CHAPTER 1

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In the thesis we concentrate our attention on the growth properties of entire and meromorphic functions and on some deficiencies of meromorphic functions. In our journey, *Nevanlinna's value distribution theory* carves our way to reach the goal. Therefore, let us first give some definitions and state some significant results of the value distribution theory.

We call a function to be meromorphic if its singularities are merely poles. A meromorphic function having no pole is an entire function. Let f be a meromorphic function in the finite complex plane $\mathbb{C}$. Also let $n(r,a;f) \equiv n(r,a)$, which is a non negative integer for each r, denote the number of $a-$*points* of f in $|z| \leq r$, counted with proper multiplicities, for a complex number a, finite or infinite. Obviously $n(r,\infty) \equiv n(r,f)$ represents the number of poles of f in $|z| \leq r$, counted with proper multiplicities. The function $N(r,a;f)$ is defined as follows:

$$N(r,a;f) = N(r,a) = \int_0^r \frac{n(t,a) - n(0,a)}{t} dt + n(0,a) \log r$$

and $N(r,\infty;f) = N(r,f)$.

The term $N(r,f)$ is called the enumerative function of f .

Next we define $\log^+ x$ as follows:

$$log^+ x = \log x \; if \;\; x \geq 1$$
$$= 0 \; if \; 0 \leq x < 1 \; .$$

The following properties are then obvious

(i) $\log^+ x \geq 0$ if $x \geq 0$,

(ii) $\log^+ x \geq \log x$ if $x > 0$,

(iii) $\log^+ x \geq \log^+ y$ if $x > y$,

(iv) $\log x = \log^+ x - \log^+ \frac{1}{x}$ if $x > 0$.

The quantity $m(r, f)$ is defined as follows:

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| d\theta \; .$$

The term $m(r, f)$ is called the proximity function of f and is a sort of average magnitude of $\log |f(z)|$ on the arcs of $|z| = r$, where $|f(z)|$ is large.

Now let us write,

$$T(r, f) = m(r, f) + N(r, f) \; .$$

The function $T(r, f)$ is called the *Nevanlinna's Characteristic* function of f. It plays an important role in the theory of meromorphic functions.

Since for any positive integer p and complex numbers $a_1, a_2, \ldots\ldots a_p$ we know that

$$\log^+ \left| \prod_{\nu=1}^{p} a_\nu \right| \leq \sum_{\nu=1}^{p} \log^+ |a_\nu|$$

$$\text{and} \quad \log^+ \left| \sum_{\nu=1}^{p} a_\nu \right| \leq \log^+ \left(p . \max_{\nu=1,2,\cdots p} |a_\nu| \right) \leq \sum_{\nu=1}^{p} \log^+ |a_\nu| + \log p \; ,$$

it is easy to show that for p meromorphic functions $f_1, f_2, \cdots f_p$,

$$m\left(r, \prod_{\nu=1}^{p} f_\nu\right) \leq \sum_{\nu=1}^{p} m(r, f_\nu)$$

$$\text{and} \quad m\left(r, \sum_{\nu=1}^{p} f_\nu\right) \leq \sum_{\nu=1}^{p} m(r, f_\nu) + \log p \; .$$

Also one can easily verify that

$$N\left(r,\prod_{\nu=1}^{p} f_\nu\right) \leq \sum_{\nu=1}^{p} N\left(r,f_\nu\right)$$
$$\text{and } N\left(r,\sum_{\nu=1}^{p} f_\nu\right) \leq \sum_{\nu=1}^{p} N\left(r,f_\nu\right) .$$

So

$$T\left(r,\sum_{\nu=1}^{p} f_\nu\right) \leq \sum_{\nu=1}^{p} T\left(r,f_\nu\right) + \log p$$
$$\text{and } T\left(r,\prod_{\nu=1}^{p} f_\nu\right) \leq \sum_{\nu=1}^{p} T\left(r,f_\nu\right) .$$

Now we state the *Poisson-Jensen formula* {p.1, [16]} in the form of the following theorem:

Theorem 1.0.1 *Suppose that f is meromorphic in $|z| \leq R$ $(0 < R < \infty)$. Also let a_μ $(\mu = 1, 2, \cdots M)$ and b_ν $(\nu = 1, 2, \cdots N)$ denote the zeros and poles of f respectively in $|z| < R$. If $z = re^{i\theta}$ $(0 < r < R)$ and if $f\left(re^{i\theta}\right) \neq 0, \infty$ then we have*

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\phi}\right)\right| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi$$
$$+ \sum_{\mu=1}^{M} \log\left|\frac{R(z - a_\mu)}{R^2 - \bar{a}_\mu z}\right| - \sum_{\nu=1}^{N} \log\left|\frac{R(z - b_\nu)}{R^2 - \bar{b}_\nu z}\right| .$$

The theorem holds good also when f has zeros and poles on $|z| = R$.

