CHAPTER - III

DUSTY FLUID FLOWS

SECTION – A

DUSTY NEWTONIAN FLUID FLOWS

J

PART ONE

ROTATIONAL MOTION OF A DUSTY VISCOUS FLUID CONTAINED IN THE SEMI-INFINITE CIRCULAR CYLINDER DUE TO AN INITIALLY APPLIED IMPULSE ON THE SURFACE *

3.1.1 Introduction

The unsteady motion of fluid resulting due to the pure rotation of a solid boundary or due to the application of a uniformly distributed shear stress along a solid boundary is of both theoretical and practical significance in fluid mechanics. A number of workers studied both steady and unsteady two-dimensional axi-symmetric rotational flow of a viscous fluid in view of its growing importance in various technical problems. Khamrui [1] analysed the slow steady motion of an infinite viscous fluid due to the rotation of a circular cylinder. Iben [2] considered the non-stationary, plane circular-symmetric flow of a viscous fluid which forms itself within as well as outside a rotating infinitely long circular cylinder. Bhattacharyya [3] studied the rotational motion produced in an enclosed fluid, contained in a circular cylinder of infinite depth. The disturbance was generated on the surface of the fluid by an impulsive couple. Later Mukherjee and Bhattacharyya [4] studied the rotational flow of viscous fluid due

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to the rotation of a circular cylinder or by the action of shearing stress on the boundary.

For quite a number of reasons, be they scientific or practical engineering ones, particles are added to fluids. Mixtures of fluid and solid lumps or particles are common in various fields of engineering—hydraulic, mechanical and chemical— and of geophysics and considerations of their motion raise many puzzling dynamical questions. Saffman [5] proposed dusty fluid model in terms of a large number density N (x,t) of undeformable spherical particles suspended in an incompressible fluid. Using the formulation of Saffman, many authors studied a number of dusty gas problems and the results were well documented in a review by Marble [6].

In the present investigation we extend the analysis of Bhattacharyya [3] to observe the effects of dust particles on the fluid flow. It is observed that the effect of mass concentration of dust particles on the flow field is to increase the velocity field of dusty fluid. This problem may have some bearing on the problems of transport of solid particles by air or water and motion of solid particles in a rocket motor exhaust.

3.1.2 Formulation of the problem

Initially the liquid and dust particles are at rest. Consider the flow of a dusty liquid in a long circular cylinder of radius 'a'. Disturbance is set up by an impulse of the shearing force on the surface. Referring the problem to cylindrical polar co-ordinates (r,θ,z), we take the z-axis along the axis of the cylinder and the origin on the surface of the fluid. The symmetry consideration gives

$$\left.\begin{array}{l} u_1 = u_3 = 0,\ v_1 = v_3 = 0,\ u_2 = u_2\ (r,z,t), \\ \\ v_2 = v_2(r,z,t) \text{ and } \dfrac{\partial}{\partial\theta} = 0\ , \end{array}\right\} \quad (3.1.1)$$

where u_2 and v_2 are circumferential velocities of liquid and dust particles respectively. Since the distribution of dust particles is uniform, the number density N of the particles equals N_0, a constant throughout the motion. Using equations (1.16) – (1.19), the linearised equations of motion of dusty fluid and that of dust particles become

$$\rho \frac{\partial u_2}{\partial t} = \mu\left(\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r}\frac{\partial u_2}{\partial r} - \frac{u_2}{r^2} + \frac{\partial^2 u_2}{\partial z^2}\right) + KN_0(v_2 - u_2), \quad (3.1.2)$$

$$m \frac{\partial v_2}{\partial t} = K\ (\ u_2 - v_2) \quad . \quad (3.1.3)$$

Initial and boundary conditions are

$$\left.\begin{array}{l} u_2 = 0 \text{ at } t \le 0 \text{ for all } z\ , \\ u_2 \longrightarrow 0 \text{ as } z \to \infty\ , \\ u_2\big|_{r=a} = 0,\ z \ge 0, \end{array}\right\} \quad (3.1.4)$$

$P_{\theta z}\big|_{z=0} = \mu\ \left(\frac{\partial u_2}{\partial z}\right)_{z=0}$ is prescribed as function of r and t, $r \le a$, $t \ge 0$.

Introducing the following non-dimensional quantities

$$u = \frac{u_2\ a}{\nu}\ , v = \frac{v_2\ a}{\nu}\ , t_1 = \frac{t}{\tau}\ , \quad r_1 = \frac{r}{(\nu\tau)^{1/2}},\quad z_1 = \frac{z}{(\nu\tau)^{1/2}}\quad .$$

$a_1 = \dfrac{a}{(\nu\tau)^{1/2}}$, $f = \dfrac{mN_0}{\rho}$ (mass concentration of dust particles),

$\tau = \dfrac{m}{K}$ (relaxation time of dust particles)

in (3.1.2) – (3.1.4), we get (dropping suffices)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} + f(v-u) \quad , \qquad (3.1.5)$$

$$\frac{\partial v}{\partial t} = (u - v) \ , \qquad (3.1.6)$$

$$u = 0 \text{ at } t \le 0 \text{ for all } z, \qquad (3.1.7)$$

$$u \longrightarrow 0 \text{ as } z \longrightarrow \infty, \qquad (3.1.8)$$

$$u\Big|_{r=a} = 0, \qquad z \ge 0 \quad , \qquad (3.1.9)$$

$$p_{\theta z}\Big|_{z=0} = \frac{\partial u}{\partial z}\Big|_{z=0} = F(r)\delta(t) \qquad \text{(prescribed)}. \qquad (3.1.10)$$

3.1.3 Method of solution

We solve the present problem by using technique of Laplace tranform. Taking Laplace transform of equations (3.1.5), (3.1.6) and using (3.1.7), we get

$$p\bar{u} = \frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} - \frac{\bar{u}}{r^2} + \frac{\partial^2 \bar{u}}{\partial z^2} + f(\bar{v} - \bar{u}) \quad , \qquad (3.1.11)$$

$$p\bar{v} = \bar{u} - \bar{v} \ , \qquad (3.1.12)$$

where

$$\bar{u} = \int_0^\infty u \, e^{-pt} dt, \qquad \bar{v} = \int_0^\infty v \, e^{-pt} dt, \quad Re(p)>0.$$

Eliminating $\bar{v}$ from (3.1.11) and (3.1.12) we get

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} - \frac{\bar{u}}{r^2} + \frac{\partial^2 \bar{u}}{\partial z^2} - \left[\frac{f}{p+1} + 1\right] p \, \bar{u} = 0. \qquad (3.1.13)$$

The solution of the above equation (3.1.13) can be expressed in the form

$$\bar{u}\,(r,z,p) = \sum_{n=1}^{\infty} \bar{u}_n \ , \qquad (3.1.14)$$

where

$$\bar{u}_n = J_1(\alpha_n r)\,\bar{\Phi}_n(z,p) \tag{3.1.15}$$

and α_n can be determined from the equation (3.1.9) as the positive root of

$$J_1(\alpha_n a) = 0. \tag{3.1.16}$$

On substituting the value of $\bar{u}_n$ from (3.1.15) in (3.1.13) we get

$$\frac{d^2\bar{\phi}_n}{dz^2} - \left\{ \alpha_n^2 + p\left(1 + \frac{f}{1+p}\right) \right\} \bar{\phi}_n = 0. \tag{3.1.17}$$

Solution of (3.1.17) subject to Laplace transform of the condition (3.1.8)

$$\bar{\phi}_n = A_n \exp\left[-z \left\{ \alpha_n^2 + p\left(1 + \frac{f}{1+p}\right) \right\}^{1/2} \right], \tag{3.1.18}$$

where A_n is independent of z for all n.

Transformed shearing stress on the surface is given by

$$\bar{p}_{\theta z}\Big|_{z=0} = F(r), \quad r \le a$$

$$= \sum_{n=1}^{\infty} C_n J_1(\alpha_n r), \tag{3.1.19}$$

where

$$C_n = 2a^{-2}\left[J_2(\alpha_n a) \right]^{-2} \int_0^a r\,F(r)\,J_1(\alpha_n r)\,dr. \tag{3.1.20}$$

Again from (3.1.14), (3.1.15) and (3.1.18) we have

$$\bar{p}_{\theta z}\Big|_{z=0} = -\sum_{n=1}^{\infty} J_1(\alpha_n r)\,A_n \sqrt{\left(\alpha_n^2 + p\left(1 + \frac{1}{1+p}\right)\right)}. \tag{3.1.21}$$

We now compare the expressions (3.1.19) and (3.1.21) to obtain

$$A_n = -C_n \left[\alpha_n^2 + p\left(1 + \frac{f}{1+p}\right) \right]^{-1/2}. \tag{3.1.22}$$

The expression for velocity becomes

$$u = -\sum_{n=1}^{\infty} C_n J_1(\alpha_n r) \mathcal{L}^{-1}\Big[\Big\{\alpha_n^2 + p\,(1 + \frac{f}{1+p})\Big\}^{-1/2}$$

$$\times \exp\Big\{ -z < \alpha_n^2 + p(1+f/(1+p)) >^{1/2} \Big\}\Big], \qquad (3.1.23)$$

where $\mathcal{L}^{-1}$ is the operator for inverse Laplace transform.

Inverse Laplace transform of (3.1.23) presents some difficulties and we restrict ourselves to calculate the velocity expression for large values of time 't' only. For large time, $p \ll 1$ and hence

$$\sqrt{\Big[\alpha_n^2 + p(1+\frac{f}{1+p})\Big]} \approx \sqrt{\Big[\alpha_n^2 + p(1+f)\Big]}\ .$$

Then

$$\mathcal{L}^{-1}\Big[\Big\{\alpha_n^2 + p(1+\frac{f}{1+p})\Big\}^{-1/2} e^{<-z\sqrt{\{\alpha_n^2 + p(1+\frac{f}{1+p})\}}>}\Big]$$

$$= \frac{1}{(1+f)^{1/2}} e^{-\frac{\alpha_n^2 t}{1+f}} \mathcal{L}^{-1}\Big[\frac{1}{p^{1/2}} e^{<-z(1+f)^{1/2}(p)^{1/2}>}\Big]$$

$$= \frac{1}{(1+f)^{1/2}} e^{-\frac{\alpha_n^2 t}{1+f}} \Big[(\pi t)^{-1/2} e^{-z^2(1+f)/4t}\Big]\ .$$

Equation (3.1.23) is then given by

$$u = \frac{-1}{\{\pi t(1+f)\}^{1/2}} \exp\{-z^2(1+f)/4t\}$$

$$\times \sum_{n=1}^{\infty} C_n J_1(\alpha_n r) \exp(-\frac{\alpha_n^2 t}{1+f})\ . \qquad (3.1.24)$$

This solution satisfies initial and boundary conditions given by (3.1.7) – (3.1.10).

3.1.4 Particular cases

CASE I. Motion due to impulsive shearing force applied within a circular area on the surface :

We take

$$F(r) = \epsilon r \quad \text{for} \quad 0 \le r \le b, \qquad = 0 \quad \text{for} \quad b < r \le a, \tag{3.1.25}$$

where ϵ is a constant.

Relation (3.1.25) corresponds to the situation where the applied force is acting within a circular area $r = b$, the rest of the surface being kept free from the impulse.

From (3.1.20) we get

$$C_n = 2 \epsilon b^2 J_2^{-2}(\alpha_n a) J_2(\alpha_n b) / \alpha_n a^2 . \tag{3.1.26}$$

On substituting the values of C_n in (3.1.24), we get the expression for velocity as

$$u = - \frac{2 \epsilon b^2}{a^2 \{\pi t (1 + f)\}^{1/2}} \exp\left\{-z^2(1 + f)/4t\right\}$$

$$\times \sum_{n=1}^{\infty} J_2(\alpha_n b) J_2^{-2}(\alpha_n a) J_1(\alpha_n r)\alpha_n^{-1} \exp\left(- \frac{\alpha_n^2 t}{1+f}\right). \tag{3.1.27}$$

In absence of dust particles ($f = 0$), the expression for velocity profile (equation (3.1.27)) becomes same as was deduced by Bhattacharyya [3] in equation (26) (when made non-dimensional).

CASE II. Flow due to applied impulsive force distributed over the circumference of the circle $r = b$; $b < a$:

We take

$$F(r) = S\,\delta(r - b) \tag{3.1.28}$$

where S is constant and δ is Dirac delta function.

