)}\right] +$$

$$+ U_+(\zeta) + V_+(\zeta) = -\exp(-i\zeta q)K_-(\zeta)\, F_1(\zeta) - U_-(\zeta) - V_-(\zeta) +$$

$$-\{\ \} + \frac{A}{\sqrt{2\pi}}\,\frac{\exp(-ik\cos\theta\, q)}{(\zeta - k\cos\theta)K_+(k\cos\theta)}\,. \qquad (33)$$

In the above equation we have written

$$U_+(\zeta) + U_-(\zeta) = e^{i\zeta(p-q)}\, F_-(\zeta)\,/\,K_+(\zeta),$$

$$V_+(\zeta) + V_-(\zeta) = A\,(2\pi)^{-1/2}\, e^{i\zeta(p-q) - ik\cos\theta\, p} / \{(\zeta - k\cos\theta)K_+(\zeta)\}.$$

In a similar way, multiply (31) by $\exp(-i\zeta p)\{K_-(\zeta)\}^{-1}$ and rearrange as

$$\frac{F_-(\zeta)}{K_-(\zeta)} + R_-(\zeta) + \frac{A}{\sqrt{2\pi}}\,\frac{e^{-ik\cos\theta\, p}}{(\zeta - k\cos\theta)K_-(\zeta)} - S_-(\zeta)$$

$$= -e^{-i\zeta p}\, K_+(\zeta)\, F_1(\zeta) - R_+(\zeta) + S_+(\zeta) \qquad (34)$$

where $R_+(\zeta) + R_-(\zeta) = e^{i\zeta(q-p)}\, F_+(\zeta)\,/\,K_-(\zeta)$.

$$S_+(\zeta) + S_-(\zeta) = A(2\pi)^{-1/2} e^{i\zeta(q-p)-i k \cos\theta\, q} / \{(\zeta - k\cos\theta)K_-(\zeta)\} .$$

The left hand side of (33) and the right hand side of (34) are regular in $t > -k_2$. The other sides are regular in $t < k_2 \cos\theta$. Assume that behaviours at infinity are such that Liouville's Theorem can be applied in the usual way to prove that each side of each equation equals zero.

We introduce the notation

$$F_+(\zeta) - (2\pi)^{-1/2} Ae^{-i k \cos\theta\, q} (\zeta - k\cos\theta)^{-1} = H_+^*(\zeta) \qquad (35,a)$$

$$F_-(\zeta) + (2\pi)^{-1/2} Ae^{-i k \cos\theta\, p} (\zeta - k\cos\theta)^{-1} = H_-(\zeta) \qquad (35,b)$$

where $H_+^*(\zeta)$ has a pole at $\zeta = k\cos\theta$ but otherwise regular in $t > -k_2$. H_- is regular in $t < k_2 \cos$. Equating the left hand side of (33) and (34) to zero and using the general decomposition theorem [cf. NOBLE (1958, p. 184] , we obtain after simplification

$$\frac{H_+^*(\zeta)}{K_+(\zeta)} + \frac{1}{2\pi i}\int_{ic-\infty}^{ic+\infty} \frac{e^{i\tau(p-q)} H_-(\tau)}{(\tau-\zeta)K_+(\tau)}\, d\tau +$$

$$+ \frac{A}{\sqrt{2\pi}} \frac{e^{-i k\cos\theta\, q}}{(\zeta - k\cos\theta)K_+(k\cos\theta)} = 0 \text{ , and}$$

$$\frac{H_-(\zeta)}{K_-(\zeta)} - \frac{1}{2\pi i}\int_{id-\infty}^{id+\infty} \frac{e^{i\tau(q-p)} H_+^*(\tau)}{(\tau-\zeta)K_-(\tau)}\, d\tau = 0 .$$

In these equations $-k_2 < d < k_2 \cos\theta$, $-k_2 < c < k_2 \cos\theta$. Assuming $0 < \theta < \pi/2$ and taking $d = -c = a$ and taking

$S_+^*(\rho) = H_+^*(\rho) + H_-(-\rho)$ and $D_+^*(\rho) = H_+^*(\rho) - H_-(-\rho)$,

we obtain from the last two equations after some manipulation

$$\frac{S_+^*(\rho)}{K_+(\rho)} - \frac{1}{2\pi i}\int_{ia-\infty}^{ia+\infty} \frac{e^{i\tau(q-p)}\, S_+^*(\tau)}{(\tau+\rho)K_-(\tau)}\, d\tau +$$

$$+ \frac{A}{\sqrt{2\pi}} \frac{e^{-i k \cos\theta\, q}}{(\rho - k\cos\theta)K_+(k\cos\theta)} = 0 \text{ , and}$$

$$\frac{D_+^*(\rho)}{K_+(\rho)} + \frac{1}{2\pi i}\int_{ia-\infty}^{ia+\infty} \frac{e^{i\tau(q-p)}\, D_+^*(\tau)}{(\tau+\rho)K_-(\tau)}\, d\tau +$$

$$+ \frac{A}{\sqrt{2\pi}} \frac{e^{-i k \cos\theta\, q}}{(\rho - k\cos\theta)K_+(k\cos\theta)} = 0.$$

These two equations are of the same type and approximate solution can be determined by the method due to JONES (1952).

A number of problems involving the diffraction of elastic waves by finite cracks or scatterers of finite size, the dimension of which are large compared to the wave length of incident wave have been treated by various authors applying Wiener-Hopf technique.

SHIH-JUNG CHANG (1971) considered the interaction by finite closed crack in an elastic medium of infinite extent when a plane dilatational harmonic wave is incident on a crack. High frequency solution is derived with the help of Wiener-Hopf technique.

WICKHAM (1980) considered the short wave radiation from rigid strip which is forced to perform rectilinear oscillation normal to and in smooth contact with a semi infinite a isotropic elastic solid. The mixed boundary value problem is reduced to Fredholm integral equation by the use of Wiener-Hopf technique.

Small time Reissner-Sagoci problem in a bimaterial elastic half space under an impulsive twist was reconsidered by GEORGE (1983). The problem is reduced to an integral equation. By the use of asymptotic analysis and application of Wiener-Hopf technique the equation is converted to Fredholm integral equation of second kind.

We now present in short another method of solving scattering problems for high frequencies viz. elastodynamic ray theory which can successfully be applied to obtain relatively simple approximations to diffracted fields of elastic waves in presence of cracks or strips of finite width in an elastic medium. Geometrical elastodynamics, Geometrical diffraction theory and uniform asymptotic theory together constitute the elastodynamic ray theory. Elastodynamic ray theory were studied in great details by KARAL and KELLER (1959). The application of ray theory to diffraction by smooth obstacles has also been investigated in some detail by RESENDE (1963).

In analogy with geometrical optics, the simplest theory for diffraction of elastic waves by cracks may be called geometrical elastodynamics (GE). In GE a crack or a strip acts as a screen, which creates a shadow zone of no motion, and zones of reflected waves. The shadow zone is bounded by all rays passing through the source point and the edge of the crack. The geometrical reflections of these rays bound the zone of reflected rays. The displacement field according to GE is of the same order of magnitude as the incident field. The GE field is however physically unrealistic, because of the discontinuities in displacement at the boundaries of the shadow gone and the zone of reflected waves.

A first correction to GE is supplied by the geometrical theory of diffraction (GTD). This correction is valid for $\omega a/C_L >> 1$ and at points $s/a > 1$ where ω is the circular frequency, a is a length dimension of the crack, C_L is the velocity of longitudinal wave and s is the distance from a crack edge. The correction provided by GTD is of the order $(\omega a/C_L)^{-1/2}$.

Basic to GTD is the fact that the incident body wave when falls on the edge of a crack gives r is e to two form of diffracted L-rays (longitudinal) and T-rays (transverse) as well as a set of R-rays (Rayleigh waves) along the crack faces. The primary diffracted rays are fans of L- and T- rays which are directly generated by an incident ray. For plane longitudinal and transverse waves, which are under arbitrary angles of incidence with a traction free semi infinite crack in an unbounded body, the displacement field due to diffracted body wave rays have been determined by ACHENBACH, GAUTSEN (1976) by asymptotic considerations. The corresponding surface wave rays have been studied by GAUTSEN, ACHENBACH and MCMAKEN (1978). When an R-ray intersects the edge of a crack, ray of reflected surface wave as well as cones of diffracted body wave rays are generated. For a plane incident surface wave, the reflection coefficients have been computed and also the cones of diffracted L-wave and T-wave have been analyzed in detail by ACHENBACH, GAUTSEN and MCMAKEN (1978).

These plane wave results in presence of a traction free semi-infinite crack in an unbounded medium are canonical solutions. In geometrical diffraction theory these canonical solutions are appropriately adjusted to account for curvature of incident wave fronts and curvature of crack edges and for finite dimensions

of the crack as discussed by GAUTSEN etal (1978).
Within the context of the GTD theory, the diffracted field at a point observation is comprised of contributions corresponding to '' primary'' diffracted body wave ray, which are directly generated by incident body wave rays, and constributions corresponding to 'secondary' diffracted body wave rays. The latter are generated by surface wave rays travelling along the crack faces. With GE and GTD, the total displacement field is of the form

$$u^t = u^g + u^d$$

where u^g is the field due to geometrical elastodynamics and u^d is the field due to geometrical theory of diffraction.
The result is still not valid at the boundaries of the shadow zone and at the boundaries of the zones of reflected waves. In a further refinement which is called uniform asymptotic theory (UAT), the fields at these boundaries are corrected. Uniform asymptotic theory in case of acoustic edge diffraction has been explained in details by LEWIS and BOERSMA (1969).
A three dimensional ray tracing algorithm is used by LANGSTON and JIA-JULEE (1983) to compute the high frequency response of an SH plane wave incident under several models of the sediment with Duamish River Valley.
Based on ray method expansion, asymptotic method is developed by SHEN (1983) for the solution of linear equations governing compressible viscous flow with free surface.
With this much of discussion on the various methods that are generally found to be useful in dealing with the mixed boundary value problems, we briefly discuss the two problems that are taken up in the first chapter.

In the first problem we have considered the rocking motion of a rigid strip on a semi-infinite elastic medium having a frictionless contact with the medium. A time harmonic displacement distribution $v_0 e^{-iwt}$ normal to the strip is prescribed where as the stress out side the strip is zero on the free surface. The mixed boundary problem is reduced to a set of dual integral equations, which is then solved by TRANTER's (1962) technique for low frequency oscillation.

In the second paper we have discussed the response of a semi-infinite elastic solid to a rotatory vibration of indentor over a circular area about a diameter. By the use of Hankel transform the solution of the problem is reduced to the solution of a pair of dual integral equation which is then solved by TRANTER's method.

The normal stress below the disc, total torque and the displacement on the free surface have been determined.

From our experience it appears that though TRANTER's method is no less powerful than the other existing methods for solving dual integral equation involving the solution of mixed boundary value problem, it has not much application in the literature.

Next we would discuss some other methods which have wide application in elastodynamic problems. One such is CAGNIARD's method (1939) which is a powerful technique and enables one to find the solution of the problems of seismic pulses or the wave propagation in an elastic medium. Two media in contact may also be dealt with when the source is in one of these media.

According to DIX (1954), Cagniard's method is not to use the standard Laplace transform inversion formula, but to use a series of transformations to beat the expression for the Laplace transform into the explicit Laplace transform integral, thus enabling one to obtain the derived solution directly out of this integral expression. An advantage of Cagniard's method over the other, is that it permits exact numerical computation of examples, whereas alternate approaches usually give approximations which are good only at large distances from the source.

As an illustration of Cagniard's method, we consider the problem of DIX (1954). It is assumed that there is a source function in a spherical cavity in an infinite medium given by

$$\varphi = \frac{1}{R} H[t - \frac{R}{\alpha}] \qquad (36)$$

for P-waves where $R = (r^2 + z^2)^{1/2}$ and H is Heaviside unit function: $H(\tau) = 0$ for $\tau \leq 0$ and $H(\tau) = 1, \tau > 0$.

If we assume the variations only with the radius r and z, the equation for φ in cylindrical coordinates is :

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial z^2} = \frac{1}{\alpha^2}\frac{\partial^2\varphi}{\partial t^2} \qquad (37)$$

where α is the P-wave velocity.

Taking the Laplace transform of (36), substituting $t-R/\alpha = \tau$ and noting that $H(\tau) = 0$, for $\tau < 0$, we have from Laplace transform of (36) and (37)

$$\frac{\bar{\varphi}}{p} = \frac{1}{pR} e^{-pR/\alpha} = \int_0^\infty G(\lambda) J_0(\lambda r)\, e^{-(\lambda^2+p^2/\alpha^2)^{1/2} z}\, d\lambda \qquad (38)$$

where p in $\bar{\phi}/p$ is included in $G(\lambda)$, and z is chosen positive. Equation (38) holds if

$$G(\lambda) = \lambda/\left[p\left(\lambda^2 + \frac{p^2}{\alpha^2} \right)^{1/2} \right] \tag{39}$$

From (36) and (39) using the substitution : $\lambda = pu$ and $1/\alpha = s$, we have to prove that

$$\frac{e^{-pR/\alpha}}{pR} = \int_0^\infty \frac{u\, J_0(pur)\, \exp[-p(u^2+s^2)^{1/2} z]}{(u^2+s^2)^{1/2}}\, du$$

$$= \int_0^\infty e^{-pt} A(r, z, t)\, dt \tag{40}$$

and that A will give the unit step function given by (36). We use the integral expression for $J_0(pur)$ and then change the order of integration to obtain (40) in the following form

$$\frac{2}{\pi} \int_0^{\pi/2} \mathrm{Re}\left[\int_0^\infty \exp[-\, p(iur \cos \theta + az)]\ \frac{u}{a}\, du \right] dw \quad . \tag{41}$$

where $a = (u^2 + s^2)^{1/2}$.

Changing the variable u in (41) by substituting

$$t' = iur \cos w + (u^2 + s^2)^{1/2} z\ , \tag{42}$$

one obtains (41) as

$$\frac{2}{\pi} \mathrm{Re} \int_0^{\pi/2} dw \int_{H_w} e^{-pt'}\ \frac{u}{a}\ \frac{\partial u}{\partial t'}\ dt' . \tag{43}$$

From the contour shown in fig (1), we obtain

$$\int_{H_w} = \int_{zs}^\infty$$

and (43) becomes

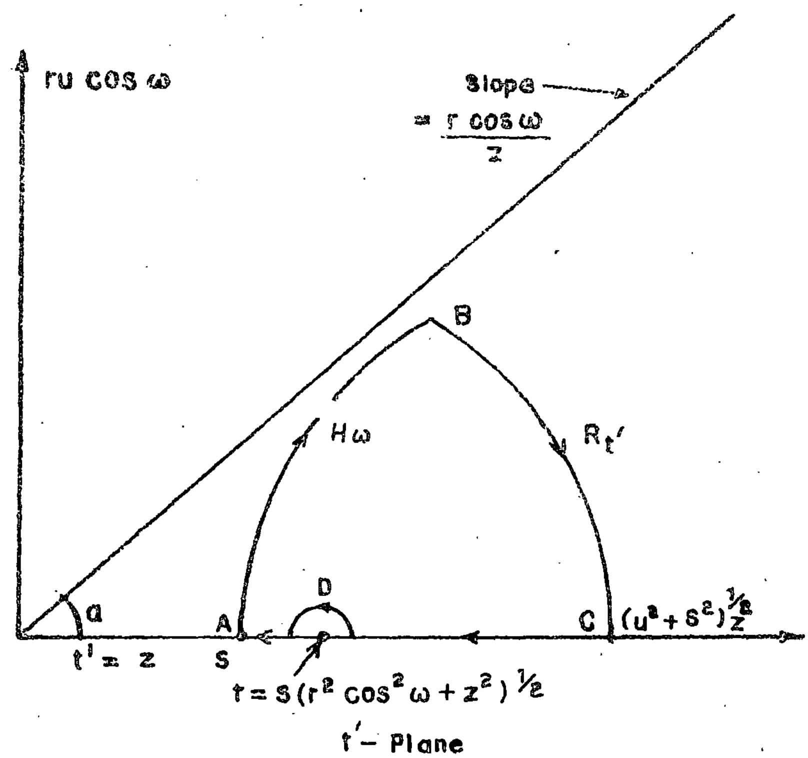

ru cos ω
Slope
= r cos ω / z
B
Hω
$R_{t'}$
D
a
A
C
$(u^2+s^2)^{1/2}_{z}$
t' = z
s
t = s(r² cos² ω + z²) ½
t' – Plane

Fig. 1.

$$\frac{2}{\pi} \mathrm{Re} \int_0^{\pi/2} dW \int_{zs}^{\infty} e^{-pt'} \frac{\partial u}{\partial t'} dt'. \qquad (44)$$

Rearranging the order of (44) , we write (44) as

$$\int_{zs}^{\infty} e^{-pt'} \left[\frac{2}{\pi} \mathrm{Re} \int_0^{\pi/2} \frac{u}{a} \frac{\partial u}{\partial t'} dW \right] dt'. \qquad (45)$$

Comparing (45) with (40), we define A as

$$A(r,z,t') = 0 \qquad \text{for } t' < zs$$

$$= \frac{2}{\pi} \mathrm{Re} \int_0^{\pi/2} \frac{u}{a} \frac{\partial u}{\partial t'} dW \quad \text{for } t' > zs; \qquad (46)$$

and we can say that we have solved our problem, because such an A satisfied the equation (40). Equation (46) is the first form of our solution. We use the substitution (42) to replace the variable w by the variable u and keep t' constant. Then the integral (46) becomes

$$A_{t'>zs} = \frac{2}{\pi} \mathrm{Re} \int_{c_{t'}} \frac{u}{a} \frac{\partial u}{\partial t'} \frac{\partial w}{\partial u} du$$

$$= \frac{2}{\pi} \mathrm{Im} \int_{c_{t'}} \frac{u\, du}{a[u^2r^2+(t'-az)^2]^{1/2}} \qquad (47)$$

where $c_{t'}$ is the corresponding integration path in u-plane. Equation (47) [cf. Mathematical aspects of seismology, Elsevier Publishing Co, New - York (1968). Markus Bath, pp.271-272] may now be written as

$$A_{t'>zs} = \frac{1}{i\pi} \int_{c_{t'}+c'_{t'}} \frac{u du}{a[u^2r^2+(t'-az)^2]^{1/2}} \qquad (48)$$

where $c'_{t'} = \bar{c}_{t'}$, i.e the integral is taken along the conjugate path but in reverse order (see fig.2).

