

CHAPTER - I

SOME CONTACT PROBLEMS IN ELASTODYNAMICS

- Paper - 1 : Forced vertical vibration of two rigid strips on a semi-infinite elastic solid.
- Paper - 2 : Diffraction of antiplane shear wave by a pair of parallel rigid strips at the interface of two bonded dissimilar elastic media.
- Paper - 3 : On steady motion of four rigid strips on the surface of a semi-infinite elastic medium.
- Paper - 4 : Diffraction of torsional elastic waves by a rigid annular disc at the bimaterial interface.
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FORCED VERTICAL VIBRATION OF TWO RIGID STRIPS ON A SEMI-INFINITE ELASTIC SOLID

1. INTRODUCTION

The study of the effect of vibrating source of pressure in different forms on the surface of an elastic medium is almost classical. Reissner (1937) and later Millar and Pursey (1954) treated the case of a uniform vibrating pressure distribution applied to a circular region on the surface of an elastic half-space. The problem of most physical interest occurs when displacement corresponding to indentation by a rigid body are prescribed over a given region and the surface tractions outside the region are zero. Analytical treatment on the dynamical response of footing and soil structure interaction are usually available in the literature only for circular and elliptical footings and infinite strip loadings. Such results are important in view of their application in the design of foundations for machinery and buildings, also in the study of the vibration of dams and large structures subjected to earthquakes. Awojobi and Grootenhuis (1965), Robertson (1966), Gladwell (1968) and others have considered the problem of circular footing in detail. Roy (1986) considered the dynamic response of an elliptical footing in frictionless contact with a homogeneous elastic half-space. For low frequency, both vertical and horizontal vibration were

treated. Low frequency solution for the vertical, horizontal and rocking vibration of an infinite strip on a semi-infinite elastic medium has been obtained by Karasudhi, Keer and Lee (1968) by reducing the governing dual integral equations into a single inhomogeneous Fredholm equation of the second kind. Wickham (1977) however removed the flaws occurring in the above paper and worked out in detail the problem of forced two dimensional oscillation of a rigid strip in smooth contact with a semi-infinite elastic medium using a different technique. To improve the dynamic models of buildings and other structures, it will be fruitful to have analytical results for foundations of more complicated nature. In this paper we have considered the problem of vertical vibration of two rigid strips in smooth contact with a semi-infinite elastic medium. The problem is also important in view of its application in the study of the vibration of an elastic medium caused by the running wheels on a railway track. The resulting mixed boundary value problem is reduced to the solution of a triple integral equation which has further been reduced to the solution of an integro differential equation. Finally iterative solution valid for low frequency has been obtained. The integral equation was solved in a manner similar to that employed by Lowengrub and Srivastava (1968) in solving static problems for two coplanar cracks in an infinite elastic medium. Jain and Kanwal (1972,1972) also used the same technique to solve the problem of diffraction of elastic waves by two coplanar Griffith cracks and also by two coplanar rigid strips in an infinite elastic medium. It may be

mentioned in this connection that recently Itu (1980) has also solved the problem of diffraction of SH-waves by two coplanar Griffith cracks in an infinite elastic medium using a different technique.

From the solution of the integral equation, we have found out stresses just below the strips and also the vertical displacement at point outside the strip on the free surface. Finally, making the distance between the strips tend to zero, we have found our results becoming identical with the results given by Wickham (1977) for the vertical vibration of a single strip on a semi-infinite elastic medium. Low frequency solution due to antiplane motion of two strips on a semi-infinite elastic medium has also been derived.

2. FORMULATION OF THE INPLANE PROBLEM

Let us consider the normal vibration of frequency ω of two rigid strips having smooth contact with a semi-infinite homogeneous isotropic elastic solid occupying the half-space $-\infty < X < \infty$, $Y \geq 0$, $-\infty < Z < \infty$. It is assumed that motion is forced by prescribed displacement distribution $v_0 e^{-i\omega t}$ normal to the two infinite strips located in the region $-a \leq X \leq -b$, $b \leq X \leq a$, $Y=0$, $|Z| < \infty$ (Fig.1), where v_0 is constant. Normalizing all lengths with respect to a and putting $b/a=c$, we find that the rigid strips are defined by $c \leq |x| \leq 1$, $y=0$, $|z| < \infty$.

Suppressing the time factor $e^{-i\omega t}$ throughout the analysis, the displacement components can be written as

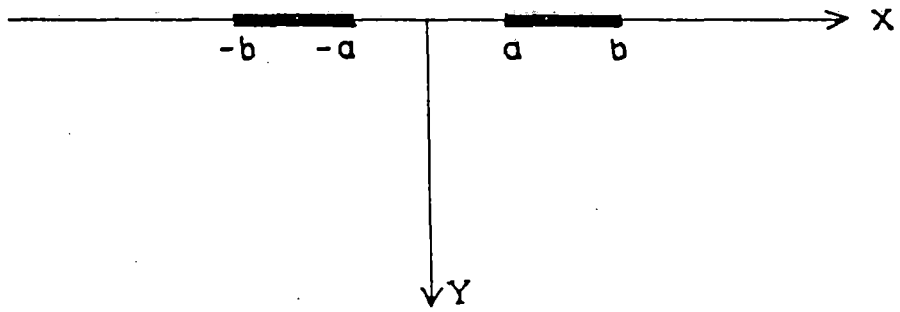


Fig 1. Geometry of the strips .

$$\begin{aligned}
 u(x,y) &= \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y} \\
 v(x,y) &= \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x} \\
 w(x,y) &= 0
 \end{aligned} \tag{1}$$

where the displacement potentials $\phi(x,y)$ and $\psi(x,y)$ satisfy the Helmholtz equations

$$\begin{aligned}
 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + k_1^2 \phi &= 0 \\
 \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + k_2^2 \psi &= 0
 \end{aligned} \tag{2}$$

where $k_1^2 = \omega^2 a^2 / c_1^2$ and $k_2^2 = \omega^2 a^2 / c_2^2$.

Consequently, the values of stress components τ_{xy} , τ_{yy} and τ_{yz} are

$$\begin{aligned}
 \tau_{xy} &= \mu \left[2 \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \right] \\
 \tau_{yy} &= -\mu \left[\left(k_2^2 + 2 \frac{\partial^2}{\partial x^2} \right) \phi - 2 \frac{\partial^2 \psi}{\partial x \partial y} \right] \\
 \tau_{zy} &= 0
 \end{aligned} \tag{3}$$

The boundary conditions are

$$\begin{aligned}
 v(x,0) &= v_0, & c \leq |x| \leq 1 \\
 \tau_{yy}(x,0) &= 0, & |x| < c, |x| > 1 \\
 \tau_{xy}(x,0) &= 0, & -\infty < x < \infty
 \end{aligned} \tag{4}$$

Solution of the Helmholtz equation given by (2) can be written as

$$\phi = \int_{-\infty}^{\infty} A(\xi) \exp(i\xi x - \gamma_1 y) d\xi \quad (5)$$

$$\psi = \int_{-\infty}^{\infty} B(\xi) \exp(i\xi x - \gamma_2 y) d\xi$$

where $\gamma_j = (\xi^2 - k_j^2)^{1/2}$, $|\xi| \geq k_j$, $j=1,2$ (6)

$$= -i(k_j^2 - \xi^2)^{1/2}, \quad |\xi| \leq k_j,$$

and $A(\xi)$, $B(\xi)$ are unknown functions to be determined from the boundary conditions.

Using the last boundary condition of (4) it can be shown that

$$B(\xi) = \frac{-2i\xi\gamma_1}{\xi^2 + \gamma_2^2} A(\xi).$$

Now the displacements and stresses given by (1) and (3) become

$$u(x, y) = \int_{-\infty}^{\infty} i\xi \left[\exp(-\gamma_1 y) - \frac{2\gamma_1\gamma_2}{\xi^2 + \gamma_2^2} \exp(-\gamma_2 y) \right] A(\xi) \exp(i\xi x) d\xi \quad (7)$$

$$v(x, y) = - \int_{-\infty}^{\infty} \gamma_1 \left[\exp(-\gamma_1 y) - \frac{2\xi^2}{\xi^2 + \gamma_2^2} \exp(-\gamma_2 y) \right] A(\xi) \exp(i\xi x) d\xi \quad (8)$$

$$\tau_{yy}(x, y) = -\mu \int_{-\infty}^{\infty} \left[(k_2^2 - 2\xi^2) \exp(-\gamma_1 y) + \frac{4\xi^2 \gamma_1 \gamma_2}{\xi^2 + \gamma_2^2} \exp(-\gamma_2 y) \right] A(\xi) \exp(i\xi x) d\xi \quad (9)$$

$$\tau_{xy}(x, y) = \mu \int_{-\infty}^{\infty} 2i\xi \gamma_1 \left[-\exp(-\gamma_1 y) + \exp(-\gamma_2 y) \right] A(\xi) \exp(i\xi x) d\xi \quad (10)$$

Next using the fact that $A(\xi)$ is an even function of ξ , and putting

$$P(\xi) = \frac{(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2}{2\xi^2 - k_2^2} A(\xi) \quad (11)$$

the first and second boundary conditions given by (4) lead to the following dual integral equations in $P(\xi)$:

$$\int_0^{\infty} P(\xi) \cos \xi x \, d\xi = 0 \quad , \quad |x| < c \quad , \quad |x| > 1 \quad (12)$$

$$\int_0^{\infty} \frac{\gamma_1 k_2^2}{(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2} P(\xi) \cos \xi x \, d\xi = \frac{1}{2} v_0 \quad , \quad c \leq |x| \leq 1 \quad (13)$$

3. SOLUTION OF THE INPLANE PROBLEM

Let us consider the solution of the integral equations (12) and (13) in the form

$$P(\xi) = \int_c^1 x_1 f(x_1^2) \cos \xi x_1 dx_1 \quad (14)$$

where $f(x_1^2)$ is an unknown function which will be determined.

The relation (12) is therefore satisfied automatically and the equation (13) becomes

$$\int_c^1 x_1 f(x_1^2) \int_0^\infty \frac{\gamma_1 k_2^2}{(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2} \cos \xi x_1 \cos \xi x dx d\xi dx_1 = \frac{1}{2} v_0$$

, $c \leq |x| \leq 1$ (15)

Using the relation

$$\frac{\sin \xi x \sin \xi x_1}{\xi^2} = \int_0^x \int_0^{x_1} \frac{wv J_0(\xi w) J_0(\xi v) dw dv}{(x^2 - w^2)^{1/2} (x_1^2 - v^2)^{1/2}}$$

the above equation converts to the form

$$\frac{d}{dx} \int_c^1 x_1 f(x_1^2) \frac{\partial}{\partial x_1} \int_0^x \int_0^{x_1} \frac{wv L_1(v, w) dw dv dx_1}{(x^2 - w^2)^{1/2} (x_1^2 - v^2)^{1/2}} = \frac{1}{2} v_0, \quad c \leq |x| \leq 1 \quad (16)$$

where

$$L_1(v, w) = \int_0^\infty \frac{\gamma_1 k_2^2}{(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2} J_0(\xi w) J_0(\xi v) d\xi \quad (17)$$

By a simple contour integration technique done in the appendix-A,

$L_1(v, w)$ can be written as

$$\begin{aligned}
L_1(v, w) = & -i\tau^2 \int_0^1 \frac{(1-\eta^2)^{1/2} (2\eta^2 - \tau^2)^2 H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{(2\eta^2 - \tau^2)^4 + 16\eta^4 (\eta^2 - 1)(\tau^2 - \eta^2)} d\eta - \\
& -4i\tau^2 \int_0^\tau \frac{\eta^2 (\eta^2 - 1)(\tau^2 - \eta^2)^{1/2} H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{(2\eta^2 - \tau^2)^4 + 16\eta^4 (\eta^2 - 1)(\tau^2 - \eta^2)} d\eta + \\
& + \pi i \tau^2 \left[\frac{(\eta^2 - 1)^{1/2} H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{Q_0'(\eta)} \right]_{\eta=\tau_0}, \quad (w > v)
\end{aligned}$$

$$= \frac{-i\tau^2}{16(1-\tau^2)} \left[\sum_{j=0}^2 p_j \int_0^1 \frac{(1-\eta^2)^{1/2} H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{\eta^2 - \tau_j^2} d\eta + \right.$$

$$\left. + \sum_{j=0}^2 s_j \int_0^\tau \frac{(\tau^2 - \eta^2)^{1/2} H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{\eta^2 - \tau_j^2} d\eta \right] +$$

$$+ \pi i \tau^2 \left[\frac{(\eta^2 - 1)^{1/2} H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{Q_0'(\eta)} \right]_{\eta=\tau_0}, \quad (w > v)$$

(18)

where $\tau = k_2/k_1 = c_1/c_2$, $Q_0(\eta) = (2\eta^2 - \tau^2)^2 - 4\eta^2(\eta^2 - 1)^{1/2}(\eta^2 - \tau^2)^{1/2}$, and τ_0 is the root of the Rayleigh wave equation $Q_0(\eta) = 0$. τ_1, τ_2 are the roots of the equation $(2\eta^2 - \tau^2)^2 + 4\eta^2(\eta^2 - 1)^{1/2}(\eta^2 - \tau^2)^{1/2} = 0$. $Q_0'(\eta)$ denote the derivative of $Q_0(\eta)$ with respect to η and

$$p_j = \frac{(2\tau_j^2 - \tau^2)}{\prod_i (\tau_j^2 - \tau_i^2)}, \quad s_j = \frac{4\tau_j^2 (\tau_j^2 - 1)}{\prod_i (\tau_j^2 - \tau_i^2)}, \quad (i, j=0, 1, 2 \text{ and } i \neq j).$$

The corresponding expression of $L_1(v, w)$ for $w < v$ follows from (18) by interchanging w and v .

For poisson ratio $\sigma = 1/4$, the values of τ , τ_0 , τ_1 and τ_2 are given by

$$\tau^2 = \frac{2(1-\sigma)}{1-2\sigma} = 3, \quad \tau_0^2 = \frac{3}{(0.9194)^2}, \quad \tau_1^2 = \frac{3}{(2+2/\sqrt{3})} \text{ and } \tau_2^2 = 3/4.$$

Hence in this case $\tau_2 < \tau_1 < 1 < \tau < \tau_0$.

Substituting the series expansion of $J_0(\)$ and $H_0^{(1)}(\)$ and evaluating the integrals arising in (18), we find after some algebraic manipulation details of which are given in the appendix-B

$$\begin{aligned} L_1(v, w) &= \frac{2}{\pi} \tau^2 \left[\left(\gamma + \log(k_1 w/2) - \frac{\pi i}{2} \right) M + N - \frac{P(w^2 + v^2)}{4} k_1^2 \log k_1 \right] + o(k_1^2) \\ &\quad , w > v \\ &= \frac{2}{\pi} \tau^2 \left[\left(\gamma + \log(k_1 v/2) - \frac{\pi i}{2} \right) M + N - \frac{P(w^2 + v^2)}{4} k_1^2 \log k_1 \right] + o(k_1^2) \\ &\quad , w < v \end{aligned} \quad (19)$$

where $\gamma = 0.5772157\dots$ is Euler's constant,

$$M = - \frac{\pi}{4(1-\tau^2)} \quad (20)$$

$$N = \frac{\pi}{32(1-\tau^2)} \left[4 \log(4/\tau) + \sum_{j=1}^2 p_j \frac{\sqrt{(1-\tau_j^2)}}{\tau_j} \tan^{-1} \frac{\sqrt{(1-\tau_j^2)}}{\tau_j} - \right]$$

$$\begin{aligned}
& - p_0 \frac{\sqrt{(\tau_0^2 - 1)}}{\tau_0} \log \left\{ \tau_0 + \sqrt{(\tau_0^2 - 1)} \right\} + \sum_{j=1}^2 s_j \frac{\sqrt{(\tau^2 - \tau_j^2)}}{\tau_j} \tan^{-1} \frac{\sqrt{(\tau^2 - \tau_j^2)}}{\tau_j} - \\
& - s_0 \frac{\sqrt{(\tau_0^2 - \tau^2)}}{\tau_0} \log \left\{ \frac{\tau_0 + \sqrt{(\tau_0^2 - \tau^2)}}{\tau} \right\} \Bigg], \quad (21)
\end{aligned}$$

$$\text{and } P = \frac{\pi}{32(1-\tau^2)} \left[\sum_{j=0}^2 p_j \left(\frac{1}{2} - \tau_j^2 \right) + \sum_{j=0}^2 s_j \left(\frac{\tau^2}{2} - \tau_j^2 \right) \right]. \quad (22)$$

Now differentiating both sides of the relation (15) with respect to x we obtain

$$\int_c^1 x_1 f(x_1^2) \int_0^\infty \frac{\gamma_1 k_2^2}{(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2} \xi \cos \xi x_1 \sin \xi x \, d\xi \, dx_1 = 0, \quad c \leq |x| \leq 1.$$

Following similar procedure as done for deriving equation (16), we obtain

$$\begin{aligned}
x \int_c^1 \frac{x_1 f(x_1^2)}{x^2 - x_1^2} \, dx_1 &= \int_c^1 x_1 f(x_1^2) \frac{\partial}{\partial x_1} \int_0^x \int_0^{x_1} \frac{wv L_2(v,w) \, dw \, dv \, dx_1}{(x^2 - w^2)^{1/2} (x_1^2 - v^2)^{1/2}} \\
& \quad , \quad c \leq |x| \leq 1 \quad (23)
\end{aligned}$$

where

$$L_2(v,w) = \int_0^\infty \left[\xi - \frac{2\gamma_1 \xi^2 (k_1^2 - k_2^2)}{(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2} \right] J_0(\xi w) J_0(\xi v) \, d\xi \quad (24)$$

For small values of k_1 and k_2 such that $k_1 = o(k_2)$, we use the contour integration technique mentioned above and get

$$\begin{aligned}
 L_2(v, w) = & 2ik_1^2(1-\tau^2) \int_0^1 \frac{(1-\eta^2)^{1/2} (2\eta^2 - \tau^2)^2 \eta^2 H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{(2\eta^2 - \tau^2)^4 + 16\eta^4 (\eta^2 - 1)(\tau^2 - \eta^2)} d\eta + \\
 & + 4ik_1^2(1-\tau^2) \int_0^\tau \frac{2\eta^4 (\eta^2 - 1)(\tau^2 - \eta^2)^{1/2} H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{(2\eta^2 - \tau^2)^4 + 16\eta^4 (\eta^2 - 1)(\tau^2 - \eta^2)} d\eta - \\
 & - 2\pi ik_2^2(1-\tau^2) \left[\frac{\eta^2 (\eta^2 - 1)^{1/2} H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{Q_0'(\eta)} \right]_{\eta=\tau_0}
 \end{aligned}$$

, $w > v$ (25)

By the procedure similar to one which led to the equation (19), (25) can be written as

$$L_2(v, w) = -\frac{4P}{\pi} (1-\tau^2) k_1^2 \log k_1 + o(k_1^2) \quad (26)$$

where P is given by (22).

Now let us consider

$$f(x_1^2) = f_0(x_1^2) + k_1^2 \log k_1 f_1(x_1^2) + o(k_1^2) \quad (27)$$

Putting the above expansion of $f(x_1^2)$ and the value of $L_2(v, w)$ given by (26) in the equation (23) and equating the coefficients of equal powers of k_1 we get

$$\int_c^1 \frac{x_1 f_0(x_1^2)}{x^2 - x_1^2} dx_1 = 0 \quad , \quad c \leq |x| \leq 1 \quad (28)$$

$$\text{and} \quad \int_c^1 \frac{x_1 f_1(x_1^2)}{x^2 - x_1^2} dx_1 = -\frac{4P}{\pi} (1-\tau^2) \int_c^1 x_1 f_0(x_1^2) dx_1 \quad , \quad c \leq |x| \leq 1 \quad (29)$$

Following Srivastava and Lowengrub (1968) the solutions of the above integral equations are

$$f_0(x_1^2) = \frac{D}{(1-x_1^2)^{1/2} (x_1^2 - c^2)^{1/2}} \quad (30)$$

and

$$f_1(x_1^2) = \frac{4}{\pi} PD(1-\tau^2) \left[\frac{x_1^2 - c^2}{1 - x_1^2} \right]^{1/2} + \frac{B}{(1-x_1^2)^{1/2} (x_1^2 - c^2)^{1/2}} \quad (31)$$

where D and B are constants which can be calculated as follows.

We substitute the value of $L_1(v, w)$ from (19) as well as the expansion of $f(x_1^2)$ obtained from (27) in the equation (16). When the coefficients of like powers of k_1 from both sides of the resulting equation are equated we get the following results :

$$\frac{d}{dx} \int_c^1 x_1 f_0(x_1^2) \frac{\partial}{\partial x_1} \int_0^x \int_0^{x_1} \frac{wv}{(x^2 - w^2)^{1/2} (x_1^2 - v^2)^{1/2}} \frac{2}{\pi} \tau^2 \left[\left(\gamma + \log(k_1 v/2) - \right. \right. \\ \left. \left. - \frac{\pi i}{2} \right) M + N \right] dw dv dx_1 = \frac{1}{2} v_0 \quad , \quad v > w \quad , \quad c \leq |x| \leq 1 \quad (32)$$

and

$$\begin{aligned}
& \frac{d}{dx} \int_c^1 x_1 f_1(x_1^2) \frac{\partial}{\partial x_1} \int_0^x \int_0^{x_1} \frac{wv}{(x^2-w^2)^{1/2} (x_1^2-v^2)^{1/2}} \frac{2}{\pi} \tau^2 \left[\left(\gamma + \log(k_1 w/2) - \right. \right. \\
& \left. \left. - \frac{\pi 1}{2} \right) M + N \right] dw dv dx_1 - \frac{\tau^2}{2\pi} P \frac{d}{dx} \int_c^1 x_1 f_0(x_1^2) \frac{\partial}{\partial x_1} \int_0^x \int_0^{x_1} \frac{wv(w^2+v^2) dw dv dx_1}{(x^2-w^2)^{1/2} (x_1^2-v^2)^{1/2}} \\
& = 0, \quad v > w, \quad c \leq |x| \leq 1 \quad (33)
\end{aligned}$$

Next, putting the values of $f_0(x_1^2)$ and $f_1(x_1^2)$ from (30) and (31) in the above two equations and integrating, the values of D and B can be obtained as :

$$D = \frac{v_0}{2\tau^2 \left[\left(\gamma + \log(k_1/2) - \frac{\pi 1}{2} + \log(1-c^2)^{1/2} \right) M + N \right]} \quad (34)$$

$$\text{and } B = \frac{2\tau^2 D^2 P}{v_0} \left[\frac{1}{2}(1+c^2) - \frac{(1-c^2)(1-\tau^2)v_0}{\pi\tau^2 D} \right] \quad (35)$$

We can now obtain the values of the vertical displacement in the plane $y=0$ from equations (8), (11) and (14) as

$$\begin{aligned}
v(x, 0) &= 2 \int_c^1 x_1 f_1(x_1^2) \int_0^\infty \frac{\gamma_1 k_2^2 \cos \xi x_1 \cos \xi x}{(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2} d\xi dx_1 \\
&= 2 \frac{d}{dx} \int_c^1 x_1 f_1(x_1^2) \frac{\partial}{\partial x_1} \int_0^x \int_0^{x_1} \frac{wv L_1(v, w) dw dv dx_1}{(x^2-w^2)^{1/2} (x_1^2-v^2)^{1/2}}
\end{aligned}$$

Next putting the values of $L_1(v, w)$ and $f(x_1^2)$ from (19) and (27) in the above equation, we obtain,

$$\begin{aligned}
 v(x, 0) = & \frac{4}{\pi} \tau^2 \left[\left\{ \left(\gamma + \log(k_1/2) - \frac{\pi i}{2} \right) M + N \right\} \frac{d}{dx} \int_c^1 x_1 f_0(x_1^2) \times \right. \\
 & \times \frac{\partial}{\partial x_1} \int_0^x \int_0^{x_1} \frac{vw \, dw \, v \, dx_1}{(x^2 - w^2)^{1/2} (x_1^2 - v^2)^{1/2}} + \\
 & + M \frac{d}{dx} \int_c^1 x_1 f_0(x_1^2) \frac{\partial}{\partial x_1} \int_0^x \int_0^{x_1} \frac{vw \begin{cases} \log v, & v > w \\ \log w, & v < w \end{cases}}{(x^2 - w^2)^{1/2} (x_1^2 - v^2)^{1/2}} \, dw \, v \, dx_1 \left. \right] + \\
 & + \frac{4\tau^2 k_1^2 \log k_1}{\pi} \left[\left\{ \left(\gamma + \log(k_1/2) - \frac{\pi i}{2} \right) M + N \right\} \frac{d}{dx} \int_c^1 x_1 f_1(x_1^2) \times \right. \\
 & \times \frac{\partial}{\partial x_1} \int_0^x \int_0^{x_1} \frac{vw \, dw \, v \, dx_1}{(x^2 - w^2)^{1/2} (x_1^2 - v^2)^{1/2}} + \\
 & + M \frac{d}{dx} \int_c^1 x_1 f_1(x_1^2) \frac{\partial}{\partial x_1} \int_0^x \int_0^{x_1} \frac{vw \begin{cases} \log v, & v > w \\ \log w, & v < w \end{cases}}{(x^2 - w^2)^{1/2} (x_1^2 - v^2)^{1/2}} \, dw \, v \, dx_1 \left. \right] - \\
 & - \frac{\tau^2}{2\pi} P \frac{d}{dx} \int_c^1 x_1 f_0(x_1^2) \frac{\partial}{\partial x_1} \int_0^x \int_0^{x_1} \frac{vw (w^2 + v^2) \, dw \, v \, dx_1}{(x^2 - w^2)^{1/2} (x_1^2 - v^2)^{1/2}} \quad (36)
 \end{aligned}$$

Substituting the values of $f_0(x_1)$ and $f_1(x_1)$ from (30) and (31) in (36) and integrating, the equation (36) yields after some

algebraic manipulation

$$\begin{aligned}
 v(x,0) &= v_0 + 2M\tau^2 \left[D + k_1^2 \log k_1 \left\{ B + \frac{2}{\pi} (1-\tau^2)(1-c^2)PD \right\} \right] \sinh^{-1} \left(\frac{c^2 - x^2}{1-c^2} \right)^{1/2} \\
 &\quad - \frac{4\tau^2 MPD(1-\tau^2)}{\pi} k_1^2 \log k_1 [(1-x^2)(c^2-x^2)]^{1/2} + o(k_1^2) \quad , \quad |x| < c \\
 &= v_0 \quad , \quad c \leq |x| \leq 1 \\
 &= v_0 + 2M\tau^2 \left[D + k_1^2 \log k_1 \left\{ B + \frac{2}{\pi} (1-\tau^2)(1-c^2)PD \right\} \right] \sinh^{-1} \left(\frac{x^2 - 1}{1-c^2} \right)^{1/2} \\
 &\quad + \frac{4\tau^2 MPD(1-\tau^2)}{\pi} k_1^2 \log k_1 [(x^2 - 1)(x^2 - c^2)]^{1/2} + o(k_1^2) \quad , \quad |x| > 1
 \end{aligned}
 \tag{37}$$

The normal stress $\tau_{yy}(x,y)$ in the plane $y=0$ just below the strips can be found from the relation (9), (11) and (14) as

$$\tau_{yy}(x,0) = 2\mu \int_c^1 x_1 f(x_1^2) dx_1 \int_0^\infty \cos \xi x \cos \xi x_1 d\xi$$

Using the result

$$\int_0^\infty \cos \xi x \cos \xi x_1 d\xi = \frac{\pi}{2} \delta(|x| - x_1)$$

and the value of $f(x_1^2)$ given by (27), the expression for stress becomes

$$\tau_{yy}(x,0) = \pi\mu \int_c^1 x_1 \left[f_0(x_1^2) + k_1^2 \log k_1 f_1(x_1^2) + o(k_1^2) \right] \delta(|x| - x_1) dx_1$$

Substituting the values of $f_0(x_1^2)$ and $f_1(x_1^2)$ from (30) and (31) respectively and integrating we finally obtain,

$$\tau_{yy}(x,0) = \frac{\pi\mu|x|}{[(1-x^2)(x^2-c^2)]^{1/2}} \left(D+Bk_1^2 \log k_1 \right) + 4\mu|x|DP(1-\tau^2) \left(\frac{x^2-c^2}{1-x^2} \right)^{1/2} \\ \times k_1^2 \log k_1 + o(k_1^2), \quad c \leq |x| \leq 1 \quad (38)$$

Now putting $c=0$ in (38) we can obtain the normal stress for a single strip, $|x| \leq 1$, $y=0$, $-\infty < z < \infty$ as

$$\tau_{yy}(x,0) = \frac{\pi\mu D}{(1-x^2)^{1/2}} + \frac{\mu}{(1-x^2)^{1/2}} k_1^2 \log k_1 [4P(1-\tau^2)Dx^2 + \pi B] + o(k_1^2)$$

where

$$D = \frac{v_0}{2\tau^2 \left[(\gamma + \log(k_1/2)) - \frac{\pi i}{2} \right] M + N}$$

and

$$B = \frac{2\tau^2 D^2 P}{v_0} \left[\frac{1}{2} - \frac{(1-\tau^2)v_0}{\pi\tau^2 D} \right]$$

Substituting $\Delta_0 = v_0/\pi^2 D$, $\beta_0 = -\tau^2/2\pi(1-\tau^2)$ and $\beta_2 = -P/\pi^2$ as done by Wickham (1977), we get

$$\tau_{yy}(x,0) = \frac{\mu v_0}{\pi \Delta_0 (1-x^2)^{1/2}} \left\{ 1 - \beta_2 k_2^2 \log k_2 \left[\frac{1}{\Delta_0} + \frac{(1-2x^2)}{\beta_0} \right] \right\} + o(k_2^2)$$

which coincides with the result obtained by G.R.Wickham (1977).

4. FORMULATION AND SOLUTION OF THE ANTIPLANE PROBLEM

For SH-wave the displacement and stress are

$$w(x, y, t) = w(x, y) e^{-i\omega t}$$

and

$$\tau_{yz}(x, y) = \mu \frac{\partial w}{\partial y}$$

As in the previous case, we write the above expressions as

$$w(x, y) = \int_{-\infty}^{\infty} \frac{Q(\xi)}{\gamma_2} \exp(i\xi x - \gamma_2 y) d\xi \quad (39)$$

and

$$\tau_{yz}(x, y) = -\mu \int_{-\infty}^{\infty} Q(\xi) \exp(i\xi x - \gamma_2 y) d\xi \quad (40)$$

where $Q(\xi)$ is an unknown function to be determined from the boundary conditions

$$w(x, 0) = w_0, \quad c \leq |x| \leq 1 \quad (41)$$

$$\tau_{yz}(x, 0) = 0, \quad |x| < c, \quad |x| > 1 \quad (42)$$

where w_0 is constant.

From equations (39) to (42) the following integral equations can be derived

$$\int_0^{\infty} \frac{Q(\xi)}{\gamma_2} \cos \xi x \, d\xi = \frac{w_0}{2}, \quad c \leq |x| \leq 1 \quad (43)$$

$$\int_0^{\infty} Q(\xi) \cos \xi x \, d\xi = 0, \quad |x| < c, \quad |x| > 1 \quad (44)$$

Substitution of

$$Q(\xi) = \int_c^1 x_1 g(x_1^2) \cos \xi x_1 dx_1 \quad (45)$$

satisfies equation (44) automatically and equation (43) then yields

$$\int_c^1 \int_0^\infty \frac{x_1 g(x_1^2)}{\gamma_2} \cos \xi x \cos \xi x_1 dx_1 = \frac{1}{2} w_0, \quad c \leq |x| \leq 1 \quad (46)$$

which can be written as

$$\frac{d}{dx} \int_c^1 x_1 g(x_1^2) \frac{\partial}{\partial x_1} \int_0^x \int_0^{x_1} \frac{wv L_3(v, w) dw dv dx_1}{(x^2 - w^2)^{1/2} (x_1^2 - v^2)^{1/2}} = \frac{1}{2} w_0, \quad c \leq |x| \leq 1 \quad (47)$$

where

$$L_3(v, w) = \int_0^\infty \frac{J_0(\xi w) J_0(\xi v)}{\gamma_2} d\xi$$

Integrating along the contour as shown in fig.2 and putting $\xi = k_2 \eta$ the above infinite integral can be reduced to the finite integral given by

$$\begin{aligned} L_3(v, w) &= i \int_0^1 \frac{J_0(k_2 \eta v) H_0^{(1)}(k_2 \eta w)}{(1 - \eta^2)^{1/2}} d\eta, \quad w > v \\ &= i \int_0^1 \frac{J_0(k_2 \eta w) H_0^{(1)}(k_2 \eta v)}{(1 - \eta^2)^{1/2}} d\eta, \quad w < v. \end{aligned}$$

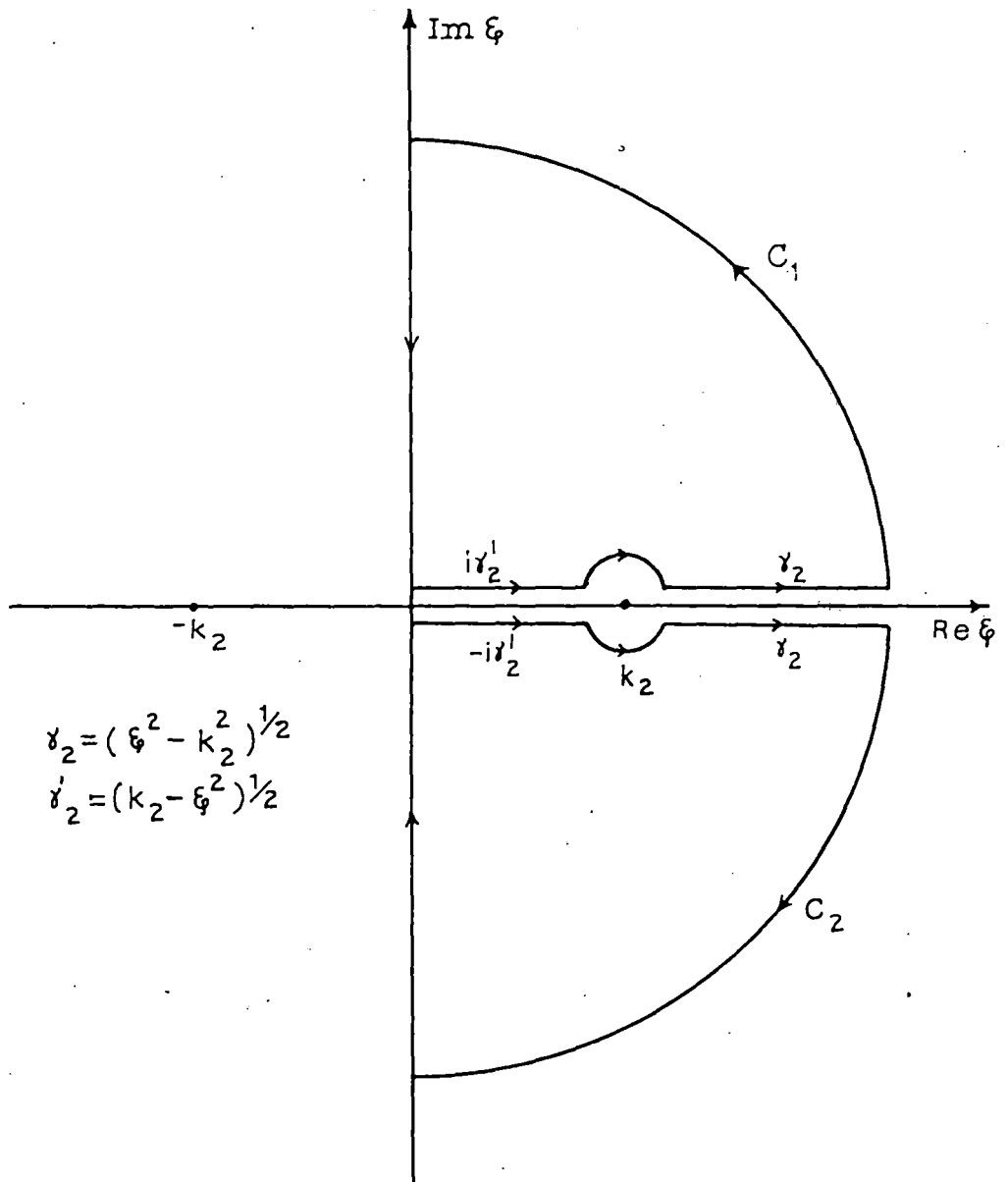


Fig. 2. The contour of integration to evaluate $L_3(v, w)$

For low frequency, $k_2 \eta w$ and $k_2 \eta v$ are small. So expanding both the Hankel function and Bessel function for small values of their arguments and integrating term by term, we obtain finally,

$$\begin{aligned} L_9(v, w) &= \left[\frac{\pi i}{2} - \gamma - \log(k_2 w/4) \right] + \frac{1}{8}(w^2 + v^2)k_2^2 \log k_2 + o(k_2^2) \quad , \quad w > v \\ &= \left[\frac{\pi i}{2} - \gamma - \log(k_2 v/4) \right] + \frac{1}{8}(w^2 + v^2)k_2^2 \log k_2 + o(k_2^2) \quad , \quad w < v \end{aligned}$$

(48)

Next, Differentiating equation (46) with respect to x and proceeding as the previous case we get

$$x \int_c^1 \frac{x_1 g(x_1^2)}{x^2 - x_1^2} dx_1 = \int_c^1 x_1 g(x_1^2) \frac{\partial}{\partial x_1} \int_0^x \int_0^{x_1} \frac{wv L_4(v, w) dw dv dx_1}{(x^2 - w^2)^{1/2} (x_1^2 - v^2)^{1/2}}$$

, $c \leq |x| \leq 1$ (49)

where $L_4(v, w) = \int_0^\infty \left[\xi - \frac{\xi^2}{\gamma_2} \right] J_0(\xi w) J_0(\xi v) d\xi$

By the procedure similar to the one which led to the derivation of equation (48) from the corresponding infinite integral, we obtain

$$\begin{aligned}
L_4(v, w) &= -ik_2^2 \int_0^1 \frac{\eta^2 J_0(k_2 \eta v) H_0^{(1)}(k_2 \eta w)}{(1-\eta^2)^{1/2}} d\eta \\
&= \frac{1}{2} k_2^2 \log k_2 + o(k_2^2)
\end{aligned} \tag{50}$$

Now considering the expansion for $g(x_1^2)$ as

$$g(x_1^2) = g_0(x_1^2) + k_2^2 \log k_2 g_1(x_1^2) + o(k_2^2) \tag{51}$$

and substituting $g(x_1^2)$ and $L_4(v, w)$ as given by (51) and (50) respectively in (49) it can be shown that

$$g_0(x_1^2) = \frac{D_1}{(1-x_1^2)^{1/2} (x_1^2 - c^2)^{1/2}} \tag{52}$$

$$\text{and } g_1(x_1^2) = -\frac{\tau^2 D_1}{2} \left[\frac{x_1^2 - c^2}{1 - x_1^2} \right]^{1/2} + \frac{B_1}{(1-x_1^2)^{1/2} (x_1^2 - c^2)^{1/2}} \tag{53}$$

where the constants D_1 and B_1 obtained from the equations (47) and (48) are found to be equal to

$$D_1 = \frac{-w_0}{\pi [\gamma + \log(k_2/4) - \frac{\pi i}{2} + \log(1-c^2)^{1/2}]} \tag{54}$$

and
$$B_1 = -\frac{\pi D_1^2}{4w_0} \left[(1+c^2) - \frac{(1-c^2)w_0}{\pi D_1} \right] \quad (55)$$

The values of the stress $\tau_{yz}(x,y)$ and displacement $w(x,y)$ in the plane $y=0$ can be found from equations (39), (40), (45), (51) to (55) and are

$$\begin{aligned} \tau_{yz}(x,0) = & \frac{-\pi\mu|x|}{[(1-x^2)(x^2-c^2)]^{1/2}} (D_1 + B_1 k_2^2 \log k_2) + \frac{\pi\mu|x|D_1}{2} \left(\frac{x^2-c^2}{1-x^2} \right)^{1/2} \times \\ & \times k_2^2 \log k_2 + o(k_2^2), \quad c \leq |x| \leq 1 \end{aligned} \quad (56)$$

and

$$\begin{aligned} w(x,0) = & w_0 - \pi \left[D_1 + k_2^2 \log k_2 \left\{ B_1 - D_1(1-c^2)/4 \right\} \right] \sinh^{-1} \left(\frac{c^2-x^2}{1-c^2} \right)^{1/2} - \\ & - \frac{\pi D_1}{4} k_2^2 \log k_2 [(1-x^2)(c^2-x^2)]^{1/2} + o(k_2^2), \quad |x| < c \end{aligned}$$

$$= w_0, \quad c \leq |x| \leq 1$$

$$= w_0 - \pi \left[D_1 + k_2^2 \log k_2 \left\{ B_1 - D_1(1-c^2)/4 \right\} \right] \sinh^{-1} \left(\frac{x^2-1}{1-c^2} \right)^{1/2} +$$

$$+ \frac{\pi D_1}{4} k_2^2 \log k_2 [(x^2-1)(x^2-c^2)]^{1/2} + o(k_2^2), \quad |x| > 1$$

(57)

5. NUMERICAL RESULTS

The vertical and the transverse displacement fields for the inplane and antiplane problems respectively for points near about the rigid strips have been depicted by means of graphs (fig.3,4) for the Poisson Solid ($\tau^2=3$). It is interesting to note from the graphs that the real part of the displacements decrease with the increase in the value of k_2 in both the cases.

