CHAPTER 1

Introduction and Preliminaries

This dissertation is divided into four chapters.

Chapter 1 introduces some basic concepts and notations we employ in this thesis and serves as a prelude to the current work. Geometric solution procedures find important applications in the problems involving location of facilities considered by us.

Chapter 2 considers the unconstrained problem of locating a single service centre in the plane in the presence of existing location points using the criterion of *minimising the maximum weighted rectilinear distance*, symmetric as well as asymmetric, and obtains the solution analytically by exploiting the geometrical structure of the problem. Asymmetric weight is typically exemplified by rush hour traffic and similar other situations.

Chapter 3 deals with the problem of locating a single service centre catering to the demands of customers distributed over a finite set of demand points in a two-dimensional space employing both the symmetric and the non-symmetric Manhattan *metric minimax* criterion. An exact solution technique, based on geometry, has been presented, under the assumption that the required centre should be situated within a convex polyhedral region.

Finally, chapter 4 focusses on the computational aspects of some of the above mentioned problems and presents algorithms having a polynomial time complexity.

1.1 Origin and development of different aspects of locational problems

Facility layout and location problems have been the subject of study for centuries. The ancient Greeks are known to have been fascinated by this subject. A version of the Euclidean distance location problem was posed by Fermat [16, 51] as a purely geometrical problem in the early seventeenth century and solved by Torricelli [36] around 1640, which may be stated thus: given three points in the plane, find a fourth one such that the sum of the distances to the three said points is a minimum. Cavalieri [36] in 1647 reviewed the problem and Jacob Steiner [16, 52], a Swiss mathematician, early in the nineteenth century, made an attempt to solve the problem posed as a classic geometry problem in the special case of equal weights while Alfred Weber [76], a German economist,in his pioneering work towards the beginning of this century,once again studied the weighted version, also known as the Steiner-Weber problem or the general Fermat problem, which consists in locating a warehouse in such a way that the total weighted distance travelled between the warehouse and a set of demand points is a minimum. The dual of this problem was solved by Fasbender [36] towards the middle of nineteenth century. But it was Kuhn [50] in 1963 who was the first to have attempted a purely mathematical approach in order to find a solution to the problem.

Although facility layout and location problems continued to receive considerable attention over the years it was only after practitioners in OR began exhibiting interest that the subject became the focal point of attraction for several disciplines.

Locational analysis deals with the study and development of methodologies seeking to determine the locations of new facilities in such a way that the users of the facilities are benefited most. By constructing suitable models which involve locating one or more new facilities, and solving them, the investigation is carried out. With the passage of time, however, the formulation of the location problem has undergone radical change.

In solving facility layout and location problems models simple rather than highly sophisticated although closely approximating the real world have been developed. In the analysis of facility layout and location problems the process of verifying if the model accurately represents the physical system under study is most important. In obtaining a solution to the problem its formulation and analysis have to be carried out at the outset.

1.1.1 Selection of criterion

The criterion of minimising some function of distance, either unweighted or weighted by some importance factor, the weight being interpreted as cost per unit of distance from a demand point to a facility, is perhaps the most natural choice. If the new facility is a factory supplying warehouses or a new machine to be located in a plant layout or a point in a network to be connected to known points in the network (the existing facilities), its location may be determined in such a way that the total cost which is directly proportional to the distances involved, will be a minimum. Sometimes, instead of minimising the total distance travelled, it may be required to minimise the maximum distance, which is actually the minimax counterpart of the more familiar Fermat problem. Such a criterion is most natural in locating some emergency facility for which the maximum delay is more important than the average or total delay incurred as a measure of effectiveness and has been rightly called the 'grease the squeaky wheel' criterion by some authors inasmuch as the objective is to minimise the effects of the worst situation, viz., the maximum cost.

1.1.2 Choice of norms

There are many a distance measure we may define on the plane. A general distance family is L_p distance defined as

$L_p(A_1, A_2) = \left(|x_1 - x_2|^p + |y_1 - y_2|^p\right)^{1/p}$, where p is the distance parameter $(1 \leq p \leq \infty)$, and (x_1, y_1), (x_2, y_2) are the coordinates of two given points A_1 and A_2. Two distance measures, however, received much attention, viz., L_1, variously called the rectilinear, rectangular or Manhattan distance and L_2, the euclidean or straight line distance. In urban location analyses travel usually takes place along an orthogonal set of streets. In problems involving machine location travel occurs along a set of rectangular aisles arranged parallel to the walls of the building where the machine is housed. In such situations rectilinear distance is the appropriate metric. Whereas for some network location or pipeline design problems or problems involving air travel euclidean distance is conveniently applied. For relatively small areas on the earth's surface, which may be treated with considerable precision as a sphere, the planar model offers a very good approximation. When the existing facilities are widely separated, the area covering these may be projected onto a plane and the location of the required facility may be determined using an appropriate location model on the plane, but only at the expense of introducing errors into the analysis. Although there exists a number of mapping techniques producing accurate projections of a sphere onto a plane, in problems involving large regions on the globe none preserves distance. On a sphere, therefore, we should apply the shortest arc distance, also called the geodesic norm, useful for global optimisation [1, 3, 5, 19, 20, 23, 32, 79]. Recently, location problems on the sphere have been the centre of much critical attention in the literature. Besides these there exist other norms notable among which are ring radial, hyper-rectilinear, block norms etc.

1.1.3 Minimax and minisum location problems in networks

Location theory evoked considerable interest after the publication of a seminal paper by Hakimi [42] who considered the general problem of locating one or many facilities on a network employing the minimax or the minisum criterion.

Problems dealing with the determination of optimal location of service centres in networks or graphs abound in practical situations. In particular, if a graph represents a road network with its nodes representing communities, one may have to optimally locate a hospital, police station, fire station or any other emergency service facility. In such situations, the optimality criterion may be the minimisation of the distance or travel time from the facility to the farthest node of the graph or the optimisation of the worst case. In a more general situation, a number of such facilities rather than a single facility, may be required to be located such that the remotest node of the graph can be reached from at least one of the facilities within a minimum distance. The problem of locating emergency facilities with a view to minimising the largest travel distance to any node from its nearest facility is naturally called the minimax location problem and the facilities the centres of the graph (Christofides [14]).

There is a different class of location problems where the objective is to minimise the sum total of the distances from the nodes to the central facility, assuming that a single such facility is to be located. The problem of locating a depot in a road network where the nodes represent customers or switching centre in a telephone network where the nodes represent subscribers, calls for such objectives. Problems of this type are consequently called minisum location problems, although the objective may be the sum of various functions of distances rather than the sum of distances. The resultant facility locations are then known as the medians of a graph.

1.1.4 The euclidean MSC problem

The minimum spanning circle problem, also known as the euclidean 1-centre problem, may be stated as the problem of covering a finite set of points in a plane with the smallest possible circle (Preparata and Shamos [67]). This is a classical problem originally posed by Sylvester[71, 72] in 1857, who while continuing the search for an efficient algorithm ultimately hit upon a graphical solution procedure only in 1860 and attributed the same to Peirce. This algorithm was rediscovered by Chrystal[15] twenty five years later. A modern account of their treatment may be found in Rademacher and Toeplitz [68]. The smallest enclosing circle thus obtained is unique and is either the circle circumscribing some three points of the set forming an acute triangle or described by two of them as diameter. Thus a finite algorithm that examines all pairs and triplets of points and determines the minimum circle enclosing the set was obtained. The complexity of this algorithm was $O(n^4)$ and Elzinga and Hearn [31, 32] suggested an improvement that would run in $O(n^2)$ time. Shamos [70] proposed an algorithm which depends on the determination of the farthest point of the Voronoi diagram requiring computational complexity of $O(n \log n)$. This amount of computational effort is at least needed for any solution algorithm. However, Megiddo [59] suggested a linear time algorithm by transforming the minimum spanning circle problem into a two - dimensional LP formulation. In the MSC problem, also called a minimax facilities location problem in Operations Research parlance, we seek a point, the required centre of the circle, the greatest distance from which to any point of the set is a minimum. The minimax criterion is most suitable in locating emergency facilities to reduce worst case response time to a minimum (Toregas et al. [74]). It has also been successfully implemented to locate a radio transmitter serving a number of discrete receivers or a radar station catering to the demands of several defence installations so that the RF power determined by the radius of the covering circle is a minimum (Nair and Chandrasekaran [65]). By treating the smallest covering circle problem as a continuous optimisation problem, a number of iterative algorithms has appeared, notable among them being Lawson [53], and Zhukhovitsky and Avdeyeva [80] algorithms. Jacobsen [48] has developed an algorithm relying on a specialised implementation of the method of feasible directions.