From (3.1.20) we get

$$C_n = 2 S b J_1 (\alpha_n b) / a^2 J_2^2 (\alpha_n a). \qquad (3.1.29)$$

On substituting the value of C_n in (3.1.24), we get the expression for velocity as

$$u = \frac{-2 S b}{a^2 (\pi t)^{1/2} (1 + f)^{1/2}} \exp \{ - z^2 (1 + f) / 4t \}$$

$$X \sum J_1(\alpha_n r) J_1(\alpha_n b) J_2^{-2}(\alpha_n a) \exp\{ - \alpha_n^2 t/(1 + f)\}. \qquad (3.1.30)$$

In case of clean fluid (f=0), the expression for velocity profile is same as that was deduced by Bhattacharyya [3] in equation (32) (when made non-dimensional).

3.1.5 Discussion

In order to illustrate the effects of mass concentration of dust particles on the flow field for two particular cases, numerical calculations are carried out and depicted in Figures 3.1 and 3.2. Figures 3.1 and 3.2 reveal that magnitude of velocity of dusty fluid increases with the increase of mass concentration of dust particles. It is clear from both the figures that maximum velocity occurs near the axis of the cylinder for a fixed f and as f increases, maximum velocity shifts towards the wall of the cylinder.

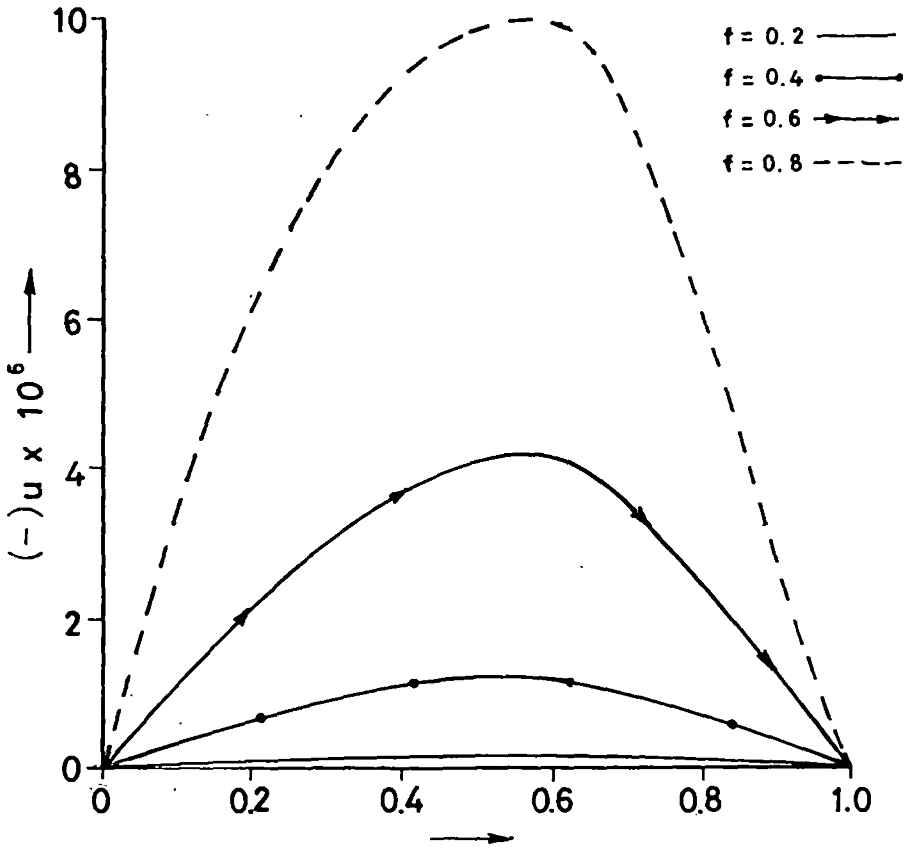

Fig.-3.1. Velocity distribution of dusty fluid for different values of f when t=1, ϵ=1, z=1, a=1, b=0.5

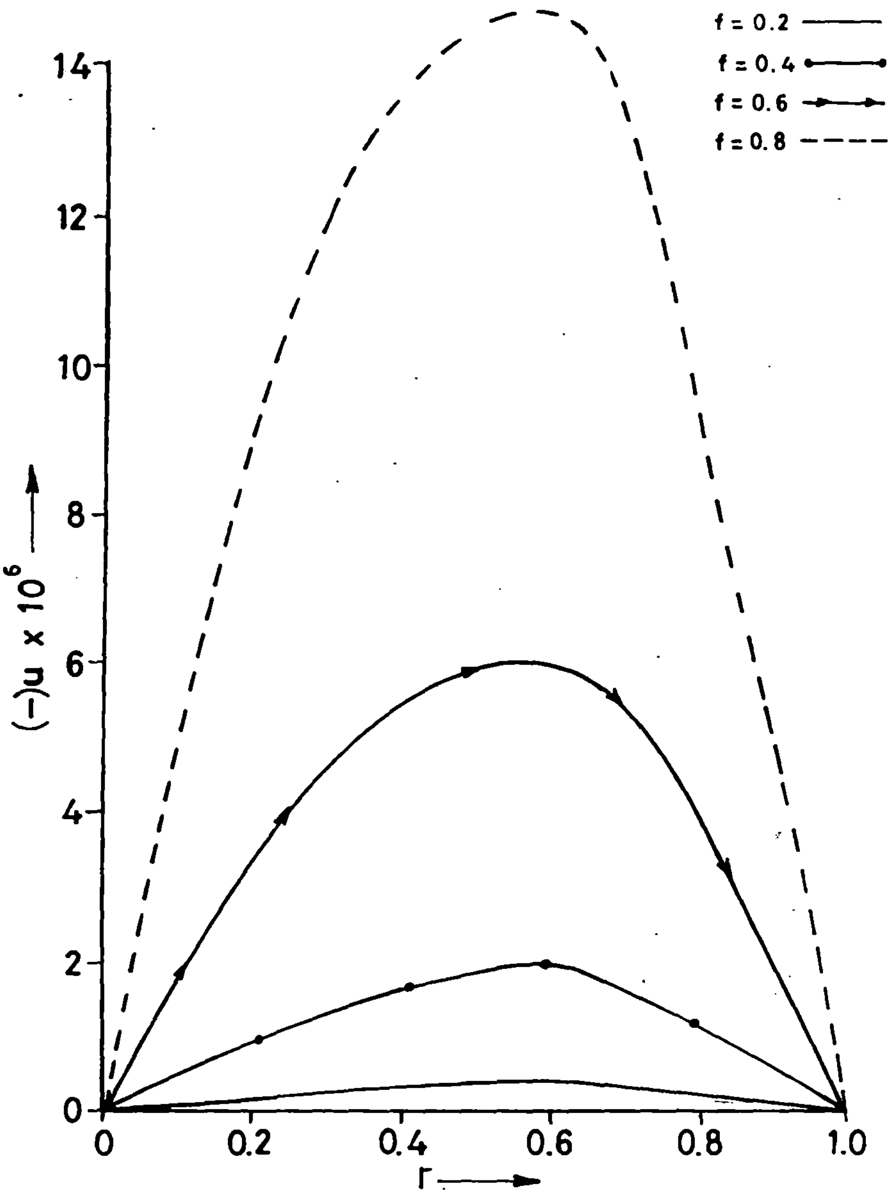

Fig.-3.2. Velocity distribution of dusty fluid for different values of f when t=1, s=1, z=1, a=1, b=0.5

EFFECT OF DUST PARTICLES ON THE FLOW OF INCOMPRESSIBLE FLUID IN A ROTATING SYSTEM

3.1.6 Introduction

The mechanical behaviour of dusty fluids has received greater attention during the recent past in the field of fluid dynamics. Problems dealing with the influence of dust particles on viscous flows find place in several branches. Some such flows are those of dissolved micromolecules, fibre suspensions, red corpuscles and other bodies in blood.

The momentum equations given by Saffman [5] characterizing the dusty fluid flow was discussed by Michael and Miller [7] for the flow in the semi-infinite space over flat plate. A comprehensive review of the dynamics of dusty gases was given by Marble [6]. Later, Ramana Prasad and Ramacharyulu [8] studied the unsteady flow of an incompressible viscous fluid with uniform distribution of dust particles between two parallel plates when one of which is impulsively stopped from the state of uniform motion. Little work seems to have been done on the flow of a dusty gas in a rotating system although this has bearing on the pollution problems as well as on the motion of aerosol over the rotating earth. Gupta and Pop [9] studied the unsteady boundary layer flow generated in a viscous dusty liquid bounded by an infinite flat plate. Here we consider the flow of viscous incompressible dusty fluid over a flat plate which is impulsively brought to rest from a state of uniform motion parallel to itself while both the liquid and the plate are in a state of rigid body rotation about an axis normal to the plate.

3.1.7 Formulation of the problem

We consider an infinite plate lying along the plane z=0 and situated in a viscous liquid in which there is a distribution of dust particles with a small bulk concentration. Initially the plate was rotating in unison with the liquid with a uniform angular velocity Ω about z-axis and was moving with a constant velocity. When the steady state was reached, the moving plate was impulsively brought to rest. The aim of the present paper is to investigate the subsequent motion.

Following Saffman [5], the momentum equations for the liquid in a rotating frame of reference are

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2} + 2\,\Omega\, v + \frac{KN_0}{\rho}\,(u' - u) , \qquad (3.1.31)$$

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial z^2} - 2\,\Omega\, u + \frac{KN_0}{\rho}\,(v' - v) \qquad (3.1.32)$$

along x and y directions respectively. Similarly, the momentum equations for dust particles along x and y directions are

$$\frac{\partial u'}{\partial t} = \frac{K}{m}\,(u - u') + 2\,\Omega\, v' , \qquad (3.1.33)$$

$$\frac{\partial v'}{\partial t} = \frac{K}{m}\,(v - v') - 2\,\Omega\, u' , \qquad (3.1.34)$$

where (u,v,0) and (u′,v′,0) are components of velocities of liquid and dust particles respectively, m is the mass of a dust particle, K is the Stokes resistance coefficient. The other symbols have their usual meanings. Equations (3.1.31) and (3.1.32) are combined as

$$\frac{\partial q}{\partial T} = \frac{\partial^2 q}{\partial Z^2} - (2i\omega + f)q + fq' , \qquad (3.1.35)$$

where

$$q = \frac{u + iv}{U} , \quad q' = \frac{u' + iv'}{U} , \; T = \frac{t}{\tau} , \; Z = \frac{z}{(\nu t)^{1/2}},$$

$\Omega\tau = \omega$, $f = \frac{mN_0}{\rho}$ is the mass concentration, $\tau = \frac{m}{K}$ is known as the relaxation time of dust.

Equations (3.1.33) and (3.1.34) are combined as

$$\frac{\partial q'}{\partial T} = q - (2i\omega + 1) q' . \qquad (3.1.36)$$

3.1.8 Steady state solution

The equations of steady state motion for fluid and dust particles are given by

$$\frac{d^2 q}{dZ^2} - (2i\omega + f)q + fq' = 0 \qquad (3.1.37)$$

and

$$q' = \frac{1}{1 + 2i\omega} q . \qquad (3.1.38)$$

It is clear from equation (3.1.38) that, in a rotating system, the fluid and dust particles do not move with same velocity whereas, in the absence of rotation, fluid and dust particles move with same velocity.

The boundary conditions are

$$\left.\begin{aligned} &q = 1 \text{ for } Z = 0 \quad \text{and} \\ &q \longrightarrow 0 \text{ as } Z \longrightarrow \infty . \end{aligned}\right\} \qquad (3.1.39)$$

The steady state velocities are given by

$$q = e^{-MZ} \tag{3.1.40}$$

and

$$q' = \frac{1}{1 + 2i\omega} e^{-MZ} \tag{3.1.41}$$

------------------with the proviso that the real part of M is taken positive where

$$M = \left(2i\omega \left(1 + \frac{f}{1 + 2i\omega}\right)\right)^{1/2}.$$

3.1.9 Flow when the moving plate is impulsively brought to rest

The fluid velocity q(Z,T) and dust velocity q'(Z,T) satisfy the equations (3.1.35) and (3.1.36) respectively. The initial conditions are

$$q(Z,0) = e^{-MZ}, \tag{3.1.42}$$

$$q'(Z,0) = \frac{1}{1 + 2i\omega} e^{-MZ}. \tag{3.1.43}$$

The boundary conditions is

$$q(0,T) = 0 \text{ for } T > 0. \tag{3.1.44}$$

The boundary conditions on dust velocity can not be prescribed as dust particles may slip on the plate z = 0.

