Chapter 2

Permutation Bigraphs: An Analogue of Permutation Graphs[1]

2.1 Introduction

An undirected graph G on n vertices is a *permutation graph* if there are two orderings of $V(G)$ such that vertices are adjacent if and only if they appear in opposite order in the two orderings. The class of permutation graphs is well studied; MathSciNet lists more than 100 papers. The definition can be restated in several well-known equivalent ways.

Theorem 2.1.1. [26, 27, 38] *The following conditions are equivalent for a graph G:*

(a) G is a permutation graph;

(b) Both G and its complement $\overline{G}$ are transitively orientable;

(c) G is the containment graph of a family of intervals in $\mathbb{R}$;

[1]A part of this chapter is co-authored with D.B.West and is communicated to J.Graph Theory for possible publication

(d) G is the comparability graph of a poset of dimension at most 2.

In this chapter, we introduce two bipartite analogues of this family, one of which we call "permutation bigraphs". We do *not* mean "bipartite permutation graphs", which are those graphs that are bipartite and are permutation graphs (discussed in [86, 87]).

An *X, Y-bigraph* is a bipartite graph with partite sets X and Y. As defined above, two orderings of $X \cup Y$ produce a permutation graph G with vertex set $X \cup Y$. There are two natural ways to generate an X, Y-bigraph contained in G. In the first, one simply deletes the edges within X and within Y.

The second model is even more restrictive about the edges retained from G. Treat the first ordering L as a reference ordering, numbering $X \cup Y$ from 1 to $|X \cup Y|$. That is, L expresses (X, Y) as a partition of $[n]$, where $n = |X \cup Y|$ and $[n]$ denotes $\{1, \ldots, n\}$. Create an edge xy for $x \in X$ and $y \in Y$ if and only if $x > y$ (as elements of $[n]$) and x occurs before y in the second ordering π. Note that x and y then form an inversion in π.

A *permutation bigraph* is an X, Y-bigraph that can be represented by vertex orderings in this second model. The graphs representable by the first model are the *bipermutation bigraphs*, where the first "bi" indicates that both orderings of the partite sets in an inversion are allowed to generate edges. Although the definition of bipermutation bigraphs may seem more natural, it turns out that the class of permutation bigraphs is better behaved.

A permutation bigraph may have many permutation representations. Indeed, when all of Y precedes all of X in L, every permutation π with X before Y yields a permutation representation of the complete bipartite graph $K_{|X|,|Y|}$. Also, interchanging and reversing L and π in a permutation representation yields another representation of the same graph.

We often specify a permutation graph by giving just one permutation; in this case the other permutation is the identity permutation on the vertex set $[n]$. To specify a permutation bigraph in this way, one must also give the partition of $[n]$ into X and Y. We may present this partition along with the permutation π of $[n]$ by putting underbars on the elements of Y and overbars on the elements of X, or simply underbars on Y.

Example 2.1.2. Let $X = \{1, 2, 4, 9, 10\}$ and $Y = \{3, 5, 6, 7, 8\}$ and consider a permutation $\pi = \{4, \underline{5}, 1, \underline{6}, \underline{7}, 9, 2, \underline{3}, 10, \underline{8}\}$. In this example the vertex 4 of X occurs before the vertex 3 of Y in π and since $4 > 3$ the pair $(4, 3)$ is an edge of the permutation bigraph $B(\pi)$. The bigraph $B(\pi)$ is given below in Figure 2.1

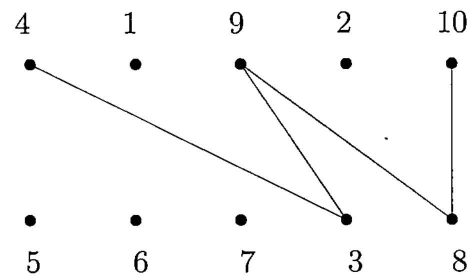

Figure 2.1: A permutation bigraph

The preference for X before Y in the inversions in π does not change which graphs are permutation bigraphs. If G is a permutation bigraph generated by L and π in this model, then reversing L and π to obtain L' and π' again generates G in the model where inversions in π' with elements of Y before those of X become edges.

Our characterizations of permutation bigraphs parallel those of permutation graphs.

Example 2.1.3. Theorem 2.1.1 characterizes permutation graphs using comparability graphs. A *poset of dimension* 2 is a partially ordered set P generated from two linear orders on its elements by putting $x < y$ if and only if x comes before y in both orders. The *comparability graph* of P is the graph whose vertices are the elements of P, with x adjacent to y if and only if $x < y$ or $y < x$ in P. Reversing one of the linear orders generating a poset of dimension 2 yields another poset of dimension 2 whose comparability graph is the complement of the first and is the permutation graph generated by the original two linear orders. Conversely, if G is a permutation graph, then the two permutations representing it give two linear orders (one reversed) that define a poset for which it is the comparability graph. □

The analogous characterization of permutation bigraphs uses a generalization of linear orders. For an X, Y-bigraph B, we call the submatrix of its adjacency matrix consisting of the rows indexed by X and the columns indexed by Y the *biadjacency matrix* or simply the *reduced adjacency matrix* of B. The matrix has no 2-by-2 permutation submatrix if and only if the neighborhoods of the vertices in one (either) partite set form a chain under inclusion. A *Ferrers bigraph* is a bipartite graph satisfying this property. The property is equivalent to having independent permutations of the rows and the columns so that the positions occupied by 1s form a Ferrers diagram in the lower left. We therefore call such a matrix a *Ferrers matrix.*

Many of these concepts arose independently for directed graphs. The directed and bipartite models are essentially equivalent because the focus is on the 0,1-matrix that records adjacency. Permuting the rows or the columns in the matrix of an X, Y-bigraph does not change the graph. Allowing non-square matrices does not essentially change the class, since adding isolated vertices does not change whether a graph is a permutation bigraph, but broadening the family in this way can simplify proofs.

