CHAPTER - I.

A NEW APPROACH TO THE NON-LINEAR ANALYSIS OF MODERATELY THICK ELASTIC PLATES.

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ABSTRACT

In this chapter a set of uncoupled differential equations in cartesian as well as in polar co-ordinates have been formulated to study the non-linear behaviours of thick plates showing the effects of shear deformation and rotatory inertia. Banerjee's hypothesis [18] along with Reissner's variational theorem [26] has been utilised in the formulation.

FORMULATION OF THE DIFFERENTIAL EQUATIONS

We consider the free vibration of thick elastic plates of thickness h. The material is transversely isotropic (such as pyrolytic graphite, for example). The origin of the co-ordinates is located at the centre of the plate. The deflections are considered to be of the same order of magnitude as the plate thickness.

So far as Reissner's variational theorem is applied, the stresses are taken in the form [26]

$$(\sigma_x, \sigma_y, \sigma_z) = \frac{1}{h}(N_x, N_y, N_{xy}) + \frac{12z}{h^3}(M_x, M_y, M_{xy}) \qquad \text{.... (1)}$$

$$(\sigma_{xz}, \sigma_{yz}) = \frac{3}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right](Q_x, Q_y) \qquad \text{.... (2)}$$

$$\sigma_z = 0 \qquad \text{.... (3)}$$

Note that the expressions of σ_{xz}, σ_{yz} are assumed to be the same form as those for small deflection case. Since free vibrations are concerned, σ_z is assumed equal to zero. The membrane stresses N_x/h, N_y/h and N_{xy}/h, involved in the expressions of σ_x, σ_y and σ_{xy} respectively, which are neglected in the linear theory as outlined in [26], can no longer be disregarded in the analysis of large deflection problems. The foregoing equations also satisfy all the stress boundary conditions.

In order to account for transverse shear deformation and rotatory inertia effects in the plate theory, where the lateral deflection is comparable with the thickness, the displacement components are assumed to be of the following expressions [38]

$$u(x,y,z,t) = u_o(x,y,t) + z\alpha(x,y,t) \qquad \ldots. \ (4)$$

$$v(x,y,z,t) = v_o(x,y,t) + z\beta(x,y,t) \qquad \ldots. \ (5)$$

$$w(x,y,z,t) = w(x,y,t) \qquad \ldots. \ (6)$$

The subscript O is used to associate with the middle surface. It should be noted that the relations involve the combined action of bending and stretching which characterizes the behaviour of the problem. However, the thickness is assumed to be unchanged during the deformation procedure, and the elements normal to the middle surface before deformation are not required to remain perpendicular to the deformed middle plane.

In view of equations (4) - (6), the strain-displacement relations for large deflection of plates are of the form :

$$\begin{aligned}
\epsilon_x &= \frac{\partial u_o}{\partial x} + z\frac{\partial \alpha}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\
\epsilon_y &= \frac{\partial u_o}{\partial y} + z\frac{\partial \beta}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\
\epsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} + z\frac{\partial \alpha}{\partial y} + z\frac{\partial \beta}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \\
\epsilon_{yz} &= \frac{1}{2}\left(\frac{\partial w}{\partial y} + \beta\right) \\
\epsilon_{xz} &= \frac{1}{2}\left(\frac{\partial w}{\partial x} + \alpha\right) \\
\epsilon_z &= 0
\end{aligned} \qquad \ldots. \ (7)$$

The membrane stress resultants in terms of strains are given by :

$$N_x = \frac{Eh}{1-\nu^2}\left(\epsilon_{x_o} + \nu \epsilon_{y_o}\right)$$

$$N_y = \frac{Eh}{1-\nu^2}\left(\epsilon_{y_o} + \nu \epsilon_{x_o}\right)$$

$$N_{xy} = \frac{Eh}{1+\nu} \epsilon_{x_o y_o} \qquad \ldots . (8)$$

where ϵ_{x_o} , ϵ_{y_o} are the normal strains of the middle surface in the x and y-directions, respectively; $\epsilon_{x_o y_o}$ is the middle surface shearing strain. From (7) it is seen that

$$\epsilon_{x_o} = \frac{\partial u_o}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\epsilon_{y_o} = \frac{\partial v_o}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$\epsilon_{x_o y_o} = \frac{1}{2}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \qquad \ldots . (9)$$

Recalling that the normal stress in the transverse direction is assumed to be zero in equation (3), the strain-stress relations for a transverse isotropic material, such as a pyrolytic graphite material, are found to be [38]

$$\epsilon_x = \frac{1}{E}\left(\sigma_x - \nu \sigma_y\right) ; \quad \epsilon_y = \frac{1}{E}\left(\sigma_y - \nu \sigma_x\right) ;$$

$$\epsilon_z = 0$$

$$\epsilon_{xy} = \frac{1}{2G} \sigma_{xy} ; \quad \epsilon_{yz} = \frac{1}{2G_c} \sigma_{yz}$$

$$\epsilon_{xz} = \frac{1}{2G_c} \sigma_{xz} \qquad \ldots . (10)$$

The Reissner's functional as outlined in [26] is of the form

$$\Psi = \int_v \left[\sigma_{ij}\epsilon_{ij} - W(\sigma_{ij})\right] dv - \int_v F_i u_i dv - \int_{si} T_i u_i ds \qquad \ldots. (11)$$

It is to be noted that the last two integrations, concerned with body forces and surface tractions, are eliminated in this problem. Now, the Reissner's functional Ψ becomes a strain-energy expression as the first term on the right-hand side of equation (11).

