

### CHAPTER III

#### PROBLEMS ON PROPAGATION OF ELASTIC WAVES IN THE PRESENCE OF TOPOGRAPHICAL IRREGULARITY

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Paper 10: Time step SH- wave transmission across a rectangular step.

Paper 11: SH- wave propagation across a vertical step in two joined elastic half- spaces

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## TIME STEP SH- WAVE TRANSMISSION ACROSS A RECTANGULAR STEP

## 1. Introduction

In recent past, several papers have appeared on propagation on elastic wave in materials with free surface having irregularities. Wolf (1967) studied the propagation of Love wave in layers with irregular boundaries using perturbation method. Adopting the representation theorem due to Knopoff (1956) to obtain a perturbed solution of the problem, Knopoff and Hudson (1964) studied the transmission of Love wave past a continental margin considering the crust to have an abrupt increase in thickness on the continental side. A series of problems involving the scattering of elastic waves by two dimensional and three dimensional topographical irregularities have been solved by Sanchez- Sesma (1979, 1982, 1983, 1985) by using a newly developed boundary method. The diffracted wave fields are constructed with linear combinations of solutions which form complete families for the wave equations and boundary conditions are then satisfied in a least square sense. Bose (1975) was also solved the transmission of SH- wave across a step like irregularity in the surface of an elastic half- space by a different method. He considered a time harmonic plane SH- wave in the form  $\exp(i(\omega t - kX))$  propagating in the direction perpendicular to the step and solving the resulting integral equation asymptotically, transmitted wave at distance far away from the step was obtained. Dutta and Mitra (1974) have presented the SH- wave motion in an elastic layer of different shear wave velocity.

In this paper, we studied the problem of transmission of time step SH- wave motion across a step like irregularity in the surface of an elastic half- space consisting of two quarter spaces of same material joined along the common boundary. Considering the incident wave in the form  $H(T-X/c)$  where  $H()$  is the Heaviside step function the problem is reduced to an integral equation by using integral transform and Green's function technique and finally using Cagniard- Dehoop's method of finding inverse Laplace transform, , transmitted field at any distances from the step on the free surface has been determined using iterative

procedure. Numerical results have been presented in the form of graph to illustrate the nature of transmission.

## 2. Formulation Of Problem

The transmission of SH- wave across a step like irregularity in the surface of an elastic half-space consisting of two quarter spaces of same material joined along the common boundary  $X=0$  has been considered here. We introduce the axes of coordinates as shown in Fig.1. Denoting the coordinates of a point in the X-Z plane by  $(X,Z)$ , we take the incident plane SH- wave as  $H(T-X/c)$ , where  $H(x)$  is the Heaviside's step function, so that the propagation is from higher side to the lower side of the step.

Let  $V_1, V_2$  be the Y- components of the displacement in the two media I and II respectively. The field equations are wave equations in two media and boundary conditions are that the shearing stress vanishes everywhere on the outer boundaries  $Z=0, X<0$ ;  $X=0, 0 \leq Z \leq H$  and  $Z=H, X>0$  the displacement and stress are continuous on the interface  $X=0, Z>H$ .

Taking Laplace transform of these wave equations and boundary conditions with respect to time T (with parameter  $p$ ), we get

$$\left[ \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} - \frac{p^2}{c^2} \right] \bar{V}_1 = 0 \quad (1)$$

$$\left[ \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} - \frac{p^2}{c^2} \right] \bar{V}_2 = 0 \quad (2)$$

$$\left[ \frac{\partial \bar{V}_2}{\partial Z} \right]_{Z=0} = 0 \quad \text{for } X < 0$$

$$\left[ \frac{\partial \bar{V}_1}{\partial Z} \right]_{Z=H} = 0 \quad \text{for } X > 0$$

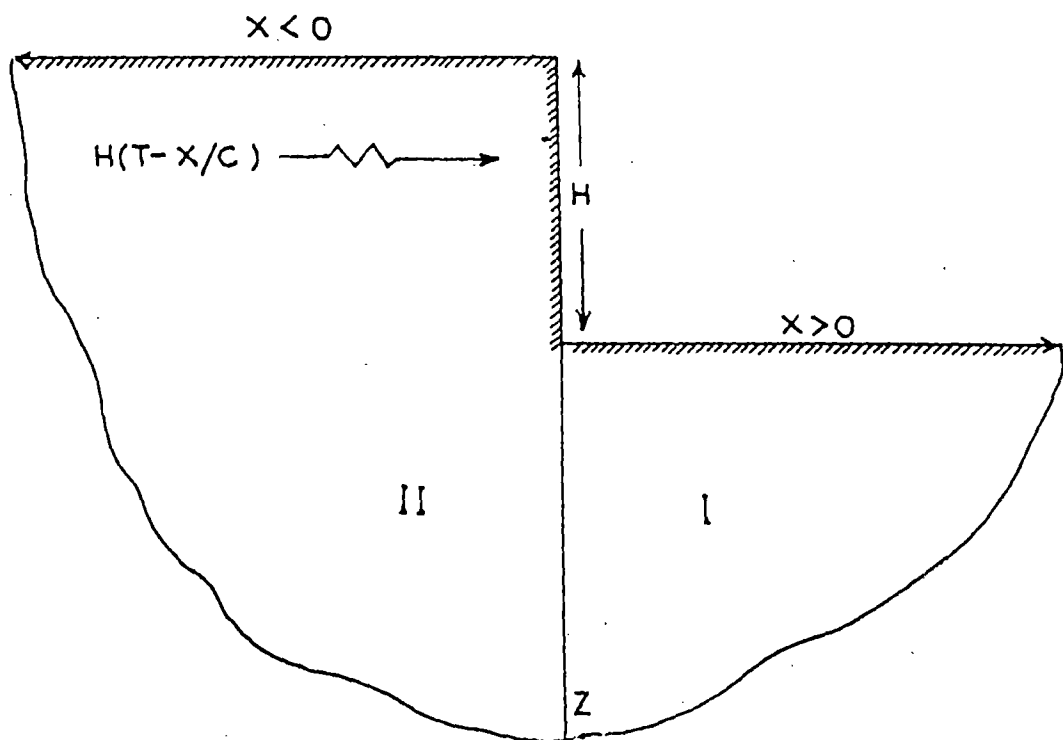


Fig.1: Geometry and coordinate system.

$$\left[ \frac{\partial \bar{V}_2}{\partial X} \right]_{X=0} = 0 \quad \text{for } 0 \leq Z \leq H \quad (3a-c)$$

$$\left[ \frac{\partial \bar{V}_2}{\partial X} \right]_{X=0} = \left[ \frac{\partial \bar{V}_1}{\partial X} \right]_{X=0} \quad \text{for } Z > H$$

$$\bar{V}_2(0, Z) = \bar{V}_1(0, Z) \quad \text{for } Z > H \quad (4a, b)$$

where  $H$  is the height of the step.

We represent transverse displacement in two domains  $X < 0$  and  $X > 0$  in the form

$$\begin{aligned} V_2 &= H(T-X/c) + H(T+X/c) + V'_2(X, Z, T), \quad X < 0, \quad Z > 0 \\ V_1 &= V_1(X, Z, T), \quad X > 0, \quad Z > H. \end{aligned} \quad (5)$$

Obviously  $\bar{V}'_2$  which is Laplace transform of  $V'_2(X, Z, T)$  satisfies the the equation

$$\left[ \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} - \frac{p^2}{c^2} \right] \bar{V}'_2 = 0 \quad (6)$$

### 3. Reduction to integral equation and its solution

We introduce Green's functions  $G_1(X, Z; R, S)$  and  $G_2(X, Z; U, W)$  in medium I and II respectively such that  $G_2(X, Z; U, W)$  is the solution of

$$\left[ \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} - \frac{p^2}{c^2} \right] G_2(X, Z; U, W) = -4\pi\delta(X-U)\delta(Z-W) \quad (7)$$

for medium II with vanishing normal derivative at  $X < 0, Z = 0$  and at  $X = 0, Z > 0$ . Similarly,  $G_1(X, Z; R, S)$  defined with reference to medium I is the solution of

$$\left[ \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} - \frac{p^2}{c^2} \right] G_1(X, Z; R, S) = -4\pi\delta(X-R)\delta(Z-S) \quad (8)$$

and satisfies the vanishing of normal derivative at  $X>0, Z=H$  and at  $X=0, Z>H$ .

