CHAPTER - I

A NOTE ON LARGE DEFLECTION ANALYSIS OF CIRCULAR SANDWICH PLATES - A NEW APPROACH

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PAPER - I

* A NOTE ON LARGE DEFLECTION ANALYSIS OF CIRCULAR SANDWICH PLATES - A NEW APPROACH

ABSTRACT

The field of sandwich construction is gaining importance in recent years as a result of improvements in manufacturing techniques. As new manufacturing methods are now being developed which make the use of sandwiches economically feasible, the collection of more research data is becoming increasingly important.

In the present analysis non-linear static and dynamic behaviour of circular sandwich plates due to large deflection have been studied by a new set of decoupled differential equations. Numerical results for different types of loading have been computed and compared to other known results.

Governing Equation

Let us first take a cylindrical polar co-ordinate system (r, θ, z); (r, θ) in the middle plane of the core and z in the thickness direction (positive downwards).

For the sake of simplicity let us consider a sandwich plate with an isotropic core as well as isotropic upper and lower faces of identical thickness while the faces respond to the bending and membrane actions of the plate, the core is assumed to transfer only shear deformations. Moreover, compared to the core thickness h, the face thickness t is supposed to be thin enough to ignore a variation of stress in the thickness direction of the faces.

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If the expressions for the strains in the i^{th} face in the r and θ directions are noted as ε_{ri} and $\varepsilon_{\theta i}$, respectively, the transverse shear strain as γ_i, the following equations hold true for each of the separate face sheets of a circular plate having a rotational symmetry :

$$\varepsilon_{ri} = \frac{\partial u_i}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2,$$

$$\varepsilon_{\theta i} = \frac{u_i}{r}, \gamma_i = 0, \qquad ..(1)$$

where u_i and w are the middle-plane displacements of the i^{th} –face sheet considered in the r and z directions respectively.

Let the stress-strain relations for each face sheet be given by

$$N_{ri} = B_i\left(\varepsilon_{ri} + \mu\varepsilon_{\theta i}\right),.\ B_i = \frac{E_i t_i}{1-\mu^2}$$

$$N_{\theta i} = B_i\left(\varepsilon_{\theta i} + \mu\varepsilon_{ri}\right), \tau_i = 0, \qquad ..(2)$$

where E_i, μ, t_i (= t) refer to Young's modulus, Poison's ratio and thickness of the i^{th} – face sheet considered, respectively. When dual subscripts are used, the first subscript refers to the direction of the strain and the second refers to the face sheet under consideration. Thus ε_{r1} signifies the strain in the radial direction in the upper face.

Let us now introduce the average values of both face strain components as

$$\varepsilon_r^m = \frac{1}{2}\left(\varepsilon_{r1} + \varepsilon_{r2}\right),\ \varepsilon_\theta^m = \frac{1}{2}\left(\varepsilon_{\theta 1} + \varepsilon_{\theta 2}\right),\ \gamma^m = \frac{1}{2}\left(\gamma_1 + \gamma_2\right) = 0, \qquad ..(3)$$

with $p = u_1 - u_2$ and $U = \dfrac{u_1 + u_2}{2}$ ..(4)

Then from (1), (3) and (4), we can write

$$\varepsilon_{r1} = \varepsilon_r^m + \frac{1}{2}\frac{\partial p}{\partial r}, \text{ and } \varepsilon_{\theta 1} = \varepsilon_\theta^m + \frac{1}{2}\frac{p}{r}$$

$$\varepsilon_{r2} = \varepsilon_r^m - \frac{1}{2}\frac{\partial p}{\partial r}, \text{ and } \varepsilon_{\theta 2} = \varepsilon_\theta^m - \frac{1}{2}\frac{p}{r} \qquad ..(5)$$

By virtue of Hook's law, strain energy per unit area of both faces for isotropic elastic materials ($E^1 = E^2$) is represented as

$$\bar{V}_o^f = \frac{Et}{1-\nu^2}[\left(\varepsilon_r^m + \nu\varepsilon_\theta^m\right)^2 + \frac{1}{4}\left(\frac{\partial p}{\partial r}\right)^2 + \frac{1}{4}\left(\frac{p}{r}\right)^2 + \frac{\nu}{2}\frac{\partial p}{\partial r}.\frac{p}{r} + \left(1-\nu^2\right)\varepsilon_\theta^{m^2}$$

$$= \frac{Et}{1-\nu^2}[I_1^{m^2} + \lambda\left(\frac{w_r^2}{2}\right)^2 + \frac{1}{4}\left(\frac{\partial p}{\partial r}\right)^2 + \frac{1}{4}\left(\frac{p}{r}\right)^2 + \frac{\nu}{2}\frac{\partial p}{\partial r}.\frac{p}{r}] \qquad ..(6)$$

where it is assumed that $(1 - \nu^2)\ \left(\varepsilon_\theta^m\right)^2 = \lambda\left(\frac{w_r^2}{2}\right)^2$. The assumption is entirely new and has a resemblance with the line of thought of Dutta and Banerjee (1991). λ is a constant to be determined from the idea of minimum potential energy and

$$I_1^m = \varepsilon_r^m + \nu\varepsilon_\theta^m = \frac{\partial U}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \nu\frac{U}{r} \qquad ..(7)$$

Moreover, the strain energy per unit area of the isotropic core due to shear becomes

$$\bar{V}_o^c = \frac{1}{2}hG'\left[\left(\frac{P}{h}\right)^2 + w_r^2 + 2\frac{P}{h}.w_r\right] \qquad ..(8)$$

In consequence, the total strain energy per unit area of sandwich circular plate is

$$\bar{V}_o = \bar{V}_o^f + \bar{V}_o^c \qquad ..(9)$$

Total strain energy is

$$E' = \iint_A \left(\bar{V}_o^f + \bar{V}_o^c\right) r\, dr\, d\theta - \iint q\, wr\, dr\, d\theta$$

$$= \iint_A [\frac{Et}{1-\nu^2}\left\{I_1^{m^2} + \lambda\left(\frac{w_r^2}{2}\right)^2 + \frac{1}{4}\left(\frac{\partial p}{\partial r}\right)^2 + \frac{1}{4}\left(\frac{p}{r}\right)^2 + \frac{\nu}{2}\frac{\partial p}{\partial r}.\frac{P}{r}\right\}$$

$$+ \frac{1}{2}hG'\left\{\left(\frac{P}{h}\right)^2 + w_r^2 + 2\frac{p}{h}.w_r\right\}]r\, dr\, d\theta - \iint q\, wr\, dr\, d\theta \qquad ..(10)$$

where q is the external distributed load acting in a direction normal to the middle surface of the sheet.

Executing the Euler's variational principle to minimize the total potential energy E' of the present elastic system of the sandwich plate, we have the following equations :

$$\frac{\partial E'}{\partial U}-\frac{\partial}{\partial r}\frac{\partial E'}{\partial U_r}=0, \quad ..(11)$$

$$\frac{\partial E'}{\partial p}-\frac{\partial}{\partial r}\frac{\partial E'}{\partial p_r}=0, \quad ..(12)$$

and $$\frac{\partial E'}{\partial w}-\frac{\partial}{\partial r}\frac{\partial E'}{\partial w_r}+\frac{\partial^2}{\partial r^2}\frac{\partial E'}{\partial w_{rr}}=0, \quad ..(13)$$

where $U_r=\frac{\partial u}{\partial r}$, $p_r=\frac{\partial p}{\partial r}$ and $w_r=\frac{\partial w}{\partial r}$

Simplifying (11), (12), (13), we get

$$I_1^m = A r^{\nu-1}, \text{ where A is a constant of integration} \quad ..(14)$$

$$\frac{Et}{1-\nu^2}\frac{d}{dr}\left(r\frac{dp}{dr}\right)-\left(\frac{Et}{1-\nu^2}\cdot\frac{1}{r}+2G'\frac{r}{h}\right)p-2G'rw_r=0, \quad ..(15)$$

and $$qr+\frac{d}{dr}\left[2Ar^{\nu}w_r+\lambda r w_r^3\right]\frac{Et}{1-\nu^2}+hG'\frac{d}{dr}(rw_r)+G'\frac{d}{dr}(rp)=0, \quad ..(16)$$