1.1.5 Generalisations of the single facility problem

The m-centre problem forms the most important class of problems in location theory and may be formally stated as follows: Given a set $D = \{d_1, d_2, \ldots, d_n\}$ of n demand points on a plane, find a set $S = \{s_1, s_2, \ldots, s_m\}$ of unknown locations of m supply points on the plane such that the furthest distance between the demand points and their closest supply points is as close as possible. Mathematically speaking,

$$\text{Minimise } z,$$

$$\text{where } z = \max_{1 \le i \le n} \{ \min_{1 \le j \le m} \{L_p(d_i, s_j)\}\}$$

Problems such as these find important applications in models concerning location of service facilities, as for example, hospitals, shopping centres, fire departments, police stations, radio or TV centres, or in many equity models in economics where the communities represent demand points. The m-centre problem may be treated as a generalisation of the 1-centre problem. This problem for general m has been considered by Aneja et al. [2], Drezner [21], Hwang et al. [47], Ko et al. [49] and Vijay [75].

1.1.6 Another generalisation of the single facility problem

The multifacility location problem concerns locating any given number of variable points representing facilities with respect to any given number of fixed points corresponding to potential users applying the minisum or the minimax criterion. Let x_i $(i = 1, 2, \quad, m)$ denote the new facilities or the so-called variable points, a_j $(j = 1, 2, \quad, n)$ the fixed points or the existing facilities, w_{1ij} and w_{2ik} the weighting constants between x_i and a_j and between x_i and x_k respectively. The weights allow for the model to discriminate in importance among distances. The problem where a minimum sum of weighted distances criterion is satisfied is the following:

$$\text{minimise } z = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij} \; L_p(x_i, a_j) + \sum_{i=1}^{m-1} \sum_{k=i+1}^{m} w_{2ik} \; L_p(x_i, x_k)$$

whereas the problem satisfying the minimax objective is given by minimise z where

$$z = \max \left\{ w_{1ij}\ L_p(x_i,\ a_j),\ 1 \le i \le m,\ 1 \le j \le n; \right.$$

$$\left. w_{2ik}\ L_p(x_i,\ x_k),\ 1 \le i \le m-1,\ i+1 \le k \le m \right\}.$$

Hakimi [42], Frank [39,40] and Goldman [41] have studied the minimax location problem in a network while Francis [35] has dealt with the same problem on a plane. Love et al. [56] and Elzinga and Hearn [33] provide solution procedures to the multifacility minimax problem using euclidean distances. Love and Morris [54] have suggested a non-linear programming approach to the problem using generalised L_p distances. However, in an urban setting the travel paths resemble more a rectangular than a straight line distance and consequently, rectilinear distances become relevant in such situations. Wesolowsky [78] has given a parametric linear programming solution to the multifacility problem using rectilinear distances. The single facility location problem can also be considered to be a special case of the multifacility location problem stated above.

1.1.7 Algorithms

There exist several algorithms for solving the minimax problem under the Euclidean norm (Blumenthal and Wahlin [5], Castells and Melville [6], Chakraborty and Chaudhuri [11], Chatelon et al. [12], Chrystal and Peirce [15], Elzinga and Hearn [31, 32], Francis [35], Hearn and Vijay [45], Jacobsen [48], Megiddo [59], Nair and Chandrasekaran [65], Oommen[66] Rademacher and Toeplitz [68] and Shamos [70]). Two papers - one by Oommen [66] and the other by Hearn and Vijay [45] - survey the literature and compare, qualitatively and computationally, the various solution procedures. Oommen has proposed a computational scheme that synthesizes three of the best known primal feasible algorithms, viz., the Chrystal-Peirce algorithm with the Chakraborty-Chaudhuri initialisation, the Jacobsen algorithm and the Castells- Melville and Francis algorithms and conjectures that the geometric algorithm has a linear time complexity. Hearn and Vijay have demonstrated that the Chrystal-Peirce algorithm with the Chakraborty-Chaudhuri starting solution is the fastest in the equiweighted case whereas for the weighted case the Elzinga-Hearn algorithm turns out to be the fastest. Chrystal-Peirce algorithm is based on primal feasibility while Elzinga-Hearn algorithm depends on dual feasibility. The primal feasibility concept to solve a minimax location problem with an efficient starting solution, was first introduced by Chakraborty and Chaudhuri [11]. The basic idea behind this method is to cover S, the set of all demand points, by a circle. The next step consists in reducing the radius of this circle so that the demand points continue to remain within the circle. The algorithm is designed in such a way that at each iteration at least one demand point could be eliminated and no future iteration would ever need any information about this point. The dual feasibility concept consists in covering any two points of S by a minimum circle. The radius of the circle is then increased at each step to accommodate more and more demand points within the circle until one gets the minimum covering circle. The basic ideas contained in our algorithm and Chakraborty-Chaudhuri's are identical. But, in order to solve the problem using rectilinear metric we have to recast the latter and effect certain changes to meet the present situation. With this end in view the spanning circle of the euclidean 1-centre problem has been replaced by the covering diamond (Francis and White [36]), defined as follows: given any point P(a, b) and any non-negative number r we define a diamond with centre P and radius r, to be denoted by D(P, r), by the set of all points X(x, y) for which $L_1(X, P) \le r$. In symbols,

$$D(P,\ r) = \left\{ X(x,\ y):\ L_1(X,\ P) \le r \right\}.$$

1.1.8 Locating an undesirable facility using the minimax criterion

There is another important class of location problems concerned with locating an obnoxious or undesirable facility that produces pollutants of the nature of radiation, noise or harmful gases, in such a way that the smallest distance or the smallest weighted distance from a given set of demand points is maximised while remaining within a prespecified region. Such problems are naturally known as maximin problems as opposed to the minimax version. Application areas include location of a noisy facility, say a school, a prescribed distance away from residential quarters, or an infectious disease hospital, an ordnance factory, a nuclear waste disposal site, a factory spewing out effluents and the like. In such situations it is imperative to locate the facility as far away as possible from the points it actually serves. Among the papers dealing with the maximin objective we may mention the ones by Dasarathy and White [17], Drezner and Wesolowsky [26 - 28], Melachrinoudis [61, 62] and Mehrez et al. [60]. For a realistic formulation of the above problem, Melachrinoudis and Cullinane [63] developed a model based on the physical laws of transfer of the unpleasant effects associated with the installation of an undesirable facility using the minimax criterion which, in this case, minimises the maximum or worst effect of the polluting facility. This model assumes that the effect of a new undesirable facility upon an existing one follows the law of inverse square of the distance between the facilities.