Further the graphs (fig.5,6) of the stress factors

$$\tau_1^* = \text{Re} \left[\frac{\tau_{yy} \left((1-x^2)(x^2-c^2) \right)^{1/2}}{\mu v_0} \right] \quad \text{and} \quad \tau_2^* = \text{Re} \left[\frac{\tau_{yz} \left((1-x^2)(x^2-c^2) \right)^{1/2}}{\mu w_0} \right]$$

versus dimensionless distance x for the inplane and the antiplane problem respectively have been plotted for points just below the rigid strips. In both the cases the magnitude of the stress factor are found to increase as one proceeds from the inner to the outer edge of the strips.

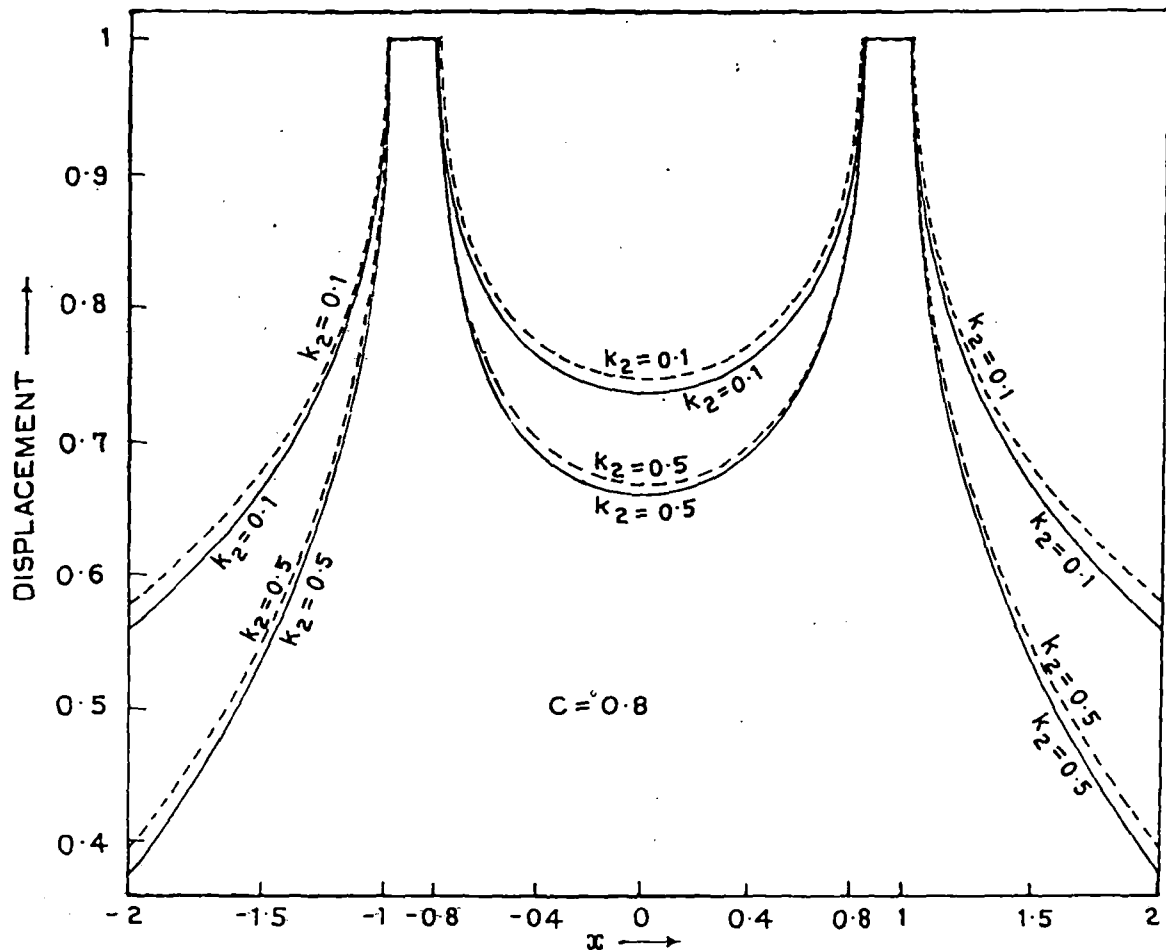


Fig. 3. Displacement vs. distance

— $\text{Re}\{v(x,0)\}$ for inplane problem, --- $\text{Re}\{w(x,0)\}$ for antiplane problem.

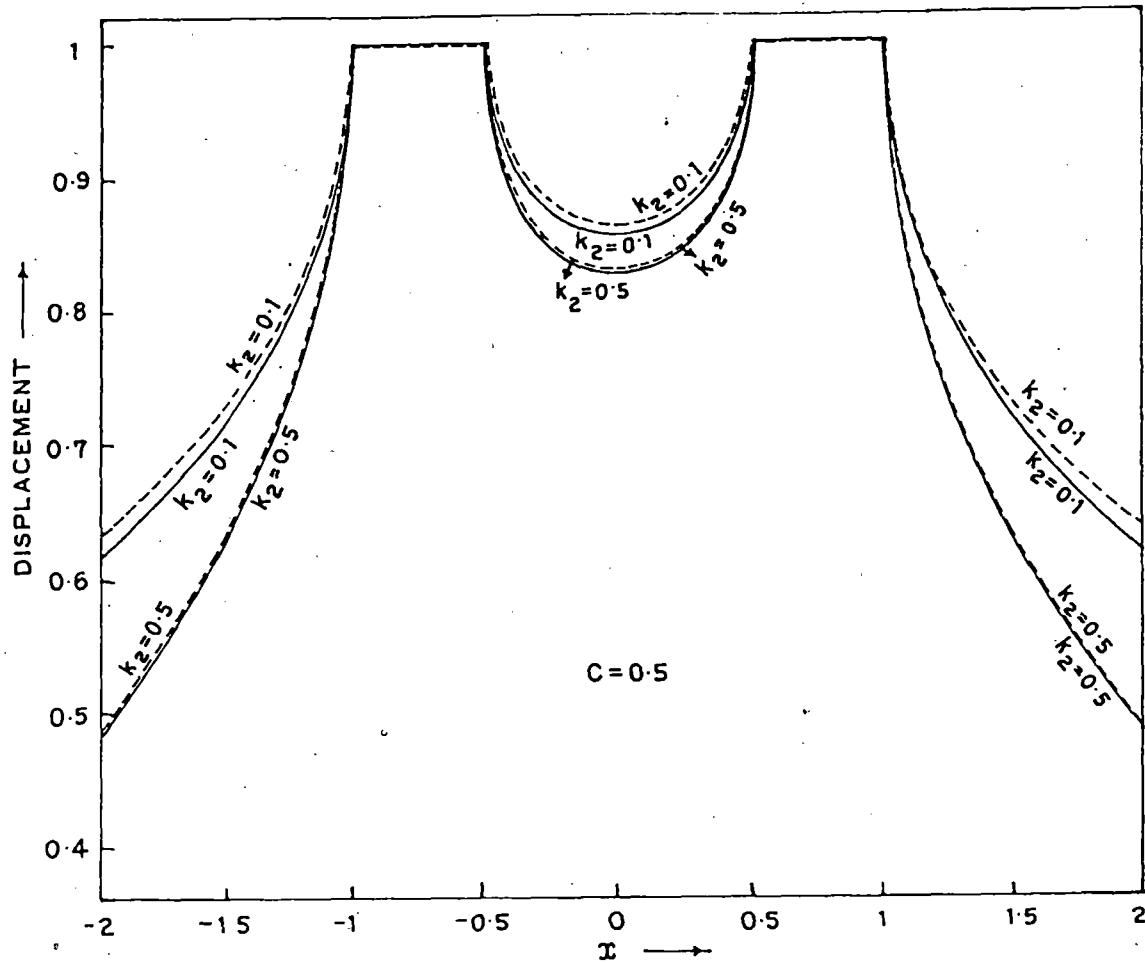


Fig. 4. Displacement vs. distance

— $\text{Re}\{v(x,0)\}$ for inplane problem, ---- $\text{Re}\{w(x,0)\}$ for antiplane problem.

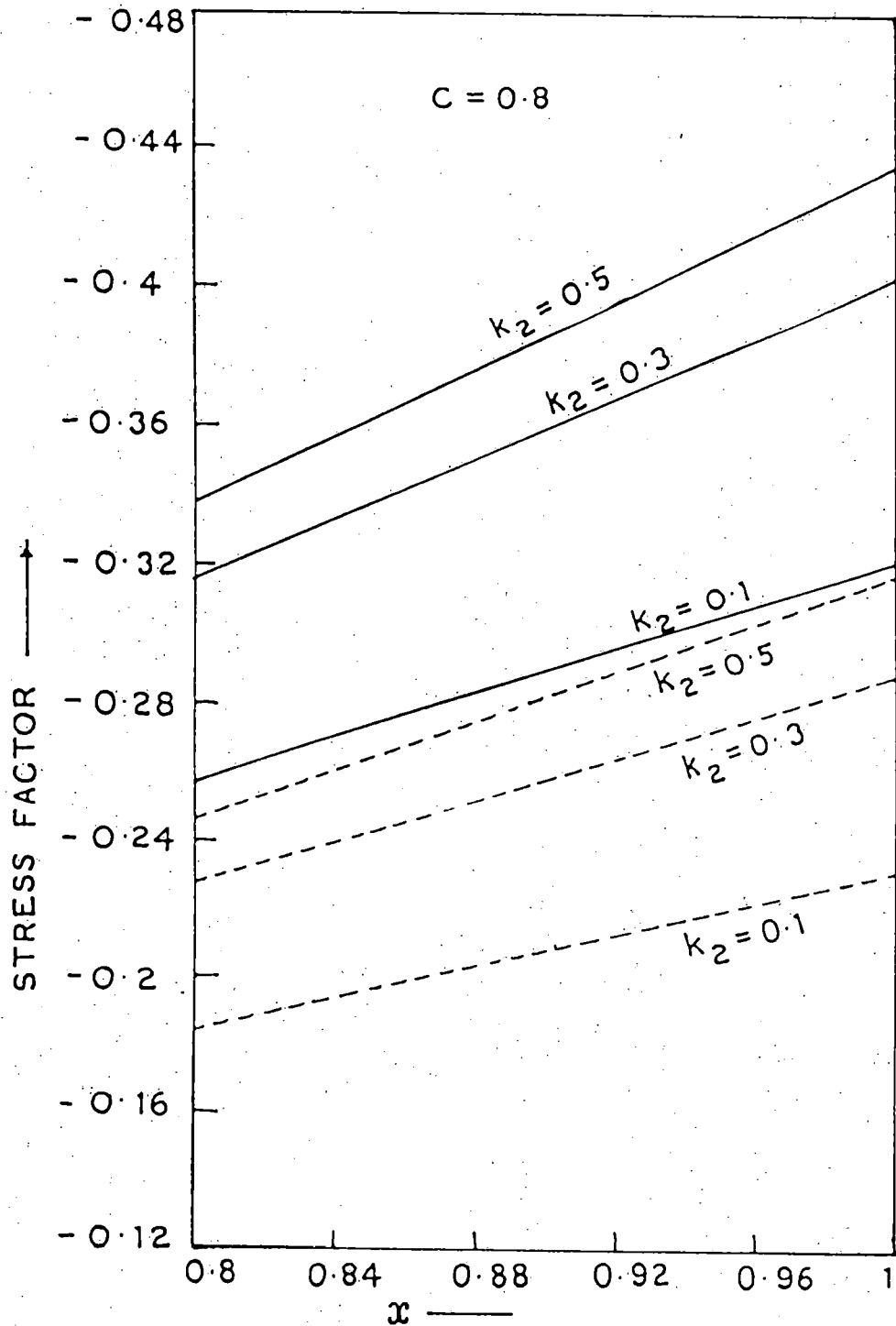


Fig. 5. Stress factor vs. displacement
 — τ_1^* for inplane problem, - - - τ_2^* for antiplane problem

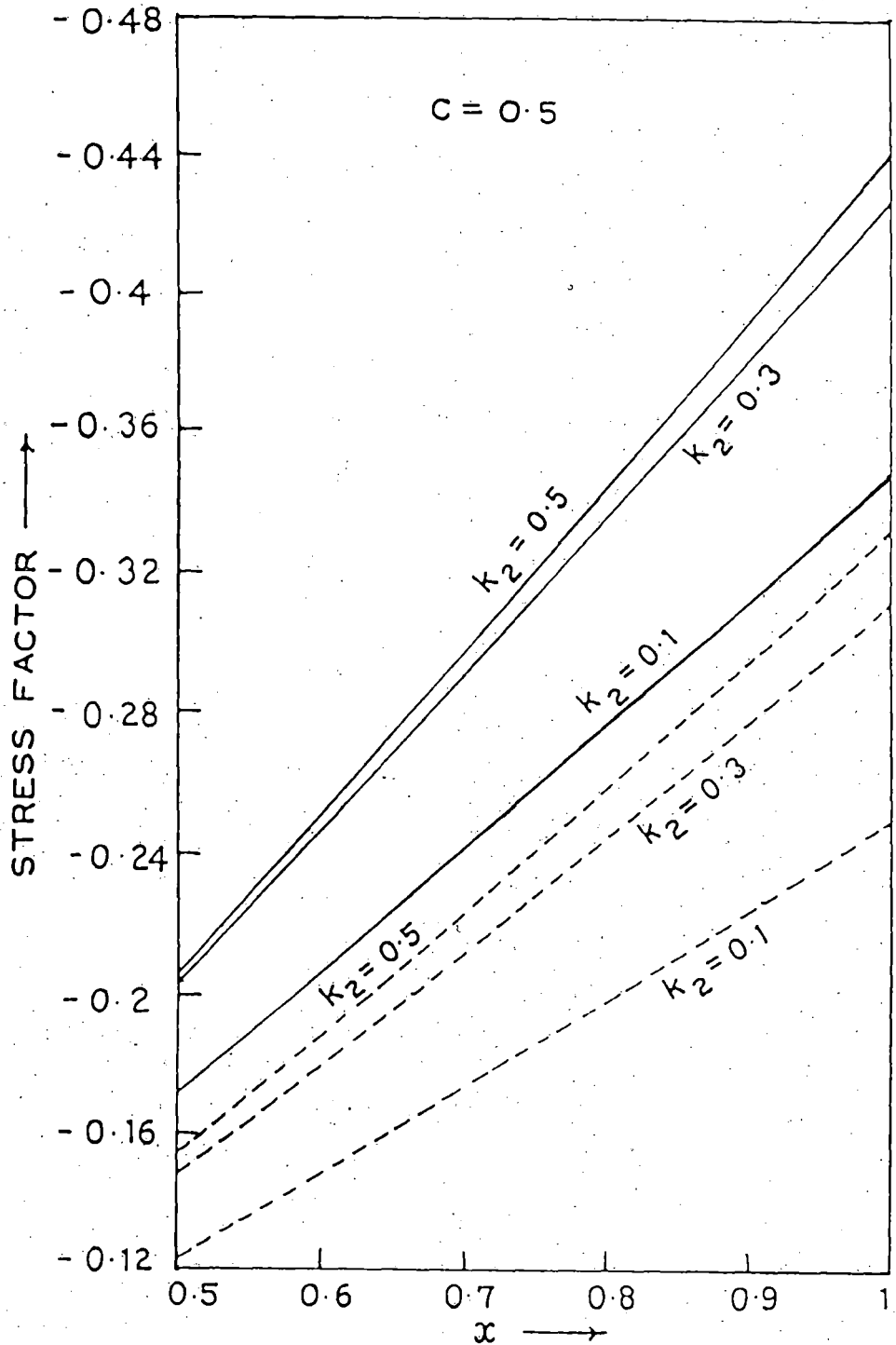


Fig. 6. Stress factor vs. displacement

— τ_1^* for inplane problem,
 - - - τ_2^* for antiplane problem

APPENDIX - A

CONTOUR INTEGRATION TO EVALUATE $L_1(v, w)$:

The kernel $L_1(v, w)$ given by equation (17) is

$$L_1(v, w) = \int_0^{\infty} \frac{\gamma_1 k_2^2}{(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2} J_0(\xi w) J_0(\xi v) d\xi \quad (A1)$$

Let us consider the following two integrals

$$I_1 = \int_{C_1} \frac{\gamma_1 k_2^2}{(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2} H_0^{(1)}(\xi w) J_0(\xi v) d\xi, \quad w > v \quad (A2)$$

$$I_2 = \int_{C_2} \frac{\gamma_1 k_2^2}{(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2} H_0^{(2)}(\xi w) J_0(\xi v) d\xi, \quad w > v \quad (A3)$$

where C_1 and C_2 are the contours of integration of I_1 and I_2 respectively as shown in fig.7.

Branch points corresponding to γ_1 and γ_2 are $\xi = \pm k_1$ and $\xi = \pm k_2$ and the pole k_R is the root of the Rayleigh wave equation

$$(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2 = 0$$

Integrating I_1 and I_2 along the contours C_1 and C_2 respectively we get,

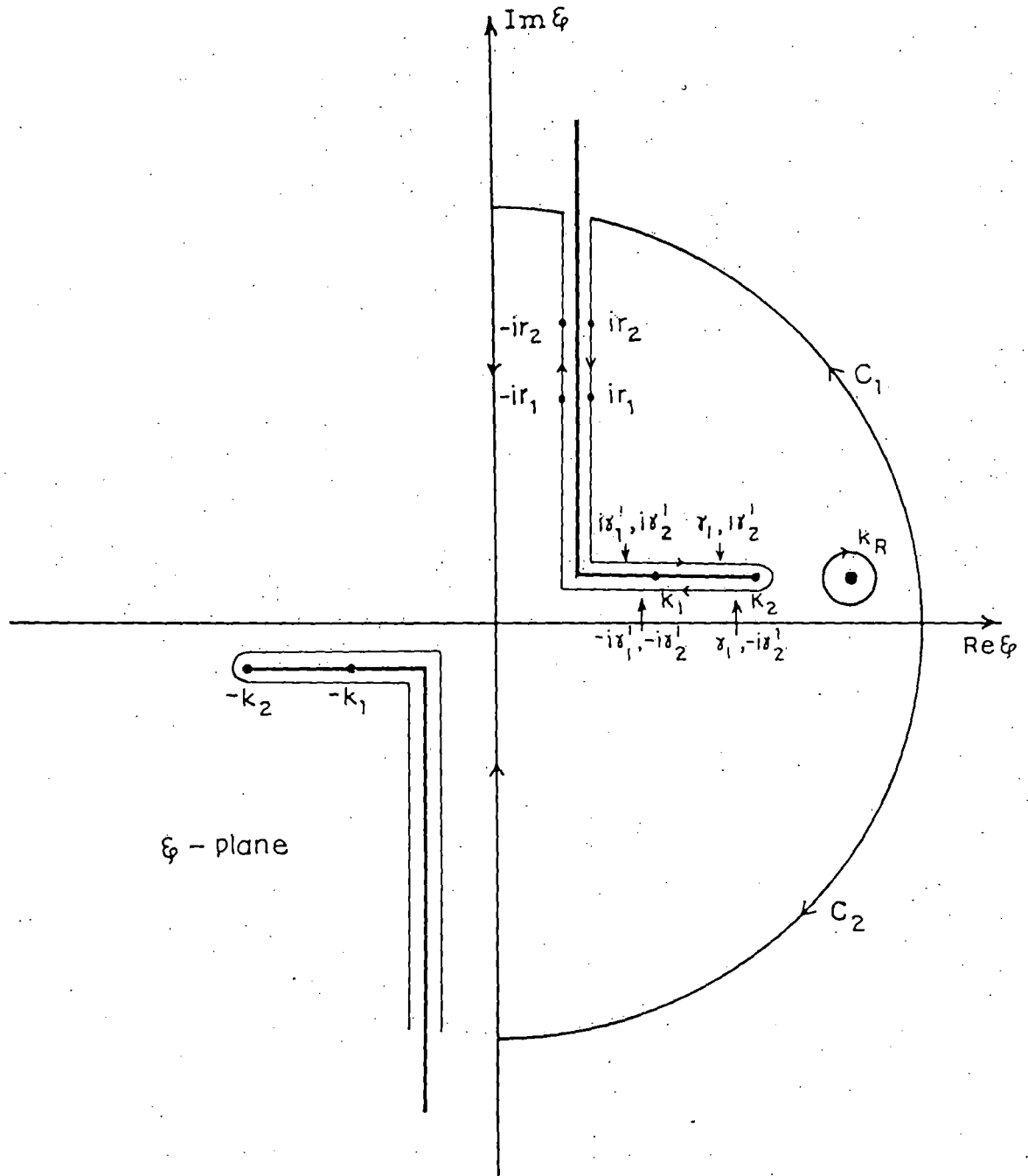


Fig. 7. The contour of integration to evaluate $L_1(u, w)$.

for contour C_1 :

$$\begin{aligned}
 & \int_0^\infty \frac{\gamma_1 k_2^2 H_0^{(1)}(\xi w) J_0(\xi v)}{(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2} d\xi + i \int_0^\infty \frac{i r_1 k_2^2 H_0^{(1)}(i\eta w) J_0(i\eta v)}{(2\eta^2 + k_2^2)^2 - 4\eta^2 r_1 r_2} d\eta + \\
 & + \int_0^{k_1} \frac{i \gamma_1' k_2^2 H_0^{(1)}(\xi w) J_0(\xi v)}{(2\xi^2 - k_2^2)^2 + 4\xi^2 \gamma_1' \gamma_2'} d\xi + \int_{k_1}^{k_2} \frac{\gamma_1 k_2^2 H_0^{(1)}(\xi w) J_0(\xi v)}{(2\xi^2 - k_2^2)^2 - 4i\xi^2 \gamma_1 \gamma_2'} d\xi + \\
 & + i \int_0^{k_1} \frac{\gamma_1' k_2^2 H_0^{(1)}(\xi w) J_0(\xi v)}{(2\xi^2 - k_2^2)^2 + 4\xi^2 \gamma_1' \gamma_2'} d\xi - \int_{k_1}^{k_2} \frac{\gamma_1 k_2^2 H_0^{(1)}(\xi w) J_0(\xi v)}{(2\xi^2 - k_2^2)^2 + 4i\xi^2 \gamma_1 \gamma_2'} d\xi \\
 & = 2\pi i \frac{\gamma_1(k_R) k_2^2 H_0^{(1)}(k_R w) J_0(k_R v)}{\phi'(k_R)} \quad (A4)
 \end{aligned}$$

and for contour C_2 :

$$\int_0^\infty \frac{\gamma_1 k_2^2 H_0^{(2)}(\xi w) J_0(\xi v)}{(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2} d\xi - \int_0^\infty \frac{r_1 k_2^2 H_0^{(2)}(-i\eta w) J_0(-i\eta v)}{(2\eta^2 + k_2^2)^2 - 4\eta^2 r_1 r_2} d\eta = 0$$

(A5)

where $\gamma_1' = (k_1^2 - \xi^2)^{1/2}$, $\gamma_2' = (k_2^2 - \xi^2)^{1/2}$

$$r_1 = (\eta^2 + k_1^2)^{1/2}, \quad r_2 = (\eta^2 + k_2^2)^{1/2}$$

$$\phi(\xi) = (2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2$$

where k_R is the root of the equation $\phi(\xi) = 0$.

Now, using the results

$$H_0^{(2)}(-i\eta w) = -H_0^{(1)}(i\eta w) \quad \text{and} \quad J_0(-i\eta v) = J_0(i\eta v)$$

in (A5) and then addition with (A4) yields

$$\begin{aligned} L_1(v, w) = & -i \int_0^{k_1} \frac{(k_1^2 - \xi^2)^{1/2} k_2^2 H_0^{(1)}(\xi w) J_0(\xi v)}{(2\xi^2 - k_2^2)^2 + 4\xi^2 (k_1^2 - \xi^2)^{1/2} (k_2^2 - \xi^2)^{1/2}} d\xi - \\ & -4i \int_{k_1}^{k_2} \frac{k_2^2 \xi^2 (\xi^2 - k_1^2) (k_2^2 - \xi^2)^{1/2} H_0^{(1)}(\xi w) J_0(\xi v)}{(2\xi^2 - k_2^2)^4 + 16\xi^4 (\xi^2 - k_1^2) (k_2^2 - \xi^2)} d\xi + \\ & + \pi i \frac{(k_R^2 - k_1^2)^{1/2} k_2^2 H_0^{(1)}(k_R w) J_0(k_R v)}{\phi'(k_R)} \end{aligned} \quad (A6)$$

Putting $\xi = k_1 \eta$, $\tau = k_2/k_1 = c_1/c_2$ and after simple calculation, (A6) takes the form

$$\begin{aligned}
L_1(v, w) = & -i\tau^2 \int_0^1 \frac{(1-\eta^2)^{1/2} (2\eta^2 - \tau^2)^2 H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{(2\eta^2 - \tau^2)^4 + 16\eta^4 (\eta^2 - 1)(\tau^2 - \eta^2)} d\eta - \\
& -4i\tau^2 \int_0^\tau \frac{\eta^2 (\eta^2 - 1) (\tau^2 - \eta^2)^{1/2} H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{(2\eta^2 - \tau^2)^4 + 16\eta^4 (\eta^2 - 1)(\tau^2 - \eta^2)} d\eta + \\
& + \pi i \tau^2 \left[\frac{(\eta^2 - 1)^{1/2} H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{Q_0'(\eta)} \right]_{\eta=\tau_0}, \quad (w > v)
\end{aligned}$$

(A7)

where $Q_0(\eta) = (2\eta^2 - \tau^2)^2 - 4\eta^2 (\eta^2 - 1)^{1/2} (\eta^2 - \tau^2)^{1/2}$ and τ_0 is the root of the Rayleigh wave equation $Q_0(\eta) = 0$. $Q_0'(\eta)$ denote the derivative of $Q_0(\eta)$ with respect to η .

Next taking

$$\Delta_0(\eta) = (2\eta^2 - \tau^2)^4 + 16\eta^4 (\eta^2 - 1)(\tau^2 - \eta^2) = 16(1 - \tau^2)(\eta^2 - \tau_0^2)(\eta^2 - \tau_1^2)(\eta^2 - \tau_2^2),$$

τ_1, τ_2 being the roots of the equation

$$(2\eta^2 - \tau^2)^2 + 4\eta^2 (\eta^2 - 1)^{1/2} (\eta^2 - \tau^2)^{1/2} = 0$$

and

$$\frac{(2\eta^2 - \tau^2)^2}{(\eta^2 - \tau_0^2)(\eta^2 - \tau_1^2)(\eta^2 - \tau_2^2)} = \frac{p_0}{\eta^2 - \tau_0^2} + \frac{p_1}{\eta^2 - \tau_1^2} + \frac{p_2}{\eta^2 - \tau_2^2}$$

$$\frac{4\eta^2(\eta^2-1)}{(\eta^2-\tau_0^2)(\eta^2-\tau_1^2)(\eta^2-\tau_2^2)} = \frac{s_0}{\eta^2-\tau_0^2} + \frac{s_1}{\eta^2-\tau_1^2} + \frac{s_2}{\eta^2-\tau_2^2}$$

so that $\sum_{j=0}^2 p_j = \sum_{j=0}^2 s_j = 4$ and

$$p_j = \frac{(2\tau_j^2 - \tau^2)}{\prod_i (\tau_j^2 - \tau_i^2)}, \quad s_j = \frac{4\tau_j^2(\tau_j^2 - 1)}{\prod_i (\tau_j^2 - \tau_i^2)}, \quad (i, j=0, 1, 2 \text{ and } i \neq j),$$

the equation (A7) can be written as

$$\begin{aligned} L_1(v, w) = & \frac{-i\tau^2}{16(1-\tau^2)} \left[\sum_{j=0}^2 p_j \int_0^1 \frac{(1-\eta^2)^{1/2} H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{\eta^2 - \tau_j^2} d\eta + \right. \\ & \left. + \sum_{j=0}^2 s_j \int_0^\tau \frac{(\tau^2 - \eta^2)^{1/2} H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{\eta^2 - \tau_j^2} d\eta \right] + \\ & + \pi i \tau^2 \left[\frac{(\eta^2 - 1)^{1/2} H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{Q_0'(\eta)} \right]_{\eta=\tau_0}, \quad (w > v). \end{aligned}$$

(A8)

APPENDIX - B

EVALUATION OF THE INTEGRALS ARISING IN THE EXPRESSION OF $L_1(v, w)$ GIVEN BY EQUATION (18) :

Expression of $L_1(v, w)$ is

$$\begin{aligned}
 L_1(v, w) = & \frac{-i\tau^2}{16(1-\tau^2)} \left[\sum_{j=0}^2 p_j \int_0^1 \frac{(1-\eta^2)^{1/2} H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{\eta^2 - \tau_j^2} d\eta + \right. \\
 & \left. + \sum_{j=0}^2 s_j \int_0^\tau \frac{(\tau^2 - \eta^2)^{1/2} H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{\eta^2 - \tau_j^2} d\eta \right] + \\
 & + \pi i \tau^2 \left[\frac{(\eta^2 - 1)^{1/2} H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v)}{Q_0(\eta)} \right]_{\eta=\tau_0}, \quad (w > v)
 \end{aligned}$$

(B1)

For low frequency i.e., for small values of the arguments $k_1 \eta w$ and $k_1 \eta v$ of $H_0^{(1)}(\)$ and $J_0(\)$ we get the following result

$$\begin{aligned}
 H_0^{(1)}(k_1 \eta w) J_0(k_1 \eta v) = & 1 + \frac{2i}{\pi} [\log(k_1 \eta w / 2) + \gamma] - \frac{i}{2\pi} \eta^2 (w^2 + v^2) k_1^2 \log k_1 + \\
 & + o(k_1^2)
 \end{aligned}$$

(B2)

where $\gamma = 0.5772157\dots$ is Euler's constant.

Substitution of this result in (B1) yields

$$L_1(v, w) = \frac{2}{\pi} \tau^2 \left[\left(\gamma + \log(k_1 w/2) - \frac{\pi i}{2} \right) M + N - \frac{P(w^2 + v^2)}{4} k_1^2 \log k_1 \right] + o(k_1^2)$$

, $w > v$ (B3)

where

$$M = \frac{1}{16(1-\tau^2)} \left[\sum_{j=0}^2 p_j \int_0^1 \frac{(1-\eta^2)^{1/2}}{(\eta^2 - \tau_j^2)} d\eta + \sum_{j=0}^2 s_j \int_0^\tau \frac{(\tau^2 - \eta^2)^{1/2}}{(\eta^2 - \tau_j^2)} d\eta \right] - \pi \left[\frac{(\eta^2 - 1)^{1/2}}{Q'_0(\eta)} \right]_{\eta=\tau_0}$$

(B4)

$$N = \frac{1}{16(1-\tau^2)} \left[\sum_{j=0}^2 p_j \int_0^1 \frac{(1-\eta^2)^{1/2} \log \eta}{(\eta^2 - \tau_j^2)} d\eta + \sum_{j=0}^2 s_j \int_0^\tau \frac{(\tau^2 - \eta^2)^{1/2} \log \eta}{(\eta^2 - \tau_j^2)} d\eta \right] - \pi \left[\frac{(\eta^2 - 1)^{1/2} \log \eta}{Q'_0(\eta)} \right]_{\eta=\tau_0}$$

(B5)

and

$$P = \frac{1}{16(1-\tau^2)} \left[\sum_{j=0}^2 p_j \int_0^1 \frac{\eta^2 (1-\eta^2)^{1/2}}{(\eta^2 - \tau_j^2)} d\eta + \sum_{j=0}^2 s_j \int_0^\tau \frac{\eta^2 (\tau^2 - \eta^2)^{1/2}}{(\eta^2 - \tau_j^2)} d\eta \right] - \pi \left[\frac{\eta^2 (\eta^2 - 1)^{1/2}}{Q'_0(\eta)} \right]_{\eta=\tau_0}$$

(B6)

The first integral of M can be written as

$$\sum_{j=0}^2 p_j \int_0^1 \frac{(1-\eta^2)^{1/2}}{(\eta^2-\tau_j^2)} d\eta = \sum_{j=1}^2 p_j \int_0^1 \frac{(1-\eta^2)^{1/2}}{(\eta^2-\tau_j^2)} d\eta - p_0 \int_0^1 \frac{(1-\eta^2)^{1/2}}{(\tau_0^2-\eta^2)} d\eta \quad (B7)$$

where $\tau_2 < \tau_1 < 1 < \tau < \tau_0$ and \int means the principal value of the integral.