To evaluate the integral (48), this is to be noted that the integrand in (48) has branch points in the u-plane. The four points c, c', Q and Q' which are branch points of the integrand and the branch cuts are shown in the figures 2 and 3. Therefore

$$\int_{c_t + c_{t'}} = \frac{1}{2} \int_{c_t + c_{t'} + D' + D}$$

As there is no pole within the contour in fig.3, so we get,

$$\int_{cc'} = \int_{c_t, c'_t, D'D} = \int_{\Gamma}$$

and $A(r,z,t)_{t'>zs}$

$$= \frac{1}{2\pi i} \lim_{\Gamma \to \infty} \int_{\Gamma} \frac{u\,du}{a\left[u^2(r^2+z^2)z^2s^2+t'^2-2t'az\right]^{1/2}} \tag{49}$$

Putting $u = R_1 e^{i\theta}$, for $0 \leq \theta \leq 2\pi$ and letting $R_1 \to \infty$, the integral is evaluated and from (49) we have

$$A(r,z,t')_{t'>zs} = \frac{1}{(r^2+z^2)^{1/2}} = \frac{1}{R} \tag{50}$$

This is what we are supposed to prove i.e A is the unit step solution of our problem for $t' > zs$, but we must prove that A is the unit step solution for $t' > Rs$ i.e. we must show that for $zs < t' < RS$, $A = 0$. We can not have any pulse before the time $t' = RS = R/\alpha$, which is the time of arrival of the pulse at a distance R from the source.

To prove that A = 0, for zs < t' < Rs we put:

$$t' = s\,(r^2c^2 + z^2)^{1/2} \quad (\, 0 < c < 1\,) \tag{51}$$

Then from (42)

$$t' = s(r^2c^2+z^2)^{1/2} = iru\cos w + z(u^2+s^2)^{1/2} \tag{52}$$

Therefore

$$u = \frac{r\,s\,z(c^2-\cos^2 w)^{1/2}}{r^2\cos^2 w + z^2} - i\,\frac{r\,s\cos w\,(r^2c^2+z^2)^{1/2}}{r^2\cos^2 w + z^2} \tag{53}$$

$c_{t'}$ starts at w = o. For this w

$$u_{w=o} = -i\,\frac{rs}{R^2}\left[(r^2c^2+z^2)^{1/2} - z(1-c^2)^{1/2}\right] \tag{54}$$

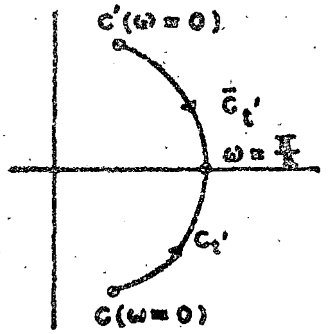
The path $c_{t'}$ runs from E down the imaginary u-axis to cos w=c and after that upto B. At B we have w=π/2 and from (53), u=rcs/z. There is no pole inside the contour shown in fig.5 and we have by Cauchy's integral theorem

$$\int_{c_{t'}} + \int_{c'_{t'}} = \int_{EE'}$$

and

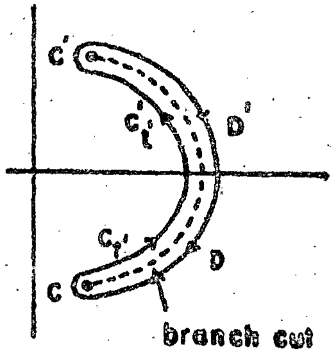
$$\int_{EE'} \frac{u\,du}{i\pi(u^2+s^2)^{1/2}\left[u^2r^2+\left\{t'-z(u^2+s^2)^{1/2}\right\}^2\right]^{1/2}} = 0$$

This proves that A = 0, for t' < Rs. Therefore we can replace zs by Rs in equation (48). This method was applied by GHOSH (1964) who considered a torsional radiator in the form of circular disc of finite radius attached to the surface of a semi-infinite isotropic medium. A twisting moment MS(t) is applied to the disk. By applying CAGNIARD's (1939) method an exact evaluation



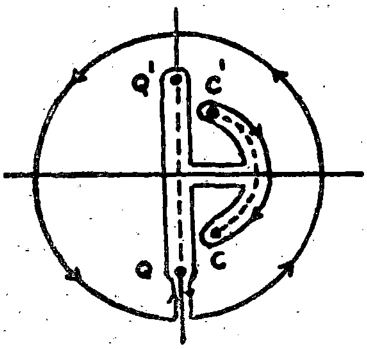
u - plane

Fig. 2.



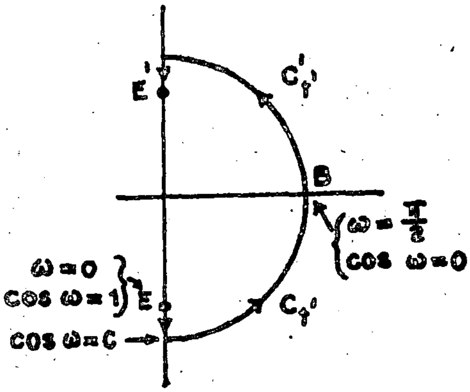
u - plane

Fig. 3.



u - plane

Fig. 4.



u - Plane

Fig. 5.

of the displacement at any point in the medium was made.

EASON and WILSON (1969) considered the displacement produced by a torsional body force situated within an elastic half-space which is bounded to a half-space of different material properties. Using CAGNIARD's transformation, displacement at points of the surface due to impulsive body force acting on the circular region in the interface was worked out in details.

GRAVIN (1956) first applied CAGNIARD's technique in two dimensional case using cartesian coordinates with some modification.

A line source is assumed to be situated at a depth (0,h) below the free surface of an elastic half space. The medium is disturbed by the emission of axially symmetric pulse from the source. The resulting disturbance at any point of the medium is determined on the medium as a function of time.

The displacement components s_x and s_y along the direction of the coordinate axes are obtained in the following form

$$\bar{s}_x(x,o) = -2ip^2\,\bar{f}(p)\int_{-\infty}^{\infty} F_x(k)\exp[-h\eta_p - ikx]\,dk \qquad (55)$$

and $$\bar{s}_y(x,o) = 2\bar{s}_{oy} + 2\bar{f}(p)\int_{-\infty}^{\infty} F_y(k)\exp[-h\eta_p - ikx]\,dk \qquad (56)$$

where bar denotes the Laplace transform to suppress the time variable t, p is the Laplace transform parameter, $\bar{f}(p)$ is the transform of a time function to be adjusted, $\eta_p = \sqrt{k^2+(p^2/v_p^2)}$; V_p is the P-wave velocity and $\bar{s}_{oy}$ is the transform of the y-displacement that would result at (x,o) in an infinite medium. $F_x(k)$ and $F_y(k)$ are respectively odd and even function of k.

Changing the variable of integration by the substitution

$u = (k\,v_p)/p$, one obtains

$$\bar{s}_{x,y}(x,o) = (o, 2\,\bar{s}_{oy}) \mp \frac{4\nu}{V_p} p\bar{f}(p)\,(o, \nu)\ X$$

$$X\ (I_m, R_e) \int_0^\infty G_{x,y}(\nu, u)\ \exp\left\{-p\left[\frac{h}{V_p}\,(u^2+1) + \frac{iux}{V_p}\right]\right\} du \qquad (57)$$

where $\nu = \sqrt{\left[\mu/(\lambda+2\mu)\right]}$, and μ are Lame's constants. A new integration variable is now defined by

$$t = \frac{h}{V_p}\,(u^2+1)^{1/2} + \frac{iux}{V_p}\ ;\ 0 \leq u \leq \infty \qquad (58)$$

This represents a conformal transformation from the u-plane to the t-plane which changes the path of integration and the positions of the singularities. The path of integration in the u-plane along the real axis from zero to infinity is made equivalent to a curve of integration passing through the origin in the t-plane, which by use of Cauchy's theorem and Jordan's lemma is finally reduced to an integral along the real t-axis from h/V_p to ∞. Thus it becomes possible to find Laplace inversion by inspection.

MITRA (1959,a) extended GARVIN's results to the case in which the source is distributed over an area. He (1960) applied GARVIN's method to find the surface displacement due to a time source when the body force is of the form

$$X = H(t)\ \frac{\partial}{\partial x}\left[\,\delta(x)\ \delta(y-h)\right],$$

and

$$= H(t)\ \frac{\partial}{\partial y}\left[\,\delta(x)\ \delta(y-h)\right].$$

MITRA (1964), using the modified CAGNIARD's method studied the uniform impulsive pressure acting over a circular portion of the surface of an elastic half space on the assumption that the surface traction on $z = o$ is

$$\tau_{zz} = P\,\delta(t)\;; \quad 0 \le r \le a$$
$$= 0 \quad ; \quad r > a$$
$$\tau_{zr} = 0$$

Another modification of CAGNIARD's method was developed by DE-HOOP (1959). The integration variable in (55) and (56) are changed by the substitution $u = (k V_P)\,/\,p$ like GARVIN. Then again a new integration variable is introduced by the substitution

$t = \frac{h}{V_P}(u^2+1)^{1/2} + i\,\frac{ux}{V_P}$ as in (58), but in this case it is assumed that t is positive and real instead of conformal mapping from u-plane to t-plane as assumed by GARVIN. As a result the path of integration with respect to u which is from $-\infty$ to ∞ along the real axis is deformed to the branch of a hyperbola whose equation is

$$u = \frac{-itx \pm h\left(t^2 - \frac{x^2+z^2}{V_P^2}\right)^{1/2}, \left(\frac{x^2+z^2}{V_P^2} < t < \infty\right)}{\frac{x^2+z^2}{V_P}}\,;$$

Hence the integration along the real axis in the u-plane may be replaced by the branch of the hyperbolic path. Consequently Laplace inversion can be obtained by inspection.

This modified method of CAGNIARD is found to be more convenient than that of GRAVIN and in recent time this method is widely used in different problems to find Laplace inversion.

GHOSH (1971) applied CAGNIARD's method as modified by DE-HOOP, to obtain displacement in the integral form due to a sudden creation of normal stress discontinuity over a circular area expanding uniformly after creation. ROY (1981) used this technique to find the displacement field due to a transient response of an elastic half-space subject to a uniform normal pressure acting over an elliptic area. MITTAL and SIDHU (1982) using DE-HOOP's version of CAGNIARD's method evaluated surface displacement due to SH-type of waves. PAL (1983) applied modified CAGNIARD's method to find the exact solution of displacement function due to the generation of SH-waves due to a stress discontinuity moving with nonuniform velocity.

Another type of problems that has to be encountered to study the dynamic behavior of an elastic solid is the response of an elastic solid to moving loads. The moving load problems which have been studied may be put into three categories:

i) steady wave motion due to a load moving with constant velocity for all time to come,

ii) transient wave motion due to a load which begins to act at certain instant and then moves with constant velocity, and

iii) transient wave motion due to a load which begins to act at certain instant and then moves in some direction with nonuniform speed.

The steady motion of a line load on the surface of an elastic half-space studied by SNEDDON [1951, cf. page-447-449], COLE and HUTH (1958) and GHOSH and GHOSH (1978) are the typical examples of the first type of problems. The transient problem of a line load, which suddenly appears on the surface and then moves with constant velocity studied by ANG (1960), is of type (ii). As a representative of the third kind of problem we refer to the study of FREUND (1972).

An analytic technique was developed by FREUND (1972) which made it possible to obtain an exact solution of a particular problem in category (iii).

For introducing the technique the author considered a line load in an unbounded elastic solid moving with nonuniform speed in a particular direction. Carterian coordinate system was used. At any time t = 0 , a line load begins to act along y-axis and the line load moves along x-direction for t > o. For any time t > o, x-coordinate of the load in given l(t). It is assumed that the function l(t) is continuous, monotone increasing function of time t and that it never acts at a single point for a finite length of time. Under the conditions mentioned the function l(t) is invertible, that is there exists a function n(x) which is the time at which the load acts at x. The functions l(t) and n(x) satisfy the following relations identically

$$l[n(x)] = t; \quad n[l(t)] = x \tag{59}$$

$$\dot{l}[n(x)]n'(x) = 1; \quad n'[l(t)]\dot{l}(t) = 1 \tag{60}$$

where dot denotes time derivative and dash denotes x-derivative. Because of the symmetry with respect to the plane z = o, the problem may be looked upon as boundary value problem for the half space z > o, with mixed boundary condition on z = o. If ϕ and ψ are dilatational and rotational displacement potentials then equations of motion are formulated as

$$\begin{aligned} \varphi_{xx} + \varphi_{zz} - a^2 \varphi_{tt} &= o \\ \psi_{xx} + \psi_{zz} - b^2 \psi_{tt} &= o \end{aligned} \tag{61}$$

where a, b are the dilataional and shear wave slowness. In case of normal loading the boundary conditions to be satisfied by the solution of (61) are

$$\sigma_{zz}(x,o,t) = -\frac{1}{2}\,\delta[x-l(t)];\ u(x,o,t) = 0 \qquad (62)$$

σ_{ij} is the stress component and u is the displacement component in x-direction.

The solution of the problem is obtained by making use of Laplace transform method. The time variable is first eleminated by application of the transform

$$\hat{\varphi}(x,z,s) = \int_0^\infty \varphi\,(x,z,t)e^{-st}\,dt. \qquad (63)$$

and next the dependence on x is avoided by taking the transform

$$\Phi(\lambda,z,s) = \int_{-\infty}^{\infty} \hat{\varphi}\,(x,z,s)e^{-s\,x}\,dx. \qquad (64)$$

Applying the transforms on the boundary conditions and keeping in mind the physical condition i.e. the solution of the transformed differential equation should remain bounded as $z \to \infty$, one obtains

$$\Phi(\lambda,z,s) = \frac{1}{\rho s^2}\,A(\lambda,s)\,\exp(-s\alpha z) \qquad (65)$$

$$\Psi(\lambda,z,s) = -(\lambda/\rho s^2\beta)A(\lambda,s)\exp(-s\beta z) \qquad (66)$$

where   is the material density and

$$\alpha = (a^2-\lambda^2)^{1/2},\ \beta = (b^2-\lambda^2)^{1/2}. \qquad (67)$$

The amplitude $A(\lambda,s)$ in (65) and (66) is the double transform of the boundary condition and is derived by making use of the relationship [cf. VANDERPOL and BREMMER (1964), p. 79].

$$\delta[x-l(t)] = n'(x)\,\delta[t-n(x)]. \qquad (68)$$

In view of (68)

$$\hat{\sigma}_{zz}(x,o,s) = \frac{1}{2}\,n'(x)\exp\,(-sn(x)\;H[n(x)]$$

and $n(o) = o$, $H[n(x)] = H[x]$. Then, applying the two sided Laplace transform,

$$\Sigma_{zz}(\lambda,o,s) = A(\lambda,s) = -\frac{1}{2}\int_{o}^{\infty} n'(x)e^{-s\lambda x}\,e^{-sn(x)}\,dx. \qquad (69)$$

The transformed solution is thus completely determined, and the potential function φ may be written as the double inversion integral $\varphi(x,z,t) =$

$$= -\frac{1}{4\pi i}\int_{B_1}\frac{1}{2\pi i}\int_{B_2}\frac{1}{s}\int_{o}^{\infty} n'(\xi)e^{s(\lambda x-\lambda\xi-\alpha z-n(\xi)+t)}\,d\xi\,d\lambda\,ds \qquad (70)$$

where B_1 and B_2 are the usual inversion paths for one sided and two sided Laplace transforms. The double integral in (70) is inverted by means of DE-HOOP's (1959) technique.