1.2 Solution procedures for some rectilinear distance planar single facility problem

Among the various solution procedures available in the literature for the rectilinear one-centre problem, we shall have the occasion now to dwell upon a few amongst these, which are geometric in nature and moreover, have a direct bearing on the present work. From a purely theoretical standpoint the complexity of geometric algorithm is of interest since it sheds new light on the intrinsic difficulty in computation. The solution procedures described here include Elzinga and Hearn algorithm [31] dealing with the equal weighted case using an innovative concept of a covering diamond (sec 1.1.7), Francis algorithm [34, 36] concerning both the unweighted and the symmetric weighted cases and Drezner and Wesolowsky algorithm [25] having substantial contribution to the asymmetric weighted problem.

Let (a_i, b_i), $i \in I = \{1, 2, \ldots, n\}$ be the location of the existing facilities or demand points and (x, y) be the proposed location of the new facility or the convenience centre.

1.2.1 Elzinga and Hearn Algorithm

Elzinga and Hearn [31] have studied four variants of the minimax location problem using geometric arguments, stated as follows:

$$\min_p \max_i [L_p(P, P_i) + k_i]$$

where p = 1 or 2 and k_i = nonnegative constant. We discuss the Elzinga-Hearn algorithm for the case for which all the $k_i = 0$ and $p = 1$. All the location points are at first covered by a rectangle by moving a line having slope -1 so as to touch at least one point situated farthest left and at least another point farthest right and doing the same thing with another line with slope +1. They next consider at most four points, one on each side of the rectangle. Four equal diamonds centred at each of these are then constructed in such a way that the radius of a diamond is less than the required minimax distance. The diamonds are now allowed to expand uniformly about their centres. Any point that belongs to all four diamonds is an optimal solution. If the rectangle has unequal adjacent sides Elzinga and Hearn have shown geometrically that the perpendicular bisector of the longer sides truncated by vertical and horizontal lines constructed through the extremities of one of the shorter sides of the rectangle constitutes the solution set.

1.2.2 Francis Algorithm

Francis [34] has considered the generalised unweighted one-centre problem in the following form:

$$\underset{(x,y) \in E^2}{\text{minimise}} \quad f(x, y)$$

$$\text{where } f(x, y) = \max_{i \in I} (|x - a_i| + |y - b_i| + g_i)$$

The inclusion of the term g_i is justified by the fact that if (x, y) be the location of an ambulance, then g_i may be interpreted as the travel distance from (a_i, b_i) to the nearest hospital.

This problem may be rewritten as:

$$\text{minimise } z$$

$$\text{subject to } |x - a_i| + |y - b_i| + g_i \le z, \ i \in I$$

By manipulating these absolute value inequalities they have, after some reductions, obtained the following linear programming problem:

$$\text{minimise } z$$

$$\begin{aligned} \text{subject to} \quad & x + y - z \le a_i + b_i - g_i \\ & x + y + z \ge a_i + b_i + g_i \\ & -x + y - z \le -a_i + b_i - g_i \\ & -x + y + z \ge -a_i + b_i + g_i \\ & i \in I \end{aligned}$$

Assuming $c_1 = \min_{i \in I} (a_i + b_i - g_i)$, $c_2 = \max_{i \in I} (a_i + b_i + g_i)$,

$$c_3 = \min_{i \in I} (-a_i + b_i - g_i), \ c_4 = \max_{i \in I} (-a_i + b_i + g_i)$$

the above LPP further reduces to

$$\text{minimise } z$$

subject to
$$\begin{aligned} x + y - z &\le c_1 \\ x + y + z &\ge c_2 \\ -x + y - z &\le c_3 \\ -x + y + z &\ge c_4 \end{aligned} \qquad (1)$$

These constraints define a rectangle, two of whose parallel sides are inclined at an angle of 45° with the x-axis while the other two make an angle of 135° with it. Let

$$c_5 = \max_{i \in I} (c_2 - c_1, c_4 - c_3).$$

Then any point belonging to the line segment joining the points $\lambda(c_1 - c_3, c_1 + c_3 + c_5)$ and $\lambda(c_2 - c_4, c_2 + c_4 - c_5)$ is a minimax location with λc_5 as the optimal objective value, where $\lambda = 1/2$.

The weighted version of the above problem considered by Francis [36] may be stated as:

$$\underset{(x,y) \in E^2}{\text{minimise}} \quad f(x,y)$$

$$\text{where } f(x, y) = \max_{i \in I} [w_i(|x - a_i| + |y - b_i|) + g_i]$$

where w_i (≥ 0) is the weight associated with (a_i, b_i) and g_i may be interpreted as the time required by user i to prepare to go to the centre.

With $M = \{i : i \in I, w_i > 0\}$, $\overline{M} = \{i : i \in I, w_i = 0\}$ the above problem may be rewritten as

$$\underset{(x,y) \in E^2}{\text{minimise}} \quad f(x,y)$$

$$\text{where } f(x,y) = \max \left\{ \max_{i \in M} [w_i(|x-a_i|+|y-b_i|)+g_i], \ \max_{i \in \overline{M}}(g_i) \right\}$$

Let $\overline{g} = \max_{i \in \overline{M}}(g_i)$. If $\overline{g} \le k$ then the inequality

$$\max_{i \in M} [w_i(|x-a_i|+|y-b_i|)+g_i] \le k$$

is clearly equivalent to

$$x + y \le a_i + b_i + (k - g_i)/w_i$$

$$x + y \ge a_i + b_i - (k - g_i)/w_i$$

$$-x + y \le -a_i + b_i + (k - g_i)/w_i$$

$$-x + y \ge -a_i + b_i - (k - g_i)/w_i$$

which are the same as (1) having g_i removed, z replaced by $(k - g_i)/w_i$ and the condition $i \in I$ substituted by $i \in M$. We now define $c_1(k)$ through $c_4(k)$ as follows:

$$c_1(k) = \min_{i \in M} \left(a_i+b_i+(k-g_i)/w_i\right), \quad c_2(k) = \max_{i \in M} \left(a_i+b_i-(k-g_i)/w_i\right)$$

$$c_3(k) = \min_{i \in M} \left(-a_i+b_i+(k-g_i)/w_i\right), \quad c_4(k)=\max_{i \in M} \left(-a_i+b_i-(k-g_i)/w_i\right)$$

The set of all (x, y) such that $f(x, y) \le k$ is given by

$$S(k) = \Big\{(x, y): c_2(k) \le x + y \le c_1(k),$$

$$c_4(k) \le -x + y \le c_3(k)\Big\}$$

which is a rectangle with a pair of parallel sides at 45° and another pair of parallel sides at 135° with the x-axis. For notational convenience, the linear transformation T and its inverse T^{-1}, have been defined as follows:

$$T(x, y) = (x + y, -x + y) = (x', y') \text{ (say) and}$$

$$T^{-1}(x, y) = \lambda(x-y, x+y), \text{ where } \lambda = 1/2.$$

The numbers α_{ij} and β_{ij} for all $1 \le i < j \le n$ are defined thus:

$$\alpha_{ij} = \max \left([w_i w_j |a'_i - a'_j| + w_i g_j + w_j g_i]/(w_i + w_j), \; g_i, g_j \right)$$

$$\beta_{ij} = \max \left([w_i \ w_j \ |b'_i - b'_j| + w_i \ g_j + w_j \ g_i]/(w_i + w_j), \; g_i, \ g_j \right)$$

Geometrically, α_{ij} represents the value of k when both the coordinates of either user are respectively greater than those of the other and β_{ij} represents the value of k in all other cases. Let p_1, p_2 be the indices for which

$$z_1 = \max_{1 \le i < j \le n} (\alpha_{ij}) = \alpha_{p_1 p_2}$$

and q_1, q_2 the indices for which

$$z_2 = \max_{1 \le i < j \le n} (\beta_{ij}) = \beta_{q_1 q_2}$$

Also let $r^* = \left(w_{p_1} \ a'_{p_1} + w_{p_2} \ a'_{p_2} + \varepsilon \ (g_{p_1} - g_{p_2})\right)/(w_{p_1} + w_{p_2})$ where $\varepsilon = -1$ when $a'_{p_1} \le a'_{p_2}$, and $\varepsilon = 1$ when $a'_{p_1} > a'_{p_2}$; and let $s^* = \left(w_{q_1} \ b'_{q_1} + w_{q_2} \ b'_{q_2} + \varepsilon \ (g_{q_1} - g_{q_2})\right)/(w_{q_1} + w_{q_2})$ where $\varepsilon = -1$ or $+1$ according as $b'_{p_1} \le b'_{p_2}$ or $b'_{p_1} > b'_{p_2}$. Then

$z_0 = \max (z_1, z_2)$ gives the minimum objective value and $T^{-1}(r^*, s^*)$ a minimax location. To decide whether a unique location or alternative locations exist the following three cases have been considered.

