CHAPTER - II

THERMAL STRESSES IN LAYERED MEDIA

Paper 1 : Stress concentration in an elastic layered media (Two layered case)

Paper 2 : Stress concentration in an elastic layered media (Three layered case)

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STRESS CONCENTRATION IN ELASTIC LAYERED MEDIA DUE TO THERMAL EFFECT (TWO LAYERED CASE)

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1.STATEMENT OF THE PROBLEM

Let the material under consideration occupy the lower half of the plane z=0 and the axis of z being taken positive when drawn into the material. Physical quantities involved here are all symmetrical about z-axis. Lower boundary of the upper layer is given by the plane $z=h_1$ and the underlying mass extended to infinity. interfaces are supposed to be perfectly rough so that stresses and displacements are continuous across it.

2.METHOD OF SOLUTION

Since the physical quantities involved in the problem are all symmetrical about z-axis, so four non-zero stress components $\sigma_r, \sigma_\theta, \sigma_z, \tau_{zr}$ and two non-zero displacements u_r, u_z are retained in terms of the stress function ϕ satisfying the differential equation [9]

$$\nabla^4\phi = \left\{ \frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right\}\phi = 0 \qquad (1)$$

The stress and displacement components are given by [17]

$$\sigma_r = \frac{\partial}{\partial z}\left[\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}\right] \qquad (2)$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left[\nu\nabla^2\phi - r^{-1}\frac{\partial\phi}{\partial r}\right] \tag{3}$$

$$\sigma_z = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \tag{4}$$

$$\tau_{zr} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \tag{5}$$

$$u_r = -\frac{1+\nu}{E}\frac{\partial^2\phi}{\partial r\partial z} \tag{6}$$

$$u_z = \frac{1+\nu}{E}\left[(1-2\nu)\nabla^2\phi + \frac{\partial^2\phi}{\partial r^2} + r^{-1}\frac{\partial\phi}{\partial r}\right] \tag{7}$$

Here the stress function ϕ due to Hankel transform [14] will take the form G, defined as

$$G(\xi, z) = \int_0^\infty r\phi(r,z)\, J_o(\xi r)\, dr, \quad (\xi>0, z>0) \tag{8}$$

where $J_o(\xi r)$ denotes the Bessel function [19] of order zero, whose inverse is given by

$$\phi(r,z) = \int_0^\infty \xi G(\xi, z)\, J_o(\xi r)\, d\xi \tag{9}$$

Applying Hankel transform on (1) and (4)-(7) we get

$$\left(\frac{d^2}{dz^2} - \xi^2 \right)^2 G(\xi, z) = 0 \tag{10}$$

$$\int_0^\infty r\sigma_z \, J_o(\xi r) \, dr = (1-\nu) \frac{d^3 G}{dz^3} - (2-\nu)\xi^2 \frac{dG}{dz} \tag{11}$$

$$\int_0^\infty r\tau_{zr} \, J_1(\xi r) \, dr = \left[\nu \frac{d^2 G}{dz^2} + (1-\nu)\xi^2 G \right] \tag{12}$$

$$\int_0^\infty r u_r \, J_1(\xi r) \, dr = \frac{1 + \nu}{E} \xi \frac{dG}{dz} \tag{13}$$

$$\int_0^\infty r u_z \, J_o(\xi r) \, dr = \frac{1 + \nu}{E} \left[(1-2\nu) \frac{d^2 G}{dz^2} - 2(1-\nu)\xi^2 G \right] \tag{14}$$