In a subsequent study of nonuniformly moving line load or a pressure step on a surface of an elastic solid as well as nonuniformly moving dislocation the above method, as shown by FREUND (1973) can be applied.

We now briefly describe the nature of the problems taken up in the second chapter and a problem of third chapter.

The second chapter of the thesis is concerned with problems of eleastic waves due to sources in the form of a ring on the surface of an isotropic elastic half space.

The first problem considered is the response of an elastic half-space to a ring source. The radius of the ring is assumed to increase with a constant velocity c less than that of shear wave velocity. A twisting impulse is prescribed in the form of $P\delta(r-ct)H(t)$. By using Laplace transform, Hankel transform and De-Hoop's version of Cagniard's method, the displacement is determined at any point of the medium in integral form. Exact evalution of displacement just after the arrival of the disturbance and displacement at any point after a sufficiently large time have also been determined.

The second problem is to study the motion produced in an isotropic elastic half-space due to impulsive torsional motion of a circular ring source located on a free surface of a homogeneous as well as in inhomogeneous medium.

The torsional motion is prescribed by $P\delta(r-a)\delta(t)$, a being the radius of the ring. In case of inhomogeneous medium inhomogeneity is prescribed by $\mu = \mu_0(1+\epsilon z)^2$ and $\rho = \rho_0(1+\epsilon z)^2$, where ρ is the material density. The method of approach is the same as that of the previous one. Graphs have been plotted for displacement on free surface as a function of time and the variation in displacement due to the presence of inhomogeneity has also been shown.

In the last problem of the second chapter, exact expressions for displacement in a homogeneous isotropic elastic half-space subjected to an impulsive torsional force over the rim of a nonuniformly expanding ring source on the free surface is obtained by CAGNIARD, DE-HOOP technique. Different wave front surface with their region of existence have been shown. The first motion responses near different wave arrivals have been obtained by a limiting process. The displacement on the free surface as a function of position of the source have been shown by means of graphs.

In the first problems of the last chapter, we have considered a concentrated line load which originates at time $t = 0$ and then moves with uniform velocity along the boundary of an isotropic inhomogeneous medium. It is assumed that the elastic parameters λ and μ and the density ρ of the medium vary according to the law

$$\lambda = \mu = \mu_0(1+\epsilon z)^2 \text{ and } \rho = \rho_0(1+\epsilon z)^2 .$$

DE-HOOP's version of CAGNIARD's method is used to find the displacement components in the integral form. An approximate evaluation of the integrals is worked out near the first arrival of the wave fronts. It is perhaps the first application of Cagniard's method in solving problems in inhomogeneous media.

We now discuss in short the necessity of studying models of the source of earthquake in elasto-dynamics though these may differ from actual phenomena.

The behavior of seismic wave propagation due to the presence of active tectonic belts, fracture or faulting, all come under the purview of elastodynamics. There has been an increasing interest in theoretical study in elastic wave motion. The relation between the earth's deeper structure and some geological formations is of great importance in revealing the genesis of mineral deposits. The rapid increase in the volume and rate of construction-work in seismically active zones makes moreurgent the need for earthquake resistant structures, high dams etc. Most important for the same purpose is the study of earthquake focal parameters and the condition of seismic wave propagation.

The faulting process is, in general, a fracture phenomena. The mechanical energy released at the fault surface is carried to the side by elastic waves propagating through the earth material contained between the source and the site. Each of these phenomena falls within the area of research in continuum mechanics and the latter in elasto dynamics. In addition, the problem of calculation of the response of building foundations to incoming earthquake waves may be regarded as an elastodynamic diffraction problem.

Seismological evidence suggests that earthquakes occur by sudden slippage of earth material across the fault surface, which may be looked upon as pre-existing weak zones of relatively small thickness. Hence if the displacement discontinuity is known across the fault surface,(in fact now it is possible to determine displacement discontinuity after the equilibrium position is restored in the surface after some disturbance) the displacement field in a considerably large area round the fault surface can be determined by the application of representation theorem of elastodynamics.

The displacement produced at any point on a free surface of an elastic half space due to displacement discontinuity across a fault surface can be calculated with the help of Green's function and the representation theorem of elasto-dynamics. The Green's function $G_{ij}(\vec{x}'|\vec{x})$ is cartesian component of displacement in x_j direction produced at a point $\vec{x}'$ of a half space due to an application of a time harmonic force of unit magnitude in x_i-direction at a point $\vec{x}$ on the free surface of a half space. It can be shown that the Fourier transform with respect to time of the displacement at the surface of the half space $x_3 = o$ due to a prescribed slip on a fault surface S may be expressed in the form [cf. MAL (1972), equation 8]

$$U_m(\vec{x}) = \int_S \left[U_i(\vec{x}')\right]_-^+ T_{ij}^m n_j \, dS$$

where $\left[U_i(\vec{x}')\right]_-^+$ is Fourier transform of the prescribed discontinuity in displacement component U_i across S, n_j is unit normal to S on the positive side and T_{ij}^m is related to G_{ij} through the equation

$$T_{ij}^m(\vec{x}' | \vec{x}) = c_{ijkl} \frac{\partial}{\partial x_l'} G_{mk} (\vec{x}' | \vec{x})$$

The calculation of the surface displacement can be carried out by taking Fourier transform of the slip function. For a given point $\vec{x}$ and frequency is $T_{ij}^m n_j$ has to be calculated at each point of the fault surface and then integrating along the fault s-urface the product of transformed slip function and $T_{ij}^m n_j$, the displacement components can be determined for different values of w.

Further, it follows from radiation pattern of first motion from earthquakes that a shearing motion occurs at the earthquake focus. To illustrate the mechanism which produces such a motion FOSSUM and FREUND (1975) considered a model of a shallow earthquake focus by a plane shear crack extending at a nonuniform rate under the action of general loading. It is assumed that the crack should remain in one plane. This is in conformity with the field observation which indicates that the directions of seismic fault extension is almost always in the plane of preexisting fault.

In the last problem of the third chapter we have considered a model, where it is assumed that a crack is developed suddenly along a horizontal line at a finite depth below the surface of the earth. The crack is assumed to move along a vertical plane upto the free surface with nonuniform velocity. Assuming the motion to be two dimensional, the surface displacement due to Rayleigh waves produced by nonuniformly moving crack has been determined.

In this connection it may be mentioned that recently, SINGH, MODDIE and HADDOW (1981) have considered the problem of finite length crack propagating with constant velocity in an infinitely long finite width strip when anti-plane shear displacements and stresses are applied to the lateral boundaries of the strip. By employing Fourier transform the solution is reduced to the solution of a pair of dual integral equation, the solutions of whi-ch is obtained directly in a closed form by making use of COOKE's (1970) result.

With this much of review work, we present the thesis chapters.

The notations used in different problems are independent of one another.

C H A P T E R I

MIXED BOUNDARY VALUE PROBLEMS

Problem 1 . Harmonic rocking of a rigid strip on a semi-infinite elastic medium.

Problem 2 . Harmonic rocking of a rigid circular indentor on an elastic half-space.

HARMONIC ROCKING OF A RIGID STRIP
ON A SEMI INFINITE ELASTIC MEDIUM

INTRODUCTION: To consider the effect of vibrating source of pressure in different form on the surface of elastic medium is almost classical. Perhaps Lamb is the pioneer on this line. The problem considered here is the rocking motion of a rigid strip of infinite length having a smooth contact with elastic medium. Generally this type of problems may be formulated so as to be governed by a set of dual integral equations. This particular problem considered here was considered by Awojobi and Grootenhuis (1965) and Awojobi (1966). They used heuristic technique of successive approximation to solve the dual integral equation. Karasudhi, Keer and Lee (1968) also considered the problem by reducing the governing dual integral equations into a single inhomogeneous Fredholm integral equation of the second kind and then solved the equation by the method of successive approximation for low frequency oscillation. But their final solution involved definite integral which were later numerically evaluated. Recently the same problem was again taken up by Wickham (1977). With the help of suitable Green's function, the mixed boundary value problem is first reduced to a Fredholm integral equation of the first kind involving dispacement boundary condition. Using Noble's (1962) method this equation has been reduced to a Fredholm integral equation of the second kind with a Kernal which is small in the low frequency limit. By the use of exact iterative solution of the integral equation of the second kind, a simple explicit long-wave asymptotic formula for the normal stress in terms of the prescribed displacement and dimensionless wave number K has been derived rigorously. Unlike Karasudi et al. the solution does not contain any integral which requires numerical evaluation. But the method of solution is a bit cumbersome.

In this paper we have reconsidered the same problem. The solution of the governing dual integral equations representing the mixed boundary value problem, is reduced to the solution of a set of linear algebraic equations following Tranter (1962) for low frequency oscillation. The asymptotic solution of normal stress below the strip is determined interms of prescribed displacement and the wave number K, where terms involving K^4 and its higher orders are neglected. The value of the reacting couple exerted by the elastic solid on the strip has also been evaluated and they are found to be in agreement with that given by Wickham.

The asymptotic solution obtained by this method is exact in the sense that it does not involve any integral requiring numerical evaluation. It appears that Tranter's technique is no less powerful than the other methods and it is less cumbrous.

FORMULATION OF THE PROBLEM: We consider the rocking vibration of frequency w of a rigid strip having a smooth contact with a semi-infinite homogeneous isotropic elastic solid occupying the half space $-\infty < x < \infty$, $y \geqslant o$, $-\infty < z < \infty$. It is assumed that the motion is prescribed by a displacement distribution $v_o e^{-iwt}$ normal to the finfinite strip $|x| \leqslant x_o$, $y = o$, $|z| < \infty$, where $v_o = \phi_o x$, ϕ_o being constant and that the tangential components of stress are zero and the normal stress is zero for

$|x| > x_o$, $y = o$, $|z| < \infty$. Thus it follows that the medium under consideration is in a state of dynamic state of plane strain satisfying the two dimensional equations of motion, given by

$$\nabla^2 \phi = \frac{1}{a^2}\frac{\partial^2 \phi}{\partial t^2}, \quad \nabla^2 \psi = \frac{1}{b^2}\frac{\partial^2 \psi}{\partial t^2} \tag{1}$$

where

$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, a and b are p- and S-wave velocities in the medium. The scalar potentials ϕ and ψ are associated with the displacement components u and v by the relations

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y} \ (2,a) \text{ and } v = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x} \tag{2,b}$$

The stress components are

$$\sigma_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{3,a}$$

$$\sigma_{yy} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y} \tag{3,b}$$

where λ and μ are Lame's constants. The boundary conditions are

$$v(x,o) = v_o e^{-iwt} = \phi_o x e^{-iwt}, \quad |x| \leqslant x_o \text{ on } y = o \tag{4,a}$$

$$\sigma_{yy}(x,o) = o, \quad |x| > x_o \text{ on } y = o, \text{ and} \tag{4,b}$$

$$\sigma_{xy}(x,o) = o \text{ every where on } y = o \tag{4,c}$$

To find the solutions of the equations (1) subject to the conditions (4-a,b,c), we make the substitution

$$\left[\varphi,\psi\right] = \left[\Phi,\Psi\right] e^{-iwt}$$

In view of the boundary conditions (4), it follows that Φ and Ψ may be taken in the form

$$\Phi = \frac{x_o^2}{2\pi} \int_{-\infty}^{\infty} A(\xi) e^{-pY} e^{-i\xi X} d\xi \qquad (5,a)$$

and

$$\Psi = \frac{x_o^2}{2\pi} \int_{-\infty}^{\infty} B(\xi) e^{-sY} e^{-i\xi X} d\xi , \qquad (5,b)$$

where $X = x/x_o$ and $Y = y/x_o$. p and s in equation (5-a,b) are

$p(\xi) = \sqrt{\xi^2 - x_o^2 h^2}, |\xi| > x_o h$ and $s(\xi) = \sqrt{\xi^2 - x_o^2 k^2}, |\xi| > x_o k$,

Where $h = w/a$ and $k = w/b$.

From equation (1) and Φ, Ψ as obtained in (5-a,b), the displacement components and the stress components may be written in the form

$$u = \frac{x_o}{2\pi} \int_{-\infty}^{\infty} \left[-i\xi A(\xi) e^{-pY} + sB(\xi) e^{-sY}\right] e^{-i\xi X} d\xi$$

$$v = -\frac{x_o}{2\pi} \int_{-\infty}^{\infty} \left[pA(\xi) e^{-pY} + i\xi B(\xi) e^{-sY}\right] e^{-i\xi X} d\xi \qquad (6)$$

$$\sigma_{xy} = \frac{\mu}{2\pi} \int_{-\infty}^{\infty} \left[2i\xi\, pA(\xi) e^{-pY} - (\xi^2 + s^2) B(\xi) e^{-sY}\right] e^{-i\xi X} d\xi$$

$$\sigma_{yy} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\left\{-\lambda\xi^2 + (\lambda + 2\mu) p^2\right\} A(\xi) e^{-pY} + 2i\mu\xi s B(\xi) e^{-sY}\right] e^{-i\xi X} d\xi$$

In the expression given in (6) in subsequent analysis the time factor e^{-iwt} has been omitted. From the boundary condition (4-c), it follows that

$$B(\xi) = 2i\xi\, pA(\xi)/\,(\xi^2 + s^2) \tag{7}$$

and the conditions (4-a,b) yield the dual integral equation

$$\int_{-\infty}^{\infty} \frac{x_o^2 k^2\, pA(\xi)}{2\xi^2 - x_o^2 k^2}\, e^{-i\xi X}\, d\xi = 2\pi\phi_o, \; |X| < 1, \tag{8-a}$$

and

$$\int_{-\infty}^{\infty} \frac{R(\xi)\, A(\xi)}{2\xi^2 - x_o^2 k^2}\, e^{-i\xi X}\, d\xi = 0, \; |X| > 1, \tag{8-b}$$

where $R(\xi) = (2\xi^2 - x_o^2 k^2)^2 - 4ps\xi^2$. This is to be noted that $A(\xi)$ is an odd function of ξ, consequently the equations (8-a,b) may be rewritten as

$$\int_{0}^{\infty} \frac{x_o^2 k^2 \sqrt{\xi}\, p(\xi) A(\xi)}{2\xi^2 - x_o^2 k^2}\, J_{\frac{1}{2}}(\xi X)\, d\xi = i\phi_o \sqrt{2\pi X}\,, \; |X| \leqslant 1 \tag{9-a}$$

and

$$\int_{0}^{\infty} \frac{R(\xi)\sqrt{\xi}\, A(\xi)}{2\xi^2 - x_o^2 k^2}\, J_{\frac{1}{2}}(\xi X)\, d\xi = 0\,, \; |X| > 1 \tag{9-b}$$

To facilitate our analysis we write equations(9-a,b) in the form

$$\int_{0}^{\infty} [1+H(\xi)]\, D(\xi)\, \xi^{-1}\, J_{\frac{1}{2}}(\xi X)\, d\xi = i\phi_o \sqrt{2\pi X}\,, |X| \leqslant 1 \tag{10-a}$$

and $\int_0^\infty D(\xi)\, J_{\frac{1}{2}}(\xi X)\, d\xi = 0, \quad |X| > 1$ (10-b)

where $D(\xi) = -(1-\eta)\dfrac{\sqrt{\xi}\, R(\xi) A(\xi)}{2\xi^2 - x_o^2 k^2}$

and $1+H(\xi) = -\dfrac{\xi x_o^2 k^2\, p(\xi)}{(1-\eta) R(\xi)}$,

$\eta = \lambda/2(\lambda+\mu)$ being Poisson's ratio. It should be noted that $H(\xi) \to 0$ as $x_o k \to 0$ and $x_o h \to 0$.