Case 1. $z_0 = z_1 = z_2$ implies $T^{-1}(r^*, s^*)$ is the unique location.

Case 2. $z_0 = z_1 > z_2$ implies that any point belonging to the line segment joining $T^{-1}(r^*, s_1)$ and $T^{-1}(r^*, s_2)$ can claim to be a minimax location where s_1 and s_2 are given by

$$s_1 = \max_{i \in I} \left\{ \left(b'_i - (z_0 - g_i)\right)/w_i \right\}, \; s_2 = \min_{i \in I} \left\{ \left(b'_i + (z_0 - g_i)\right)/w_i \right\}$$

Case 3. $z_0 = z_2 > z_1$ implies that any point within the line segment defined by $T^{-1}(r_1, s^*)$ and $T^{-1}(r_2, s^*)$ is an optimal location, r_1 and r_2 being given by

$$r_1 = \max_{i \in I} \left\{ \left[a'_i - (z_o - g_i)\right]/w_i \right\}, \quad r_2 = \min_{i \in I} \left\{ \left[a'_i + (z_o - g_i)\right]/w_i \right\}$$

1.2.3 Drezner and Wesolowsky Algorithm

We now discuss Drezner and Wesolowsky's method [25] of solution of the asymmetric rectilinear minimax problem. Let the distance between the proposed location (x, y) and the demand point (a_i, b_i) to be denoted by $d_i(x, y)$, be defined as follows:

$$d_i(x, y) = d_i(x) + d_i(y) \text{ where}$$

$$d_i(x) = \begin{cases} E_i|x-a_i| & \text{if } x \geq a_i \\ W_i|x-a_i| & \text{if } x < a_i \end{cases} \quad \text{and } d_i(y) = \begin{cases} N_i|y-b_i| & \text{if } y \geq b_i \\ S_i|y-b_i| & \text{if } y < b_i \end{cases}$$

E_i, N_i, W_i and S_i being the four weights to east, north, west and south respectively.

The problem considered by Drezner and Wesolowsky is as follows:

$$\underset{(x,y) \in E^2}{\text{minimise}} \left\{ f(x,y) = \max_{i \in I} \{d_i(x, y)\} \right\}$$

This problem can be restated as a linear programming problem involving 3 variables and 4n constraints as shown below:

$$\text{minimise } z$$

$$\text{subject to} \quad E_i(x - a_i) + N_i(y - b_i) \leq z$$
$$W_i(a_i - x) + N_i(y - b_i) \leq z$$
$$E_i(x - a_i) + S_i(b_i - y) \leq z$$
$$W_i(a_i - x) + S_i(b_i - y) \leq z$$

The method of solution proposed by Drezner and Wesolowsky consists in finding a set of three demand points in such a way that the solution to this 3-point problem coincides with that of the original problem. By a lemma it has been shown that the optimal point must lie in the smallest rectangle obtained by drawing sides parallel to the coordinate axes containing all the demand points. Consequently, the solution point for the 2-point problem will lie in the rectangle constructed with these two points as opposite vertices. As a result, the solution point in this case is the point of intersection of one of the sides of the rectangle and the line of equal distances from the two given points i and j, whose equation is given by

$$X_i\,|x - a_i| + Y_i\,|y - b_i| = X_j\,|x - a_j| + Y_j\,|y - b_j|$$

where $X_r = \begin{cases} E_r, \text{if } x \geq a_r \\ W_r, \text{otherwise} \end{cases}$ and $Y_r = \begin{cases} N_r, \text{if } y \geq b_r \\ S_r, \text{otherwise} \end{cases}$, $r = i, j$.

Not all the four points are feasible. From the set of feasible points the one with the minimum objective is to be chosen.

The 3-point problem has been next decomposed into three 2-point problems. A solution point (x^*, y^*) to a 2-point problem involving i, j (say), is also a solution to a 3-point problem characterised by i, j, k iff $d_i\ (x^*, y^*) \geq d_k\ (x^*, y^*)$. Otherwise, the solution (x_o, y_o) to the 3-point problem must satisfy

$$d_i(x_o, y_o) = d_j(x_o, y_o) = d_k(x_o, y_o).$$

In such a situation (x_o, y_o) will be the point of intersection between any two lines of equal distances with respect to i, j and i, k (say), which will lie in one of the sub-rectangles enclosing the three points. The algorithm given by Drezner and Wesolowsky solves the above problem using an iterative technique and may be described as under:

Step 1. Choose any three points from the set of given demand points and solve the 3-point problem. Obtain $f(x^1, y^1)$ the objective value corresponding to the solution point (x^1, y^1) and go to step 2.

Step 2. Calculate $d^r = d_j(x^r, y^r)$ the maximum distance at the rth iteration from (x^r, y^r) to (a_j, b_j) and the corresponding $f(x^r, y^r)$ and go to step 3.

Step 3. If $d^r = f(x^r, y^r)$ then (x^r, y^r) is the optimal solution; stop. Else go to step 4.

Step 4. Introduce (a_j, b_j) by dropping one of the three points in such a way that the three retained points correspond to the maximum objective value $f(x^{r+1}, y^{r+1})$, increment r by one and go to step 2.

1.3 The present methodology

The complexity of the rectilinear one-centre problem increases as the number of demand points increases. But the algorithm we are about to describe easily yields exact solution even for large size problems and as far as we know no existing algorithm uses geometric concepts for the constrained case. Before we proceed to describe our method of solution let us say a few words about the usefulness of developing a specialised algorithm. Although simplex method is quite efficient in solving minimax location problems and there are readily available LP solvers capable of solving medium to large-scale problems, we give below the reasons for working out yet another algorithm. The current solution procedure requires storing of three vectors with n components each in the symmetric weighted case whereas standard LP packages need three 4n-vectors occupying that many memory locations. As a result, the former can handle a problem having 3500 data points even on a PC. In contrast, solving a problem having 1000 points with one of the above mentioned packages makes memory management more cumbersome. The results obtained by running the Pascal program of the algorithm on a 486 PC AT after randomly generating various sets of data points in the range of 1500 to 3500 are summarised below. The number of iterations never exceeded four.