Using the results

$$\int_0^1 \frac{(1-z^2)^{1/2}}{z^2-a^2} dz = -\frac{\pi}{2} \quad (a < 1)$$

and

$$\int_0^1 \frac{(1-z^2)^{1/2}}{a^2-z^2} dz = \frac{\pi}{2} - \frac{(a^2-1)^{1/2}}{a} \frac{\pi}{2} \quad (a > 1)$$

in (B7) we get,

$$\sum_{j=0}^2 p_j \int_0^1 \frac{(1-\eta^2)^{1/2}}{(\eta^2-\tau_j^2)} d\eta = -\frac{\pi}{2} \left[4 - \frac{p_0 (\tau_0^2-1)^{1/2}}{\tau_0} \right] \quad (B8)$$

Similarly, the second integral of M is found to be evaluated as

$$\sum_{j=0}^2 s_j \int_0^{\tau} \frac{(\tau^2-\eta^2)^{1/2}}{(\eta^2-\tau_j^2)} d\eta = -\frac{\pi}{2} \left[4 - \frac{s_0 (\tau_0^2-\tau^2)^{1/2}}{\tau_0} \right] \quad (B9)$$

Now, third term (denoted by D_o) of M given in (B4) can be written as

$$D_o = -\pi \left[\frac{(\eta^2 - 1)^{1/2}}{Q_o'(\eta)} \right]_{\eta=\tau_o}$$

$$= -\pi \left[\frac{((2\eta^2 - \tau^2)^2 + 4\eta^2(\eta^2 - \tau^2)^{1/2}(\eta^2 - 1)^{1/2})(\eta^2 - 1)^{1/2}}{\Delta_o'(\eta)} \right]_{\eta=\tau_o}$$

(B10)

where $\Delta_o'(\eta)$ is the derivative of $\Delta_o(\eta)$ with respect to η .

Let us consider

$$\frac{(\eta^2 - 1)^{1/2} (2\eta^2 - \tau^2)^2}{\Delta_o'(\eta)} = \frac{(\eta^2 - 1)^{1/2} (2\eta^2 - \tau^2)^2}{16(1 - \tau^2)(\eta^2 - \tau_o^2)(\eta^2 - \tau_1^2)(\eta^2 - \tau_2^2)}$$

$$= \frac{(\eta^2 - 1)^{1/2}}{16(1 - \tau^2)} \left[\frac{p_o}{\eta^2 - \tau_o^2} + \frac{p_1}{\eta^2 - \tau_1^2} + \frac{p_2}{\eta^2 - \tau_2^2} \right]$$

Multiplying both sides of the above equation by $(\eta^2 - \tau_o^2)/2\eta$ and putting $\eta = \tau_o$, we finally obtain,

$$\left[\frac{(\eta^2 - 1)^{1/2} (2\eta^2 - \tau^2)^{1/2}}{\Delta_o'(\eta)} \right]_{\eta=\tau_o} = \frac{1}{32(1 - \tau^2)} \left[\frac{p_o (\tau_o^2 - 1)^{1/2}}{\tau_o} \right] \quad (B11)$$

Similarly,

$$\left[\frac{4\eta^2 (\eta^2 - \tau^2)^{1/2} (\eta^2 - 1)}{\Delta_o(\eta)} \right]_{\eta=\tau_o} = \frac{1}{32(1-\tau^2)} \left[\frac{s_o (\tau_o^2 - \tau^2)^{1/2}}{\tau_o} \right] \quad (\text{B12})$$

Adding (B11) and (B12) D_o can be obtained as

$$D_o = \frac{-\pi}{32(1-\tau^2)} \left[\frac{p_o (\tau_o^2 - 1)^{1/2}}{\tau_o} + \frac{s_o (\tau_o^2 - \tau^2)^{1/2}}{\tau_o} \right] \quad (\text{B13})$$

Using (B8), (B9) and (B13) in (B4), M can be written as

$$M = - \frac{\pi}{4(1-\tau^2)} \quad (\text{B14})$$

Following the same procedure and using the results

$$\int_0^1 \frac{(1-z^2)^{1/2} \log z}{z^2 - a^2} dz = \frac{\pi}{2} \left[\frac{(1-a^2)^{1/2}}{a} \tan^{-1} \frac{(1-a^2)^{1/2}}{a} + \log 2 \right], \quad (a < 1)$$

$$\int_0^1 \frac{(1-z^2)^{1/2} \log z}{a^2 - z^2} dz = \frac{\pi}{2} \left[\frac{(a^2 - 1)^{1/2}}{a} \log \left\{ 1 + \frac{(a^2 - 1)^{1/2}}{a} \right\} - \log 2 \right], \quad (a > 1)$$

$$\int_0^1 \frac{z^2 (1-z^2)^{1/2}}{z^2 - a^2} dz = \frac{\pi}{2} \left(\frac{1}{2} - a^2 \right), \quad (a < 1)$$

$$\int_0^1 \frac{z^2(1-z^2)^{1/2}}{a^2-z^2} dz = -\frac{\pi}{2} \left(\frac{1}{2} - a^2 + a(a^2-1)^{1/2} \right), \quad (a>1)$$

$$\pi \left[\frac{(\eta^2-1)^{1/2} \log \eta}{Q_0(\eta)} \right]_{\eta=\tau_0} = \frac{\pi}{32(1-\tau^2)} \left[\frac{p_0(\tau_0^2-1)^{1/2}}{\tau_0} + \frac{s_0(\tau_0^2-\tau^2)^{1/2}}{\tau_0} \right] \log \tau_0$$

$$\pi \left[\frac{\eta^2(\eta^2-1)^{1/2}}{Q_0(\eta)} \right]_{\eta=\tau_0} = \frac{\pi}{32(1-\tau^2)} \left[p_0 \tau_0 (\tau_0^2-1)^{1/2} + s_0 \tau_0 (\tau_0^2-\tau^2)^{1/2} \right]$$

the values of N and P from (B5) and (B6) are evaluated and are given by

$$\begin{aligned} N = & \frac{\pi}{32(1-\tau^2)} \left[4 \log(4/\tau) + \sum_{j=1}^2 p_j \frac{\sqrt{(1-\tau_j^2)}}{\tau_j} \tan^{-1} \frac{\sqrt{(1-\tau_j^2)}}{\tau_j} \right. \\ & - p_0 \frac{\sqrt{(\tau_0^2-1)}}{\tau_0} \log \left\{ \tau_0 + \sqrt{(\tau_0^2-1)} \right\} + \sum_{j=1}^2 s_j \frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j^2} \tan^{-1} \frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j^2} \\ & \left. - s_0 \frac{\sqrt{(\tau_0^2-\tau^2)}}{\tau_0} \log \left\{ \frac{\tau_0 + \sqrt{(\tau_0^2-\tau^2)}}{\tau} \right\} \right], \end{aligned} \quad (B15)$$

$$\text{and } P = \frac{\pi}{32(1-\tau^2)} \left[\sum_{j=0}^2 p_j \left(\frac{1}{2} - \tau_j^2 \right) + \sum_{j=0}^2 s_j \left(\frac{\tau^2}{2} - \tau_j^2 \right) \right] \quad (B16)$$

DIFFRACTION OF ANTIPLANE SHEAR WAVE BY A PAIR OF PARALLEL RIGID STRIPS AT THE INTERFACE OF TWO BONDED DISSIMILAR ELASTIC MEDIA

1. INTRODUCTION

The problems of diffraction of elastic waves by a cracks or by inclusions are of considerable importance in view of their application in Seismology and Geophysics. The study becomes more relevant if the cracks or inclusions are located at the interface of layered media. Following Mal (1970) low frequency solution of the interaction of antiplane shear waves by a Griffith crack at the interface of two bonded dissimilar half spaces has been derived by Srivastava et al (1980). Bostrom (1987) adopted a different technique to solve the same problem. He followed a procedure similar to that of Krenk and Schmidt (1982) and reduced the problem to the solution of a Fredholm integral equation of first kind with the crack opening displacement as the unknown. The corresponding problem of diffraction of antiplane shear wave by a finite rigid strip at the interface has been treated by Palaiya and Majumder (1981). As regards the dynamic crack or strip problems, research has mainly been confined to the case of a single crack or a strip because of the severe mathematical complexity encountered in solutions for two or more cracks or

strips. Jain and Kanwal (1972,1972) however presented low frequency solution for the diffraction of elastic waves by coplanar Griffith cracks and also by two coplanar rigid strips located in an infinite isotropic homogeneous elastic medium. Recently Itu (1980) reconsidered the elastodynamic problem involving diffraction by two Griffith cracks in an infinite elastic medium and used a different technique to solve the problem. He expands the surface displacement in a series of functions which is automatically zero outside of the cracks and uses the Schmidt (1982) method to solve the resulting integral equation.

In the present paper we have considered the problem of diffraction of elastic waves by a pair of coplanar rigid strips between two homogeneous elastic half spaces for the case of antiplane strain. The resulting triple integral equation has been reduced to the solution of an integro differential equation, approximate solution of which has been obtained following the method of Lowengrub and Srivastava (1968). These solutions have been used to obtain approximate values of the displacement field and also the stress intensity factors at the edges of the strips. Making the distance between the inner edges of the strips tend to zero, the diffraction problem for a single rigid strip has been obtained. Even this result of the limiting case appears to have been presented here for the first time.

2. FORMULATION OF THE PROBLEM

We consider the problem of low frequency scattering of antiplane shear waves by two rigid strips situated parallel to each other, at the interface of two bonded dissimilar elastic half spaces. The strips are located in the region $-a \leq X \leq -b$, $b \leq X \leq a$, $Z=0$, $|Y| < \infty$ (fig.1). Normalizing all lengths with respect to a and putting $b/a=c$, we find that the rigid strips are defined by $c \leq |x| \leq 1$, $|y| < \infty$, $z=0$ at the interface of the half-spaces $z \leq 0$ and $z \geq 0$. Let an antiplane shear wave given by $v'_0 \exp(im_2 z - \omega t)$, where $m_2 = \omega a / c_2$ and v'_0 a constant, be incident normally on the strips. Henceforth the time factor $e^{-i\omega t}$ will be suppressed throughout the analysis. The non-vanishing components of displacement and stresses are

$$v = v(x, z)$$

$$\tau_{xy} = \tau_{xy}(x, z) \quad (1)$$

and $\tau_{yz} = \tau_{yz}(x, z)$

The boundary conditions are

$$v_1(x, 0) = v_2(x, 0) = -v_0, \quad c \leq |x| \leq 1 \quad (2)$$

$$v_1(x, 0) = v_2(x, 0), \quad |x| < c, \quad |x| > 1 \quad (3)$$

$$\tau_{yz}^{(1)}(x, 0) = \tau_{yz}^{(2)}(x, 0), \quad |x| < c, \quad |x| > 1 \quad (4)$$

where $v_0 = 2\mu_2 m_2 v'_0 / (\mu_1 m_1 + \mu_2 m_2)$.

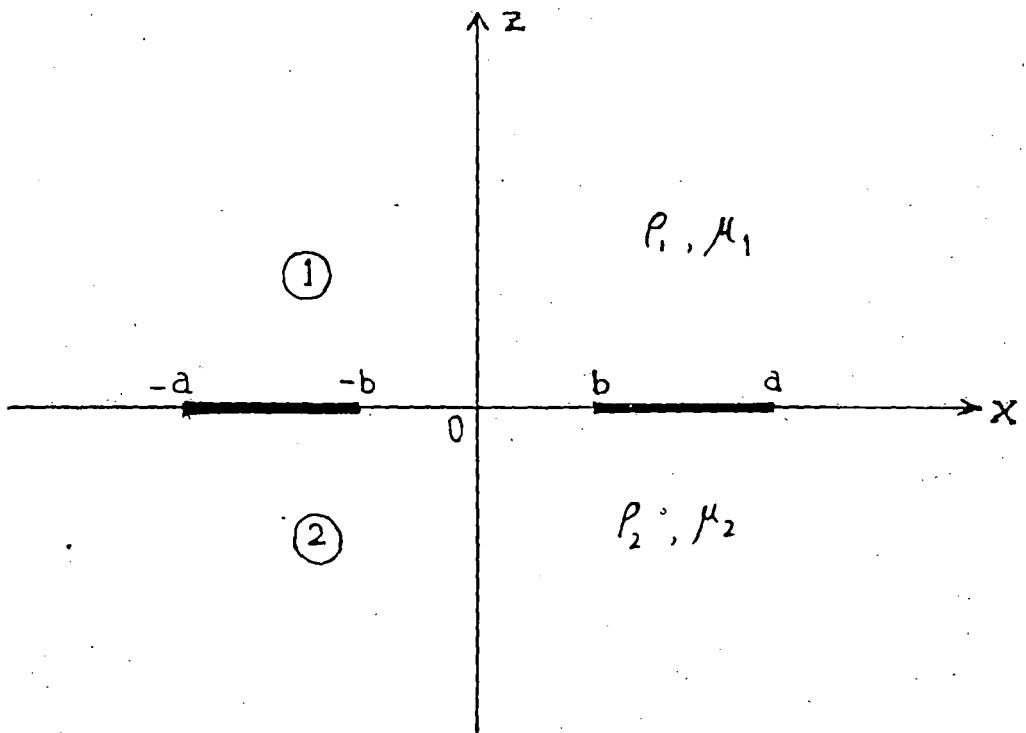


Fig. 1. The geometry of the strips.

The displacement v_j satisfies the equation

$$\frac{\partial^2 v_j}{\partial x^2} + \frac{\partial^2 v_j}{\partial y^2} + m_j^2 v_j = 0 \quad (5)$$

where $m_j = \omega a / c_j$ ($j=1,2$), c_j being the shear wave velocity. The suffices 1 and 2 are used to denote the values of the corresponding quantities in the upper and lower half-spaces respectively. Without any loss of generality we assume that $c_2 > c_1$. The solution of the equation (5) can be written as

$$v_j(x, z) = \int_0^{\infty} A_j(\xi) \exp(-\beta_j |z|) \cos \xi x \, d\xi \quad (6)$$

where

$$\begin{aligned} \beta_j &= (\xi^2 - m_j^2)^{1/2}, & \xi > m_j \\ &= -i(m_j^2 - \xi^2)^{1/2}, & \xi < m_j \quad (j=1,2), \end{aligned} \quad (7)$$

$A_1(\xi)$ and $A_2(\xi)$ are unknown functions to be determined from the boundary conditions.

Now the stress component τ_{yz} is given by

$$\tau_{yz}^{(j)}(x, z) = (-1)^j \mu_j \int_0^{\infty} \beta_j A_j(\xi) \exp(-\beta_j |z|) \cos \xi x \, d\xi \quad (8)$$

3. DERIVATION OF THE INTEGRAL EQUATIONS

The boundary conditions (2) and (3) imply that $v_1(x, 0) = v_2(x, 0)$ for

all values of x and thus from (6) we get

$$A_1(\xi) = A_2(\xi) \quad (9)$$

Again the boundary conditions (2) and (4) lead to the following integral equations :

$$\int_0^{\infty} A_1(\xi) \cos \xi x \, d\xi = -v_0, \quad c \leq |x| \leq 1 \quad (10)$$

$$\int_0^{\infty} (\mu_1 \beta_1 + \mu_2 \beta_2) A_1(\xi) \cos \xi x \, d\xi = 0, \quad |x| < c, \quad |x| > 1 \quad (11)$$

Putting $(\mu_1 \beta_1 + \mu_2 \beta_2) A_1(\xi) = P(\xi)$ (12)

the equations (10) and (11) transform into the following set of equations involving $P(\xi)$:

$$\int_0^{\infty} \frac{P(\xi)}{(\mu_1 \beta_1 + \mu_2 \beta_2)} \cos \xi x \, d\xi = -v_0, \quad c \leq |x| \leq 1 \quad (13)$$

$$\int_0^{\infty} P(\xi) \cos \xi x \, d\xi = 0, \quad |x| < c, \quad |x| > 1 \quad (14)$$

4. SOLUTION OF THE PROBLEM

Let us consider the solution of the integral equations (13) and (14) in the form

$$P(\xi) = \int_c^1 t f(t^2) \cos \xi t \, dt \quad (15)$$

where $f(t^2)$ is an unknown function to be determined.

The relation (14) is therefore satisfied automatically and the equation (13) becomes

$$\int_c^1 t f(t^2) \int_0^\infty \frac{\cos \xi x \cos \xi t}{(\mu_1 \beta_1 + \mu_2 \beta_2)} \, d\xi \, dt = -v_0, \quad c \leq |x| \leq 1 \quad (16)$$

Using the relation

$$\frac{\sin \xi x \sin \xi t}{\xi^2} = \int_0^x \int_0^t \frac{w v J_0(\xi w) J_0(\xi v) \, dv \, dw}{(x^2 - w^2)^{1/2} (t^2 - v^2)^{1/2}} \quad (17)$$

the above equation converts to the form

$$\frac{d}{dx} \int_c^1 t f(t^2) \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{w v L_1(v, w) \, dv \, dw \, dt}{(x^2 - w^2)^{1/2} (t^2 - v^2)^{1/2}} = -v_0, \quad c \leq |x| \leq 1 \quad (18)$$

$$\text{where } L_1(v, w) = \int_0^\infty \frac{J_0(v\xi) J_0(w\xi)}{(\mu_1 \beta_1 + \mu_2 \beta_2)} \, d\xi \quad (19)$$

By a simple contour integration technique (Srivastava et al, 1980)

$$L_1(v, w) = \frac{i}{\mu_1} \left[\int_0^1 \frac{J_0(m_2 \eta v) H_0^{(1)}(m_2 \eta w)}{(\tau^2 - \eta^2)^{1/2} + \mu(1 - \eta^2)^{1/2}} \, d\eta + \int_1^\tau \frac{(\tau^2 - \eta^2)^{1/2} J_0(m_2 \eta v) H_0^{(1)}(m_2 \eta w)}{\mu^2 (\eta^2 - 1) + (\tau^2 - \eta^2)} \, d\eta \right], \quad w > v \quad (20)$$

where $\tau = m_1/m_2$, $\mu = \mu_2/\mu_1$.

Substituting the series expansion of $J_0(\)$ and $H_0^{(1)}(\)$ the integrals arising in (20) have been evaluated assuming that $1 < \tau < \mu$.

We find after some algebraic manipulation,

$$L_1(v, w) = \frac{2}{\pi(\mu_1 + \mu_2)} \left[\left\{ \gamma + \log(m_2 w/2) - \frac{i\pi}{2} \right\} M + N - \frac{(w^2 + v^2)}{4} R m_2^2 \log m_2 \right] +$$

$$+ o(m_2^2), \quad w > v$$

$$= \frac{2}{\pi(\mu_1 + \mu_2)} \left[\left\{ \gamma + \log(m_2 v/2) - \frac{i\pi}{2} \right\} M + N - \frac{(w^2 + v^2)}{4} R m_2^2 \log m_2 \right] +$$

$$+ o(m^2), \quad w < v \quad (21)$$

where

$$M = -(1+\mu) \left[\int_0^1 \frac{d\xi}{(\tau^2 - \xi^2)^{1/2} + \mu(1 - \xi^2)^{1/2}} + \int_1^\tau \frac{(\tau^2 - \xi^2)^{1/2} d\xi}{\mu^2(\xi^2 - 1) + (\tau^2 - \xi^2)} \right] \quad (22)$$

$$N = -(1+\mu) \left[\int_0^1 \frac{-\log \xi \cdot d\xi}{(\tau^2 - \xi^2)^{1/2} + \mu(1 - \xi^2)^{1/2}} + \int_1^\tau \frac{(\tau^2 - \xi^2)^{1/2} \log \xi \cdot d\xi}{\mu^2(\xi^2 - 1) + (\tau^2 - \xi^2)} \right] \quad (23)$$

$$R = -(1+\mu) \left[\int_0^1 \frac{\xi^2 d\xi}{(\tau^2 - \xi^2)^{1/2} + \mu(1 - \xi^2)^{1/2}} + \int_1^\tau \frac{\xi^2 (\tau^2 - \xi^2)^{1/2} d\xi}{\mu^2(\xi^2 - 1) + (\tau^2 - \xi^2)} \right] \quad (24)$$

Now M can be written as

$$M = -(1+\mu) \left[-\mu \int_0^1 \frac{(1 - \xi^2)^{1/2} d\xi}{\mu^2(\xi^2 - 1) + (\tau^2 - \xi^2)} + \int_0^\tau \frac{(\tau^2 - \xi^2)^{1/2} d\xi}{\mu^2(\xi^2 - 1) + (\tau^2 - \xi^2)} \right]$$

$$= -(1+\mu) \left[-\frac{\mu}{(1-\mu^2)} \int_0^1 \frac{(1-\xi^2)^{1/2} d\xi}{\frac{\tau^2 - \mu^2}{1-\mu^2} - \xi^2} + \frac{1}{(1-\mu^2)} \int_0^1 \frac{(1-\eta^2)^{1/2} d\eta}{\frac{\tau^2 - \mu^2}{\tau^2(1-\mu^2)} - \eta^2} \right]$$

Using the result

$$\int_0^1 \frac{(1-z^2)^{1/2} dz}{a^2 - z^2} = \frac{\pi}{2} \left[1 - \frac{(a^2-1)^{1/2}}{a} \right] \quad (a>1)$$

M can be finally expressed as

$$M = -\frac{\pi}{2} \quad (25)$$

Similarly, using the results

$$\int_0^1 \frac{(1-z^2)^{1/2} \log z}{a^2 - z^2} dz = \frac{\pi}{2} \left[\frac{(a^2-1)^{1/2}}{a} \log \left(1 + \frac{(a^2-1)^{1/2}}{a} \right) - \log 2 \right] \quad (a>1)$$

$$\int_0^1 \frac{z^2 (1-z^2)^{1/2} dz}{a^2 - z^2} = -\frac{\pi}{2} \left[\frac{1}{2} - a^2 + a(a^2-1)^{1/2} \right] \quad (a>1)$$

in (23) and (24), N and R can be obtained as

$$N = \frac{-\pi}{2(\mu-1)} \left[\mu \frac{\sqrt{\tau^2-1}}{\sqrt{\mu^2-\tau^2}} \tan^{-1} \left\{ \frac{(\tau^2-1)^{1/2} (\mu^2-\tau^2)^{1/2}}{(\tau^2+\mu)} \right\} - \log \tau - (\mu-1) \log 2 \right] \quad (26)$$

$$R = -\frac{\pi(\tau^2+\mu)}{4(1+\mu)} \quad (27)$$

and $\gamma = 0.5772157\dots$ is Euler's constant.

Now differentiating both sides of the equation (16) with respect to x and making some rearrangements and finally using the result (17), we obtain,

$$x \int_c^1 \frac{tf(t^2)dt}{x^2-t^2} = \int_c^1 tf(t^2) \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{wvL_2(v,w)dvdw}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} dt, \quad c \leq |x| \leq 1 \quad (28)$$

where
$$L_2(v,w) = \int_0^\infty \xi H(\xi) J_0(\xi v) J_0(\xi w) d\xi \quad (29)$$

and
$$H(\xi) = \frac{\mu_1(\beta_1 - \xi) + \mu_2(\beta_2 - \xi)}{\mu_1\beta_1 + \mu_2\beta_2} \quad (30)$$

For small values of m_1 and m_2 , we use the contour integration technique mentioned above to evaluate the integral given by (29) as :

$$L_2(v,w) = -i(1+\mu)m_2^2 \left[\int_0^1 \frac{\eta^2 J_0(m_2 \eta v) H_0^{(1)}(m_2 \eta w)}{(\tau^2 - \eta^2)^{1/2} + \mu(1 - \eta^2)^{1/2}} d\eta + \int_1^\tau \frac{\eta^2 (\tau^2 - \eta^2)^{1/2} J_0(m_2 \eta v) H_0^{(1)}(m_2 \eta w)}{\mu^2 (\eta^2 - 1) + (\tau^2 - \eta^2)} d\eta \right], \quad w > v, \quad (31)$$

Following the similar process as done to derive the relation (21), (31) can be written as

$$L_2(v,w) = -\frac{2}{\pi} R m_2^2 \log m_2 + o(m_2^2) \quad (32)$$

where R is given by (27).

Let us consider the solution of (28) as

$$f(t^2) = f_0(t^2) + m^2 \log m_2 f_1(t^2) + o(m_2^2) \quad (33)$$

Substituting the above expression of $f(t^2)$ and the value of $L_2(v, w)$ in (28) and equating the coefficients of equal powers of m_2 we get

$$\int_c^1 \frac{t f_0(t^2)}{x^2 - t^2} dt = 0, \quad c \leq |x| \leq 1 \quad (34)$$

$$\text{and} \quad \int_c^1 \frac{t f_1(t^2)}{x^2 - t^2} dt = -\frac{2}{\pi} R \int_c^1 t f_0(t^2) dt, \quad c \leq |x| \leq 1 \quad (35)$$

From the paper of Srivastava and Lowengrub (1968) we know that the solution to the integral equation

$$\frac{2}{\pi} \int_a^b \frac{t h(t^2)}{t^2 - y^2} dt = p(y), \quad y \in (a, b)$$

is given by

$$h(t^2) = -\frac{1}{\pi} \left[\frac{t^2 - a^2}{b^2 - t^2} \right]^{1/2} \int_a^b \left[\frac{b^2 - y^2}{y^2 - a^2} \right]^{1/2} \frac{2yp(y) dy}{y^2 - t^2} + \frac{C}{(t^2 - a^2)^{1/2} (b^2 - t^2)^{1/2}}$$

where C is an arbitrary constant.

Applying the above result in the integral equations (34) and (35), the solutions $f_0(t^2)$ and $f_1(t^2)$ are obtained as

$$f_0(t^2) = \frac{D_1}{(1-t^2)^{1/2} (t^2-c^2)^{1/2}} \quad (36)$$

$$\text{and } f_1(t^2) = \frac{2}{\pi} RD_1 \left[\frac{t^2-c^2}{1-t^2} \right]^{1/2} + \frac{D_2}{(1-t^2)^{1/2} (t^2-c^2)^{1/2}} \quad (37)$$

where D_1 and D_2 are constants.

The constants are determined by putting the value of $L_1(v,w)$ from (21) and the value of $f(t^2)$ obtained from (33), (36) and (37) in the equation (18). Equating the coefficients of like powers of m_2 from both sides of the resulting equation we obtain

$$D_1 = \frac{-(\mu_1 + \mu_2)v_0}{\left[\left(\gamma + \log(m_2/2) - \frac{\pi i}{2} \right) M + N + M \log(1-c^2)^{1/2} \right]} \quad (38)$$

$$\text{and } D_2 = - \frac{D_1^2 R}{(\mu_1 + \mu_2)v_0} \left[\frac{1}{2}(c^2+1) + \frac{(\mu_1 + \mu_2)(1-c^2)v_0}{\pi D_1} \right] \quad (39)$$

Now, the displacement $v_1 = v_2 = v(x,z)$ in the plane $z=0$ is obtained from (6), (12) and (15) as

$$\begin{aligned} v(x,0) &= \int_0^\infty A_1(\xi) \cos(\xi x) d\xi \quad , \quad |x| > 1 \quad , \quad |x| < c \\ &= \frac{d}{dx} \int_c^1 t f(t^2) \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{wvL_1(v,w)dvdw dt}{(x^2-w^2)^{1/2} (t^2-v^2)^{1/2}} \quad , \quad |x| > 1 \quad , \quad |x| < c \end{aligned}$$

Substituting the values of $L_1(v,w)$ and $f(t^2)$ from (21) and (33) in

the above expression we get

$$\begin{aligned}
 v(x,0) &= \frac{2}{\pi(\mu_1 + \mu_2)} \frac{d}{dx} \int_c^1 t f_0(t^2) \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{vw}{(x^2 - w^2)^{1/2} (t^2 - v^2)^{1/2}} \times \\
 &\quad \times \left\{ \left[\gamma + \log(m_2 w/2) - \frac{i\pi}{2} \right] M + N \right\} dv dw dt + \\
 &+ \frac{2}{\pi(\mu_1 + \mu_2)} m_2^2 \log m_2 \left[\frac{d}{dx} \int_c^1 t f_1(t^2) \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{vw}{(x^2 - w^2)^{1/2} (t^2 - v^2)^{1/2}} \times \right. \\
 &\quad \times \left. \left\{ \left[\gamma + \log(m_2 w/2) - \frac{i\pi}{2} \right] M + N \right\} dv dw dt - \right. \\
 &\quad \left. - \frac{R}{4} \frac{d}{dx} \int_c^1 t f_0(t^2) \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{vw(v^2 + w^2) dv dw dt}{(x^2 - w^2)^{1/2} (t^2 - v^2)^{1/2}} \right] + o(m_2^2)
 \end{aligned}$$

Using (36) and (37) in the above equation and integrating term by term $v(x,0)$ can be finally obtained as

$$\begin{aligned}
 v(x,0) &= -v_0 + \frac{M}{(\mu_1 + \mu_2)} \left[D_1 + m_2^2 \log m_2 \left\{ D_2 + \frac{(1-c^2)RD_1}{\pi} \right\} \right] \sinh^{-1} \left(\frac{c^2 - x^2}{1-c^2} \right)^{1/2} - \\
 &\quad - \frac{MRD_1 m_2^2 \log m_2}{\pi(\mu_1 + \mu_2)} \left\{ (c^2 - x^2)(1-x^2) \right\}^{1/2} + o(m_2^2), \quad |x| < c \\
 &= -v_0, \quad c \leq |x| \leq 1 \\
 &= -v_0 + \frac{M}{(\mu_1 + \mu_2)} \left[D_1 + m_2^2 \log m_2 \left\{ D_2 + \frac{(1-c^2)RD_1}{\pi} \right\} \right] \sinh^{-1} \left(\frac{x^2 - 1}{1-c^2} \right)^{1/2} + \\
 &\quad + \frac{MRD_1 m_2^2 \log m_2}{\pi(\mu_1 + \mu_2)} \left\{ (x^2 - c^2)(x^2 - 1) \right\}^{1/2} + o(m_2^2), \quad |x| > 1
 \end{aligned}$$

(40)

The difference of the stress components on the lower and upper surfaces of the strips is found from equations (8), (12) and (15) and is given by

$$\begin{aligned}
 \tau_{yz}^{(2)}(x,0) - \tau_{yz}^{(1)}(x,0) &= \int_0^{\infty} (\mu_1 \beta_1 + \mu_2 \beta_2) A_1(\xi) \cos(\xi x) d\xi \\
 &= \int_c^1 t f(t^2) \int_0^{\infty} \cos(\xi x) \cos(\xi t) d\xi dt \\
 &= \frac{\pi}{2} \int_c^1 t f(t^2) \delta(x-t) dt \\
 &= \frac{\pi}{2} x f(x^2) \quad , \quad c \leq |x| \leq 1
 \end{aligned}$$

After putting the values of $f(x^2)$ from (33), (36), (37) and integrating, the difference of the stress components finally becomes

$$\begin{aligned}
 \tau_{yz}^{(2)}(x,0) - \tau_{yz}^{(1)}(x,0) &= \frac{\pi x}{2[(1-x^2)(x^2-c^2)]^{1/2}} \left[D_1 + m_2^2 \log m_2 \times \right. \\
 &\quad \left. \times \left\{ \frac{2}{\pi} R D_1 (x^2 - c^2) + D_2 \right\} \right] \quad (41)
 \end{aligned}$$

Now, putting $\mu_1 = \mu_2 = \mu_0$, $v_0 = 1$ and omitting $m_2^2 \log m_2$ order term we get from (40) and (41) the displacement and stress components for single medium as

$$v(x,0) = -1 - \frac{1}{2[q_2 + \log(1-c^2)^{1/2}]} \begin{cases} \sinh^{-1} \left(\frac{c^2 - x^2}{1-c^2} \right)^{1/2}, & |x| < c \\ 0, & c \leq |x| \leq 1 \\ \sinh^{-1} \left(\frac{x^2 - 1}{1-c^2} \right)^{1/2}, & |x| > 1 \end{cases}$$

and

$$\tau_{yz}^{(j)}(x,0) = \frac{\mp \mu_0 |x|}{[q_2 + \log(1-c^2)^{1/2}] [(1-x^2)(x^2-c^2)]^{1/2}} + o(m_2^2), \quad c \leq |x| \leq 1$$

$$\text{where } q_2 = \gamma + \log(m_2/4) - \frac{\pi i}{2},$$

which coincide with the results obtained by Jain and Kanwal (1972).

Now defining the stress intensity factors by the relations

$$K_1 = \text{Lt}_{x \rightarrow 1^-} (1-x)^{1/2} \left[\frac{\tau_{yz}^{(2)}(x,0) - \tau_{yz}^{(1)}(x,0)}{(\mu_1 + \mu_2) v_0} \right], \quad c|x| < 1$$

$$K_c = \text{Lt}_{x \rightarrow c^+} (x-c)^{1/2} \left[\frac{\tau_{yz}^{(2)}(x,0) - \tau_{yz}^{(1)}(x,0)}{(\mu_1 + \mu_2) v_0} \right], \quad c|x| < 1$$

we obtain from (41)

$$K_1 = \frac{\pi}{2\sqrt{2}(1-c^2)^{1/2}} \left[D_1 + m_2^2 \log m_2 \left\{ \frac{2}{\pi} R D_1 (1-c^2) + D_2 \right\} \right] \quad (42)$$

$$\text{and } K_c = \frac{\pi\sqrt{c}}{2\sqrt{2}(1-c^2)^{1/2}} \left[D_1 + m_2^2 \log m_2 D_2 \right] \quad (43)$$

Putting $c=0$ we get the displacement $v^0(x,0)$ and stress intensity factor K_1^0 from (40) and (41) for the single strip as :

$$v^0(x,0) = -v_0 + \frac{M}{(\mu_1 + \mu_2)} \left[D_1^0 + m_2^2 \log m_2 \left\{ D_2^0 + \frac{RD_1^0}{\pi} \right\} \right] \sinh^{-1}(x^2-1)^{1/2} +$$

$$+ \frac{|x| MRD_1^0 m_2^2 \log m_2}{\pi(\mu_1 + \mu_2)} (x^2-1)^{1/2} + o(m_2^2) \quad , \quad |x| > 1$$

$$= -v_0 \quad , \quad |x| \leq 1 \quad (44)$$

$$\text{and} \quad K_1^0 = \frac{\pi}{2\sqrt{2}} \left[D_1^0 + m_2^2 \log m_2 \left\{ \frac{2}{\pi} RD_1^0 + D_2^0 \right\} \right] \quad (45)$$

$$\text{where} \quad D_1^0 = \frac{-(\mu_1 + \mu_2) v_0}{\left[\left(\gamma + \log(m_2/2) - \frac{\pi i}{2} \right) M + N \right]} \quad (46)$$

$$\text{and} \quad D_2^0 = - \frac{(D_1^0)^2 R}{(\mu_1 + \mu_2) v_0} \left[\frac{1}{2} + \frac{(\mu_1 + \mu_2) v_0}{\pi D_1^0} \right] \quad (47)$$

5. NUMERICAL RESULTS

While calculating numerical results, the displacement and the stress intensity factors have been obtained for the following set of materials :

Aluminium	$\rho_1 = 2.7 \text{ gm/cm}^3$,	$\mu_1 = 2.63 \times 10^{11} \text{ dyne/cm}^2$
Wrought iron	$\rho_2 = 7.8 \text{ gm/cm}^3$,	$\mu_2 = 7.7 \times 10^{11} \text{ dyne/cm}^2$

The displacement field in the interface near about the rigid strips has been depicted by means of graphs. It is interesting to note that the magnitude of the real part of the displacement increases with the increase in the value of the wave number m_2 . The variation of the displacement with c , the separating distance between the strips has also been shown by means of graphs. Further the graphs of the real part of the stress intensity factors at both the edges of the strips versus dimensionless wave number m_2 for several values of c have been plotted to show the nature of the variation of the stress intensity factors with different parameters.