For $z = 0$, we obtain Jensen's formula

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\phi}\right)\right| d\phi + \sum_{\mu=1}^{M} \log\frac{|a_\mu|}{R} - \sum_{\nu=1}^{N} \log\frac{|b_\nu|}{R} ,$$

provided that $f(0) \neq 0, \infty$.

If f has a zero of order λ or a pole of order $-\lambda$ at $z = 0$ such that $f = c_\lambda z^\lambda + \cdots$ then Jensen's formula takes the form

$$\log|c_\lambda| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\phi}\right)\right| d\phi + \sum_{\mu=1}^{M} \log\frac{|a_\mu|}{R} - \sum_{\nu=1}^{N} \log\frac{|b_\nu|}{R} - \lambda \log R \; .$$

This tiresome modification is one of the minor irritations of the theory. Generally we shall assume that our functions behave in such a way that the terms in the *Jensen's* formula do not become infinite in our use of that formula knowing that exceptional cases can also be treated.

When f has no $a-$*points* (*i.e.*, the roots of the equation $f = a$) at $z = 0$, then it follows from *Riemann-Stieltjes* integral that

$$\sum_{0<a_\nu\le r} \log\frac{r}{|a_\nu|} = \int_0^r \frac{n(t,a)}{t} dt \; ,$$

where a_ν's are the $a-$*points* of f in $|z| \le r$.

Again since $N(r,0;f) = N\left(r,\frac{1}{f}\right)$, from Jensen's formula we get,

$$\log|f(0)| = m(R,f) - m\left(R,\frac{1}{f}\right) + N(R,f) - N\left(R,\frac{1}{f}\right)$$

$$\text{i.e., } T(R,f) = T\left(R,\frac{1}{f}\right) + \log|f(0)| \; .$$

For any finite complex number 'a' let us denote by $m(r,a)$ the function $m\left(r,\frac{1}{f-a}\right)$ and $m(r,\infty) = m(r,f)$. Now we express Nevanlinna's First Fundamental theorem in the following form:

Theorem 1.0.2 ({p.6, [16]}) *Let f be a meromorphic function in $\mathbb{C}$ and a be any complex number, finite or infinite, then*

$$m(r,a) + N(r,a) = T(r,f) + O(1) \; .$$

This result shows the remarkable symmetry exhibited by a meromorphic function in its behaviour relative to different complex number a, finite or infinite. The sum $m(r,a) + N(r,a)$ for different values of a maintains a total, given by the quantity $T(r,f)$ which is invariant up to a bounded additive term.

One part of this invariant sum, the quantity $N(r,a;f)$ hints how densely the roots of the equation $f = a$ are distributed in the average in the disc $|z| < r$. The larger the number of $a-points$ the faster this counting function for $a-points$ grows with r.

The first term $m(r,a)$ which is defined to be the mean value of $\log^{+}\left|\frac{1}{f-a}\right|$ $\left(or\ \log^{+}|f|\ \ if\ a=\infty\right)$ on the circle $|z| = r$, receives a remarkable contribution only from those arcs or the circle where the functional values differ very little from the given value 'a'. The magnitude of the proximity function can thus be considered as a measure for the mean deviation on the circle $|z| = r$ of the functional value f from the value a.

If the $a-points$ of a meromorphic function are relatively scarce for a certain complex number a, this fact finds expression analytically in the relatively slow growth of the function $N(r,a)$ as $r \longrightarrow \infty$; in the extreme case where a is a *Picard's exceptional value* of the function (so that $f \neq a$ in $\mathbb{C}$), $N(r,a)$ is identically zero. But this fact on $a-points$ finds a compensation.