The *intersection* of two X, Y-bigraphs G and H with the same vertex set is the X, Y-bigraph whose matrix has 1 precisely where the matrices of G and H both have 1 (under a fixed naming of rows and columns). In Section 2.2 we characterize permutation bigraphs: a bipartite graph is a permutation bigraph if and only if it is the intersection of two Ferrers bigraphs. Furthermore, as we will show in Section 2.2 that these graphs are also the complements of two-clique circular-arc graphs, where a *circular-arc graph* is the intersection graph of a family of arcs on a circle, and a *two-clique* graph is a graph whose vertices can be covered by two complete subgraphs. We will discuss these results again in Chapter 5. Examples in Section 2.3 compare permutation bigraphs to other related classes.

The characterization using Ferrers bigraphs enables us to describe subfamilies of the permutation bigraphs that have been studied in other contexts by their permutation representations. An *interval bigraph* is an X, Y-bigraph representable by giving each vertex an interval in $\mathbb{R}$ so that vertices $x \in X$

and $y \in Y$ are adjacent if and only if their intervals intersect. Thus an interval X, Y-bigraph arises from an interval graph with vertex set $X \cup Y$ by deleting the edges within X and within Y, just as bipermutation bigraphs arise from permutation graphs. An *indifference bigraph* is an interval bigraph having an interval representation in which all intervals have the same length.

Interval bigraphs have many known characterizations (see [75, 47], etc.). The key characterization related to permutation bigraphs is that a bipartite graph is an interval bigraph if and only if it is the intersection of two Ferrers bigraphs whose union is a complete bipartite graph. Thus every interval bigraph is a permutation bigraph. In Section 2.4 we obtain necessary and sufficient conditions on the defining permutations for a permutation bigraph to be an interval bigraph, and we similarly characterize the indifference bigraphs.

Finally, in Section 2.5 we interpret permutation bigraphs in terms of comparability graphs. Our most difficult result in this chapter is a direct proof of a characterizarion of posets whose comparability graphs are permutation graphs using permutation bigraphs. This characterization was previously expressed in [78], but there the difficult direction relied on a result of Bouchet [4] characterizing the dimension of posets. The proof here uses only a simpler and better known theorem of Cogis [13], plus our results from Section 2.2.

2.1.1 Associted permutation bigraph

Given a permutation bigraph $B = B(X, Y, E)$ defined by π describe below three other associated bigraphs such that all the four bigrphs are pairwise edge disjoint and their union is the complete bipartite graph $K_{|X|,|Y|}$.

1. Let B^*(X, Y E*) be the bigraph formed from π by subtracting each element from $n + 1$ and then reversing the order of elements. The resulting permutation is denoted by π^* and the bigraph B^* is called the conjugate bigraph of B. We will use this idea when we characterize an interval bigraph interms of its permutation bigraph.

2. Let B_1 be the bigraph whose permutationn is the reverse of the permutation π.

3. Let B_1^* be the conjugate of B_1.

Example 2.1.4. Let X $=$ $(x_1, x_2, x_3, x_4, x_5) = (3, 9, 8, 6, 4)$ and Y $=$ $(y_1, y_2, y_3, y_4, y_5) = (7, 10, 2, 5, 1)$ and let π $=$ $(\underline{7}, 3, \underline{10}, 9, \underline{2}, 8, \underline{5}, \underline{1}, 6, 4)$. The bigraph B and its three associated bigraphs are given in Figure 2.2.

It can be verified that the three associated bigraphs described above can be obtained from B by imposing different conditions on the reference labeling and the permutation π.

1. The bigraph B^*(X, Y, E*) is obtained with the conditions

   (i) $x < y$ as elements of $[n]$

   (ii) y occurs before x in π

2. B_1(X, Y, E_1) is the graph for which

   (i) $y < x$

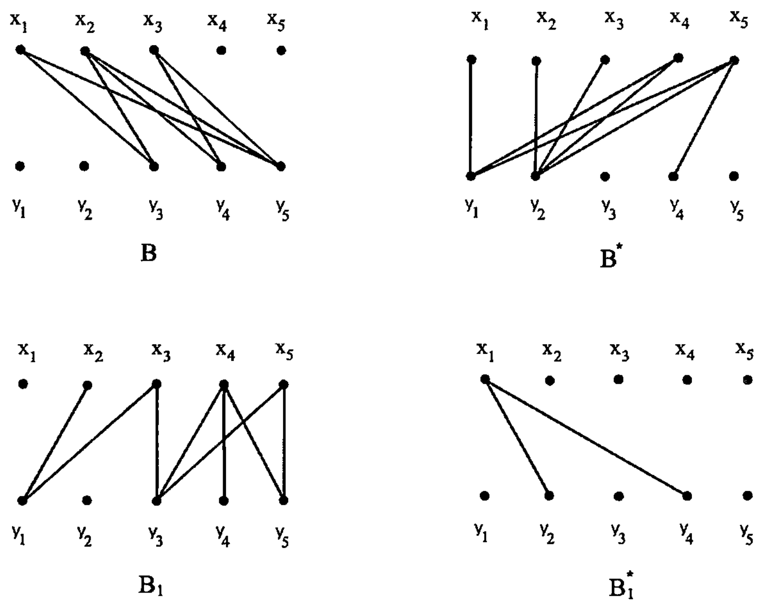

Figure 2.2: A permutation bigraph and its associates

(ii) y occurs before x in π.

3. $B_1{}^*$(X, Y, E$_1{}^*$) is the graph with the condition

(i) $x < y$

(ii) x occurs before y in π.