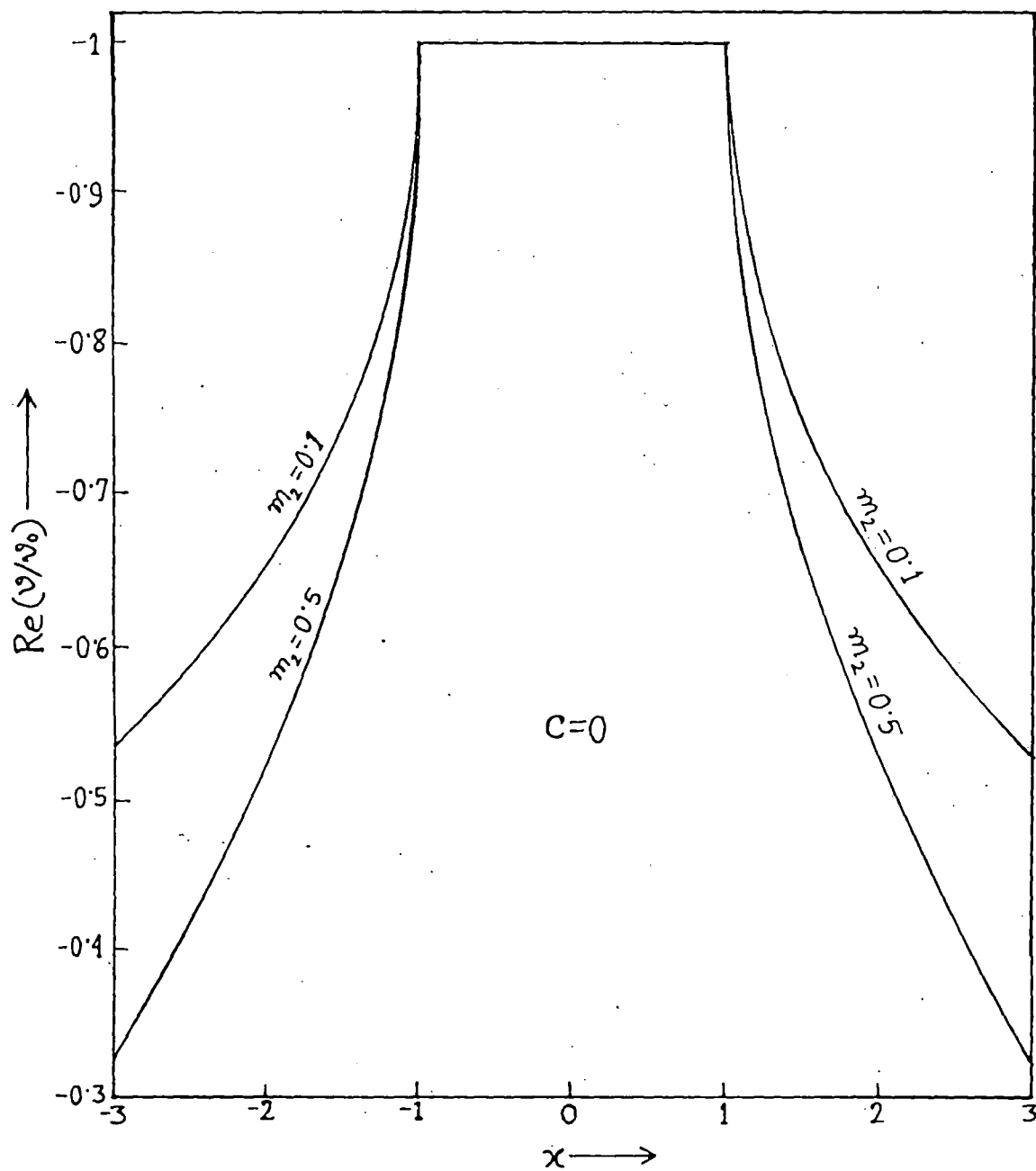


Fig. 2. $\text{Re}(v/v_0)$ versus distance for $c=0$ (single strip).

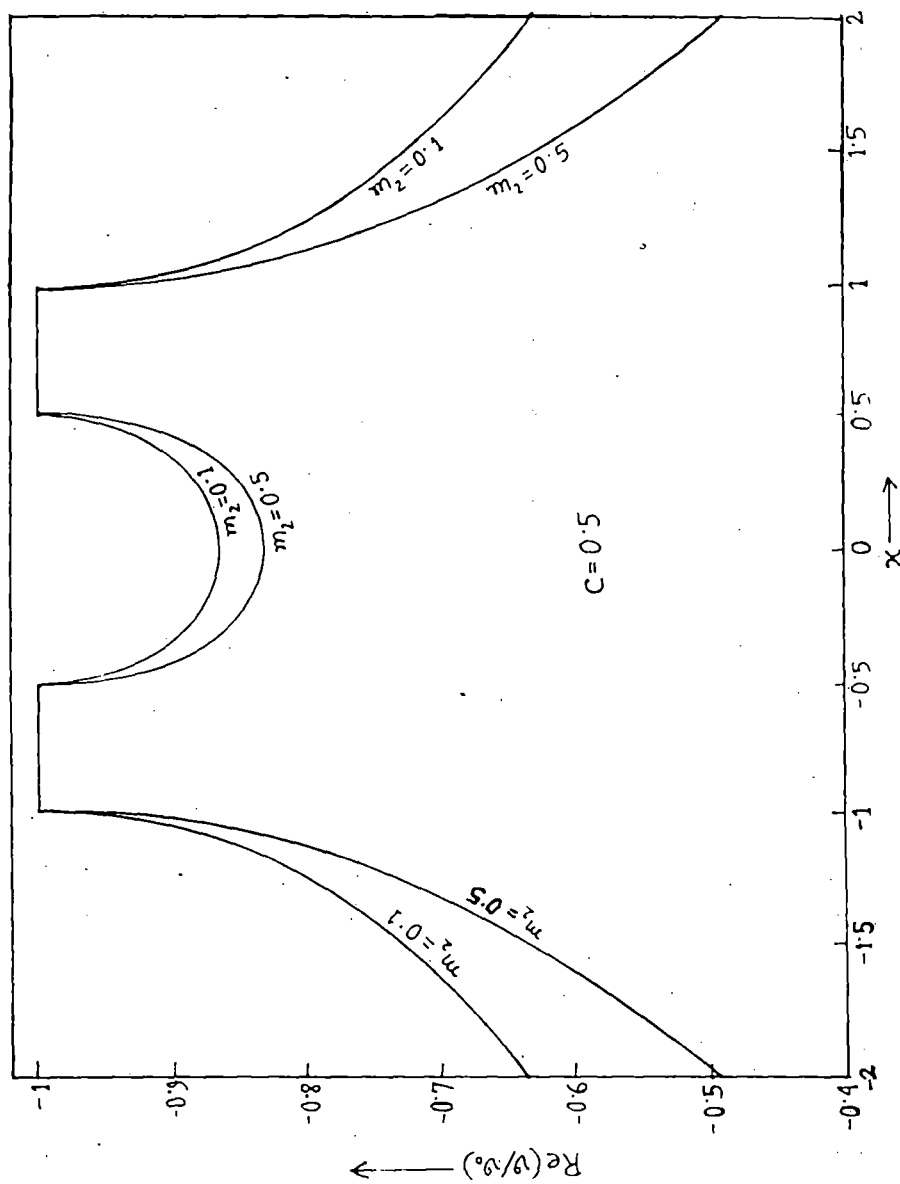


Fig. 3 $Re(v/v_0)$ versus distance for $c = 0.5$

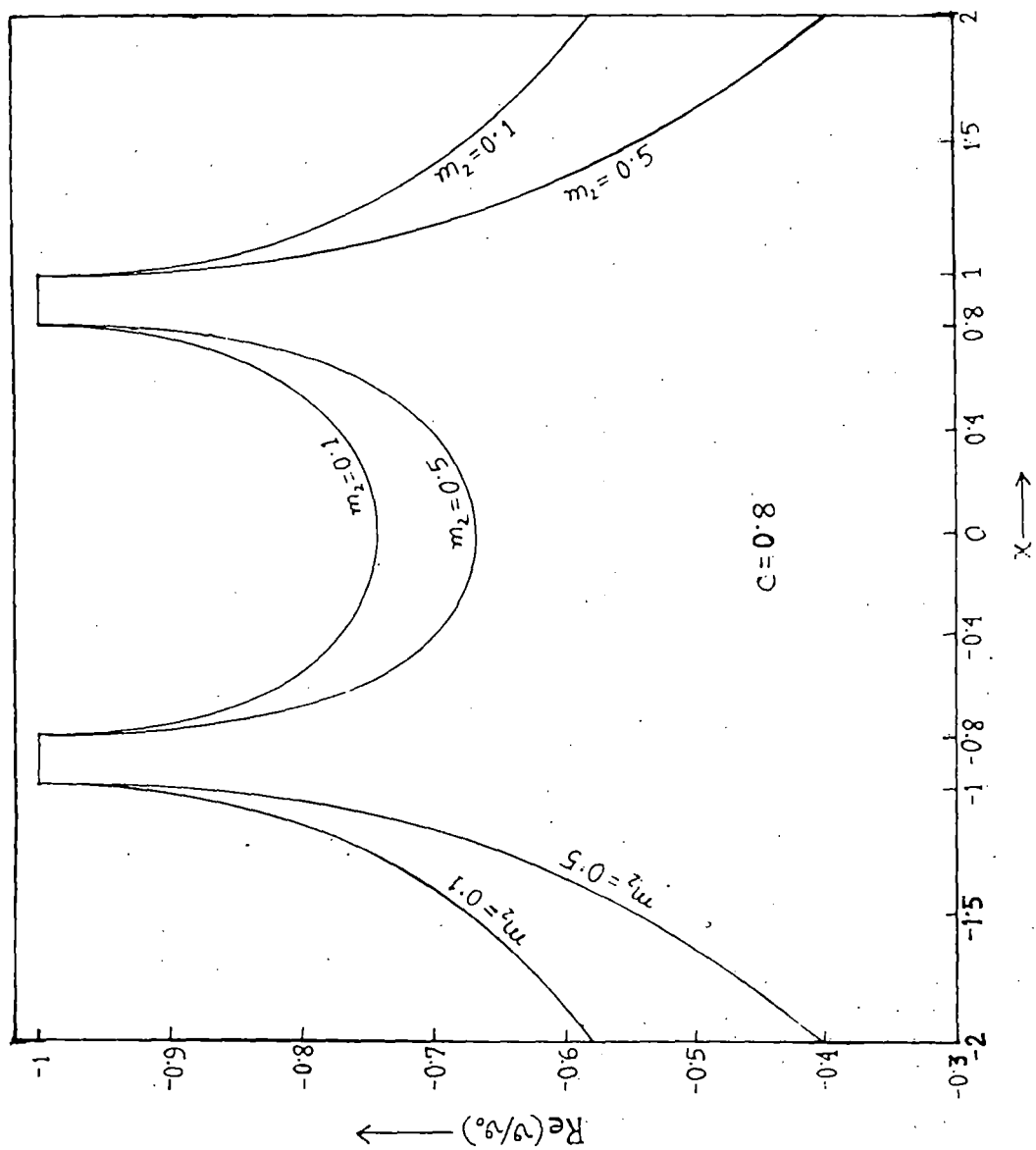


Fig 4 $\text{Re}(v/v_0)$ versus distance for $c=0.8$

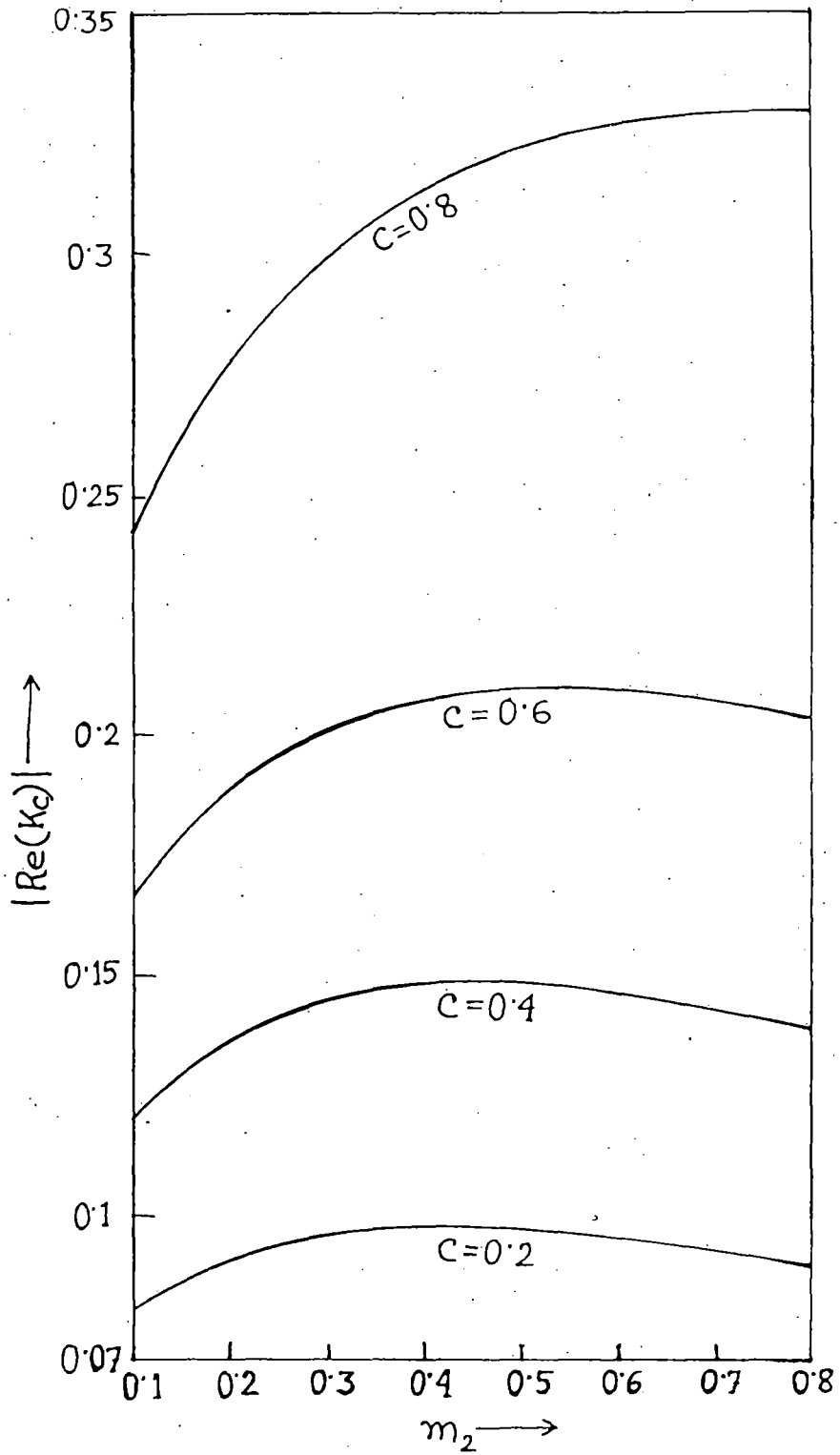


Fig. 5. $|\text{Re}(K_c)|$ versus m_2 , the nondimensional wave number.

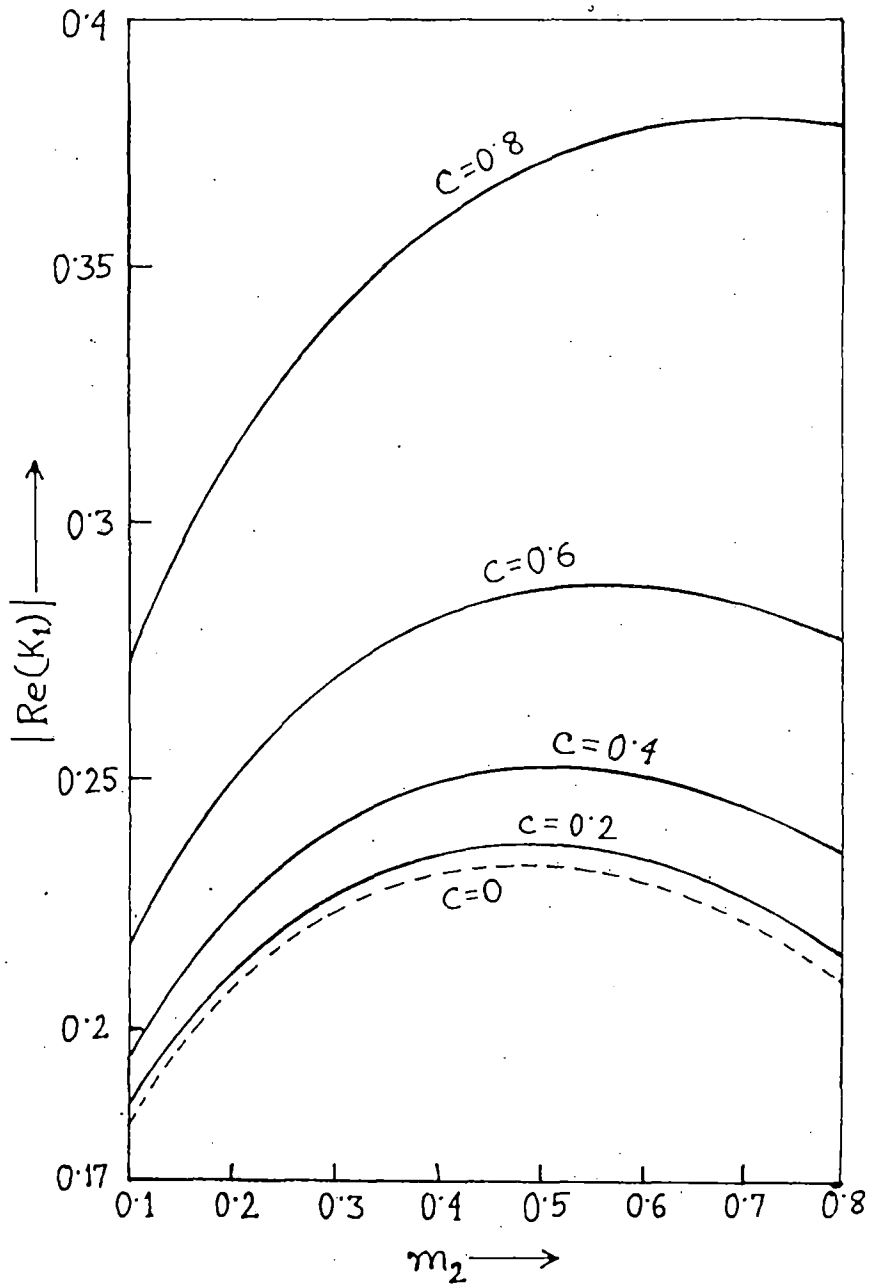


Fig. 6. $|Re(K_1)|$ versus m_2 , the nondimensional wave number.

ON STEADY MOTION OF FOUR RIGID STRIPS ON THE SURFACE OF A SEMI-INFINITE ELASTIC MEDIUM

1. INTRODUCTION

Recently, the problems of diffraction of elastic waves by cracks or inclusions have aroused attention in the field of fracture mechanics in view of their application in Seismology and Geophysics. Study of a single Griffith crack as well as two parallel and coplanar Griffith cracks have been made by Mal (1970) and Jain et al (1972), Itou (1980). The corresponding problems of diffraction by a single and two parallel rigid strips have been solved by Wickham (1977), Palaiya et al (1981) and Jain et al (1972), Mandal et al (1992) respectively. In most of the cases the problems have been solved by the integral equation technique. But the solution of interesting problems involving the scattering of elastic waves by more than two coplanar Griffith cracks or strips are still lacking. The statical problem of three coplanar cracks in an infinite transversely isotropic medium has been studied by Dhawan et al (1978). Using integral equation method and Hilbert transform the stress distribution and displacement have been derived in closed form. The interesting problem of interaction between a Griffith crack and two rigid inclusions has been discussed by Matysiak et al (1986). They considered that the crack

had been opened out to the prescribed shape and the normal stresses at the crack tips and at the ends of the inclusions have been analysed for the crack openings in the shape of an ellipse and that of two symmetrical parabolas.

In our case, we have considered the two dimensional problems of diffraction of elastic waves by four coplanar parallel rigid strips moving steadily on the free surface of a semi-infinite isotropic elastic medium. By Fourier transform the five part mixed boundary value problem has been reduced to the solution of a set of four integral equations. Following the technique, developed by Srivastava and Lowengrub (1970), the quadruple integral equations have been solved. The normal stress under the strips and displacement outside the strips are derived in closed form. The effect of stress intensity factors at the edges of the strips is shown by means of graphs. Also letting the strip velocity tend to zero the results for statical problem have been prescribed in this paper as a particular case.

2. FORMULATION OF THE PROBLEM

Consider a semi-infinite elastic medium on which four rigid strips are moving steadily in the X - direction with constant velocity v . Strips are assumed to be in smooth contact with the semi-infinite medium and the vertical displacement just under the strips are assumed to be prescribed. In terms of the displacement potentials, non vanishing displacement components are given by

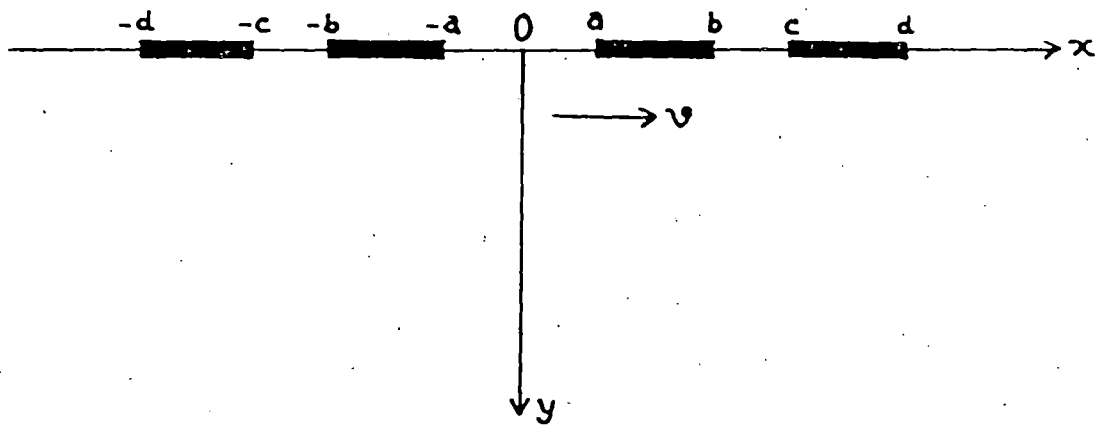


Fig. 1. Geometry of the strips.

$$u_1 = \frac{\partial \phi}{\partial X} - \frac{\partial \psi}{\partial Y} \quad , \quad u_2 = \frac{\partial \phi}{\partial Y} + \frac{\partial \psi}{\partial X} \quad (1)$$

where ϕ and ψ satisfy the following equations

$$\frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial Y^2} = \frac{1}{c_1^2} \frac{\partial^2 \phi}{\partial t^2} \quad (2)$$

$$\frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Y^2} = \frac{1}{c_2^2} \frac{\partial^2 \psi}{\partial t^2}$$

where $c_1^2 = \frac{\lambda + 2\mu}{\rho}$, $c_2^2 = \frac{\mu}{\rho}$.

It is convenient to shift the origin of co-ordinate at $X=vt$. New co-ordinate axes (x,y) are parallel to the fixed ones (X,Y) (fig.1).

Therefore putting $x=X-vt$, $y=Y$ we obtain from (1) to (2)

$$u_1 = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y} \quad , \quad u_2 = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x} \quad (3)$$

and

$$\beta_1^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad (4)$$

$$\beta_2^2 \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

where $\beta_1^2 = 1 - v^2/c_1^2$ and $\beta_2^2 = 1 - v^2/c_2^2$.

The location of the strips referred to moving system of co-ordinates are $a \leq |x| \leq b$, $c \leq |x| \leq d$, $y=0$, $|z| < \infty$.

In terms of ϕ and ψ the non vanishing stress components are

$$\tau_{xy}(x,y) = \mu \left[2 \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \right] \quad (5)$$

$$\tau_{yy}(x,y) = -\mu \left\{ (1+\beta_2^2) \frac{\partial^2 \phi}{\partial x^2} - 2 \frac{\partial^2 \psi}{\partial x \partial y} \right\}$$

The boundary conditions are

$$u_2 = v_0, \quad a \leq |x| \leq b, \quad c \leq |x| \leq d, \quad y=0 \quad (6)$$

$$\tau_{xy} = 0, \quad -\infty < x < \infty, \quad y=0 \quad (7)$$

$$\tau_{yy} = 0, \quad |x| < a, \quad b < |x| < c, \quad |x| > d \quad (8)$$

where v_0 is constant.

Solutions of the equations (4) are given by

$$\phi = \int_0^\infty A_1(\xi) e^{-\beta_1 \xi y} \cos \xi x \, d\xi \quad (9)$$

$$\psi = \int_0^\infty A_2(\xi) e^{-\beta_2 \xi y} \sin \xi x \, d\xi$$

where $A_1(\xi)$ and $A_2(\xi)$ are unknown functions to be determined from the boundary conditions.

From the boundary condition (7) we get

$$A_2(\xi) = \frac{2\beta_1}{(1+\beta_2^2)} A_1(\xi) \quad (10)$$

Now the displacement and stress components are

$$u_1(x, y) = -\int_0^\infty \xi \left[e^{-\beta_1 \xi y} - \frac{2\beta_1 \beta_2}{(1+\beta_2^2)} e^{-\beta_2 \xi y} \right] A_1(\xi) \sin \xi x \, d\xi \quad (11)$$

$$u_2(x, y) = \int_0^\infty \xi \left[-\beta_1 e^{-\beta_1 \xi y} + \frac{2\beta_1}{(1+\beta_2^2)} e^{-\beta_2 \xi y} \right] A_1(\xi) \cos \xi x \, d\xi \quad (12)$$

$$\tau_{xy}(x, y) = \mu \int_0^\infty 2\beta_1 \xi^2 \left[e^{-\beta_1 \xi y} - e^{-\beta_2 \xi y} \right] A_1(\xi) \sin \xi x \, d\xi \quad (13)$$

$$\tau_{yy}(x, y) = -\mu \int_0^\infty \xi^2 \left[-(1+\beta_2^2) e^{-\beta_1 \xi y} + \frac{4\beta_1 \beta_2}{(1+\beta_2^2)} e^{-\beta_2 \xi y} \right] A_1(\xi) \cos \xi x \, d\xi \quad (14)$$

Putting $A(\xi) = \xi^2 A_1(\xi)$ (15)

the boundary conditions (6) and (8) lead to the following quadruple integral equations (assuming that $v \neq v_R$, where v_R is the Rayleigh wave velocity).

$$\int_0^{\infty} A(\xi) \cos \xi x \, d\xi = 0 \quad , \quad |x| < a \quad (15)$$

$$\int_0^{\infty} \frac{A(\xi)}{\xi} \cos \xi x \, d\xi = p_0 \quad , \quad a \leq |x| \leq b \quad (16)$$

$$\int_0^{\infty} A(\xi) \cos \xi x \, d\xi = 0 \quad , \quad b < |x| < c \quad , \quad |x| > d \quad (17)$$

$$\int_0^{\infty} \frac{A(\xi)}{\xi} \cos \xi x \, d\xi = p_0 \quad , \quad c \leq |x| \leq d \quad (18)$$

where

$$p_0 = \frac{1 + \beta_2^2}{\beta_1 (1 - \beta_2^2)} v_0 \quad (19)$$

3. SOLUTION OF THE QUADRUPLE INTEGRAL EQUATIONS

Let us consider the solutions of the integral equations (15) - (18) in the form

$$A(\xi) = \int_a^b \frac{h(t^2)}{t} \{1 - \cos \xi t\} dt + \int_c^d \frac{g(u^2)}{u} \{1 - \cos \xi u\} du \quad (20)$$

where $h(t^2)$ and $g(u^2)$ are unknown functions.

This choice of $A(\xi)$ automatically satisfies the equations (15) and (17). Substituting the value of $A(\xi)$ from (20) into (16) and using the relation (Gradshteyn and Ryzhik, 1965)

$$\int_0^{\infty} \frac{\cos \xi y (1 - \cos \xi u)}{\xi} d\xi = \frac{1}{2} \log \left| 1 - \frac{u^2}{y^2} \right| \quad (21)$$

we obtain

$$\frac{1}{2} \int_a^b \frac{h(t^2)}{t} \log \left| 1 - \frac{t^2}{x^2} \right| dt + \frac{1}{2} \int_c^d \frac{g(u^2)}{u} \log \left| 1 - \frac{u^2}{x^2} \right| du = p_0, \quad a \leq |x| \leq b \quad (22)$$

On differentiation with respect to x , the above equation yields

$$\int_a^b \frac{th(t^2)}{x^2 - t^2} dt = - \int_c^d \frac{ug(u^2)}{x^2 - u^2} du, \quad a \leq |x| \leq b$$

from which applying Hilbert transform (Srivastava et al, 1970) we get $h(t^2)$ as

$$h(t^2) = - \frac{2}{\pi} \frac{(t^2 - a^2)^{1/2}}{(b^2 - t^2)} \int_c^d \frac{ug(u^2) (u^2 - b^2)^{1/2}}{u^2 - t^2 (u^2 - a^2)} du + \frac{D_1}{\sqrt{(t^2 - a^2)(b^2 - t^2)}} \quad (23)$$

, $a \leq t \leq b$

where D_1 is an arbitrary constant to be determined.

Next, using the relations (20) and (21) in (18) we obtain

$$\frac{1}{2} \int_a^b \frac{h(t^2)}{t} \log \left| 1 - \frac{t^2}{x^2} \right| dt + \frac{1}{2} \int_c^d \frac{g(u^2)}{u} \log \left| 1 - \frac{u^2}{x^2} \right| du = p_0, \quad c \leq |x| \leq d \quad (24)$$

Differentiating both sides of the above equation with respect to x and substituting the value of $h(t^2)$ from (23), we obtain

$$\int_c^d \frac{ug(u^2)}{x^2-u^2} du - \frac{2}{\pi} \int_a^b \frac{t}{x^2-t^2} \left(\frac{t^2-a^2}{b^2-t^2} \right)^{1/2} \int_c^d \frac{ug(u^2)}{u^2-t^2} \left(\frac{u^2-b^2}{u^2-a^2} \right)^{1/2} dudt +$$

$$+ D_1 \int_a^b \frac{t dt}{(x^2-t^2) \sqrt{(t^2-a^2)(b^2-t^2)}} = 0, \quad c \leq x \leq d.$$

Using the results

$$\int_a^b \frac{t dt}{(x^2-t^2) \sqrt{(t^2-a^2)(b^2-t^2)}} = \frac{\pi}{2} \frac{1}{\sqrt{(x^2-a^2)(x^2-b^2)}}$$

and

$$\int_a^b \frac{t}{(u^2-t^2)(x^2-t^2)} \left(\frac{t^2-a^2}{b^2-t^2} \right)^{1/2} dt$$

$$= \frac{\pi}{2} \frac{1}{(x^2-u^2)} \left[-\sqrt{\frac{(x^2-a^2)}{(x^2-b^2)}} + \sqrt{\frac{(u^2-a^2)}{(u^2-b^2)}} \right]$$

the above equation can be written as

$$\int_c^d \frac{ug(u^2)}{u^2-x^2} \left(\frac{u^2-b^2}{u^2-a^2} \right)^{1/2} du = \frac{\pi}{2} \frac{D_1}{(x^2-a^2)}, \quad c \leq u \leq d$$

which after Hilbert transform and on use of the result

$$\int_c^d \frac{x}{(x^2-u^2)(x^2-a^2)} \left(\frac{d^2-x^2}{x^2-c^2} \right)^{1/2} dx = -\frac{\pi}{2} \frac{1}{(u^2-a^2)} \sqrt{\frac{(d^2-a^2)}{(c^2-a^2)}},$$

yields

$$g(u^2) = D_1 \left[\frac{d^2 - a^2}{c^2 - a^2} \right]^{1/2} \left[\frac{u^2 - c^2}{d^2 - u^2} \right]^{1/2} \frac{1}{\sqrt{(u^2 - b^2)(u^2 - a^2)}} +$$

$$+ \frac{D_2}{\sqrt{(u^2 - c^2)(d^2 - u^2)}} \left[\frac{u^2 - a^2}{u^2 - b^2} \right]^{1/2}, \quad c \leq u \leq d \quad (25)$$

where D_2 is another arbitrary constant.