From (16), we get

$$r\frac{dp}{dr}=-p-h\frac{d}{dr}(rw_r)-\frac{qr}{G'}-\frac{d}{dr}\left[2Ar^{\nu}w_r+\lambda r w_r^3\right]\frac{Et}{G'\left(1-\nu^2\right)} \quad ..(17)$$

From (15) and (17), we obtain that

$$\frac{Et}{1-\nu^2}\frac{dp}{dr}+\frac{hEt}{1-\nu^2}\frac{d^2}{dr^2}(rw_r)+\frac{Et}{G'\left(1-\nu^2\right)}\frac{d}{dr}(qr)$$

$$+\left(\frac{Et}{1-\nu^2}\right)^2\frac{1}{G'}\frac{d^2}{dr^2}(2Ar^{\nu}w_r+\lambda r w_r^3)+\left(\frac{Et}{1-\nu^2}\cdot\frac{1}{r}+2G'\frac{r}{h}\right)p+2G'rw_r=0$$

..(18)

From (17), we have

$$\frac{dp}{dr}=-\frac{p}{r}-\frac{h}{r}\frac{d}{dr}(r\,w_{,r})-\frac{q}{G'}-\left[\frac{d}{dr}\left(2Ar^{2}\,w_{,r}+\lambda r\,w_{,r}^{3}\right)\right]\frac{Et}{G'(1-\nu^{2})} \quad ..(19)$$

Using (19), we can write (18) as

$$rp=-\frac{h}{2G'}\Big[\frac{Et}{G'(1-\nu^{2})}\Big[\frac{d}{dr}(qr)-q\Big]+\frac{1}{G'}\Big(\frac{Et}{1-\nu^{2}}\Big)^{2}\frac{d^{2}}{dr^{2}}(2Ar^{\nu}\,w_{,r}+\lambda r\,w_{,r}^{3})$$

$$-\frac{1}{G'r}\Big(\frac{Et}{1-\nu^{2}}\Big)^{2}\frac{d}{dr}(2Ar^{\nu}\,w_{,r}+\lambda r\,w_{,r}^{3})-\frac{Eth}{(1-\nu^{2})r}\frac{d}{dr}(r\,w_{,r})$$

$$+\frac{Eth}{1-\nu^{2}}\frac{d^{2}}{dr^{2}}(r\,w_{,r})+2G'r\,w_{,r}\Big] \quad ..(20)$$

Eliminating (r p) from (20) and (16), we obtain finally that

$$\frac{Bh}{r}\Big[\frac{d^{3}}{dr^{3}}(r\,w_{,r})-\frac{d}{dr}\Big\{\frac{1}{r}\frac{d}{dr}(r\,w_{,r})\Big\}\Big]+\frac{B}{G'r}\frac{d}{dr}\Big(r\frac{dq}{dr}\Big)-2\frac{q}{h}$$

$$+\frac{B^{2}}{G'r}\Big[2A\Big\{\frac{d^{3}}{dr^{3}}(r^{\nu}\,w_{,r})-\frac{d}{dr}\frac{1}{r}\frac{d}{dr}((r^{\nu}\,w_{,r})\Big\}+\lambda\Big\{\frac{d^{3}}{dr^{3}}(r\,w_{,r}^{3})$$

$$-\frac{d}{dr}\frac{1}{r}\frac{d}{dr}((r\,w_{,r}^{3})\Big\}\Big]-2\frac{B}{hr}\frac{d}{dr}[2Ar^{\nu}\,w_{,r}+\lambda r\,w_{,r}^{3}]=0 \quad ..(21)$$

Boundary Conditions

If the edges are clamped, the boundary conditions are

$W = 0,\ w_{r} = 0$ for $r = a$ .. (22)

Solution of the problem

For the solution of the equation (21) satisfying the boundary conditions (22), we assume

$w = w_{0}\,(1 - r^{2} / a^{2})^{2}$ .. (23)

Introducing (23) in (14) and assuming suitable solution for U compatible with the boundary conditions (22) and carrying out desired integration on the plate, we get

$A=\{(\nu+1)\,w_{0}^{2}\}/(3a^{\nu+1})$ .. (24)

Introducing (23) into (21) and carrying out the required integration and eliminating A with the help of (24), we obtain finally for the **immovable edge** that

$$(w_0 / h)^3 + 2.1202979\,(w_0 / h) - 1.7862905\ \{ qa^4 / (Eh^4) \} = 0 \qquad ..\ (25)$$

For the **movable edge,** we put A = 0 in (21) and with the help of (23), we obtain that

$$(w_0 / h)^3 + 15.896471\,(w_0 / h) - 13.392632\ \{ qa^4 / (Eh^4) \} = 0 \qquad ..\ (26)$$

For $\lambda = 0$, we get the equation (27), which corresponds to the equation if Berger's method is applied :

$$(w_0 / h)^3 + 1.8707727\,(w_0 / h) - 1.5760723\ \{ qa^4 / (Eh^4) \} = 0 \qquad ..\ (27)$$

Numerical Results :

Table-I shows the numerical results of central deflections of circular plates obtained from (25) for **immovable edge** and from (26) for **movable edge,** where geometry of plates and material constants are taken as

$$a = 0.254\text{m},\ t = 0.635 \times 10^{-3}\text{m},\ h = 1.7135 \times 10^{-2}\text{m},\ E = 7347.201 \times 10^{6}\text{kg} / \text{m}^2,$$

$$G' = 4218.4884 \times 10^3\ \text{kg} / \text{m}^2,\ \nu = 0.03 \text{ and } \lambda = 0.09 \text{ and } Q = qa^4 / Eh^4 \quad ..\ (28)$$

Table – I

Q	Immovable Edge (P.S.)	Immovable Edge[14]	Movable Edge (P.S.)	Movable Edge[14]
	w_0 / h	w_0 / h	w_0 / h	w_0 / h
0.256	0.211	0.211	0.215	Absurd
0.556	0.431	0.427	0.462	Absurd
0.856	0.613	0.604	0.700	Absurd
1.154	0.763	0.748	0.923	Absurd
2.540	1.240	1.205	1.783	Absurd
3.540	1.474	1.427	2.258	Absurd
4.540	1.660	1.607	2.652	Absurd
5.540	1.821	1.759	2.988	Absurd
6.540	1.960	1.892	3.283	Absurd

Vibrations Under Dynamic Loading

Let us now consider free vibrations of sandwich circular plates. Adding the potential energy given in equation (9) to the kinetic energy of the plate, one may form the Lagrangian function and then applying. Hamilton's principle, the following equation (29) is obtained (neglecting in-plane inertia) through Euler's variational principle :

$$-\frac{Bh}{r}[\frac{\partial^3}{\partial r^3}(rw_{,r})-\frac{\partial}{\partial r}\{\frac{1}{r}\frac{\partial}{\partial r}(r\,w_{,r})\}]-\frac{B^2}{G'r}[2I_1^m\{\frac{\partial^3}{\partial r^3}(r^v\,w_{,r})-\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(r^v\,w_{,r})$$

$$+\lambda\{\frac{\partial^3}{\partial r^3}(rw_{,r}^3)-\frac{\partial}{\partial r}\{\frac{1}{r}\frac{\partial}{\partial r}(r\,w_{,r}^3)\}\}]+2\frac{B}{hr}\frac{\partial}{\partial r}[2\,I_1^m\,r^v\,w_{,r}+\lambda\,r\,w_{,r}^3]$$

$$-(\rho_1\,t_1+\rho_2\,h)\frac{\partial^2 w}{\partial t^2}=0 \qquad ..(29)$$

and $$I_1^m=A\,F(t)\,r^{v-1}=\frac{\partial U}{\partial r}+\frac{1}{2}(\frac{\partial w}{\partial r})^2+\frac{vU}{r} \qquad ..(30)$$

where ρ_1 = surface density, ρ_2 = core density, t_1 = face thickness, h = core thickness, f(t) is a function of time and A is a constant. We assume w in the following form compatible with the boundary conditions (22) for fundamental mode of vibration, where $F^2(t) = f(t)$:

$$w=w_0\,(1-r^2/a^2)^2\,F(t) \qquad ..(31)$$

The defining equation (30) of I_1^m is integrated over the plate and yields the same result as obtained in (24)

$$A=\{(v+1)\,w_0^2\}/(3a^{v+1}) \qquad ..(32)$$

Inserting (31) into (29) and taking account of (32), one gets the final equation (33) in the following form for the **immovable edge** :

$$\ddot{F}(t)+\frac{1881.5783}{(\rho_1\,t_1+\rho_2\,h)}\{F(t)\}+\frac{4023.104}{(\rho_1\,t_1+\rho_2\,h)}(\frac{w_0}{2h})^2\{F(t)\}^3=0 \qquad ..(33)$$