Let $\bar{q}$ and $\bar{q}'$ be the Laplace transforms of q and q' respectively, defined by

$$\bar{q} = \int_0^\infty e^{-pt} q(Z,T)dT,$$

$$\bar{q}' = \int_0^\infty e^{-pt} q'(Z,T)dT, \quad Re(p) > 0.$$

With the help of equations (3.1.42) and (3.1.43), the Laplace transforms of equations (3.1.35) and (3.1.36) (after rearrangement) give

$$\frac{d^2\bar{q}}{dZ^2} - N^2\,\bar{q} = -\,K'\,e^{-MZ} \quad , \tag{3.1.45}$$

$$\bar{q}\,' = \frac{1}{(1 + p + 2i\omega\,)}\ [\,\bar{q} + \frac{e^{-MZ}}{1 + 2i\omega}\,] \ , \tag{3.1.46}$$

where

$$N^2 = (p + 2i\omega\,)\ \frac{1 + p + f + 2i\omega}{1 + p + 2i\omega}$$

and

$$K' = 1 + \frac{f}{(1 + 2i\omega\,)\ (1 + p + 2i\omega\,)} \quad .$$

The transformed boundary conditions are

$$\left.\begin{aligned} &\bar{q} = 0 \text{ on } Z = 0 \quad \text{and} \\ &\bar{q} \longrightarrow 0 \text{ as } Z \longrightarrow \infty \quad . \end{aligned}\right\} \tag{3.1.47}$$

The solution of equation (3..1.45) subject to the boundary conditions in (3.1.47) is given by

$$\bar{q} = \frac{K'}{M^2 - N^2}\ (\,e^{-NZ} - e^{-MZ}\,) \tag{3.1.48}$$

with the proviso that the real part of N is taken positive.

Using equation (3.1.48) in equation (3.1.46) we find the velocity of dust particles at the plate z = 0 as

$$q'(0,T) = \frac{1}{1 + 2i\omega}\ \mathcal{L}^{-1}\ [\frac{1}{(1 + p + 2i\omega\,)}\,] \ , \tag{3.1.49}$$

where $\mathcal{L}^{-1}$ is the inverse Laplace transform operator. Evaluation of the inverse transform of (3.1.49) and separating the resulting expression into real and imaginary parts, we have

$$\frac{u'(0,T)}{U} = \frac{e^{-T}}{1 + 4\omega^2}\ [\,\cos 2\omega T - 2\omega \sin 2\omega T\,] \ , \tag{3.1.50}$$

$$\frac{v'(0,T)}{U} = \frac{-\,e^{-T}}{1 + 4\omega^2}\ [2\omega \cos 2\omega T + \sin 2\omega T\,] \quad . \tag{3.1.51}$$

Inversion of (3.1.48) presents some difficulties and we restrict ourselves to large values of time T which correspond to small values of p.

Inverting (3.1.48), we have

$$q(Z,T) = \frac{(1+2i\omega)^2+f}{(1+2i\omega)(1+f+2i\omega)} \left[\frac{\eta \exp\{-(2i\omega T+\eta^2/4T)\}}{2(\pi T)^{1/2}(2i\omega T-\eta^2/4T)}\right], \quad (3.1.52)$$

where

$$\eta = \sqrt{1+\frac{f}{1+2i\omega}}\,.Z = \frac{1}{(1+4\omega^2)^{1/2}}(A_3-iB_3)Z\,.$$

Separating (3.1.52) into real and imaginary parts, we have

$$\frac{u}{U} = A(A_7 A_8 + B_7 B_8), \quad (3.1.53)$$

$$\frac{v}{U} = A(A_7 B_8 - A_8 B_7), \quad (3.1.54)$$

where

$$A = \frac{2Z\exp\{-A_4/4T(1+\omega^2)\}}{\{(1+f-4\omega^2)^2+4\omega^2(2+f)^2\}(A_5^2+B_5^2)}\sqrt{\frac{(1+4\omega^2)T}{\pi}},$$

$$A_1 = Z^2(1+f) - 16\omega^2T^2,$$

$$B_1 = 8\omega T^2 + 2\omega Z^2,$$

$$A_2 = -Z^2(1+f) - 16\omega^2T^2,$$

$$B_2 = 8\omega T^2 - 2\omega Z^2,$$

$$A_3 = \frac{1}{(2)^{1/2}}\left[((1+f+4\omega^2)^2+4f^2\omega^2)^{1/2} + (1+f+4\omega^2)\right],$$

$$B_3 = \frac{1}{(2)^{1/2}}\left[((1+f+4\omega^2)^2+4f^2\omega^2)^{1/2} - (1+f+4\omega^2)\right],$$

$$A_4 = A_1 + 2\omega B_1 ,$$

$$B_4 = B_1 - 2\omega A_1 ,$$

$$A_5 = A_2 + 2\omega B_2 ,$$

$$B_5 = B_2 - 2\omega A_2 ,$$

$$A_6 = A_3 A_5 - B_3 B_5 ,$$

$$B_6 = B_3 A_5 + A_3 B_5 ,$$

$$A_7 = A_6 \cos B_4 - B_6 \sin B_4 ,$$

$$B_7 = A_6 \sin B_4 + B_6 \cos B_4 ,$$

$$A_8 = (1 + f - 4\omega^2)^2 + 8\omega^2 (2 + f) ,$$

$$B_8 = -2\omega f (1 + f - 4\omega^2) .$$

3.1.10 Discussion

From the expression (3.1.50) and (3.1.51) for the velocity of dust particles at the plate wall, it may be remarked that inertial oscillations take place with very small amplitude.

We have calculated numerically the values of u and v from the expressions in (3.1.53) and (3.1.54) for different values of f and graphs are drawn to show the effects of dust particles on velocity profile of liquid particle in the directions of x and y respectively. Figures 3.3 and 3.4 show that motion is highly of oscillating nature. Also, flow reversal takes place in different layers both for u and v. Amplitude of oscillation steadily increases with the distance from the plate upto a certain distance and then rapidly decreases and tends to zero as it is expected. As f increases, the change of phase of oscillations is quicker near the plate wall.

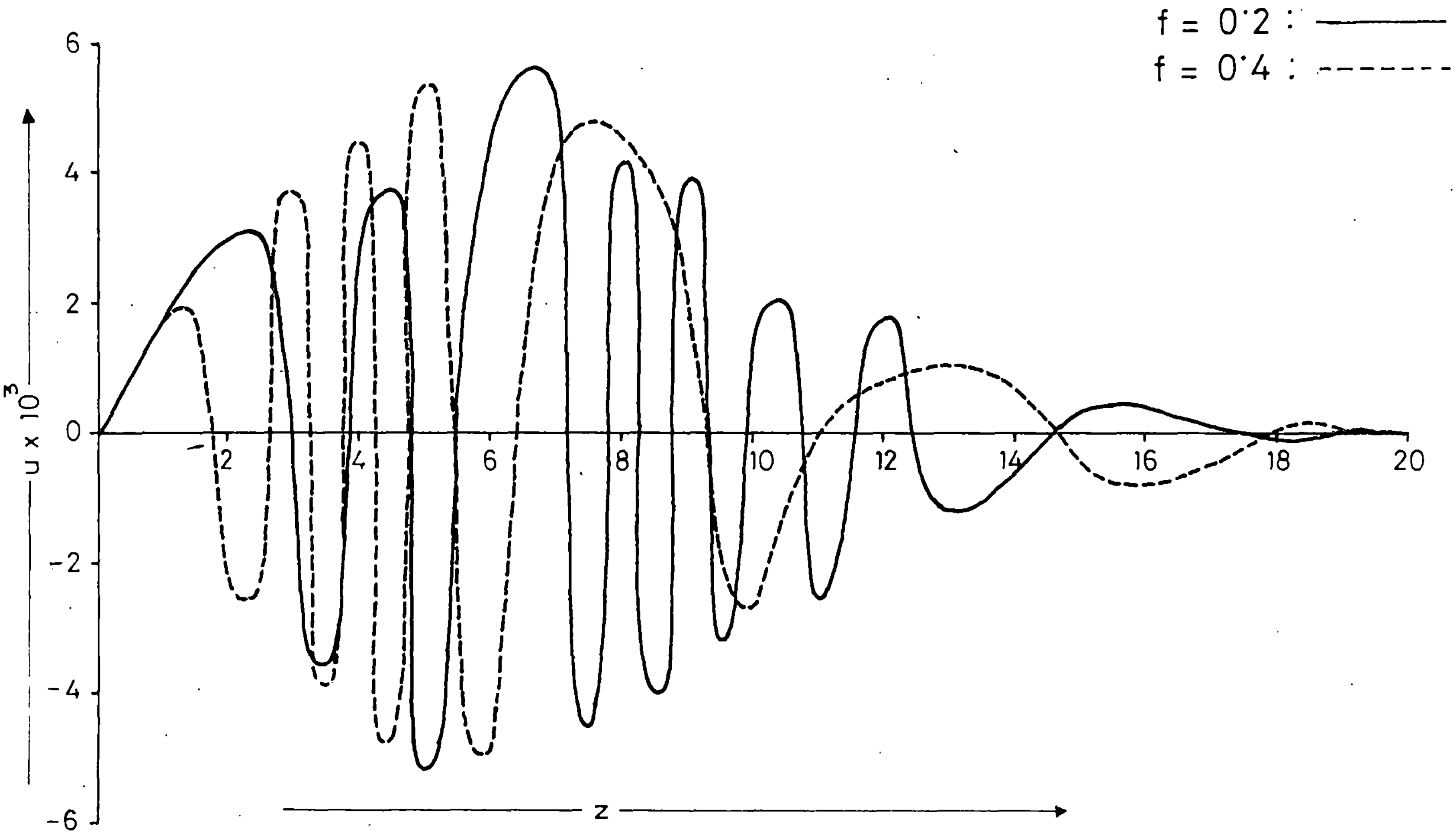

Fig. 3·3. Plot of $u \times 10^3$ against z for different values of f when $t = 40$, $\omega = 1$.