Substituting the value of $g(u^2)$ from (25) in (23), $h(t^2)$ takes the form

$$h(t^2) = -\frac{2D_1}{\pi} \left[\frac{t^2 - a^2}{b^2 - t^2} \right]^{1/2} \left[\frac{d^2 - a^2}{c^2 - a^2} \right]^{1/2} \int_c^d \frac{u}{(u^2 - t^2)(u^2 - a^2)} \left[\frac{u^2 - c^2}{d^2 - u^2} \right]^{1/2} du -$$

$$- \frac{2D_2}{\pi} \left[\frac{t^2 - a^2}{b^2 - t^2} \right]^{1/2} \int_c^d \frac{u du}{(u^2 - t^2)\sqrt{(u^2 - c^2)(d^2 - u^2)}} + \frac{D_1}{\sqrt{(t^2 - a^2)(b^2 - t^2)}} \quad a \leq t \leq b$$

Again, using the results

$$\int_c^d \frac{u}{(u^2 - t^2)(u^2 - a^2)} \left[\frac{u^2 - c^2}{d^2 - u^2} \right]^{1/2} du$$

$$= \frac{\pi}{2} \frac{1}{(t^2 - a^2)} \left[-\sqrt{\frac{(c^2 - t^2)}{(d^2 - t^2)}} + \sqrt{\frac{(c^2 - a^2)}{(d^2 - a^2)}} \right]$$

and

$$\int_c^d \frac{u du}{(u^2 - t^2)\sqrt{(u^2 - c^2)(d^2 - u^2)}} = \frac{\pi}{2} \frac{1}{\sqrt{(d^2 - t^2)(c^2 - t^2)}}$$

in the above expression, $h(t^2)$ can be found out in the form

$$h(t^2) = D_1 \left(\frac{d^2 - a^2}{c^2 - a^2} \right)^{1/2} \left(\frac{c^2 - t^2}{d^2 - t^2} \right)^{1/2} \frac{1}{\sqrt{(t^2 - a^2)(b^2 - t^2)}} -$$

$$- \frac{D_2}{\sqrt{(d^2 - t^2)(c^2 - t^2)}} \left(\frac{t^2 - a^2}{b^2 - t^2} \right)^{1/2}, \quad a \leq t \leq b \quad (26)$$

Now in order to determine the unknown constants D_1 and D_2 , occurring in the expression of $g(u^2)$ and $h(t^2)$ given by (25) and (26) respectively, we multiply the equation (22) by

$$\frac{x}{\sqrt{(x^2 - a^2)(b^2 - x^2)}} \text{ and integrate with respect to } x \text{ from } a \text{ to } b \text{ and}$$

then using the result

$$\int_a^b \frac{x}{\sqrt{(x^2 - a^2)(b^2 - x^2)}} \log \left| 1 - \frac{z^2}{x^2} \right| dx = \begin{cases} \pi \log \left[\frac{\sqrt{(a^2 - z^2)} + \sqrt{(b^2 - z^2)}}{a + b} \right], & 0 < z < a \\ \frac{\pi}{2} \log \left(\frac{b-a}{b+a} \right), & a \leq z \leq b \\ \pi \log \left[\frac{\sqrt{(z^2 - a^2)} + \sqrt{(z^2 - b^2)}}{a + b} \right], & z > b \end{cases} \quad (27)$$

we obtain,

$$\frac{\pi}{2} \log \left(\frac{b-a}{b+a} \right) \int_a^b \frac{h(t^2)}{t} dt + \pi \int_c^d \frac{g(u^2)}{u} \log \left[\frac{\sqrt{(u^2 - a^2)} + \sqrt{(u^2 - b^2)}}{a + b} \right] du = \pi p_0$$

which after substituting the values of $h(t^2)$ and $g(u^2)$ from (26) and (25) finally takes the form

$$D_1 X_1 + D_2 X_2 = p_0 \quad (28)$$

where

$$X_1 = \frac{1}{2} \log \left(\frac{b-a}{b+a} \right) \frac{(c^2-b^2)}{b^2(c^2-a^2)} \left(\frac{d^2-a^2}{d^2-b^2} \right)^{1/2} \Pi \left(\frac{\pi}{2}, \frac{c^2(b^2-a^2)}{b^2(c^2-a^2)}, r \right) + \left(\frac{d^2-a^2}{c^2-a^2} \right)^{1/2} M_1 \quad (29)$$

$$X_2 = -\frac{1}{2} \log \left(\frac{b-a}{b+a} \right) \frac{1}{\sqrt{(d^2-b^2)(c^2-a^2)}} \left\{ \left(1 - \frac{a^2}{c^2} \right) F \left(\frac{\pi}{2}, r \right) - \frac{a^2(c^2-b^2)}{b^2 c^2} \times \right. \\ \left. \times \Pi \left(\frac{\pi}{2}, \frac{c^2(b^2-a^2)}{b^2(c^2-a^2)}, r \right) \right\} + M_2 \quad (30)$$

$$M_1 = \int_c^d \frac{1}{u} \log \left(\frac{\sqrt{(u^2-a^2)} + \sqrt{(u^2-b^2)}}{a+b} \right) \left(\frac{u^2-c^2}{d^2-u^2} \right)^{1/2} \frac{du}{\sqrt{(u^2-b^2)(u^2-a^2)}} \quad (31)$$

$$M_2 = \int_c^d \frac{1}{u} \log \left(\frac{\sqrt{(u^2-a^2)} + \sqrt{(u^2-b^2)}}{a+b} \right) \left(\frac{u^2-a^2}{u^2-b^2} \right)^{1/2} \frac{du}{\sqrt{(u^2-c^2)(d^2-u^2)}} \quad (32)$$

F and Π are elliptic integrals of first and third kind respectively

and
$$r = \sqrt{\frac{(d^2-c^2)(b^2-a^2)}{(d^2-b^2)(c^2-a^2)}}$$

Next multiplying the equation (24) by $\frac{x}{\sqrt{(d^2-x^2)(x^2-c^2)}}$ and

integrating with respect to x from c to d and then following the

same procedure as that while deriving equation (28), we obtain

$$D_1 X_3 + D_2 X_4 = p_0 \quad (33)$$

where

$$X_3 = \frac{1}{2} \left(\frac{d^2 - a^2}{c^2 - a^2} \right)^{1/2} \log \left(\frac{d-c}{d+c} \right) \frac{(c^2 - b^2)}{b^2 \sqrt{(d^2 - b^2)(c^2 - a^2)}} \left\{ \Pi \left(\frac{\pi}{2}, \frac{b^2(d^2 - c^2)}{c^2(d^2 - b^2)}, r \right) - F \left(\frac{\pi}{2}, r \right) \right\} + \left(\frac{d^2 - a^2}{c^2 - a^2} \right)^{1/2} M_3 \quad (34)$$

$$X_4 = \frac{1}{2} \log \left(\frac{d-c}{d+c} \right) \frac{1}{b^2 \sqrt{(d^2 - b^2)(c^2 - a^2)}} \left\{ (b^2 - a^2) F \left(\frac{\pi}{2}, r \right) + \frac{a^2(c^2 - b^2)}{c^2} \times \right. \\ \left. \times \Pi \left(\frac{\pi}{2}, \frac{b^2(d^2 - c^2)}{c^2(d^2 - b^2)}, r \right) \right\} - M_4 \quad (35)$$

$$M_3 = \int_a^b \frac{1}{t} \log \left(\frac{\sqrt{(c^2 - t^2)} + \sqrt{(d^2 - t^2)}}{d + c} \right) \left(\frac{c^2 - t^2}{d^2 - t^2} \right)^{1/2} \frac{dt}{\sqrt{(t^2 - a^2)(b^2 - t^2)}} \quad (36)$$

$$M_4 = \int_a^b \frac{1}{t} \log \left(\frac{\sqrt{(c^2 - t^2)} + \sqrt{(d^2 - t^2)}}{d + c} \right) \left(\frac{t^2 - a^2}{b^2 - t^2} \right)^{1/2} \frac{dt}{\sqrt{(d^2 - t^2)(c^2 - t^2)}} \quad (37)$$

From equations (28) and (33), D_1 and D_2 can be found to be

$$D_1 = \frac{p_0 (X_4 - X_2)}{(X_1 X_4 - X_2 X_3)} \quad , \quad D_2 = \frac{p_0 (X_3 - X_1)}{(X_2 X_3 - X_1 X_4)} \quad (38)$$

4. STRESS INTENSITY FACTORS AND DISPLACEMENT

The normal stress $\tau_{yy}(x,y)$ in the plane $y=0$ just below the strips can be found from the relation (14), (15) and (20) as

$$\begin{aligned} \tau_{yy}(x,0) &= \mu \frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{1+\beta_2^2} \frac{d}{dx} \left[\int_a^b \frac{h(t^2)}{t} \int_0^\infty \frac{\sin \xi x}{\xi} \{1 - \cos \xi t\} d\xi dt + \right. \\ &\quad \left. + \int_c^d \frac{g(u^2)}{u} \int_0^\infty \frac{\sin \xi x}{\xi} \{1 - \cos \xi u\} d\xi du \right] \quad , \quad a \leq x \leq b \quad , \quad c \leq x \leq d \\ &= 2\mu \frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{1+\beta_2^2} \frac{d}{dx} \left[\int_a^b \frac{h(t^2)}{t} \int_0^\infty \frac{\sin(\xi x) \sin^2(\xi t/2)}{\xi} d\xi dt + \right. \\ &\quad \left. + \int_c^d \frac{g(u^2)}{u} \int_0^\infty \frac{\sin(\xi x) \sin^2(\xi u/2)}{\xi} d\xi du \right] \quad , \quad a \leq x \leq b \quad , \quad c \leq x \leq d \end{aligned}$$

Using the result (Gradsteyn and Ryzhik, 1965)

$$\begin{aligned} \int_0^\infty \frac{\sin^2(ax) \sin(bx)}{x} &= \frac{\pi}{2} \quad , \quad 0 < b < 2a \\ &= \frac{\pi}{8} \quad , \quad b = 2a \\ &= 0 \quad , \quad b > 2a \quad , \end{aligned}$$

$\tau_{yy}(x,0)$ can be found to be given by

$$\tau_{yy}(x,0) = -\frac{\mu\pi}{2} \frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{1+\beta_2^2} \frac{h(x^2)}{x}, \quad a \leq x \leq b$$

$$= -\frac{\mu\pi}{2} \frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{1+\beta_2^2} \frac{g(x^2)}{x}, \quad c \leq x \leq d$$

Putting the values of $h(x^2)$ and $g(x^2)$ from (26) and (25), the normal stress $\tau_{yy}(x,0)$ finally can be obtained as

$$\tau_{yy}(x,0) = -\frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{(1+\beta_2^2)} \frac{\pi\mu}{2x \sqrt{(x^2-a^2)(b^2-x^2)}} \left\{ D_1 \left(\frac{d^2-a^2}{c^2-a^2} \right)^{1/2} \times \right.$$

$$\left. \times \left(\frac{c^2-x^2}{d^2-x^2} \right)^{1/2} - \frac{D_2 (x^2-a^2)}{\sqrt{(d^2-x^2)(c^2-x^2)}} \right\}, \quad a \leq |x| \leq b \quad (39)$$

and

$$\tau_{yy}(x,0) = -\frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{(1+\beta_2^2)} \frac{\pi\mu}{2x \sqrt{(x^2-c^2)(d^2-x^2)}} \left\{ D_1 \left(\frac{d^2-a^2}{c^2-a^2} \right)^{1/2} \times \right.$$

$$\left. \times \frac{(x^2-c^2)}{\sqrt{(x^2-b^2)(x^2-a^2)}} + D_2 \left(\frac{x^2-a^2}{x^2-b^2} \right)^{1/2} \right\}, \quad c \leq |x| \leq d \quad (40)$$

Now the stress intensity factors at the edges of the strips can be defined as

$$K_a = \text{Lt}_{x \rightarrow a+0} \left| \frac{d \tau_{yy}(x, 0)}{\pi \mu v_0} \left(\frac{x-a}{d} \right)^{1/2} \right|$$

$$K_b = \text{Lt}_{x \rightarrow b-0} \left| \frac{d \tau_{yy}(x, 0)}{\pi \mu v_0} \left(\frac{b-x}{d} \right)^{1/2} \right|$$

$$K_c = \text{Lt}_{x \rightarrow c+0} \left| \frac{d \tau_{yy}(x, 0)}{\pi \mu v_0} \left(\frac{x-c}{d} \right)^{1/2} \right|$$

and

$$K_d = \text{Lt}_{x \rightarrow d-0} \left| \frac{d \tau_{yy}(x, 0)}{\pi \mu v_0} \left(\frac{d-x}{d} \right)^{1/2} \right|$$

which can be expressed as

$$K_a = \frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{\beta_1(1-\beta_2^2)} \frac{(d)^{1/2}}{(2a)^{3/2}(b^2-a^2)^{1/2}} \frac{(X_4 - X_2)}{(X_1 X_4 - X_2 X_3)} \quad (41)$$

$$K_b = \frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{\beta_1(1-\beta_2^2)} \frac{(d)^{1/2}}{(2b)^{3/2}(b^2-a^2)^{1/2}} \left\{ \left(\frac{d^2-a^2}{c^2-a^2} \right)^{1/2} \left(\frac{c^2-b^2}{d^2-b^2} \right)^{1/2} (X_4 - X_2) + \right. \\ \left. + \frac{(b^2-a^2)(X_3 - X_1)}{\sqrt{(d^2-b^2)(c^2-b^2)}} \right\} \frac{1}{(X_1 X_4 - X_2 X_3)} \quad (42)$$

$$K_c = \frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{\beta_1(1-\beta_2^2)} \frac{(d)^{1/2}}{(2c)^{3/2}(d^2-c^2)^{1/2}} \left(\frac{c^2-a^2}{c^2-b^2} \right)^{1/2} \frac{(X_3-X_1)}{(X_2X_3-X_4X_1)} \quad (43)$$

$$K_d = \frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{\beta_1(1-\beta_2^2)} \frac{(d)^{1/2}}{(2d)^{3/2}(d^2-c^2)^{1/2}} \left\{ \frac{(d^2-c^2)(X_4-X_2)}{\sqrt{(d^2-b^2)(c^2-a^2)}} - \left(\frac{d^2-a^2}{d^2-b^2} \right)^{1/2} (X_3-X_1) \right\} \frac{1}{(X_1X_4-X_2X_3)} \quad (44)$$

The vertical displacement $u_2(x, y)$ in the plane $y=0$ outside the strips is obtained from (12), (15), (20) and (26) and is given by the expression

$$\begin{aligned} u_2(x, 0) = & \frac{\beta_1}{2} \left(\frac{1-\beta_2^2}{1+\beta_2^2} \right) \left[\int_a^b \frac{1}{t} \log \left| 1 - \frac{t^2}{x^2} \right| \left\{ D_1 \left(\frac{d^2-a^2}{c^2-a^2} \right)^{1/2} \left(\frac{c^2-t^2}{d^2-t^2} \right)^{1/2} \right. \right. \\ & \times \frac{1}{\sqrt{(t^2-a^2)(b^2-t^2)}} - \left. \left. \frac{D_2}{\sqrt{(d^2-t^2)(c^2-t^2)}} \left(\frac{t^2-a^2}{b^2-t^2} \right)^{1/2} \right\} dt + \right. \\ & + \int_c^d \frac{1}{u} \log \left| 1 - \frac{u^2}{x^2} \right| \left\{ D_1 \left(\frac{d^2-a^2}{c^2-a^2} \right)^{1/2} \left(\frac{u^2-c^2}{d^2-u^2} \right)^{1/2} \frac{1}{\sqrt{(u^2-b^2)(u^2-a^2)}} \right. \\ & \left. \left. + \frac{D_2}{\sqrt{(u^2-c^2)(d^2-u^2)}} \left(\frac{u^2-a^2}{u^2-b^2} \right)^{1/2} \right\} du \right], \quad 0 < x < a, \quad b < x < c, \quad x > d \end{aligned}$$

Now letting $v \rightarrow 0$ we can obtain expressions of normal stress $\tau_{yy}^{(0)}(x,0)$ and displacement $u_2^{(0)}(x,0)$ from (39), (40) and (45) corresponding to the statical case as

$$\begin{aligned} \tau_{yy}^{(0)}(x,0) &= -\frac{\pi\mu}{x} \left(\frac{C_1^2}{C_2^2} - 1 \right) \frac{1}{\sqrt{(x^2-a^2)(b^2-x^2)}} \left\{ C_1 \left(\frac{d^2-a^2}{c^2-a^2} \right)^{1/2} \times \right. \\ &\quad \times \left. \left(\frac{c^2-x^2}{d^2-x^2} \right)^{1/2} - \frac{C_2(x^2-a^2)}{\sqrt{(d^2-x^2)(c^2-x^2)}} \right\}, \quad a \leq |x| \leq b \\ &= -\frac{\pi\mu}{x} \left(\frac{C_1^2}{C_2^2} - 1 \right) \frac{1}{\sqrt{(x^2-c^2)(d^2-x^2)}} \left\{ C_1 \left(\frac{d^2-a^2}{c^2-a^2} \right)^{1/2} \times \right. \\ &\quad \times \left. \frac{(x^2-c^2)}{\sqrt{(x^2-b^2)(x^2-a^2)}} + C_2 \left(\frac{x^2-a^2}{x^2-b^2} \right)^{1/2} \right\}, \quad c \leq |x| \leq d \end{aligned} \quad (46)$$

$$\begin{aligned} u_2^{(0)}(x,0) &= -\frac{C_1^2}{2C_2^2} \left[\int_a^b \frac{1}{t} \log \left| 1 - \frac{t^2}{x^2} \right| \left\{ C_1 \left(\frac{d^2-a^2}{c^2-a^2} \right)^{1/2} \left(\frac{c^2-t^2}{d^2-t^2} \right)^{1/2} \times \right. \right. \\ &\quad \times \left. \left. \frac{1}{\sqrt{(t^2-a^2)(b^2-t^2)}} - \frac{C_2}{\sqrt{(d^2-t^2)(c^2-t^2)}} \left(\frac{t^2-a^2}{b^2-t^2} \right)^{1/2} \right\} dt + \right. \end{aligned}$$

$$\begin{aligned}
& + \int_c^d \frac{1}{u} \log \left| 1 - \frac{u^2}{x^2} \right| \left\{ C_1 \left(\frac{d^2 - a^2}{c^2 - a^2} \right)^{1/2} \left(\frac{u^2 - c^2}{d^2 - u^2} \right)^{1/2} \frac{1}{\sqrt{(u^2 - b^2)(u^2 - a^2)}} + \right. \\
& \left. + \frac{C_2}{\sqrt{(u^2 - c^2)(d^2 - u^2)}} \left(\frac{u^2 - a^2}{u^2 - b^2} \right)^{1/2} \right\} du, \quad 0 < x < a, \quad b < x < c, \quad x > d \quad (47)
\end{aligned}$$

where C_1 and C_2 are given by D_1 and D_2 replacing p_0 by $-(c_2^2/c_1^2)v_0$.

5. NUMERICAL RESULTS AND DISCUSSIONS

Stress intensity factors at the edges of the strips have been evaluated numerically and have been depicted by means of graphs. Accordingly, all the lengths have been made dimensionless with respect to d . Substituting $\frac{a}{d} = d_1$, $\frac{b}{d} = d_2$ and $\frac{c}{d} = d_3$, the stress intensity factors at the four edges of the strips viz. K_a , K_b , K_c and K_d have been plotted against v/c_2 for various values of the strip length parameters. It is found that whatever be the lengths of the strips, stress intensity factors at the four edges of the strips decrease with the increase in the value of v/c_2 . From the graphs, it may be noted further that with the decrease in length of the inner strip which might be done either by increasing d_1 or by decreasing the value of d_2 , the stress intensity factor K_a at the innermost edge gradually decreases whereas the stress intensity factor at the other edges show just the opposite character (fig.2 - fig.9).

Also the decrease in the value of the length of the outer strip, which might be done by increasing the value of d_3 , causes a decrease in the value of the stress intensity factor K_a and increase in the values of the stress intensity factor K_b, K_c and K_d (fig.10 - fig.13) from which an interesting conclusion might be drawn that the presence of the inner strip suppresses the stress intensity factors at both the edges of the outer strip whereas the presence of the outer strip suppresses the stress intensity factor at the outer edge of the inner strip but increases the stress intensity factor at its inner edge.

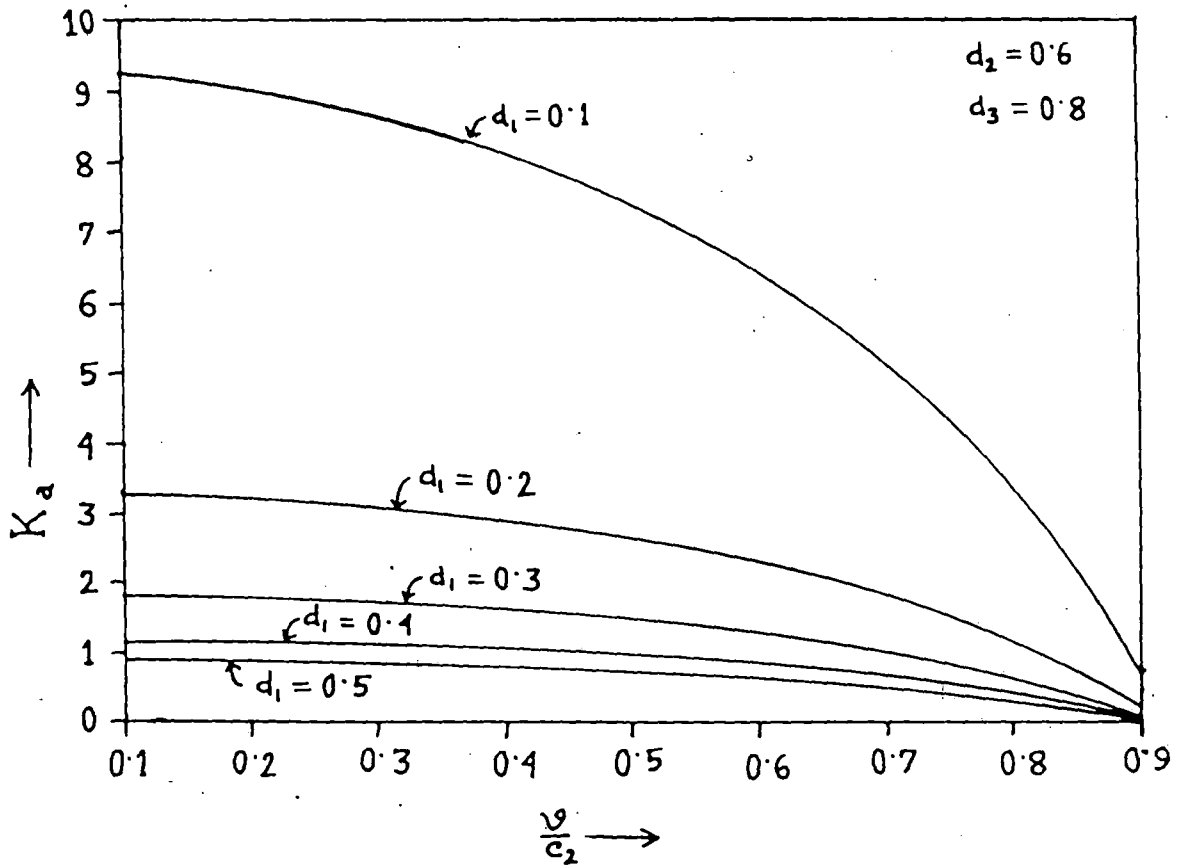


Fig. 2. Stress intensity factor K_d vs. v/c_2 ($d_2 = 0.6, d_3 = 0.8$).

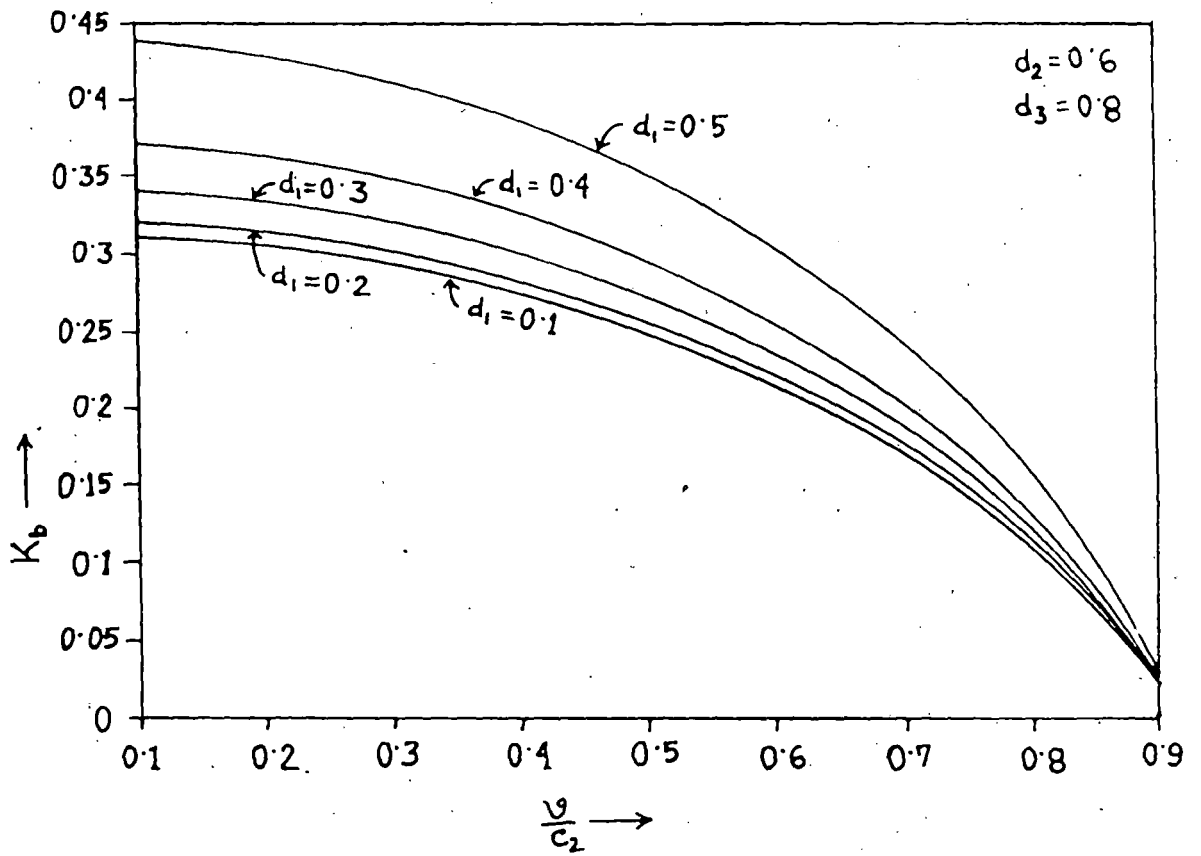


Fig. 3 Stress intensity factor K_b vs. v/c_2 ($d_2=0.6$, $d_3=0.8$).

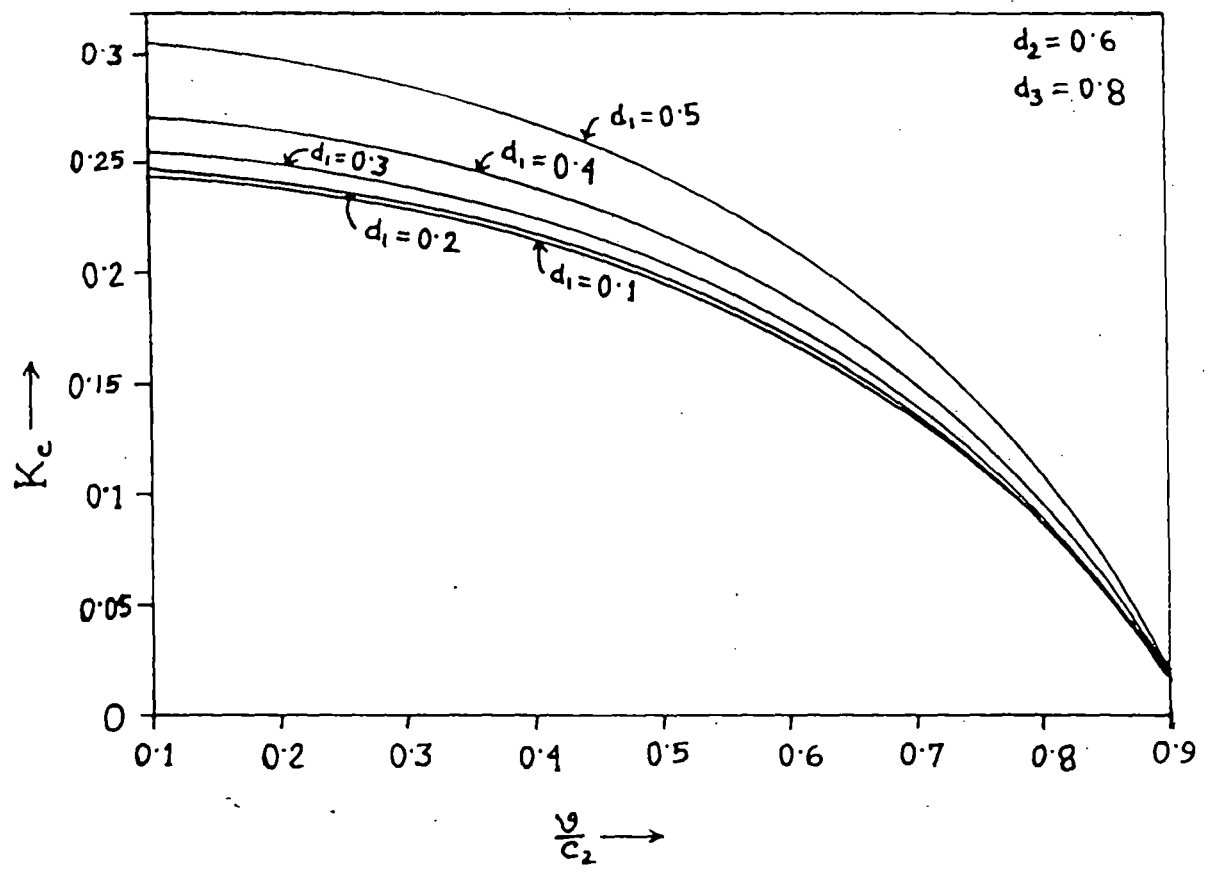


Fig. 4 Stress intensity factor K_c vs. v/c_2 ($d_2=0.6, d_3=0.8$).

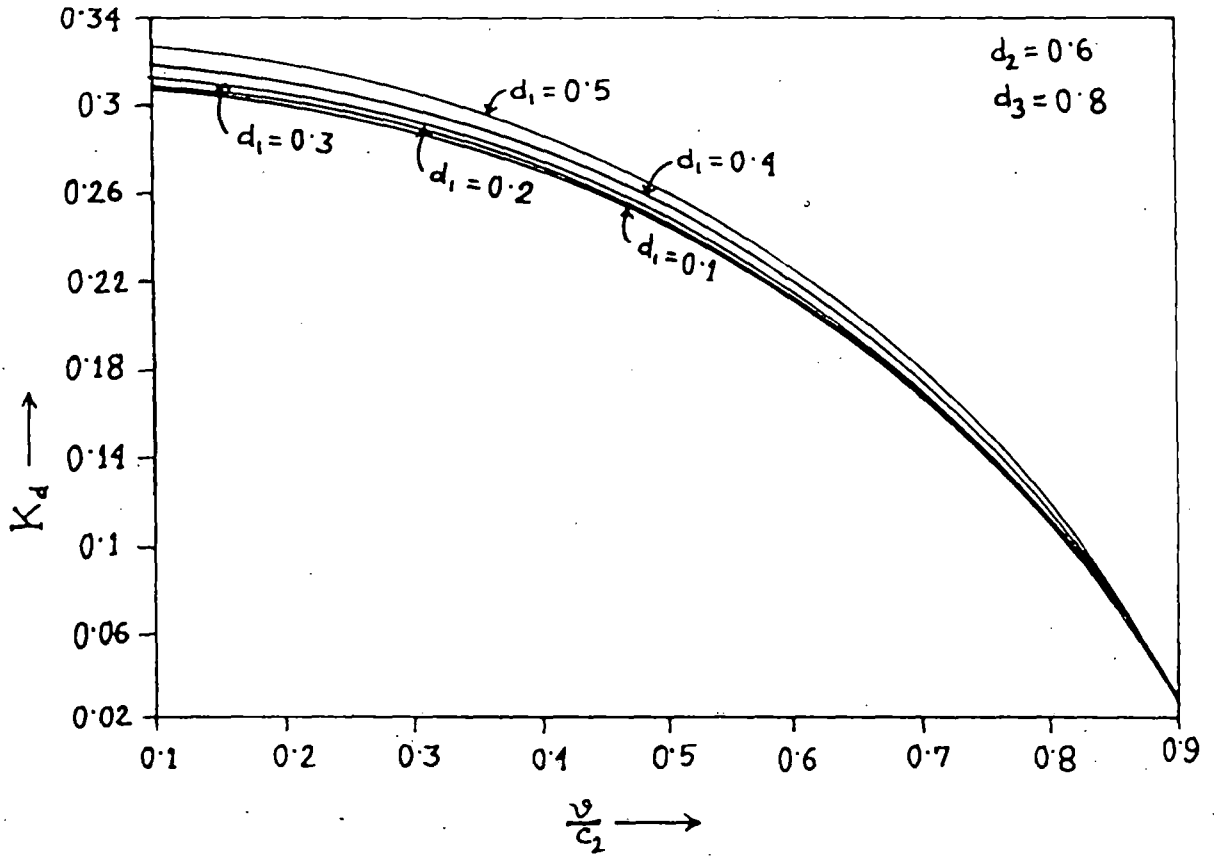


Fig. 5. Stress intensity factor K_d vs. v/c_2 ($d_2 = 0.6, d_3 = 0.8$).

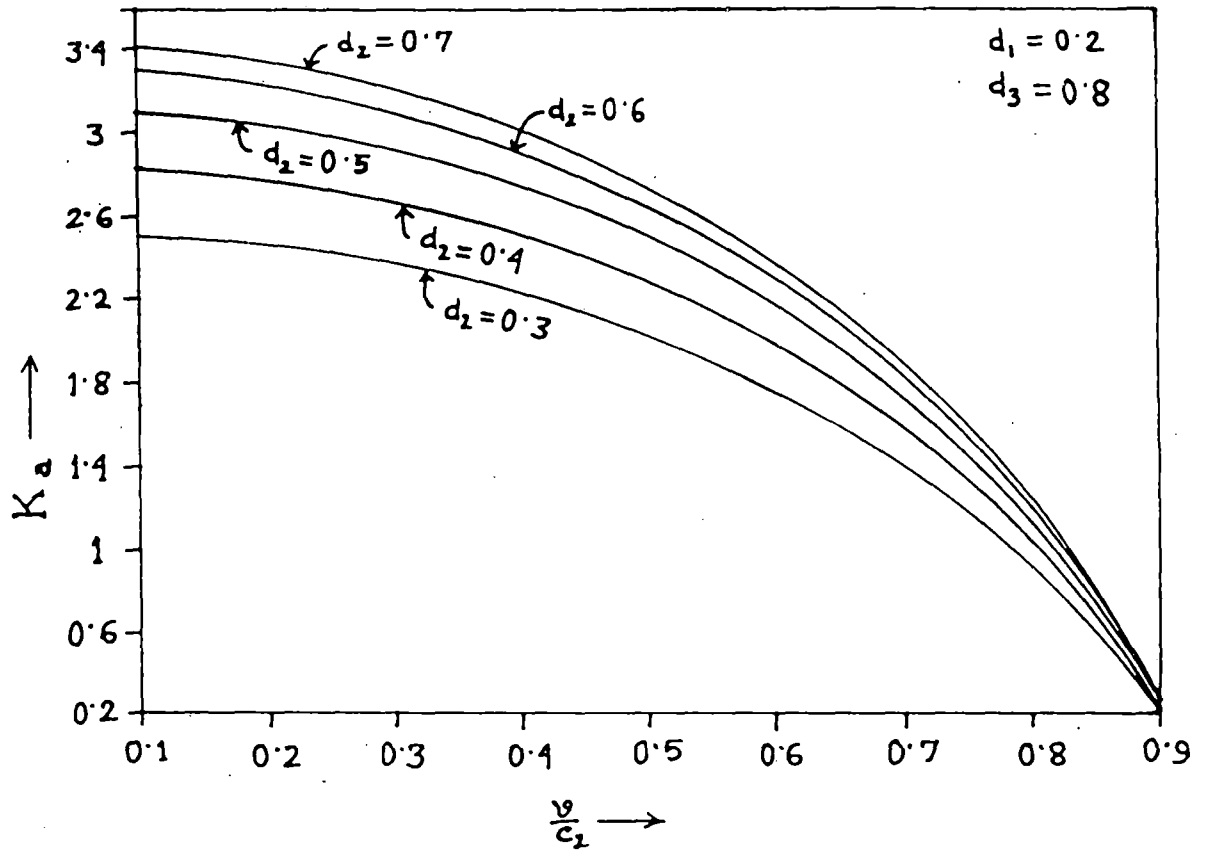


Fig. 6. Stress intensity factor K_d vs. v/c_2 ($d_1 = 0.2$, $d_3 = 0.8$)

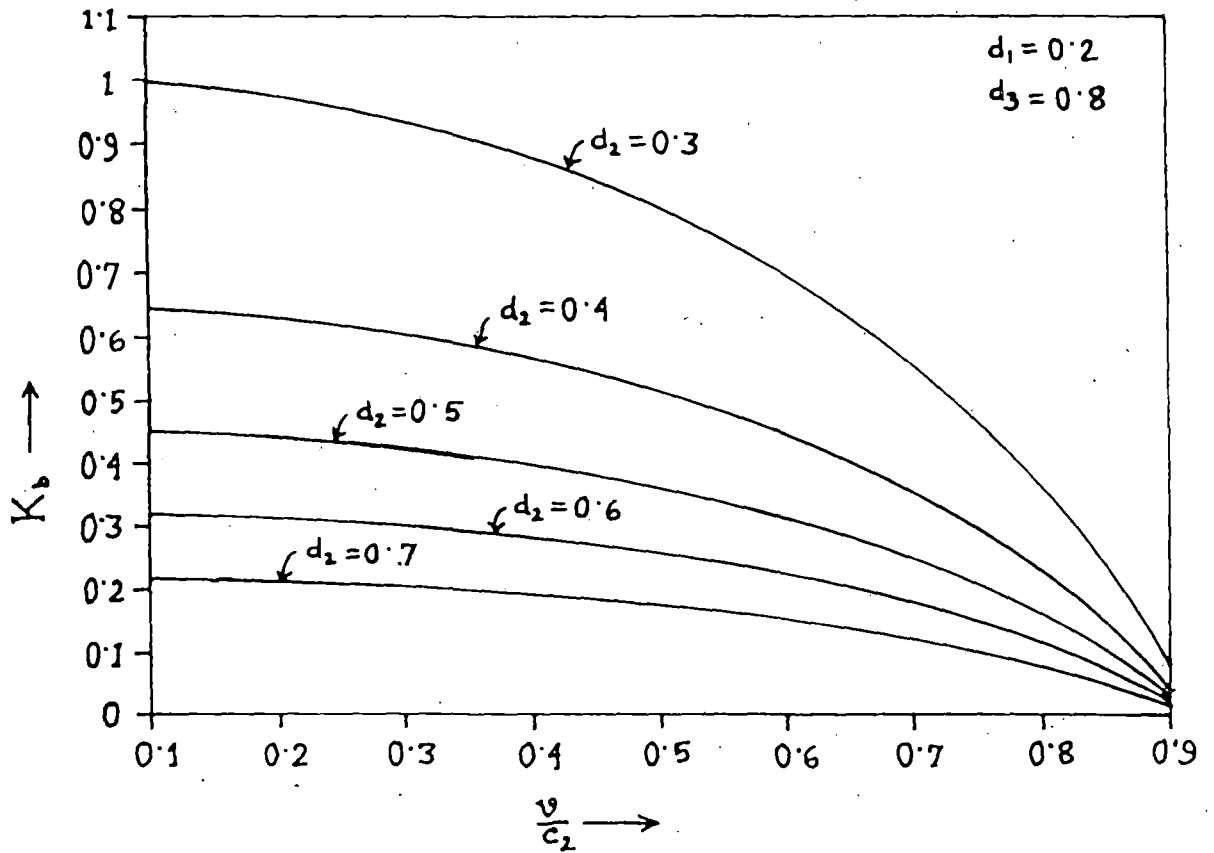


Fig. 7. Stress intensity factor K_b vs. v/c_2 ($d_1 = 0.2$, $d_3 = 0.8$).