Now multiplying equation (7) by  $\bar{V}'_2$  and equation (6) by  $G_2$ , then subtracting the resulting second equation from the first, we obtain

$$\bar{V}'_2 \nabla^2 G_2 - G_2 \nabla^2 \bar{V}'_2 = -4\pi \bar{V}'_2(X, Z) \delta(X-U) \delta(Z-W)$$

where  $\nabla^2$  is the Laplacian operator in the X-Z plane.

Integration of this over the region of medium II yields

$$\begin{aligned} -4\pi \bar{V}'_2(U, W) &= \iint \left[ \bar{V}'_2 \nabla^2 G_2 - G_2 \nabla^2 \bar{V}'_2 \right] dX dZ \\ &= \int_{\Gamma} \left[ \bar{V}'_2 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial \bar{V}'_2}{\partial n} \right] d\Gamma \end{aligned}$$

where  $\Gamma$  is the boundary of the medium II and  $n$  is outward drawn normal to it.

$$\text{Therefore, } 4\pi \bar{V}'_2(U, W) = \int_H^{\infty} G_2(0, Z; U, W) \left[ \frac{\partial \bar{V}'_2}{\partial X} \right]_{X=0} dZ \quad (9)$$

and a similar application of Green's theorem to medium I yields

$$4\pi \bar{V}'_1(R, S) = - \int_H^{\infty} G_1(0, Z; R, S) \left[ \frac{\partial \bar{V}'_1}{\partial X} \right]_{X=0} dZ \quad (10)$$

Substitution of (9) and (10) into the equation (4b) yields with the aid of (4a) and (5) the integral equation

$$\int_H^{\infty} \left[ G_2(0, Z; 0, W) + G_1(0, Z; 0, W) \right] \left[ \frac{\partial \bar{V}'_1}{\partial X} \right]_{X=0} dZ = -8\pi/p \quad (11)$$

To evaluate the Green's Function for the medium I and II Fourier cosine transform with respect to Z has been taken. With this application of Fourier cosine transform, equation (7) reduces to ordinary equation viz

$$d^2 G_2^c / dX^2 - (\alpha^2 + p^2 / c^2) G_2^c = -4\pi \cos(\alpha W) \delta(X-U)$$

where

$$G_2^c = \int_0^\infty G_2(X, Z; U, W) \cos(\alpha Z) dZ.$$

Solution of the above differential equation can be taken as

$$\begin{aligned} G_2^c &= A \cosh(\beta X) e^{\beta U}, & X \geq U \\ &= A \cosh(\beta U) e^{\beta X}, & X \leq U \end{aligned}$$

where the unknown constant A is determined as follows

$$A\beta e^{\beta U} [\sinh(\beta U) - \cosh(\beta U)] = -4\pi \cos(\alpha W)$$

$$\text{or } A = 4\pi \cos(\alpha W) / \beta$$

Hence taking inverse Fourier cosine transform we find

$$\begin{aligned} G_2(X, Z; U, W) &= 8 \int_0^\infty \frac{e^{\beta U}}{\beta} \cosh(\beta X) \cos(\alpha W) \cos(\alpha Z) d\alpha & 0 \leq X \leq U \\ &= 8 \int_0^\infty \frac{e^{\beta X}}{\beta} \cosh(\beta U) \cos(\alpha W) \cos(\alpha Z) d\alpha & -\infty < X \leq U \end{aligned} \quad (12)$$

$$\text{where } \beta^2 = \alpha^2 + p^2 / c^2.$$

Again introducing Fourier cosine transform defined by

$$G_1^c = \int_0^\infty G_1(X, Z; R, S) \cos \alpha(Z-H) d(Z-H)$$

in equation (8), we obtain,

$$\begin{aligned}
 G_1(X, Z; R, S) &= 8 \int_0^\infty \frac{e^{-\beta R}}{\beta} \cosh(\beta X) \cos \alpha(S-H) \cos \alpha(Z-H) d\alpha \quad 0 \leq X \leq R \\
 &= 8 \int_0^\infty \frac{e^{-\beta X}}{\beta} \cosh(\beta R) \cos \alpha(S-H) \cos \alpha(Z-H) d\alpha \quad R \leq X < \infty \quad (13)
 \end{aligned}$$

On substitution of these values of Green's functions in the integral equation (11), it takes the form

$$\int_H^\infty \left( \frac{\partial \bar{V}_1}{\partial X} \right)_{X=0} dz \int_0^\infty \left[ \frac{\cos \alpha(W-H) \cos \alpha(Z-H)}{\beta} + \frac{\cos \alpha W \cos \alpha Z}{\beta} \right] d\alpha = -\pi/p \quad (14)$$

Substituting  $W=wH$ ,  $Z=zH$ , we obtain form (14)

$$\begin{aligned}
 H \int_1^\infty \left( \frac{\partial \bar{V}_1}{\partial X} \right)_{X=0} dz \int_0^\infty \frac{\cos \alpha H(w-1) \cos \alpha H(z-1)}{\beta} d\alpha &= -\pi/p - H \int_1^\infty \left( \frac{\partial \bar{V}_1}{\partial X} \right)_{X=0} dz \\
 &\quad \times \int_0^\infty \frac{\cos(\alpha H w) \cos(\alpha H z)}{\beta} d\alpha
 \end{aligned}$$

On inversion with respect to  $\alpha$ , it takes the form

$$\begin{aligned}
 H \int_1^\infty \left( \frac{\partial \bar{V}_1}{\partial X} \right)_{X=0} \frac{\cos \alpha H(z-1)}{\beta} dz &= -\frac{2H}{p} \int_1^\infty \cos \alpha H(w-1) dw - \frac{2H^2}{\pi} \int_1^\infty \left( \frac{\partial \bar{V}_1}{\partial X} \right)_{X=0} dz \\
 &\quad \times \int_0^\infty \frac{\cos(\alpha H z)}{\beta(\tau)} d\tau \int_1^\infty \cos(\tau H w) \cos \alpha H(w-1) dw \quad (15)
 \end{aligned}$$

where  $\beta(\tau)$  is obtained from  $\beta$  by replacing  $\alpha$  by  $\tau$ ,

Next using the results

$$\int_1^\infty \cos \alpha H(w-1) dw = \pi \delta(\alpha)/H \quad (16)$$

and 
$$\int_1^{\infty} \sin \alpha H(w-1) dw = 1/\alpha H \quad (17)$$

it can be easily shown that

$$\int_1^{\infty} \cos \tau H w \cos \alpha H(w-1) dw = \frac{\pi}{2} \cos(\tau H) \left[ \delta\{(\tau+\alpha)H\} + \delta\{(\tau-\alpha)H\} \right] - \frac{\tau \sin(\tau H)}{H(\tau^2 - \alpha^2)} \quad (18)$$

where  $\delta(x)$  is the dirac  $\delta$ - function.