SOLUTION OF THE DUAL INTEGRAL EQUATION: To find the solution of the dual integral equations (10-a,b) following Tranter (1962) we assume

$$D(\xi) = \sqrt{\xi} \sum_{m=0}^{\infty} a_m J_{1+2m}(\xi) \quad (11)$$

So that the equation (10-b) is automatically satisfied. The coefficients a_m are to be so chosen that the form of $D(\xi)$ as assumed in (11) satisfies the equation (10-a). So we must have

$$\sum_{m=0}^{\infty} a_m \int_0^\infty [1+H(\xi)]\, \xi^{-1/2} J_{1+2m}(\xi) J_{\frac{1}{2}}(\xi X)\, d\xi = i\phi_o \sqrt{2\pi X}, \quad |X| \leq 1 \quad (12)$$

Multiplying equation (12) by $X^{3/2}(1-X^2)^{-1/2}\ \mathcal{F}(1, \frac{3}{2}, X^2)$

where n is a positive integer or zero and $\mathcal{F}_n$ is a Jacobi polynomial of degree n and then integrating with respect to X from 0 to 1, one obtains

$$\sum_{m=0}^{\infty} a_m \int_0^{\infty}[1+H(\xi)]\xi^{-1} J_{1+2m}(\xi)J_{1+2n}(\xi)d\xi = E(\tfrac{1}{2},n,\tfrac{1}{2}) = E_n \text{ (say)}, \qquad (13)$$

where $E_n = 4i\phi_o \dfrac{\Gamma(n+\frac{3}{2})}{\Gamma(n+\frac{1}{2})} \displaystyle\int_0^1 x^2(1-x^2)^{-1/2} \mathcal{F}_n(1,\tfrac{3}{2},x^2)dx.$

Using the result that $\mathcal{F}_o(1,\frac{3}{2},x^2) = 1$ we obtain from orthogonality relation of Jacobi polynomial

$$E_n = \tfrac{1}{2}\pi\phi_o \text{ or } 0 \text{ according as } n = o \text{ or } n \neq o. \qquad (14)$$

Since $\displaystyle\int_0^{\infty} \xi^{-1} J_{1+2m}(\xi)J_{1+2n}(\xi)d\xi = o$ for $m \neq n$

$= (2+4n)^{-1}$ for $m = n$,

wo we obtain from (13)

$$a_n + \sum_{m=o}^{\infty} L_{mn}a_m = (2+4n)E_n \qquad (15)$$

where $L_{mn} = (2+4n)\displaystyle\int_0^{\infty} H(\xi)\xi^{-1} J_{1+2m}(\xi)J_{1+2n}(\xi)d\xi \qquad (16)$

Equation (15) gives us an infinite set of algebraic equations for the determination of the coefficients a_m.

Usihg the generalisation of Neumann's integral [cf. Watson(1958)p.150]

$$J_{1+2m}(\xi)J_{1+2n}(\xi) = \frac{2}{\pi}\int_0^{\pi/2} J_{2+2m+2n}(2\xi\cos\theta)\cos 2(m-n)\theta\, d\theta$$

in the equation (16) and changing the order of integration one obtains

$$L_{mn} = \frac{4(1+2n)}{\pi} \int_0^{\pi/2} A_{mn} \cos 2(m-n)\,\theta\, d\,\theta, \qquad (17)$$

where $$A_{mn} = \int_0^{\infty} H(\xi)\,\xi^{-1}\, J_{2+2m+2n}\,(2\xi\cos\theta)\,d\xi$$

$$= -\,I - \frac{1}{2(m+n+1)} \qquad (18)$$

and $$I = \int_0^{\infty} \frac{x_o^2\, k^2\, p(\xi)}{(1-\eta)R(\xi)}\, J_{2+2m+2n}\,(2\xi\cos\theta)\,d\xi \qquad (19)$$

EVALUATION OF THE INFINITE INTEGRAL BY METHOD OF CONTOUR INTEGRATION: To evaluate the integral I, we put $\xi = x_o hx$ and take $k/h = \tau$, then

$$I = \int_0^{\infty} \frac{\tau^2 q_1\,(x)}{(1-\eta)Q\,(x)}\, J_{2+2m+2n}(2\,x_o hx\,\cos\theta)dx, \qquad (20)$$

where $Q(\,x\,) = (2x^2 - \tau^2\,)^2 - 4x^2\, q_1(x)q_2(x)$ and

$$q_1(x) = (x^2 - 1)^{1/2},\quad q_2\,(x) = (x^2 - \tau^2\,)^{1/2}\,.$$

Taking

$$Q_o(x) = (2x^2 - \tau^2\,)^2 + 4x^2\, q_1(x)q_2(x) \text{ and}$$

$\Delta_o(x) = (2x^2 - \tau^2\,)^4 - 16x^4\, q_1^2\,(x)q_2^2(x)$, the integral I may be written in the form

$$I = \int_0^{\infty} \frac{\tau^2\, q_1(x)Q_o(x)}{(1-\eta)\,\Delta_o(x)}\, J_{2+2m+2n}(2x_o\, hx\,\cos\theta)dx. \qquad (21)$$

For our convenience we replace the Bessel function of the first kind by Hankel function given by

$$J_{2+2m+2n}(\ ) = \frac{1}{2}\left[H^{(1)}_{2+2m+2n}(\ ) + H^{(2)}_{2+2m+2n}(\ )\right],$$

where $H^{()}_{\nu}$ denotes Hankel function. Consequently, $I = I_1 + I_2$ where I_1 and I_2 are integrals involving $H^{(1)}_{2+2m+2n}(2x_o\ hx \cos\theta)$ and $H^{(2)}_{2+2m+2n}(2x_o\ hx \cos\theta)$ respectively.

Thus for I_1 and I_2 we consider the integrals

$$J_{1,2} = \frac{1}{2(1-\eta)} \int_{\Gamma_{1,2}} \frac{\tau^2 q_1(z) Q_o(z)}{\Delta_o(z)} H^{(1,2)}_{2+2m+2n}(2x_o\ hz \cos\theta)dz$$

where Γ_1 and Γ_2 are the contours in the first and fourth quadrants of the complex z-plane, as shown in the figure 1. Consistent sign of the double valued functions $q_1(z)$ and $q_2(z)$ are shown in the Fig.1. The branch points (1,0) and $(\tau,0)$, the poles $(\tau_j,0)$ for $j = 0,1,2$ which are the zeroes of the function $\Delta_o(z)$ and the origin at which the Hankel function fails to have finite value are all avoided by semi circular indentations, in order to ensure that the integrands are analytic within and on the contour. It is found that the integrals satisfy Jordon's lemma and therefore the contribution to the integrals from the infinitely distant parts of the contours is zero.

After integration round the contours Γ_1 and Γ_2 and adding we obtain finally,

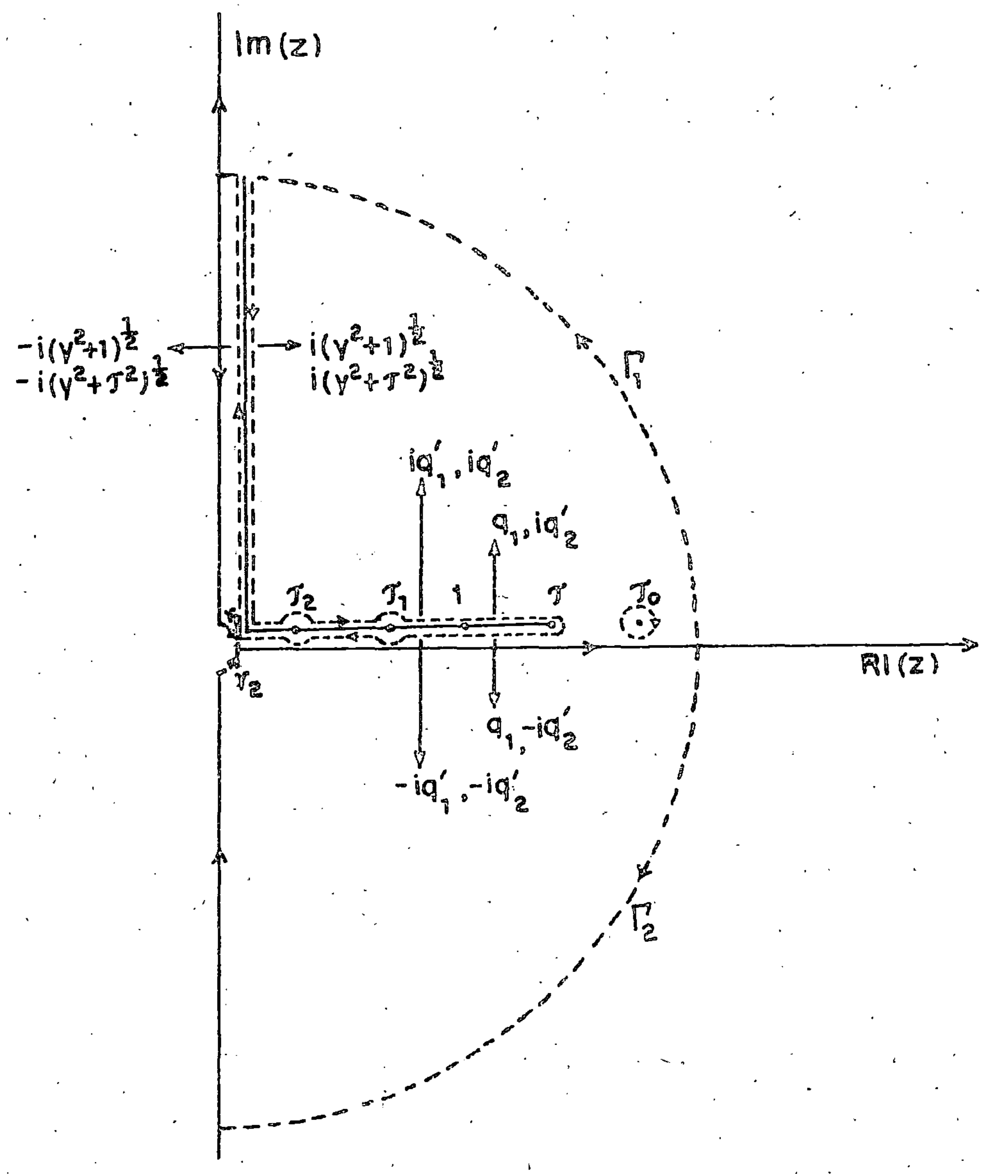


Fig.1- Contours of integration Γ_1 and Γ_2 in the complex z-plane for Poisson's ratio $\eta = 1/4$

$$I = I_1 + I_2$$

$$= \frac{i\pi\tau^2}{1-\eta} D_o H^{(1)}_{2+2m+2n}(2 x_o h\tau_o \cos\theta) - \frac{i\tau^2}{1-\eta}\int_0^1 M_1(x) H^{(1)}_{2+2m+2n}(2x_o hx \cos\theta)dx -$$

$$- \frac{i\tau^2}{1-\eta}\int_0^\tau M_2(x) H^{(1)}_{2+2m+2n}(2 x_o hx \cos\theta)dx -$$

$$- \frac{\tau^2}{2(1-\eta)} \oint_{\gamma_1} \frac{q_1(z)Q_o(z)}{\Delta_o(z)} H^{(1)}_{2+2m+2n}(2 x_o hz \cos\theta)\, dz -$$

$$- \frac{\tau^2}{2(1-\eta)} \oint_{\gamma_2} \frac{q_1(z)Q_o(z)}{\Delta_o(z)} H^{(2)}_{2+2m+2n}(2 x_o hz \cos\theta)\, dz \qquad (22)$$

In (22) notations used are given below

$$D_o = \frac{q_1(\tau_o)(2\tau_o^2 - \tau^2)^2 + 4\tau_o^2\, q_1^2(\tau_o) q_2(\tau_o)}{\Delta_o'(\tau_o)}$$, ($\Delta_o'(x)$ is derivative of $\Delta_o(x)$ with respect to x)

$$M_1(x) = \frac{q_1'(x)(2x^2 - \tau^2)^2}{\Delta_o(x)}, \quad (1 - x^2)^{1/2} = q_1'(x), \text{ (say)}$$

$$M_2(x) = \frac{4x^2\, q_1^2(x)\, q_2'(x)}{\Delta_o(x)}, \quad (\tau^2 - x^2)^{1/2} = q_2'(x), \text{ (say).}$$

In order to simplify (22), we make use of the series expansion for $H^{(1,2)}_{2+2m+2n}(z)$ in the form

$$H^{(1,2)}_{2+2m+2n}(z) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{z}{2}\right)^{2r+2m+2n+2} \Big/ \left[r!\ \Gamma(r+2m+2n+3)\right] \pm$$

$$\pm \frac{i}{\pi}\left[\left\{2 \sum_{r=0}^{\infty} (-1)^r \left(\frac{z}{2}\right)^{2r+2m+2n+2} \Big/ \left[r!\ \Gamma(r+2m+2n+3)\right]\right\} \log\left(\frac{z}{2}\right) -\right.$$

$$- \sum_{r=0}^{2m+2n+1} \left(\frac{z}{2}\right)^{2r-2m-2n+2} \times \frac{(2m+2n+1-r)!}{r!} -$$

$$\left. - \sum_{l=0}^{\infty} (-1)^l \left(\frac{z}{2}\right)^{2m+2n+2+2l} \times \frac{\{\chi(2m+2n+1+3)+\chi(l+1)\}}{l!\ (2m+2n+2+l)!}\right], \quad (23)$$

where $\chi(\ )$ is Euler's function. When $H^{(1,2)}_{2+2m+2n}(\ )$ occurring on the right hand side of (22) are replaced by the corresponding series mentioned above, it is found that the terms involving $(2\, x_o h)^{2r-2m-2n-2}$ for which $(2r-2m-2n-2) < o$ on the right hand side of equation (22) vanish. This can be proved by intergrating the complex function

$$f(z) = \frac{T^2}{2(1-\eta)} (2\, x_o h \cos\theta)^{2r-2m-2n-2} \frac{q_1(z) Q_o(z)}{\Delta_o(z)} z^{2r-2m-2n-2}$$

along the contours Γ_1 and Γ_2 as shown in fig.1. and then subtracting. Again when $2r - 2m - 2n - 2 = o$, origin is not the singularity of the Hankel functions. In this case the contours considered slightly differ from that of the contours shown in fig.1, Here the identation round the origin is not required and it is found after integration of the function $f(z)$ round the modified contours Γ_1 and Γ_2 and subtracting the sum of the terms on right hand side of (22) which do not involve any power of $(2\,x_o h)$ is equal to $^{-1}/2(m+n+1)$. Finally when $(2r - 2m - 2n - 2) > o$, origin is not the singularity of Henkel functions and hence considering the same contour where the indentation at the origin is deleted, it is found that the terms involving $(2x_o h)^{2r-2m-2n-2}$ on the right hand side of (22) is not zero because the contribution to the integrals from the infinitely distent parts of the contours do not vanish in this case. Thus with the help of (22) and (18) we may write

$$A_{mn} = - \frac{i\,\pi\tau^2}{1-\eta} D_o \bar{H}^{(1)}_{2+2m+2n}(2\,x_o hx \cos\theta) +$$

$$+ \frac{i\,\tau^2}{1-\eta} \int_o^1 M_1(x)\, \bar{H}^{(1)}_{2+2m+2n}(2\,x_o hx \cos\theta)dx +$$

$$+ \frac{i\tau^2}{1-\eta} \int_o^{\tau} M_2(x)\, \bar{H}^{(1)}_{2+2m+2n}(2\,x_o\, hx \cos\theta)\, dx \qquad (24)$$

where
$$\bar{H}^{(1)}_{2+2m+2n}(z) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{z}{2}\right)^{2r+2m+2n+2} / \left[r!\,\Gamma(r+2m+2n+3)\right] +$$

$$+ \frac{i}{\pi}\left[\left\{2 \sum_{r=0}^{\infty} (-1)^r \left(\frac{z}{2}\right)^{2r+2m+2n+2} / \left[r!\,\Gamma(r+2m+2n+3)\right]\right\} \log\left(\frac{z}{2}\right) -\right.$$