The function deviates in the mean slightly from the value a in question, the corresponding proximity function $m(r,a)$ will be relatively large, so that the sum $m(r,a) + N(r,a)$ reaches the magnitude $T(r,f)$ the characteristic function of f.

For an entire function f, $N(r,f) = 0$ and so $T(r,f) = m(r,f)$, *i.e.,* in the case of an entire function, the *Nevanlinna's characteristic* function and the proximity function are same.

Let us consider that f be an entire function, *i.e.,* a function of a complex variable regular in the whole finite complex plane $\mathbb{C}$. By *Taylor's* theorem such a function has an everywhere convergent power series expansion as

$$f = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots \tag{1.0.1}$$

which forms a natural generalization of the polynomials.

The degree of a polynomial which is equal to its number of zeros, estimate the rate of growth of the polynomial as the independent variable move without bound. So the more zeros, the greater is the growth.

An analogous property that relate the set of zeros and the growth of a function can be developed for arbitrary entire functions.

Establishing relations between the distribution of the zeros of an entire function and its asymptotic behaviour as $z \longrightarrow \infty$ enriched most of the classical results of the theory of entire functions. The classical investigations of *Borel, Hadamard* and *Lindelöf* are of this kind.

To characterise the growth of an entire function and the distribution of its zeros a special growth scale called maximum modulus function of f on $|z| = r$ has been introduced as $M(r) \equiv M(r, f) = \max_{|z|=r} |f(z)|$.

It plays an important role in the theory of entire functions. Since by *Liouville's* theorem a bounded entire function is constant, it follows that for non-constant f the maximum modulus function $M(r)$ is unbounded.

The following theorem is worth mentioning.

Theorem 1.0.3 ({Th 1, p.5, [35]}) *The maximum of the modulus of a function f, which is regular in a closed connected region D, bounded by one or more curves C, is attained on the boundary.*

This theorem implies that when f is an entire function, $M(r)$ is a non-decreasing function of r for all values of r. Using the uniform continuity of f in any closed region and the above theorem, *i.e.*, the value $M(r)$ is attained by f on $|z| = r$, it follows that $M(r)$ is a continuous function of r. Also $M(r)$ is differentiable in adjacent intervals {Theorem 10, p.27, [35]}. In view of *Hadamard's* theorem {Theorem 9, p.20, [35]} we know that $\log M(r)$ is a continuous, convex and ultimately increasing function of $\log r$.

For an entire function f the study of the comparative growth properties of $T(r, f)$ and $\log M(r, f)$ is a popular problem among the researchers.

Now we mention a fundamental inequality relating $T(r, f)$ and $\log M(r, f)$.

Theorem 1.0.4 ({p. 18, [16]}) *If f is regular for $|z| \leq R$ then*

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R+r}{R-r} T(R, f), \; 0 \leq r < R .$$

In case of a transcendental entire function f, $M(r)$ grows faster than any positive power of r. Thus in order to estimate the growth of transcendental entire functions we choose a comparison function e^{rk}, $k > 0$ that grows more rapidly than any positive power of r.

More precisely f is said to be a function of finite order if there exists a positive constant k such that $\log M(r) < r^k$ for all sufficiently large values of r $(r > r_0(k), \; say)$. The infimum of such k's is called the order of f. If no such k exists, f is said to be of infinite order.

For example, the order of the function $\exp z$ is 1 *i.e.*, finite but that of $\exp^{[2]} z$ is infinite where $\exp^{[2]} z = \exp(\exp z)$.

Let ρ be the order of f. It can be easily shown that the order ρ of f has the following alternative definition

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} .$$

The lower order λ of f is defined as follows

$$\lambda = \liminf_{r\to\infty} \frac{\log\log M(r)}{\log r} .$$

Clearly $\lambda \leq \rho$.

If in particular, for a function f, $\lambda = \rho$, then f is said to be of regular growth.

For example, a polynomial or the functions $\exp z, \cos z$ *etc.* are of regular growth.

With known order $\rho\,(0 < \rho < \infty)$ the growth of an entire function can be characterised more precisely by the type of the function. The number τ given by

$$\tau = \limsup_{r\to\infty} \frac{\log M(r)}{r^{\rho}}, \;\; 0 < \rho < \infty$$

is called the type of f.

