Chapter 1

INTRODUCTION

The world, we live in, is highly complex. A small seed, almost spherical in shape, and so may be considered to be geometrically simple, under right conditions, slowly grows, in successive steps, into a sampling and then gradually into a fully grown plant, with many flowers and new generations seeds. This well known mundane example can be considered to represent a model for the paradigm of complexity. The salient features of a complex system consist mainly of nonlinearity, scale invariance, self-similarity and so on. By nonlinearity we mean in this thesis that the governing dynamical principle inducing evolution of the system concerned may be described by one or more of nonlinear differential equation(s) and/or similar other equations and processes. Because of scale invariance the relevant dynamical variables can be represented by power laws of the form t^{α}, where t is a real variable and α is a constant. As a consequence, the growth of a complex system is expected to have influences from many different scales of a dynamical variable. Self-similarity finally means roughly "a part resembling exactly (or approximately) similar to the whole". More analytical definition is given latter. Fractals, a very active area of contemporary research in the field of nonlinear sciences, may be said to represent an example of a class of complex systems. Although there is still no generally acceptable definition, fractals are generally considered to be those subsets of the Euclidean space R^n which are highly irregular, nonsmooth and also enjoy some sort of scale invariance and self-similarity. Further, generation of such fractal subsets of R^n admit some non-linear process(es). Various natural objects and processes are known to reflect fractal like self-similarity and scale invariance. For example, large scale galaxy distribution, cloud boundaries, topographical surfaces of the planet earth, coastlines, turbulence in fluid, stock market fluctuations, structures of mammalian hearts and lungs and so on [1, 2]. The interest in the study of self-similarity and scale invariance of the global and local structures of the nature - ranging from the macroscopic cosmological scales down to the microscopic finer scales - is gaining momentum over the last few decades from extensive work of several mathematicians and physicists throughout the world [1, 2, 3, 4, 5].

Appearance of fractal like irregular (pathological) subsets in (real/complex) analysis dates back to the later half of nineteenth century when various examples of no-where non-differentiable continuous curves were studied. Weierstras's construction provided one such early example. Weierstras's curve $f(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin \lambda^k t$, $1 < s < 2$, $\lambda > 1$ also enjoys self-similarity on all scales as represented by the scaling law $f(\lambda^{-1}t) \approx \lambda^{s-2} f(t)$, $\lambda >> 1$ [3]. As a result, a smaller portion of the said curve when suitably magnified will resemble the original curve. For about three decades after the construction of such functions, these were still considered to be rather pathological cases without any practical and/or analytical interest. In the recent years, the attitude has changed considerably. It has been realized that irregular sets provide a much better representation of many natural phenomena than the figures of classical (Euclidean/non-Euclidean) geometry. Perrin was the first physicist who pointed out their applications in the real physical world. His ground breaking work on the Brownian motion showed that trajectory of the diffusive Brownian particles are nowhere differentiable and have fractional dimension 3/2 [6]. In the fluid systems, passive scalars adverted by a turbulent fluid have isoscalar surfaces which are highly irregular. In dynamical systems attractors of some systems, for instance, the Lorenz attractor, are found to be continuous but nowhere differentiable [7]. Over the last few decades it has also become clear that the occurrence of chaos in a deterministic dynamical system such as logistic map for a suitable range of values of the control parameter requires formation of Cantor sets dynamically in a region of the so called strange (chaotic) attractor [6].

A Cantor set is a totally disconnected, compact and perfect subset of the real line. Cantor set is an example of a self-similar fractal set that arises, as indicated above, in various fields of applications. The chaotic attractors of a number of one dimensional maps; such as the logistic maps, tent map, turn out to be topologically equivalent to Cantor sets [8]. Cantor set also arises in electrical communications [1], in biological systems [2], and diffusion processes [9, 10]. Recently there have been a lot of interest in developing a framework of analysis on a Cantor like fractal sets [6, 11, 12, 13]. Because of the disconnected nature, methods of ordinary real analysis break down on a Cantor set. Various approaches based on the fractional derivatives [14, 15] and the measure theoretic harmonic analysis [16], functional analysis, probability theory [6] have already been considered at length in the literature. However, a simpler intuitively appealing approach is still considered to be welcome.