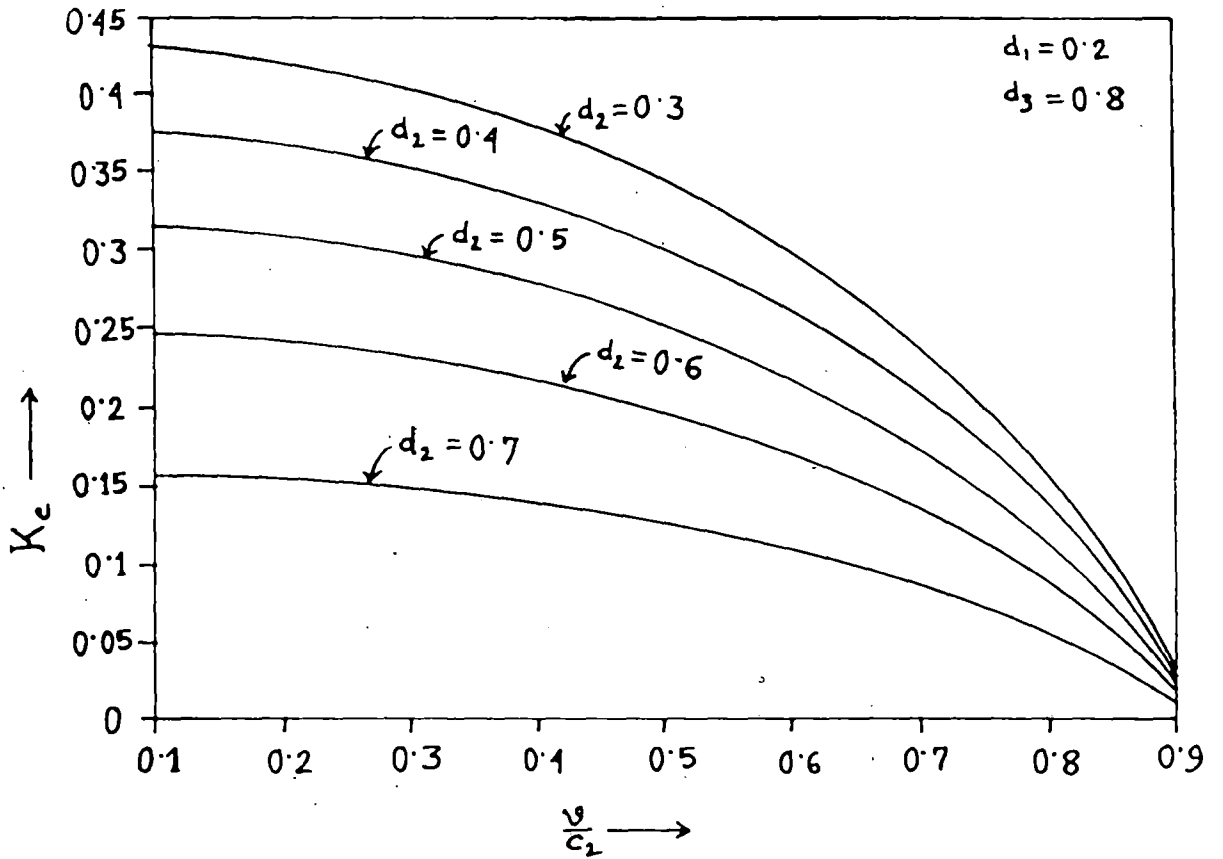


Fig. 8. Stress intensity - factor K_c vs. v/c_2 ($d_1=0.2, d_3=0.8$).

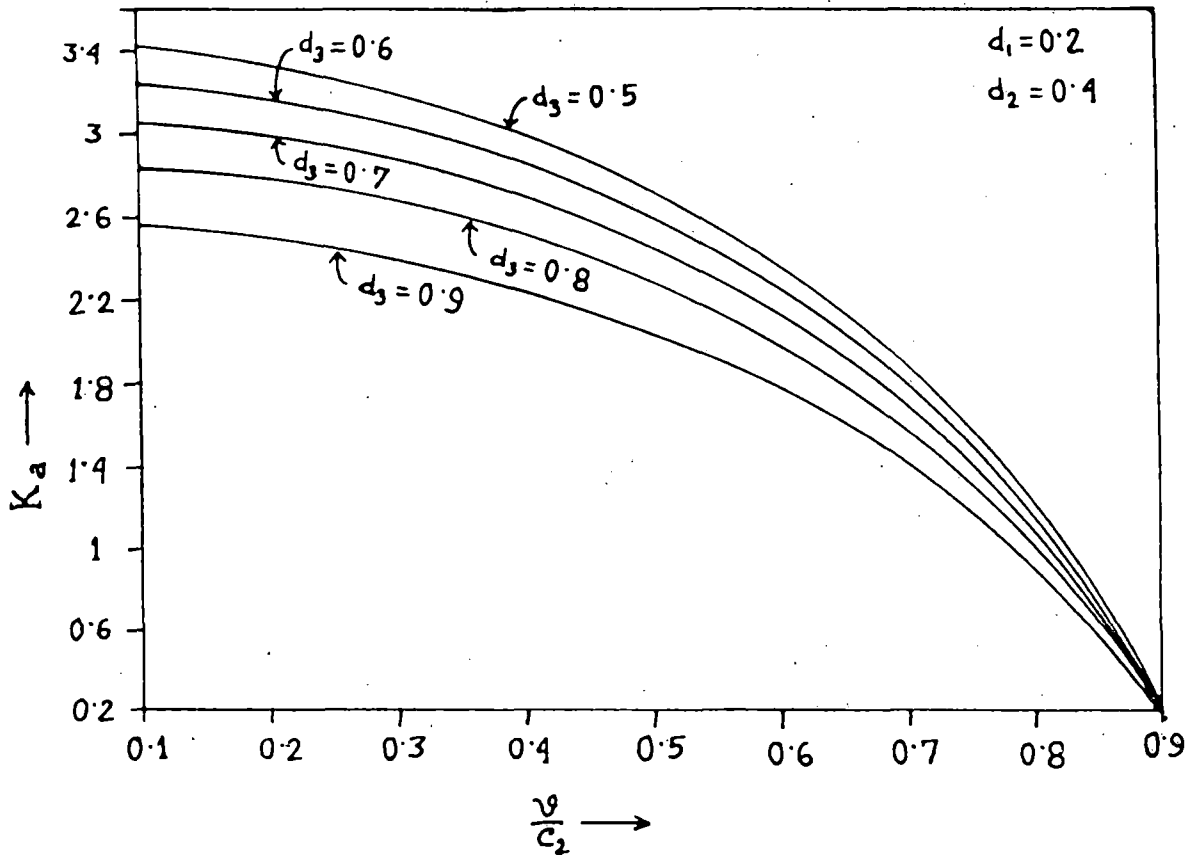


Fig. 10. Stress intensity factor K_a vs. v/c_2 ($d_1 = 0.2$, $d_2 = 0.4$).

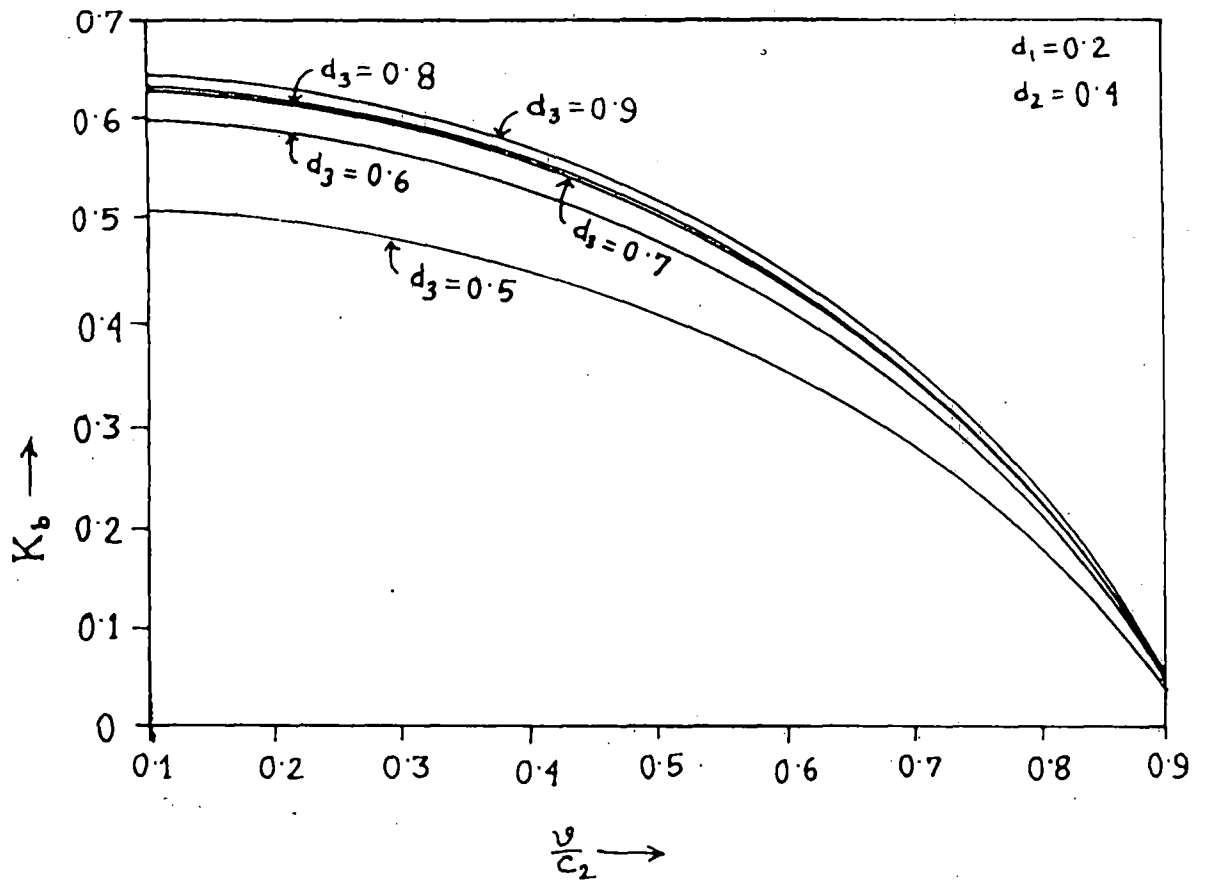


Fig. 11. Stress intensity factor K_b vs. v/c_2 ($d_1 = 0.2, d_2 = 0.4$)

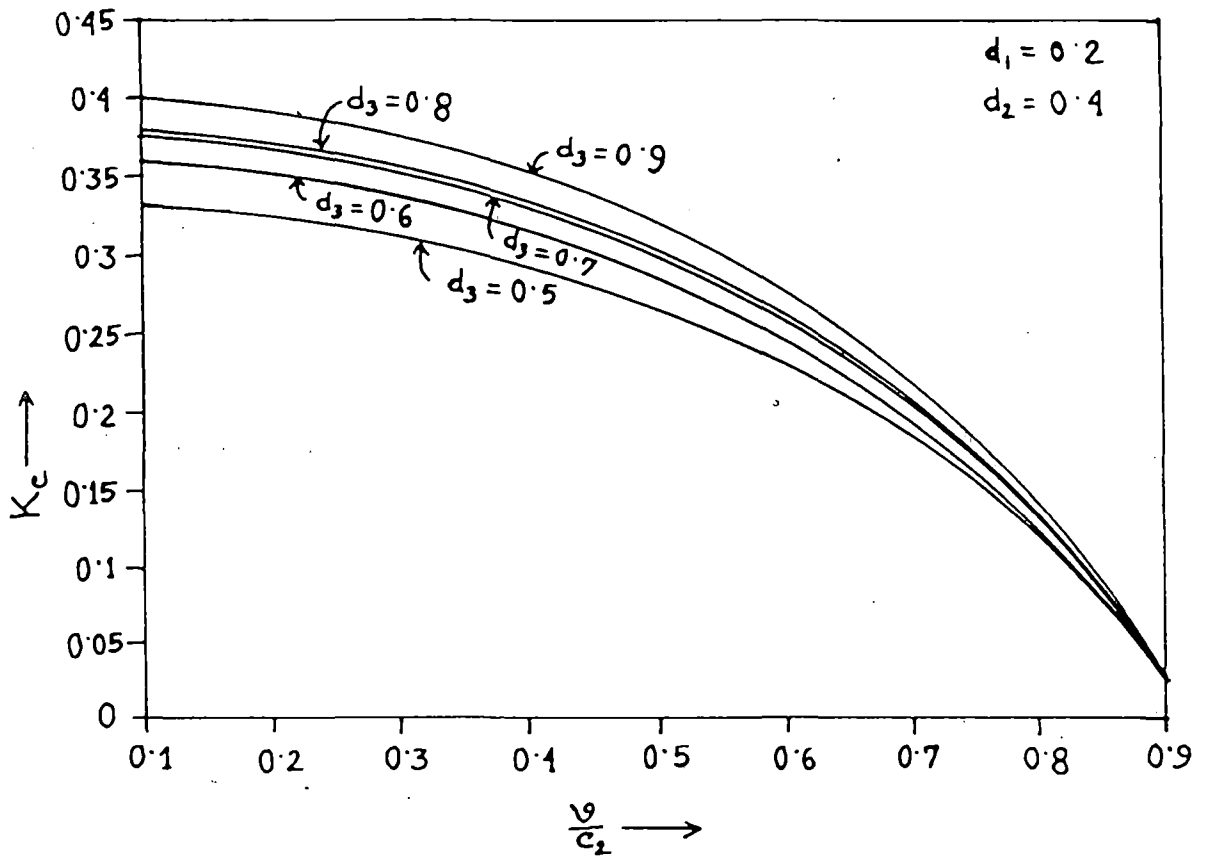


Fig. 12. Stress intensity factor K_c vs. v/c_2 ($d_1 = 0.2, d_2 = 0.4$).

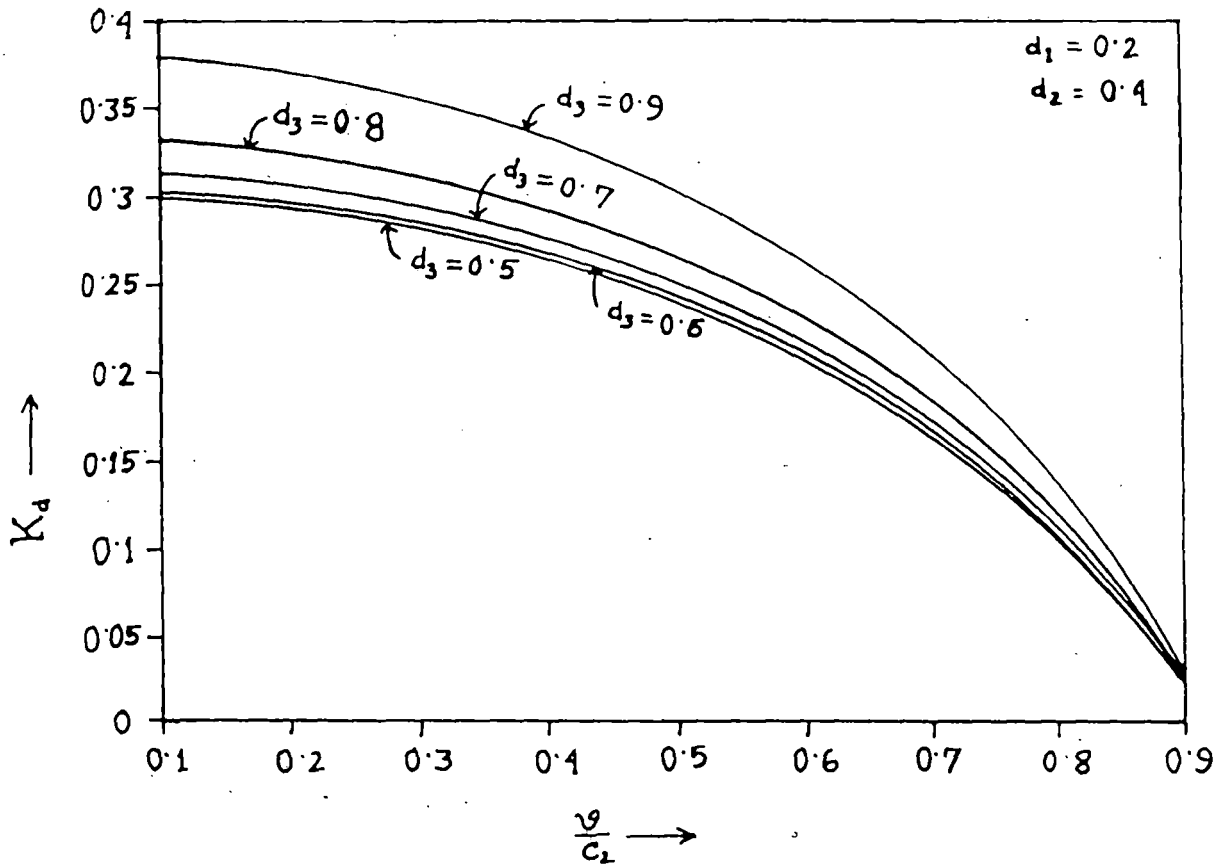


Fig. 13. Stress intensity factor K_d vs. v/c_2 ($d_1=0.2, d_2=0.4$).

DIFFRACTION OF TORSIONAL ELASTIC WAVES BY A RIGID ANNULAR DISC AT THE BIMATERIAL INTERFACE

1. INTRODUCTION

The study of the problems involving diffraction of elastic waves by cracks or inclusions are of considerable importance in view of their extensive applications in mechanical engineering and also in seismology and geophysics. If the cracks or inclusions are located at the interface of layered media, the study becomes more relevant. The extensive use of composite materials in modern technology has evoked interest in the wave propagation problems in layered media with interfacial discontinuities. Onder et al (1975) studied the diffraction of plane SH-wave obliquely incident on a rigid half plane lying at the interface of two dissimilar semi-infinite elastic media. Following Mal (1970), problem of interaction of antiplane shear wave by a Griffith crack at the interface of two bonded dissimilar elastic half spaces has been treated by Srivastava et al (1980). Bostrom (1987) also treated the same problem following a procedure similar to that of Krenk and Schmidt (1982). The corresponding problem of diffraction of antiplane shear wave by a finite rigid strip at the bimaterial interface has been treated by Palaiya and Majumder (1981). The problem of diffraction of transient torsional shear waves by a penny shaped crack at the interface of two bonded dissimilar elastic half spaces has been investigated by Ueda et al (1983). As

regards the dynamic crack or strip problems, research has mainly been confined to the case of a single crack or strip of finite width or of circular in shape. These are two part mixed boundary value problems which are usually reduced to solutions of dual integral equations. But the solution of interesting problems involving the diffraction of elastic waves by annular discs or cracks at the bimaterial interface which give rise to three part mixed boundary value problems are still lacking.

However recently the problems involving the diffraction of torsional waves by flat annular crack in an infinite elastic medium have been studied by Shindo (1979,1981), the problems are reduced to that of solving singular integral equation of first kind which were later solved by following the technique of Erdogan (1965,1969). The problem of diffraction of acoustic wave by a soft annular disc was studied by Thomas (1965). Following the method of Williams (1963) the three part mixed boundary value problem was reduced to a set of integral equation which was solved by an iterative procedure for low frequency. The same technique was followed by Jain and Kanwal (1970) to study the problem of torsional oscillations of an elastic half space due to annular disc. In this paper we have discussed the problem of diffraction of torsional wave by a rigid annular disc at the interface of two bonded dissimilar elastic media. Applying the method developed by Williams (1963) and used subsequently by Thomas (1965) and Jain et al (1970), the three part mixed boundary value problem has been reduced to the solution of a set of integral equations. The

solutions of these integral equations are obtained iteratively for low frequency and small values of the ratio of the inner and outer radii of the disc. These solutions are used to determine the jump in stresses across the annular disc and stress intensity factors at both the edges of the disc. Torque and Far field amplitudes in both the media have also been deduced. The effect of normalised frequency, material properties and geometric parameters in stress intensity factors and far field amplitude are shown graphically.

2. FORMULATION OF THE PROBLEM

Let us consider the torsional vibration of frequency ω of an annular rigid disc of inner and outer radii b and a respectively lying at the interface of two bonded dissimilar elastic half spaces. The region occupied by the annular disc is defined by $z=0$ and $b \leq r \leq a$ in a cylindrical polar co-ordinate system (r, θ, z) as shown in the fig.1. Let an antiplane shear wave given by $\Omega_2 r e^{ik_2(z-c_2 t)}$, where Ω_2 is a constant, $k_2 = \omega/c_2$, and $c_2 = \sqrt{\mu_2/\rho_2}$, the shear wave velocity in medium 2, be incident normally on the disc. Henceforth the time factor $e^{-i\omega t}$ will be suppressed through the analysis.

The only non-vanishing θ -component of the displacement V_j and the non-vanishing stresses $\tau_{r\theta}^{(j)}$, $\tau_{z\theta}^{(j)}$ ($j=1,2$) due to the scattered field are independent of θ and are given by

$$V_j = V_j(r, z, t) = v_j(r, z) e^{-i\omega t} \quad (1)$$

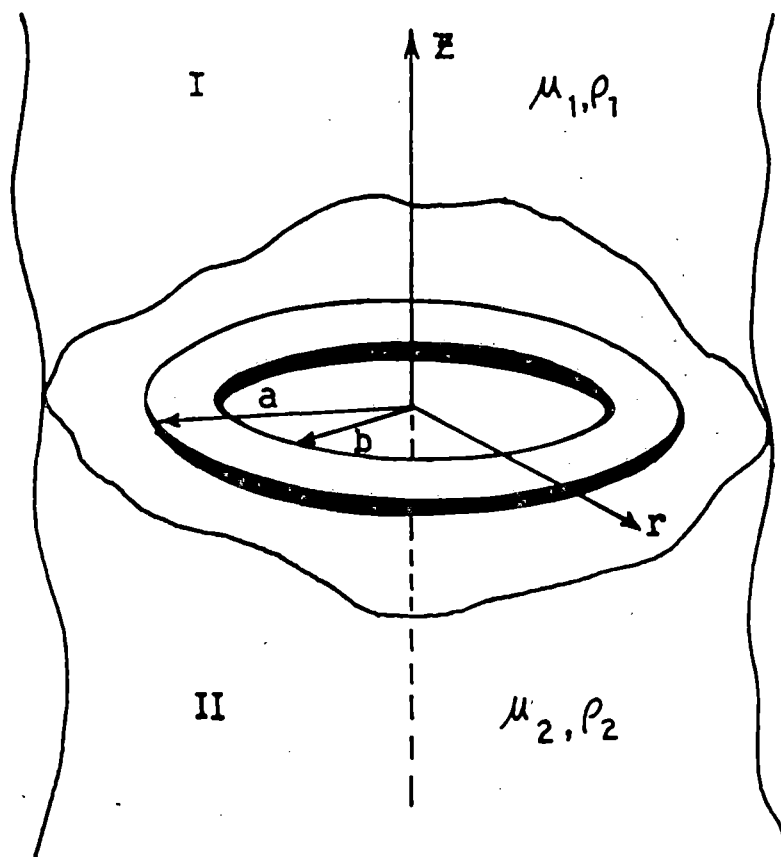


Fig.1. Geometry of the annular disc.

$$\tau_{r\theta}^{(j)} = \tau_{r\theta}^{(j)}(r, z) = \mu_j \left(\frac{\partial v_j}{\partial r} - \frac{v_j}{r} \right) \quad (2)$$

$$\tau_{z\theta}^{(j)} = \tau_{z\theta}^{(j)}(r, z) = \mu_j \frac{\partial v_j}{\partial z}$$

where μ_j ($j=1,2$) are the shear modulus of the elastic materials. The suffices 1 and 2 are used to denote the values of the corresponding quantities in the upper and lower half spaces respectively. Without any loss of generality we assume that $c_2 > c_1$. The displacement V_j satisfies the equation

$$\frac{\partial^2 V_j}{\partial r^2} + \frac{1}{r} \frac{\partial V_j}{\partial r} - \frac{V_j}{r^2} + \frac{\partial^2 V_j}{\partial z^2} = \frac{\rho_j}{\mu_j} \frac{\partial^2 V_j}{\partial t^2} \quad (3)$$

where ρ_j ($j=1,2$) are the density of the elastic materials.

Putting $V_j = v_j(r, z)e^{-i\omega t}$, equation (2.3) and the boundary conditions at the interface $z=0$, take the form

$$\frac{\partial^2 v_j}{\partial r^2} + \frac{1}{r} \frac{\partial v_j}{\partial r} - \frac{v_j}{r^2} + \frac{\partial^2 v_j}{\partial z^2} + k_j^2 v_j = 0 \quad (4)$$

$$v_1(r, 0) = v_2(r, 0) = -\Omega r, \quad b \leq r \leq a \quad (5)$$

$$v_1(r, 0) = v_2(r, 0), \quad 0 \leq r < b, \quad a < r < \infty \quad (6)$$

$$\tau_{z\theta}^{(1)}(r, 0) = \tau_{z\theta}^{(2)}(r, 0), \quad 0 \leq r < b, \quad a < r < \infty \quad (7)$$

where $k_j = \omega/c_j$, $c_j = \sqrt{(\mu_j/\rho_j)}$ and $\Omega = 2\Omega_2 \mu_2 k_2 / (\mu_1 k_1 + \mu_2 k_2)$.

The solution of the equation (4) can be written as

$$v_j(r, z) = \int_0^{\infty} A_j(\xi) \exp(-\gamma_j |z|) J_1(\xi r) d\xi \quad (8)$$

where

$$\begin{aligned} \gamma_j &= (\xi^2 - k_j^2)^{1/2}, \quad \xi > k_j \\ &= -i(k_j^2 - \xi^2)^{1/2}, \quad \xi < k_j \end{aligned}$$

and $A_j(\xi)$ ($j=1,2$) are functions of ξ to be determined from the boundary conditions.

Therefore, the stress components are

$$\tau_{z\theta}^{(1)}(r, z) = -\mu_1 \int_0^{\infty} \gamma_1 A_1(\xi) \exp(-\gamma_1 |z|) J_1(\xi r) d\xi, \quad z \geq 0 \quad (9)$$

$$\tau_{z\theta}^{(2)}(r, z) = \mu_2 \int_0^{\infty} \gamma_2 A_2(\xi) \exp(-\gamma_2 |z|) J_1(\xi r) d\xi, \quad z \leq 0$$

Now using the boundary conditions (2.5), (2.6) and (2.7) and assuming that $\tau_{z\theta}^{(2)}(r, 0) - \tau_{z\theta}^{(1)}(r, 0) = f(r)$, $b \leq r \leq a$, we obtain the integral equation

$$\int_b^a t f(t) dt \int_0^{\infty} \frac{\xi}{(\mu_1 \gamma_1 + \mu_2 \gamma_2)} J_1(\xi r) J_1(\xi t) d\xi = -\Omega r, \quad b \leq r \leq a \quad (10)$$

$$\text{where } A_1(\xi) = A_2(\xi) = \frac{\xi}{(\mu_1 \gamma_1 + \mu_2 \gamma_2)} \int_b^a t f(t) J_1(\xi t) dt \quad (11)$$

3. METHOD OF SOLUTION

In order to solve the integral equation (10), we apply the technique developed by Williams (1963) for solving integral equations arising in three part boundary value problems. The same technique was also applied by Thomas (1965) and Jain et al (1970) in order to solve scattering problem by annular disc. Following Kanwal (1971), the kernel of the integral equation (10) is split into two kernels as follows:

$$(\mu_1 + \mu_2) \int_0^{\infty} \frac{\xi}{(\mu_1 \gamma_1 + \mu_2 \gamma_2)} J_1(\xi r) J_1(\xi t) d\xi = K_1(r, t) + K_2(r, t) \quad (12)$$

where

$$K_1(r, t) = \int_0^{\infty} J_1(\xi r) J_1(\xi t) d\xi \quad (13)$$

$$K_2(r, t) = \int_0^{\infty} M(\xi, \gamma_1, \gamma_2) J_1(\xi r) J_1(\xi t) d\xi \quad (14)$$

$$M(\xi, \gamma_1, \gamma_2) = \frac{\mu_1(\xi - \gamma_1) + \mu_2(\xi - \gamma_2)}{\mu_1 \gamma_1 + \mu_2 \gamma_2} \quad (15)$$

The equation (2.10) then takes the form

$$\int_b^a tf(t)K_1(r, t)dt = -(\mu_1 + \mu_2)\Omega r - \int_b^a tf(t)K_2(r, t)dt, \quad b \leq r \leq a \quad (16)$$

Next, consider two functions $f_1(r)$ and $f_2(r)$ such that

$$f_1(r) + f_2(r) = \begin{cases} 0 & , \quad 0 \leq r < b \\ f(r) & , \quad b \leq r \leq a \\ 0 & , \quad a < r < \infty \end{cases} \quad (17)$$

As a result, the equation (5) reduces to two integral equations given by

$$\int_0^{\infty} t f_1(t) K_1(r, t) dt = -(\mu_1 + \mu_2) \Omega r - \int_0^{\infty} t f_1(t) K_2(r, t) dt, \quad 0 < r < a \quad (18)$$

$$\text{and} \quad \int_0^{\infty} t f_2(t) K_1(r, t) dt = - \int_0^{\infty} t f_2(t) K_2(r, t) dt, \quad b < r < \infty \quad (19)$$

The procedure adopted by Williams (1963) and Thomas (1965) is followed to solve these integral equations. Using the results

$$\begin{aligned} J_n(pr) &= \left(\frac{2p}{\pi}\right)^{1/2} \frac{1}{r^n} \int_0^r \frac{J_{n-1/2}(pw) w^{n+1/2}}{(r^2 - w^2)^{1/2}} dw \\ &= \left(\frac{2p}{\pi}\right)^{1/2} r^n \int_r^{\infty} \frac{J_{n+1/2}(pw) w^{-(n-1/2)}}{(w^2 - r^2)^{1/2}} dw \end{aligned}$$

$$\text{and} \quad \int_0^{\infty} p J_{\mu}(pw) J_{\mu}(pv) dp = \delta(w-v) / (wv)^{1/2}$$

we have the following relations

$$\int_0^{\infty} K_1(t, r) t f(t) dt = \frac{2}{\pi r} \int_0^r \frac{w^2 dw}{(r^2 - w^2)^{1/2}} \int_w^{\infty} \frac{f(t) dt}{(t^2 - w^2)^{1/2}}, \quad 0 < r < a$$

$$= \frac{2r}{\pi} \int_r^{\infty} \frac{w^{-2} dw}{(w^2 - r^2)^{1/2}} \int_0^w \frac{t^2 f(t) dt}{(w^2 - t^2)^{1/2}}, \quad b < r < \infty \quad (20)$$

and

$$K_2(t, r) = \frac{2}{\pi t r} \int_0^r \int_0^t \frac{L_1(v, w) w v dv dw}{(r^2 - w^2)^{1/2} (t^2 - v^2)^{1/2}}, \quad 0 < r < a$$

$$= \frac{2 t r}{\pi} \int_r^{\infty} \int_t^{\infty} \frac{L_2(v, w) dv dw}{w v (w^2 - r^2)^{1/2} (v^2 - t^2)^{1/2}}, \quad b < r < \infty \quad (21)$$

where

$$L_1(v, r) = (v r)^{1/2} \int_0^{\infty} \xi M(\xi, \gamma_1, \gamma_2) J_{1/2}(\xi v) J_{1/2}(\xi r) d\xi \quad (22)$$

and

$$L_2(v, r) = (v r)^{1/2} \int_0^{\infty} \xi M(\xi, \gamma_1, \gamma_2) J_{3/2}(\xi v) J_{3/2}(\xi r) d\xi \quad (23)$$

Substituting the relations (20) and (21) in (18) we get

$$\frac{2}{\pi r} \int_0^r \frac{w dw}{(r^2 - w^2)^{1/2}} w \int_w^{\infty} \frac{f_1(t) dt}{(t^2 - w^2)^{1/2}} = -(\mu_1 + \mu_2) \Omega r -$$

$$- \frac{2}{\pi r} \int_0^r \frac{w dw}{(r^2 - w^2)^{1/2}} \int_0^{\infty} f_1(t) dt \int_0^t \frac{v L_1(v, w) dv}{(t^2 - v^2)^{1/2}}, \quad 0 < r < a$$

which after changing the order of integration can be written as

$$\frac{2}{\pi r} \int_0^r \frac{w dw}{(r^2 - w^2)^{1/2}} w \int_w^\infty \frac{f_1(t) dt}{(t^2 - w^2)^{1/2}} = -(\mu_1 + \mu_2) \Omega r -$$

$$- \frac{2}{\pi r} \int_0^r \frac{w dw}{(r^2 - w^2)^{1/2}} \int_0^\infty L_1(v, w) dv v \int_v^\infty \frac{f_1(t) dt}{(t^2 - v^2)^{1/2}}, \quad 0 < r < a \quad (24)$$

In view of the above equation, we assume

$$r \int_r^\infty \frac{f_1(t) dt}{(t^2 - r^2)^{1/2}} = \begin{cases} S_1(r) & , \quad 0 < r < a \\ -T_1(r) & , \quad a < r < \infty \end{cases} \quad (25)$$

Use of the relation (25) in (24) yields

$$\frac{2}{\pi r} \int_0^r \frac{w S_1(w) dw}{(r^2 - w^2)^{1/2}} = -(\mu_1 + \mu_2) \Omega r - \frac{2}{\pi r} \int_0^r \frac{w G(w) dw}{(r^2 - w^2)^{1/2}} \quad (26)$$

$$\text{where} \quad G(w) = \int_0^a S_1(v) L_1(v, w) dv - \int_a^\infty T_1(v) L_1(v, w) dv \quad (27)$$

In order to apply Abel's transform the equation (26) can be written as

$$\int_0^r \frac{w [S_1(w) + G(w)]}{(r^2 - w^2)^{1/2}} dw = -\frac{\pi}{2} (\mu_1 + \mu_2) \Omega r^2$$

and after taking Abel's transform and substituting the value of $G(w)$ from (27) we obtain the following integral equation :

$$S_1(r) + \int_0^a L_1(v, r) S_1(v) dv = -2\Omega (\mu_1 + \mu_2) r + \int_a^\infty L_1(v, r) T_1(v) dv$$

$$, \quad 0 < r < a \quad (28)$$

Again substituting the relations (20) and (21) in (19) and following the same procedure as is done to deduce the integral equation (28), another integral equation can be derived as

$$S_2(r) + \int_b^{\infty} L_2(v, r) S_2(v) dv = \int_0^b L_2(v, r) T_2(v) dv, \quad b < r < \infty \quad (29)$$

where it is assumed that

$$\frac{1}{r} \int_0^r \frac{t^2 f_2(t) dt}{(r^2 - t^2)^{1/2}} = \begin{cases} -T_2(r) & , \quad 0 < r < b \\ S_2(r) & , \quad b < r < \infty \end{cases} \quad (30)$$

Now, using Abel's transform in (25) and (30), the functions $f_1(t)$ and $f_2(t)$ are found to be

$$f_1(t) = -\frac{2}{\pi} \frac{d}{dt} \left[\int_t^a \frac{S_1(u) du}{(u^2 - t^2)^{1/2}} - \int_a^{\infty} \frac{T_1(u) du}{(u^2 - t^2)^{1/2}} \right], \quad 0 < t < a \quad (31)$$

and

$$f_2(t) = \frac{2}{\pi t^2} \frac{d}{dt} \left[-\int_0^b \frac{u^2 T_2(u) du}{(t^2 - u^2)^{1/2}} + \int_b^t \frac{u^2 S_2(u) du}{(t^2 - u^2)^{1/2}} \right], \quad t > b \quad (32)$$

Further, by the help of the relation (17), $T_1(r)$ and $T_2(r)$ can be written from (25) and (30) as

$$T_1(r) = r \int_r^{\infty} \frac{f_2(t) dt}{(t^2 - r^2)^{1/2}}, \quad a < r < \infty$$

$$T_2(r) = \frac{1}{r} \int_0^r \frac{t^2 f_1(t) dt}{(r^2 - t^2)^{1/2}}, \quad 0 < r < b$$

in which putting the values of $f_1(t)$ and $f_2(t)$ from (31) and (32) and using the results

$$\int_r^\infty \frac{dt}{t(t^2 - r^2)^{1/2}(t^2 - u^2)^{3/2}} = \frac{\sqrt{\pi}}{2r^2 \Gamma(5/2)(r^2 - u^2)} {}_2F_1(1/2, 1; 5/2; u^2/r^2), \quad u < r \quad (33)$$

$$\int_0^r \frac{t^3 dt}{(r^2 - t^2)^{1/2}(u^2 - t^2)^{3/2}} = \frac{\sqrt{\pi} r^3}{2\Gamma(5/2)u(u^2 - r^2)} {}_2F_1(1/2, 1; 5/2; r^2/u^2), \quad u > r \quad (34)$$

we get the following two integral equations

$$T_1(r) = l_2(r) + \frac{1}{\sqrt{\pi}r\Gamma(5/2)} \int_0^b \frac{u^2 T_2(u) {}_2F_1(1/2, 1; 5/2; u^2/r^2)}{(r^2 - u^2)} du, \quad a < r < \infty \quad (35)$$

$$T_2(r) = l_1(r) + \frac{r^2}{\sqrt{\pi}\Gamma(5/2)} \int_a^\infty \frac{T_1(u) {}_2F_1(1/2, 1; 5/2; r^2/u^2)}{u(u^2 - r^2)} du, \quad 0 < r < b \quad (36)$$

where

$$l_1(r) = -\frac{2}{\pi r} \int_0^r \frac{t^2 dt}{(r^2 - t^2)^{1/2}} \frac{d}{dt} \int_t^a \frac{S_1(u) du}{(u^2 - t^2)^{1/2}}, \quad 0 < r < b \quad (37)$$

$$I_2(r) = \frac{2r}{\pi} \int_r^\infty \frac{dt}{t^2(t^2-r^2)^{1/2}} \frac{d}{dt} \int_b^t \frac{u^2 S_2(u) du}{(t^2-u^2)^{1/2}}; \quad a < r < \infty \quad (38)$$

Assuming that $\alpha = k_2 a$, $\beta = k_2 b$ and $\lambda = b/a$ are small, the unknown functions $S_1(r)$, $S_2(r)$, $T_1(r)$, $T_2(r)$ which are solutions of integral equations (28), (29), (35) and (36) are obtained approximately following iterative process. Using the result that

$${}_2F_1(1/2, 1; 5/2; r^2/u^2) = \frac{3u}{4r^3} \left\{ 2ur - (u^2 - r^2) \log \left(\frac{u+r}{u-r} \right) \right\}, \quad r < u$$

equations (35) and (36) become

$$T_1(ar) = I_2(ar) + \frac{1}{\pi} \int_0^1 T_2(bu) \left\{ \frac{2\lambda r}{(r^2 - \lambda^2 u^2)} - \frac{1}{u} \log \left(\frac{r+\lambda u}{r-\lambda u} \right) \right\} du, \quad 1 < r < \infty \quad (39)$$

and

$$T_2(br) = I_1(br) + \frac{1}{\pi \lambda r} \int_1^\infty T_1(au) \left\{ \frac{2\lambda ur}{(u^2 - \lambda^2 r^2)} - \log \left(\frac{u+\lambda r}{u-\lambda r} \right) \right\} du, \quad 0 < r < 1 \quad (40)$$

Next, we assume that $\alpha = o(\lambda)$ so that $\beta = \alpha\lambda = o(\alpha^2)$.