Computing inverses of (11)-(14) we get

$$\sigma_z = \int_0^\infty \xi \left[(1-\nu) \frac{d^3 G}{dz^3} - (2-\nu)\xi^2 \frac{dG}{dz} \right] J_o(\xi r) \, d\xi \tag{15}$$

$$\tau_{zr} = \int_0^\infty \xi^2 \left[\nu \frac{d^2 G}{dz^2} + (1-\nu)\xi^2 G \right] J_1(\xi r) \, d\xi \tag{16}$$

$$u_r = \frac{1+\nu}{E}\int_0^{\infty} \xi^2 \frac{dG}{dz} J_1(\xi r)\, d\xi \tag{17}$$

$$u_z = \frac{1+\nu}{E}\int_0^{\infty} \left[\xi(1-2\nu)\frac{d^2G}{dz^2} - 2(1-\nu)\xi^2 G\right] J_o(\xi r)\, d\xi \tag{18}$$

Let the solution of (10) be given by

$$G(\xi, z) = (A+Bz)(2\sinh\xi z + e^{-\xi z}) + (C+Dz)(2\cosh\xi z - e^{\xi z})$$

where A, B, C and D are functions of ξ to be determined from the suitable boundary conditions.

3.Solution Of The Problem

To determine the stresses, the potential of thermoelastic displacement ψ, related by the equations

$$\frac{\partial\psi}{\partial r} = (u_r)_T$$

$$\frac{\partial\psi}{\partial z} = (u_z)_T \tag{19}$$

is considered in the steady state of the temperature field given by

[3]

$$\nabla^2 T = 0 \tag{20}$$

Since upper layer and the underlying mass have different thermal properties, different potential displacement functions are chosen. From the stress-strain relations of thermo-elasticity and the equation of equilibrium [17], we have

$$\nabla^2 \psi_i = \beta_i T \tag{21}$$

where

$$\beta_i = \frac{1+\nu_i}{1-\nu_i} \alpha_i$$

Applying Hankel transform on (20) and (21) we obtain

$$\left(\frac{d^2}{dz^2} - \xi^2 \right) M(\xi, z) = 0 \tag{22}$$

and

$$\left(\frac{d^2}{dz^2} - \xi^2 \right) L_i(\xi, z) = \beta_i M(\xi, z), \quad i=1,2 \tag{23}$$

where $$M(\xi, z) = \int_0^\infty r T J_o(\xi r)\, dr \tag{24}$$

$$L_i(\xi, z) = \int_0^\infty r \psi_i J_o(\xi r)\, dr \tag{25}$$

The corresponding thermal stress components $(\sigma_z)_T, (\tau_{zr})_T$ and displacement components $(u_r)_T, (u_z)_T$ are given by

$$(\sigma_z)_{T_i} = 2\mu_i\left\{ \frac{\partial^2\psi_i}{\partial z^2} - \nabla^2\phi_i \right\} \tag{26}$$

$$(\tau_{zr})_{T_i} = 2\mu_i \frac{\partial^2\psi_i}{\partial r \partial z} \tag{27}$$

$$(u_r)_{T_i} = \frac{\partial\psi_i}{\partial r} \tag{28}$$

$$(u_z)_{T_i} = \frac{\partial\psi_i}{\partial z} \tag{29}$$

Now we consider the transformations of (26)-(29) into relation involving $L_i(\xi, z)$ and $M(\xi, z)$. The results of computations are

$$\int_0^\infty r(\sigma_z)_{T_i} J_o(\xi r)\, dr = 2\mu_i \xi^2 L_i(\xi, z) \tag{30}$$

$$\int_0^\infty r(\tau_{zr})_{T_i} J_1(\xi r)\, dr = -2\mu_i \xi \frac{d}{dz} L_i(\xi, z) \tag{31}$$

$$\int_0^\infty r(u_r)_{T_i} J_1(\xi r)\ dr = -\xi\ L_i(\xi, z) \tag{32}$$

$$\int_0^\infty r(u_z)_{T_i} J_o(\xi r)\ dr = \frac{d}{dz}\ L_i(\xi, z) \tag{33}$$