$$- \sum_{r=m+n+2}^{2m+2n+1} \left(\frac{z}{2}\right)^{2r-2m-2n-2} \times \frac{(2m+2n+1-r)!}{r!} -$$

$$\left. - \sum_{l=0}^{\infty} (-1)^l \left(\frac{z}{2}\right)^{2m+2n+2+2l} \times \frac{\{\chi(2m+2n+l+3)+\chi(l+1)\}}{l!\,(2m+2n+2+l)!}\right], \qquad (25)$$

for $m,n = 0,1,2, \ldots\ldots\ldots\ldots$

DETERMINATION OF THE COEFFICIENTS a_n : For low frequency of vibration i.e for small (x_0k) and (x_0h) if we neglect terms involving $(x_0k)^4$ and higher order of (x_0k) in the expansion of $\bar{H}^{(1)}_{2+2m+2n}(2\,x_0hx\cos\theta)$ and $\bar{H}^{(1)}_{2+2m+2n}(2\,x_0h\;{}_0\cos\theta)$ occurring in A_{mn} of (24), we find

$$(1-\eta)A_{00} = \left\{(x_0k)^2 \log(x_0k)\right\} B_1 \cos^2\theta + (x_0k)^2 \cos^2\theta \left\{\left(-\frac{i\pi}{2} + \log\cos\theta - \right.\right.$$
$$\left.\left. - \log\tau - \frac{\chi(3)+\chi(1)}{2}\right) B_1 + B_2\right\} - \frac{(x_0k)^4 \log(x_0k)}{3\tau^2} B_3 \cos^4\theta\,,$$

$$(1-\eta)A_{10} = (1-\eta)A_{01} = -\frac{(x_0k)^2}{6} B_1 \cos^2\theta + \frac{(x_0k)^2 \log(x_0k)}{12\tau^2} B_3 \cos^4\theta \text{ and}$$

$$\underset{(m+n>1)}{(1-\eta)A^{*}_{mn}} = -\frac{(x_0k)^2}{(m+n)(m+n+1)(m+n+2)} B_1 \cos^2\theta, \qquad (26)$$

where $B_1 = D_o \tau_o^2 - \frac{1}{\pi}\int_o^1 M_1(x)x^2\,dx - \frac{1}{\pi}\int_o^{\tau} M_2(x)\,x^2\,dx$,

$$B_2 = D_o \tau_o^2 \log \tau_o - \frac{1}{\pi}\int_o^1 M_1(x)x^2 \log x\,dx - \frac{1}{\pi}\int_o^{\tau} M_2(x)x^2 \log x\,dx,$$

and $B_3 = D_o \tau_o^4 - \frac{1}{\pi}\int_o^1 M_1(x)x^4\,dx - \frac{1}{\pi}\int_o^{\tau} M_2(x)x^4\,dx.$

When A_{oo}, A_{10} etc from (26) are substituted in (17) and then integrated over θ, it is found that

$$(1-\eta)L_{oo} = \left\{K^2 \log K\right\} B_1 + K^2\left\{B_1\left(-\frac{i\pi}{2} + \frac{1}{2} - \log 2\tau - \frac{\chi(3)+\chi(1)}{2}\right) + B_2\right\} - \\ -\left\{K^4 \log K\right\}\frac{B_3}{4\tau^2} \qquad (27)$$

$$(1-\eta)L_{10} = -K^2\frac{B_1}{12} + \left\{K^4 \log K\right\}\frac{B_3}{24\tau^2}$$

$L_{01} = 3\,L_{10}$ and $L_{mn} = o$ for $m + n > 1$, where $x_o k = K$ is the dimensionless wave number.

Equation (15) can be solved, iteratively to determine a_n when values of L_{mn} are known. Using the values of L_{mn} given by (27) we obtain a_n in the following form where terms involving K^4 and its higher orders have been neglected

$$a_o = i\pi\phi_o\Big[1 - \frac{1}{1-\eta}\Big\{B_1K^2\log K + \Big[B_1\Big(-\frac{i\pi}{2} + \frac{1}{2} - \log 2T - \frac{3-4\nu}{4}\Big) + B_2\Big]K^2 -$$

$$- \frac{B_1^2}{1-\eta}K^4(\log K)^2 - \Big[\frac{B_3}{4T^2} + 2\frac{B_1}{1-\eta}\Big\{B_1\Big(-\frac{i\pi}{2} + \frac{1}{2} - \log 2T - \frac{3-4\nu}{4}\Big) + B_2\Big\}\Big]K^4\log K\Big\}\Big]$$

(28)

$$a_1 = \frac{i\pi\,\phi_o}{4(1-\eta)}\Big[B_1K^2 - \Big(\frac{B_3}{2} + \frac{B_1^2}{1-\eta}\Big)K^4\log K\Big]$$

and $a_n = o$ for $n > 2$.

where $\nu = 0.5772157$ .... is the Euler's constant.

RESULTS: With the evaluation of B_i's for $i = 1,2,3$ (which have been shown in the appendix), it follows from (11) that $D(\xi)$ is known which is the solution of the dual integral equations given in (10-a,b).

i) Noting that $A(\xi)$ is an odd function of ξ, with the help of last equation of (6) and (7), normal stress on $y = o$ is given by

$$(\sigma_{yy})_{y=o} = \frac{-1}{\pi}\int_0^\infty\Big[-\lambda x_o^2 h^2 + 2\mu(\xi^2 + x_o^2h^2) - \frac{4\mu ps\xi^2}{\xi^2 + s^2}A(\xi)\sin\xi X\Big]d\xi, \qquad (29)$$

when $A(\xi)$ is replaced in terms of $D(\xi)$ in (29) one obtains

$$(\sigma_{yy})_{y=o} = \frac{i\mu}{\pi(1-\eta)}\int_0^\infty\Big[(\xi^2+s^2) - \frac{4bs\xi^2}{\xi^2+s^2}\Big]\frac{\xi^2+s^2}{\sqrt{\xi}R(\xi)}D(\xi)\sin\xi X\,d\xi.$$

When $D(\xi)$ is replaced by the infinite series given by the equation (11), we have

$$(\sigma_{yy})_{y=o} = \frac{i\mu}{\pi(1-\eta)}\left[a_o \int_o^\infty J_1(\xi)\sin\xi X \, d\xi + a_1 \int_o^\infty J_3(\xi)\sin\xi X \, d\xi + \ldots\ldots\right]$$

$$= \frac{i\mu}{\pi(1-\eta)}\left[\frac{a_o X}{\sqrt{1-X^2}} + \frac{a_1(3X-4X^3)}{\sqrt{1-X^2}}\right], |X| < 1$$

because $a_n = o$, for $n > 2$. Substituting the values of the constants a_o, a_1 we finally obtain

$$(\sigma_{yy})_{y=o} = \frac{\mu\phi_o}{1-\eta}\frac{X}{\sqrt{(1-X^2)}}\Big[1 - \frac{B_1}{1-\eta}K^2\log K + \frac{1}{1-\eta}\Big\{B_1\Big(\frac{i\pi}{2} + \log 2\tau - \nu + 1 - X^2\Big) - B_2\Big\}K^2 +$$

$$+ \frac{B_1^2}{(1-\eta)^2}K^4\log^2 K + \frac{1}{1-\eta}\Big\{\frac{B_1^2}{1-\eta}\Big(-i\pi - \frac{5}{4} - 2\log 2\tau + 2\nu + X^2\Big) +$$

$$+ \frac{2B_1 B_2}{1-\eta} + \frac{B_3}{4}\Big(\frac{1}{\tau^2} - \frac{3}{2} + 2X^2\Big)\Big\}K^4\log K\Big], \quad |X| < 1$$

ii) The value of the reactive couple exerted by the elastic material on the strip is

$$G = 2\int_o^{x_o}(\sigma_{yy})_{y=o}\, x\, dx = V(g_1 - ig_2)$$

where $V = \mu\,\phi_o\, x_o^2\,\pi$,

$$g_1 = -\frac{1}{2(1-\eta)}\Big\{1 - \frac{B_1}{1-\eta}K^2\log K + \frac{1}{1-\eta}\Big[B_1\Big(\log 2\tau + \frac{1}{4} - \nu\Big) - B_2\Big]K^2 + \frac{B_1^2}{(1-\eta)^2}K^4\log^2 K -$$

$$- \frac{1}{1-\eta}\Big[\frac{B_1^2}{1-\eta}\Big(\frac{1}{2} + 2\log 2\tau - 2\nu\Big) - \frac{2B_1B_2}{1-\eta} - \frac{B_3}{4\tau^2}\Big]K^4\log K \qquad (30)$$

and

$$g_2 = -\frac{1}{2(1-\eta)}\Big\{-\frac{\pi B_1}{2(1-\eta)}K^2 + \frac{\pi B_1^2}{(1-\eta)^2}K^4\log K\Big\}.$$

For Poisson's ration $\eta = \frac{1}{4}$, τ^2 is equal to 3. τ_0, τ_1 and τ_2 which are the roots of $\Delta_o(x) = o$ are then given by

$$\tau_o^2 = \frac{3}{2-2\ /\ \sqrt{3}}, \quad \tau_1^2 = \frac{3}{2+2\ /\ \sqrt{3}} \quad \text{and} \quad \tau_2^2 = 3/4.$$

With these values of τ_i, values of the constants B_1, B_2 and B_3 are determined and they are

$B_1 = -\ {}^{\cdot}28125$, $B_2 = -\ {}^{\cdot}1507729$ and $B_3 = -\ {}^{\cdot}8828124$.

For different values of K, we compare values of $|\varepsilon_1|$ and $|\varepsilon_2|$ with corresponding values determined by Wickham (5)

K	Wickham		values from formula (30)	
	$\|\varepsilon_1\|$	$\|\varepsilon_2\|$	$\|\varepsilon_1\|$	$\|\varepsilon_2\|$
0.143	.655	.008	.655	.008
0.2578	.638	.026	.639	.024
0.4368	.609	.075	.616	.066

Numerical values calculated from (30) differ from the values obtained by Wickham due to the fact that Wickham's results contain terms upto K^2 where as in our results terms upto (K^4 log K) have been retained.

APPENDIX

Evaluation of B_1, B_2 and B_3.

We have

$$B_1 = D_o \tau_o^2 - \frac{1}{\pi} \int_o^1 M_1(x)x^2\,dx - \frac{1}{\pi} \int_o^{\tau} M_2(x)x^2\,dx \qquad \text{(A1)}$$

Splitting the integrands of the integrals of (A1) into partial fractions, we write

$$\int_o^1 M_1(x)x^2\,dx = \frac{1}{16(1-\tau^2)} \sum_{j=o}^{2} P_j \int_o^1 \frac{q_1'(x)\,x^2}{x^2-\tau_j^2}\,dx$$

$$= \frac{1}{16(1-\tau^2)}\left[\sum_{j=o}^{2} P_j \int_o^1 q_1'(x)dx + \sum_{j=1}^{2} \tau_j^2 P_j \int_o^1 \frac{q_1'(x)}{x^2-\tau_j^2}\,dx - \tau_o^2 P_o \int_o^1 \frac{q_1'(x)}{\tau_o^2-x^2}\,dx\right]$$

$$= \frac{\pi}{32(1-\tau^2)}\left[\sum_{j=o}^{2} P_j\left(\frac{1}{2} - \tau_j^2\right) + P_o\,\tau_o\left(\tau_o^2 - 1\right)^{1/2}\right] \qquad \text{(A2)}$$

and

$$\int_o^{\tau} M_2(x)x^2\,dx = \frac{1}{16(1-\tau^2)} \sum_{j=o}^{2} S_j \int_o^{\tau} \frac{q_2'(x)}{x^2-\tau_j^2}\,x^2\,dx$$

$$= \frac{1}{16(1-\tau^2)}\left[\sum_{j=o}^{2} S_j \int_o^{\tau} q_2'(x)dx \sum_{j=1}^{2} \tau_j^2 S_j \int \frac{q_2'(x)}{x^2-\tau_j^2}\,dx - \tau_o^2 S_o \int_o^{\tau} \frac{q_2'(x)}{\tau_o^2-x^2}\,dx\right]$$

$$= \frac{\pi}{32(1-\tau^2)}\left[\sum_{j=o}^{2} S_j\left(\frac{\tau^2}{2} - \tau_j^2\right) + S_o\,\tau_o\left(\tau_o^2 - \tau^2\right)^{1/2}\right], \qquad \text{(A3)}$$

where $\int$ denotes the principal value of the integral. Since by partial fraction

$$\frac{(2x^2 - \tau^2)^2}{\Delta_o(x)} = \frac{1}{16(1-\tau^2)} \sum_{j=o}^{2} \frac{P_j}{x^2 - \tau_j^2} \quad \text{and}$$

$$\frac{4x^2(x^2-1)}{\Delta_o(x)} = \frac{1}{16(1-\tau^2)} \sum_{j=o}^{2} \frac{S_j}{x^2 - \tau_j^2}, \text{ so we obtain}$$

after algebraic simplification

$$D_o \, \tau_o^2 = \frac{\tau_o^2(2\tau_o^2 - \tau^2)^2 q_1(\tau_o) + 4\tau_o^4 q_2(\tau_o) q_1^2(\tau_o)}{\Delta_o'(\tau_o)}$$

$$= \frac{1}{32(1-\tau^2)} \left[P_o \tau_o q_1(\tau_o) + S_o \tau_o q_2(\tau_o)\right]. \qquad (A4)$$

Hence it follows by the use of (A2),(A3) and (A4)

$$B_1 = \frac{1}{32(1-\tau^2)} \sum_{j=o}^{2} \left[P_j(\tau_j^2 - \tfrac{1}{2}) + S_j(\tau_j^2 - \frac{\tau^2}{2})\right].$$

Next we consider

$$B_2 = D_o \, \tau_o^2 \log \tau_o - \frac{1}{\pi} \int_o^1 M_1(x) x^2 \log x \, dx - \frac{1}{\pi} \int_o^{\tau} M_2(x) x^2 \log x \, dx$$

By (A4) it follows that

$$D_o \, \tau_o^2 \log \tau_o = \frac{1}{32(1-\tau^2)} \left[P_o \tau_o q_1(\tau_o) + S_o \tau_o q_2(\tau_o)\right] \log \tau_o \qquad (A5)$$

By splitting the integrands of integrals of B_2 as before one obtains,

$$\int_0^1 M_1(x)x^2 \log x \, dx = \frac{1}{16(1-\tau^2)} \sum_{j=0}^{2} P_j \int_0^1 \frac{q_1'(x).x^2 \log x}{x^2 - \tau_j^2} dx$$

$$= \frac{1}{16(1-\tau^2)} \Big[\sum_{j=0}^{2} P_j \int_0^1 q_1'(x) \log x \, dx + \sum_{j=1}^{2} \tau_j^2 P_j \int_0^1 \frac{q_1'(x)\log x}{x^2 - \tau_j^2} dx - P_0 \tau_0^2 \int_0^1 \frac{q_1'(x) \log x}{\tau_0^2 - x^2} dx \Big]$$

$$= \frac{\pi}{32(1-\tau^2)} \Big[\sum_{j=0}^{2} P_j \Big\{ (\tau_j^2 - \frac{1}{2}) \log 2 - \frac{1}{4} \Big\} + \sum_{j=1}^{2} P_j \ \tau_j \ q_1'(\tau_j) \tan^{-1} \frac{q_1'(\tau_j)}{\tau_j} -$$

$$- P_0 \ \tau_0 \ q_1(\tau_0) \log \frac{\tau_0 + q_1(\tau_0)}{\tau_0} \Big] \qquad \text{(A6)}$$

and

$$\int_0^\tau M_2(x) x^2 \log x \, dx = \frac{1}{16(1-\tau^2)} \sum_{j=0}^{2} S_j \int_0^\tau \frac{q_2'(x) \ x^2 \log x}{x^2 - \tau_j^2} dx$$

$$= \frac{1}{16(1-\tau^2)} \Big[\sum_{j=0}^{2} S_j \int_0^\tau q_2'(x) \log x \, dx + \sum_{j=1}^{2} \tau_j^2 S_j \int_0^\tau \frac{q_2'(x) \log x}{x^2 - \tau_j^2} dx -$$

$$- \tau_0^2 S_0 \int_0^\tau \frac{q_2'(x) \log x}{\tau_0^2 - x^2} dx \Big]$$

$$= \frac{\pi}{32(1-\tau^2)} \Big[\sum_{j=0}^{2} S_j \Big\{ (\frac{\tau^2}{2} - \tau_j^2) \log \frac{\tau}{2} - \frac{\tau^2}{4} \Big\} + \sum_{j=1}^{2} S_j \ \tau_j q_2'(\tau_j) \tan^{-1} \frac{q_2'(\tau_j)}{\tau_j} +$$

$$+ S_0 \ \tau_0 \ q_2(\tau_0) \log \tau - S_0 \ \tau_0 \ q_2(\tau_0) \log \frac{\tau_0 + q_2(\tau_0)}{\tau_0} \Big] \qquad \text{(A7)}$$