Using equation (18) in equation (15) and on simplification it reduces to the form

$$\begin{aligned} H \int_1^{\infty} \left[ \frac{\partial \bar{V}_1}{\partial X} \right]_{X=0} \frac{\cos \alpha H(z-1)}{\beta} dz &= -\frac{\pi \delta(\alpha)}{p} + \frac{H}{2} \int_1^{\infty} \left[ \frac{\partial \bar{V}_1}{\partial X} \right]_{X=0} \frac{\sin(\alpha H z) \sin(\alpha H)}{\beta} dz \\ + \frac{H}{\pi} \int_1^{\infty} \left[ \frac{\partial \bar{V}_1}{\partial X} \right]_{X=0} \int_0^{\infty} \frac{\cos(\tau H z) \sin(\tau H)}{\beta(\tau)(\tau^2 - \alpha^2)} \tau d\tau &\quad (19) \end{aligned}$$

#### 4. Evaluation of Displacement

Substituting the value of  $G_1(0, Z; R, S)$  from equation (19) in equation (10) we obtain

$$4\pi \bar{V}_1(X, S) = -8H \int_0^{\infty} e^{-\beta X} \cos \alpha H(s-1) d\alpha \int_1^{\infty} \left[ \frac{\partial \bar{V}_1}{\partial X} \right]_{X=0} \frac{\cos \alpha H(z-1)}{\beta} dz$$

where  $S=sH$ . Inserting the result (19) in the above expression we find

$$\bar{V}_1(X, S) = \frac{e^{-pX/c}}{p} - \frac{H}{\pi} \int_0^{\infty} e^{-\beta X} \cos \alpha H(s-1) d\alpha \int_1^{\infty} \left[ \frac{\partial \bar{V}_1}{\partial X} \right]_{X=0} \frac{\cos \alpha H(z-1)}{\beta} dz +$$

$$\begin{aligned}
& + \frac{2H^2}{\pi^2} \int_0^\infty e^{-\beta X} \cos \alpha H(s-1) d\alpha \int_0^\infty \frac{d\tau}{\beta(\tau)} \int_0^\infty \cos(\tau H w) \cos \alpha H(w-1) dw \times \\
& \times \int_0^\infty \left[ \frac{\partial \bar{V}}{\partial X} \right]_{X=0} \cos(\tau H z) dz \quad (20)
\end{aligned}$$

We can compute  $\bar{V}_1(X, S)$  from equation (20) by iterative process. We take the first iterate of (20) as

$$\bar{V}_1(X, S) = \frac{e^{-pX/c}}{p} \quad (21)$$

Deriving  $\left[ \frac{\partial \bar{V}}{\partial X} \right]_{X=0}$  from (21) and substituting it in the second term on the right hand side of (20), the second term, with the aid of (16) becomes  $\frac{e^{-pX/c}}{2p}$ .

Similarly, with the aid of (16) and (17), the third term on the right hand side of (20) becomes

$$-\frac{e^{-pX/c}}{2p} + \frac{2H^2}{\pi^2 c} \int_1^\infty dw \int_0^\infty e^{-\beta X} \cos \alpha H(s-1) \cos \alpha H(w-1) d\alpha \int_0^\infty \frac{\cos(\tau H w) \cos(\tau H)}{\tau \beta(\tau)} d\tau$$

Thus the second iterate is

$$\begin{aligned}
\bar{V}_1(X, S) & = \frac{e^{-pX/c}}{p} + \frac{2H^2}{\pi^2 c} \int_1^\infty dw \int_0^\infty e^{-\beta X} \cos \alpha H(s-1) \cos \alpha H(w-1) d\alpha \times \\
& \times \int_0^\infty \frac{\cos(\tau H w) \cos(\tau H)}{\tau \beta(\tau)} d\tau \quad (22)
\end{aligned}$$

In order to find the displacement on the free surface, the next task is to put  $s=1$  and obtain Laplace inversion of the right hand side of (22).

$$\text{Now } \int_0^{\infty} \frac{\cos(\tau H w) \sin(\tau H)}{\tau \beta(\tau)} d\tau = \frac{1}{4ip} \int_{-\infty}^{\infty} \frac{e^{ipH(w+1)v} - e^{ipH(w-1)v}}{v \sqrt{v^2 + 1/c^2}} dv$$

where  $\tau = pv$ .

Deforming the path of integration to the path parallel to the imaginary  $v$ - axis round the branch point  $v=1/c$ , the above integral can be reduced to the form

$$\frac{H}{2p} \left[ (w-1) \int_{H(w-1)/c}^{\infty} \frac{e^{-pT} dT}{T \sqrt{T^2 - H^2(w-1)^2/c^2}} - (w+1) \int_{H(w+1)/c}^{\infty} \frac{e^{-pT} dT}{T \sqrt{T^2 - H^2(w+1)^2/c^2}} \right] \quad (23)$$

Further

$$\int_0^{\infty} e^{-\beta X} \cos \alpha H(w-1) d\alpha = \frac{1}{2} \int_{-\infty}^{\infty} p e^{-p[\sqrt{u^2 + c^{-2}} X - iuH(w-1)]} du$$

which by the well known Cagniard- Dehoop transformation reduces to the form

$$pX \int_{R/c}^{\infty} \frac{T e^{-pT}}{R^2 \sqrt{T^2 - R^2/c^2}} dT \quad (24)$$

where  $R^2 = X^2 + H^2(w-1)^2$ .

Substituting the results (23) and (24) on the right hand side of (22) and taking the Laplace inverse transform, we obtain

$$V_1(X, H, T) = H(T-X/C) + \frac{XH^2}{\pi^2 c} \left[ H(T-X/C) \int_1^{1 + \frac{C^2 T^2 - X^2}{2HCT}} \frac{w-1}{R^2} dw \times \right]$$

$$\begin{aligned}
 & \times \int_{H(w-1)/c}^{T-R/C} \frac{(T-\sigma) d\sigma}{\sigma \sqrt{[(T-\sigma)^2 - R^2/C^2][\sigma^2 - H^2(w-1)^2/c^2]}} - H\left(T - \frac{X+2H}{C}\right) \times \\
 & \times \int_1^{1 + \frac{(CT-2H)^2 - X^2}{2H(CT-2H)}} \frac{w+1}{R^2} dw \int_{H(w+1)/c}^{T-R/C} \frac{(T-\sigma) d\sigma}{\sigma \sqrt{[(T-\sigma)^2 - R^2/C^2][\sigma^2 - H^2(w+1)^2/c^2]}}
 \end{aligned} \quad (25)$$

which represents the displacement due to transmitted wave in the free surface of the medium 1.

Introducing  $X=Hx$ ,  $T=Ht/c$ ,  $\sigma=H\sigma'/c$  and writing

$A=t+\sqrt{x^2+(w-1)^2}$ ,  $B=t-\sqrt{x^2+(w-1)^2}$ ,  $D=w-1$ ,  $E=-D$  so that  $A>B>D>E$ ; we get from (25)

$$\begin{aligned}
 V_1(x, 1, t) = & H(t-x) + \frac{x}{\pi^2} \left[ H(t-x) \int_1^{1 + \frac{t^2 - x^2}{2t}} \frac{w-1}{x^2 + (w-1)^2} dw \left\{ \frac{2}{\sqrt{\alpha_1 \beta_1}} \times \right. \right. \\
 & \times \left. \left. \left[ \frac{2t}{D} \Pi(\pi/2, -\delta_1/\beta_1, q_1) - \left( \frac{t}{D} + 1 \right) F(\pi/2, q_1) \right] \right\} - H(t-x+2) \times \right. \\
 & \times \int_1^{1 + \frac{(t-2)^2 - x^2}{2(t-2)}} \frac{w+1}{x^2 + (w-1)^2} dw \left\{ \frac{2}{\sqrt{\alpha_2 \beta_2}} \left[ \frac{2t}{D} \Pi(\pi/2, -\delta_2/\beta_2, q_2) - \left( \frac{t}{D} + 1 \right) \times \right. \right. \\
 & \left. \left. \times F(\pi/2, q_2) \right] \right\} dw \left. \right] \quad (26)
 \end{aligned}$$

where

$$A-D = \alpha_1, \quad B-E = \beta_1, \quad A-E = \tau_1, \quad B-D = \delta_1$$

$$A-\bar{D} = \alpha_2, \quad B-\bar{E} = \beta_2, \quad A-\bar{E} = \tau_2, \quad B-\bar{D} = \delta_2$$

$$q_i = \sqrt{\frac{\delta_i \tau_i}{\alpha_i \beta_i}}, \quad i=1,2$$

and  $\bar{D}, \bar{E}$  are obtained from  $D, E$  respectively by replacing  $w-1$  by  $w+1$  and  $\Pi(\phi, n, k), F(\phi, k)$  are elliptic integral of third and first kind respectively.