Stating from B we get the three associated bigraphs B^*, B_1, $B_1{}^*$. If instead of B, we start with any one of B^*, B_1, $B_1{}^*$, then it can be seen that its three associates are the remaining three of the quadruple $\{B, B^*, B_1, B_1{}^*\}$.

2.2 Characterization of Permutation Bigraphs

An *interval containment bigraph* is an X, Y-bigraph representable by assigning each vertex an interval in $\mathbb{R}$ so that vertices $x \in X$ and $y \in Y$ are adjacent if and only if the interval for y contains the interval for x. The alternative model putting x and y adjacent whenever the interval for either contains the interval for the other yields *interval bicontainment bigraphs*; the first "bi" allows both directions of containment. The ordered version corresponds to a digraph model that will be useful in Section 2.5. In [78], it was proved that a bipartite graph B is an interval containment bigraph if and only if it is the intersection of two Ferrers bigraphs; this follows easily from a useful characterization of Ferrers bigraphs.

Ferrers bigraphs (see [78, 97]) are characterized by the existence of an ordering f of the vertex set $X \cup Y$ so that xy is an edge if and only if $f(x) > f(y)$. Given a permutation of the rows and columns of the matrix A of the graph such that the 1-entries appear as a Ferrers diagram in the lower left corner, the separation between the 1-entries and 0-entries forms a "stair" that crosses each row and column once. The entry in row x and column y is below the stair if and only if the row for x is crossed after the column for y is crossed. The separation of the entries by a stair is equivalent to the existence of the function f.

To express a Ferrers X,Y-bigraph B as a permutation bigraph, let the reference ordering L consist of all of Y followed by all of X, giving X the higher numbers. Obtain π from the stair ordering f discussed above by writing the elements in decreasing order of their value under f. Now B is precisely the permutation bigraph represented by L and π.

To illustrate, consider the Ferrers bigraph (X, Y) whose reference labelling L and function f is given in Figure 2.3.

The defining permutation of the bigraph is $\pi = (7, \underline{3}, 6, 5, \underline{2}, 4, \underline{1})$. In con-

	1	2	3
4	1		
5	1	1	
6	1	1	
7	1	1	1

(a) Labelling L

	1	3	6
2	1		
4	1	1	
5	1	1	
7	1	1	1

(b) Function f

Figure 2.3: The two orderings of a Ferrers bigraph

tinuation of the study of bigraphs of Ferrers dimension at most 2(cf. Theo 1.4.1), below we prove that permutation bigraphs are precisely the bigraphs of Ferrers dimension at most 2.

Theorem 2.2.1. *For an X, Y-bigraph B, the following conditions are equivalent:*

(a) B is a permutation bigraph;

(b) B is an interval containment bigraph;

(c) B is the intersection of two Ferrers bigraphs.

Also, B is a bipermutation bigraph if and only if it is an interval bicontainment bigraph.

Proof. Note first that in representations that generate edges by containment, all intervals may be assumed to contain the origin. This holds because expanding two intervals outward from their centers by the same amount (added to the upper endpoints and subtracted from the lower endpoints) does not change whether one contains the other or which contains the other. After sufficient expansion, all intervals contain the origin. Hence we consider interval containment or interval bicontainment representations of this form for bipartite graphs.

Since containment is determined by the order of the endpoints, not the distance between them, we may further assume (for an n-vertex graph) that the right endpoints are $[n]$ and the left endpoints are the negatives of these points.

Such a set of intervals corresponds to two orderings L and π of the vertices. Define the orderings so that the left endpoint for vertex v is $-(n+1)+L(v)$ and the right endpoint is $\pi^{-1}(v)$. That is, L numbers the vertices in left-to-right order of the left endpoints, and π lists them in left-to-right order of the right endpoints. Now the interval for u contains the interval for v if and only if u and v appear in opposite order in L and π, with v appearing first in π. This proves the equivalence of (a) and (b) and proves the final statement.

For the equivalence of (b) and (c) (see [77]), consider intervals containing the origin. Let L be the left-endpoint ordering as before, and let π' be the reverse of the right-endpoint ordering. These are the stair-orderings for two Ferrers bigraphs. The interval for y contains the interval for x if and only if x follows y in each of those orderings, which is true if and only if xy is an edge in each of the two Ferrers bigraphs. □

As a supplement, we provide a direct proof of $(c) \Longrightarrow (a)$ below.

Proof. Let $B = F_1 \cap F_2$ where $F_1(X, Y, E_1)$ and $F_2(X, Y, E_2)$ are two Ferrers bigraphs.

Order the vertices of X and Y of F_1 and of F_2 so that its edges form the lower left corner of its biadjacency matrix of Figure 2.4a and the upper right corner of the biadjacency matrix of Figure 2.4b. The stair partition of the

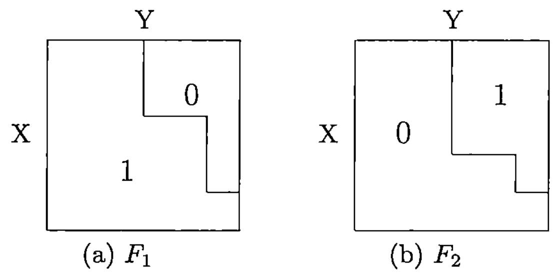

Figure 2.4: B=$F_1 \cap F_2$

matrix of F_1 from the upper left corner to the lower right corner separating 1's from 0's provide us with a linear order of vertices of X ∪ Y and we treat this ordering as the reference ordering L on X ∪ Y. Note that for $x \in$ X and $y \in$ Y, $xy \in E_1$ if and only if $x > y$ in the ordering L as elemens of $[n]$. Next, the stair partition of the matrix of F_2 provide us with a linear order of its vertices and let π denote the permutation of X ∪ Y as elements of $[n]$ obtained from this order. It is clear that for $x \in$ X and $y \in$ Y, $xy \in E_2$ if and only if x occurs before y in the permutation π.