In this case using (A5),(A6) and (A7), we obtain after some algebraic simplification

$$B_2 = \frac{1}{32(1-\tau^2)}\left[\sum_{j=0}^{2} \left\{ P_j\left[\frac{1}{4} - \left(\tau_j^2 - \frac{1}{2}\right)\log 2\right] + S_j\left[\frac{\tau^2}{4} - \left(\frac{\tau^2}{2} - \tau_j^2\right)\log\frac{\tau}{2}\right]\right\} - \right.$$

$$- \sum_{j=1}^{2} \left\{ P_j \; \tau_j \; q_1'(\tau_j)\tan^{-1}\frac{q_1'(\tau_j)}{\tau_j} + S_j \; \tau_j \; q_2'(\tau_j)\tan^{-1}\frac{q_2'(\tau_j)}{\tau_j} + \right.$$

$$\left. + P_o \; \tau_o \; q_1(\tau_o)\log(\tau_o + q_1(\tau_o)) + S_o \; \tau_o \; q_2(\tau_o)\log\frac{\tau_o + q_2(\tau_o)}{\tau}\right].$$

Finally to obtain B_3 we proceed as before and we have by (A4)

$$D_o \; \tau_o^4 = \frac{1}{32(1-\tau^2)}\left[P_o \; \tau_o^3 \; q_1(\tau_o) + S_o \; \tau_o^3 \; q_2(\tau_o)\right], \qquad \text{(A8)}$$

$$\int_o^1 M_1(x)x^4 \, dx = \frac{\pi}{32(1-\tau^2)}\left[\frac{1}{2} + \sum_{j=o}^{2} P_j \; \tau_j^2 \left(\frac{1}{2} - \tau_j^2\right) + P_o \; \tau_o^3 \; q_1(\tau_o)\right] \qquad \text{(A9)}$$

$$\int_o^\tau M_2(x)x^4 \, dx = \frac{\pi}{32(1-\tau^2)}\left[\frac{\tau^4}{2} + \sum_{j=o}^{2} S_j \; \tau_j^2\left(\frac{\tau^2}{2} - \tau_j^2\right) + S_o \; \tau_o^3 \; q_2(\tau_o)\right]. \qquad \text{(A10)}$$

Consequently, we have from (A8), (A9) and (A10)

$$B_3 = \frac{1}{32(1-\tau^2)}\left[-\frac{1}{2}(1+\tau^4) + \sum_{j=o}^{2} \tau_j^2\left\{P_j\left(\tau_j^2 - \frac{1}{2}\right) + S_j\left(\tau_j^2 - \frac{\tau^2}{2}\right)\right\}\right].$$

HARMONIC ROCKING OF A RIGID CIRCULAR INDENTOR ON AN ELASTIC HALF-SPACE

INTRODUCTION: Study of elastic waves due to different types of oscillations of an indentor or of a vibrating punch on the surface of a homogeneous, isotropic elastic medium has become almost classical. Robertson (1964) extended the static problem of indentation of a semi-infinite elastic solid by a rigid circular disc solved by Sneddon (1951) and derived the solution due to a forced vertical vibration of the disk for low frequencies. Zakorko and Rostovtsev (1965) considered the effect of sinusoidal load transmitted through a weight less rigid circular punch, where a single equation is used which contained directly the sought for pressure through an unknown function which is then transformed to an integral equation of the second kind and then solved iteratively. Robertson (1967) determined the stress distribution due to impinging on the surface of a penny-shaped crack of a plane longitudinal wave, harmonic in time. He adopted the same technique of reducing the dual integral equation to Fredholm integral equation and solving by the method of iteration. Gladwell (1968) discussed the response of a semi-infinite elastic solid to a rotatory vibration of an indentor, over a circular area about a diameter, on the free surface. There also dual integral equations are converted to Fredholm integral equation of second kind and for low frequency oscillation, the solution is determined by iteration.

In this paper, the authors reconsider the problem of Gladwell (1968). By using Hankel transform the solution of the problem has been reduced to the solution of dual integral equations and then, following the technique of Tranter, to the solutions of an infinite set of algebraic equations. There is not much reference to the method of

Tranter in the literature involving the solution of dual integral equations. The authors find that the method prescribed by Tranter is no less powerful than the other methods used in this connection.

The normal stress below the disk, total torque, and the displacement on the free surface at large distance compared to the wavelength have been determined. Unlike Gladwell (1968), the results derived are exact in the sense that they do not involve any integral. The graphs of the normalised stress below the disc against the normalised distance from the centre of the disc have been plotted.

FORMULATION OF THE PROBLEM AND REDUCTION TO DUAL INTEGRAL EQUATIONS:

We consider a harmonic oscillation of frequency ω of a rigid circular disc of radius r_o about a diameter. The disc having a frictionless contact, is indented on the free surface of a semi-infinite homogeneous isotropic elastic solid occupying the half-space $-\infty < x < \infty$, $-\infty < y < \infty$ and $z > o$, z-axis being drawn into the medium. The indentor is assumed to produce rotatory vibration of amplitude φ_o about one of its diameters which is taken to be the y-axis.

The equations of motion in the cylindrical polar co-ordinates are

$$\nabla^2 \varphi = \frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} \text{ and } \quad \nabla^2\psi = \frac{1}{b^2}\frac{\partial^2\psi}{\partial t^2} \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$ and φ, ψ are scalar

potentials associated with dilatational and rotational parts of the displacement, a and b are P- and S- wave velocities respectively.

The nonvanishing components of displacement are

$$u_r = \frac{\partial\varphi}{\partial r} + \frac{\partial^2\psi}{\partial r\partial z},\ u_\theta = \frac{1}{r}\left(\frac{\partial\varphi}{\partial\theta} + \frac{\partial^2\psi}{\partial\theta\partial z}\right),$$

$$u_z = \frac{\partial\varphi}{\theta z} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{b^2}\frac{\partial^2\psi}{\partial t^2} \qquad (2)$$

and those of the stress are

$$\sigma_{rz} = \mu\left[2\frac{\partial^2\varphi}{\partial r\partial z} + 2\frac{\partial^3\psi}{\partial r\partial z^2} - \frac{1}{b^2}\frac{\partial^3\psi}{\partial r\partial t^2}\right],$$

$$\sigma_{\theta z} = \mu\left[\frac{2}{r}\frac{\partial^2\varphi}{\partial\theta\partial z} + \frac{2}{r}\frac{\partial^3\psi}{\partial\theta\partial z^2} - \frac{1}{b^2 r}\frac{\partial^3\psi}{\partial\theta\partial t^2}\right], \qquad (3)$$

$$\sigma_{zz} = \mu\left[\frac{1}{b^2}\frac{\partial^2\varphi}{\partial t^2} + 2\left(\frac{\partial^2\varphi}{\partial z^2} - \frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial^3\psi}{\partial z^3} - \frac{1}{b^2}\frac{\partial^3\psi}{\partial z\partial t^2}\right)\right].$$

The boundary conditions to be satisfied on $z = o$, are

$$u_z = \varphi r_o \cos\theta\ ;\ r \leq r_o \qquad (4a)$$

$$\sigma_{zz} = o \qquad ;\ r > r_o \qquad (4b)$$

$$\sigma_{rz} = o = \sigma_{\theta z} \text{ every where on } z = o. \qquad (4c)$$

To find the solutions of the equations(1) subject to the conditions (4), we make the substitution

$$[\varphi, \psi] = [\bar{\Phi}(r,\theta,z), \bar{\Psi}(r, \theta, z)]e^{iwt}.$$

In view of the boundary condition (4a), $\bar{\Phi}$ and $\bar{\Psi}$ are taken in the form

$$\bar{\Phi} = r_o^2 \cos\theta \int_0^\infty \xi A(\xi) J_1(r\xi) e^{-pz}\, d\xi \qquad (5a)$$

and $\bar{\Psi} = r_o^3 \cos\theta \int_0^\infty \xi B(\xi) J_1(\rho\xi) e^{-sZ}\, d\xi$ (5b)

where $\rho = r/r_o$ and $Z = z/r_o$. p and s in equations (5a - b) are

$p(\xi) = \sqrt{\xi^2 - r_o^2 h^2}, \quad \xi > r_o h$ and $s(\xi) = \sqrt{\xi^2 - r_o^2 k^2}, \quad \xi > r_o k$

$= i\sqrt{r_o^2 h - \xi^2}, \quad r_o h > \xi \qquad = i\sqrt{r_o^2 k^2 - \xi^2}, \quad r_o k > \xi$

where h = w/a and k = w/b. With the help of equations (1), (2) and from $\bar{\Phi}$, Ψ as obtained in (5), the following expressions for the components of displacement and those of required stress components are obtained as

$$u_r = r_o \cos\theta \int_0^\infty \xi \frac{\partial J_1(\rho\xi)}{\partial\rho} \left\{ A(\xi)\, e^{-pZ} - sB(\xi)\, e^{-sZ} \right\} d\xi,$$

$$u_\theta = \frac{r_o^2}{r} \sin\theta \int_0^\infty \xi J_1(\rho\xi) \left\{ -A(\xi)\, e^{-pZ} + sB(\xi) e^{-sZ} \right\} d\xi,$$

$$u_z = r_o \cos\theta \int_0^\infty \xi J_1(\rho\xi) \left\{ -pA(\xi) e^{-pZ} + \xi^2 B(\xi) e^{-sZ} \right\} d\xi, \qquad (6)$$

$$\sigma_{rz} = \mu \cos\theta \int_0^\infty \xi \frac{\partial J_1(\rho\xi)}{\partial\rho} \left\{ -2pA(\xi) e^{-pZ} + (2\xi^2 - r_o^2 k^2) B(\xi) e^{-sZ} \right\} d\xi,$$

$$\sigma_{\theta z} = \mu \frac{r_o}{r} \sin\theta \int_0^\infty \xi J_1(\rho\xi) \left\{ 2pA(\xi) e^{-pZ} - (\xi^2 + s^2) B(\xi) e^{-sZ} \right\} d\xi,$$

$$\sigma_{zz} = \mu \cos\theta \int_0^\infty \xi J_1(\rho\xi) \left\{ A(\xi)\, (\xi^2 + s^2)\, e^{-pZ} - 2sB(\xi)\, \xi^2\, e^{-sZ} \right\} d\xi.$$

In the expressions given in (6) and in subsequent analysis the time factor e^{-iwt} has been omitted. From the boundary condition (4c) we have

$$B(\xi) = 2\, pA(\xi)/(\xi^2 + s^2) \qquad (7)$$

and the conditions (4,a-b) give rise to the dual integral equations

$$-\frac{1}{(1-\eta_1)}\int_0^\infty \frac{r_o^2k^2\,p(\xi)}{R(\xi)}\,D(\xi)\,J_1(\rho\xi)d\xi = \phi_o\rho \;,\; o \le \rho \le 1 \tag{8}$$

and $$\int_0^\infty D(\xi)J_1(\rho\xi)d\xi = o, \quad \rho > 1$$

where $R(\xi) = (2\xi^2 - r_o^2k^2)^2 - 4ps\xi^2$,

$$D(\xi) = -(1-\eta_1)\frac{A(\xi)R(\xi)}{2\xi^2 - r_o^2k^2}, \tag{9}$$

and $\eta_1 = \lambda/2(\lambda + \mu)$ is poisson's ratio, λ and μ being Lame's constants.

Equations (8) can be written in the form

$$\int_0^\infty \left\{1 + H(\xi)\right\}\xi^{-1}\,D(\xi)J_1(\rho\xi)\,d\xi = \phi_o\rho \;,\; o \le \rho \le 1 \tag{10}$$

and $$\int_0^\infty D(\xi)J_1(\rho\xi) = o, \quad \rho > 1$$

where $1 + H(\xi) = -\dfrac{\xi r_o^2k^2\,p(\xi)}{(1-\eta_1)\,R(\xi)}$ and it can be whown that

$H(\xi) \to o$ as $r_oh \to o$ and $r_ok \to o$.

SOLUTION OF THE DUAL INTEGRAL EQUATIONS: Following Tranter we take

$$D(\xi) = \xi^{1/2} \sum_{m=0}^{\infty} a_m J_{2m+\frac{3}{2}}(\xi), \tag{11}$$

so that the second equation of (10) is automatically satisfied. The coefficients a_m are to be so chosen that the form of $D(\xi)$ as assumed in (11) satisfies the first of equations (10). For such a choice of a_m we must have

$$\sum_{m=0}^{\infty} a_m \int_0^{\infty} [1+H(\xi)]\xi^{-\frac{1}{2}} J_{2m+\frac{3}{2}}(\xi) J_1(\rho\xi)\, d\xi = \varphi_0 \rho \quad , 0 \leq \rho \leq 1 \tag{12}$$

multiplying the equation (12) by $\rho^2(1-\rho^2)^{-1/2} \mathcal{J}_n(\frac{3}{2}, 2, \rho^2)$, where n is a positive integer or zero and $\mathcal{J}_n$ is a Jacobi polynomial of degree n and then integrating with respect to ρ from o to 1, we obtain

$$\sum_{m=0}^{\infty} a_m \int_0^{\infty} [1+H(\xi)]\xi^{-1} J_{2m+\frac{3}{2}}(\xi) J_{2n+3/2}(\xi)\, d\xi = E(1, n, \tfrac{1}{2}) = E_n \text{ (say)} \tag{13}$$

$$\text{where } E_n = \frac{\sqrt{2}\,\Gamma(2+n)}{\Gamma 2\,\Gamma(n+\frac{1}{2})} \varphi_0 \int_0^1 \rho^3 (1-\rho^2)^{-1/2} \mathcal{J}_n(\tfrac{3}{2}, 2, \rho^2)\, d\rho.$$

Noting that $\mathcal{J}_0(\frac{3}{2}, 2, \rho^2) = 1$, we have from orthogonality relation of Jacobi polynomial

$$E_n = \frac{2\sqrt{2}\,\phi_0}{3\sqrt{\pi}} \text{ or o according as } n = o \text{ or } n \neq o. \tag{14}$$

$$\text{Since } \int_0^{\infty} \xi^{-1} J_{2m+\frac{3}{2}}(\xi) J_{2n+\frac{3}{2}}(\xi)\, d\xi = o, \qquad \text{for } m \neq n$$
$$= (3+4n)^{-1} \text{ for } m = n$$

so we have from equation (13)

$$a_n + \sum_{m=0}^{\infty} L_{mn}\, a_m = (3 + 4n)\, E_n \tag{15}$$

where $$L_{mn} = (3 + 4n) \int_0^{\infty} H(\xi)^{-1}\, J_{2m+\frac{3}{2}}(\xi) J_{2n+\frac{3}{2}}(\xi)\, d\xi \tag{16}$$

Equation (15) gives an infinite set of algebraic equations for the determination of the coefficients a_m.

