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On maximum compactness bound in relativistic theory

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This article reviews recent developments in estimating the maximum compactness bound of a compact star in the General Theory of Relativity and some extended theories of gravity.

I. MAXIMUM COMPACTNESS BOUND IN GTR

The General Theory of Relativity (GTR), developed by Einstein over a hundred years ago, is still considered the most modern theory of gravity. The theory has seen tremendous success in terms of its predictability of various tests of gravity, including the most recent detection of gravitational waves and imaging of black holes.

One of the consequences of GTR is the existence of black holes. The compactness vis-a-vis gravitational potential of a black hole is so high that nothing (not even light) can escape from it. A black hole is characterized by its event horizon [1]. A pertinent question naturally arises: What is the upper bound on the compactness of a self-gravitating star without an event horizon? In GTR, the maximum compactness bound is the well-known Buchdahl bound [2]. The largest compactness bound can be written as $M/R < 4/9$, where M is the entire mass confined within a boundary R. This is implied by Buchdahl's demonstration that there can be no uniform density stars with radii lower than $9/8M$. Even though the bound was obtained for a uniform-density star, it turns out that bound holds even for stellar configurations where density decreases radially outward.

Following Buchdahl's pioneering work, many investigators analyzed the limiting compactness bound for a wide variety of matter compositions under varied conditions. An interesting development in this direction is the estimation of the maximum compactness bound by incorporating charge into the matter composition. For the Einstein-Maxwell system, Giuliani and Rothman [3], amongst others, put forward a new compactness bound which, in the absence of charge, reduces to Buchdahl bound. Using a different technique, we have a compactness bound that is identical to the one that Giuliani and Rothman previously got. In our approach, we first developed a new charged stellar model, which reduces to the Schwarzschild interior solution in the absence of charge. By demanding that physical quantities like pressure and energy density must not diverge in this model, we arrived at the compactness bound [4]

$$\frac{M}{R} = \frac{8/9}{(1 + \sqrt{1 - \frac{8\alpha^2}{9}})}, \tag{1}$$

where $\alpha^2 = Q^2/M^2$ and Q is the total charge. The above result offers an upper bound on the charge-to-mass ratio for a horizon-less stellar composition as $Q^2/M^2 \leq 9/8$ and maximum compactness bound $M/R \leq 4/9 < 1$, where no charge exists. We have obtained the maximum compactness bound by superimposing the electromagnetic field on a uniform-density fluid distribution, which might be considered a generalization of the Buchdahl bound to electromagnetic fields.

The incorporation of local anisotropy at the interior of a self-gravitating star is another area of considerable interest in the analysis of relativistic compact stars. Due to various physical processes, local anisotropy is expected to develop inside a compact star [5–8]. For an anisotropic stellar configuration, of particular interest is the estimation of compactness bound obtained by Sharma *et al.* [9, 10]

$$\frac{M}{R} \leq \frac{2(k-2)}{(5k-9)}, \tag{2}$$

where the parameter K specifies anisotropy. Interestingly, for $K = 0$ (zero anisotropy) in the model, one regains the Buchdahl bound. How anisotropy affects the physical characteristics of anisotropy and maximum compactness bound, is detailed in Ref. [9, 10] and references therein.

One of the key achievements of GTR is its geometric interpretation in terms of finding physically meaningful solutions to Einstein's field equations. Tolman IV solution is one such example that can be used as a viable model for interpreting compact stars such as neutron stars [11]. Subsequently, the solution has been generalized by many investigators to accommodate a more realistic compact star equation of state (EOS). Such improved versions of Tolman IV solutions can accommodate EOS and stiffness as input parameters [12–14]. Purely from the stability point of view, we have examined how such factors can influence a star's maximum compactness bound, the details of which are available in Ref. [15, 16].

II. MAXIMUM COMPACTNESS BOUND IN SOME EXTENDED THEORIES OF GRAVITY

Despite the remarkable success of Einstein's gravity in predicting many tests of gravity, the theory faces several challenges on many fronts. Consequently, many alternative theories of gravity have been developed, some of which have been shown to be relevant in the context of many unresolved issues in cosmology, in particular. It is, therefore, worthwhile to analyze the limiting compactness in the domain of such modifications. This section outlines some of our recent probes in this direction.