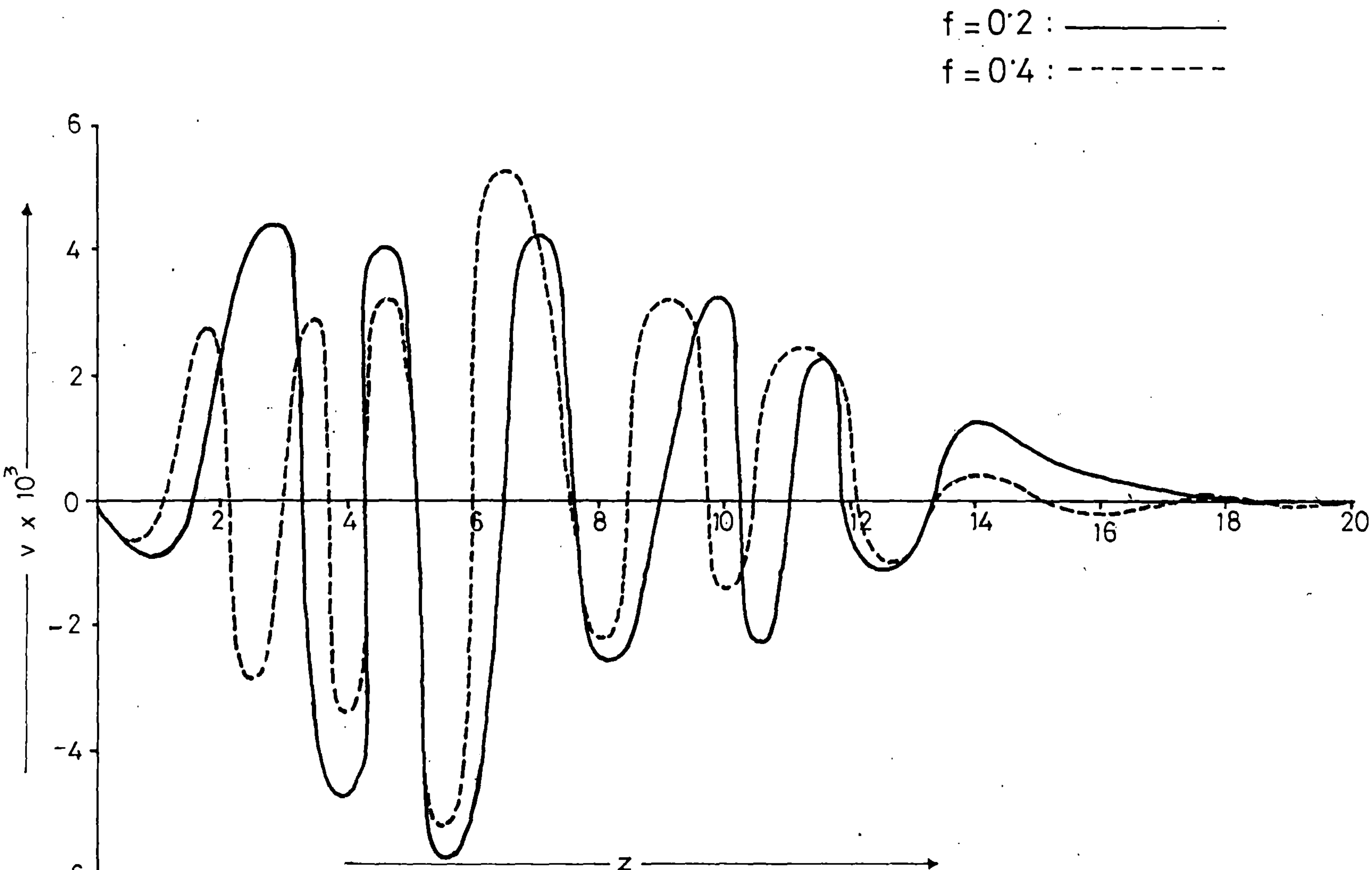

Fig. 3·4. Plot of $v \times 10^3$ against z for different values of f when $t = 40$, $\omega = 1$.

SECTION - B

DUSTY NON-NEWTONIAN FLUID FLOWS

UNSTEADY FLOW OF DUSTY VISCO-ELASTIC LIQUID BETWEEN TWO OSCILLATING PLATES *.

3.2.1 Introduction

Interests in problems of flow of a dusty gas i.e. a mixed system of fluid and particles have increased enormously in recent years. A model equation describing the motion of such mixed system has been given by Saffman [5]. Based upon Saffman's model numerous authors [6,10,11] investigated a number of dusty gas flow problems in different situations.

There is another class of flow problems which concern with the study of the flow of dusty non-Newtonian fluids such as latex particles in emulsion paints, reinforcing particles in polymer melts and rock crystals in molten lava. However, the study on this class of problems and rheological aspects of such flows have not received much attention. Little work is reported in literature [12,13,14,15] on the flow of dusty non-Newtonian fluid, although this has some bearing on the problems of petroleum industry and chemical engineering interest. This consideration provides motivation for the present study.

In the present investigation, we consider the unsteady laminar flow of a visco-elastic liquid [16] containing uniformly

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small solid particles between two infinitely extended parallel plates when the lower plate is at rest and the upper one begins oscillating harmonically in its own plane. The velocity fields for the liquid and dust particles are obtained explicitely by using the technique of Laplace transform. The effect of elastic element in the liquid, the mass concentration and the relaxation time of dust particle on the velocity profiles are studied graphically. The skin-friction at the lower plate wall and the total volume-flow in between the plates are also obtained.

This problem is likely to have some industrial and chemical engineering applications on the problems of transport of solid particles suspended in visco-elastic fluids through channels.

3.2.2 Basic equations and their solutions

We suppose that the dusty visco-elastic liquid fills the space between two infinite parallel flat plates at a distance h apart. The lower plate is kept at rest and the upper one begins to perform harmonic oscillations with a frequency w in its own plane. In our analysis we take a co-ordinate system such that the x-axis coincides with the lower fixed plate and the z-axis is perpendicular to it. The dust particles are assumed to be spherical in shape and uniform in size and the number density of dust particles is taken as constant throughout the flow and let it be N_0. Since the plates are infinite, the velocities will depend on z and time t only.

Using equations (1.12) - (1.19) we get the equations of motion of dusty visco-elastic liquid as (dropping dashes)

$$(1 + \alpha \frac{\partial}{\partial t}) \frac{\partial u}{\partial t} = R \frac{\partial^2 u}{\partial z^2} + \frac{f}{\tau} (1 + \alpha \frac{\partial}{\partial t})(v - u), \qquad (3.2.1)$$

$$\frac{\partial v}{\partial t} = \frac{1}{\tau} (u - v), \qquad (3.2.2)$$

where u and v are velocities of liquid and dust particles respectively in the direction of x and $u' = \frac{u}{h\,w}$,

$v' = \frac{v}{hw}$, $t' = tw$, $z' = \frac{z}{h}$, $\alpha = \lambda_0 w$, $R = \frac{\nu}{h^2 w}$,

f(mass concentration) = $\frac{m\ N_0}{\rho}$, τ(relaxation time)= $\frac{m\ w}{K}$.

The initial and boundary conditions in non-dimensional form are

$t \leq 0$:

$$u = \frac{\partial u}{\partial t} = 0 \text{ for all } z, \qquad (3.2.3)$$

$t > 0$:

$$\left.\begin{array}{ll} u = a \sin t & \text{at } z = 1, \\ u = 0 & \text{at } z = 0. \end{array}\right\} \qquad (3.2.4)$$

Taking Laplace transforms of (3.2.1) and (3.2.2), using (3.2.3), we get

$$p(1 + \alpha\ p)\ \bar{u} = R \frac{\partial^2 \bar{u}}{\partial z^2} + \frac{f}{\tau} (1 + \alpha\ p)\ (\bar{v} - \bar{u}\ ), \qquad (3.2.5)$$

$$v = \frac{1}{1 + p\,\tau} \bar{u}\ , \qquad (3.2.6)$$

where

$$\bar{u} = \int_0^\infty u e^{-pt} dt, \quad \bar{v} = \int_0^\infty v e^{-pt} dt \quad ,$$

$Re(p) > 0$.

Transformed boundary conditions are

$$\left.\begin{aligned} \bar{u} &= \frac{a}{1+p^2} \quad \text{at } z = 1, \\ \bar{u} &= 0 \quad\quad\quad \text{at } z = 0. \end{aligned}\right\} \tag{3.2.7}$$

Substituting (3.2.6) into (3.2.7) we get

$$R \frac{\partial^2 \bar{u}}{\partial z^2} - \bar{u} \left[\frac{F + p\tau}{p\tau + 1} \right] p(1 + \alpha p) = 0, \tag{3.2.8}$$

where $f+1 = F$.

Thus the solution of (3.2.8) is

$$\bar{u} = A \cosh Mz + B \sinh Mz, \tag{3.2.9}$$

where

$$M^2 = \frac{(F + p\tau)\, p(1 + \alpha p)}{R(p\tau + 1)} .$$

On using boundary conditions in (3.2.7) and taking inverse transform, we get

$$u = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{a}{1+p^2} \frac{\sinh Mz}{\sinh M} e^{pt} dp , \tag{3.2.10}$$

where γ is greater than the real part of the singularities of the integrand in (3.2.10). The integrand is an integral function of p and has singularities at $p = \pm i$ and at the zeros of sinh M. Calculating the residues and simplifying further, we obtain the expression for u as

$$u = \frac{a}{E_3^2 + E_4^2} \left[(E_1E_4 - E_2E_3)\cos t + (E_1E_3 + E_2E_4)\sin t \right]$$
$$+ 2 R \pi \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} (-1)^{n+1} \frac{n}{1 + p_n^{i\,2}} Q(p_n^{\,i}) \sin(n\pi z)\, e^{p_n^i t} , \tag{3.2.11}$$

where

$$E_1 = \sin N_2 z \cosh N_1 z \quad ,$$
$$E_2 = \cos N_2 z \sinh N_1 z \quad ,$$
$$E_3 = \sin N_2 \cosh N_1 \quad ,$$
$$E_4 = \cos N_2 \sinh N_1 \quad ,$$

$$N_1, N_2 = \frac{1}{2}\Big[\big[\{(F - \tau\alpha) + \tau(\tau + F\alpha)\}^2 + \{\tau(F - \tau\alpha) - (\tau + F\alpha)^2\}\big]^{1/2}$$
$$\pm \left\{ \tau(F - \tau\alpha) - (F\alpha + \tau) \right\} \Big]^{1/2} X \frac{1}{\{R(1 + \tau^2)\}^{1/2}} ,$$

and p_n^i 's are those roots of the cubic equation

$$p_n^{i^3} \cdot \tau\alpha + p_n^{i^2}(\tau + \alpha F) + p_n^i(F + n^2\pi^2 R\tau) + R n^2 \pi^2 = 0 , \tag{3.2.12}$$

which are having negative real parts,

$$Q(p_n^i) = (1 + p_n^i\tau)^2 / \Big[\Big\{ p_n^i \tau (1 + \alpha p_n^i) + (1 + 2\alpha p_n^i) \; X (F + \tau p_n^i) \Big\} - \tau p_n^i (F + \tau p_n^i)(1 + \alpha p_n^i) \Big] . \tag{3.2.13}$$

By the convolution theorem, we obtain from (3.2.6)

$$v = \frac{a}{E_3^2 + E_4^2} \frac{\tau}{\tau^2 + 1} \Big[(E_1E_3 + E_2E_4) \left\{ \left(\frac{1}{\tau} \sin t - \cos t\right) + e^{-t/\tau} \right\}$$
$$+ (E_1E_4 - E_2E_3) \left\{ \left(\sin t + \frac{1}{\tau} \cos t\right) - \frac{1}{\tau} e^{-t/\tau} \right\} \Big]$$
$$+ 2R\pi \sum_{n=0}^{\infty} \sum_{i=1}^{3} \Big[\frac{(-1)^{n+1}}{p_n^i\tau + 1} \frac{n}{p_n^{i^2} + 1} Q(p_n^i) \sin(n\pi z)$$
$$X \left(e^{p_n^i t} - e^{-t/\tau}\right) \Big] . \tag{3.2.14}$$

The dimensionless shearing stress τ_p at the lower plate due to the dusty visco-elastic liquid is

$$\tau_p = \left[\left(1 - \alpha \frac{\partial}{\partial t}\right) \frac{\partial u}{\partial z} \right]_{z=0}$$

$$= \frac{a}{E_3^2 + E_4^2} \Big[\cos t \left\{ (E_1 E_4 - E_2 E_3) - \alpha (N_2 E_3 + N_1 E_4) \right\}$$

$$+ \sin t \left\{ (E_1 E_3 + E_2 E_4) + \alpha (N_2 E_4 - N_1 E_3) \right\} \Big]$$

$$+ 2 R \pi \sum_{n=0}^{\infty} \sum_{i=1}^{3} \Big[(-1)^{n+1} \cdot \frac{n^2 \pi}{(p_n^{i^2} + 1)} Q(p_n^i) e^{p_n^i t} (1 - \alpha p_n^i). \qquad (3.2.15)$$

The volume flow of dusty visco-elastic liquid discharged per unit breadth of the plate is given by

$$Q_v = 2 \int_0^1 u \, dz = \frac{2a}{(N_1^2 + N_2^2)(E_3^2 + E_4^2)} \Big[(E_4 \cos t + E_3 \sin t)$$

$$X(N_1 \sin N_2 \sinh N_1 - N_2 \cos N_2 \cosh N_1 + 2 N_2)$$

$$+(E_4 \sin t - E_3 \cos t) (N_1 \cos N_2 \sinh N_1 + N_2 \sin N_2 \cosh N_1) \Big]$$

$$- 2 \pi R \cdot \sum_{n} \sum_{i=1}^{3} \frac{1}{(1 + p_n^{i^2}) \pi} \cdot Q(p_n^i) \cdot e^{p_n^i t} , \qquad (3.2.16)$$

where n = 1, 3, 5,..... .