From above it is clear that $xy \in E = E_1 \cap E_2$ if and only if (i) $x > y$ as elements of $[n]$ and (ii) x occurs before y in the permutation π. This means that the permutation π defines the bigraph B. □

The above result helps us to find a permutation of the vertices of a permutation bigraph.

Consider the bigraph B of the following Figure 2.5. This bigraph is the intersection of two Ferrers bigraphs whose biadjacency matrices are given in Figure 2.6. The labeling L of X ∪ Y obtained from Figure 2.6a is

x	v_2	v_4	u_1	u_4	v_3	u_2	u_3	v_1
L	1	2	3	4	5	6	7	8

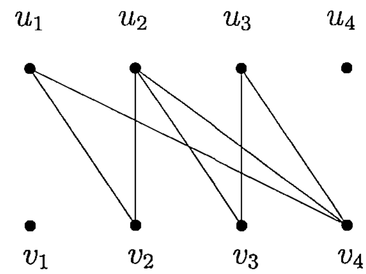

Figure 2.5: A permutation bigraph B

	v_2	v_4	v_3	v_1
u_1	1	1		
u_4	1	1		
u_2	1	1	1	
u_3	1	1	1	

(a) F_1

	v_1	v_2	v_3	v_4
u_1	1	1	1	1
u_2		1	1	1
u_3			1	1
u_4				

(b) F_2

Figure 2.6: Biadjacency matrices of F_1 and F_2

The permutation π obtained from Figure 2.6b is

$$\begin{aligned}\pi &= (u_1, \underline{v_1}, u_2, \underline{v_2}, u_3, \underline{v_3}, \underline{v_4}, u_4) \\ &= (3, \underline{8}, 6, \underline{1}, 7, \underline{5}, \underline{2}, 4)\end{aligned}$$

Cogis [13] obtained a simple and easily tested criterion for a bipartite graph to be the intersection of two Ferrers bigraphs. He used the language of digraphs, but the characterization describes the corresponding 0,1-matrix.

Definition 2.2.2. (Cogis) In a 0,1-matrix, the two zeros in a 2-by-2 permutation submatrix form an *obstruction* or *couple*. For a 0,1-matrix D, the *associated graph* $H(D)$ has vertex set equal to the positions with 0 in D; two vertices are adjacent in $H(D)$ if and only if they form a couple in D.

Cogis proved that a bipartite graph with matrix D is the intersection of two Ferrers bigraphs if and only if $H(D)$ is bipartite. Since deleting one edge from a complete bipartite graph yields a Ferrers bigraph, every bipartite graph G is the intersection of a family of Ferrers bigraphs. The *Ferrers dimension* of a bipartite graph G (or $0,1$-matrix D) is the minimum number of Ferrers bigraphs (or Ferrers matrices) whose intersection is G (or D). Hence Theorem 2.2.1 states that the permutation bigraphs are the bipartite graphs with Ferrers dimension at most 2.

The equivalence to interval containment bigraphs yields another characterization; the proof relies on a result of Spinrad [85] described in [47].

Theorem 2.2.3. *A graph is an interval containment bigraph if and only if its complement is a two-clique circular-arc graph.*

Proof. In an interval containment representation of B where each assigned interval contains the origin, view the intervals as arcs whose union occupies half of a circle. Complement the arc for each vertex of Y to obtain a new family of arcs. The arcs for X contain the point arising from the origin, and the arcs for Y contain the opposite point on the circle, so the intersection graph G of these arcs is a two-clique circular-arc graph.

We claim that G is the complement of B. The arcs within X are pairwise intersecting, and similarly for Y. For $x \in X$ and $y \in Y$, we have $xy \in B$ if and only if the interval I_y for y contains the interval I_x for x, in which case the arc for y is disjoint from the arc for x. Otherwise, on at least one side of the origin the endpoint of I_y is closer to the origin than that of I_x, and on that side the arc for y intersects the arc for x.

Conversely, Spinrad [85] showed that if X and Y are disjoint cliques covering the vertices of a two-clique circular-arc graph, then there is a circular-arc representation and points a and b such that the arcs for X contain a, the arcs for Y contain b, and no arc contains both. Complementing the arcs for Y in this representation reverses the construction above. □

Combining Theorems 1.4.1, 2.2.1, 2.2.3, we get the following result which gives a comprehensive view of the equivalent forms of bigrphs of Ferrers dimension at most 2.

Theorem 2.2.4. *For an X, Y-bigraph B, the following conditions are equivalent:*

1) B is of Ferrers dimension at most 2;

2) The rows and columns of its biadjacency matrix can be permuted independently so that no 0 has a 1 both to its right and below;

3) The associated graph $H(B)$ of couples of B is bipartite;

4) B is a permutation bigraph;

5) B is an interval containment bigraph;

6) The graph complement of B is a 2-clique circular-arc graph.

We will however provide a direct proof of the equivalence of (1) and (6) again in Chapter 5, when we will look at the result as an extension of the characterization of an interval bigraph in terms of 2-clique circular-arc graph by Hell and Huang [47].

2.3 Examples

Example 2.3.1. *Bipartite permutation graphs form a proper subclass of permutation bigraphs.* Steiner [87] proved that the bipartite permutation graphs are precisely the indifference bigraphs, which by [75] are the intersections of two Ferrers bigraphs whose union is complete. Thus Theorem 2.2.1 implies that every bipartite permutation graph is a permutation bigraph.