No. of data points	Average running time in secs.	Maximum no. of iterations
1500	0.36	2
2000	0.48	2
2500	0.65	2
3000	0.83	3
3500	0.92	4

With the above preliminaries let us now give a brief account of the solution procedure for the rectilinear minimax problem. The locus of a point (x, y), whose weighted rectilinear distances from two given points P_i and P_j are equal, is given by

$$l_i(x - a_i) + m_i(y - b_i) = l_j(x - a_j) + m_j(y - b_j) \qquad (2)$$

where

$$l_r = \begin{cases} u_r^+ \text{ if } x \geq a_r \\ -u_r^- \text{ otherwise} \end{cases} \text{ and } m_r = \begin{cases} v_r^+ \text{ if } y \geq b_r \\ -v_r^- \text{ otherwise} \end{cases}, \; r = i \text{ or } j$$

in the asymmetric weighted case. By asymmetric weight we mean that with every location point is associated four different weights corresponding to the four principal directions viz., left, right, up and down, with respect to a pair of mutually perpendicular lines. The weights u_r^+, u_r^-, v_r^+, v_r^- are considered positive. In the symmetric case $u_r^+ = u_r^- = v_r^+ = v_r^- = w_r$ and if, additionally, the weights are equal, $w_r = 1$. This locus has been shown to be a closed polygon in the weighted case with at most six sides enclosing the demand point having greater weight. Here by greater weight in the asymmetric case we mean $u_i^{\pm} > u_j^{\pm}$ and $v_i^{\pm} > v_j^{\pm}$. The locus reduces to an open polygon in the absence of weights. This locus will be subsequently called an equipolygon EP(i-j). The optimal solution with respect to a pair of demand points P_i, P_j has been found to lie within or on the rectangle drawn with these two points as opposite vertices, to be called $R(P_i, P_j)$ henceforth. Our method clearly obtains the direction of descent i.e., the directed edge of the equipolygon along which the objective does not increase, leading to the boundary of the rectangle.

1.3.1 The solution procedure for the unconstrained case

Although any point may be chosen as the starting point, we take, for convenience, one of the vertices of the smallest rectangle SR whose sides are papallel to the axes of coordinates enclosing all the demand points as the initial solution point. In particular, we have taken the vertex at the rightmost bottom corner P_o(xmax, ymin) as the starting solution point. We next determine the weighted farthest demand point, say P_i, from P_o. By traversing an L-shaped path joining P_o to (a_i, ymin) to P_i, to be denoted by L(P_o, P_i), we reach a point T(p, q) equidistant from P_i and at least another demand point, say P_j. T represents L, M or N in case $a_i < a_j$ as shown in figure 1a). In case $a_i > a_j$, T denotes L or N (see figure 1b). In the figures from 1a to 3d and 6a, 6b any two opposite corners of the rectangle ABCD denote P_i, P_j. From T we continue moving in the direction of descent of the equipolygon EP(i-j) until a point E is obtained so that any one of the following possibilities is true:

(i) E is on the boundary of R(P_i, P_j) and no P_k, $k \in I\setminus\{i,j\}$, is as far away from E as P_i or P_j.

(ii) E is equidistant from three points P_i, P_j and P_k.

In case (i) E is optimal. In case (ii) if E falls within or

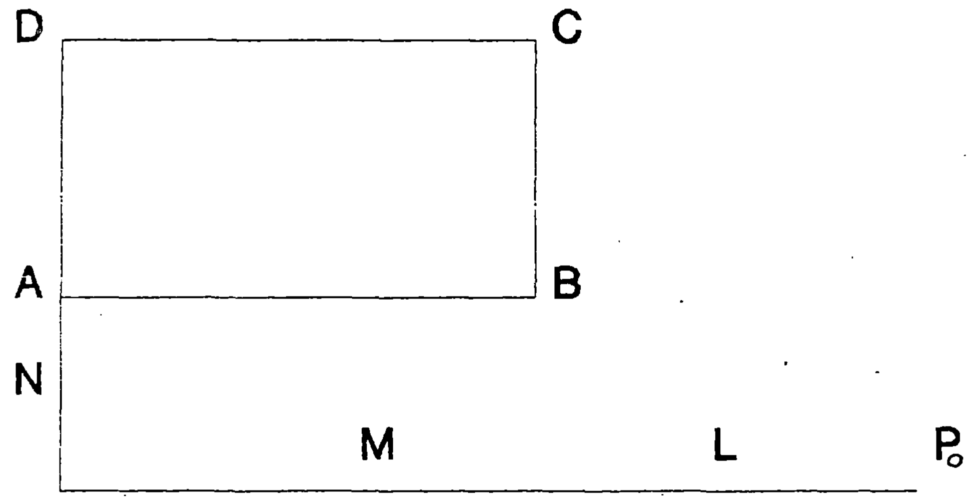

Figure 1a

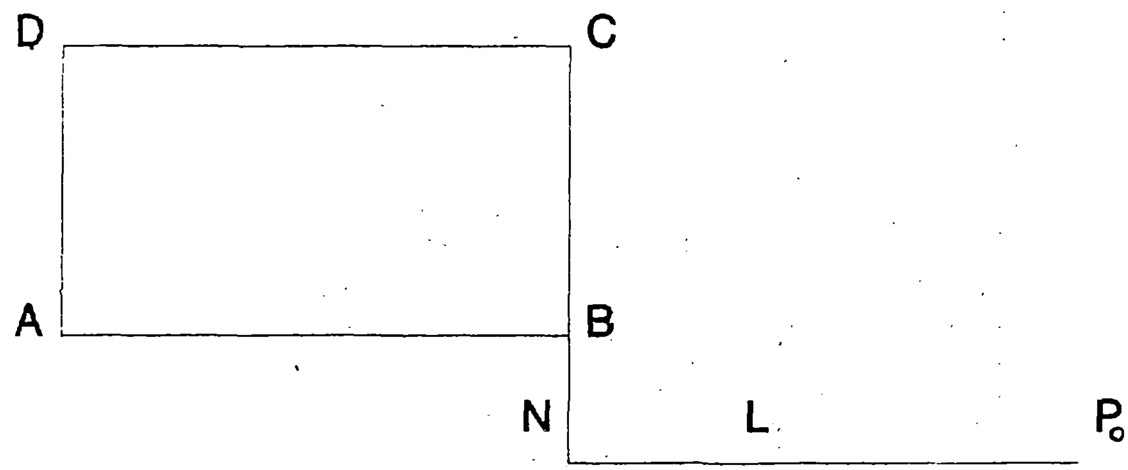

Figure 1b

on the boundary of $R(P_i, P_k)$ or $R(P_j, P_k)$ then E is optimal. Otherwise, the two points needed for the next iteration can be obtained by means of a well defined selection rule stated below. If more than three points are equidistant from E then by a repeated application of the procedure for case (ii) the pair of points required for the next iteration may be easily obtained.

Determination of the point E is carried out in two steps: first, *finding the direction of descent at T* and second, *obtaining the point of intersection of the edge of EP(i-j) containing T and EP(i-k)*, $k \in I \setminus \{i, j\}$.

Finding the direction of descent at T :

Let us denote the difference of the weighted rectilinear distances of P_i and P_j from a point X by diff(X, P_i, P_j). Refer to figures 2a and 2b for the case for which abscissa of T is greater than both a_i and a_j. If diff(A, P_i, P_j) and diff(B, P_i, P_j) be of the same sign then the descent direction TV will be given as in figure 2a; otherwise, TV will be given as in figure 2b, where V is the vertex of EP(i-j) lying on the edge containing T.