3.2.3 Discussion

The present analysis reveals that the solution contains three pertinent non-dimensional parameters viz. α (elastic parameter of the liquid particle), τ (relaxation time of dust particle), and F (= (f+1) , f is mass concentration of dust particle). The behaviour of these parameters, therefore, yields a physical insight into the problem. Numerical computation is made to observe the effects of these parameters on velocity profiles, skin-friction at the lower plate and volume flow in between the plates.

Figures 3.5 and 3.6 depict the velocity profiles of liquid and dust particles against z for different values of mass concentration and elastic parameter when τ is fixed . It is interesting to note that both u and v increase with the increase in f in the case of Newtonian fluid ($\alpha = 0$) but behave in a reverse fashion when the fluid is visco-elastic. It is also to be noted that velocity profiles of dusty fluid and dust particles decrease with the increase in elastic parameter. Figure 3.7 reveals that u decreases with the increase in relaxation time τ (with F fixed) when $\alpha = 0$ but in the presence of elastic element u increases with increase in τ. Figure 3.8 shows that the effect of relaxation time is to increase the velocity of dust particles irrespective of whether the fluid is Newtonian or non-Newtonian. It is also to be remarked that the influence of relaxation time (τ) is more on the velocity of dust particles than that of dusty liquid. But the mass concentration has very little effect on both u and v irrespective of whether the fluid is Newtonian or non-Newtonian.

Finally, for some representative values of F and τ, skin-friction at the lower plate and the volume flow in between the plate walls are calculated numerically for different values of α. Table 3.1 reveals that the magnitude of shear stress and total flux increase with the increase in the value of elastic parameter.

Table 3.1 : Shear stress (τ_p) at the lower plate and volume flow (Q_v) in between the plates.

	τ_p			Q_v		
α	t = 5	t = 10	t = 15	t = 5	t = 10	t = 15
1.0	−1.2426252	0.2952193	1.4101103	−0.4795342	−0.2721468	0.3251387
2.0	−1.5263597	1.1345457	2.1700154	−0.4795486	−0.2722571	0.3252040
2.25	−1.5834031	1.3425213	2.3920024	−0.4795531	−0.2722841	0.3252201

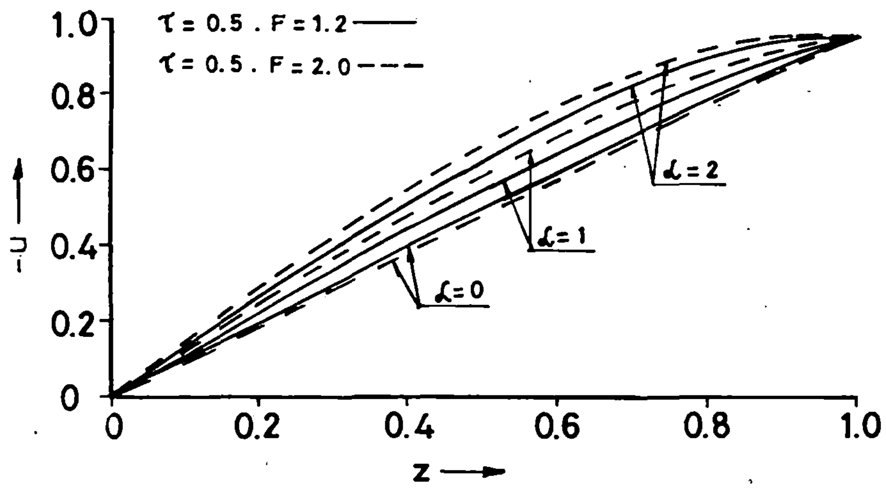

Fig.- 3.5. Velocity profile of dusty fluid (u) against z at t = 5 & $\tau = 0.5$

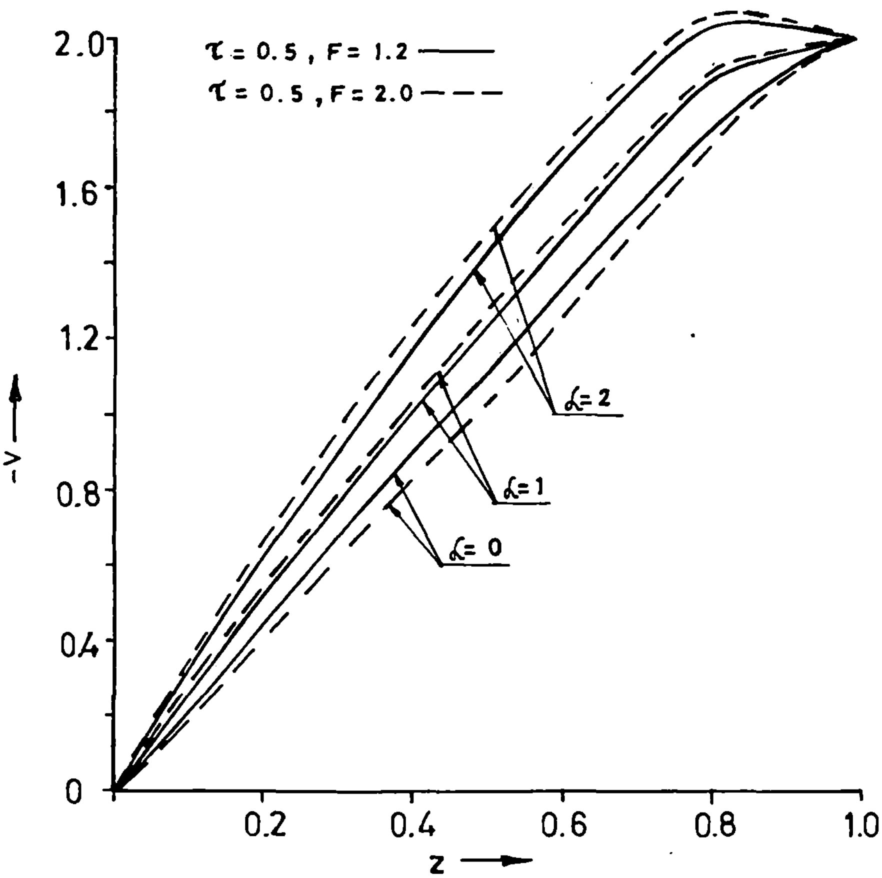

Fig.-3.6. Velocity profile of dust particles (v) against z at t = 5 & τ = 0.5

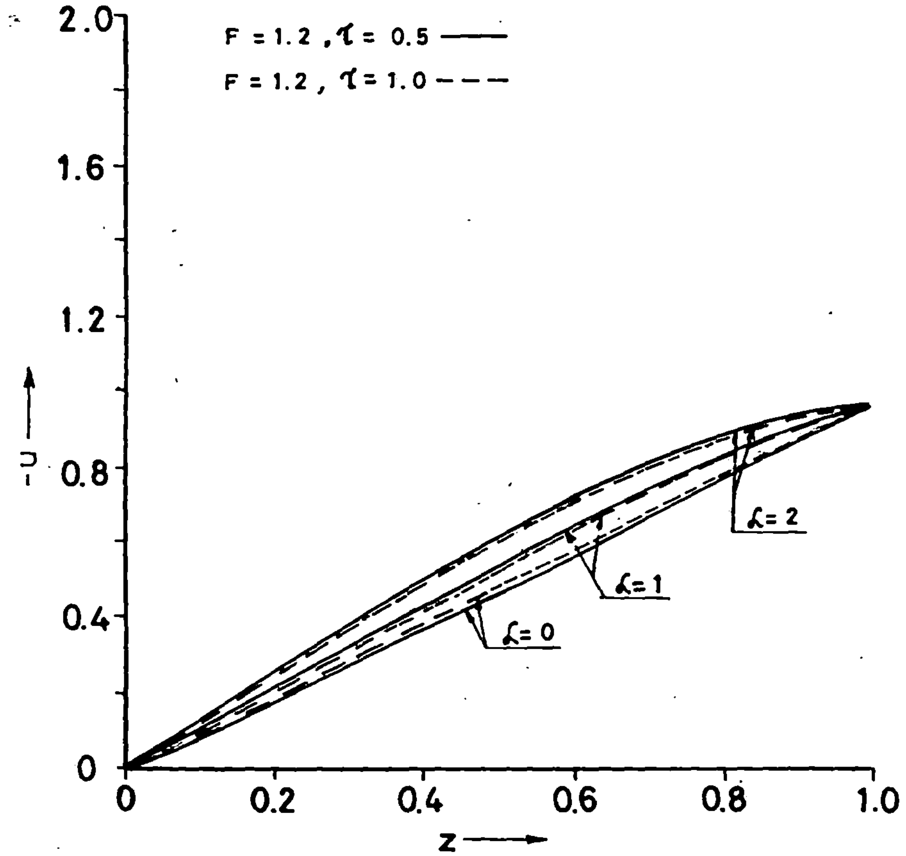

Fig.-3.7. Velocity profile of dusty fluid (u) against z at t = 5 & F = 1.2

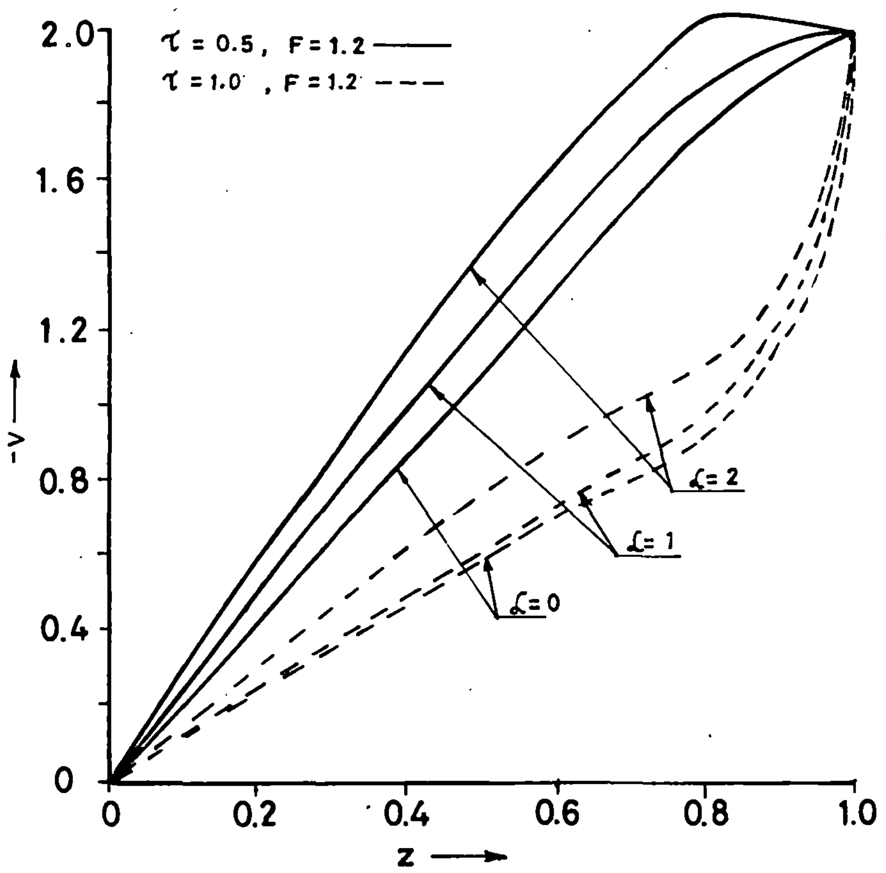

Fig.-3.8. Velocity profile of dust particles (v) against z at t = 5 & F = 1.2.

UNSTEADY AXISYMMETRIC ROTATIONAL FLOW OF DUSTY ELASTICO-VISCOUS LIQUID*

3.2.4 Introduction

The study of rotational transient flow of non-Newtonian fluids in both semi-infinite field and bounded field is of practical need for certain industrial processes to have the description of fluid mechanical phenomena exhibited by non-Newtonian materials. Srivastava [17] and Tandon [18] analysed the propagation of small disturbances in an Oldroyd fluid contained in a semi-infinite circular cylinder due to the slow rotation of a disc at the base. Srivastava [17] considered the radius of the disc to be same as the radius of the cylinder while Tandon [18] considered it to be smaller. Rao and Rao [19] investigated the rectilinear oscillations of a circular cylinder about a mean position along a diameter in an infinitely extended micropolar fluid. Tandon and Chandra [20] discussed the unsteady motion inside and outside of an infinite cylinder which suddenly starts rotating impulsively about an axis in an incompressible Oldroyd's two-parametric fluid, not three-parametric one as claimed by the authors. Recently Mukherjee and Mukherjee [21] considered the unsteady axisymmetric rotational flow of

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elastico-viscous liquid due to the time-dependent rotation of a circular cylinder.