The substitution of equations (7) and (10) into equation (11) gives

$$\begin{aligned}\Psi = \iiint_{-h/2}^{+h/2} \Bigg\{ & \sigma_x \left[\frac{\partial u_o}{\partial x} + z\frac{\partial \alpha}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] \\ & + \sigma_y \left[\frac{\partial v_o}{\partial y} + z\frac{\partial \beta}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] \\ & + \sigma_{xy}\left[\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} + z\left(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x}\right) + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right] \\ & + \sigma_{yz}\left(\frac{\partial w}{\partial y} + \beta\right) + \sigma_{xz}\left(\frac{\partial w}{\partial x} + \alpha\right) \\ & - \frac{1}{2}\Bigg[\frac{1}{E}\left(\sigma_x^2 + \sigma_y^2\right) - \frac{2\nu}{E}\sigma_x\sigma_y + \frac{2(1+\nu)}{E}\sigma_{xy}^2 \\ & + \frac{1}{G_c}\left(\sigma_{xz}^2 + \sigma_{yz}^2\right)\Bigg]\Bigg\} dx\, dy\, dz \qquad \ldots. (12)\end{aligned}$$

Integrating equation (12) with respect to z and using equations (1), (2), (8) and (9) in the preceding, we obtain [38]

$$\Psi = \iint \Big\{ \frac{Eh}{2(1-\nu^2)} \left[\bar{I}_e^2 - 2(1-\nu)\,\bar{II}_e \right] + M_x \frac{\partial \alpha}{\partial x} + M_y \frac{\partial \beta}{\partial y} + M_{xy}\left(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x}\right) + Q_x\left(\frac{\partial w}{\partial x} + \alpha\right) + Q_y\left(\frac{\partial w}{\partial y} + \beta\right) - \frac{1}{2E}\left[\frac{12}{h^3}\left(M_x^2 + M_y^2\right) - \frac{24 M_x M_y \nu}{h^3}\right] - \frac{3}{5G_c h}\left(Q_x^2 + Q_y^2\right) \Big\}\, dx\, dy \qquad \text{.... (13)}$$

where $\bar{I}_e$, $\bar{II}_e$ are the first and second invariants of the middle surface strains. These are

$$\bar{I}_e = \epsilon_{x_0} + \epsilon_{y_0} \; ; \; \bar{II}_e = \epsilon_{x_0}\epsilon_{y_0} - \epsilon_{x_0 y_0}^2 \qquad \text{.... (14)}$$

Let us now apply Banerjee's hypothesis and rewrite the expression for Ψ in the following form :

$$\Psi' = \iint \Big\{ \frac{Eh}{2(1-\nu^2)} \left(\bar{\bar{I}}_e^2 + \frac{\lambda}{4}\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\}^2 \right) + M_x \frac{\partial \alpha}{\partial x} + M_y \frac{\partial \beta}{\partial y} + M_{xy}\left(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x}\right) + Q_x\left(\frac{\partial w}{\partial x} + \alpha\right) + Q_y\left(\frac{\partial w}{\partial y} + \beta\right) - \frac{1}{2E}\left[\frac{12}{h^3}\left(M_x^2 + M_y^2\right) - \frac{24 M_x M_y \nu}{h^3}\right] - \frac{3}{5G_c h}\left(Q_x^2 + Q_y^2\right) \Big\}\, dx\, dy \qquad \text{.... 15(a)}$$

where

$$\bar{\bar{I}}_e = \frac{\partial u_o}{\partial x} + \nu \frac{\partial v_o}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad \ldots\ldots \quad 15(b)$$

and λ is a constant depending on Poisson's ratio of the plate material [18].

The Kinetic energy equation after integrating through the thickness is

$$T = \iint \left\{ \frac{\rho h}{2}\left[\left(\frac{\partial u_o}{\partial t}\right)^2 + \left(\frac{\partial v_o}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] + \frac{\rho h^3}{24}\left[\left(\frac{\partial \alpha}{\partial t}\right)^2 + \left(\frac{\partial \beta}{\partial t}\right)^2\right]\right\} dx\, dy \qquad \ldots\ldots \quad (16)$$

In order to derive the equation of motion we now apply Hamilton's principle in conjunction with the strain-energy equation given by (15) as well as the kinetic energy given by (16). Therefore we have to minimise the integral,

$$\Phi = \int_{t_1}^{t_2} (\Psi - T)\, dt \qquad \ldots\ldots \quad (17)$$

Taking the variation of Φ, equating it to zero and eliminating M_x, M_y, M_{xy}, etc. we get the following set of decoupled differential equations governing the vibrations of the plates : *