If the initial conditions are F(0) = 1, $\dot{F}(0)=0$, at t = 0, then the solution of (33) can be written as

$$\frac{\omega_1^*}{\omega_1} = [1+ \frac{\mu_2}{\mu_1} (\frac{w_0}{2h})^2]^{1/2}, \quad .. (34)$$

where ω_1^* and ω_1 are the non-linear to linear frequencies and

$$\mu_1 = \frac{1881.5783}{\rho_1 t_1 + \rho_2 h},$$

$$\mu_2 = \frac{4023.1040}{\rho_1 t_1 + \rho_2 h} \quad .. (35)$$

For the **movable edge,** the equation (33) will reduce to

$$\ddot{F}(t) + \frac{1881.5783}{(\rho_1 t_1 + \rho_2 h)} F(t) + \frac{473.4580}{(\rho_1 t_1 + \rho_2 h)} \cdot (\frac{w_0}{2h})^2 . \{F(t)\}^3 = 0 \quad .. (36)$$

with the above initial conditions, the ratio of the non-linear to linear frequencies is given by

$$\frac{\omega_2^*}{\omega_2} = [1+0.2516 (\frac{w_0}{2h})^2]^{1/2}, \quad .. (37)$$

For **immovable edge** in the case of Berger's approximation, the relation (34) will take the following form :

$$\frac{\omega_B^*}{\omega_B} = [1 +1.8865 (\frac{w_0}{2h})^2]^{1/2}, \quad .. (38)$$

It is already pointed out that no result for **movable edge** can be obtained in the case of Berger's approximation.

Numerical Results

Table-II shows the numerical results of the variation of non-dimensional frequency ratios with the dimensionless amplitudes for various values of the parameters already defined in (28) in the cases of **immovable** and **movable** edge conditions in the present study (P.S.) and compared with the results of Berger (1955).

Table – II

	Immovable Edge (P.S.)	Immovable Edge[14]	Movable Edge (P.S.)	Movable Edge[14]
$w_0 / 2h$	ω_1^* / ω_1	ω_B^* / ω_B	ω_2^* / ω_2	ω_B^* / ω_B
0	1	1	1	Absurd
0.5	1.23876	1.21311	1.03097	Absurd
1.0	1.77148	1.69898	1.11876	Absurd
1.5	2.41057	2.29013	1.25144	Absurd
2.0	3.09073	2.92337	1.41651	Absurd
2.5	3.78991	3.57640	1.60390	Absurd
3.0	4.49926	4.24013	1.80676	Absurd
3.5	5.21463	4.91016	2.02042	Absurd
4.0	5.93384	5.58426	2.24178	Absurd
4.5	6.65564	6.26112	2.46890	Absurd

Observations and Conclusions :

(1) From the given tables and curves it is observed that the present study yields larger values of the estimated parameters than those obtained from known theoretical analysis. It is also well-known that experimental results always show greater values than those obtained in theoretical analysis. Hence the method shown in the present study is more acceptable for the practical purpose than other approaches.

(2) From the uncoupled equations presented in Kamiya's (1976) results of immovable edges can only be obtained, but the present study yields better results for immovable edge conditions and acceptable results for movable edges. This is certainly an advantage.

(3) From the same cubic equation, results both for movable as well as for immovable edge conditions can be obtained. This is an additional advantage of the present analysis.

(4) Accuracy of Kamiya's (1976) method depends on a correction factor which is a function of the plate geometry. This correction factor will vary according to plate geometry. But the present analysis does not depend on any correction factor. Moreover, the uncoupled differential equation proposed in the present study is simple, accurate and thus has been able to fill up the void in the literature of the non-linear theory of sandwich circular plates.