### 5. Numerical results.

To show the nature of the motion, we have evaluated numerically the displacement curve for transmitted wave. The results shown in Fig.2 depicted the variation of transmitted wave versus dimensionless time  $t$  for different values of  $x$ . Due to the presence of the step, the value of the displacement is found to increase initially at the arrival of the displacement an abrupt decrease after a small interval of time. Also, as expected from physical stand point, the initial increment in the value of the displacement is found to decrease gradually with the increase in distance from the step.

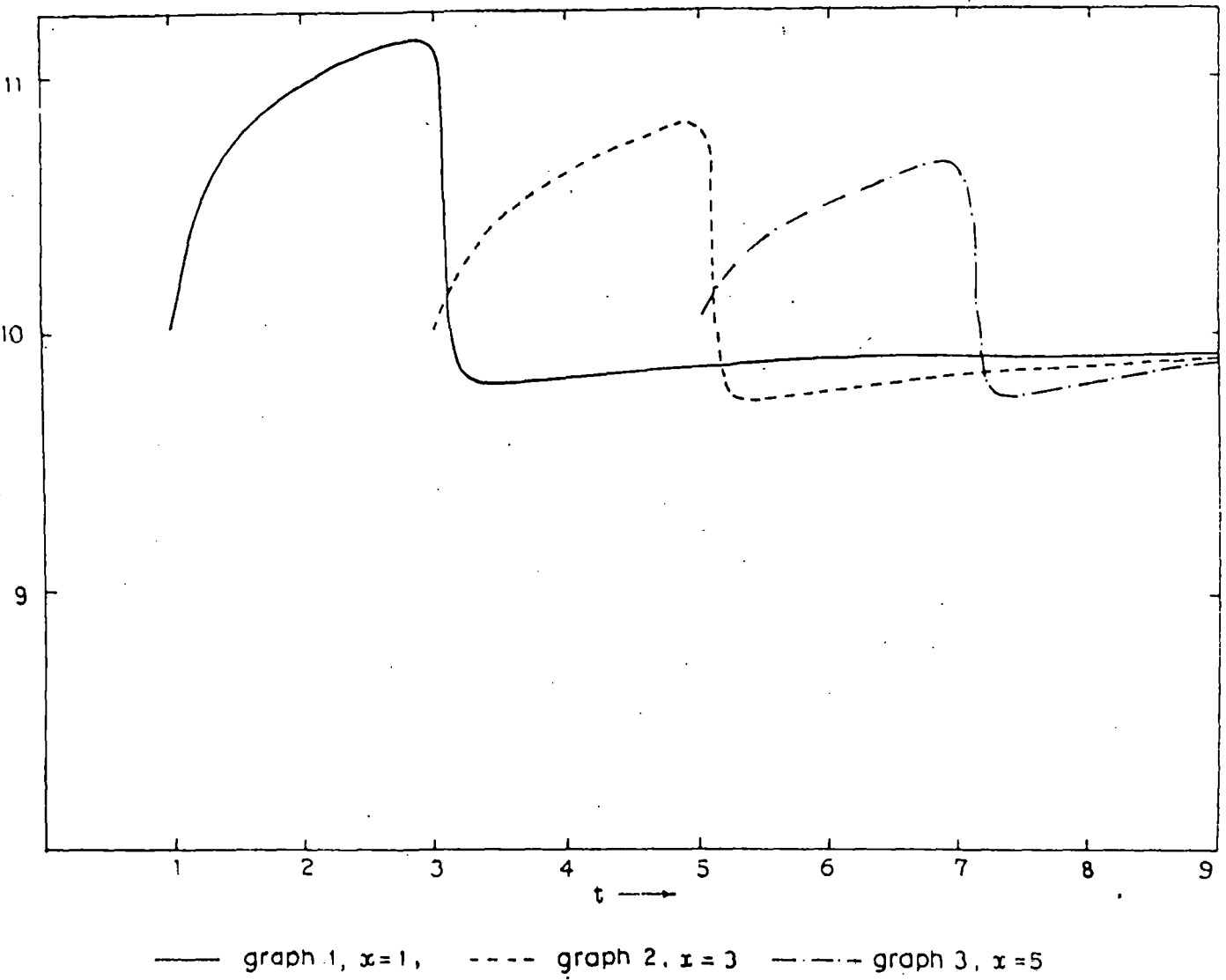


Fig.2: Variation of transverse displacement with time.

# SH WAVE PROPAGATION ACROSS A VERTICAL STEP IN TWO JOINED ELASTIC HALF SPACES

## 1. Introduction

The problem of propagation of elastic waves in the presence of surface irregularities has been studied by several investigators. Abubakar (1963) studied the effect of an irregular surface with an isolated irregularity like a trough or ditch on incident harmonic P- and SV-waves. Propagation of Love wave in an elastic layer having an irregular boundary overlying a rigid half-space has been treated by Wolf (1967) using perturbation technique. The transmission of elastic waves across a step like irregularity in the surface of an elastic half-space is of great importance in seismology in connection with the propagation of waves from ocean basins to continental regions and vice versa. Knopoff and Hudson (1964) studied the transmission of Love waves past a continental margin considering the crust to have an abrupt increase in thickness on the continental side. The transmission of SH-waves across a step like irregularity in the surface of an elastic half space was also considered by Bose (1975). Sato (1961) discussed the problem of propagation of Love wave in an elastic layer of variable thickness overlying a semi-infinite elastic medium. Approximate expressions for the transmission and reflection factors are obtained by the application of a method based on Wiener-Hopf technique.

In this paper, we consider the propagation of SH- wave in a medium consisting of two welded quarter spaces of different materials and having a step change in elevation at the vertical interface. The problem reduces to an integral equation by using transform and Green's function method and finally applying the method of steepest descent, transmitted and reflected fields at large distance from the step have been determined. It may be mentioned in this connection that the problem of transient shear wave in a half-space composed of two joined elastic quarter spaces of different

materials are subjected to time varying shear tractions at the free surface, parallel to plane of juncture has been solved by Achenbach (1969). Datta and Mitra (1974) also considered SH-wave propagation in a composite elastic medium consisting of an elastic quarter space in welded contact with a uniform layer of different shear wave velocity.

## 2. Formulation Of The Problem

We consider two quarter spaces of different materials joined along the common boundary  $X=0$  in such a way that there is a step change in elevation at the free surface. We consider the axes of coordinate as shown in Fig 1. Denoting the coordinate of a point in the X-Z plane by  $(X,Z)$ , we take the incident plane SH-wave as  $\exp[i(\omega t - K_2 X)]$  where  $K_2 = \omega/c_2$ , so that the propagation is from higher side to the lower side of the step.