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$$\nabla^4 W + \frac{6}{5(1-\nu^2)} \cdot k\left(\frac{E}{Gc}\right) \frac{\bar{\alpha}^2 h^2}{12} \tau^2(t) \nabla^2 \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)$$

$$+ \frac{3\lambda}{5(1-\nu^2)} k\left(\frac{E}{Gc}\right) \nabla^2 \left[\nabla^2 w \left\{ \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right\} \cdot \right.$$

$$\left. + 2\left\{ \frac{\partial^2 w}{\partial x^2} \left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial^2 w}{\partial y^2} \left(\frac{\partial w}{\partial y}\right)^2 \right\} + 4 \frac{\partial^2 w}{\partial x \partial y} \cdot \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \right]$$

$$- \frac{6}{5} \cdot \frac{\rho}{Gc} \cdot \frac{\partial^2}{\partial t^2} \left(\nabla^2 w \right) - \bar{\alpha}^2 \tau^2(t) \left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right]$$

$$- \frac{6\lambda}{h^2} \left[\nabla^2 w \left\{ \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right\} \right.$$

$$\left. + 2\left\{ \frac{\partial^2 w}{\partial x^2} \left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial^2 w}{\partial y^2} \left(\frac{\partial w}{\partial y}\right)^2 \right\} + 4 \frac{\partial^2 w}{\partial x \partial y} \cdot \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \right]$$

$$+ \frac{12}{h^2 C_p^2} \cdot \frac{\partial^2 w}{\partial t^2} = 0 \qquad \text{.... 18(a)}$$

The coupling parameter $\bar{\alpha}^2$ is given by

$$\frac{\bar{\alpha}^2 h^2}{12} \tau^2(t) = \frac{\partial u_o}{\partial x} + \nu \frac{\partial v_o}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad \text{.... 18(b)}$$

where $\tau(t)$ is a function of time and $C_p = \left[\frac{E}{\rho(1-\nu^2)}\right]^{1/2}$ is the speed of wave propagation along the surface of the plates.

For movable edge condition $\bar{\alpha} = 0$ and for immovable edge condition $u_0 = v_0 = 0$ at the boundary.

In polar co-ordinates the above set of equations are transformed into the following form : *

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right]$$

$$+ \frac{6}{5(1-\nu^2)} k\left(\frac{E}{G_c}\right) \frac{\bar{\alpha}^2 h^2}{12} \cdot r^{\nu-1}\left[\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r}\right]\tau^2(t)$$

$$+ \frac{3\lambda}{5(1-\nu^2)} k\left(\frac{E}{G_c}\right)\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]\left[\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)\left(\frac{\partial w}{\partial r}\right)^2 +\right.$$

$$\left. 2\frac{\partial^2 w}{\partial r^2}\left(\frac{\partial w}{\partial r}\right)^2\right] - \frac{6}{5}\frac{\rho}{G_c}\cdot\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right]$$

$$- \bar{\alpha}^2\tau^2(t) r^{\nu-1}\left[\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\cdot\frac{\partial w}{\partial r}\right]$$

$$- \frac{6\lambda}{h^2}\left[\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial w}{\partial r}\right)\left(\frac{\partial w}{\partial r}\right)^2 + 2\frac{\partial^2 w}{\partial r^2}\left(\frac{\partial w}{\partial r}\right)^2\right]$$

$$+ \frac{12}{h^2 c_p^2}\cdot\frac{\partial^2 w}{\partial t^2} = 0 \qquad \text{.... 19(a}$$

where

$$r^{\nu-1}\frac{\bar{\alpha}^2 h^2}{12}\tau^2(t) = \frac{\partial u_0}{\partial r} + \nu\frac{u_0}{r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 \qquad \text{.... 19(b}$$

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For movable edge condition $\bar{\alpha} = 0$ and for immovable edge condition $u_0 = 0$ at the boundary. Equations18(a) - 19(b) will be utilised to study the nonlinear behaviours of different elastic plates.

BERGER'S EQUATIONS :

To derive Berger's equations, as outlined by Wu and Vinson [38], let us now recall our attention to the strain energy expression given by equation (13). If Π_e is neglected in the expression (13) we shall arrive at the following set of decoupled equations by using the same procedure as adopted in the previous case.

$$\left[1 + \frac{\bar{\alpha}^2 h^2}{10(1-\nu^2)} \left(\frac{E}{G_c}\right) \tau^2(t) \right] \nabla^4 w - \bar{\alpha}^2 \tau^2(t) (\nabla^2 w)$$

$$- \frac{6}{5} \cdot \frac{\rho}{G_c} \cdot \frac{\partial^2}{\partial t^2} (\nabla^2 w) + \frac{12}{h^2 C_p^2} \cdot \frac{\partial^2 w}{\partial t^2} = 0 \qquad \text{.... } 20(a)$$

where

$$\bar{I}_e = \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{\bar{\alpha}^2 h^2}{12} \tau^2(t) \qquad \text{.... } 20(b)$$

Equations 20(a) and 20(b) are well-known Berger's equations on the thick plate theory and will now be utilised to investigate the nonlinear behaviours of different elastic plates.