The converse does not hold. Lin and West[54] gave a forbidden subgraph characterization for the indifference bigraphs (i.e., the bipartite permutation graphs) within the class of bipartite graphs with Ferrers dimension 2, using the three matrices below. From this and Theorem 2.2.1, a permutation bigraph is a bipartite permutation graph if and only if it has no induced subgraph whose matrix is one of these three. □

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Example 2.3.2. *Every bipartite permutation graph is both a permutation bigraph and a bipermutation bigraph.* Ferrers dimension 2 implies that bipartite permutation graphs are permutation bigraphs; in fact, the same permutation works in both models. Let G be a permutation graph specified by π relative to the identity permutation of $[n]$. If G is bipartite, then π has no decreasing 3-term sublist. Let X consist of all entries that begin an inversion in π, and put the remaining entries in Y. Now G is an X,Y-permutation bigraph, with Y containing the isolated vertices. No inversions occur within X or Y, and for each edge the endpoint in X occurs before the endpoint in Y.

Also, π expresses G as a bipermutation bigraph, because π has no inversions consisting of an element of Y before an element of X. $\square$

Example 2.3.3. *Not all bipermutation bigraphs are permutation bigraphs.* To distinguish these families, consider the 6-cycle C_6. (We use C_n, P_n, K_n for the cycle, path, and complete graph with n vertices, respectively.) Applying the result of Cogis [13] with $H(C_6) = K_3$, the 6-cycle C_6 has Ferrers dimension 3. Hence C_6 is not a permutation bigraph and not a bipartite permutation graph, and C_6 is a forbidden induced subgraph for such graphs.

However, C_6 is a bipermutation bigraph. Consider the partition of [6] with $X = \{\overline{1}, \overline{4}, \overline{6}\}$ and $Y = \{\underline{2}, \underline{3}, \underline{5}\}$. Let $\pi = (\overline{4}, \underline{2}, \overline{6}, \underline{5}, \overline{1}, \underline{3}\}$. In the bipermutation model it does not matter which partite set occurs first in an inversion, so the resulting bipermutation bigraph is a 6-cycle with vertices $\underline{2}, \overline{1}, \underline{5}, \overline{6}, \underline{3}, \overline{4}$ in order.

Even without using Ferrers dimension, the definition implies directly that C_6 is not a permutation graph. Let π be a permutation of [6] expressing C_6 as a permutation graph with partite sets X and Y. Since X and Y are independent, each is in increasing order in π. If 1 and 6 are in one part, say X, then 1 precedes 6 in π, with two vertices of Y before 1 and two vertices of Y after 6; this requires $|Y| \geq 4$. Hence we may assume $6 \in X$ and $1 \in Y$.

Since 1 is the first element of Y and 6 is the last element of X in π, these two vertices have degree 3 if 6 precedes 1; hence π has the pattern $(\overline{X}, \overline{X}, \underline{1}, \overline{6}, \underline{Y}, \underline{Y})$. If $2 \in Y$, then 2 has degree 3; if $2 \in X$, then 2 has degree 1. Hence all cases are eliminated, and C_6 is not a permutation graph. $\square$

Example 2.3.4. *Bipermutation bigraphs are edge-disjoint unions of permu-*

tation bigraphs. Let (L, π) be a representation of a bipermutation bigraph G. The same pair generates a permutation bigraph contained in G, keeping only the edges for pairs having opposite order in L and π such that the element of X comes first in the pair in π. To obtain the other edges in G, exchange L and π and generate another permutation bigraph.

It seems hard to convert this to a matrix characterization of bipermutation bigraphs. □

Example 2.3.5. *The comparability bigraph.* The *comparability digraph* $D(P)$ of a poset P is the orientation of its comparability graph obtained by putting $uv \in D(P)$ if $u \geq v$ in P; note that there is a loop at each vertex. With X and Y being two copies of the elements of P, the *comparability bigraph* $B(P)$ of P is the X,Y-bigraph B such that $xy \in E(B)$ for $x \in X$ and $y \in Y$ if and only if $x \geq y$ in P.

Note that $D(P)$ is obtained from $B(P)$ by orienting edges from X to Y and merging the two copies of each element, and $B(P)$ is obtained from $D(P)$ by splitting each vertex into a vertex of X inheriting the outgoing edges and a vertex of Y inheriting the incoming edges.

If the comparability graph $G(P)$ is a permutation graph, then there is a numbering $1, \ldots, n$ of the elements of P and a permutation π of $[n]$ such that elements are comparable in P if and only if they are equal or form an inversion in π, with the first (larger) element of each inversion above the second in P.

Given such a representation, we express $B(P)$ as a permutation bigraph. Let $X = \{\bar{i} \colon i \in [n]\}$ and $Y = \{\underline{i} \colon i \in [n]\}$, numbered as in the numbering for $G(P)$. Let the reference ordering L be $(\underline{1}, \overline{1}, \ldots, \underline{n}, \overline{n})$. Define π' from π by expanding each entry i in π into the consecutive pair $\overline{i}, \underline{i}$ in π'. Now $\overline{i}$ and $\underline{j}$ are adjacent in the resulting permutation bigraph if and only if $i = j$ or i and j are adjacent in $G(P)$ with $i > j$ in the numbering of P. This makes $\overline{i}$ and $\underline{j}$ adjacent if and only if $i \geq j$ in P.

We conclude that if $G(P)$ is a permutation graph, then $B(P)$ is a permutation bigraph. In Section 2.5 we prove the converse, thereby showing that the comparability graph of a poset is a permutation graph if and only if its comparability bigraph is a permutation bigraph. □

2.4 Interval Bigraphs and Indifference Bigraphs

Let $\mathcal{F}$ be the family of bipartite graphs with Ferrers dimension at most 2. Since interval bigraphs (and indifference bigraphs) lie in $\mathcal{F}$, one can describe these subfamilies by appropriately restricting any characterization of $\mathcal{F}$. For example, Theorem 2.2.3 characterizes $\mathcal{F}$ as the complements of two-clique circular-arc graphs, and Hell and Huang [47] proved that within this family the interval bigraphs are the graphs whose complement has a circular-arc representation with no two arcs covering the entire circle.