The boundaries  $Z=0, X<0$ ;  $Z=H, X>0$  and  $X=0, 0 \leq Z \leq H$  are assumed to be stress free. Omitting the time factor  $\exp(i\omega t)$ , let  $V_1(X,z), V_2(X,Z)$  be the SH-wave displacement component in two media (I) and (II) respectively in Y-direction which is perpendicular to the plane of the paper.

The field equations are wave equations in the two media and boundary conditions are that (i) The outer boundary is stress free and (ii) the displacement and stress are continuous on the interface  $X=0, Z>H$ .  $\mu, \rho, c$  are assumed to be the modulus of rigidity, density and shear wave velocity with appropriate suffix for each of the two media.

Introducing the dimensionless quantities

$$x = \frac{X}{H}, \quad z = \frac{Z}{H}$$

We get from the wave equations and boundary conditions

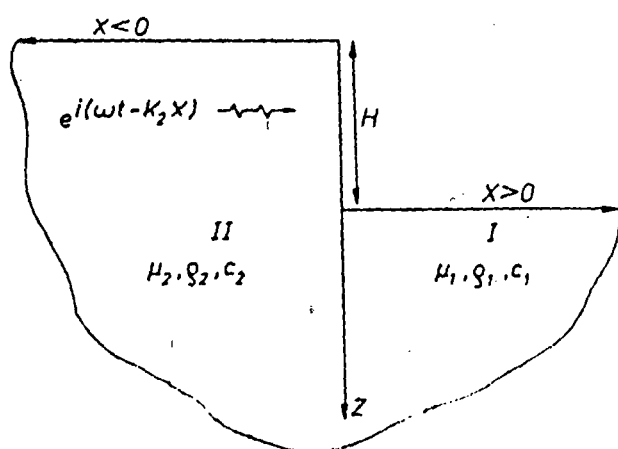


Fig.1: Geometry and coordinate system.

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_1^2 \right] v_1 = 0$$

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_2^2 \right] v_2 = 0 \quad (2.1)$$

$$\mu_2 \frac{\partial v_2}{\partial x} \Big|_{x=0} = 0 \text{ for } 0 < z < 1$$

$$\mu_2 \frac{\partial v_2}{\partial z} \Big|_{z=0} = 0 \text{ for } x < 0 \quad (2.2)$$

$$\mu_1 \frac{\partial v_1}{\partial z} \Big|_{z=1} = 0 \text{ for } x > 0 \quad (2.3)$$

$$\mu_1 \frac{\partial v_1}{\partial x} \Big|_{x=0} = \mu_2 \frac{\partial v_2}{\partial x} \Big|_{x=0} = 0 \text{ for } z > 1$$

$$v_1(0, z) = v_2(0, z) \quad \text{for } z > 1 \quad (2.4)$$

where  $H$  is the height of the step and  $k_i^2 = \omega^2 H^2 / c_i^2$ ,  $V_i(X, Z) = v_i(x, z)$ ,  $i=1, 2$ . We represent transverse displacement in the two domains  $x < 0$  and  $x > 0$  in the form

$$\begin{aligned} v_2 &= 2 \cos k_2 x + v'_2(x, z) & x < 0, z > 0 \\ v_1 &= v_1(x, z) & x > 0, z > 1 \end{aligned} \quad (2.5)$$

### 3. Reduction To Integral Equation and Its Solution

We introduce Green's functions  $G_1(x, z; r, s)$  and  $G_2(x, z; u, v)$  for the medium (I) and (II) respectively such that  $G_2(x, z; u, v)$  is the solution of

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_2^2 \right] G_2(x, z; u, v) = -4\pi \delta(x-u) \delta(z-v) \quad (3.1)$$

for medium (II) with vanishing normal derivative at  $x < 0, z = 0$  and at

$x=0, z>0$ . Similarly,  $G_1(x, z; r, s)$  is the solution of

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_1^2 \right] G_1(x, z; r, s) = -4\pi\delta(x-r)\delta(z-s) \quad (3.2)$$

and satisfies the vanishing of the normal derivative at  $x>0, z=1$  and at  $x=0, z>1$ .

From equation (2.1b) and (3.1) we obtain by applying Green's theorem to the medium (II) and using appropriate boundary condition

$$4\pi v_2'(u, v) = \int_1^\infty G_2(0, z; u, v) \left[ \frac{\partial v_2'}{\partial x} \right]_{x=0} dz \quad (3.3)$$

and similar application of Green's theorem to the medium (I) yields

$$4\pi v_1(r, s) = - \int_1^\infty G_1(0, z; r, s) \left[ \frac{\partial v_1}{\partial x} \right]_{x=0} dz \quad (3.4)$$

Substitution of (3.3) and (3.4) into the equation (2.4b) yields with the aid of (2.4a) and equation (2.5) the integral equation

$$\int_1^\infty \left[ G_1(0, z; 0, v) + \frac{\mu_1}{\mu_2} G_2(0, z; 0, v) \right] \left[ \frac{\partial v_1}{\partial x} \right]_{x=0} dz = -8\pi \quad (3.5)$$

The expression of the Green's function for the medium (II) will now be derived using Fourier cosine transform with respect to  $z$ ; which reduces the determination of  $G_2(x, z; u, v)$  to that of a Green's function for an ordinary differential equation. Accordingly taking Fourier cosine transform defined by

$$G_2^c = \int_0^\infty G_2(x, z; u, v) \cos(\alpha z) dz, \text{ we obtain from the equation (3.1)}$$

$$\frac{d^2 G_2^c}{dx^2} - (\alpha^2 - k_2^2) G_2^c = -4\pi \cos(\alpha v) \delta(x-u)$$

from which we obtain in a straight forward manner

$$\begin{aligned}
 G_2(x, z; u, v) &= 8 \int_0^{\infty} \frac{e^{\beta_2 u}}{\beta_2} \cosh(\beta_2 x) \cos(\alpha v) \cos(\alpha z) \, d\alpha & u \leq x \leq 0 \\
 &= 8 \int_0^{\infty} \frac{e^{\beta_2 x}}{\beta_2} \cosh(\beta_2 u) \cos(\alpha v) \cos(\alpha z) \, d\alpha & -\infty < x \leq u
 \end{aligned} \tag{3.6}$$

where  $\beta_2^2 = \alpha^2 - k_2^2$ .

Again introducing Fourier cosine transform defined by

$$G_1^c = \int_0^{\infty} G_1(x, z; r, s) \cos \alpha(z-1) d(z-1), \quad \text{we obtain from equation (3.2)}$$

$$\frac{d^2 G_1^c}{dx^2} - (\alpha^2 - k_1^2) G_1^c = -4\pi \cos \alpha(s-1) \delta(x-r)$$

from which we obtain easily

$$\begin{aligned}
 G_1(x, z; r, s) &= 8 \int_0^{\infty} \frac{e^{-\beta_1 r}}{\beta_1} \cosh(\beta_1 x) \cos \alpha(s-1) \cos \alpha(z-1) \, d\alpha & 0 \leq x \leq r \\
 &= 8 \int_0^{\infty} \frac{e^{-\beta_1 x}}{\beta_1} \cosh(\beta_1 r) \cos \alpha(s-1) \cos \alpha(z-1) \, d\alpha & r \leq x \leq \infty
 \end{aligned} \tag{3.7}$$

where  $\beta_1^2 = \alpha^2 - k_1^2$ .