Between two functions of same order one can be characterised to be of greater growth if its type is greater. The quantities ρ, λ and τ are extensively used in the study of growth properties of f. At this stage we note the following definition.

Definition 1.0.1 ({p. 16, [16]}) *Let $S(r)$ be a real and non-negative function increasing for $0 \leq r_0 \leq r < \infty$. The order k and lower order λ of the function $S(r)$ are defined as*

$$k = \limsup_{r\to\infty} \frac{\log S(r)}{\log r}$$

and

$$\lambda = \liminf_{r\to\infty} \frac{\log S(r)}{\log r} .$$

Moreover if $0 < k < \infty$, we set

$$c = \limsup_{r \to \infty} \frac{S(r)}{r^k}$$

and distinguish the following possibilities:

(a) $S(r)$ has maximal type if $c = +\infty$;

(b) $S(r)$ has mean type if $0 < c < +\infty$;

(c) $S(r)$ has minimal type if $c = 0$;

(d) $S(r)$ has convergence class if

$$\int_{r_0}^{\infty} \frac{S(t)}{t^{k+1}} dt \text{ converges} .$$

Now we state the following theorem.

Theorem 1.0.5 ({p. 18, [16]}) *If f is an entire function then the order k of the function $S_1(r) = \log^+ M(r, f)$ and $S_2(r) = T(r, f)$ is the same. Further if $0 < k < \infty$, $S_1(r)$ and $S_2(r)$ belong to the same classes $(a), (b), (c)$ and (d).*

Also we note that $S_1(r)$ and $S_2(r)$ have the same lower order.

A function f meromorphic in the plane is said to have order ρ, lower order λ and maximal, minimal, mean type or convergence class if the function $T(r, f)$ has this property. For entire functions these coincide by the above theorem with the corresponding definition in terms of $M(r, f)$ which is classical. The type of a meromorphic function f is defined by

$$\tau = \limsup_{r \to \infty} \frac{T(r, f)}{r^\rho}, \; 0 < \rho < \infty .$$

We know that the order and the lower order of an entire function f and its derivative are equal. The same result holds for a meromorphic function also.

After revealing the important symmetry property of a meromorphic function f, which is expressed in the first fundamental theorem through the invariance of the sum $m(r, a) + N(r, a)$, it is natural to attempt for a more careful investigation of the relative strength of two terms in the sum, of the proximity component $m(r, a)$ and of the counting component $N(r, a)$. Individual results have been obtained in this direction {p.234,[35]}:

1. Picard's theorem shows that the counting function for a non constant meromorphic function in the finite complex plane can vanish for atmost two values of a.

2. For a meromorphic function of finite non-integral order there is atmost one Picard's exceptional value.

3. That the counting function $N(r,a)$ is in general *i.e.*, for the great majority of the values of a, large in comparison to the proximity function.

We now state Nevanlinna's Second Fundamental theorem.

Theorem 1.0.6 ({p. 31, [16]}) *Suppose that f is a non-constant meromorphic function in $|z| \leq r$. Let $a_1, a_2, \cdots a_q$ $(q \geq 2)$ be distinct finite complex numbers, $\delta > 0$ and suppose that $|a_\mu - a_\nu| \geq \delta$ for $1 \leq \mu < \nu \leq q$. Then*

$$m(r,\infty;f) + \sum_{\nu=1}^{q} m(r,a_\nu,f) \leq 2T(r,f) - N_1(r) + S(r,f)$$

where $N_1(r)$ is positive and is given by

$$N_1(r) = N\left(r,\frac{1}{f'}\right) + 2N(r,f) - N(r,f') \text{ and}$$

$$S(r,f) = m\left(r,\frac{f'}{f}\right) + m\left\{r,\sum_{\nu=1}^{q}\frac{f'}{(f-a_\nu)}\right\} + q\log^{+}\frac{3q}{\delta} + \log 2 + \log\frac{1}{|f'(0)|},$$

with modifications if $f(0) = 0$ or ∞ and $f'(0) = 0$.

The quantity $S(r,f)$ will in general play the role of an unimportant error term. The combination of this fact with the above theorem yields the second fundamental theorem.

The following theorem gives an estimation of $S(r)$.