Let the abscissa of T lie between a_i and a_j. If the product of the above differences for A and B be negative then the descent direction TV is given as shown in figures 3a, 3b; otherwise, the abscissa of the demand point with the smaller weight is assigned to that of the point V and the ordinate calculated from the equation (2) of the equipolygon

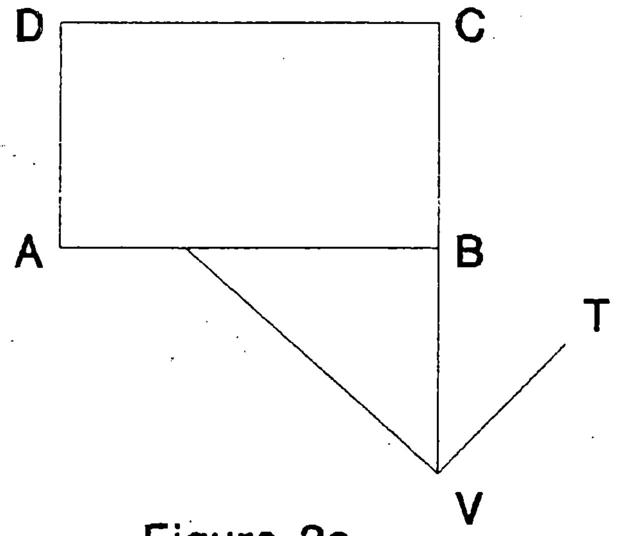

Figure 2a

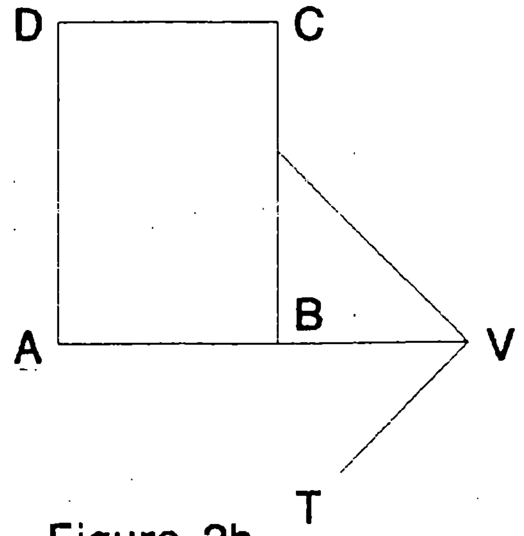

Figure 2b

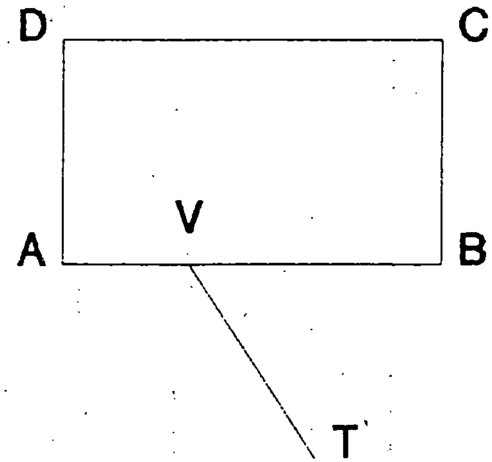

Figure 3a

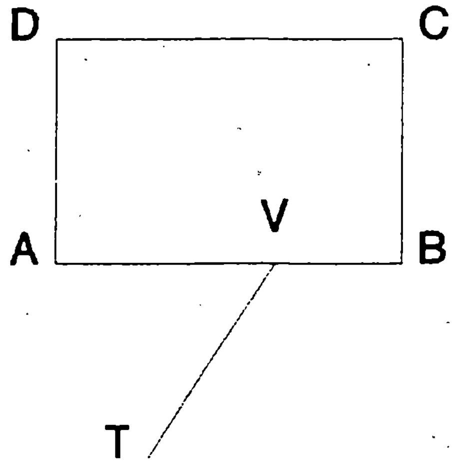

Figure 3b

(see figures 3c, 3d). If the ordinate of T lies between b_i and b_j then the position of V may be obtained in a similar manner. It should be clearly borne in mind that to reach the boundary of $R(P_i, P_j)$ we have to traverse at most three edges of EP(i-j); see figures 3c, 3d.

Obtaining the point of intersection of the edge of EP(i-j) containing T and EP(i-k) :

If $\mathrm{diff}(V, P_i, P_k) \le 0$, for some $k \in I \setminus \{i, j\}$, then
V_k := EP(i-j) $\cap$ EP(i-k) exists; and TE = min $\{TV_k\}$
else
if V is on the boundary of the rectangle then E $\longleftarrow$ V
else T $\longleftarrow$ V and repeat the above procedure to get E.

The line segment TV of EP(i-j) may intersect either (I) $x=a_k$ and $y=b_k$ or (II) $x=a_k$ or (III) $y=b_k$ or (IV) none of the above. For case (I) see figures 4a, 4b; for cases (II),(III) and (IV) refer to figures 4c, 4d and 4e respectively. Let TG = s_1 and TH = s_2. Figure 4a corresponds to the case for which $s_1 < s_2$; figure 4b represents the case where $s_1 > s_2$. Let us consider $s_1 < s_2$. We try to obtain the point V_k first by ascertaining if it belongs to the line segment TG. If it does then no more search is necessary. Otherwise, we replace TG by GH and repeat the same procedure. If V_k is still not found we have to do the same thing with GH replaced by HV. In any case at most three searches are required to get V_k. For $s_1 > s_2$ all the above steps are needed to be performed