DETERMINATION OF THE COEFFICIENTS a_m : Using the generalisation of Neumann's Integral [cf. Watson (1958) p 150]

$$J_{2m+\frac{3}{2}}(\xi)\, J_{2n+\frac{3}{2}}(\xi) = \frac{2}{\pi} \int_0^{\pi/2} J_{2m+2n+3}(2\xi\cos\alpha)\cos 2(m-n)\alpha\, d\alpha$$

$$= \frac{4}{\pi^2} \int_0^{\pi/2} \cos 2(m-n)\alpha\, d\alpha \int_0^{\pi/2} \sin(2m+2n+3)\beta\ \sin(2\xi\cos\alpha\sin\beta)\, d\beta$$

in the equation (16) and changing the order of integration we obtain

$$L_{mn} = \frac{4(3 + 4n)}{\pi^2} \int_0^{\pi/2} \cos 2(m-n)\alpha\, d\alpha \int_0^{\pi/2} \sin(2m+2n+3)\beta\, d\beta\ \times$$

$$\times \int_0^{\infty} H(\xi)\xi^{-1} \sin(2\xi\cos\alpha\sin\beta)\, d\xi . \tag{17}$$

The last integral in (17) may be written in the form

$$\int_0^{\infty} H(\xi)\xi^{-1} \sin(2\xi\cos\alpha\sin\beta)\, d\xi = -\frac{P}{(1-\eta_1)} - \frac{\pi}{2} \tag{18}$$

where $P = \int_0^\infty \frac{r_o^2 k^2 p(\xi)}{R(\xi)} \sin(2\xi \cos\alpha \sin\beta)\, d\xi$. Following the method of Lapwood [cf. Ewing et al (1957) p.49], the integral in (18) is found to be $P = D_o e^{-2 i r_o g \cos\alpha \sin\beta}$

$$- \int_0^{r_o h} \frac{r_o^2 k^2 \sqrt{r_o^2 h^2 - \xi^2}\; e^{-i\xi \cos\alpha \sin\beta}}{(2\xi^2 - r_o^2 k^2)^2 + 4\xi^2 \sqrt{r_o^2 h^2 - \xi^2}\sqrt{r_o^2 k^2 - \xi^2}}\, d\xi$$

$$- \int_{r_o h}^{r_o k} \frac{4 r_o^2 k^2 \xi^2 (\xi^2 - r_o^2 h^2)\sqrt{r_o^2 k^2 - \xi^2}\; e^{-2i\xi \cos\alpha \sin\beta}}{(2\xi^2 - r_o^2 k^2)^4 + 16\xi^4(\xi^2 - r_o^2 h^2)(r_o^2 k^2 - \xi^2)}\, d\xi \tag{19}$$

where $g = w/c$, c is Rayleigh wave velocity and

$$D_o = \left[\frac{\pi r_o^2 k^2 p(\xi)}{R'(\xi)}\right]_{\xi = r_o g} \tag{20}$$

Substituting $\xi = r_o gx$ in the integrals of (19) and expanding the exponential term in power series, we obtain

$$P = \sum_{l=o}^{\infty} \frac{(-2 i r_o g \cos\alpha \sin\beta)^l}{l!} M_l \tag{21}$$

where $M_l = D_o - \frac{c^2}{b^2} \int_o^{c/a} \frac{\sqrt{\frac{c^2}{a^2} - x^2}\; x^l}{(2x^2 - \frac{c^2}{b^2})^2 + 4x^2\sqrt{\frac{c^2}{a^2} - x^2}\sqrt{\frac{c^2}{b^2} - x^2}}\, dx -$

$$- 4\frac{c^2}{b^2} \int_{c/a}^{c/b} \frac{(x^2 - c^2/a^2)(c^2/b^2 - x^2)^{1/2}\; x^{l+2}}{(2x^2 - \frac{c^2}{b^2})^4 + 16x^4(x^2 - \frac{c^2}{a^2})(\frac{c^2}{b^2} - x^2)}\, dx \tag{22}$$

After substitution of the value of P from (21), (18) becomes

$$\int_0^\infty H(\xi)\xi^{-1} \sin (2\xi \cos \alpha \sin \beta) d\xi = \sum_{l=1}^\infty (-2ir_o g \cos\alpha \sin\beta)^l \frac{M_l}{l!}, \qquad (23)$$

since $M_o = -\frac{\pi}{2}(1-\eta_1)$, by (A3).

Substituting in (17), the result obtained in (23) and then integrating term by term one obtains

$$L_{mn} = (-1)^{m+n} \frac{3+4n}{(1-\eta_1)} I_{mn}, \quad \text{where}$$

$$I_{mn} = \sum_{l=1}^\infty \frac{(-i)^l \, l! \, (r_o g/2)^l \, M_l}{\Gamma(\frac{1}{2}+m-n+1)\,\Gamma(\frac{1}{2}-m+n+1)\,\Gamma(\frac{1}{2}+m+n+\frac{5}{2})\,\Gamma(\frac{1}{2}-m-n-\frac{1}{2})}. \qquad (24)$$

Using the fact that when $-\frac{1}{2}+m+n+\frac{3}{2}$ is not a negative integer or zero,

$$\frac{1}{\Gamma(\frac{1}{2}-m-n-\frac{1}{2})} = \frac{(-1)^{m+n+1}}{\pi} \cos \frac{1}{2}\pi \, \Gamma(-\frac{1}{2}+m+n+\frac{3}{2}),$$ we write below some particular values of I_{mn} for different values of m,n to calculate L_{mn}.

$$I_{mm} = -\frac{2}{\pi}\left[\frac{2M_2(r_o g)^2}{(4m+5)(4m+3)(4m+1)} - \frac{2i}{3}\frac{M_3(r_o g)^3}{\Gamma(2m+4)\,\Gamma(1-2m)} - \frac{3\pi}{16}\frac{M_4(r_o g)^4}{\Gamma(2m+\frac{9}{2})\,\Gamma(\frac{3}{2}-2m)} + \cdots\cdots\cdots\right]$$

$$I_{m(m+1)} = \frac{2}{\pi}\left[\frac{M_2(r_o g)^2}{(4m+7)(4m+5)(4m+3)} + \frac{4M_4(r_o g)^4}{(4m+9)(4m+7)(4m+5)(4m+3)(4m+1)} + \cdots \right]$$

$$I_{m(m+2)} = -\frac{2}{\pi} \frac{M_4(r_o g)^4}{(4m+11)(4m+9)(4m+7)(4m+5)(4m+3)} + \cdots\cdots .$$

For low frequency oscillation, $r_o h$ and $r_o k$ are small, consequently $r_o g$ is also a small quantity. Retaining terms upto fourth power of $(r_o g)$ in L_{mn}, we obtain with the help of equation (25)

$$L_{oo} = \frac{2}{5}\frac{M_2}{M_o}(r_o g)^2 - \frac{1}{3}\frac{M_3}{M_o}(r_o g)^3 - \frac{6}{35}\frac{M_4}{M_o}(r_o g)^4 ,$$

$$L_{o1} = \frac{1}{15}\frac{M_2}{M_o}(r_o g)^2 + \frac{4}{135}\frac{M_4}{M_o}(r_o g)^4 ,$$

$$L_{1o} = \frac{1}{35}\frac{M_2}{M_o}(r_o g)^2 + \frac{4}{315}\frac{M_4}{M_o}(r_o g)^4 , \qquad (26)$$

$$L_{11} = \frac{2}{45}\frac{M_2}{M_o}(r_o g)^2 + \frac{2}{495}\frac{M_4}{M_o}(r_o g)^4 ,$$

$$L_{o2} = \frac{1}{945}\frac{M_4}{M_o}(r_o g)^4 , \quad L_{20} = \frac{1}{3465}\frac{M_4}{M_o}(r_o g)^4$$

The values of the constants a_m are determined from (15) with the help of (14) and (26). Retaining terms upto the fourth power of $(r_o g)$, we find

$$a_o = 2\sqrt{\frac{2}{\pi}}\, \rho_o \left[1 - \frac{2}{5}\frac{M_2}{M_o}(r_o g)^2 + \frac{1}{3}\frac{M_3}{M_o}(r_o g)^3 + \frac{2}{5}\left\{ \frac{3}{7}\frac{M_4}{M_o} + \frac{17}{42}\left(\frac{M_2}{M_o}\right)^2 \right\}(r_o g)^4 \right]$$

$$a_1 = -\frac{2}{15}\sqrt{\frac{2}{\pi}}\, \rho_o \left[\frac{M_2}{M_o}(r_o g)^2 + \frac{4}{9}\left\{ \frac{M_4}{M_o} - \left(\frac{M_2}{M_o}\right)^2 \right\} (r_o g)^4 \right]$$

$$a_2 = -\frac{2}{945}\sqrt{\frac{2}{\pi}}\, \rho_o \left[\frac{M_4}{M_o} - \left(\frac{M_2}{M_o}\right)^2 \right](r_o g)^4, \quad a_3 = a_4 = \ldots\ldots = o.$$

RESULTS: i) From the last equation of (6), (7), (9) and (11) the normal stress below the disc is

$$(\sigma_{zz})_{z=0} = -\frac{\mu \cos\theta}{(1-\eta_1)} \sum_{m=0}^{\infty} a_m \int_0^{\infty} \sqrt{\xi}\, J_1(\rho\xi) J_{2m+\frac{3}{2}}(\xi)\, d\xi \qquad (27)$$

for $\rho < 1$, the value of the integral in (27) is given by [8, p - 401]

$$\int_0^{\infty} \sqrt{\xi}\, J_1(\rho\xi) J_{2m+\frac{3}{2}}(\xi)\, d\xi$$

$$= \frac{\rho\sqrt{2}\,\Gamma(m+2)}{\Gamma(m+\frac{1}{2})}\, {}_2F_1\left(m+2, \tfrac{1}{2}-m; 2; \rho^2\right) = \frac{\rho\sqrt{2}\,\Gamma(m+2)}{\Gamma(m+\frac{1}{2})}(1-\rho^2)^{-1/2} \times$$

$$\times\; {}_2F_1\left(-m, m+\tfrac{3}{2}; 2; \rho^2\right). \qquad (28)$$

Therefore,
$$(\sigma_{zz})_{z=0} = 2\mu \cos\theta \frac{\varphi_o}{M_o} \frac{\rho}{\sqrt{(1-\rho^2)}} \Big[1 - \frac{2}{3}\frac{M_2}{M_o}(1-\frac{\rho^2}{2})(r_o g)^2 +$$

$$+ \frac{1}{3}\frac{M_3}{M_o}(r_o g)^3 + \frac{2}{45}\Big\{(1+4\rho^2 - \frac{\rho^4}{2})\frac{M_4}{M_o} + (\frac{13}{2} - 4\rho^2 + \frac{\rho^4}{2})(\frac{M_2}{M_o})^2\Big\}(r_o g)^4\Big]. \tag{29}$$

ii) Total torque about y-axis is

$$T = -\int_0^{2\pi}\int_0^{r_o} (\sigma_{zz})_{z=0}\, r^2 \cos\theta \, dr d\theta$$

$$= -\frac{4\pi\mu r_o^3}{3}\frac{\varphi_o}{M_o}\Big[1 - \frac{2}{5}\frac{M_2}{M_o}(r_o g)^2 + \frac{1}{3}\frac{M_3}{M_o}(r_o g)^3 + \frac{1}{105}\Big\{18\frac{M_4}{M_o} + 17(\frac{M_2}{M_o})^2\Big\}(r_o g)^4\Big].$$

This result agrees with result obtained by Gladwell [1968].

iii) We have from (6), (7), (9) and (11), the displacement components on the free surface far out side the disc as

$$(u_r)_{z=0} = -\frac{r_o \cos\theta}{(1-\eta_1)} \sum_{m=0}^{\infty} a_m \int_0^{\infty} \xi^{3/2} \frac{2\xi^2 - r_o^2 k^2 - 2ps}{R(\xi)} J_{2m+\frac{3}{2}}(\xi) \times$$

$$\times \Big\{J_0(\rho\xi) - \frac{1}{\rho\xi} J_1(\rho\xi)\Big\} d\xi, \tag{30}$$

$$(u_\theta)_{z=0} = \frac{r_o^2}{r}\frac{\sin\theta}{(1-\eta_1)} \sum_{m=0}^{\infty} a_m \int_0^{\infty} \sqrt{\xi}\, \frac{2\xi^2 - r_o^2 k^2 - 2ps}{R(\xi)} J_{2m+\frac{3}{2}}(\xi) J_1(\rho\xi) d\xi \tag{31}$$

$$(u_z)_{z=0} = -\frac{r_o \cos\theta}{(1-\eta_1)} \sum_{m=0}^{\infty} a_m \int_0^{\infty} \sqrt{\xi}\, \frac{r_o^2 k^2\, p}{R(\xi)}\, J_{2m+\frac{3}{2}}(\xi) J_1(\rho\xi) d \quad . \tag{32}$$

The integrals in the above expressions can be evaluated by Lapwood's method as in the appendis. But to obtain more information about bodily P- and S-wave the following method is adopted. To evaluate the integral in (32) i.e.

$$I_z = \int_0^{\infty} \sqrt{\xi}\, \frac{r_o^2 k^2\, p}{R(\xi)}\, J_{2m+\frac{3}{2}}(\xi)\, J_1(\rho\xi) d\xi \,, \tag{33}$$

we write $J_1(\rho\xi)$ in the form

$$J_1(\rho\xi) = \frac{1}{2}\left[H_1^{(1)}(\rho\xi) + H_1^{(2)}(\rho\xi)\right].$$

In the complex $\zeta\ (=\xi + i\eta)$- plane, we draw cuts parallel to the imaginary axis joining $\pm\, r_o h$ with $\pm\,(r_o h - i\infty)$, $\pm\, r_o k$ with $\pm\,(r_o k - i\,\infty)$ making the factors $\sqrt{\zeta - r_o h}$, $\sqrt{\zeta + r_o h}$ of $\sqrt{\zeta^2 - r_o^2 h^2}$ and $\sqrt{\zeta - r_o k}$, $\sqrt{\zeta + r_o k}$ of $\sqrt{\zeta^2 - r_o^2 k^2}$ single valued. Integrating the part of (33) consisting of $H_1^{(1)}(\rho\xi)$ along the boundary of the fourth quadrant with the cuts, then adding the integrals, we obtain

$$I_z = -\, i\, D_o \sqrt{(r_o g)}\, J_{2m+\frac{3}{2}}(r_o g)\, H_1^{(2)}(r_o g) -$$

$$-4ie^{-i\pi/4} \int_0^{\infty} \frac{(r_o k - i\eta)^{5/2}\, r_o^2 k^2 \sqrt{(2r_o k - i\eta)}\left\{(r_o k - i\eta)^2 - r_o^2 h^2\right\}}{\left[2(r_o k - i\eta)^2 - r_o^2 k^2\right]^4 + 16\, i\eta (r_o k - i\eta)^4 (2r_o k - i\eta)\left[(r_o k - i\eta)^2 - r^2 h^2\right]} \times$$

$$\times J_{2m+\frac{3}{2}}(r_o k - i\eta)\, H_1^{(2)}(rk - i\rho\eta)\sqrt{\eta}\, d\eta -$$

$$-ie^{-i\pi/4}\int_0^{\infty}\frac{\sqrt{(r_o h - i\eta)r_o^2k^2}\,\sqrt{(2r_o h - i\eta)}\,\left[2(r_o h - i\eta)^2 - r_o^2k^2\right]^2}{\left[2(r_o h - i\eta)^2 - r_o^2k^2\right]^4 + 16i\eta(r_o h - i\eta)^4(2r_o h - i\eta)\left[(r_o h - i\eta)^2 - r_o^2k^2\right]}\times$$

$$\times J_{2m+\frac{3}{2}}(r_o h - i\eta)\, H_1^{(2)}(r\, h - i\rho\eta)\sqrt{\eta}\, d\eta \cdot \qquad (34)$$

First term on the right hand side of (34) arises due to pole at $(r_o g)$, second and third terms form integrals along loops round the branch cuts at $r_o k$ and $r_o h$ respectively.