However, studies on dusty non-Newtonian fluid flows and rheological aspects of such flows have not received much attention though the studies of dusty non-Newtonian fluid flows are likely to have some industrial and chemical engineering applications on the problems of polluted oil extraction, polymer extrution and paint spraying. Based upon the theoretical model proposed by Saffman [5], Srivastava [12] analysed the unsteady flow of dusty Rivlin-Ericksen fluid through a channel. Bagchi and Maiti [14] studied the unsteady flow of dusty elastico-viscous liquid through a channel with arbitrary time-varying pressure gradient.

Here we study the rotational flow of dusty elastico-viscous liquid. The expressions for the velocity fields of the liquid and the dust particles are obtained explicitely. The effect of elastic element in the liquid, the mass concentration and the relaxation time of dust particles on the velocity profiles of liquid and dust particles are studied graphically. This problem is likely to have some bearing on the problems of transport of solid particles suspended in non-Newtonian fluids through pipes.

3.2.5 Mathematical formulation of the problem

In the present problem, it is assumed that the particles are spherical in shape and uniform in size and the bulk-concentration (concentration by volume) of dust is very small. Following Saffman, it is assumed that steady Stokes law of resistance between the particles and fluid is applicable. However the mass concentration of dust can be of the order of unity by allowing the ratio of the density of the dust and fluid to be large. For sufficiently small particles, the velocity of sedimentation will be small compared with a characteristic velocity of the flow and can be neglected.

Initially the liquid and dust particles are at rest. We consider the flow of a dusty elastico-viscous liquid in an infinitely long circular cylinder of radius 'a' which oscillates with constant frequency about the axis of the cylinder. In the cylindrical polar co-ordinate system (r,θ,z), the z-axis is chosen along the axis of the cylinder. The physics of the problem suggests

$$u_i \equiv (0,u(r,t),0)$$

and

$$v_i \equiv (0,v(r,t),0),$$

where u_i and v_i are the local velocity vectors of liquid and dust particles respectively.

Using equations (1.12) – (1.19), we get the equations of motion of dusty elastico-viscous liquid [16] as

$$(1 + \lambda_0 \frac{\partial}{\partial t})\frac{\partial u}{\partial t} = \nu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} \right) + \frac{KN_0}{\rho} (1 + \lambda_0 \frac{\partial}{\partial t})(v - u), \tag{3.2.17}$$

$$\frac{\partial v}{\partial t} = \frac{K}{m} (u - v) , \tag{3.2.18}$$

where ν is the kinematic viscosity of the liquid and the number density of dust particles is $N = N_0$, a constant throughout the motion, λ_0 is the relaxation time, K is the Stokes resistance coefficient.

Initial and boundary conditions for the problem are

$$\left.\begin{aligned} u(r,t) &= \frac{\partial u(r,t)}{\partial t} = 0 , \\ v(r,t) &= \frac{\partial v(r,t)}{\partial t} = 0 , \end{aligned}\right\} \text{ at } t = 0 \text{ and for all } r \qquad (3.2.19)$$

$$\left.\begin{aligned} &u = u_0\, e^{-i\Omega t} \text{ on } r=a , \\ &u \text{ is finite on } r=0 , \end{aligned}\right\} t > 0 \qquad (3.2.20)$$

where u_o is the characteristic velocity and Ω is the imposed oscillation.

Using the non-dimensional variables

$$\bar{u} = \frac{u}{u_o} , \ \bar{v} = \frac{v}{u_o} , \ \bar{r} = \frac{r}{a} , \ \bar{t} = \frac{t\nu}{a^2} , \ \bar{\Omega} = \frac{\Omega a^2}{\nu}$$

equations (3.2.17) – (3.2.20) in non-dimensional form are written as (dropping bars)

$$(1 + \alpha \frac{\partial}{\partial t})\frac{\partial u}{\partial t} = (\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}) + \beta(1 + \alpha \frac{\partial}{\partial t})(v - u), \qquad (3.2.21)$$

$$\frac{\partial v}{\partial t} = \frac{1}{\tau}(u - v), \qquad (3.2.22)$$

$$\left.\begin{aligned} u &= \frac{\partial u}{\partial t} = 0, \\ v &= \frac{\partial v}{\partial t} = 0, \end{aligned}\right\} \text{ at } t = 0 \text{ and for all } r. \qquad (3.2.23)$$

$$\left.\begin{aligned} &u = e^{-i\Omega t} \text{ on } r=1, \\ &u \text{ is finite on } r = o, \end{aligned}\right\} t > 0 \qquad (3.2.24)$$

where

$\alpha \left[= \frac{\lambda_0 \nu}{a^2} \right]$ the non-dimensional parameter,

$f \left[= \frac{m\, N_0}{\rho} \right]$ the mass concentration of dust particles,

$\tau\left(=\dfrac{m\,\nu}{Ka^2}\right)$ the dimensionless relaxation time of dust particles and $\beta = \dfrac{f}{\tau}$.

3.2.6 Solution of the problem

Using the Laplace transform technique in equations (3.2.21) and (3.2.22) subject to initial and boundary conditions in equations (3.2.23) and (3.2.24), it turns out that the expressions for the velocity profile of liquid and dust particles can be represented by the Laplace inversion integral in the form

$$u = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{(p+i\,\Omega)}\;\frac{I_1\left[r\left\{\dfrac{p(1+\alpha p)(1+f+p\tau)}{(p\tau+1)}\right\}^{1/2}\right]}{I_1\left[\left\{\dfrac{p(1+\alpha p)(1+f+p\tau)}{(p\tau+1)}\right\}^{1/2}\right]}.e^{pt}\,dp\;, \quad (3.2.25)$$

$$v = \frac{1}{\tau}\,e^{-t/\tau}\int_0^t u(r,\lambda)e^{\lambda/\tau}\,d\lambda \;, \quad (3.2.26)$$

where γ is greater than the real parts of the singularities of the integrand and $Re(p) > 0$.

On evaluating equations (3.2.25) and (3.2.26), we have the expressions for the velocity profile of liquid as

$$u = \frac{I_1[rA]}{I_1[A]}\,e^{-i\Omega t} - 2\sum_n\sum_j \frac{\beta_n}{Q(p_{nj})}\,\frac{1}{(p_{nj}+i\Omega)} \times \frac{J_1(\gamma\beta_n)}{J_0(\beta_n)}\,e^{p_{nj}t} \quad (3.2.27)$$

and of dust particles as

$$v = \frac{I_1[rA]}{I_1[A]} \frac{e^{-i\Omega t} - e^{-t/\tau}}{(1 - i\Omega\tau)} - 2 \sum_n \sum_j \frac{\beta_n}{Q(p_{nj})} \frac{1}{(p_{nj} + i\Omega)}$$

$$\times \frac{J_1(r\beta_n)}{J_0(\beta_n)} \frac{1}{(1 + p_{nj}\tau)} \left(e^{p_{nj}t} - e^{-t/\tau} \right), \tag{3.2.28}$$

where

$$A = \left\{ \frac{-i\,\Omega\,(1 - i\,\alpha\,\Omega\,)(1 + f - i\,\Omega\,\tau\,)}{(1 - i\,\Omega\,\tau\,)} \right\}^{1/2},$$

β_n's are the roots of

$$J_1(\beta) = 0 \tag{3.2.29}$$

and p_{nj}'s are the roots of the cubic equation

$$\frac{p_n(1 + \alpha\, p_n)(1 + f + p_n\tau\,)}{(\,p_n\,\tau + 1)} = -\beta_n^2\,, \quad n = 0,1,2,\ldots, \tag{3.2.30}$$

$$Q(p_{nj}) = \Big[\Big\{ (\,1 + 2\,\alpha\, p_{nj})(1 + f + p_{nj}\tau) + p_{nj}\tau(\,1 + \alpha\, p_{nj}) \Big\}$$

$$\times\, (p_{nj}\tau + 1) - \tau\, p_{nj}(\,1 + \alpha\, p_{nj})(1 + f + p_{nj}\tau\,)\Big] / (1+\tau\, p_{nj})^2. \tag{3.2.31}$$

It is clear from equation (3.2.30) that all roots of p_n (for any $n = 0,1,2,\ldots.$) are either negative or one negative and other two complex. From the physics of the problem we consider those values of p_{nj} in equations (3.2.27) and (3.2.28) for which $e^{p_{nj}t} \longrightarrow 0$ as $t \longrightarrow \infty$.

The non-dimensional skin-friction on the wall of the cylinder is given by

$$\tau_{r\theta}\Big|_{r=1} = (1 + i\alpha\Omega)\Big[\frac{A I_1'[A]}{I_1[A]} - 1\Big] e^{-i\Omega t}$$

$$- 2 \sum_n \sum_j \frac{\beta_n^2(1 - \alpha\, p_{nj}\,)}{Q(p_{nj})\,(p_{nj} + i\,\Omega\,)}\, e^{p_{nj}t}\,. \tag{3.2.32}$$

It is evident from equations (3.2.27) and (3.2.28) that velocity of liquid and dust particles become same as the relaxation time tends to zero, i.e., when the dust particles become very fine. In absence of elastic parameter and dust particles, the expression for the velocity profile of liquid particles is same as that obtained by Mukherjee and Bhattacharyya [4] (if it is made dimensionless).

3.2.7 Discussion

The analysis of the present study reveals that the solution contains three pertinent flow parameters, viz., α (the dimensionless elastic parameter), f (the mass concentration of dust particles) and τ (the relaxation time of dust particles). The behaviour of these parameters, therefore, yeilds a physical insight into the problem. Keeping this in view, the numerical computations of real part of equations (3.2.27) – (3.2.29) are carried out to represent graphically the velocity fields, skin-friction at the plate walls for different values of α, f, τ.

The velocity of liquid and dust particles are depicted in figures 3.9 – 3.12 against r for different values of α, f and τ. Figures 3.9 and 3.10 show the effect of f on u and v (with τ fixed) while figures 3.11 and 3.12 depict the variation of u and v due to the change of relaxation time of dust particles (with f fixed) for different values of elastic parameter. From figures 3.9 and 3.11, it is seen that u increases with increasing α for fixed τ and f, i.e., the effect of elastic element in the liquid is to increase the velocity of liquid particles. Also it is observed that both mass concentration (f) and relaxation time (τ) increase the velocity of liquid for any α. Figure 3.10 shows that flow occurs in reverse direction, (i.e., in the direction of decreasing θ) for α = 0,1,2 and τ = 0.5. As f increases both forward flow and back flow exist and the region of forward flow increases with the increase in f for any value of α. It is evident from figure 3.12 that as τ increases, the magnitude of the velocity of dust particles increases with fixed f.

Table 3.2 shows that the magnitude of skin-friction increases with the increase in elastic parameter for $f=0.2$, $\tau=0.5$ at $t = 5$. The negative values of skin-friction indicate that the shearing stress acts in the decreasing θ direction at $t = 5$.

Table 3.2 : Skin-friction on the wall at t=5, f=0.2 and τ =0.5.