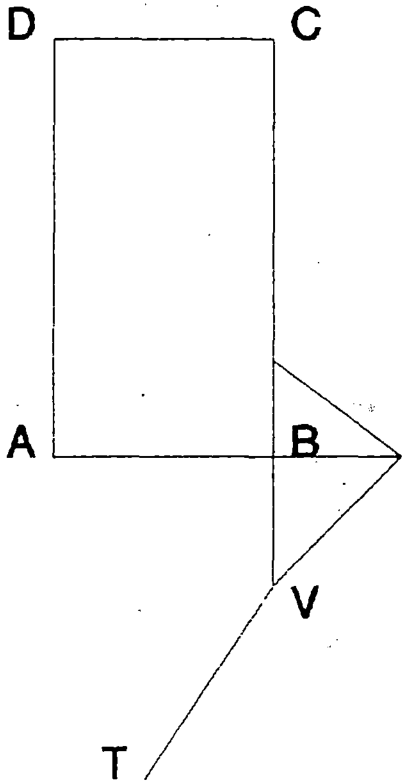

Figure 3c

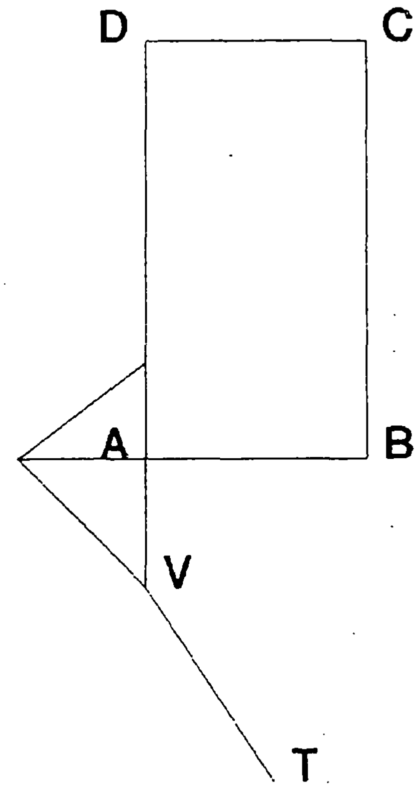

Figure 3d

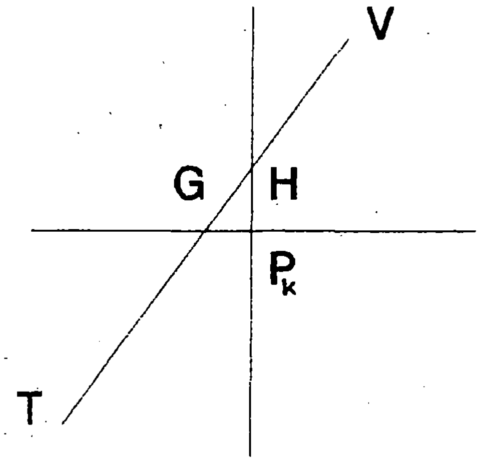

Figure 4a

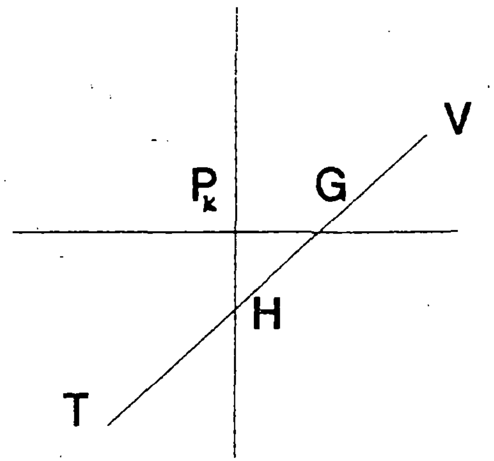

Figure 4b

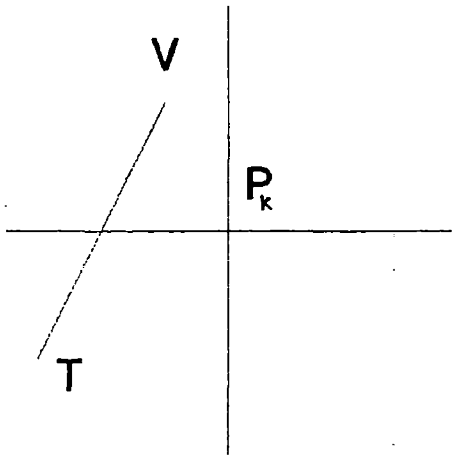

Figure 4c

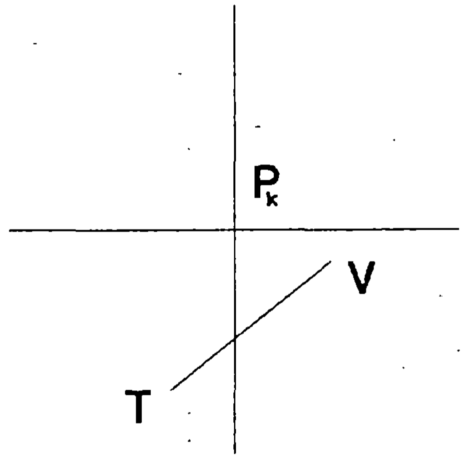

Figure 4d

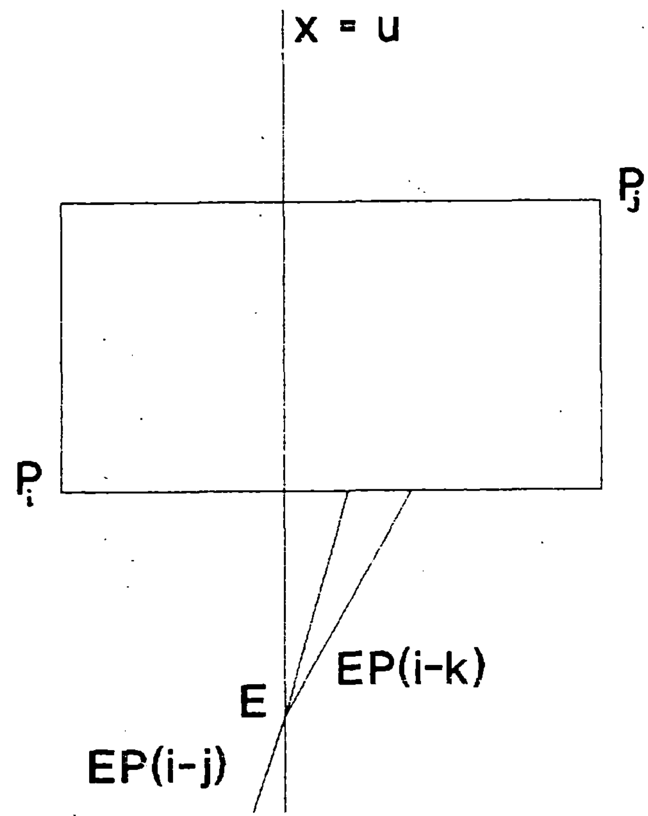

Figure 5

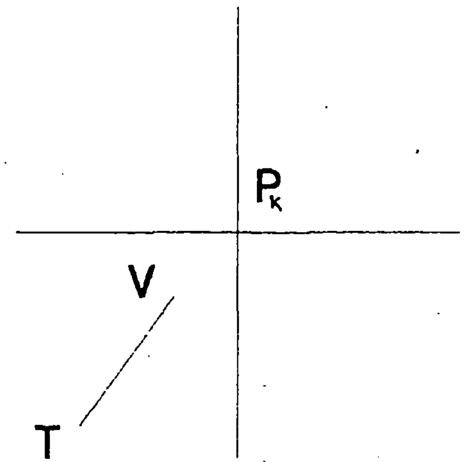

Figure 4e

taking the segments TH, HG, GV in that order. All the other cases are disposed of in a similar manner. It should be observed that cases (II) and (III) require at most two searches and case (IV) one. Although in all the figures 4a through 4e the point T has been shown to lie in the third quadrant with respect to P_k, the above arguments apply equally well had we chosen the point T in any one of the remaining three quadrants. We now describe the search technique. For an illustration let us again refer to case (I). If diff(G, P_i, P_k) < 0 then the point of intersection V_k of EP(i-j) and EP(i-k) exists within the line segment TG and V_k is given by the convex combination of the points T and G satisfying equation (2). Otherwise, if there is still some segment left to be searched we apply the above criterion to the next segment.

We now go on to describe *the criterion for selecting the pair of points needed for the next iteration when three or more equipolygons meet at E(u, v) and E is non-optimal.*

Selection Rule :

Let S_1 = {P_k| the weighted rectilinear distance from E of P_k = the weighted rectilinear distance of P_i or P_j from E; $k \neq i,j$}.

All the points belonging to S_1 must be on one side of x = u (y = v). Take the points P_i, P_j and another point $P_k \in S_1$. If these three points lie on one side of x = u (y = v) then

if two of them be on the same side of y = v (x = u) then the point corresponding to the smaller of the two weights associated with these latter points is to be retained for the next iteration after relabelling them as P_i, P_j

else perform the next iteration after excluding the point with the maximum weight.

Justification of the selection Rule:

To give a concrete example let us take P_j, P_k on one side of x = u. We want to prove that *P_k has a smaller weight compared to P_j*. If possible, let us suppose that weight of P_k is greater. Since T is then outside EP(j-k) the weighted rectilinear distance of P_j is less than that of P_k contradicting primal feasibility.

We next propose to show that the direction of descent at E of EP(i-k) is pointed outside EP(i-j) when the weight of P_i is greater than that of P_j and inside, otherwise. From what has just now been proved, weight of P_j > weight of P_k implies weight of P_i > weight of P_k. Hence, by primal feasibility and also by virtue of the fact that an equipolygon encloses the greater weight, it is evident that T is outside EP(i-k). As E is the point of intersection of EP(i-j) and EP(i-k) it immediately follows that the direction of descent at E of EP(i-k) is pointed outside EP(i-j). See figure 5.

In case of the latter let us take a point $T_1 \in$ EP(i-k) in the neighbourhood of E and opposite to the direction of descent of EP(i-j) at E. Therefore, T_1 must be outside EP(i-j) in order to maintain primal feasibility thereby proving our assertion.

From the above we conclude that if we follow the direction of descent of EP(i-k) at E the weighted distances of P_i, P_k while remaining equal in relation to each other, continually diminish but this distance remains greater than that of P_j.

When P_i, P_j and P_k belong to the same quadrant with respect to E the three equipolygons coincide. Although we may select any equipolygon, without loss of generality we can exclude the point having the maximum weight from the purview of the following iteration.

Finding the stretch containing the set of optimal solutions:

Two cases are to be considered here depending on the position of the optimal point E. When E is within R(P_i, P_j) and a third point P_k, as far away from E as P_i or P_j in the weighted rectilinear distance sense, is available. First the symmetric case. Following the edge of EP(i-j) through E the objective will remain unaltered. But for P_k it will decrease in one direction only which, therefore, has to be chosen as the direction of descent. T $\longleftarrow$ E and proceeding exactly as the method described above for getting E we shall get TE as the required stretch. If E is on the boundary of R(P_i, P_j) then we obtain a unique direction of descent which will give the stretch by applying the same arguments as has been put forward in this section. In the asymmetric case, the same arguments once again hold. But we must bear in mind that the values of the objective function at various points on the edge of EP(i-j) lying within R(P_i, P_j) are not, in general, the same.