Theorem 1.0.7 ({p. 34, [16]}) *Let f be a meromorphic function and not constant in $|z| < R_0 \leq \infty$ and that $S(r,f)$ is defined as in the above theorem. Then we have*

(*i*) *If $R_0 = +\infty$, $S(r,f) = O\{\log T(r,f)\} + O(\log r)$ as $r \to \infty$ through all values if f has finite order and as $r \to \infty$ outside a set E of finite linear measure otherwise.*

(ii) If $0 < R_0 < \infty$,

$$S(r,f) = O\left\{\log^+ T(r,f) + \log \frac{1}{R_0 - r}\right\} \text{ as } r \to R_0$$

outside a set E *such that* $\int_E \frac{dr}{R_0 - r} < +\infty$.

Further there is a point r *outside* E *for which* $\rho < r < \rho'$ *provided that* $0 < R - \rho' < e^{-2}(R - \rho)$.

Consequently we get the following result.

Theorem 1.0.8 ({p. 41, [16]}) *Let* f *be meromorphic and non-constant in* $|z| \leq R_0$. *Then*

$$\frac{S(r,f)}{T(r,f)} \to 0 \qquad (*)$$

as $r \to R_0$ *with the following provisions :*

(a) $(*)$ *holds without restrictions if* $R_0 = +\infty$ *and* f *is of finite order in the plane.*

(b) If f *has infinite order in the plane,* $(*)$ *still holds as* $r \to \infty$ *outside a certain exceptional set* E *of finite length. Here* E *depends only on* f.

(c) If $R_0 < +\infty$ *and*

$$\limsup_{r \to \infty} \frac{T(r,f)}{\log\{\frac{1}{R_0 - r}\}} = +\infty ,$$

then $(*)$ *holds as* $r \to R_0$ *through a suitable sequence* r_n, *which depends on* f *only.*

This theorem points out why $S(r,f)$ plays the role of an unimportant error term.

Let f be meromorphic and non constant in the plane. We shall denote by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. Also we shall denote by a, a_0, a_1 *etc.* functions meromorphic in the plane and satisfying $T(r,a) = S(r,f)$ as $r \to \infty$. Now we introduce *Milloux's* theorem which is important in studying the properties of the derivatives of meromorphic functions.

Theorem 1.0.9 ({p. 55, [16]}) *Let l be a positive integer and $\psi = \sum_{\nu=0}^{l} a_\nu f^{(\nu)}$. Then $m\left(r, \frac{\psi}{f}\right) = S(r,f)$ and $T(r,\psi) \leq (l+1)T(r,f) + S(r,f)$.*

Milloux showed that in the second fundamental theorem we can replace the counting functions for certain roots of $f(z) = a$ by roots of the equation $\psi(z) = b$, where ψ is given as in the above theorem. In this connection we state the following theorem.

Theorem 1.0.10 ({p.57, [16]}) *Let f be meromorphic and non constant in the plane and $\psi = \sum_{\nu=0}^{l} a_\nu f^{(\nu)}$ where l is a positive integer. If ψ is non constant then*

$$T(r,f) \leq \overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) - N_0\left(r, \frac{1}{\psi'}\right) + S(r,f)$$

where in $N_0\left(r, \frac{1}{\psi'}\right)$ only zeros of ψ' not corresponding to the repeated roots of $\psi = 1$ are to be considered.

Now we set

$$\delta(a) = \delta(a;f) = \liminf_{r\to\infty} \frac{m(r,a)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r,f)}.$$

$$\Theta(a) = \Theta(a;f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a)}{T(r,f)}$$

where $\overline{N}(r,a) \equiv \overline{N}(r,a;f)$ is the counting function for distinct $a - points$,

$$\theta(a) = \theta(a;f) = \liminf_{r\to\infty} \frac{N(r,a) - \overline{N}(r,a)}{T(r,f)}.$$

Evidently given $\varepsilon\,(>0)$ we have for sufficiently large values of r,

$$N(r,a) - \overline{N}(r,a) > \{\theta(a) - \varepsilon\}\, T(r,f)$$
$$N(r,a) < \{1 - \delta(a) + \varepsilon\}\, T(r,f)$$

and hence

$$\overline{N}(r,a) < \{1 - \delta(a) - \theta(a) + 2\varepsilon\}\, T(r,f)$$

so that

$$\Theta(a) \geq \delta(a) + \theta(a) \ .$$

The quantity $\delta(a)$ is known as the deficiency of the value a and $\theta(a)$ is called the index of multiplicity. Evidently $\delta(a)$ is positive only if there are relatively few roots of the equation $f(z) = a$, while $\theta(a)$ is positive if there are relatively many multiple roots.