Since we have shown that $\mathcal{F}$ is the family of permutation bigraphs, our goal now is to characterize the permutation representations that generate interval bigraphs or indifference bigraphs. From [75], a bipartite graph is an interval bigraph if and only if it is the intersection of two Ferrers bigraphs whose union is complete. Also, the proof of Theorem 2.2.1 converts Ferrers bigraphs whose intersection is B into a permutation representation of B.

We use these two ideas to characterize the permutation representations of interval bigraphs.

We recall from Subsection 2.1.1 that for a permutation π of $[n]$, the *conjugate permutation* π^* is formed by subtracting each element from $n+1$ and then reversing the order of the elements. Given an X,Y-bigraph F, we write $\overline{F}$ for the binary complement of F; the matrices of F and $\overline{F}$ sum to the all-1 matrix. When F is a Ferrers matrix, also $\overline{F}$ is a Ferrers matrix, and the correspondence between their stair orderings suggested by the matrices below is the basis of the characterization.

$$
\begin{array}{c|cccc|c}
 & y_1 & y_2 & y_3 & y_4 & \\
x_1 & 1 & 0 & 0 & 0 & 2 \\
x_2 & 1 & 1 & 1 & 0 & 5 \\
x_3 & 1 & 1 & 1 & 1 & 7 \\
 & 1 & 3 & 4 & 6 &
\end{array}
\qquad
\begin{array}{c|cccc|c}
 & y_4 & y_3 & y_2 & y_1 & \\
x_3 & 0 & 0 & 0 & 0 & 1 \\
x_2 & 1 & 0 & 0 & 0 & 3 \\
x_1 & 1 & 1 & 1 & 0 & 6 \\
 & 2 & 4 & 5 & 7 &
\end{array}
$$

$$L = (\underline{1}, 2, \underline{3}, \underline{4}, 5, \underline{6}, 7) \qquad L^* = (1, \underline{2}, 3, \underline{4}, \underline{5}, 6, \underline{7})$$

$$L = (y_1, x_1, y_2, y_3, x_2, y_4, x_3) \qquad L^* = (x_3, y_4, x_2, y_3, y_2, x_1, y_1)$$

Theorem 2.4.1. *An n-vertex permutation bigraph is an interval bigraph if and only if it has a permutation representation (L, π) such that the permutation bigraph represented by the conjugate permutations (L^*, π^*) has no edges.*

Proof. Let B be a permutation bigraph and the intersection of Ferrers bigraphs F_1 and F_2 with stair orderings f_1 and f_2. As in Theorem 2.2.1, form a permutation representation of B by letting the reference ordering L put $X \cup Y$ in the same order as f_1 and π put $X \cup Y$ in the reverse order to f_2. As noted in Theorem 2.2.1, reversing this construction yields two Ferrers bigraphs whose intersection is B from any permutation representation (L, π)

of B.

When F is a Ferrers bigraph with stair-ordering f, the complementary bigraph $\overline{F}$ has x and y adjacent when $f(x) < f(y)$. To permute the matrix so that the 1s for $\overline{F}$ form a Ferrers diagram in the lower left, we reverse the rows and reverse the columns. In order to read the stair in the new matrix from upper left to lower right, we must also reverse the numbering of the elements. This is clearest when we write the stair ordering using vertex names instead of just $[n]$; the stair ordering for $\overline{F}$ is then the reverse of the stair ordering for F. When expressed using a partition of the identity permutation on $[n]$, reversal amounts to subtracting the numbers from $n+1$. Therefore, when (L, π) generates the permutation bigraph $F_1 \cap F_2$ on $[n]$, we conclude that (L^*, π^*) generates the permutation bigraph $\overline{F}_1 \cap \overline{F}_2$.

Now, recall that B is an interval bigraph if and only if F_1 and F_2 can be chosen so that $F_1 \cup F_2$ is complete, which is equivalent to $\overline{F}_1 \cap \overline{F}_2$ having no edges. The equivalent condition for the permutation representation (L, π) of B generated by F_1 and F_2 is that the permutation bigraph with the conjugate representation (L^*, π^*) has no edges. □

The matrices of indifference bigraphs were characterized (in the language of digraphs) in [77]. A *monotone consecutive arrangement* of a $0,1$-matrix consists of independent permutations of the rows and the columns and a labeling of each 0-entry as R or C such that every entry above or rightward of an R is R and every entry below or leftward of a C is C. Thus the 1-entries are consecutive in each row and in each column, and the ends of these intervals of 1-entries behave monotonically across the columns or down the rows.

Theorem 2.4.2. [77] *A bipartite graph is an indifference bigraph if and only if its matrix has a monotone consecutive arrangement.*

A monotone consecutive arrangement expresses the bipartite complement as the union of disjoint Ferrers bigraphs, so such a graph is an interval bigraph. To characterize indifference bigraphs among permutation bigraphs, we translate the conditions for a monotone consecutive arrangement into conditions on the resulty permutation representation.

Theorem 2.4.3. *A permutation bigraph B is an indifference bigraph if and only if it has a permutation representation (L, π) such that*
(1) the permutation bigraph generated by (L^*, π^*) has no edges, and
(2) each partite set appears in increasing order in π.

Proof. Let B be an X, Y-bigraph that is a permutation bigraph. By Theorem 2.2.1, B is the intersection of Ferrers bigraphs F_1 and F_2; let f_1 and f_2 be the corresponding stair orderings. As in Theorem 2.4.1, there is a permutation representation (L, π) of B in which L puts $X \cup Y$ in the same order as f_1 and π puts $X \cup Y$ in the reverse order to f_2, and (L^*, π^*) generates no edges if and only if $F_1 \cup F_2$ is complete.