Assuming $rk > rh >> 1$, the integrals in (34) can be evaluated asymptotically at points far out side the disc. Hankel functions are expanded asymptotically. Retaining the first term in the expansion of Hankel functions, the integrals arising in (34) are of the form

$$\int_0^{\infty}\sqrt{\eta}\; G(\eta)\, e^{-\rho\eta}\, d\eta = \frac{\lceil 3/2}{\rho^{3/2}}\, G(o) + \frac{\lceil 5/2}{\rho^{5/2}}\, G'(o) + \frac{\lceil 7/2}{\rho^{7/2}}\,\frac{G''(o)}{2!}$$

Keeping the leading terms again, the asymptotic values of the displacement on the free surface at points far from the centre of the disc is determined and it is found to be

$$(u_z)_{z=o} = \frac{r_o\cos\theta}{(1-\eta_1)}\sum_{m=o}^{\infty} a_m\left[i\sqrt{\tfrac{2}{\pi}}\, D_o e^{-i(rg+\frac{\pi}{4})} J_{2m+\frac{3}{2}}(r_o g)\sqrt{\frac{r_o}{r}} + \right.$$

$$+ 4e^{-irk} \frac{K^2}{(r_o k)^{3/2}} J_{2m+\frac{3}{2}}(r_o k)(\frac{r_o}{r})^2 + e^{-irh} \frac{S(2b^2-a^2)}{aK(r_o k)^{3/2}} J_{2m+\frac{3}{2}}(r_o h)(\frac{r_o}{r})^2 \Big]$$

Similarly we obtain

$$(u_r)_{z=o} = - \frac{r_o \cos\theta}{(1-\eta_1)} \sum_{m=o}^{\infty} a_m \Big[\sqrt{\frac{2}{\pi}} D_o Q \, e^{-i(rg+\frac{\pi}{4})} J_{2m+\frac{3}{2}}(r_o g)(1 - \frac{i}{rg}) \sqrt{\frac{r_o}{r}} -$$

$$- 2ie^{-irk} \frac{K}{(r_o k)^{3/2}} J_{2m+\frac{3}{2}}(r_o k) \, (1 - \frac{i}{rk}) \, (\frac{r_o}{r})^2 +$$

$$+ 2e^{-irh} \frac{S}{(r_o k)^{3/2}} J_{2m+\frac{3}{2}}(r_o h)(b - \frac{ia}{rk})(\frac{r_o}{r})^2 \Big]$$

and

$$(u_\theta)_{z=o} = \frac{r_o \sin\theta}{(1-\eta_1)} \sum_{m=o}^{\infty} a_m \Big[i\sqrt{\frac{2}{\pi}} \, e^{-i(rg+\frac{\pi}{4})} \frac{D_o Q}{r_o g} J_{2m+\frac{3}{2}}(r_o g)(\frac{r_o}{r})^{3/2} +$$

$$+ 2e^{-irk} \frac{K}{(r_o k)^{5/2}} J_{2m+\frac{3}{2}}(r_o k)(\frac{r_o}{r})^3 + 2ie^{-irh} \frac{aS}{(r_o k)^{5/2}} J_{2m+\frac{3}{2}}(r_o h)(\frac{r_o}{r})^3 \Big]$$

where $K = \dfrac{\sqrt{a^2 - b^2}}{a}$, $S = \dfrac{a^3 \sqrt{ab\,(a^2 - b^2)}}{(2b^2 - a^2)^3}$ and

$$Q = \frac{a\,(2b^2 - c^2) - 2b\sqrt{a^2 - c^2}\sqrt{b^2 - c^2}}{c^2 \sqrt{a^2 - c^2}} \, .$$

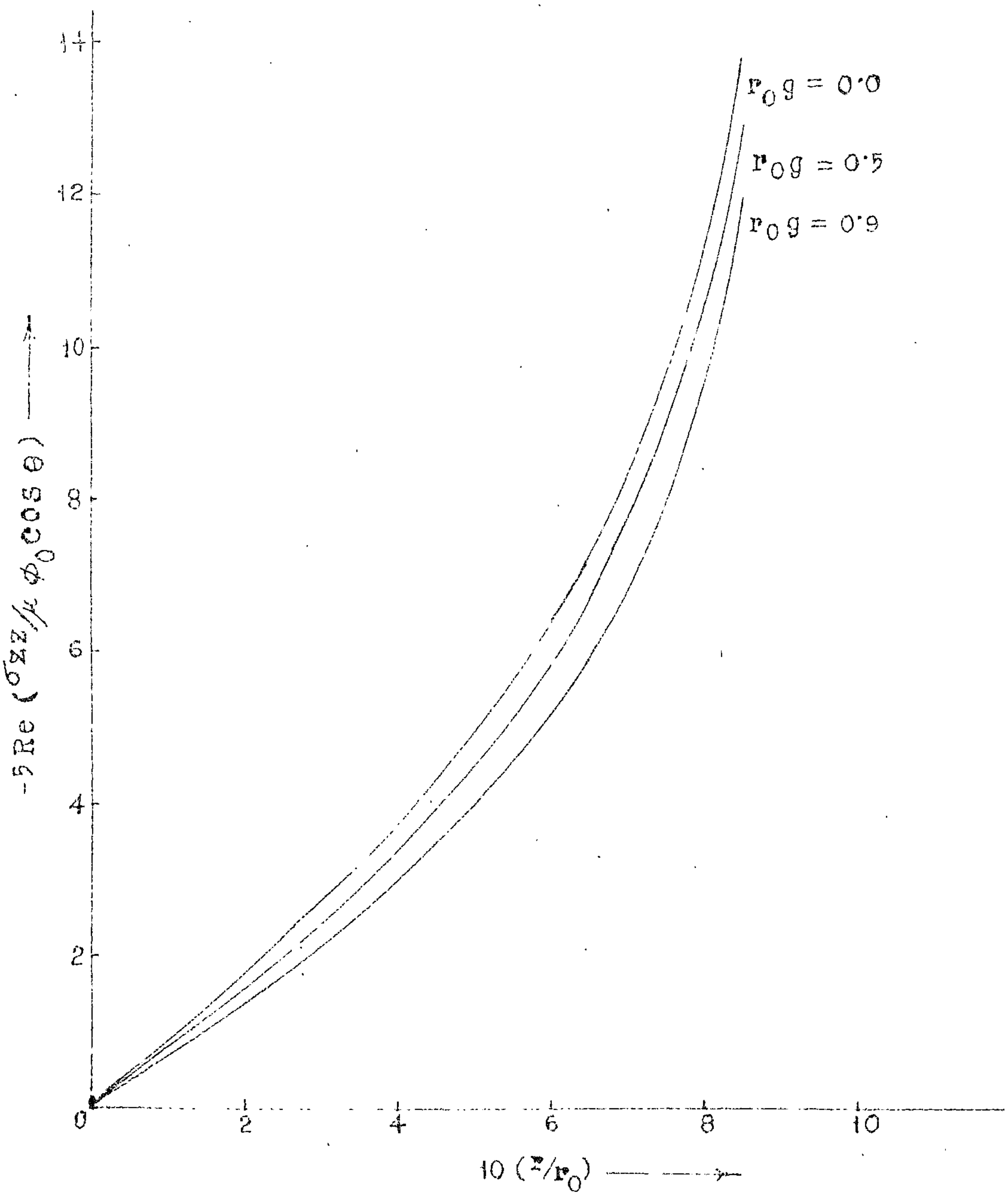
iv) From equation (28)

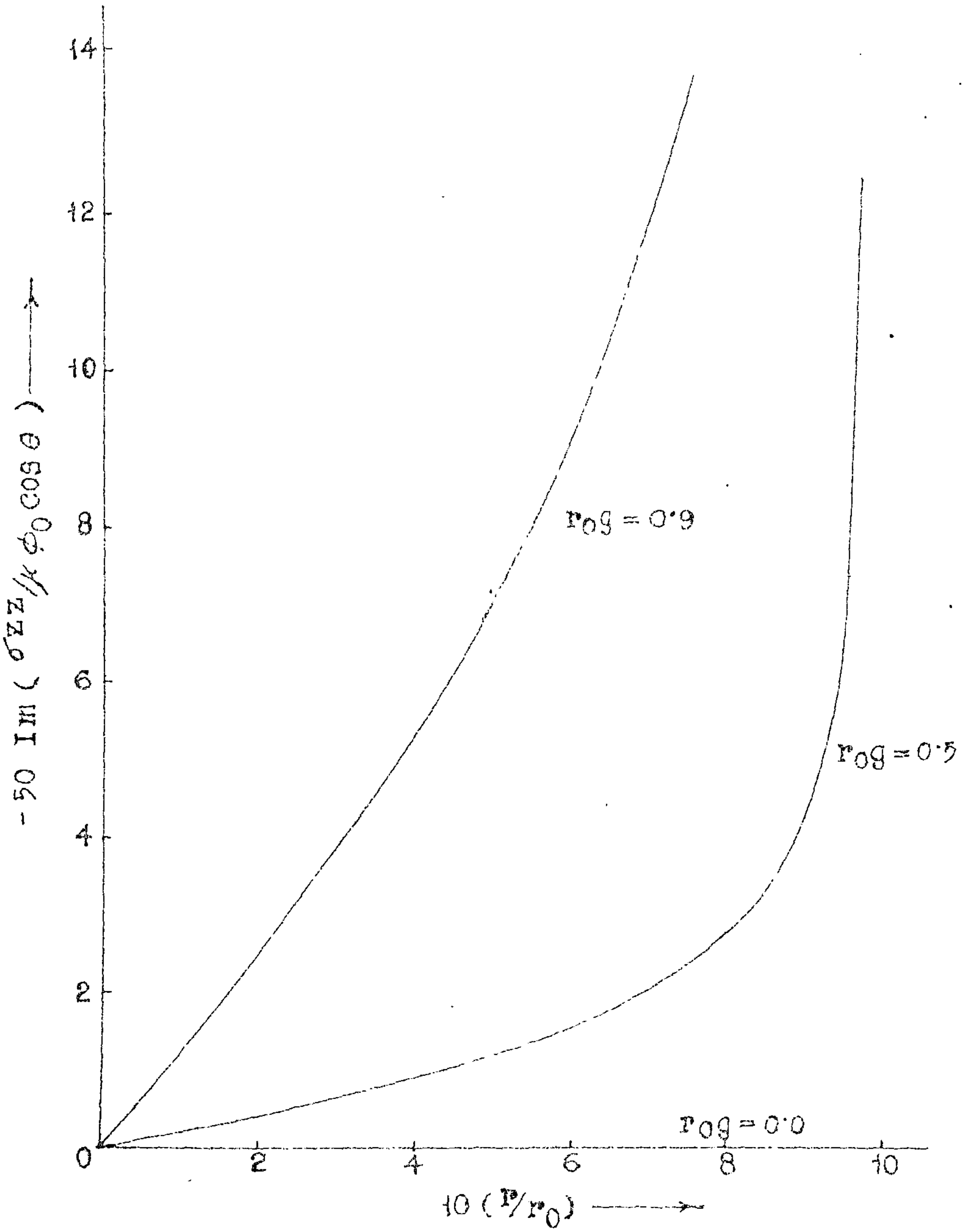
$$\frac{Re(\sigma_{zz})_{z=0}}{\mu \cos\theta\, \varphi_0} = \frac{2}{M_0}\frac{\rho}{\sqrt{1-\rho^2}}\Big[1-\frac{2}{3}\frac{M_2}{M_0}\left(1-\frac{\rho^2}{2}\right)(r_0 g)^2+\frac{2}{45}\Big\{\left(1+4\rho^2-\frac{\rho^4}{2}\right)\frac{M_4}{M_0}+$$

$$+\left(\frac{13}{2}-4\rho^2+\frac{\rho^4}{2}\right)\left(\frac{M_2}{M_0}\right)^2\Big\}(r_0 g)^4\Big]$$

and $$\frac{Im(\sigma_{zz})_{z=0}}{\mu \cos\theta\, \varphi_0} = \frac{2}{3M_0}\frac{\rho}{\sqrt{1-\rho^2}}\frac{M_3}{M_0}(r_0 g)^3$$

are plotted against $\rho(<1)$ for values of $r_0 g$ = 0·0, 0·5 and 0·9. It is found that the absolute value of stress just below the disk increases with the increase of distance of the point from the centre of the disc where as at a point below the disc the absolute value of real part of stress decreases and imaginary part of stress increases with the increase of the frequency of oscillation of the disc.


$r_0 g = 0{\cdot}0$
$r_0 g = 0{\cdot}5$
$r_0 g = 0{\cdot}9$
$-5\,\mathrm{Re}\,(\sigma_{zz}/\mu\,\phi_0 \cos\theta)$
$10\,(z/r_0)$
0
2
4
6
8
10
12
14



- 50 Im (σzz/μ Φ0 cos θ) →
14
12
10
8
6
4
2
0
2
4
6
8
10
10 (r/r0) →
r0g = 0·9
r0g = 0·5
r0g = 0·0

APPENDIX

To find the value of M_o, following Lapwood, we consider the function

$$F(\zeta)= \frac{r_o^2k^2p(\zeta)}{R(\zeta)}$$ in the region R_1 and R_2 of the

complex ζ ($= \xi + i\eta$)- plane bounded by the curves as shown in the Fig. A1. By Cauchy's residue theorem, we have, when the function F( ) is integrated along the curve bounding the region R_1

$$\int_0^\infty F(\xi)d\xi + \int_0^\infty \frac{r_o^2k^2\sqrt{\eta^2 + r_o^2h^2}}{R(i\eta)} d\eta - i(1-\eta_1)\frac{\pi}{2} = 0 \qquad (A1)$$

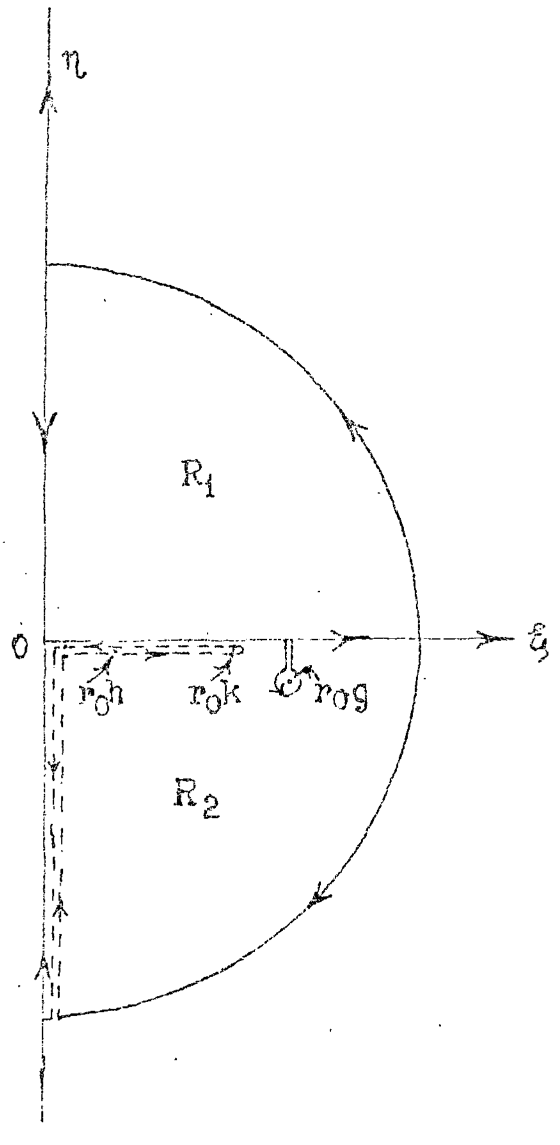
and integrating along the curve bounding the region R_2

$$\int_0^\infty F(\xi)d\xi + \int_0^\infty \frac{r_o^2k^2\sqrt{\eta^2 + r_o^2h^2}}{R(-i\eta)} d\eta + i(1-\eta_1)\frac{\pi}{2} -$$

$$- 2i\int_0^{r_oh} \frac{r_o^2k^2\sqrt{r_o^2h^2 - \xi^2}}{(2\xi^2-r_o^2k^2)^2 + 4\xi^2\sqrt{r_o^2h^2-\xi^2}\sqrt{r_o^2k^2-\xi^2}} d\xi -$$

$$-8i\int_{r_oh}^{r_ok} \frac{r_o^2k^2(\xi^2 - r_o^2h^2)\sqrt{r_o^2k^2-\xi^2}\,d\xi}{(2\xi^2-r_o^2k^2)^4 + 16\xi^4(\xi^2-r_o^2h^2)(\xi^2-r_o^2k^2)} = -2iD_o \qquad (A2)$$

The terms $- i\,(1-\eta_1)\,\pi/2$ and $i\,(1-\eta_1)\pi/2$ occuring on the left hand side of equations (A1) and (A2) respectively are the


η
R1
0
ξ
r0h
r0k
r0g
R2

values of the integrals on the large circular arcs (Fig. A1) in the first and fourth quadrants. $-2D_o i$ on the right hand side of (A2) is the contribution from the residue at $(r_o g)$ where D_o is given by (20).

Subtracting (A2) from (A1) and dividing by 2i we obtain

$$- (1-\eta_1)\pi/2 = D_o - \frac{c^2}{b^2} \int_o^{c/a} \frac{\sqrt{\frac{c^2}{b^2} - x^2}\, dx}{(2x^2 - \frac{c^2}{b^2})^2 + 4x^2 \sqrt{\frac{c^2}{a^2} - x^2} \sqrt{\frac{c^2}{b^2} - x^2}} -$$

$$- \frac{4c^2}{b^2} \int_{c/a}^{c/b} \frac{x^2 (x^2 - \frac{c^2}{a^2}) \sqrt{\frac{c^2}{b^2} - x^2}\, dx}{(2x^2 - \frac{c^2}{b^2})^4 + 16x^4 (x^2 - \frac{c^2}{a^2})(\frac{c^2}{b^2} - x^2)} \qquad (A3)$$

$= M_o$, defined in (22).