In order to solve the equation (28), we rewrite it as

$$S_1(ar) + a \int_0^1 L_1(av, ar) S_1(av) dv = -2\Omega(\mu_1 + \mu_2) ar + a \int_1^\infty L_1(av, ar) T_1(av) dv$$

$0 < r < 1 \quad (41)$

$$\text{Now we put } S_1(ar) = X(ar) + Y(ar) \quad (42)$$

so that equation (41) yields a pair of integral equations given by

$$X(ar) = -2\Omega(\mu_1 + \mu_2)ar - a \int_0^1 L_1(av, ar)X(av)dv, \quad 0 < r < 1 \quad (43)$$

and

$$Y(ar) = a \int_1^\infty L_1(av, ar)T_1(av)dv - a \int_0^1 L_1(av, ar)Y(av)dv, \quad 0 < r < 1 \quad (44)$$

The kernel $L_1(av, ar)$ given by (22) can be converted to an expression involving finite integrals by the application of the contour integration technique followed by Srivastava et al (1980) and is given by

$$aL_1(av, ar) = i(1+\mu)\alpha^2(vr)^{1/2} \left[\int_0^1 \frac{\eta^2 J_{1/2}(\alpha\eta r) H_{1/2}^{(1)}(\alpha\eta v)}{\mu(1-\eta^2)^{1/2} + (\sigma^2 - \eta^2)^{1/2}} d\eta + \int_1^\sigma \frac{\eta^2 (\sigma^2 - \eta^2)^{1/2} J_{1/2}(\alpha\eta r) H_{1/2}^{(1)}(\alpha\eta v)}{\mu^2(\eta^2 - 1) + (\sigma^2 - \eta^2)} d\eta \right], \quad v > r$$

where $\sigma = k_1/k_2$, $\mu = \mu_2/\mu_1$. For $v < r$, v and r are to be interchanged. Next, expanding the Bessel and Hankel functions in series for small values of their arguments and integrating, assuming that $\mu > \sigma > 1$, the above expression can be written as (details are given in appendix - A)

$$aL_1(av, ar) = \alpha^2 r M_1 + i\alpha^3 r v M_2 - \frac{\alpha^4 (3v^2 r + r^3)}{6} M_3 - \frac{i\alpha^5 (v^3 r + r^3 v)}{6} M_4 + \frac{\alpha^6 (5v^4 r + 10r^3 v^2 + r^5)}{120} M_5 + \frac{i\alpha^7 (3v^5 r + 10r^3 v^3 + 3r^5 v)}{360} M_6 + o(\alpha^8), \quad v > r$$

$$\begin{aligned}
&= \alpha^2 v M_1 + i \alpha^3 r v M_2 - \frac{\alpha^4 (3r^2 v + v^3)}{6} M_3 - \frac{i \alpha^5 (r^3 v + v^3 r)}{6} M_4 + \\
&\quad + \frac{\alpha^6 (5r^4 v + 10v^3 r^2 + v^5)}{120} M_5 + \frac{i \alpha^7 (3r^5 v + 10v^3 r^3 + 3v^5 r)}{360} M_6 + \\
&\quad + o(\alpha^8), \quad v < r \quad (45)
\end{aligned}$$

$$\text{where } M_1 = \frac{\sigma^2 + \mu}{2(\mu + 1)} \quad (46)$$

$$M_2 = \frac{2}{\pi(\mu - 1)} \left[\frac{\sigma^3 - \mu + \mu^2 - \sigma^2}{3} \frac{\mu^2 - \sigma^2}{\mu^2 - 1} \left\{ (\mu - \sigma) + \mu \frac{\sigma^2 - 1}{\sqrt{\mu^2 - 1}} \log \left(\frac{\sigma \sqrt{\mu^2 - 1} + \mu \sqrt{\sigma^2 - 1}}{\sqrt{\mu^2 - 1} + \sqrt{\sigma^2 - 1}} \right) \right\} \right] \quad (47)$$

$$M_3 = \frac{1}{(\mu - 1)} \left[\frac{1}{8} (\sigma^4 - \mu) + \left(\frac{\mu^2 - \sigma^2}{\mu^2 - 1} \right) \left\{ \frac{\sigma^2 - \mu}{2} + \frac{\mu^2 - \sigma^2}{\mu + 1} \right\} \right] \quad (48)$$

$$\begin{aligned}
M_4 = \frac{2}{\pi(\mu - 1)} \left[\frac{2(\sigma^5 - \mu)}{15} + \frac{\mu^2 - \sigma^2}{\mu^2 - 1} \left[\frac{\sigma^3 - \mu}{3} + \frac{\mu^2 - \sigma^2}{\mu^2 - 1} \left\{ (\mu - \sigma) + \mu \frac{\sigma^2 - 1}{\sqrt{\mu^2 - 1}} \right. \right. \right. \\
\left. \left. \left. \times \log \left(\frac{\sigma \sqrt{\mu^2 - 1} + \mu \sqrt{\sigma^2 - 1}}{\sqrt{\mu^2 - 1} + \sqrt{\sigma^2 - 1}} \right) \right\} \right] \right] \quad (49)
\end{aligned}$$

$$M_5 = \frac{1}{(\mu - 1)} \left[\frac{\sigma^6 - \mu}{16} + \left(\frac{\mu^2 - \sigma^2}{\mu^2 - 1} \right) \left\{ \frac{1}{8} (\sigma^4 - \mu) + \left(\frac{\mu^2 - \sigma^2}{\mu^2 - 1} \right) \left\{ \frac{\sigma^2 - \mu}{2} + \frac{\mu^2 - \sigma^2}{\mu + 1} \right\} \right\} \right] \quad (50)$$

$$M_6 = \frac{2}{\pi(\mu-1)} \left[\frac{8}{105}(\sigma^7 - \mu) + \frac{\mu^2 - \sigma^2}{\mu^2 - 1} \left\{ \frac{2(\sigma^5 - \mu)}{15} + \frac{\mu^2 - \sigma^2}{\mu^2 - 1} \left(\frac{\sigma^3 - \mu}{3} + \frac{\mu^2 - \sigma^2}{\mu^2 - 1} \times \right. \right. \right. \\ \left. \left. \left. \times \left\{ (\mu - \sigma) + \mu \frac{\sqrt{\sigma^2 - 1}}{\sqrt{\mu^2 - 1}} \log \left(\frac{\sigma \sqrt{\mu^2 - 1} + \mu \sqrt{\sigma^2 - 1}}{\sqrt{\mu^2 - 1} + \sqrt{\sigma^2 - 1}} \right) \right\} \right\} \right] \quad (51)$$

Substituting the value of $L_1(ar, av)$ given by (45) in (43) and using iterative method, an approximate value of $X(ar)$ for low frequency can be derived as follows :

$aL_1(ar, av)$ given by (45) is rewritten as

$$aL_1(ar, av) = \alpha^2 L_{12} + \alpha^3 L_{13} + \alpha^4 L_{14} + \alpha^5 L_{15} + \alpha^6 L_{16} + \alpha^7 L_{17} \quad (52)$$

where

$$L_{12} = M_1 r, \quad v > r \\ = M_1 v, \quad v < r$$

$$L_{13} = iM_2 rv$$

$$L_{14} = - \frac{(3v^2 r + r^3)}{6} M_3, \quad v > r \\ = - \frac{(3r^2 v + v^3)}{6} M_3, \quad v < r$$

$$L_{15} = - \frac{i(v^3 r + r^3 v)}{6} M_4$$

$$L_{16} = \frac{5v^4r + 10v^2r^3 + r^5}{120} M_5, \quad v > r$$

$$= \frac{5r^4v + 10r^2v^3 + v^5}{120} M_5, \quad v < r$$

$$L_{17} = \frac{i(3v^5r + 10v^3r^3 + 3r^5v)}{360} M_6$$

Also let $X(ar) = X_0(ar) + \alpha X_1(ar) + \alpha^2 X_2(ar) + \alpha^3 X_3(ar) + \alpha^4 X_4(ar) +$

$$+ \alpha^5 X_5(ar) + \alpha^6 X_6(ar) + \alpha^7 X_7(ar) + o(\alpha^8) \quad (53)$$

where $X_i(ar)$, ($i=0,1,\dots,7$) are to be determined.

Now putting the values of $X(ar)$ and $aL_1(ar,av)$ given by (53) and (52) in (43) and equating the coefficient of like powers of α from both sides we obtain,

$$X_0(ar) = -2a\Omega(\mu_1 + \mu_2)r \quad (54)$$

$$X_1(ar) = 0 \quad (55)$$

$$X_2(ar) = -\int_0^1 L_{12} X_0(av) dv \quad (56)$$

$$X_3(ar) = -\int_0^1 [L_{12} X_1(av) + L_{13} X_0(av)] dv \quad (57)$$

$$X_4(ar) = -\int_0^1 [L_{12} X_2(av) + L_{13} X_1(av) + L_{14} X_0(av)] dv \quad (58)$$

$$X_5(ar) = - \int_0^1 \left[L_{12} X_3(av) + L_{13} X_2(av) + L_{14} X_1(av) + L_{15} X_0(av) \right] dv \quad (59)$$

$$X_6(ar) = - \int_0^1 \left[L_{12} X_4(av) + L_{13} X_3(av) + L_{14} X_2(av) + L_{15} X_1(av) + \right. \\ \left. + L_{16} X_0(av) \right] dv \quad (60)$$

$$X_7(ar) = - \int_0^1 \left[L_{12} X_5(av) + L_{13} X_4(av) + L_{14} X_3(av) + L_{15} X_2(av) + \right. \\ \left. + L_{16} X_1(av) + L_{17} X_0(av) \right] dv \quad (61)$$

Substituting the value of $X_0(ar)$ and L_{12} in (56) and integrating, $X_2(ar)$ is found to be

$$X_2(ar) = \frac{a\Omega}{3} M_1 (\mu_1 + \mu_2) (3r - r^3) \quad (62)$$

Similarly replacing the necessary unknowns by their corresponding iterated values in (57) to (61) and integrating we obtain

$$X_3(ar) = \frac{2}{3} ia\Omega (\mu_1 + \mu_2) M_2 r \quad (63)$$

$$X_4(ar) = - \frac{a\Omega}{60} (\mu_1 + \mu_2) \left[(M_1^2 - M_3) r^5 - 10(M_1^2 - M_3) r^3 - 5(M_1^2 - 3M_3) r \right] \quad (64)$$

$$X_5(ar) = \frac{ia\Omega}{180} (\mu_1 + \mu_2) \left[20(M_1 M_2 - M_4) r^3 - 12(9M_1 M_2 + M_4) r \right] \quad (65)$$

$$X_6(ar) = \frac{a\Omega}{180}(\mu_1 + \mu_2) \left[\frac{1}{14}(2M_1M_3 - M_1^2 - M_5)r^7 + \frac{3}{2}(M_1^2 - 2M_1M_3 + M_5)r^5 + \frac{5}{2}(M_1^3 + 2M_1M_3 + 3M_5)r^3 + \frac{1}{2}(94M_1M_3 - 29M_1^3 + 80M_2^2 + 5M_5)r \right] \quad (66)$$

$$X_7(ar) = \frac{ia\Omega}{180}(\mu_1 + \mu_2) \left[(M_1^2M_2 - M_1M_4 - M_2M_3 + M_6)r^5 + 2(3M_1M_4 - 9M_1^2M_2 + 5M_2M_3 + M_6)r^3 + \frac{1}{7}(269M_1^2M_2 + 109M_1M_4 + 249M_2M_3 + 3M_6)r \right] \quad (67)$$

The values of $X_i(ar)$, $i=0,1,\dots,7$ from (54), (55), (62) - (67) are substituted in (52) and arranging the terms in ascending powers of r , $X(ar)$ can be rewritten as

$$X(ar) = a\Omega(\mu_1 + \mu_2) [p_1(\alpha)r + p_3(\alpha)r^3 + p_5(\alpha)r^5 + p_7(\alpha)r^7 + o(\alpha^8)] \quad (68)$$

where

$$p_1(\alpha) = -2 + M_1\alpha^2 + \frac{2i}{3}M_2\alpha^3 + \frac{1}{12}(M_1^2 - 3M_3)\alpha^4 + \frac{1}{360}(94M_1M_3 - 29M_1^3 + 80M_2^2 + 5M_5)\alpha^6 - \frac{i}{15}(9M_1M_2 + M_4)\alpha^5 + \frac{i}{1260}(269M_1^2M_2 + 109M_1M_4 + 249M_2M_3 + 3M_6)\alpha^7$$

$$p_3(\alpha) = -\frac{1}{3}M_1\alpha^2 + \frac{1}{6}(M_1^2 - M_3)\alpha^4 + \frac{i}{9}(M_1M_2 - M_4)\alpha^5 + \frac{1}{72}(M_1^3 + 2M_1M_3 + 3M_5)\alpha^6 + \frac{i}{90}(3M_1M_4 - 9M_1^2M_2 + 5M_2M_3 + M_6)\alpha^7$$

$$p_5(\alpha) = -\frac{1}{60}(M_1^2 - M_3) \alpha^4 + \frac{1}{120}(M_1^3 - 2M_1 M_3 + M_5) \alpha^6 + \frac{1}{180}(M_1^2 M_2 - M_1 M_4 - M_2 M_3 + M_6) \alpha^7$$

$$p_7(\alpha) = \frac{1}{2520}(2M_1 M_3 - M_1^3 - M_5) \alpha^6 .$$

Next, replacing r by br , equation (29) can be written as

$$S_2(br) + b \int_1^\infty L_2(bv, br) S_2(bv) dv = b \int_0^1 L_2(bv, br) T_2(bv) dv, \quad 1 < r < \infty \quad (69)$$

Following the same procedure as done for the evaluation of $L_1(av, ar)$, $L_2(bv, br)$ given by (23) can be evaluated to the form

$$bL_2(bv, br) = i(1+\mu)\beta^2(vr)^{1/2} \left[\int_0^1 \frac{\eta^2 J_{3/2}(\beta\eta r) H_{3/2}^{(1)}(\beta\eta v)}{\mu(1-\eta^2)^{1/2} + (\sigma^2 - \eta^2)^{1/2}} d\eta + \int_1^\sigma \frac{\eta^2 (\sigma^2 - \eta^2)^{1/2} J_{3/2}(\beta\eta r) H_{3/2}^{(1)}(\beta\eta v)}{\mu^2(\eta^2 - 1) + (\sigma^2 - \eta^2)} d\eta \right], \quad v > r$$

For $v < r$, v and r are to be interchanged.

For low frequency $bL_2(bv, br)$ is now reduced to the following form after using the series expansions of Bessel and Hankel functions.

$$bL_2(bv, br) = \alpha^2 \lambda^2 \left[\frac{1}{3} M_1 \frac{r^2}{v} + o(\alpha^4) \right], \quad v > r$$

$$= \alpha^2 \lambda^2 \left[\frac{1}{3} M_1 \frac{v^2}{r} + o(\alpha^4) \right], \quad v < r \quad (70)$$

The functions which occur in the integral equations (43), (28), (37), (36), (29), (38), (35) and (44) are calculated by iterative process in the following order

$$X, S_1, l_1, T_2, S_2, l_2, T_1, Y, S_1$$

Iterative procedure is followed in order to obtain the following results sufficiently accurate correct upto the order of (α^7)

$$l_1(br) = \frac{8\Omega(\mu_1 + \mu_2)ar^2\lambda^2}{3\pi} \left[-1 + \frac{2}{3}M_1\alpha^2 + \frac{1}{3}M_2\alpha^3 - \frac{2}{5}\lambda^2r^2 + o(\alpha^4) \right], \quad 0 < r < 1 \quad (71)$$

$$T_2(br) = l_1(br) + o(\alpha^7), \quad 0 < r < 1 \quad (72)$$

$$S_2(br) = -\frac{8\Omega(\mu_1 + \mu_2)aM_1\alpha^2\lambda^4}{45\pi} \left[\frac{1}{r} + o(\alpha^2) \right], \quad 1 < r < \infty \quad (73)$$

$$l_2(ar) = -\frac{16\Omega(\mu_1 + \mu_2)aM_1\alpha^2\lambda^5}{45\pi^2} \left[\frac{1}{r} + o(\alpha^2) \right], \quad 1 < r < \infty \quad (74)$$

$$T_1(ar) = \frac{16\Omega(\mu_1 + \mu_2)a\lambda^5}{45\pi^2} \left[-\frac{1}{r} M_1\alpha^2 + 2 \left\{ -1 + \frac{2}{3} M_1\alpha^2 - \frac{2}{7} \lambda^2 \right\} \frac{1}{r^3} - \frac{12\lambda^2}{7} \frac{1}{r^5} + o(\alpha^3) \right], \quad 1 < r < \infty \quad (75)$$

$$Y(ar) = -\frac{16\Omega(\mu_1 + \mu_2)aM_1\alpha^2\lambda^5}{45\pi^2} \left[r + o(\alpha) \right], \quad 0 < r < 1 \quad (76)$$

$$S_1(ar) = X(ar) - \frac{16\Omega(\mu_1 + \mu_2)aM_1 r \alpha^2 \lambda^5}{45\pi^2} + o(\alpha^8), \quad 0 < r < 1 \quad (77)$$

Detailed derivation of the above expressions have been done in appendix-B.

4. STRESS DIFFERENCE ACROSS THE ANNULAR DISC, TORQUE AND FAR FIELD AMPLITUDE

The jump of the stresses at the annular disc is given by

$$\begin{aligned} \tau(r, 0, t) = \tau(r) e^{-i\omega t} &= \tau_{z\theta}^{(2)}(r, 0, t) - \tau_{z\theta}^{(1)}(r, 0, t) = f(r) e^{-i\omega t} \\ &, \quad b \leq r \leq a, \quad r=0 \\ &= f_1(r) + f_2(r) \quad (\text{supressing } e^{-i\omega t}). \end{aligned}$$

Putting the values of $f_1(r)$ and $f_2(r)$ from (31) and (32) in the above expression we obtain

$$\begin{aligned} \tau(r) &= \frac{2}{\pi} \left[\frac{d}{dr} \left\{ -\int_r^a \frac{S_1(u) du}{(u^2 - r^2)^{1/2}} + \int_a^\infty \frac{T_1(u) du}{(u^2 - r^2)^{1/2}} \right\} + \right. \\ &\quad \left. + \frac{1}{r^2} \frac{d}{dr} \left\{ -\int_0^b \frac{u^2 T_2(u) du}{(r^2 - u^2)^{1/2}} + \int_b^r \frac{u^2 S_2(u) du}{(r^2 - u^2)^{1/2}} \right\} \right], \quad b \leq r \leq a \quad (78) \end{aligned}$$

Finally, substitution of the values of $S_1(u)$, $T_1(u)$, $T_2(u)$ and

$S_2(u)$ from (77), (75), (72) and (73) respectively in (78), yields after integration,

$$\begin{aligned} \tau(r) = & \frac{2(\mu_1 + \mu_2)\Omega}{\pi} \left[\frac{\nu_1}{(1-\nu_1^2)^{1/2}} \left\{ -2 + \left(\frac{2M_1}{3}(1-\nu_1^2) + \frac{2M_1}{3} \right) \alpha^2 + \frac{2i}{3} M_2 \alpha^3 + \right. \right. \\ & + \left. \left(\frac{1}{30}(7M_1^2 - 12M_3) - \frac{4}{15}(M_1^2 - M_3)(1-\nu_1^2) - \frac{2}{45}(M_1^2 - M_3)(1-\nu_1^2)^2 \right) \alpha^4 - \right. \\ & - i \left(\frac{2}{45}(11M_1 M_2 + 4M_4) + \frac{2}{9}(M_1 M_2 - M_4)(1-\nu_1^2) \right) \alpha^5 + \left. \left(\frac{1}{1260}(386M_1 M_3 - 74M_1^3 + \right. \right. \\ & + 280M_2^2 + 101M_5) - \frac{1}{630}(37M_1^3 + 80M_1 M_3 + 114M_5)(1-\nu_1^2) + \frac{1}{1260}(24M_1^3 + 78M_1 M_3 + \\ & + 87M_5)(1-\nu_1^2)^2 - \frac{2}{1575}(2M_1 M_3 - M_1^3 - M_5)(1-\nu_1^2)^3 \left. \right) \alpha^6 + i \left(\frac{1}{630}(75M_1^2 M_2 + \right. \\ & + 72M_1 M_4 + 156M_2 M_3 + 12M_6) + \frac{2}{135}(M_1^2 M_2 - M_1 M_4 - M_2 M_3 + M_6)(1-\nu_1^2)^2 - \frac{2}{45}(M_1 M_4 - \\ & - 4M_1^2 M_2 + 2M_2 M_3 + M_6)(1-\nu_1^2) \left. \right) \alpha^7 - \frac{16M_1 \alpha^2 \lambda^5}{45\pi^2} \left. \right\} + \frac{16\lambda^5}{45\pi^2} \left\{ - \frac{M_1 \alpha^2}{\nu_1} \left(- \frac{\sin^{-1} \nu_1}{\nu_1} + \right. \right. \\ & + (1-\nu_1^2)^{-1/2} \left. \right) + \frac{1}{\nu_1^3} \left(-1 + \frac{2}{3} M_1 \alpha^2 - \frac{2}{7} \lambda^2 \right) \left(- \frac{3 \sin^{-1} \nu_1}{\nu_1} + (1-\nu_1^2)^{1/2} + 2(1-\nu_1^2)^{-1/2} \right) - \\ & - \frac{3\lambda^2}{14\nu_1^5} \left(- \frac{15 \sin^{-1} \nu_1}{\nu_1} - 2(1-\nu_1^2)^{3/2} + 9(1-\nu_1^2)^{1/2} + 8(1-\nu_1^2)^{-1/2} \right) \left. \right\} - \end{aligned}$$

$$\begin{aligned}
& -\frac{2\lambda}{3\pi} \left\{ 2 \left[-1 + \frac{2}{3}M_1 \alpha^2 + \frac{1}{3}M_2 \alpha^3 \right] \left[3\nu_2 \sin^{-1} \left(\frac{1}{\nu_2} \right) - \frac{(\nu_2^2 - 1)^{1/2}}{\nu_2} - 2\nu_2 (\nu_2^2 - 1)^{-1/2} \right] - \right. \\
& \left. - \frac{\lambda^2 \nu_2^2}{5} \left[15\nu_2 \sin^{-1} \left(\frac{1}{\nu_2} \right) + \frac{2(\nu_2^2 - 1)^{3/2}}{\nu_2^3} - \frac{9(\nu_2^2 - 1)^{1/2}}{\nu_2} - 8\nu_2 (\nu_2^2 - 1)^{-1/2} \right] - \right. \\
& \left. - \frac{8M_1 \alpha^2 \lambda^4}{45\pi \nu_1 (\nu_2^2 - 1)^{1/2}} + o(\alpha^6) \right] , \quad b \leq r \leq a \quad (79)
\end{aligned}$$

where $\nu_1 = r/a$ and $\nu_2 = r/b$

Substituting $\alpha=0$ and $\lambda=0$ in (79) the jump in the statical stress across the rigid circular disc of radius a embedded at the bimaterial interface is easily found to be

$$\tau_0(r) = - \frac{4(\mu_1 + \mu_2)\Omega}{\pi} \frac{\nu_1}{(1 - \nu_1^2)^{1/2}}$$

so that the stress intensity factor at the edge of the circular disc in the statical case is

$$K_0 = \lim_{r \rightarrow a^-} \left[(1 - \nu_1^2)^{1/2} \tau_0(r) \right] = - \frac{2\sqrt{2}(\mu_1 + \mu_2)\Omega}{\pi} .$$

Therefore, in our dynamical problem involving annular disc, stress intensity factors at the outer and inner edges of the disc defined by

$$K_a^* = \frac{K_a}{K_0} = \text{Lt}_{r \rightarrow a-} \left[\frac{\tau(r) (1-\nu_1)^{1/2}}{K_0} \right]$$

and

$$K_b^* = \frac{K_b}{K_0} = \text{Lt}_{r \rightarrow b+} \left[\frac{\tau(r) (\nu_2 - 1)^{1/2}}{K_0} \right]$$

are given by

$$K_a^* = -\frac{1}{2} \left[\left\{ -2 + \frac{2}{3} M_1 \alpha^2 + \frac{1}{30} (7M_1^2 - 12M_3) \alpha^4 + \frac{1}{1260} (386M_1 M_3 - 74M_1^3 + 280M_2^2 + 101M_5) \alpha^6 - \right. \right. \\ \left. \left. - \frac{32\lambda^5}{45\pi^2} \left(1 + \frac{1}{3} M_1 \alpha^2 + \frac{8}{7} \lambda^2 \right) \right\} + i \left\{ \frac{2}{3} M_2 \alpha^3 - \frac{2}{45} (11M_1 M_2 + 4M_4) \alpha^5 + \frac{1}{630} (75M_1^2 M_2 + \right. \right. \\ \left. \left. + 72M_1 M_4 + 156M_2 M_3 + 12M_6) \alpha^7 \right\} \right] \quad (80)$$

$$\text{and } K_b^* = -\frac{1}{2} \left[\left\{ \frac{8\lambda}{3\pi} \left(-1 + \frac{2}{3} M_1 \alpha^2 \right) - \frac{16\lambda^3}{15\pi} - \frac{8M_1 \alpha^2 \lambda^3}{45\pi} \right\} + i \frac{8\lambda}{9\pi} M_2 \alpha^3 \right] \quad (81)$$

The torque of the shear stress on the annular disc is represented by the expression

$$T = 2\pi \int_b^a r^2 \tau(r) dr \quad (82)$$

which can be written after putting the value of $\tau(r)$ given by (79) and integrating as follows.

$$\begin{aligned}
T = \frac{4}{3}(\mu_1 + \mu_2)\Omega a^3 & \left[-4 + \frac{8}{5}M_1\alpha^2 + \frac{4i}{3}M_2\alpha^3 + \frac{1}{105}(37M_1 - 72M_3)\alpha^4 - \frac{4i}{15}(4M_1M_2 + M_4)\alpha^5 + \right. \\
& + \frac{1}{4725}(2704M_1M_3 + 2100M_2^2 - 650M_1^3 + 472M_5)\alpha^6 + \frac{i}{1575}(491M_1^2M_2 + 328M_1M_4 + \\
& \left. + 720M_2M_3 + 36M_6)\alpha^7 + \frac{64\lambda^5}{15\pi^2}\left(1 - \frac{4}{3}M_1\alpha^2 + \frac{4}{7}\lambda^2\right) + o(\alpha^8) \right] \quad (83)
\end{aligned}$$

Next, in order to deduce the far field amplitude of the displacement in both the media, we substitute the value of $A_j(\xi)$ in equation (8) and obtain

$$v_j(r, z) = \int_b^a t f(t) dt \int_0^\infty \frac{\xi}{(\mu_1\gamma_1 + \mu_2\gamma_2)} J_1(\xi r) J_1(\xi t) \exp(-\gamma_j |z|) d\xi \quad (84)$$

Evaluating the integral with respect to ξ by the method of steepest descent for large values of $\sqrt{(r^2 + z^2)}$, we obtain finally for $z > 0$

$$v_1(r, \theta, z) = F_1(\theta) \frac{e^{ik_1 R_1}}{R_1} + O\left(\frac{1}{R_1^2}\right) \quad \text{as } R_1 \rightarrow \infty \quad (85)$$

where $r = R_1 \cos\theta$, $|z| = R_1 \sin\theta$

$$F_1(\theta) = - \frac{1 \cos\theta}{\sigma \mu_1 \sin\theta + \mu_2 \sqrt{(1 - \sigma^2 \cos^2\theta)}} G_1(\theta), \quad \text{for } |\cos\theta| < \frac{1}{\sigma}$$

$$= - \frac{\sigma \sin \theta [\mu_2 \sqrt{(\sigma^2 \cos^2 \theta - 1)} + i \sigma \mu_1 \sin \theta]}{\sigma^2 \mu_1^2 \sin^2 \theta + \mu_2^2 (\sigma^2 \cos^2 \theta - 1)} G_1(\theta), \quad \text{for } |\cos \theta| > \frac{1}{\sigma}$$

(86)

and $G_1(\theta) = \frac{2\Omega a^2 (\mu_1 + \mu_2) \sigma \alpha \cos \theta}{\pi} \left[-\frac{2}{3} + \frac{4}{15} M_1 \alpha^2 + \frac{2i}{9} M_2 \alpha^3 + \frac{1}{630} (37M_1^2 - 72M_3) \alpha^4 - \right.$

$$-\frac{2i}{45} (4M_1 M_2 + M_4) \alpha^5 + \frac{1}{2835} (256M_1 M_3 - 65M_1^3 + 210M_2^2 + 40M_5) \alpha^6 +$$

$$+ \frac{i}{9450} (491M_1^2 M_2 + 328M_1 M_4 + 720M_2 M_3 + 36M_6) \alpha^7 - \frac{\sigma^2 \alpha^2 \cos^2 \theta}{6} \left\{ -\frac{2}{5} + \right.$$

$$+ \frac{16}{105} M_1 \alpha^2 + \frac{2i}{15} M_2 \alpha^3 + \frac{1}{1890} (73M_1^2 - 136M_3) \alpha^4 - \frac{2i}{1575} (82M_1 M_2 + 23M_4) \alpha^5 \left. \right\} +$$

$$+ \frac{\sigma^4 \alpha^4 \cos^4 \theta}{120} \left\{ -\frac{2}{7} + \frac{20}{189} M_1 \alpha^2 + \frac{2i}{21} M_2 \alpha^3 \right\} + \frac{\sigma^6 \alpha^6 \cos^6 \theta}{22680} +$$

$$\left. + \frac{32\lambda^5}{45\pi^2} \left\{ 1 - \frac{4}{3} M_1 \alpha^2 + \frac{4}{7} \lambda^2 + \frac{1}{6} \sigma^2 \alpha^2 \cos^2 \theta \right\} \right] \quad (87)$$

Also for $z < 0$,

$$v_2(r, \phi, z) = F_2(\phi) \frac{e^{ik_2 R_2}}{R_2} + O\left(\frac{1}{R_2^2}\right) \quad \text{as } R_2 \rightarrow \infty \quad (88)$$

where $r = R_2 \cos \phi$, $|z| = R_2 \sin \phi$

$$F_2(\phi) = - \frac{i \sin \phi}{\mu_1 \sqrt{(\sigma^2 - \cos^2 \phi)} + \mu_2 \sin \phi} G_2(\phi) \quad (89)$$

and $G_2(\phi)$ is obtained by replacing θ by ϕ and also σ by 1 in $G_1(\phi)$.

5. NUMERICAL RESULTS AND DISCUSSION

Numerical results have been calculated to study the variations of the dynamic stress intensity factors with the normalized frequency α at both the outer and inner edges of the annular disc situated at the bimaterial interface for different values of the ratio of the inner and outer radii of the annular disc for the following two sets of materials.

First set :

Aluminium	$\rho_1 = 2.7 \text{ gm/cm}^3$	$\mu_1 = 2.63 \times 10^{11} \text{ dyne/cm}^2$
Wrought iron	$\rho_2 = 7.8 \text{ gm/cm}^3$	$\mu_2 = 7.7 \times 10^{11} \text{ dyne/cm}^2$

Second set :

Copper	$\rho_1 = 8.96 \text{ gm/cm}^3$	$\mu_1 = 4.5 \times 10^{11} \text{ dyne/cm}^2$
Steel	$\rho_2 = 7.6 \text{ gm/cm}^3$	$\mu_2 = 8.32 \times 10^{11} \text{ dyne/cm}^2$

The dynamic stress intensity factors are normalized by the static solution $K_o = -2\sqrt{2}(\mu_1 + \mu_2)\Omega/\pi$ for the penny shaped rigid disc.

It is interesting to note that for both the two set of materials

stress intensity factor at the outer edge changes appreciably with the normalized frequency α and gradually decreases with the increase of α but in the case of inner edge, the stress intensity factor decrease very slowly with the increase in the values of the normalized frequency. It may further be noted that change in the values of the stress intensity factor with increase in the values of λ is more prominent at the inner edge than that at the outer edge. We also note from fig.2 and fig.3 that stress intensity factors for the two sets of materials are nearly the same for low frequency and increase gradually with the increase in frequency. Far field amplitudes defined by $F_1(\theta)$ and $F_2(\phi)$ in the upper and lower medium $z>0$ and $z<0$ respectively for fixed R have been plotted in fig.4 - fig.7 against their arguments for different values of the normalized frequency α and λ , the ratio of the inner and outer radii of the annular disc for two different sets of materials.