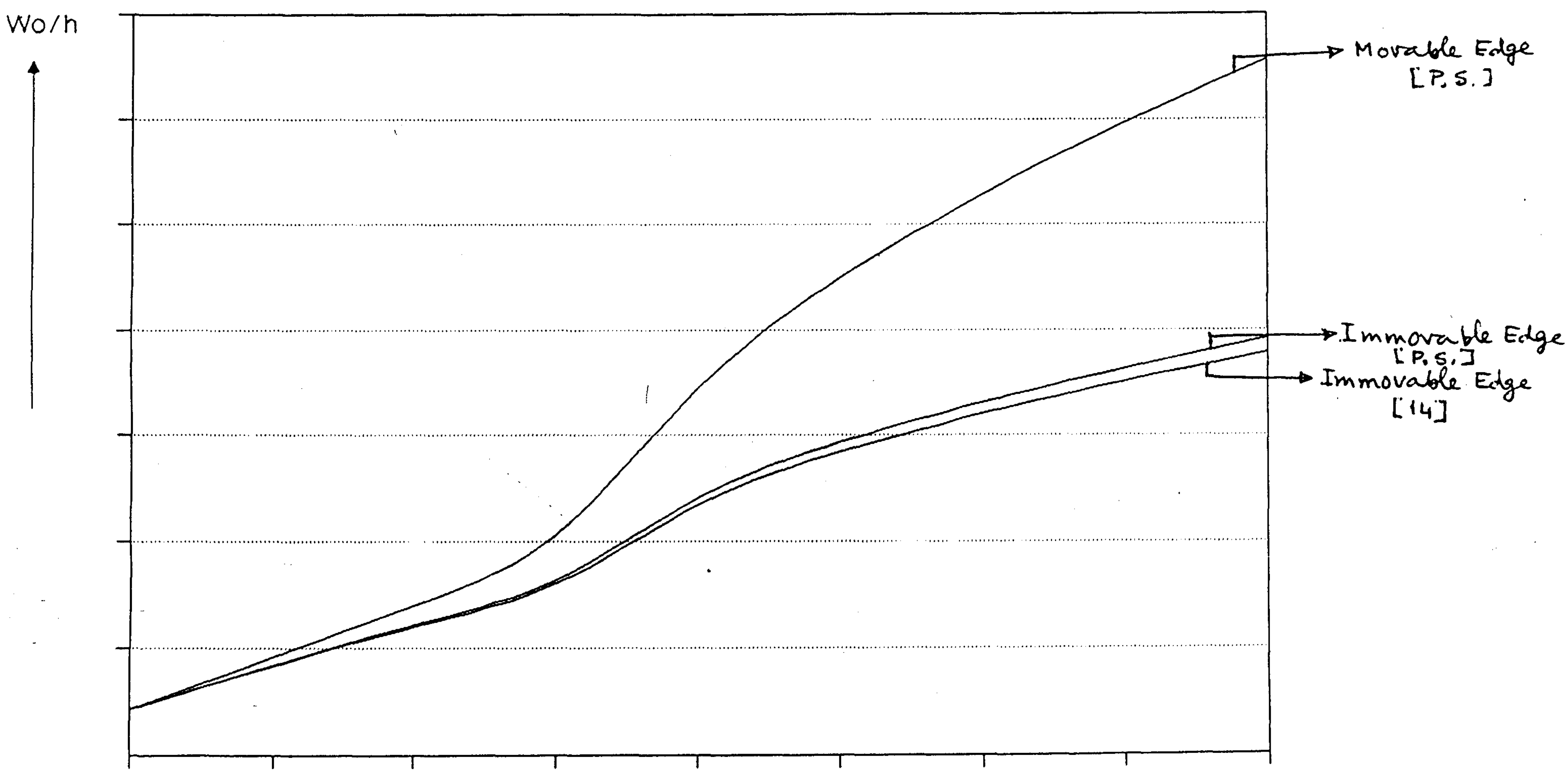

Fig. 1

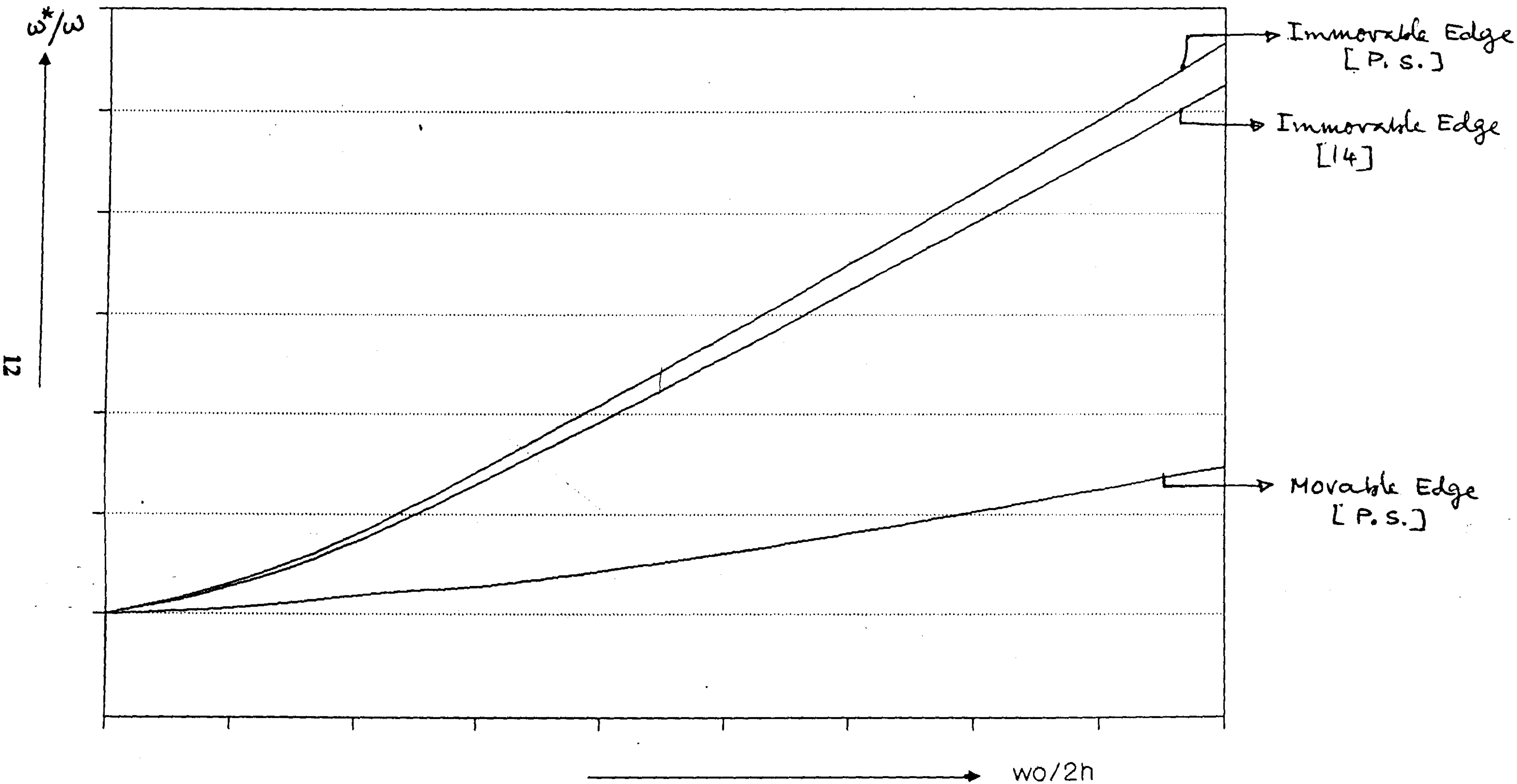

Fig. 2

PAPER - II

* BENDING OF A SANDWICH CIRCULAR PLATE UNDER LARGE DEFLECTION

ABSTRACT

The problem of large deflection of sandwich plates has been investigated by several authors. Outstanding research workers in this field are Reissner (1948), Hoff (1950), Eringen (1951), Wang (1952), Cheng (1962), Alwan (1964, 1967), Nowinski and Ohnabe (1973), Kamiya (1976) and Dutta and Banerjee (1995).

This paper deals with the detailed study of bending of sandwich circular plate under large deflection following Berger's technique. Numerical results can be computed and the corresponding graphs can also be drawn from the obtained results. The reduction of the present results to the classical results for the ordinary plate has been found to be in excellent agreement.

Governing Equations :

Let us take a polar co-ordinate system (r, θ, z) in the middle plane of the core; (r, θ) in the middle plane of the core, z in thickness direction (positive downwards). For the sake of simplicity, let us consider a sandwich plate with an isotropic core as well as isotropic upper and lower faces of identical thickness. While the faces respond to the bending and membrane actions of the plate, the core is assumed to transfer only shear deformations. Moreover, compared to the core thickness h, the face thickness t is supposed to be thin enough to ignore a variation of stress in the thickness direction of faces.

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If the expressions for the strains in the ith-face sheet in the r and θ directions are noted as $\in_{ri}, \in_{\theta i}$ respectively, the transverse shear strain as γ_i, equations (39) hold true for each of the separate face sheets of a circular plate having a rotational symmetry :

$$\in_{ri} = \frac{\delta u_i}{\delta r} + \frac{1}{2}\left(\frac{\delta w}{\delta r}\right)^2,$$

$$\in_{\theta i} = \frac{u_i}{r}, \gamma_i = 0, \qquad ..(39)$$

where u_i and w middle-plane displacements of the ith-face sheet considered in the r and z directions.

Let the stress-strain relations for the each face sheet be given by the equations (40) :

$$N_{ri} = B_i\left(\in_{ri} + \mu \in_{\theta i}\right),$$

$$N_{\theta i} = B_i\left(\in_{\theta i} + \mu \in_{ri}\right),$$

$$\tau_i = 0, B_i = E_i t_i / \left(1-\mu^2\right), \qquad ..(40)$$

where E_i, μ and t_i (= t) refer to Young's modulus, Poisson's ratio and thickness of the ith-face sheet respectively. When dual subscripts are used, the first subscript refers to the direction of the strain and the second refers to the face sheet under consideration. Thus $\in_{r1}$ signifies the strain in the radial direction in the upper face. Let us now introduce the averaged value of both face strain components as :

$$\in_r^m = \frac{1}{2}\left(\in_{r1} + \in_{r2}\right), \in_\theta^m = \frac{1}{2}\left(\in_{\theta 1} + \in_{\theta 2}\right),$$

$$\gamma^m = \frac{1}{2}\left(\gamma_1 + \gamma_2\right) \qquad ..(41)$$

with $p = u_1 - u_2$ and $U = (u_1 + u_2)/2$.