If the matrix of B has a monotone consecutive arrangement, then the row and column permutations of the matrices of F_1 and F_2 exhibiting Ferrers diagrams in the lower left are reversals of each other. That is, in the stair ordering f_2, both X and Y are ordered in reverse of their ordering in the stair ordering f_1. Since f_2 is reversed to form π, the numbers assigned by L to X appear in increasing order in π, and similarly for Y.

Conversely, if a permutation representation (L, π) satisfies this increasing subsequence condition for X and Y, transforming back to Ferrers bigraphs via the stair orderings yields two Ferrers bigraphs whose row orderings (and column orderings) are reverses. If (L^*, π^*) generates no edges, then the union of these two Ferrers bigraphs is complete, and hence the orderings yield a monotone consecutive arrangement for the matrix of B. □

As mentioned earlier, Steiner [87] proved that an X,Y-bigraph is an indifference bigraph if and only if it is a bipartite permutation graph. Theorem 2.4.3 allows us to express bipartite permutation graphs as a special case of permutation bigraphs.

Corollary 2.4.1. *If B is an indifference bigraph with permutation representation (L, π) from a monotone consecutive arrangement, then (L, π) also represents B as a permutation graph.*

Proof. By Theorem 2.4.3, each partite set (X or Y) occurs in the same order in L and π. Therefore, as a permutation graph with reference order L, no edges are generated within X or within Y by π. Also, since the union of the corresponding Ferrers bigraphs F_1 and F_2 is complete, there is no $x \in X$ and $y \in Y$ with y after x in L and x after y in π (that would put (x, y) above the stair in both F_1 and F_2). Hence the only inversions in π relative to L are those that generate edges in B, and the permutation graph arising from (L, π) is B itself. □

2.5 Comparability graphs and Permutation bigraphs

As mentioned in Example 2.3.5, we prove here that if the comparability bigraph of a poset P is a permutation bigraph, then the comparability graph of P is a permutation graph. With Example 2.3.5, this completes the proof that these conditions are equivalent.

We use digraphs and posets associated with families of intervals. An *interval containment poset* is a poset P representable by giving each element an interval in $\mathbb{R}$ so that $x \leq y$ in P if and only the interval for y contains the interval for x. In Section 2.2 we defined an *interval containment bigraph* to be an X, Y-bigraph representable by assigning each vertex an interval in $\mathbb{R}$ so that vertices $x \in X$ and $y \in Y$ are adjacent if and only if the interval for y contains the interval for x. The natural digraph analogue is an *interval containment digraph*, defined to be a digraph D representable by assigning intervals S_w and T_w to each vertex w so that $uv \in E(D)$ if and only if $S_u \subseteq T_v$.

The relationship between interval containment digraphs and interval containment bigraphs is like that between the comparability digraph and comparability bigraph of a poset: splitting the vertices of an interval containment digraph D into copies in X and Y yields an interval containment bigraph, represented by assigning to the copies of w in X and Y the intervals S_w and T_w in an interval containment representation of D.

Given an interval containment representation of a poset P, with interval I_w assigned to element w, letting $S_w = T_w = I_w$ expresses the comparability digraph as an interval containment digraph; we are given $x \leq y$ in P if and only if $I_x \subseteq I_y$, which is now equivalent to $S_x \subseteq T_y$. This proves necessity in Theorem 2.5.2. Sufficiency is not immediate, because there is no immediate way to reverse the transformation when $S_w \neq T_w$.

In [78], sufficiency was proved indirectly using the result of Bouchet [4] that the order dimension of a poset equals the Ferrers dimension of its comparability digraph. Our argument below uses only the result of Cogis, which we state first. We use the term *bipartition* for the vertex partition given by a proper 2-coloring of a bipartite graph. For a digraph D, recall the notion of the *associated graph* $H(D)$ from Definition 2.2.2. We give a more complete statement of the theorem of Cogis.

Theorem 2.5.1. **(Cogis** [13]) *Let $H(D)$ be the associated graph of a digraph D, and let H' be the subgraph of $H(D)$ consisting of the nontrivial components of $H(D)$. If $H(D)$ is bipartite, with $\mathbf{I}$ denoting its set of isolated vertices, then there is a bipartition $\{\mathbf{R}, \mathbf{C}\}$ of H' such that the positions corresponding to $\mathbf{R} \cup \mathbf{I}$ form a Ferrers digraph F_1, the positions corresponding to $\mathbf{C} \cup \mathbf{I}$ form a Ferrers digraph F_2, and $\overline{D} = F_1 \cup F_2$.*

Theorem 2.5.2. [78] *A poset is an interval containment poset if and only if its comparability digraph is an interval containment digraph.*

Proof. We have noted necessity of the condition. For sufficiency, let P be a poset whose comparability digraph D is an interval containment digraph; we prove that P is an interval containment poset. As noted in Theorem 2.2.1, we may consider an interval containment representation of D in which all intervals contain the origin. The left endpoints and right endpoints both give stair orderings to show that D has Ferrers dimension at most 2, as in the equivalence of (b) and (c) in Theorem 2.2.1.

Since D has Ferrers dimension at most 2, and complements of Ferrers digraphs are Ferrers digraphs, the 0-positions in D can be expressed as the union of two Ferrers digraphs. Since the isolated vertices of $H(D)$ form no couples, they can be included in both Ferrers digraphs. Put the nonisolated vertices into $\mathbf{R}$ or $\mathbf{C}$ when they lie in the first or second Ferrers digraph, respectively. Now $H(D)$ is bipartite, $\{\mathbf{R}, \mathbf{C}\}$ is a bipartition of the subgraph H' of nonisolated vertices, and the positions in $\mathbf{R} \cup \mathbf{I}$ and $\mathbf{C} \cup \mathbf{I}$ form Ferrers digraphs. This is the trivial direction of Cogis' result. We study the resulting coloring in more detail.