1.3.2 The solution procedure for the constrained case

If the smallest rectangle SR containing all the demand points belongs to the convex polyhedral region CP then the unconstrained case we have already discussed is obtained. If the right hand bottom corner of SR $\in$ CP then as usual we take P_o as the starting point; otherwise, P_o is chosen arbitrarily within CP. In any case we obtain the weighted farthest point P_i from P_o and then move along L(P_o, P_i) until a point T is obtained from which P_i and another point P_j are at the same weighted rectangular distance. We denote by Y the point where the direction of movement meets the boundary.

If T $\notin$ CP then Y $\longleftarrow$ ∂CP $\cap$ L(P_o, P_i) and the boundary criterion given below is applied.

Else we follow the direction of descent of EP(i-j) so as to encounter a point E equidistant from P_i, P_j and at least one more point P_k. Now either E $\in$ CP or E $\notin$ CP.

If E $\in$ CP then

if E satisfies the criterion discussed in sec 1.3.1

then E is optimal

else apply the selection rule discussed in sec 1.3.1, drop a point from P_i, P_j and repeat this step.

Else $Y \longleftarrow \partial CP \cap EP(i-j)$ and we apply the boundary criterion. If no such E is available then

if we can move upto the point V on $\partial R(P_i, P_j)$ then V is optimal

else $Y \longleftarrow \partial CP \cap EP(i-j)$ and the boundary criterion is applied.

Boundary criterion: We now present the procedure that determines the direction governing the movement on reaching a point Y on ∂CP. For this purpose let us introduce the concept of the *Cone of descent direction* which will be found to be useful in our subsequent discussion. Let $H_i(Y)$ denote the halfspace at Y defined by the isoline of P_i through Y containing P_i. Then $H_i(Y) \cap CP$ is the cone of descent direction provided no $EP(i-j)$, $j \in I \setminus \{i\}$, passes through Y. Otherwise, $H_i(Y) \cap H_j(Y) \cap CP$ is the cone of descent direction as shown in figures 6a, 6b. Let $\mathscr{C}$ represent this cone. If $\mathscr{C} = \{Y\}$ then Y is the unique optimal solution. The portion of ∂CP constituting an extreme direction of $\mathscr{C}$ will give the direction of the next movement. If P be any point lying within $EP(i-j)$ the weighted rectilinear distance from P to the demand point having greater weight is less than the weighted rectilinear distance from P to the other point whereas if P is outside $EP(i-j)$ this property is reversed.

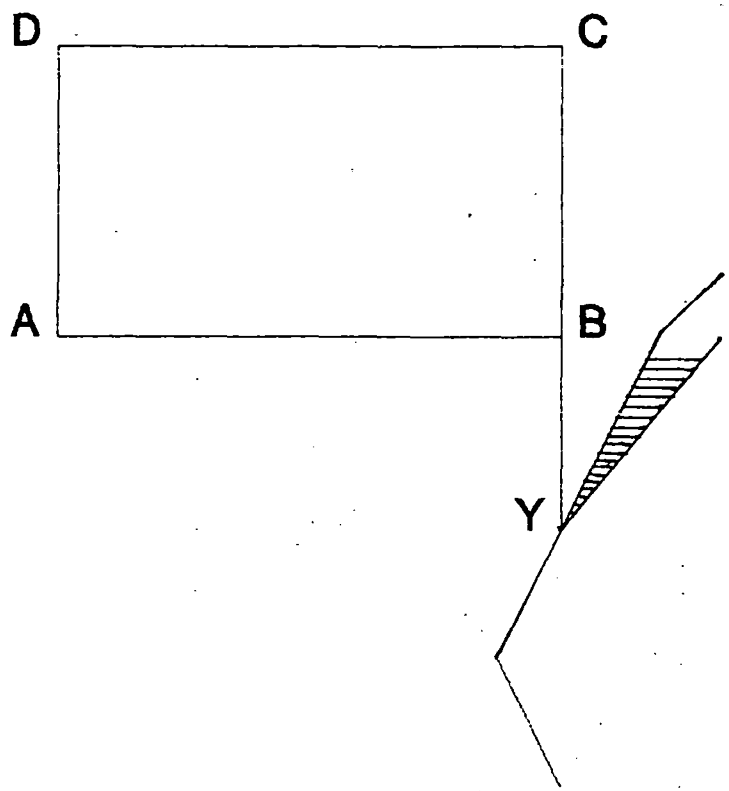

Figure 6a

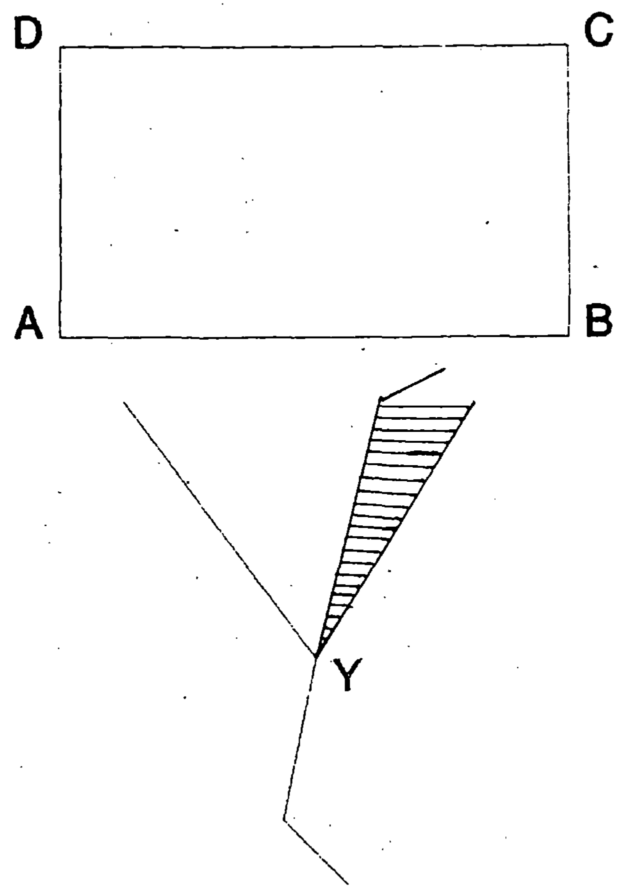

Figure 6b

We, therefore, have the following criterion.

If the direction of movement along the boundary of CP is towards the interior of EP(i-j) then drop the demand point with the greater weight; else drop the point corresponding to the smaller weight. It is worth mentioning in this connection that the interior or exterior of EP(i-j) is determined by comparing the gradients of the concerned edge of the equipolygon and the extreme direction of $\mathcal{E}$.

Moving along an extreme direction of $\mathcal{E}$ any one of the cases enumerated below may become true:

(i) a point P on ∂CP equidistant from at least two demand points is found;

(ii) an extreme point V of ∂CP is attained;

(iii) a point P situated at the intersection of ∂CP and a line through P_i drawn parallel to either coordinate axis is obtained.

In case (i) if the equipolygon with respect to a suitably chosen pair of demand points is directed towards the interior of CP then move along this side of the equipolygon; else get $\mathcal{E}$ afresh and decide on the proper course of action.

In case (ii) if V satisfies the condition of optimality then stop; else drop the currently active edge of ∂CP, select the next edge, obtain $\mathcal{E}$ de novo and repeat the above steps.

In case (iii) determine $\mathcal{E}$ at P with respect to the isoline of P_i, different from the earlier one, and perform the above actions.