Then from (39) and (41), we have

$$\in_{r1} = \in_r^m + \frac{1}{2}\frac{\delta p}{\delta r}, \in_{\theta 1} = \in_\theta^m + \frac{1}{2}\frac{p}{r},$$

$$\in_{r2} = \in_r^m - \frac{1}{2}\frac{\delta p}{\delta r}, \in_{\theta 2} = \in_\theta^m - \frac{1}{2}\frac{p}{r}, \qquad ..(42)$$

By virtue of Hooke's law, the strain energy per unit area of both faces for isotropic elastic materials ($E_1 = E_2 = E$) is represented as :

$$\bar{V}_o^f = \frac{Et}{I-\mu^2}\left[\left(\in_r^m + \in_\theta^m\right)^2 + 2\mu \in_r^m \in_\theta^m + \frac{1}{4}\left\{\left(\frac{\delta p}{\delta r}\right)^2 + \left(\frac{p}{r}\right)^2 + 2\mu\frac{\delta p}{\delta r}\right\}\right] \qquad ..(43)$$

Now, if we rewrite the equation (43) by introducing two invariants of the average strains

$$I_1^m = \in_r^m + \in_\theta^m, \quad \text{and} \quad I_2^m = \in_r^m \in_\theta^m - \gamma^{m^2}, \qquad ..(44)$$

we obtain

$$\bar{V}_o^f = \frac{Et}{I-\mu^2}\left[I_1^{m^2} + 2\mu I_2^m + \frac{1}{4}\left\{\left(\frac{\delta p}{\delta r}\right)^2 + \left(\frac{p}{r}\right)^2 + 2\mu\frac{\delta p}{\delta r}\frac{p}{r}\right\}\right] \qquad ..(45)$$

Furthermore, the strain energy per unit area of the isotropic core due to the shear becomes

$$\bar{V}_o^c = \frac{1}{2}hG'\left[\left(\frac{p}{h}\right)^2 + \left(\frac{\delta w}{\delta r}\right)^2 + 2\frac{p}{h}\left(\frac{\delta w}{\delta r}\right)\right] \qquad ..(46)$$

In consequence, the total strain energy per unit area of the sandwich plate is

$$\bar{V}_o = \bar{V}_o^f + \bar{V}_o^c \qquad ..(47)$$

According to Berger's line of thought, we ignore I_2^m appearing in equations (45) and (47). Executing the variational calculus so as to minimize the total P.E. of the present elastic system of the sandwich plate, we arrive at the following three equations after some lengthy calculations :

$$I_1^m = \frac{\delta U}{\delta r} + \frac{U}{r} + \frac{1}{2}\left(\frac{\delta w}{\delta r}\right)^2 = \frac{\alpha^2 h^2}{2}, \text{ say} \qquad ..(48)$$

where α is an integration constant ;

$$\frac{Et}{1-\mu^2}\left[\frac{d}{dr}\left(r\frac{\delta p}{\delta r}\right)-\frac{p}{r}\right]-2G'\left[\left(\frac{p}{h}+\frac{dw}{dr}\right)r\right]=0, \qquad ..(49)$$

and

$$\frac{d}{dr}\left[2\alpha^2\frac{Et}{1-\mu^2}\left(r\frac{dw}{dr}\right)\right]+hG'\frac{d}{dr}\left(r\frac{dw}{dr}\right)+G'\frac{d}{dr}(r\,p)=0 \qquad ..(50)$$

Eliminating p from (49) and (50), we get

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}-\beta^2 w\right)=0 \qquad ..(51)$$

where

$$\beta^2=2\alpha^2h^2/\left[h^2+2\alpha^2h^2\ Et/\left\{G'\left(1-\mu^2\right)\right\}\right] \qquad ..(52)$$

In the previous derivation external and boundary force terms and lateral load are thoroughly excluded for abbreviation and (51) holds good except at the concentrated load. Thus, the derived equations (48) and (51) are approximation to the governing equations for large deflections of circular sandwich plates in the context of reduced Berger's method. From (48), it is seen that I_2''' is constant and this result is analogous to Berger's original theory on conventional plates.

Again considering the radial stress and shearing stress on a concentric circular ring of radius r and concentrated load P at the centre, we have, as r $\to$ 0,

$$D\frac{d}{dr}\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)=\frac{Et}{1-\mu^2}\left[\frac{dU}{dr}+\mu\frac{U}{r}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2\right]\frac{dw}{dr}+\frac{P}{2\pi r} \qquad ..(53)$$

where $D = Eth^2/(1-\mu^2)$.

Substituting (48) into (53), we have the relation

$$Dr\frac{d}{dr}\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}-\beta^2w\right)=D(\alpha^2-\beta^2)\,r\frac{dw}{dr}-\frac{Et}{1+\mu}\mu\frac{dw}{dr}+\frac{P}{2\pi} \qquad ..(54)$$

as $r \to 0$. At the centre of the plate, both U and dw / dr will be zero. Hence we have

$$\lim \; [Dr \frac{d}{dr}\left(\frac{d^2 w}{dr^2}+\frac{1}{r}\frac{dw}{dr}-\beta^2 w\right)]=\frac{P}{2\pi} \qquad ..(55)$$

A Circular Sandwich Plate Under Concentrated Load At The Centre With Clamped Edge :

Let the radius of the plate be a. Then for a clamped edge, the boundary conditions will be

$$w=\frac{dw}{dr}=0,\; U=0,\; when\; r=a. \qquad ..(56)$$

For the solution of (51), let us take w in the form

$$w=AI_o(\beta r)+B\left[K_o(\beta r)+\log\left(\frac{r}{a}\right)\right]+C, \qquad ..(57)$$

where $I_o(\beta r)$ and $K_o(\beta r)$ are respectively the modified Bessel functions of the first and the second kind of order zero. Substituting (57) in (55), we have

$$B=-P/(2\pi D\beta^2) \qquad ..(58)$$

Using the boundary conditions (56) for w and the relations

$$I_o'(x)=I_1(x), K_o'(x)=-K_1(x),$$

$$I_1(x)K_o(x)+I_o(x)K_1(x)=\frac{1}{x}, \qquad ..(59)$$

we get

$$A=\frac{\beta a K_1(\beta a)-1}{\beta a I_1(\beta a)}B,\; C=\frac{I_o(\beta a)-1}{\beta a I_1(\beta a)}B. \qquad ..(60)$$

With these values of A, B, and C, w (r) is given by

$$w(r)=\frac{-P}{2\pi D\beta^2 a I_1(\beta a)}[\beta a K_1(\beta a)I_o(\beta r)+K_o(\beta r)I_1(\beta a)$$

$$+\beta a I_1(\beta a)\log\frac{r}{a} - I_o(\beta r) + I_o(\beta a) - 1\,] \qquad ..(61)$$

To find the displacement U, we introduce (57) in (48) and integrate to obtain

$$Ur = \frac{\alpha^2 h^2}{4} r^2 - \frac{A^2}{2}[\frac{1}{2}\beta^2 r^2\{I_1^2(\beta r) - I_o^2(\beta r)\} + \beta r I_1(\beta r) I_o(\beta r)$$

$$-\frac{B^2}{2}[\frac{1}{2}\beta^2 r^2\{K_1^2(\beta r) - K_o^2(\beta r)\} - \beta r K_1(\beta r) K_o(\beta r)\,]$$

$$+AB[\frac{1}{2}\beta^2 r^2\{I_1(\beta r)K_1(\beta r) + I_o(\beta r)K_o(\beta r)\}$$

$$-\frac{1}{2}\beta r\{I_1(\beta r)K_o(\beta r) - I_o(\beta r)K_1(\beta r)\}]$$

$$-ABI_o(\beta r) - B^2 K_o(\beta r) - \frac{B^2}{2}\log r + F, \qquad ..(62)$$

where A and B are given by (58) and (60).