On substituting the values of  $G_1(0, z; 0, v)$  and  $G_2(0, z; 0, v)$  from equations (3.7) and (3.6) in the equation (3.5) we obtain

$$\int_1^{\infty} \left[ \frac{\partial v_1}{\partial x} \right]_{x=0} dz \int_0^{\infty} \left[ \frac{\cos \alpha(v-1) \cos \alpha(z-1)}{\beta_1} + \frac{\mu_1}{\mu_2} \frac{\cos(\alpha v) \cos(\alpha z)}{\beta_2} \right] d\alpha = -\pi$$

$$\int_1^{\infty} \left[ \frac{\partial v_1}{\partial x} \right]_{x=0} dz \int_0^{\infty} \frac{\cos \alpha(v-1) \cos \alpha(z-1)}{\beta_1} d\alpha = -\pi - \frac{\mu_1}{\mu_2} \int_1^{\infty} \left[ \frac{\partial v_1}{\partial x} \right]_{x=0} dz \times$$

$$\times \int_0^{\infty} \frac{\cos(\alpha v) \cos(\alpha z)}{\beta_2} d\alpha$$

Taking Fourier cosine inverse transform with respect to  $\alpha$ , we get

$$\int_1^{\infty} \left[ \frac{\partial v_1}{\partial x} \right]_{x=0} \frac{\cos \alpha (v-1)}{\beta_1} dz = -2 \int_1^{\infty} \cos \alpha (v-1) dv - \frac{2\mu_1}{\pi\mu_2} \int_1^{\infty} \left[ \frac{\partial v_1}{\partial x} \right]_{x=0} dz$$

$$\times \int_0^{\infty} \frac{\cos(\tau z)}{\beta_2(\tau)} d\tau \int_1^{\infty} \cos(\tau v) \cos \alpha (v-1) dv \quad (3.8)$$

where  $\beta_2(\tau)$  is obtained from  $\beta_2$  by replacing  $\alpha$  by  $\tau$ .

Next using the results that

$$\int_1^{\infty} \cos \alpha (v-1) dv = \pi \delta(\alpha) \quad (3.9)$$

$$\int_1^{\infty} \sin \alpha (v-1) dv = \frac{1}{\alpha} \quad (3.10)$$

it can be easily shown that

$$\int_1^{\infty} \cos(\tau v) \cos \alpha (v-1) dv = \frac{\pi}{2} \cos \tau [\delta(\tau+\alpha) + \delta(\tau-\alpha)] - \frac{\tau \sin \tau}{\tau^2 - \alpha^2} \quad (3.11)$$

where  $\delta(x)$  is the Dirac  $\delta$ -function.

Using these results and after a little algebraic manipulation it can be easily shown that equation (3.8) reduces to the form

$$\int_1^{\infty} \left[ \frac{\partial v_1}{\partial x} \right]_{x=0} \cos \alpha (z-1) dz = - \frac{2\pi\mu_2\beta_1\beta_2\delta(\alpha)}{\mu_1\beta_1 + \mu_2\beta_2} + \frac{\mu_1\beta_1}{\mu_1\beta_1 + \mu_2\beta_2} \int_1^{\infty} \left[ \frac{\partial v_1}{\partial x} \right]_{x=0}$$

$$\times \sin(\alpha z) \sin \alpha dz + \frac{2\mu_1\beta_1\beta_2}{\pi(\mu_1\beta_1 + \mu_2\beta_2)} \int_1^{\infty} \left[ \frac{\partial v_1}{\partial x} \right]_{x=0} dz \int_0^{\infty} \frac{\tau \cos(\tau z) \sin \tau}{\beta_2(\tau)(\tau^2 - \alpha^2)} d\tau \quad (3.12)$$

## 4. Evaluation Of Displacement

Substituting the value of  $G_1(0, z; r, s)$  from equation (3.7) in equation (3.4) and then using the result (3.12), the displacement in the medium I is obtained in the form

$$v_1(x, s) = \frac{2\mu_2 k_2}{\mu_1 k_1 + \mu_2 k_2} e^{-ik_1 x} - \frac{2\mu_1}{\pi} \int_0^\infty \frac{e^{-\beta_1 x} \cos \alpha (s-1)}{\mu_1 \beta_1 + \mu_2 \beta_2} d\alpha \times$$

$$\times \int_1^\infty \left[ \frac{\partial v_1}{\partial x} \right]_{x=0} \sin(\alpha z) \sin \alpha dz - \frac{4\mu_1}{\pi^2} \int_0^\infty \frac{\beta_2 e^{-\beta_1 x} \cos \alpha (s-1)}{\mu_1 \beta_1 + \mu_2 \beta_2} d\alpha \int_1^\infty \left[ \frac{\partial v_1}{\partial x} \right]_{x=0} dz$$

$$\times \int_0^\infty \frac{\tau \cos(\tau z) \sin \tau}{\beta_2 (\tau) (\tau^2 - \alpha^2)} d\tau \quad (4.1)$$

We can compute  $v_1(x, s)$  iteratively solving equation (4.1) and using asymptotic values of integrals arising in the right hand side of (4.1) for large values of  $x$ .

The first iterate is

$$v_1(x, s) = \frac{2\mu_2 k_2}{\mu_1 k_1 + \mu_2 k_2} e^{-ik_1 x} \quad (4.2)$$

which is obviously the displacement in medium (I) in the absence of step change in elevation.

Deriving  $\left[ \frac{\partial v_1}{\partial x} \right]_{x=0}$  from the first iterate given in (4.2) and using this in the right hand side of (4.1) with the aid of (3.10) it can be easily show that second term in the right hand side of (4.1) takes the form.

$$I_1 = \frac{2ik_1 k_2 \mu_1 \mu_2}{\pi (\mu_1 k_1 + \mu_2 k_2)} \int_0^\infty \frac{e^{-\beta_1 x} \cos \alpha (s-1)}{\alpha (\mu_1 \beta_1 + \mu_2 \beta_2)} \sin 2\alpha d\alpha \quad (4.3)$$

Which, for large values of  $x$ , can be evaluated asymptotically by the method of steepest descent. therefore, for large  $x$ , we find

$$I_1 \sim \frac{4\mu_1 k_1 \mu_2 k_2}{(\mu_1 k_1 + \mu_2 k_2)^2} \left[ \frac{k_1}{2\pi x} \right]^{1/2} \exp(\pi i/4 - ik_1 x) \quad (4.4)$$

Similarly, with the aid of (4.2) and the result given in (3.9), the third term in the right hand side of (4.1) reduces to the form

$$I_2 = \frac{8i\mu_1 k_1 \mu_2 k_2}{\pi^2 (\mu_1 k_1 + \mu_2 k_2)} \int_0^\infty \frac{\sin^2 \tau}{\beta_2(\tau)} d\tau \int_0^\infty \frac{\beta_2 e^{-\beta_1 x} \cos \alpha(s-1)}{(\mu_1 \beta_1 + \mu_2 \beta_2)(\alpha^2 - \tau^2)} d\alpha \quad (4.5)$$

In order to evaluate asymptotically for large values of  $x$ , integrals of the type

$$I'_1 = \int_0^\infty \frac{f(\alpha)}{(\alpha^2 - \tau^2)} e^{-\beta_1 x} d\alpha$$

we have to take into account the residue at the singularity  $\alpha = \tau$  in addition to the integral along the steepest descent. Thus we get

$$I'_1 \sim \pi i \frac{f(\tau)}{2\tau} e^{-\beta_1(\tau)x} - \left[ \frac{\pi k_1}{2x} \right]^{1/2} \frac{f(0)}{\tau^2} \exp(\pi i/4 - ik_1 x) \quad (4.6)$$