The present thesis is a part of an ongoing project that aims at developing a scale invariant analytical framework that would be suitable to construct a rigorous analysis on fractal subsets of R^n. In the present thesis, in particular, *we formulate a scale invariant analysis on Cantor like fractal subsets of* R. Since a Cantor set C is a totally disconnected, compact, perfect subset of R, the ordinary analysis of R can not be meaningfully extended over C, i.e., when a real variable x is assumed to live and undergo changes only over the points of C. More specifically, the concept of a derivative in the sense of rate of change of a dynamic quantity, namely, a function of time when time is supposed to vary over a Cantor set, (say)[1] can not be formulated consistently on such a set. The general trend in the literature is to bypass defining derivatives directly on such sets, by taking recourse to technically more involved approaches based on geometric measure theory [3], harmonic analysis [7], functional analysis on noncommutative [17] spaces, probability theory [6] and so on. The present scale invariant analysis utilizing the concepts of *relative and scale invariant infinitesimals* is not only simpler than the other contemporary approaches but also offers an elegant avenue extending the well known differential calculus of R over a Cantor set C in a conceptually

[1]The possibility of a time variation on a Cantor like fractal set is considered in Continuous Time Random Walk theories of statistical mechanics [9].

appealing manner. Recall that ordinary measure theoretic arguments can essentially establish an analytic statement on R upto a *Lebesgue measure zero set* only. Our analysis, on the otherhand, succeeds in deducing results which are valid *everywhere* in R. For instance, a Cantor function $\phi(x)$ can be defined classically as one which satisfies $\frac{d\phi}{dx} = 0$ *almost everywhere* in [0,1]. In the present scale invariant approach a Cantor function is shown to be locally constant *everywhere* in [0,1]. Further, the global variability of a Cantor function is shown to get exposed in a double logarithmic scale $\log\log x^{-1}$. We also define and study some evolutionary equation on such a Cantor set. The present approach rests on a novel extension of the usual ultrametric structure of a Cantor set into an inequivalent class of ultrametrics using a seemingly new concepts of relative infinitesimals that are shown to exist in the gaps of infinitesimally small neighbourhoods of 0, considered as an element of a Cantor set $C \subset [0, 1]$. In short, the present thesis represents a body of analytic results which are of interdisciplinary in nature involving various topics such as Cantor like fractal sets, nonstandard analysis, nonarchimedean spaces, real analysis, measure theory etc.

1.1 Main Results of the Thesis

In chapter 2, the salient features of several key notions such as fractals, ultrametric spaces, nonstandard analysis and Cantor sets, which will be useful in the subsequent development of the new analysis, are reviewed briefly.

In chapter 3, the basic concepts of relative infinitesimals and scale invariant infinitesimals are introduced (defined) and discussed in detail. Next, we introduce the novel definition of a scale invariant absolute value, which is shown to assign a nontrivial ultrametric valuation to such a scale invariant infinitesimal, thus raising the corresponding set of infinitesimals into an ultrametric space. We show that this nonarchimedean valuation, essentially, is defined by a suitable Cantor function associated with the original Cantor set. We then study some basic properties of topology and analytic structures on this ultrametric space of scale invariant infinitesimals. The definitions of limit, continuity and differentiability are formulated. The ultrametric and the corresponding induced topology are shown to represent respectively inequivalent classes in comparison to the natural ultrametric on a Cantor set. Next, we explain how this ultrametric structure is carried over to the entire Cantor set, thereby inducing an associated ultrametric structure in the said set. The chapter ends with a discussion of a *valued measure* that arise naturally in the above ultrametric space generalizing the standard metric Lebesgue measure. The valued measure turns out to give rise to directly the finite, nonzero Hausdorff s-measure of the underlying Cantor set when s denotes the Hausdorff dimension of the set.

In chapter 4, several explicit examples, namely, the middle third Cantor set, middle α- Cantor set and (p,q) Cantor set are reexamined in the light of present scale invariant analytic framework. To clarify the basic analytic ingredients, namely, the relative infinitesimals and associated absolute values, we present here an *independent* set of arguments detailing the *origin, actual role, and significance* of the above concepts of relative infinitesimals and associated valuations in the context of a family of homogeneous Cantor sets [24, 25, 26]. It is shown that the singleton set of the zero of the real line is replaced by a nontrivial zero measure set of relative infinitesimals, which are supposed to live in an inverted Cantor set, defined as the collection of the closure of the gaps of the original Cantor set in the neighbourhood of 0. Further more, it is verified explicitly, in each of the above distinct cases, that the valued infinitesimals induce a finer structure in the neighbourhood of each Cantor point, leading to a multiplicative structure defined on a Cantor set. The nonarchimedean valuation realized here as an appropriate Cantor function is next interpreted as a locally constant function satisfying the equation $\frac{dv}{dx} = 0$. Although locally constant in the neighbourhood of a point, such a function , nevertheless, can enjoy global variability. The chapter ends with a discussion of the global variability of the locally constant function in the usual topology [25].