α	$\tau_{r\theta}\|_{r=1}$
1.0	− 0.0273578
1.5	− 0.053213
2.0	− 0.912364
2.5	− 1.563151
3.0	− 2.251595

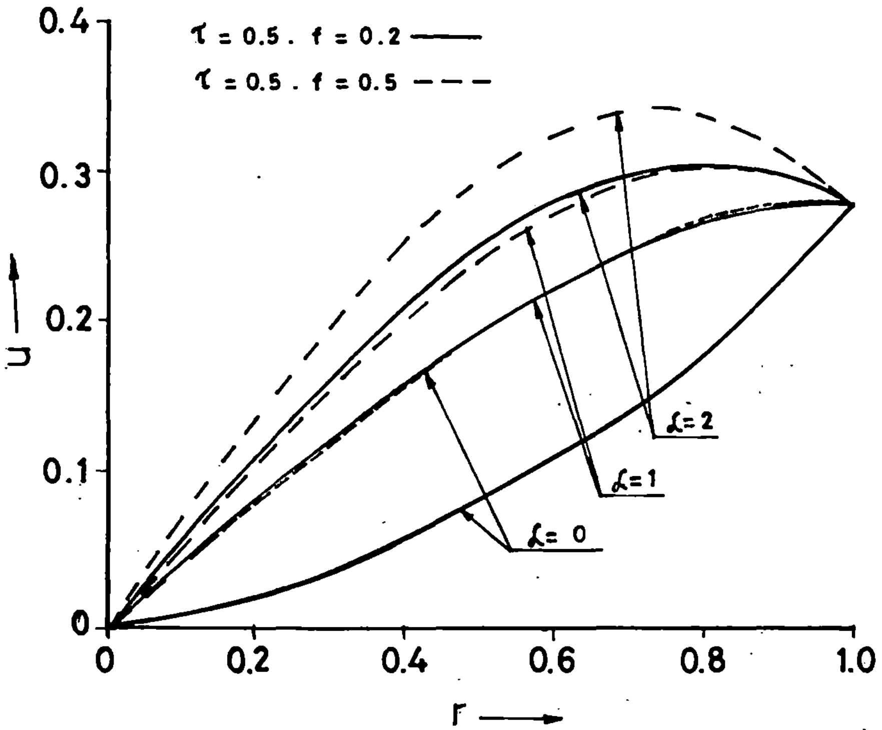

Fig.-3.9. Velocity profile of liquid particles at $t = 5$ when $\tau = 0.5$

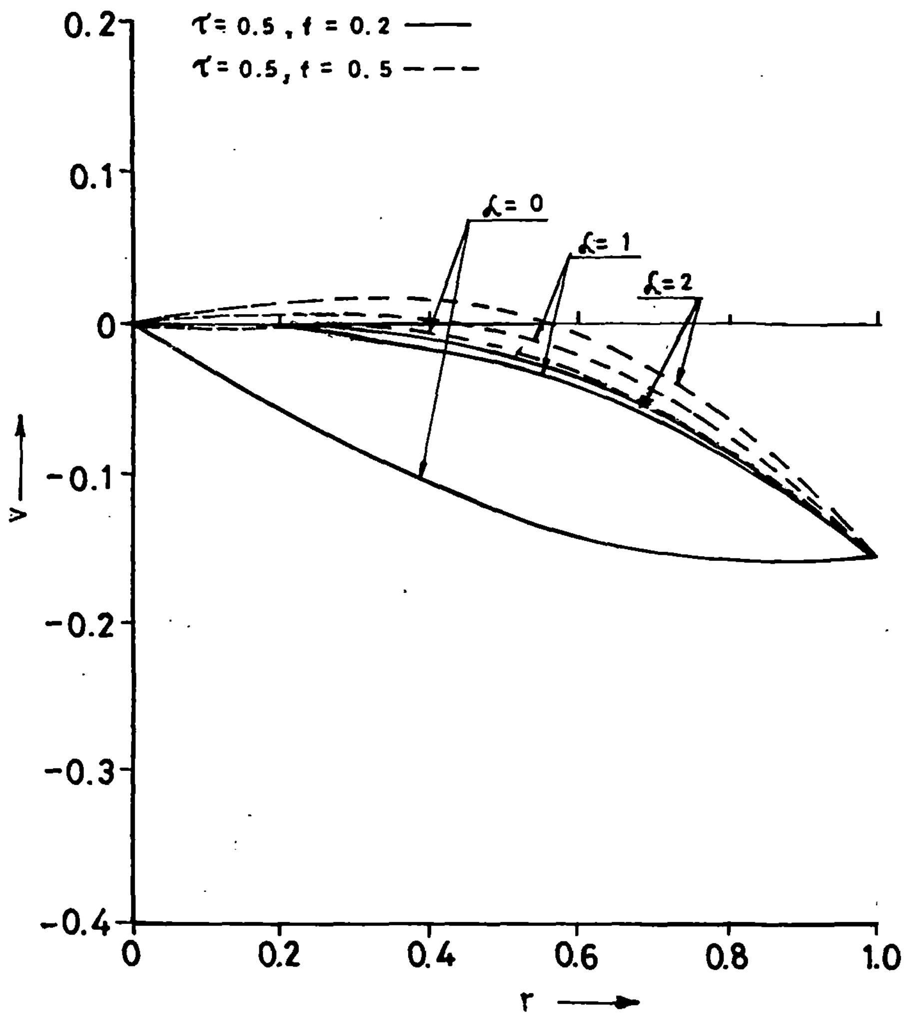

Fig.-3.10. Velocity profile of dust particles at t=5 when $\tau = 0.5$

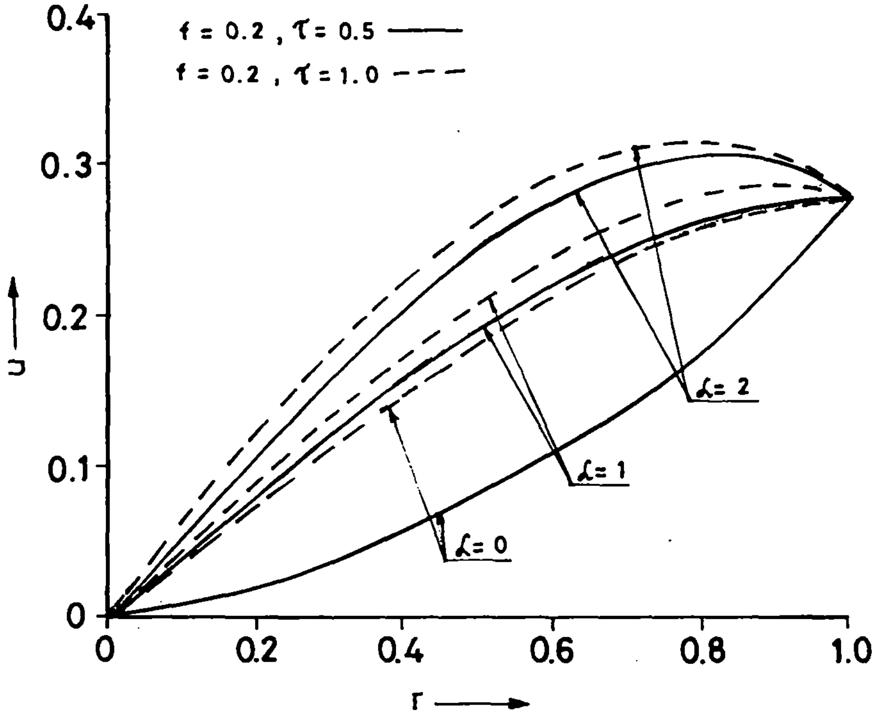

Fig.-3.11. Velocity profile of liquid particles at t = 5 when f = 0.2

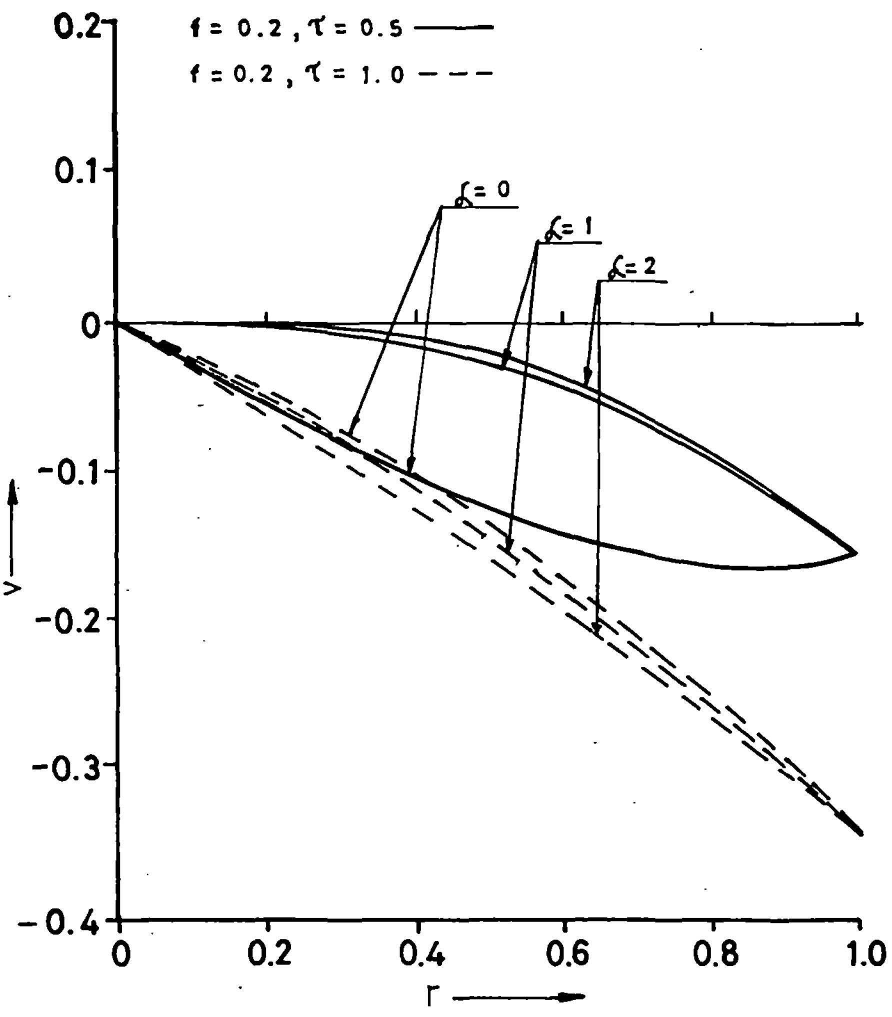

Fig.-3.12. Velocity profile of dust particles at t = 5 when f = 0.2

PART THREE

UNSTEADY FLOW OF A DUSTY ELASTICO-VISCOUS LIQUID IN THE EKMAN LAYER *

3.2.8 Introduction

Multiphase flow problems are of current interest in fluid dynamics. In particular, fluid mechanical problems involving gas particle mixtures arise in various fields of engineering. Based on the theoretical model proposed by Saffman [5], many authors investigated a number of dusty gas flow problems in various geometries. Regarding the plate problems, Michael and Miller [7], Liu [22,23], Healy and Yang [24] investigated a number of dusty gas flow problems. But little attention is paid to the flow of a dusty gas in rotating system although this has some bearing on the pollution problem as well as on the motion of aerosol over the rotating earth. Recently, Gupta and Pop [9] investigated the unsteady boundary layer flow in a rotating viscous liquid bounded by an infinite flat plate when there is a suspension of dust particles in the liquid. Very recently Jana et al. [25] investigated the unsteady flow in the Ekman layer of an elastico-viscous liquid.

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In the present investigation, we extend the analysis of Gupta and Pop [9] to cover a wider class of elastico-viscous liquid, viz. Walters' liquid B' (with short memory) [26] and, in particular, to observe (qualitatively) the effects of elastic element and mass concentration on the flow field.

3.2.9 Mathematical formulation and asymptotic analysis

We consider an infinite plate coinciding with the plane z=0 and rotating in unison with a dusty elastico-viscous liquid occupying the region $z > 0$ with a uniform angular velocity Ω about the z-axis for time $t \leq 0$. At time $t > 0$, the plate starts moving with a uniform velocity U in its own plane relative to the rotating frame of reference. The horizontal homogeneity of the problem demands that the physical quantities depend on z and t only. The equation of continuity of the liquid then gives $w \equiv 0$ everywhere in the flow, where (u,v,w) are components of the liquid velocity at a point.

The equation of continuity of dust particles is given as

$$\frac{\partial N}{\partial t} + (N v_i')_{,i} = 0 ,$$

where N is the number density of dust particle and v_i' (u ', v ', 0) is the velocity of dust particle. Since the distribution of dust particles is uniform, the number density of the particles $N = N_0$, a constant throughout the motion.