Using this result in(4.5). for large values of  $x$ , we obtain

$$I_2 \sim - \frac{4\mu_1 k_1 \mu_2 k_2}{(\mu_1 k_1 + \mu_2 k_2)^2} \left[ \frac{k_1}{2\pi x} \right]^{1/2} M \exp(\pi i/4 - ik_1 x) \quad (4.7)$$

where

$$M = \frac{2ik_2}{\pi} \int_0^{\infty} \frac{\sin^2 \tau}{\tau^2 \beta_2(\tau)} d\tau \quad (4.8)$$

$$= \frac{2}{\pi k_2} \int_0^1 \frac{\sin^2 k_2 t}{t^2 \sqrt{1-t^2}} dt + \frac{2i}{\pi k_2} \int_1^{\infty} \frac{\sin^2 k_2 t}{t^2 \sqrt{t^2-1}} dt$$

$$= J - iY$$

Following Bose (1975) it can be shown that

$$J = \int_0^{2k_2} J_0(z) dz - J_1(2k_2) \quad \text{and} \quad Y = \int_0^{2k_2} Y_0(z) dz - Y_1(2k_2) - \frac{1}{\pi k_2}$$

Thus from (4.1), (4.4) and (4.7), the second iterate, for large  $x$ , is

$$v_1(x, z) = \frac{2\mu_2 k_2}{\mu_1 k_1 + \mu_2 k_2} \left[ 1 + \frac{2(1-M)\mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2} \left\{ \frac{k_1}{2\pi x} \right\}^{1/2} e^{\pi i/4} \right] e^{-ik_1 x} \quad (4.9)$$

If we neglect terms of order  $1/x$ , the higher order iterates yield the same expression.

Now in order to find out displacement component due to reflected wave in the medium (II), we rewrite the equation (3.12) with the aid of equation (2.4a) and (2.5a) as

$$\int_1^{\infty} \left[ \frac{\partial v'_2}{\partial x} \right]_{x=0} \cos \alpha(z-1) dz = - \frac{2\mu_1 \beta_1 \beta_2 \delta(\alpha)}{\mu_1 \beta_1 + \mu_2 \beta_2} + \frac{\mu_1 \beta_1}{\mu_1 \beta_1 + \mu_2 \beta_2} \int_1^{\infty} \left[ \frac{\partial v'_2}{\partial x} \right]_{x=0}$$

$$\times \sin(\alpha z) \sin \alpha dz + \frac{2\mu_1 \beta_1 \beta_2}{\pi(\mu_1 \beta_1 + \mu_2 \beta_2)} \int_1^{\infty} \left[ \frac{\partial v'_2}{\partial x} \right]_{x=0} dz \int_0^{\infty} \frac{\tau \cos(\tau z) \sin \tau}{\beta_2(\tau) (\tau^2 - \alpha^2)} d\tau$$

Taking inverse Fourier cosine transform with respect to  $z$ , we get

$$\left[ \frac{\partial v'_2}{\partial x} \right]_{x=0} = -\frac{2i\mu_1 k_1 k_2}{\mu_1 k_1 + \mu_2 k_2} + \frac{2}{\pi} \int_0^\infty \frac{\mu_1 \beta_1}{\mu_1 \beta_1 + \mu_2 \beta_2} \cos \alpha(z-1) d\alpha \times$$

$$\int_1^\infty \left[ \frac{\partial v'_2}{\partial x} \right]_{x=0} \sin(\alpha u) \sin \alpha \, du + \frac{4}{\pi^2} \int_0^\infty \frac{\mu_1 \beta_1 \beta_2}{(\mu_1 \beta_1 + \mu_2 \beta_2)} \cos \alpha(z-1) d\alpha \int_1^\infty \left[ \frac{\partial v'_2}{\partial x} \right]_{x=0} du$$

$$\times \int_0^\infty \frac{\tau \cos(\tau u) \sin \tau}{\beta_2(\tau)(\tau^2 - \alpha^2)} d\tau \quad (4.10)$$

Thus substitution of equation (4.10) in equation (3.3) with the aid of equation (3.6) yields

$$v'_2(x, v) = \frac{-2\mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2} e^{ik_2 x} + \frac{4i\mu_1 k_1 k_2}{\pi(\mu_1 k_1 + \mu_2 k_2)} \int_0^\infty \frac{e^{\beta_2 x} \cos(\alpha v) \sin \alpha}{\alpha \beta_2} d\alpha$$

$$+ \frac{1}{\pi} \int_0^\infty \frac{\mu_1 \beta_1 e^{\beta_2 x} \cos(\alpha v)}{\beta_2(\mu_1 \beta_1 + \mu_2 \beta_2)} \sin 2\alpha \, d\alpha \int_1^\infty \left[ \frac{\partial v'_2}{\partial x} \right]_{x=0} \sin(\alpha u) \, du - \frac{4}{\pi^2} x$$

$$\int_0^\infty \frac{\alpha e^{\beta_2 x} \cos(\alpha v) \sin \alpha}{\beta_2} d\alpha \int_0^\infty \frac{\beta'_1 \mu_1 \sin \alpha'}{(\mu_1 \beta'_1 + \mu_2 \beta'_2)(\alpha^2 - \alpha'^2)} d\alpha' \int_1^\infty \left[ \frac{\partial v'_2}{\partial x} \right]_{x=0} \sin(\alpha' u) \, du +$$

$$+ \frac{4\mu_1}{\pi^2} \int_0^\infty \frac{\beta_1 e^{\beta_2 x} \cos(\alpha v)}{(\mu_1 \beta_1 + \mu_2 \beta_2)} \cos \alpha \, d\alpha \int_0^\infty \frac{\tau \sin \tau}{\beta_2(\tau)(\tau^2 - \alpha^2)} d\tau \int_1^\infty \left[ \frac{\partial v'_2}{\partial x} \right]_{x=0} \cos(\tau u) \, du -$$

$$- \frac{8\mu_1}{\pi^2} \int_0^\infty \frac{\alpha e^{\beta_2 x} \cos(\alpha v) \sin \alpha}{\beta_2} d\alpha \int_0^\infty \frac{\beta'_1 \beta'_2}{(\mu_1 \beta'_1 + \mu_2 \beta'_2)(\alpha^2 - \alpha'^2)} d\alpha' \int_0^\infty \frac{\tau \sin \tau}{\beta_2(\tau)(\tau^2 - \alpha'^2)} d\tau \times$$

$$x \int_1^{\infty} \left[ \frac{\partial v'_2}{\partial x} \right]_{x=0} \cos(\alpha u) du \quad (4.11)$$

where  $\beta'_1$  and  $\beta'_2$  are obtained from  $\beta_1$  and  $\beta_2$  by replacing  $\alpha$  by  $\alpha'$ .

Now to solve equation (4.11) iteratively, we take the first iteration as

$$v'_2(x, v) = \frac{-2\mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2} e^{ik_2 x} + \frac{4i\mu_1 k_1 k_2}{\pi(\mu_1 k_1 + \mu_2 k_2)} \int_0^{\infty} \frac{e^{\beta_2 x} \cos(\alpha v) \sin \alpha}{\alpha \beta_2} d\alpha \quad (4.12)$$

$$\text{i.e., } \left[ \frac{\partial v'_2}{\partial x} \right]_{x=0} = -\frac{2ik_2 \mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2} \quad (4.13)$$

The asymptotic evaluation of (4.12) for large values of  $x$  by the method of steepest descent yield the first iterate as

$$v'_2(x, v) = \frac{-2\mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2} \left[ 1 - \left\{ \frac{2k_2}{-\pi x} \right\}^{1/2} e^{\pi i/4} \right] e^{ik_2 x} \quad (4.14)$$