Using the boundary condition for U, we have

$$F = \frac{B^2}{2}\left\{\frac{1}{2} - \log a + \frac{I - I_o(\beta a)}{\beta a I_1(\beta a)} - \frac{1}{2}\left[\frac{I - I_o(\beta a)}{I_1(\beta a)}\right]^2\right\} - \frac{\alpha^2 h^2 a^2}{4} \qquad ..(63)$$

To determine the constant α or β, we shall use the condition that U = 0 (since $u_1 = u_2 = 0$) as $r \to 0$, which gives

$$\frac{B^2}{2}\left\{-\frac{1}{2} + \gamma + \log\frac{\beta}{2} - \frac{\beta a K_1(\beta a) - 1}{\beta a I_1(\beta a)}\right\} + F = 0 \qquad ..(64)$$

From (63) and (64), we can write

$$\left(\frac{Pa^2}{\pi Dh}\right)^2 = \frac{2(\alpha\beta^2 a^3)^2}{\gamma + \log\frac{\beta a}{2} - \frac{I_o(\beta a) + \beta a K_1(\beta a) - 1}{\beta a K_1(\beta a)} - \frac{1}{2}\left[\frac{I_o(\beta a) - 1}{I_1(\beta a)}\right]^2} \qquad ..(65)$$

where γ is an Euler constant. The equation (65) will determine the constant α or β. As $\beta \to 0$ (i.e. $\alpha \to 0$), the equation (51), (52) and (55) reduce to corresponding equations for the problem of small deflections of sandwich circular plates under concentrated load at the centre obtained by Wang (1952). Taking the limit of (61) as $\beta \to 0$, we have

$$w(r)=\frac{P}{8\pi D}\left[-r^2\log\frac{a}{r}+\frac{1}{2}\left(a^2-r^2\right)\right] \qquad ..(66)$$

which is the expression for small deflection of circular sandwich plate under concentrated load at the centre. The deflection w (r) in (61) will be maximum at the centre of the plate f and it is given by

$$w_o=\frac{P}{2\pi D\beta^2}\left\{\gamma+\log\frac{a\beta}{2}-\frac{I_o(\beta a)+\beta a\,K_1(\beta a)-2}{\beta a I_1(\beta a)}\right\} \qquad ..(67)$$

The results corresponding to the supported edges can be obtained following the similar procedure. The numerical results for deflection w (r) can be easily obtained from equations (61) and (65).

From the equations (52) and (65), we can write

$$\left\{\frac{Pa^2}{\pi Dh}\right\}^2=\frac{(\beta a)^6}{[1-c(\beta a)^2][\gamma+\ln\{(\beta a)/2\}-\dfrac{I_0(\beta a)+(\beta a)K_1(\beta a)-1}{(\beta a)K_1(\beta a)}}-\frac{I_0(\beta a)-1}{2I_1(\beta a)}$$

where

$$c=\frac{Et}{a^2G'(1-\mu^2)} \quad \text{and} \quad \gamma=0.5772157....$$

For different values of $P'=(\frac{Pa^2}{\pi Dh})^2$, we can get the corresponding values of (βa), which has been calculated with the help of MATLAB as follows:

βa	7	8	9	10	11
P'	6.42	1.01	0.19	0.04	0.01

PAPER – III

* A NOTE ON THE LARGE DEFLECTION OF HEATED SANDWICH CIRCULAR PLATES - A NEW APPROACH

ABSTRACT

The problem of large deflections of sandwich plates without thermal loading has been investigated by several authors but such problems for heated sandwich plates are very few in the literature. Only few papers can be located in this area, of which the papers of Kamiya (1978) and Roy, Dutta and Banerjee [1993] are of worth mentioning.

The aim of the present analysis is to propose a new set of uncoupled differential equations for the study of circular sandwich plates under thermal loading by using a new and simple approximation. Moreover, the failure of Berger's analysis has been overcome by the present method. The analysis of clamped and simply supported circular sandwich plates with movable as well as immovable edges has been carried out in detail. Numerical results obtained from the present study have been plotted graphically for some special values of the parameters involved. The results have been compared to those available from other methods.

GOVERNING EQUATIONS

Considered here a cylindrical polar co-ordinate system (r, θ, z) in the middle plane of the core; (r, θ) being in the middle plane of the core and z the thickness direction taken to be positive downwards.

For the sake of illustration of the present method considered here a circular sandwich plate with isotropic core as well as isotropic lower and upper faces of identical thickness. While the faces respond to the bending and membrane actions of the plate, the core is assumed to transfer only shear deformations. Moreover, compared to the core

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thickness h, the face thickness t_1 is supposed to be thin enough ($h >> t_1$) to ignore a variation of stress in the thickness direction of the faces.

By virtue of Hooke's law, the total strain energy of both faces of circular sandwich plate of isotropic elastic material in polar coordinates, as obtained by Kamiya (1978), is represented below :

$$\bar{V}_o^f = B\iint_A [(\xi_r^m + v_1 \xi_\theta^m)^2 + 1/4\{(\partial p/\partial r)^2 + (p/r)^2\} + v_1/2\, \partial p/\partial r . p/r + (1-v_1^2)(\xi_\theta^m)^2 +$$

$$\{1/2hG_c\{(p/h)+w_r\}^2/B - 2(1+v_1)\alpha_1 - \{T_m(\xi_r^m + \xi_\theta^m) + T_c/4\,(\partial p/\partial r + p/r)\}\}]rd\theta dr \qquad ..(68)$$

where

$B = E_1 t_1 (1-v_1^2)$, $T_m = (T^u + T^1)/2, T_d = (T^u - T^1)$ and

$p = u_1 - u_2$; T^u and T^1 are temperatures at the upper and lower faces respectively.

One now can assume that

$$(I - v_1^2)(\xi_\theta^m)^2 = \lambda\, 1/2(dw/dr)^2 \qquad ..(69)$$

This assumption is really new and can be easily justified through the discussion of Sinha Ray and Banerjee (1985) and has a resemblance with the line of thought of Dutta and Banerjee (1993), λ is a parameter to be determined from the idea of minimum potential energy and other physical considerations. Let us further assume that

$$I_1^m = \xi_r^m + v_1 \xi_\theta^m = du/dr + 1/2(dw/dr)^2 + v_1\, u/r, \qquad ..(70)$$

in case of rotational symmetry. Furthermore, the strain energy per unit area of the isotropic core due to shear becomes

$$\bar{V}_o^c = 1/2h\, G_c[\,(p/h)^2 + (dw/dr)^2 + 2(p/h)dw/dr], \qquad ..(71)$$

Introducing (69) and (70) in (68), and taking into account of (71), the total strain energy of the entire circular sandwich plate with isotropic core is

$$V_T = B \iint_A [(I_1^m)^2 + \lambda (w_r/2)^2 + 1/4\{(\partial p/\partial r)^2 + (p/r)^2 + 2v/r.p\ \partial p/\partial r\}$$

$$+(1/2h)(G_c/B)\{(p/h) + w_r\}^2$$

$$-2(1+v_1)\alpha_1\{T_m(I_1^m + \sqrt{\lambda}(1-v_1)/(1-v_1^2)(w_1^2/2) + T_d/2\,(\partial p/\partial r + p/r)\}]rd\,\theta\,dr,$$

.. (72)

where w_r is the derivative of w with respect to r .

Executing the variational principle of Euler so as to minimize the total potential energy of the present elastic system of the sandwich plate, we arrive at the following differential equations :

$$r.d/dr\ \{I_1^m - (1+v_1)\,a_1\,T_m\} - (v_1 - 1)\{I_1^m - (1+v_1)a_1\ T_m\} = 0 \quad .. (73)$$

$$d/dr(r\,dp/dr) - (1/r + 2G_c/B.r/h)\ p - 2G_c/B.rw_r - (1+v_1)\,a_1\,r\ d/dr\ T_d = 0$$

.. (74)

$$d/dr[2rw_r\{I_1^m - (1+v_1)a_1\ T_m\} + \lambda r w_r^3 - 2\sqrt{\{\lambda(1-v_1^2)\}}a_1\,T_m(rw_r)$$

$$+r.hG_c/B(w_r + p/h)] = 0$$

.. (75)