In chapter 5, another *independent* analysis is presented on the derivation of a smooth multiplicative representation of an element of a Cantor set. This is expected to offer new *insights* into the *mechanism of smoothening* of a Cantor function at the points of Cantor set. The analysis is based on the standard classical analysis arguments exposing the nondifferentiability of a Cantor function $\phi(x)$ at $x \in C$. Our scale invariant analysis leading to the above results are presented again in the context of the classical middle third Cantor set, as well as in the (p, q) type Cantor set [24, 25].

In chapter 6, some new results leading to the differential jump measure on a Cantor Set are presented. It exposes the *precise* nature of variability of a nontrivial valuation. The ordinary limit $x \to 0$ on the real line R is shown to extend over to a sublinear limit $x \log x^{-1} \to 0$, when x is assumed to vary over a Cantor set. Further, the incremental measure of smooth self similar jump processes is determined. It corresponds to the multiplicative increment which is realized as a smooth measure and may be considered to contribute an independent component in the ordinary measure of R [28].

An interesting new phenomenon, called the *growth of measure* is studied in chapter 7 [26]. Using the reparametrisation invariance of the valuation it is shown how the scale factors of a Lebesgue measure zero Cantor set might get *deformed* leading to a *deformed* Cantor set with a positive measure. The definition of a new *valuated exponent* is introduced which is shown to yield the fatness exponent in the case of a positive measure (fat) Cantor set. Here, we also study a class of Cantor set having identical Hausdorff dimensions and thickness. However, the higher order valuated exponent, introduced here, may be exploited to distinguish such sets.

In chapter 8, a class of an exact, higher order derivative discontinuous (nonsmooth) solutions to the simplest scale invariant ordinary differential equation $t\frac{d\tau}{dt} = \tau$ is derived using a novel iteration procedure revealing the possible presence of a nontrivial selfsimilar multiplicative structure in such a solution [27]. The new class of solutions are shown to break the reflection symmetry of original differential equation. The existence of such non-trivial solutions, which can be put in a rigorous setting in the context of a nonstandard model of real analysis, can be interpreted in an extended framework of calculus accommodating (random) inversions as a valid mode of changes over and above the usual mode of linear increments in the real analysis.

In chapter 9, a few interesting applications of the above class of nonsmooth solutions are presented in the context of some selected topics of nonlinear dynamical systems [27]. First, we discuss how the class of nonsmooth solutions might lead to a new paradigm in realizing and reinterpreting randomness that appears so abundantly in nonlinear deterministic models. Next, we argue that the reflection asymmetry of the class of nonsmooth solutions may be reinterpreted as a novel framework to understand the origin of *time asymmetry* in any evolutionary process [47, 49]. The origin and genesis of universally present *flicker* $(1/f)$ noise in diverse natural, biological , financial processes is still considered to be a riddle by many authors [53, 57]. In Sec.9.4, we discuss the relevance of nonsmooth solutions to the flicker noise problem. Because of the presence of multiscale stochastic behaviours, the nonsmooth solutions naturally become relevant in understanding flicker noise. In the final two subsections, we show how a derivation of the q-exponential power law dynamics of the sensitivity to initial conditions of a logistic map in the edge of chaos can be formulated in the present framework [53, 54]. We further show how a hyperbolic type distribution arise naturally at the asymptotic late time $(t \to \infty)$ limit even from a normally distributed variate.

In chapter 10, we show that the above scale free differential equation which is actually not defined on a Cantor set, even in the usual (ultrametric) sense, is raised to an equation which is well defined on a Cantor

Set C [25]. The derivation becomes possible as every point of a Cantor set C is replaced by the closure of collection of gaps of another Cantor Set $\widetilde{C}$, called an inverted Cantor set, where the relative infinitesimals are supposed to live in. We also rederive local constancy of a Cantor function and the valuation is realized now as the so called nonsmooth solutions of the said scale invariant equation.

In the concluding chapter 11, we summarize our main results and also indicate briefly how the present formalism may be extended further.