The second iterate is obtained by inserting the value of  $\left[ \frac{\partial v'_2}{\partial x} \right]_{x=0}$  given in the equation (4.13) on the right hand side of equation (4.11) and using the results

$$\int_1^{\infty} \sin(\alpha z) dz = \frac{\cos \alpha}{\alpha} \quad \text{and} \quad \int_1^{\infty} \cos(\alpha z) dz = \pi \delta(\alpha) - \frac{\sin \alpha}{\alpha}$$

and then evaluating the integrals on the right hand side of (4.11) by the method steepest descent for large value of  $x$ . Thus we obtain the second iterate as

$$v'_2(x, v) = \frac{-2\mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2} \left[ 1 - 2 \left( 1 - \frac{(1-M)\mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2} \right) \left\{ \frac{k_2}{-2\pi x} \right\}^{1/2} e^{\pi i/4} \right] e^{ik_2 x} \quad (4.15)$$

where  $M$  is given in (4.8)

Thus from equation (2.5a) and (4.15) we get

$$v_2(x, v) = e^{-ik_2 x} + \left[ \frac{\mu_2 k_2 - \mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2} + \frac{4\mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2} \left( 1 - \frac{(1-M)\mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2} \right) \sqrt{\frac{k_2}{-2\pi x}} e^{\pi i/4} \right] e^{ik_2 x} \quad (4.16)$$

### 5. Numerical Results and Discussions

To investigate the nature of the motion, we have evaluated numerically the increment in amplitude due to the step for both transmitted wave and reflected waves. The results are shown in the form of graphs showing the variation of  $\sqrt{x} \Delta V_{1T}$  with  $k_2$  for different values of  $\mu_2/\mu_1$  in Fig.2 for transmitted wave and the variation  $\sqrt{-x} \Delta V_{2R}$  with  $k_2$  in Fig.3 for reflected wave, where

$$\Delta V_{1T} = |v_1| - \frac{2\mu_2 k_2}{\mu_1 k_1 + \mu_2 k_2} \quad \text{and} \quad \Delta V_{2R} = |v_{2R}| - \frac{\mu_2 k_2 - \mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2}$$

where  $v_{2R}$  is the reflected part of  $v_2$ .

The value of  $\sqrt{x} \Delta V_{1T}$  is found to increase gradually with the increase in the value of  $\mu_2/\mu_1$  and for all values of  $\mu_2/\mu_1$ , it is found that the maximum value of  $\sqrt{x} \Delta V_{1T}$  occurs at  $k_2 = 0.75$ . It is also observed from Fig.2 that  $\sqrt{x} \Delta V_{1T}$  is positive for all values of  $k_2$  and  $\mu_2/\mu_1$ , that is the amplitude of transmitted wave is always greater than that of transmitted wave in the absence of of the step. Moreover, with the increase in the values of  $k_2$  the graphs show an undulating character with decreasing amplitude.

From Fig.3, we see that the value of  $\sqrt{-x} \Delta V_{2R}$  gradually decreases with the increase in the value of  $\mu_2/\mu_1$  and shows a gradual increase as the value of  $k_2$  increases.

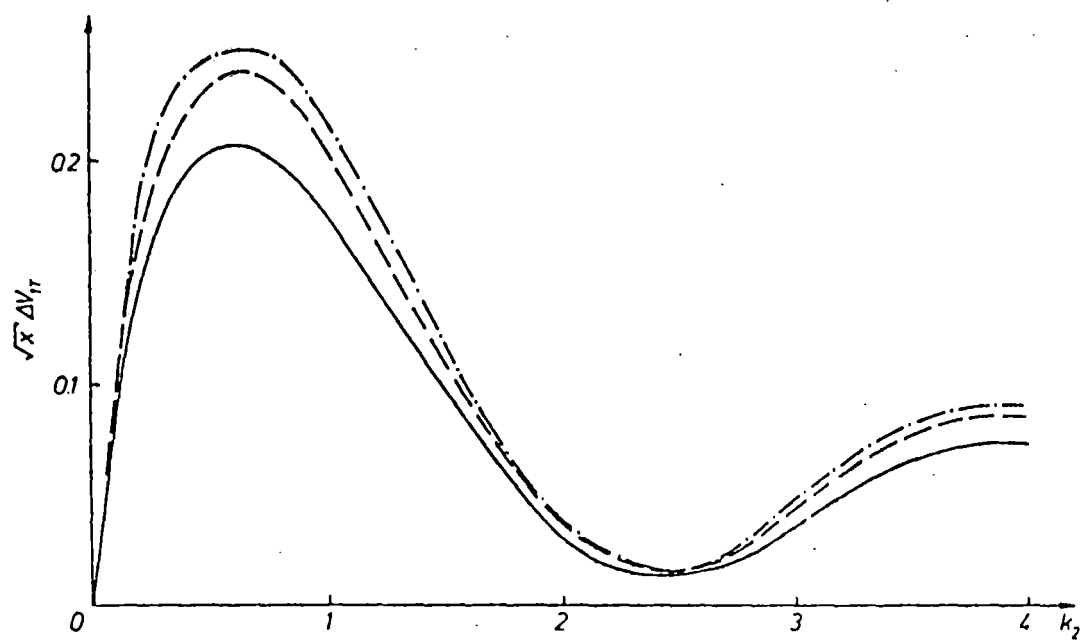


Fig. 2: Variation of  $\sqrt{x} \Delta V_{IT}$  with  $k_2$ .

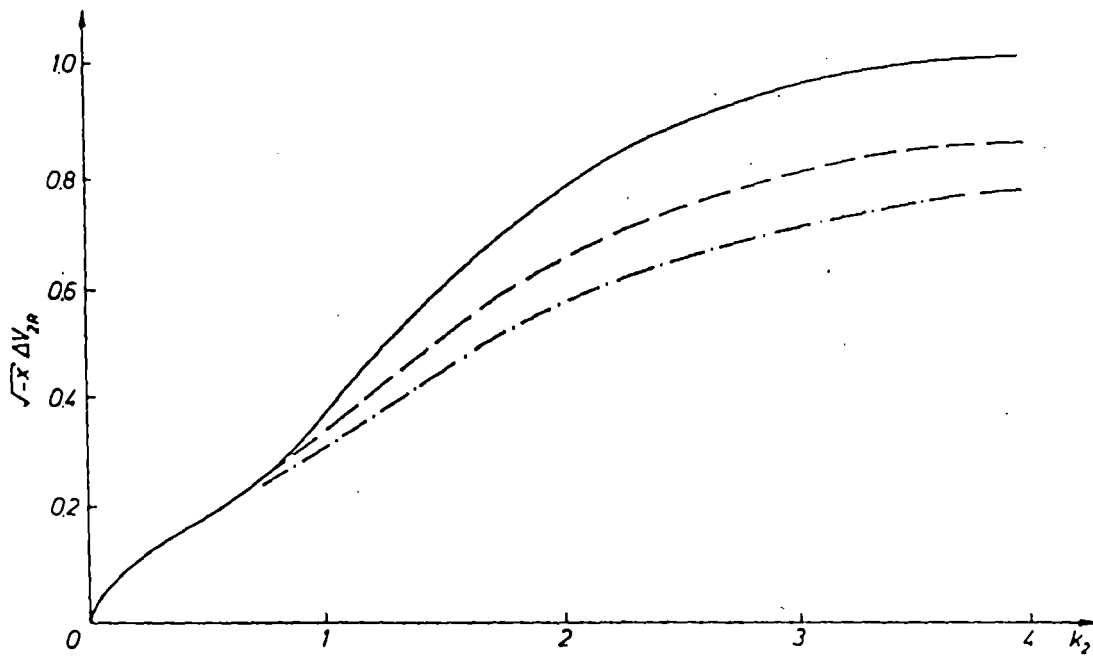


Fig. 3: Variation of  $\sqrt{-x} \Delta V_{2R}$  with  $k_2$ .