From the equation (73), we can write

$$\{I_1^m - (1+v_1)\alpha_1\ T_m\} = A\,r^{v_1 - 1} \quad .. (76)$$

for an immovable edge, where A is an integration constant. Eliminating p from the equations (74) and (75) and introducing the equation (76) we can obtain the following differential equation in terms of the normal deflection w :

$$h[d^3/df^3(fw_f) - d/df\{1/f\ d/df(fw_f)\}] + B/G_c[d^3/df^3\{2Aa^{v_1-1}\ f_1^v\ w_f$$

$$+\lambda/a^2\ f\,w_f^3 - 2\sqrt{\{\lambda(1-v_1^2)\}}$$

$$\{\alpha_1\ T_m f\ w_f\}] - d/df \frac{1}{f}.\frac{d}{df}\{2Aa^{v_1-1}\ f^v\ w_f + \lambda/a^2\ f\,w_f^3 - 2\ \{\lambda(1-v_1^2)\}\alpha_1\ T_m f\ w_f\}] -$$

$$2a^2/hd/df\{2Aa^{v_1-1} f^v w_f + \lambda/a^2 f w_f^3 - 2\sqrt{\{\lambda(1-v_1^2)\}}\ \alpha_1 T_m f w_f\}]$$

$$+ (1+v_1)\,\alpha_1 a^2\, d/df\,(f\, dT_d/df)=0$$

where $f = r/a$ .. (77)

It is to be noted in this connection that for movable edges, we have the condition

$$I_1^m - (1+v_1)\alpha_1 T_m = 0 \quad .. (78)$$

BOUNDARY CONDITIONS

a) If the edges are clamped, the boundary conditions are

$u = 0$, $p = 0$, and $w = 0 = w_f$ for $f = 1$ .. (79)

b) If the edges are simply supported, the boundary conditions are

$$u=0,\ p=0,\ and\ w=0=d^2 w/df^2 + v_1/f\,.dw/df\quad for\ f=1 \quad .. (80)$$

SOLUTION OF THE PROBLEM :

(a) Clamped Edge : For the solution of the equation (77) satisfying the boundary conditions (79), we assume w in the following form :

$$w = w_o(1-f^2)^2 \quad .. (81)$$

Furthermore, in the present analysis, the following temperature distributions at each face are assumed to be

$$T_m = (T'' + T^1)/2 = \bar{T}_m = \text{constant and } T_d = T'' - T^1 = T_o(I-f^2)^2 \quad .. (82)$$

To fmd the expression for A, the equation (76) is integrated over the entire area of the plate so that

$$A = \{(v_1+1)/(a+1)\}\{(w_o^2/3 - \alpha_1 T_m a^2/2)\} \quad .. (83)$$

It may be mentioned here that the expressions for u and p are suitably chosen to satisfy the boundary conditions and they have no contribution in the calculation of (83).

Let us now pay our attention to the most vital equation (77). Substituting (81), (82) and (83) into the equation (77) and applying Galerkin's procedure, we arrive at a cubic equation for the determining the deflection w_o for immovable edges in the form

$$C_1 (w/h)^3 + C_2 (w/h) + C_3 = 0 \qquad ..(84)$$

where C_1 , C_2 and C_3 are known constants.

To find the equation for the movable edges, we utilize the condition (78) and , after some simplification, obtain the result

$$(w_o/h)^3 \{1(B/a^2G_c 6.36519 + 0.72746/h)\}$$

$$+(w_o/h)\{(1+u_1)(a/h)^2 a_1 T_m\}\{(-)9.75236B/G_c a^2$$

$$+ 0.81272/h \sqrt{\lambda (1-v_1^2)/1+v_1} + 4.87619/h\}$$

$$-\{(1+v_1)(a/h)^2 a_1 T_m\} T_o/T_m 0.20317/h = 0$$

$$..(85)$$

It is interesting to note that the result obtained by Kamiya (1978) following Berger's method can be easily deduced from (84), if we put $\lambda = 0$ and make some skillful manipulations to arrive at the result :

$$(w_o/h)^3 (32/3\, B/a^2G_c + 8/9h) + (w_o/h)[8/h - \{(1+v_1)(a/h)^2 a_1 T_m\}/$$

$$1+v_1 (16B/a^2 G_c + 4/3h)] - \{(1+v_1)a^2/h^2 a_1 T_m\} 1/3h\, T_o/T_m = 0$$

$$..(86)$$

(b) Supported Edge :

For the solution of the equation (77) satisfying the boundary conditions (80), we assume w in the following form :

$$w = w_o\{1 - 2Pf^2 + Qf^4\} \qquad ..(87)$$

where $P = (3+v_1)/(5+v_1)$ *and* $Q = (1+v_1)/(5+v_1)$

Let us assume the temperature distribution as :

$$T_m = (T^u + T^1)/2 = \overline{T}_m = \text{constant, and } T_d = (T^u - T^1) = T_o\{1 - 2Pf^2 + Qf^4\} \quad .. (88)$$

Substituting (87) and (88) into (77) and applying Galerkin's principle, keeping in mind the result (83), a cubic equation for the determination of the deflection w_o for the immovable edges can be obtained and therefrom result corresponding to Berger's approximation can conveniently be deduced.

NUMERICAL RESULTS :

Figs. 1 – 4 show numerical results of the maximum deflections of a heated circular sandwich plate of isotropic material with isotropic inner core and with immovable as well as movable clamped and supported edges respectively. The geometries of the plate and material constants are identical with those used in the investigation of Nowinski and Ohnabe (1973) namely,

$a = 0.254\,m,\ t_1 = 0.635\,x 10^{-3}\,m,$ $h = 1.735 \times 10^{-2}$ m ,

$E_1 = 7347.201 \times 10^6$ kg/m^2 , $G_c = 4218.4884 \times 10^3$ kg/m^2, $u_1 = 0.3$ and 1=0.09

and $\alpha_1\, T_m\, (1+\alpha_1)(\alpha/h)^2 = 1$ [assumed for simplicity] .. (89)

DISCUSSIONS :

From the graphs it is observed that the present study yields larger values of the estimated parameters than those obtained from known theoretical analysis by Alwan (1964) and other approximate method by Berger (1955).

It is also well-known that experimental results always show greater values than those obtained in theoretical analysis. Hence the method shown in the present study is more acceptable for the practical purpose.

From the uncoupled equations presented in Berger's (1955) analysis, results of immovable edges can only be obtained but the present study yields accurate results both for movable as well as immovable edge conditions. This is certainly an advantage.

Accuracy of Kamiya's (1978) method depends on a correction factor which is a function of the plate geometry. This correction factor will vary according to plate geometry. But the present analysis does not depends on any correction factor. Moreover, the uncoupled differential equation proposed in the present study is simple and more accurate compared to other approximate methods and thus has been able to fill-up the void in the literature of the non-linear theory of sandwich plates.

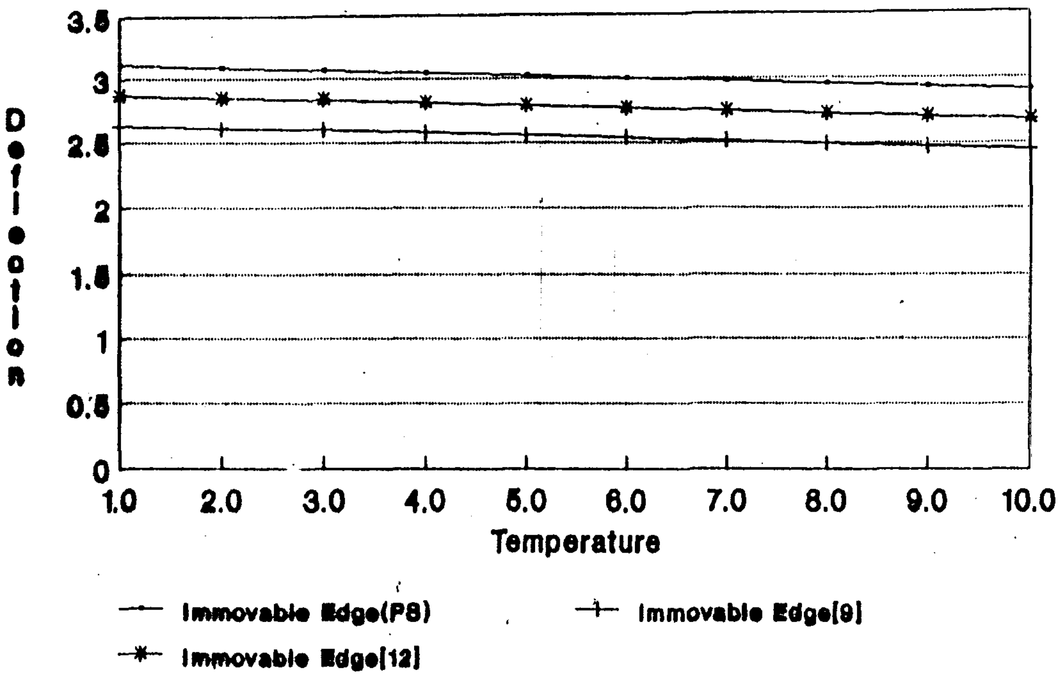

Fig.1 : Clamped Edge(Immovable)