Einstein's gravity naturally leads to the modified $f(R)$ gravity in which the Einstein-Hilbert action incorporates higher-order curvature terms. Taking into account the non-minimal coupling of the matter field with the geometry is, in fact, a more radical approach that results in $f(R,T)$ theory, where R is the Ricci scalar and T is the trace of the energy-momentum tensor. In one of our recent works, we first generated a realistic stellar solution in $f(R,T)$ gravity theory. Utilizing the model, we have developed a technique to calculate the maximum compactness bound in the $f(R,T)$ theory. Assuming a linear modification, where χ serves as a dimensionless coupling parameter and thereupon $f(R,T) = R + 2\chi T$, our method provides the maximum compactness bound in the following form [17]:

$$\frac{M}{R} = \frac{\frac{32\pi}{9}\left[\frac{2}{(8\pi-\chi)} - \frac{\chi}{(8\pi-\chi)^2}\right]}{1 + \frac{4\pi}{(8\pi-\chi)}\sqrt{\frac{16\alpha^2}{9}(\frac{3\chi}{8\pi} - 2) + (\frac{\chi}{4\pi} - 2)^2}}. \tag{3}$$

It's intriguing to observe, the general bound [Eq. (3)] contains the bound [Eq. (2)] obtained earlier by Giuliani and Rothman and Sharma *et al* for a relativistic charged sphere in Einstein's gravity ($\chi = 0$).

In pursuing alternative theories of gravity, $f(Q)$ gravity has also gained considerable interest in recent years. While GTR involves Riemannian geometry, a non-Riemannian geometry incorporates torsion and non-metricity as additional geometrical properties of the spacetime. In modified $f(Q)$ gravity, the non-metricity Q is the mediator of gravitational interactions. $f(Q)$ gravity has gained considerable attention in recent years in terms of understanding its cosmological and astrophysical implications. Assuming that the remodeling is linear in Q, such that $f(Q) = aQ + b$, where a and b are constants, we have developed a stellar model in (Q) gravity and analyzed its distinctive features. Moreover, the limiting compactness bound under such modifications has been computed, which has the form [18]

$$\frac{M}{R} \leq \frac{2(3a(1+K) + a^2(2K+1))}{4a^2K - 9(1+K)}, \tag{4}$$

where a deviation from spherical geometry is represented by the curvature parameter K[19]. For $K = 0$ and $\alpha = -1$, the associated geometry in this treatment is spherical, and the model reduces to an interior solution obtained by Schwarzschild describing a homogeneous fluid sphere. Obviously, by setting $a = -1$ and $K = 0$, one regains the Buchdahl bound $\frac{M}{R} \leq \frac{4}{9}$.

The maximum compactness bound has also been studied in higher dimensional spacetime [20–23]. Compactness bound in $D \geq 4$ dimensional spacetime is defined as

$$u_{n+2} = \frac{C_{n+2}}{2R^{n-1}}, \tag{5}$$

where C_{n+2} is given by

$$C_{n+2} = \frac{\pi^{\frac{-(n+1)}{2}}\Gamma(\frac{n+1}{2})\kappa_{n+2}M_{tot}}{n}. \tag{6}$$

The link between the physical mass M and the star's total mass M_{tot} is as follows:

$$M = \frac{M_{tot}}{R^{n-2}}, \tag{7}$$

thereby for $n = 2$, we have $\frac{2GM}{c^4}$ for C_4.

In a spacetime of universal dimensions $D(= n + 2)$, Leon and Cruz [22] and Chanda and Sharma [23] have independently shown that for a star possessing a constant density, the compactness bound can be obtained in the form

$$u_{n+2} \leq \frac{2n}{(n+1)^2}. \tag{8}$$

In order for the solution to reduce to Schwarzschild's interior solution in $D \geq 4$ dimensions in the isotropic case, we assumed anisotropy in our formalism. In this model, anisotropy has been correlated to the curvature parameter K,which describes how, when embedded in a 4-dimensional Euclidean space, the $t =$ constant hypersurface of the corresponding spacetime deviates from spherical geometry, [19]. The constraint in four dimensions with $n = 2$ simplifies to the standard bound $u_{n+2} \leq 4/9$, which is the four-dimensional Buchdahl limit. Compactness bound in $D \geq 4$ dimensional spacetime has been obtained in the form [23]

$$\begin{aligned} 4K^2u_{n+2}^3 + 4Ku_{n+2}^2(Kn + n + 1) \\ +u_{n+2}((K+1)^2n^2 + 2Kn - 8K(K+1) + 2n + 1) \\ -2(K+1)(K(n-2) + n) \leq 0, \end{aligned} \tag{9}$$

giving the Buchdahl compactness bound for an anisotropic star's higher dimensional counterpart. With $D = 4$ and $K = 0$, the Buchdahl compactness limit is restored.

III. CONCLUDING REMARKS

The accurate estimation of compactness bound has tremendous importance in the studies of compact stars. Such studies help us understand how physical factors such as charge and anisotropy influence the compactness bound in GTR and extended theories of gravity. A star's mass and size rely on the equation of state (EOS) and different physical processes at the interior of a star. If the EOS is known, one integrates the Tolman-Oppenheimer-Volkoff equations to obtain a self-gravitating compact star's mass-radius $(M - R)$ relationship. Conversely, accurate mass and radius estimation helps us constrain the EOS. In the current era of multi-messenger astronomy, it is possible to get a more accurate estimation of the stellar observable than ever before. Such measurements will not only help us constrain the EOS of a compact star, similar studies are expected to help us get a better insight into the correct theory of gravity.

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