(i) If x and y are incomparable in P, then positions (x, y) and (y, x) in the matrix of D contain 0. Since D is reflexive, these positions form a couple. Hence they have distinct colors in any bicoloration of $H(D)$.

(ii) If $x < y$ in P, then $xy \in E(D)$. Hence position (x, y) is 1 and position (y, x) is 0 in the matrix. If position (y, x) forms a couple with (u, v), then positions (u, x) and (y, v) are 1. Now $u \leq x < y \leq v$, so transitivity of P requires $u < v$, and position (u, v) is 1, a contradiction. We conclude that (y, x) is an isolated vertex in $H(D)$.

Let $\mathbf{E}$ be the set of positions containing 1 in the matrix of D. We have shown that $\mathbf{E}$, $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{I}$ partition the positions. Position (x, x) lies in E. For $x \neq y$, we have

$$(x, y) \in \mathbf{E} \Leftrightarrow (y, x) \in \mathbf{I} \text{ and } (x, y) \in \mathbf{R} \Leftrightarrow (y, x) \in \mathbf{C}. \tag{2.1}$$

We next obtain an ordering of P such that when the rows and the columns of the matrix of D are simultaneously given this ordering, $\mathbf{C} \cup \mathbf{I}$ occupies the lower triangle of positions below the diagonal, while $\mathbf{R} \cup \mathbf{E}$ occupies the diagonal and the positions above it.

Since $\mathbf{C} \cup \mathbf{I}$ is a Ferrers digraph, there exist orderings of the rows and columns of D so that $\mathbf{C} \cup \mathbf{I}$ occupies the positions of a Ferrers diagram in the lower left. The complement $\mathbf{R} \cup \mathbf{E}$ then occupies a Ferrers diagram in the upper right.

The positions corresponding to the loops xx now occupy the positions of the 1s in a permutation matrix. Let σ be the corresponding permutation, with the loop in row i occurring in column $\sigma(i)$. Since $\mathbf{R} \cup \mathbf{E}$ is a Ferrers diagram in the upper right, every position (r, s) such that $r \leq i$ and $s \geq \sigma(i)$ lies in $\mathbf{R} \cup \mathbf{E}$. These positions are those *generated by* σ. Since they lie in $\mathbf{R} \cup \mathbf{E}$, there are at most $\binom{n}{2} + n$ of them. We show that only the identity permutation generates this few positions.

If σ is not the identity permutation, then there is some greatest i such that $\sigma(i) < i$. Let $j = \sigma(i)$ and $k = \sigma^{-1}(i)$. Let τ agree with σ except for $\tau(i) = i$ and $\tau(k) = j$; that is, j and i are switched when the word form of σ is modified to obtain the word form of τ. Every position generated by τ is also generated by σ, but position (i, j) is generated by σ and not by τ. Repeating this argument shows that every nonidentity permutation generates more positions than the identity permutation.

Since $\mathbf{R} \cup \mathbf{E}$ contains all positions generated by its loops, and (2.1) implies $|\mathbf{R} \cup \mathbf{E}| = \binom{n}{2} + n$, we conclude that the loops must appear along the diagonal. The resulting common ordering of the rows and columns is a numbering f

of the elements P by 1 through n such that $x < y$ in P implies $f(x) < f(y)$.

From this ordering, we obtain an interval containment representation of P. Let the right endpoint of the interval I_x for x be $f(x)$. For the left endpoints, we define g mapping P into $\{-1, \ldots, -n\}$. At step i, among the current minimal elements, let the one with the rightmost right endpoint be assigned $-i$ as its left endpoint. Delete this element and continue.

If $x < y$ in P, then because f is a linear extension we have $f(x) < f(y)$. Also, in the assignment of left endpoints, y cannot receive a (negative) left endpoint before x; hence $g(y) < g(x)$, and $I(x) \subseteq I(y)$.

If the representation fails, then there exist x and y incomparable in P with $I_x \subseteq I_y$. Let x be the element with smaller interval in such a pair, and let y be a minimal element among those incomparable to x whose intervals contain I_x. Since $f(x) < f(y)$, the construction procedure requires that y is not a minimal remaining element when $g(x)$ is assigned. Hence y is then above some currently minimal element z. Since x is chosen now in preference to z, we have $f(z) < f(x)$. We have $f(x) < f(z) < f(y)$, but x is incomparable to y and z.

With the given common ordering of rows and column, in which $\mathbf{C} \cup \mathbf{I}$ consists of the positions below the diagonal, $\mathbf{R} \cup \mathbf{E}$ consists of those on the diagonal and above, and the diagonal corresponds to the loops, we have obtained the submatrix below, which contradicts that $\mathbf{R} \cup \mathbf{I}$ is a Ferrers matrix. Hence we have successfully constructed a representation, and P is an interval containment poset. $\square$

	z	x	y
z	1	R	1
x		1	R
y			1

Finally, we remark that relative to posets there is a correspondence betwee permutation graphs and permutation bigraphs. That is, the comparability graph of a poset is a permutation graph if and only if its comparability bigraph is a permutation bigraph.

Corollary 2.5.1. *For a poset P, the following conditions are equivalent.*

(a) The comparability graph of P is a permutation graph.

(b) The comparability graph of P is an interval containment graph.

(c) P is an interval containment poset.

(d) The comparability digraph of P is an interval containment digraph.

(e) The comparability bigraph of P is an interval containment bigraph.

(f) The comparability bigraph of P is a permutation bigraph.

Proof. Equivalence of (a),(b),(c) follows from Theorem 2.1.1, as elaborated in Example 2.1.3. Equivalence of (c) and (d) is Theorem 2.5.2. Equivalence of (d) and (e) is Example 2.3.5. Equivalence of (e) and (f) is Theorem 2.2.1.□