Using (1.6) and (1.10) in equations (1.18) and (1.19), we get the equations of motion for the liquid and dust particle in a rotating frame of reference as

$$\frac{\partial u}{\partial t} - 2 \Omega v = \nu \left(1 - \frac{K_0}{\rho\nu} \frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial z^2} + \frac{K N_0}{\rho}(u' - u) , \qquad (3.2.33)$$

$$\frac{\partial v}{\partial t} + 2\,\Omega\,u = \nu\left(1 - \frac{K_0}{\rho\nu}\frac{\partial}{\partial t}\right)\frac{\partial^2 v}{\partial z^2} + \frac{K\,N_0}{\rho}(v' - v)\,, \qquad (3.2.34)$$

$$\frac{\partial u'}{\partial t} - 2\,\Omega\,v' = \frac{K}{m}(u - u')\,, \qquad (3.2.35)$$

$$\frac{\partial v'}{\partial t} + 2\,\Omega\,u' = \frac{K}{m}(v - v')\,, \qquad (3.2.36)$$

where m is the mass of a dust particle, K is Stokes resistance coefficient, K_0 the elastic coefficient, $\nu = \frac{\eta}{\rho}$, η the limiting viscosity at small rates of shear and ρ the density. Equations (3.2.33) – (3.2.36) are combined as

$$\frac{\partial q}{\partial T} - \left(1 - K_1\frac{\partial}{\partial T}\right)\frac{\partial^2 q}{\partial Z^2} + (2i\omega + f)q - fq' = 0, \qquad (3.2.37)$$

$$\frac{\partial q'}{\partial T} + (2i\omega + 1)q' - q = 0, \qquad (3.2.38)$$

where

$$q = \frac{u + iv}{U}\;(= u_1 + iv_1),\; q' = \frac{u' + iv'}{U}\;(= u'_1 + iv'_1)\;,$$

$$T = \frac{t}{\tau}\,,\; Z = \frac{z}{(\nu\tau)^{1/2}}\,,\; \tau = \frac{m}{K}\,,\; \Omega\,\tau = \omega,\; K_1 = \frac{K_0}{\rho\nu\tau} \text{ and}$$

$f = \frac{mN_0}{\rho}$ is the mass concentration of dust particles and τ is the relaxation time of dust particles.

The initial and boundary conditions are

$$q = q' = 0 \quad \text{for } T \le 0, \qquad (3.2.39)$$

$$q = 1 \text{ at } Z = 0 \text{ for } T > 0, \qquad (3.2.40)$$

$$q \longrightarrow 0 \text{ as } Z \longrightarrow \infty\,. \qquad (3.2.41)$$

Taking Laplace transforms of (3.2.37) and (3.2.38) and using (3.2.39), we get

$$(1 - K_1 p)\frac{d^2\bar{q}}{dZ^2} - (2i\omega + f + p)\bar{q} + f\bar{q}' = 0, \qquad (3.2.42)$$

$$(p + 2 i \omega + 1) \bar{q}' = \bar{q} \ , \tag{3.2.43}$$

where

$$\bar{q}(Z,p) = \int_0^\infty q(Z,T) \, e^{-pT} \, dT \ ,$$

$$\bar{q}'(Z,p) = \int_0^\infty q'(Z,T) \, e^{-pT} \, dT \ , \ Re(p) > 0.$$

Eliminating $\bar{q}'$ from equations (3.2.42) and (3.2.43) and then solving for $\bar{q}$ with the help of transformed boundary condition, we have

$$\bar{q}(Z,p) = \frac{1}{p} e^{-MZ} \ , \tag{3.2.44}$$

where

$$M = \frac{1}{(1 - K_1 p)^{1/2}} \left[p + 2 i \omega + f - \frac{f}{1 + p + 2 i \omega} \right]^{1/2}$$

with the proviso that real part of M is taken positive.

The dust velocity at the plate Z = 0 is given by equations (3.2.43) and (3.2.44) as

$$q'(0,T) = \mathcal{L}^{-1} \left[\frac{1}{p(p + 1 + 2 i \omega)} \right] . \tag{3.2.45}$$

Taking inverse transformation of equation (3.2.45) and separating real and imaginary parts, we have

$$u'_1(0,T) = \frac{1}{1+ 4\omega^2} \left[1-e^{-T}\cos 2\omega T+2\omega e^{-T}\sin 2\omega T \right] , \tag{3.2.46}$$

$$v'_1(0,T) = \frac{1}{1+ 4\omega^2} \left[e^{-T}\sin 2\omega T-2\omega(1-e^{-T}\cos 2\omega T) \right] . \tag{3.2.47}$$

It is interesting to note from equations (3.2.46) and (3.2.47) that the dust particles do not stick to the plate but move relative to it. However, such a velocity slip is compatible with the assumption that the bulk concentration of the dust particles is small since we can then imagine that the particles nearest to the plate will in general be several particle diameters away from it. It is also remarked that the velocity of dust particles on the plate is unaffected by the presence of elastic parameter of the fluid.

To investigate the asymptotic nature of the solution for large time, we assume $p << 1$. Equation (3.2.44) can then be approximated by

$$\bar{q}\,(Z,p) = \frac{1}{p} e^{-[A(\,p + B/A)]^{1/2}\, Z}, \qquad (3.2.48)$$

where

$$A = \alpha + i\beta\,, \qquad (3.2.49)$$

$$B = 2\, i\, \omega \left(1 + \frac{f}{1 + 2\, i\, \omega} \right), \qquad (3.2.50)$$

so that $\alpha = (1 + 4\omega^2 + 4\,\omega^2 f\, K_1)\; /\; (1 + 4\,\omega^2)$,

$\beta = 2\,\omega\, K_1 (1 + 4\omega^2 + f\,)\; /\; (1 + 4\,\omega^2)$,

Equation (3.2.48) gives, on using the table of the inverse Laplace transforms due to Campbell and Foster [27],

$$q\,(Z,T) = \frac{1}{2} \Big[\, e^{\sqrt{B}\, Z} .\mathrm{erfc} \left(\frac{\sqrt{A}}{2\sqrt{T}}\, Z + \frac{\sqrt{B\,T}}{\sqrt{A}} \right) + e^{-\sqrt{B}\, Z}$$
$$\times\ \mathrm{erfc} \left(\frac{\sqrt{A}}{2\sqrt{T}}\, Z - \frac{\sqrt{B\,T}}{\sqrt{A}} \right) \Big] \,. \qquad (3.2.51)$$

It is known that

$$\mathrm{erfc}\,(Z) \simeq Z^{-1}\, \pi^{-1/2} \exp\,(-Z^2) \text{ as } |\,Z\,| \longrightarrow \infty\,, \quad (3.2.52)$$

$$\mathrm{erfc}\,(-Z) = 2 - \mathrm{erfc}\,(Z).$$

Using equations (3.2.51) and (3.2.52) and separating $q(Z,T)$ into real and imaginary parts, we get the asymptotic expressions for $u_1(Z,T)$ and $v_1(Z,T)$ for large T as

$$u_1(Z,T) = e^{-Z\alpha_1} \cos \beta_1 Z - \frac{Z}{2\sqrt{\pi T}} \cdot \frac{1}{(\alpha_5^2 + \beta_5^2)}$$

$$X \left\{(\alpha_2\alpha_5 + \beta_2\beta_5) \cos \alpha_4 + (\beta_2\alpha_5 - \alpha_2\beta_5) \sin \alpha_4 \right\}$$

$$X \exp \left\{ - (\frac{\alpha}{4}\frac{Z^2}{T} + (\alpha_3^2 - \beta_3^2) T)\right\} , \qquad (3.2.53)$$

$$v_1(Z,T) = e^{-Z\alpha_1} \sin \beta_1 Z - \frac{Z}{2\sqrt{\pi T}} \cdot \frac{1}{(\alpha_5^2 + \beta_5^2)}$$

$$X \left\{(\beta_2\alpha_5 - \alpha_2\beta_5) \cos \alpha_4 - (\alpha_2\alpha_5 + \beta_2\beta_5) \sin \alpha_4 \right\}$$

$$X \exp \left\{ - (\frac{\alpha Z^2}{4T} + (\alpha_3^2 - \beta_3^2) T) \right\} , \qquad (3.2.54)$$

where

$$\alpha_1, \beta_1 = \left(\frac{\omega}{1 + 4\omega^2}\right)^{1/2} \left[\left\{(1 + 4\omega^2 + f)^2 + 4 \omega^2 f^2 \right\} \pm 2\omega f\right]^{1/2},$$

$$\alpha_2, \beta_2 = \frac{1}{(2(1 + 4\omega^2))^{1/2}} \left[\left\{(1 + 4\omega^2 + 4\omega^2 fK_1)^2 + 4\omega^2 K_1^2 \right.\right.$$

$$\left.\left. X (1+f+4\omega^2)^2\right\}^{1/2} \pm (1 + 4\omega^2 + 4\omega^2 fK_1) \right]^{1/2},$$

$$\alpha_3 = (\alpha_1\alpha_2 + \beta_1\beta_2) / (\alpha_2^2 + \beta_2^2),$$

$$\beta_3 = (\beta_1\alpha_2 - \alpha_1\beta_2) / (\alpha_2^2 + \beta_2^2),$$

$$\alpha_4 = (\beta + 2 \alpha_3\beta_3 T) ,$$

$$\beta_4 = 0,$$

$$\alpha_5 = (\alpha_3^2 - \beta_3^2)T - \alpha Z^2/4T,$$

$$\beta_5 = 2\alpha_3\beta_3 T - \beta Z^2/4T.$$

The dimensionless skin-friction can be calculated from equation (3.2.51) as

$$- \left.\frac{\partial q}{\partial Z}\right|_{Z=0} = (\alpha_1 + i\beta_1).\mathrm{erf}((\alpha_3 + i\beta_3)\sqrt{T})$$

$$+ \frac{\alpha_2 + i\beta_2}{\sqrt{\pi T}} e^{-BT/A} . \qquad (3.2.55)$$

3.2.10 Discussion

The first terms in equations (3.2.53) and (3.2.54) represent the velocity components for the Ekman boundary layer (modified by the presence of dust particles) on the plate, which is established in the final steady state. The boundary layer thickness is clearly of order $(\alpha_1)^{-1}$ and gradually it becomes thinner with the increase in f, the mass concentration. This distribution is independent of K_1. Another distinctive feature of the above asymptotic solution is that the second terms in equations (3.2.53) and (3.2.54) confirm the existence of inertial oscillations which decay exponentially with time. The effect of rotation menifests itself through these oscillations with frequency, $(2\alpha_3\beta_3)$. The effects of elastic element, mass concentration on the frequency of inertial oscillations are shown in table 3.3 (a,b). It reveals that the frequency decreases with increase in either mass concentration or elastic element in the liquid.

As $T \longrightarrow \infty$, the equation (3.2.55) gives the steady state skin-friction as $(\alpha_1 + i\beta_1)$. In absence of dust parameter, it reduces to $(1 + i)\,\omega^{1/2}$, which agrees with the classical result for Ekman spiral near a plate in a rotating frame. It is of interest to have an estimate of the time which elapses from the start of the plate in the rotating frame till the steady state is reached. It is clear from the equation (3.2.55) that the steady state is reached after a time T_0 where

$$\operatorname{erf}((\alpha_3 + i\beta_3)T_0) \simeq 1 .$$

Since $\operatorname{erf}(x) \simeq 1$ when $|x| \gtrsim 2$, it follows that

$$T_0 \simeq \frac{2}{(\alpha_3^2 + \beta_3^2)^{1/2}} .$$

The effects of mass concentration and elastic parameter on T_0 can be seen from table 3.4 (a,b). It reveals that the effect of mass concentration is to decrease the time to reach the steady state while it increases as the elastic element increases.

Table 3.3:(a) Effect of elastic element on frequency of inertial oscillations when $f = 0.1$, $\omega = 1.0$.

K_1	0.5	1.0	1.5	2.0	2.5
$2\alpha_3\beta_3$	1.4004176	0.8837880	0.6260497	0.4809395	0.4193873

(b) Effect of mass concentration on frequency of inertial oscillations when $K_1 = 0.5$, $\omega = 1.0$.

f	0.10	0.20	0.30	0.35	0.40
$2\alpha_3\beta_3$	1.4004176	1.3872846	1.3747694	1.3699121	1.3609122

Table 3.4:(a) Effect of elastic element on the time (T_0) to reach the steady state when $f = 0.1$, $\omega = 1.0$.

K_1	0.5	1.0	1.5	2.0	2.5
T_0	1.5906671	1.9306756	2.2767133	2.6301002	3.0001219

(b) Effect of mass concentration on the time (T_0) to reach the steady state when $K_1 = 0.5$, $\omega = 1.0$.

f	0.10	0.15	0.20	0.25	0.30
T_0	1.5906671	1.5851219	1.5768748	1.5751921	1.5728127

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