It may be noted that both in the upper and lower medium for the two sets of materials, amplitudes $F_1(\theta)$ and $F_2(\phi)$ respectively increase gradually from θ and ϕ equal to zero, attain maximum values and then gradually decrease to zero at θ and ϕ equal to 90° . The values of the angle at which maxima are attained are found to depend on the material properties and not on the values of the frequency and λ . On the other hand if the material properties are kept fixed, maximum values of the far field amplitude are found to depend on the normalized frequency α and λ , which is equal to the ratio of the inner and outer radii of the annular disc.

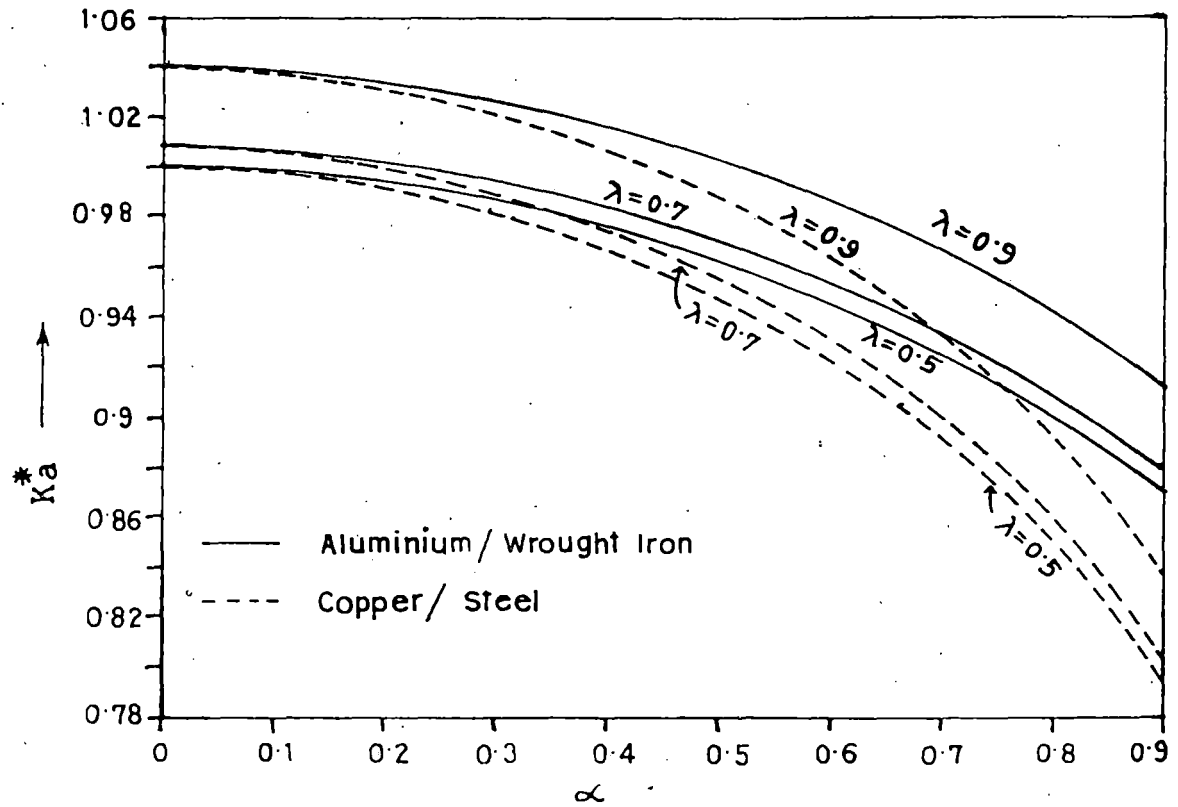


Fig. 2. Stress intensity factor vs. normalized frequency (outer edge)

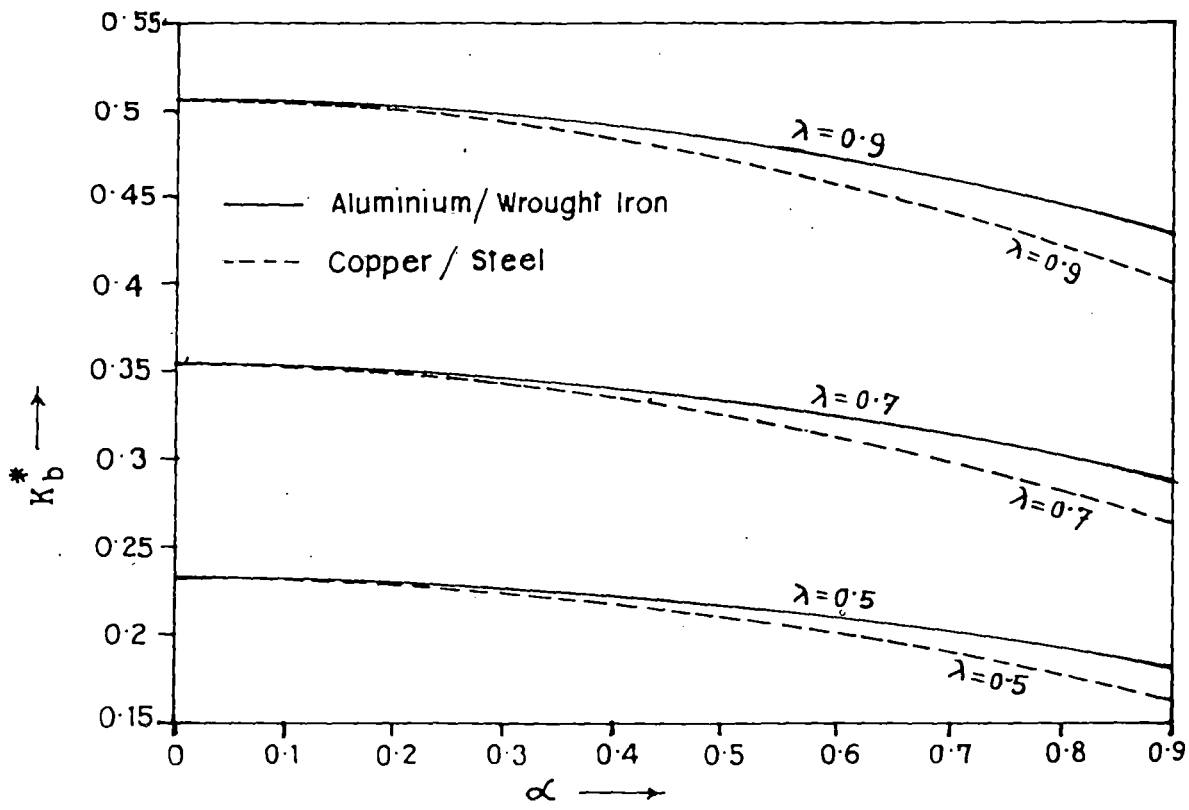


Fig. 3. Stress intensity factor vs. normalized frequency. (Inner edge)

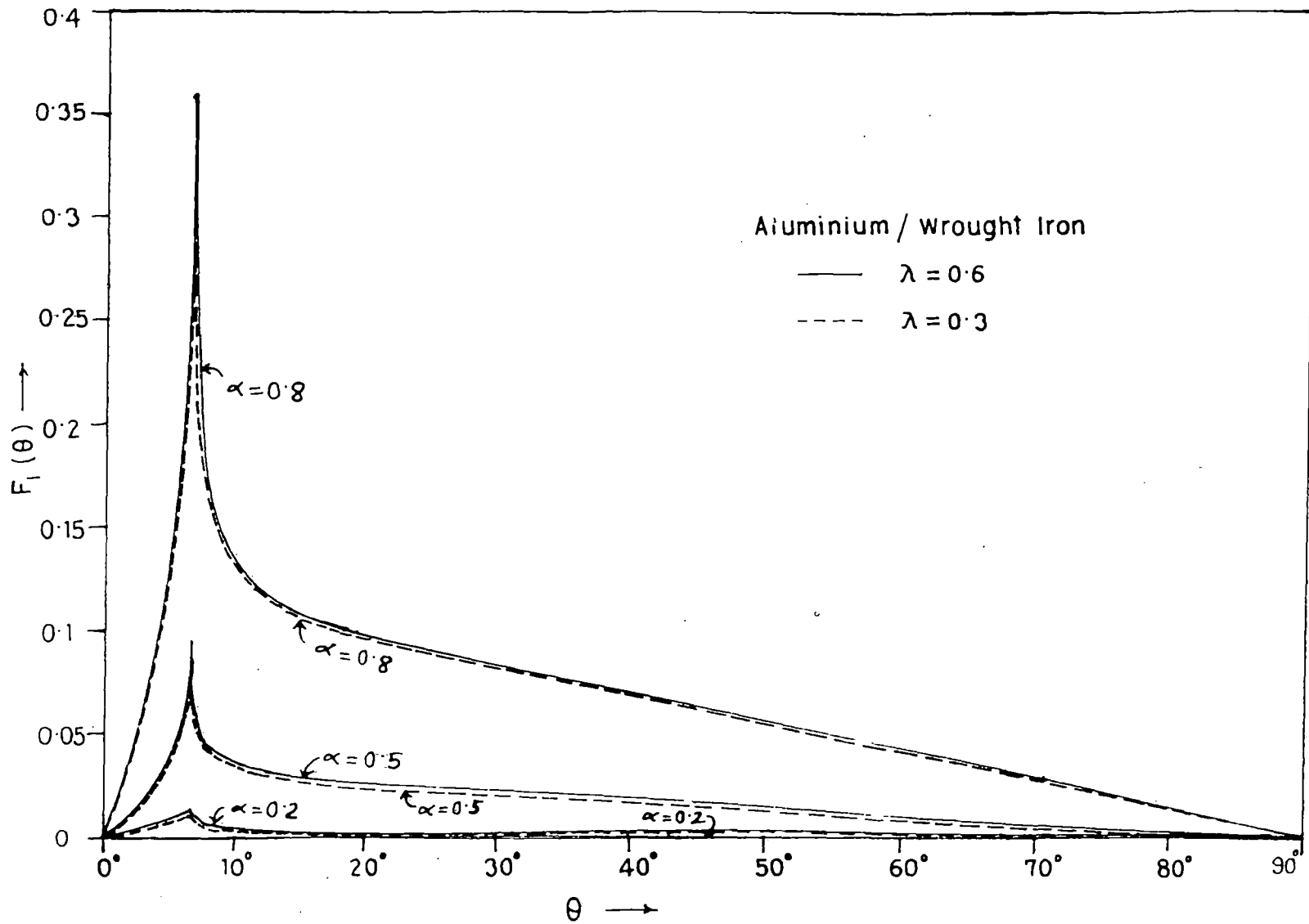


Fig. 4. Farfield amplitude $F_1(\theta)$ vs. argument θ of the amplitude for upper medium ($z > 0$)

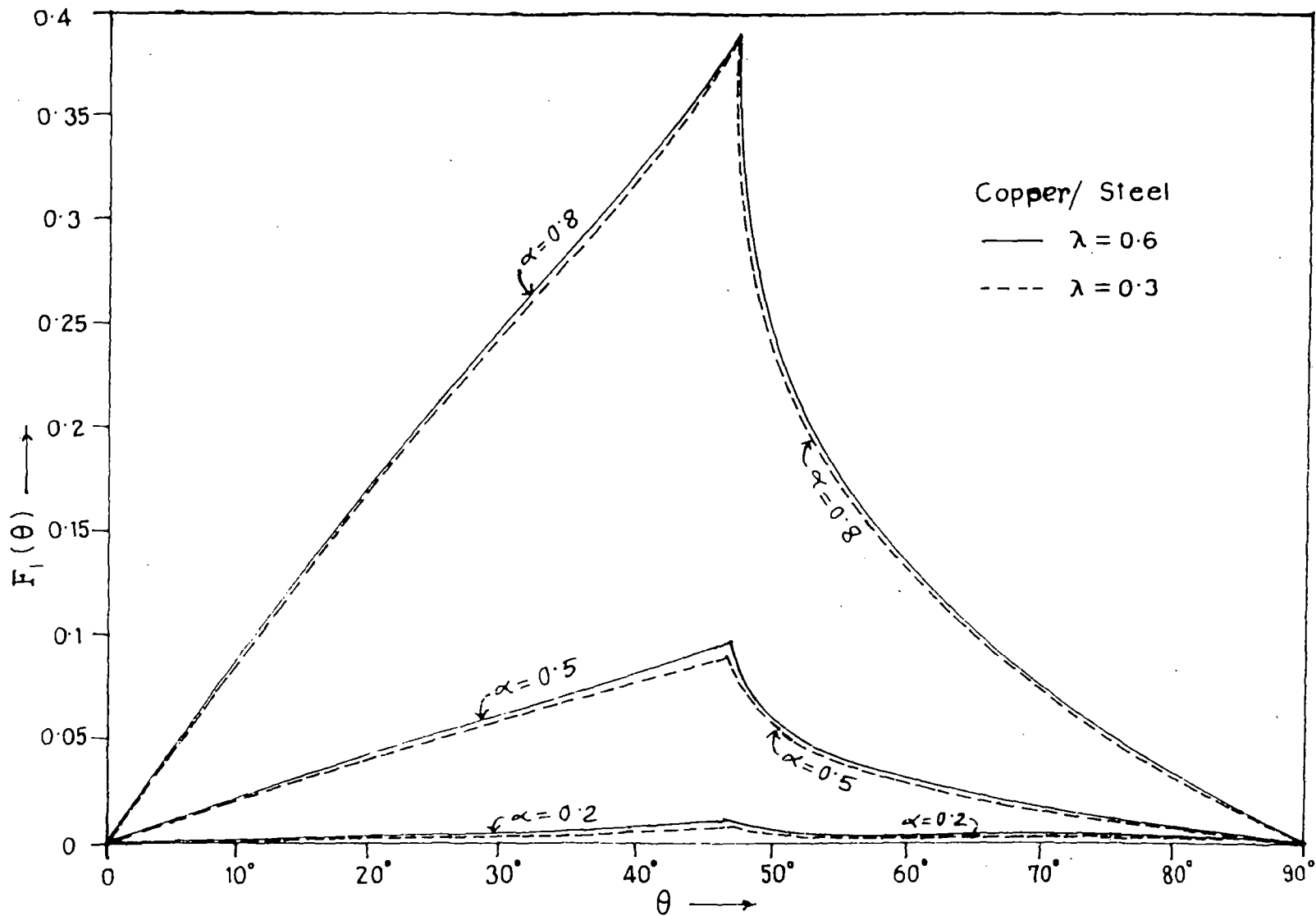


Fig. 5. Farfield amplitude $F_1(\theta)$ vs. argument θ of the amplitude for upper medium ($z > 0$)

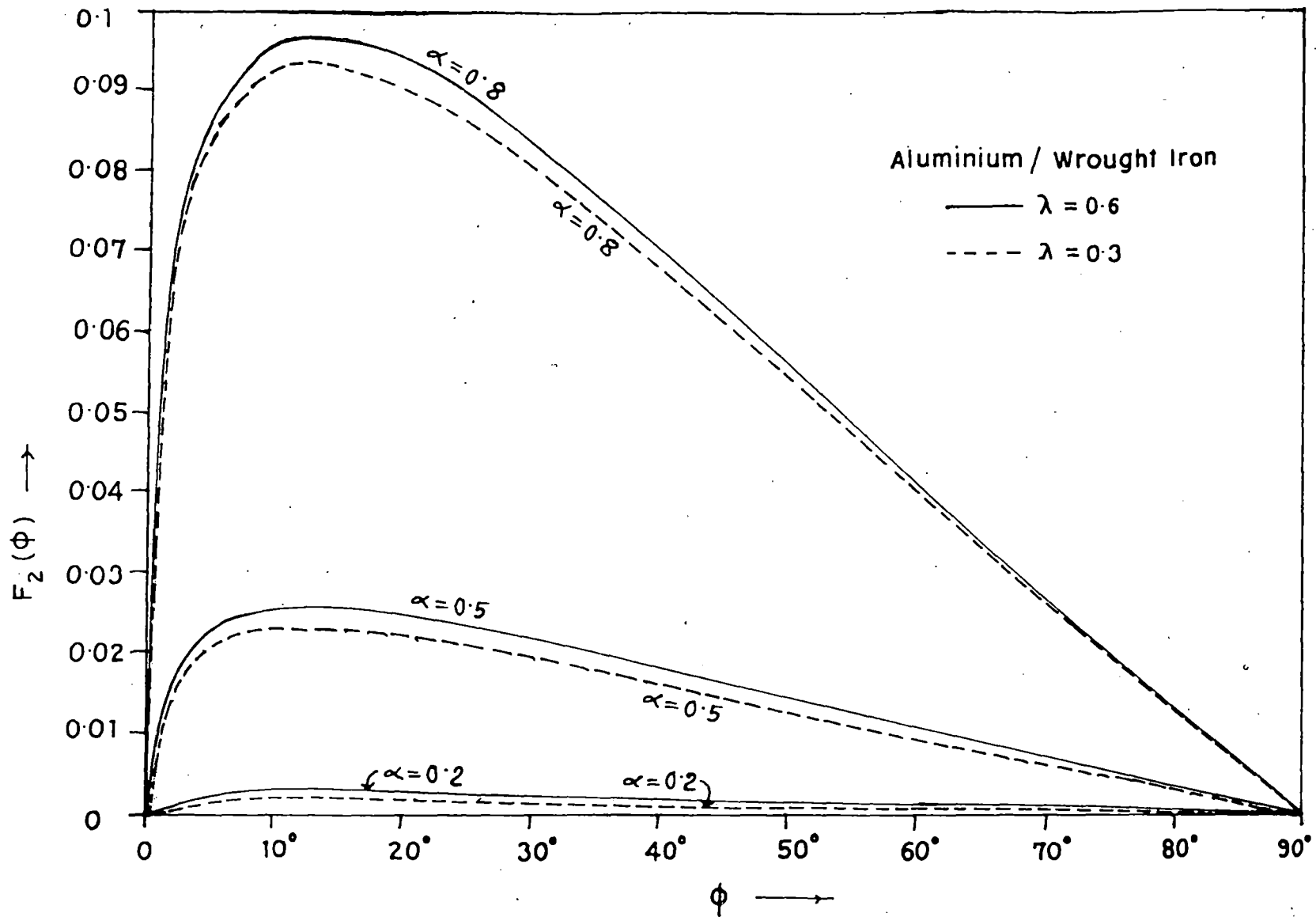


Fig. 6. Farfield amplitude $F_2(\phi)$ vs. argument ϕ of the amplitude for lower medium ($z < 0$)

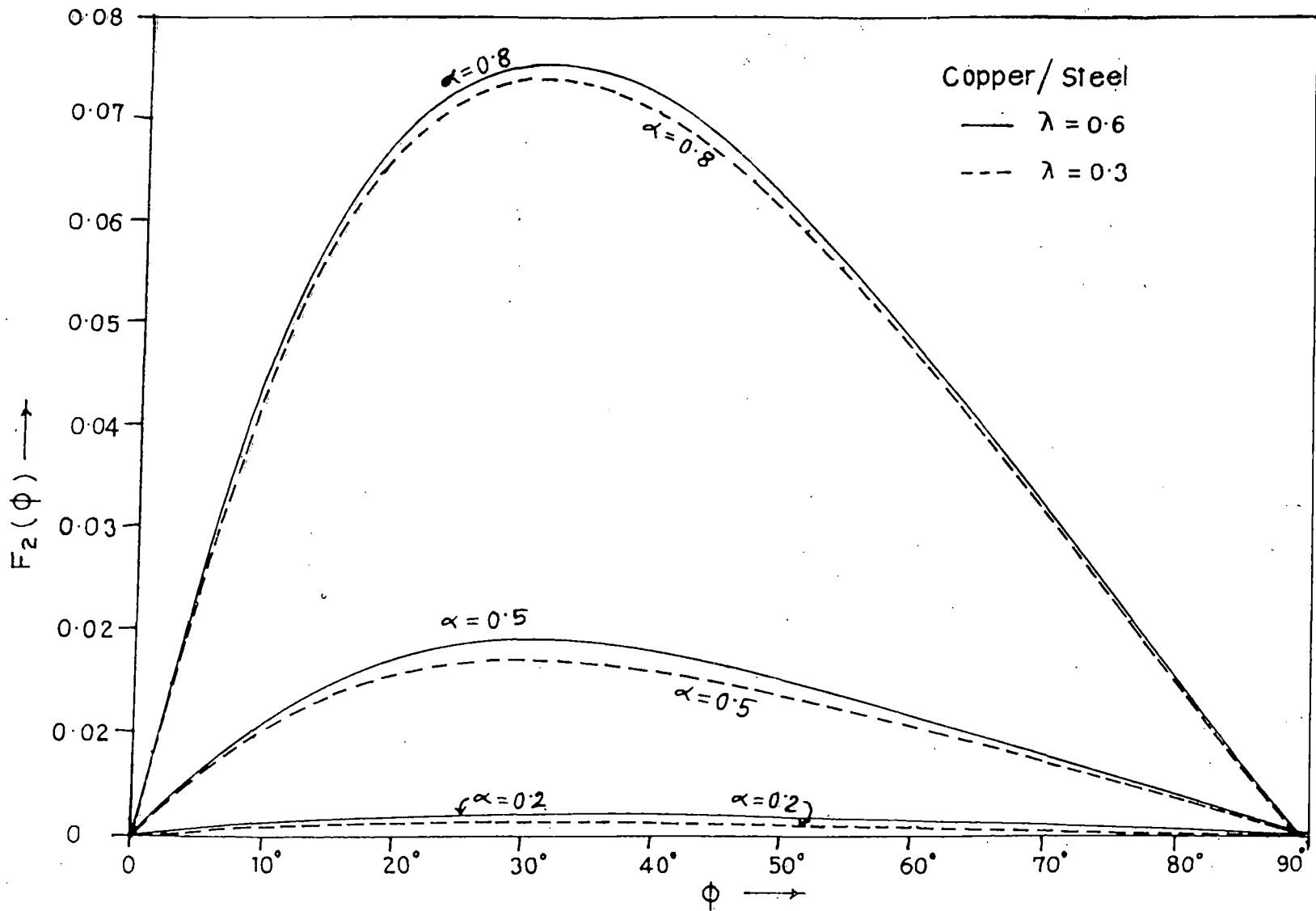


Fig. 7. Farfield amplitude $F_2(\phi)$ vs. argument ϕ of the amplitude for lower medium ($z < 0$)

APPENDIX - A

EVALUATION OF $aL_1(av, ar)$:

$$\begin{aligned}
 aL_1(av, ar) = i(1+\mu)\alpha^2(vr)^{1/2} & \left[\int_0^1 \frac{\eta^2 J_{1/2}(\alpha\eta r) H_{1/2}^{(1)}(\alpha\eta v)}{\mu(1-\eta^2)^{1/2} + (\sigma^2 - \eta^2)^{1/2}} d\eta + \right. \\
 & \left. + \int_1^\sigma \frac{\eta^2 (\sigma^2 - \eta^2)^{1/2} J_{1/2}(\alpha\eta r) H_{1/2}^{(1)}(\alpha\eta v)}{\mu^2(\eta^2 - 1) + (\sigma^2 - \eta^2)} d\eta \right], \quad v > r \quad (A1)
 \end{aligned}$$

For small values of arguments expanding the Bessel and Hankel functions we get

$$\begin{aligned}
 J_{1/2}(\alpha\eta r) H_{1/2}^{(1)}(\alpha\eta v) = \frac{2}{\pi} \sqrt{\frac{r}{v}} & \left[-i + \alpha\eta v + \frac{i\alpha^2 \eta^2}{6} (3v^2 + r^2) - \frac{\alpha^3 \eta^3}{6} (v^2 + r^2)v - \right. \\
 & \left. - \frac{i\alpha^4 \eta^4}{120} (5v^4 + 10r^2 v^2 + r^4) + \frac{\alpha^5 \eta^5}{360} (3v^5 + 10r^2 v^3 + 3r^4 v) + o(\alpha^6) \right]
 \end{aligned}$$

Putting this expansion in (A1) $aL_1(av, ar)$ can be evaluated as

$$\begin{aligned}
 aL_1(av, ar) = \alpha^2 r M_1 + i\alpha^3 r v M_2 - \frac{\alpha^4 (3v^2 r + r^3)}{6} M_3 - \frac{i\alpha^5 (v^3 r + r^3 v)}{6} M_4 + \\
 + \frac{\alpha^6 (5v^4 r + 10r^3 v^2 + r^5)}{120} M_5 + \frac{i\alpha^7 (3v^5 r + 10r^3 v^3 + 3r^5 v)}{360} M_6 + \\
 + o(\alpha^8), \quad v > r
 \end{aligned}$$

$$\begin{aligned}
&= \alpha^2 v M_1 + i \alpha^3 r v M_2 - \frac{\alpha^4 (3r^2 v + v^3)}{6} M_3 - \frac{i \alpha^5 (r^3 v + v^3 r)}{6} M_4 + \\
&\quad + \frac{\alpha^6 (5r^4 v + 10v^3 r^2 + v^5)}{120} M_5 + \frac{i \alpha^7 (3r^5 v + 10v^3 r^3 + 3v^5 r)}{360} M_6 + \\
&\quad + o(\alpha^8), \quad v < r \quad (A2)
\end{aligned}$$

where

$$M_i = \frac{2}{\pi} (1+\mu) \left[\int_0^\sigma \frac{\eta^{(i+1)} (\sigma^2 - \eta^2)^{1/2}}{\mu^2 (\eta^2 - 1) + (\sigma^2 - \eta^2)} d\eta - \mu \int_0^1 \frac{\eta^{(i+1)} (1 - \eta^2)^{1/2}}{\mu^2 (\eta^2 - 1) + (\sigma^2 - \eta^2)} d\eta \right]$$

$i=1, 2, \dots, 6$ (A3)

Now

$$M_1 = \frac{2}{\pi} (1+\mu) \left[\int_0^\sigma \frac{\eta^2 (\sigma^2 - \eta^2)^{1/2}}{\mu^2 (\eta^2 - 1) + (\sigma^2 - \eta^2)} d\eta - \mu \int_0^1 \frac{\eta^2 (1 - \eta^2)^{1/2}}{\mu^2 (\eta^2 - 1) + (\sigma^2 - \eta^2)} d\eta \right]$$

(A4)

Without any loss of generality we assume $\mu > \tau > 1$.

The first integral in the expression of M_1 is

$$\begin{aligned}
&\int_0^\sigma \frac{\eta^2 (\sigma^2 - \eta^2)^{1/2}}{\mu^2 (\eta^2 - 1) + (\sigma^2 - \eta^2)} d\eta \\
&= \frac{1}{(\mu^2 - 1)} \int_0^\sigma (\sigma^2 - \eta^2)^{1/2} d\eta + \frac{\mu^2 - \sigma^2}{(\mu^2 - 1)^2} \int_0^\sigma \frac{(\sigma^2 - \eta^2)^{1/2}}{\eta^2 - \frac{\mu^2 - \sigma^2}{\mu^2 - 1}} d\eta \\
&= \frac{\sigma^2}{2(\mu^2 - 1)} \frac{\pi}{2} - \frac{\mu^2 - \sigma^2}{(\mu^2 - 1)^2} \frac{\pi}{2} \quad (A5)
\end{aligned}$$

and the second integral is

$$\begin{aligned}
 & \int_0^1 \frac{\eta^2 (1-\eta^2)^{1/2}}{\mu^2 (\eta^2 - 1) + (\sigma^2 - \eta^2)} d\eta \\
 &= \frac{1}{(\mu^2 - 1)} \int_0^1 (1-\eta^2)^{1/2} d\eta + \frac{\mu^2 - \sigma^2}{(\mu^2 - 1)^2} \int_0^1 \frac{(1-\eta^2)^{1/2}}{\eta^2 - \frac{\mu^2 - \sigma^2}{\mu^2 - 1}} d\eta \\
 &= \frac{1}{2(\mu^2 - 1)} \frac{\pi}{2} - \frac{\mu^2 - \sigma^2}{(\mu^2 - 1)^2} \frac{\pi}{2}
 \end{aligned} \tag{A6}$$

Putting the results (A5) and (A6) in (A4) and simplifying, M_1 can be obtained as

$$M_1 = \frac{\sigma^2 + \mu}{2(\mu + 1)} \tag{A7}$$

Similarly, $M_i (i=2,3,\dots,6)$ can be calculated and they are found to be given by

$$M_2 = \frac{2}{\pi(\mu - 1)} \left[\frac{\sigma^3 - \mu + \mu^2 - \sigma^2}{3} + \frac{\mu^2 - \sigma^2}{\mu^2 - 1} \left\{ (\mu - \sigma) + \mu \sqrt{\frac{\sigma^2 - 1}{\mu^2 - 1}} \log \left(\frac{\sigma \sqrt{\mu^2 - 1} + \mu \sqrt{\sigma^2 - 1}}{\sqrt{\mu^2 - 1} + \sqrt{\sigma^2 - 1}} \right) \right\} \right] \tag{A8}$$

$$M_3 = \frac{1}{(\mu - 1)} \left[\frac{1}{8}(\sigma^4 - \mu) + \left(\frac{\mu^2 - \sigma^2}{\mu^2 - 1} \right) \left\{ \frac{\sigma^2 - \mu}{2} + \frac{\mu^2 - \sigma^2}{\mu + 1} \right\} \right] \tag{A9}$$

$$M_4 = \frac{2}{\pi(\mu-1)} \left[\frac{2(\sigma^5 - \mu)}{15} + \frac{\mu^2 - \sigma^2}{\mu^2 - 1} \left(\frac{\sigma^3 - \mu}{3} + \frac{\mu^2 - \sigma^2}{\mu^2 - 1} \left\{ (\mu - \sigma) + \mu \frac{\sqrt{\sigma^2 - 1}}{\sqrt{\mu^2 - 1}} \times \right. \right. \right. \\ \left. \left. \left. \times \log \left(\frac{\sigma \sqrt{\mu^2 - 1} + \mu \sqrt{\sigma^2 - 1}}{\sqrt{\mu^2 - 1} + \sqrt{\sigma^2 - 1}} \right) \right\} \right) \right] \quad (A10)$$

$$M_5 = \frac{1}{(\mu-1)} \left[\frac{\sigma^6 - \mu}{16} + \left(\frac{\mu^2 - \sigma^2}{\mu^2 - 1} \right) \left\{ \frac{1}{8} (\sigma^4 - \mu) + \left(\frac{\mu^2 - \sigma^2}{\mu^2 - 1} \right) \left(\frac{\sigma^2 - \mu}{2} + \frac{\mu^2 - \sigma^2}{\mu + 1} \right) \right\} \right] \quad (A11)$$

$$M_6 = \frac{2}{\pi(\mu-1)} \left[\frac{8}{105} (\sigma^7 - \mu) + \frac{\mu^2 - \sigma^2}{\mu^2 - 1} \left\{ \frac{2(\sigma^5 - \mu)}{15} + \frac{\mu^2 - \sigma^2}{\mu^2 - 1} \left(\frac{\sigma^3 - \mu}{3} + \frac{\mu^2 - \sigma^2}{\mu^2 - 1} \times \right. \right. \right. \\ \left. \left. \left. \times \left\{ (\mu - \sigma) + \mu \frac{\sqrt{\sigma^2 - 1}}{\sqrt{\mu^2 - 1}} \log \left(\frac{\sigma \sqrt{\mu^2 - 1} + \mu \sqrt{\sigma^2 - 1}}{\sqrt{\mu^2 - 1} + \sqrt{\sigma^2 - 1}} \right) \right\} \right) \right\} \right] \quad (A13)$$

APPENDIX - B

$l_1(r)$ is given by

$$l_1(r) = -\frac{2}{\pi r} \int_0^r \frac{t^2 dt}{(r^2 - t^2)^{1/2}} \frac{d}{dt} \int_t^a \frac{S_1(u) du}{(u^2 - t^2)^{1/2}}, \quad 0 < r < b \quad (B1)$$

Taking $S_1(u) = X(u)$ as a first approximation from (42) (B1) can be written as

$$l_1(r) = -\frac{2}{\pi r} \int_0^r \frac{t^2 dt}{(r^2 - t^2)^{1/2}} \frac{d}{dt} \int_{t/a}^1 \frac{X(ay) dy}{(y^2 - t^2/a^2)^{1/2}}, \quad 0 < r < b \quad (B2)$$

Substituting $X(ay) = a\Omega(\mu_1 + \mu_2) [p_1(\alpha)y + p_3(\alpha)y^3]$,

(neglecting α^4 and higher powers of α)

where

$$p_1(\alpha) = -2 + M_1 \alpha^2 + \frac{2i}{3} M_2 \alpha^3 + o(\alpha^4)$$

$$p_3(\alpha) = -\frac{1}{3} M_1 \alpha^2 + o(\alpha^4)$$

in (B2) and integrating we obtain

$$l_1(r) = \frac{2\Omega}{\pi r} (\mu_1 + \mu_2) \left[-2 + \frac{4}{3} M_1 \alpha^2 + \frac{2i}{3} M_2 \alpha^3 \right] \frac{r^3}{2a} \left[\frac{4}{3} + \frac{8r^2}{15a^2} \right] - \\ - \frac{4\Omega}{3\pi r a^2} (\mu_1 + \mu_2) M_1 \alpha^2 \frac{8r^5}{15a}, \quad 0 < r < b$$

Replacing r by br and putting $b/a=\lambda$ in the above expression we obtain,

$$l_1(br) = \frac{8\Omega(\mu_1 + \mu_2)ar^2\lambda^2}{3\pi} \left[-1 + \frac{2}{3}M_1\alpha^2 + \frac{1}{3}M_2\alpha^3 - \frac{2}{5}\lambda^2 r^2 + o(\alpha^4) \right], \quad 0 < r < 1 \quad (B3)$$

Next,

$$T_2(br) = l_1(br) + \frac{1}{\pi\lambda r} \int_1^\infty T_1(au) \left\{ \frac{2\lambda ur}{(u^2 - \lambda^2 r^2)} - \log \left(\frac{u+\lambda r}{u-\lambda r} \right) \right\} du, \quad 0 < r < 1$$

Neglecting higher order terms of α , $T_2(br)$ is found to be

$$T_2(br) = l_1(br) + o(\alpha^7), \quad 0 < r < 1 \quad (B4)$$

Replacing r by br , (29) can be rewritten as

$$S_2(br) + b \int_1^\infty L_2(bv, br) S_2(bv) dv = b \int_0^1 L_2(bv, br) T_2(bv) dv, \quad 1 < r < \infty \quad (B5)$$

First approximation of the above integral equation yields

$$S_2(br) = b \int_0^1 L_2(bv, br) T_2(bv) dv, \quad 1 < r < \infty$$

in which substituting the value of $T_2(bv)$ from (B4) and

$$bL_2(bv, br) = \alpha^2 \lambda^2 \left[\frac{1}{3}M_1 \frac{v^2}{r} + o(\alpha^4) \right], \quad v < r$$

we get

$$S_2(\text{br}) = - \frac{8\Omega(\mu_1 + \mu_2)aM_1\alpha^2\lambda^4}{45\pi} \left[\frac{1}{r} + o(\alpha^2) \right], \quad 1 < r < \infty \quad (\text{B6})$$

Similarly, first approximation of other functions from their respective integral equations can be derived and are given by

$$I_2(\text{ar}) = - \frac{16\Omega(\mu_1 + \mu_2)aM_1\alpha^2\lambda^5}{45\pi^2} \left[\frac{1}{r} + o(\alpha^2) \right], \quad 1 < r < \infty \quad (\text{B7})$$

$$T_1(\text{ar}) = \frac{16\Omega(\mu_1 + \mu_2)a\lambda^5}{45\pi^2} \left[-\frac{1}{r} M_1\alpha^2 + 2 \left\{ -1 + \frac{2}{3} M_1\alpha^2 - \frac{2}{7} \lambda^2 \right\} \frac{1}{r^3} - \frac{12\lambda^2}{7} \frac{1}{r^5} + o(\alpha^3) \right], \quad 1 < r < \infty \quad (\text{B8})$$

$$Y(\text{ar}) = - \frac{16\Omega(\mu_1 + \mu_2)aM_1\alpha^2\lambda^5}{45\pi^2} \left[r + o(\alpha) \right], \quad 0 < r < 1 \quad (\text{B9})$$

$$S_1(\text{ar}) = X(\text{ar}) - \frac{16\Omega(\mu_1 + \mu_2)aM_1r\alpha^2\lambda^5}{45\pi^2} + o(\alpha^3), \quad 0 < r < 1 \quad (\text{B10})$$

where $X(\text{ar})$ is given by (68).