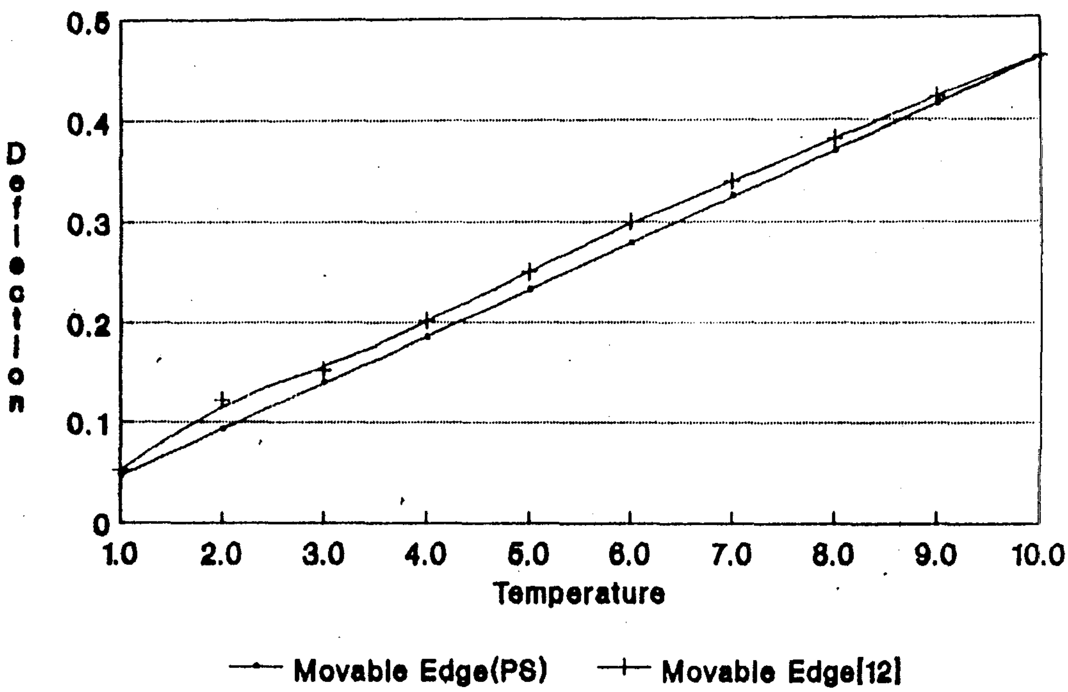

Fig.2 : Clamped Edge (Movable)

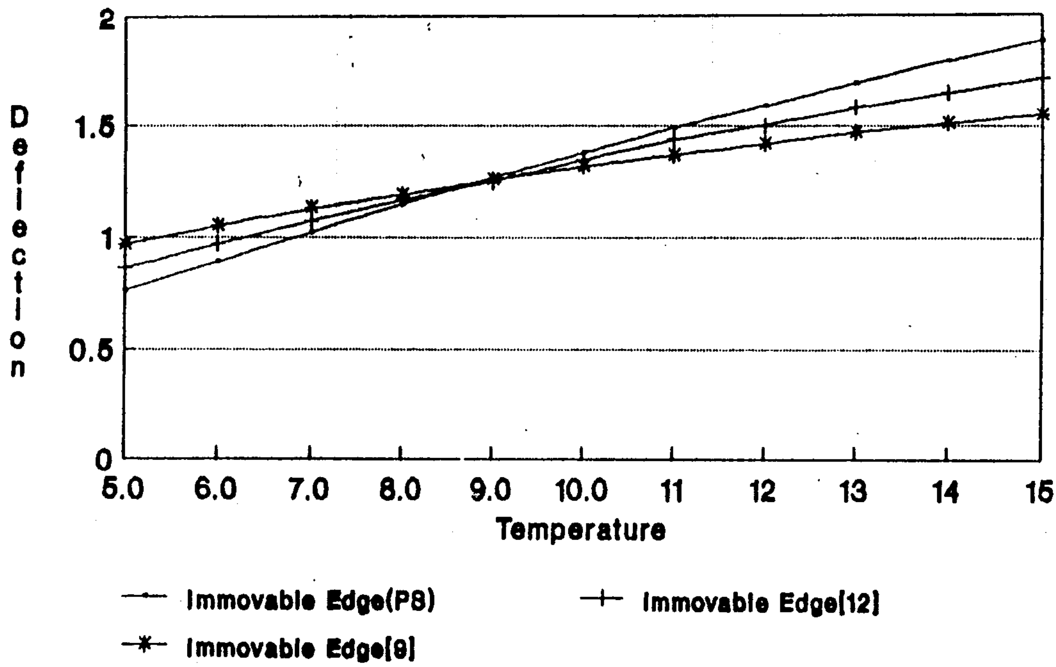

Fig.3 : Supported Edge(Immovable)

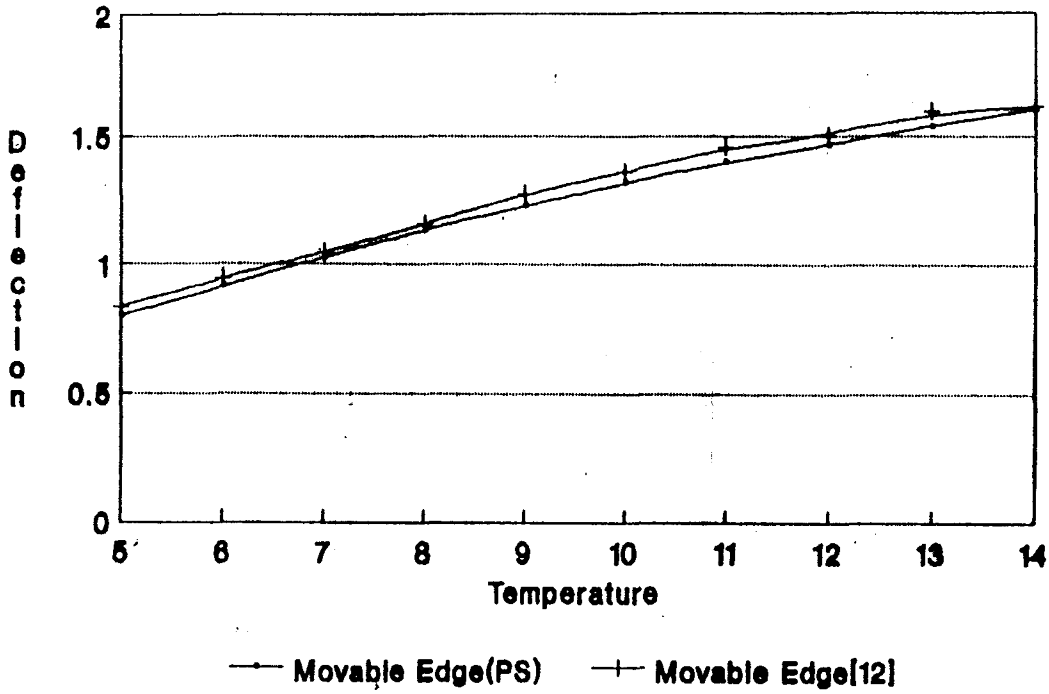

Fig.4 : Supported Edge (Movable)