Let us now state a fundamental theorem called *Nevanlinna's theorem on deficient values.*

Theorem 1.0.11 ({p.43, [16]}) *Let f be a non constant meromorphic function defined on the plane. Then the set of values of a for which $\Theta(a) > 0$ is countable and we have on summing over all values a*

$$\sum_a \{\delta(a) + \theta(a)\} \leq \sum_a \Theta(a) \leq 2 \ .$$

The magnitude of the deficiency $\delta(a)$ lies in the closed unit interval $[0,1]$ and it gives us a very accurate measure for the relative density of the points where the function f assumes the value a in question. The larger the deficiency is, the more rare are the latter points. The deficiency reaches its maximum value 1 when the latter have been very sparsely distributed, as for example, in the extreme case where the value a is a *Picard exceptional value i.e.,* a complex number which is not assumed by the function f. We shall call every value of vanishing deficiency $\delta(a)$, a normal value in contrast to the deficient values for which $\delta(a)$ is positive.

We know from *Picard's theorem* that a meromorphic function can have at most two *Picard exceptional values.* This theorem follows easily from *Nevanlinna's theorem on deficient values* because as we have stated before that for a *Picard exceptional value* a, $\delta(a) = 1$.

The quantity

$$\Delta(a; f) = 1 - \liminf_{r \to \infty} \frac{N(r,a)}{T(r,f)} = \limsup_{r \to \infty} \frac{m(r,a)}{T(r,f)}$$

gives another measure of deficiency and is called the *Valiron deficiency.*

Clearly $0 \leq \delta(a; f) \leq \Delta(a; f) \leq 1$.

Apart from **Chapter 1** the thesis consists of seven chapters.

- **Chapter 2** is concerned with the study of comparative growth properties of composite entire or meromorphic functions and differential monomials generated by one of the factors improving some earlier results. The results of this chapter have been published in **International Mathematical Forum, see [9].**

- In **Chapter 3** we investigate the comparative growth of composite entire or meromorphic functions and wronskians generated by one of the factors which improves some earlier results. The results of this chapter have been published in **International Journal of Mathematical Analysis, see [10].**

- In **Chapter 4** we discuss about the comparative growth properties of composite entire or meromorphic functions and differential polynomials generated by one of the factors. Also we study the relationship between the L-order (L-type) of a meromorphic function and that of a differential polynomial generated by it. The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [7] and in **International Journal of Pure and Applied Mathematics,** see **[13].**

- In **Chapter 5** we compare the maximum term of composition of two entire functions with their corresponding left and right factors on the basis of (p, q) th order, $L-(p, q)$ th order and $L^*-(p, q)$ th order where p, q are positive integers with $p > q$ and $L = L(r)$ is a slowly changing function. The results of this chapter have been published in **International Journal of Pure and Applied Mathematics, see [11].**

- In **Chapter 6** we compare the relative (k, n) Valiron defect with the relative (k, n) Nevanlinna defect of a meromorphic function where k and n are both non negative integers. The results of this chapter have been published in **International Mathematical Forum, see [6].**

- In **Chapter 7** we compare the relative Valiron defect with the relative Nevanlinna defect of wronskians generated by a transcendental meromorphic function. The results of this chapter have been published in **International Journal of Contemporary Mathematical Sciences, see [12].**

- In **Chapter 8** we consider two meromorphic functions having common roots and find some relations involving their relative deficiencies. The results of this chapter have been published in **International Journal of Pure and Applied Mathematics, see [8].**

When we write Theorem $a.b.c$ (or Corollary $a.b.c$ or Equation $a.b.c$ *etc.*) where a, b and c are positive integers, we mean the c *th* theorem (or c *th* corollary or c *th* equation *etc.*) in the b *th* section of the a *th* chapter. Individual chapters have been presented in such a way that they are more or less independent of the other chapters. The references to books and journals have been classified as bibliography and are given at the end of the thesis.

The author of the thesis is thankful to the authors of various papers and books which have been consulted during the preparation of the entire thesis.

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