Applying Hankel inverse transform, we get

$$(\sigma_z)_{T_i} = 2\mu_i \int_0^\infty \xi^3 L_i(\xi, z) J_o(\xi r)\ d\xi \tag{34}$$

$$(\tau_{zr})_{T_i} = -2\mu_i \int_0^\infty \xi^2\ \frac{d}{dz}\ L_i(\xi, z) J_1(\xi r)\ d\xi \tag{35}$$

$$(u_r)_{T_i} = -\int_0^\infty \xi^2 L_i(\xi, z) J_1(\xi r)\ d\xi \tag{36}$$

$$(u_z)_{T_i} = \int_0^\infty \xi\ \frac{d}{dz}\ L_i(\xi, z) J_o(\xi r)\ d\xi \tag{37}$$

Let the solution of (23) be

$$L_i(\xi, z) = \frac{\beta_i}{2\xi} A_o(1-z)e^{-\xi z} \tag{38}$$

then the solution of (22) is

$$M(\xi, z) = A_o e^{-\xi z} \tag{39}$$

where A_o is a function of ξ.

Applying (38) we have from (30)-(33)

$$\int_0^\infty r(\sigma_z)_{T_i} J_o(\xi r)\ dr = \mu_i \beta_i \xi A_o (1-z) e^{-\xi z} \tag{40}$$

$$\int_0^\infty r(\tau_{zr})_{T_i} J_1(\xi r)\ dr = -\mu_i \beta_i A_o (z\xi - \xi - 1) e^{-\xi z} \tag{41}$$

$$\int_0^\infty r(u_r)_{T_i} J_1(\xi r)\ dr = -\frac{1}{2} \beta_i A_o (1-z) e^{-\xi z} \tag{42}$$

$$\int_0^\infty r(u_z)_{T_i} J_o(\xi r)\ dr = \frac{1}{2\xi} \beta_i (z\xi - \xi - 1) A_o e^{-\xi z} \tag{43}$$

Taking solution of (10) for different layers [59] as:

For the upper layer

$$G_1(\xi, z) = (A_1 + B_1 z)(2\sinh \xi z + e^{-\xi z}) + (C_1 + D_1 z)(2\cosh \xi z - e^{\xi z}) \tag{44}$$

For the underlying mass

$$G_2(\xi, z) = (A_2 + B_2 z) e^{-\xi z} + (C_2 + D_2 z) e^{\xi z}, \quad (\xi > 0, z > 0) \tag{45}$$

It is to be noted that stress and displacements in the underlying mass vanish as z tends to infinity. So, the components of stress and displacement for the upper layer obtained from (11)-(14) as

$$\int_0^\infty r(\sigma_z)_1 \, J_o(\xi r)\, dr =$$

$$= (1-2\nu_1)\xi^2 B_1(2\sinh \xi z + e^{-\xi z}) + (1-2\nu_1)\xi^2 D_1(2\cosh \xi z - e^{\xi z})$$

$$+\xi^3(A_1+B_1 z)(e^{-\xi z}-2\cosh \xi z) + \xi^3(C_1+D_1 z)(e^{\xi z}-2\sinh \xi z) \qquad (46)$$

$$\int_0^\infty r(\tau_{zr})_1 J_1(\xi r)dr = 2\nu_1\xi^2 B_1(2\cosh \xi z - e^{-\xi z}) + 2\nu_1\xi^2 D_1(2\sinh \xi z - e^{\xi z})$$

$$+\xi^3(A_1+B_1 z)(e^{-\xi z}+2\sinh \xi z)+\xi^3(C_1+D_1 z)(2\cosh \xi z - e^{\xi z}) \qquad (47)$$

$$\int_0^\infty r(u_r)_1 \, J_1(\xi r)dr = \frac{1+\nu_1}{E_1}\,\xi\left[B_1(2\sinh \xi z + e^{-\xi z}) + D_1(2\cosh \xi z - e^{\xi z})\right.$$

$$\left.+\xi(A_1+B_1 z)(2\cosh \xi z - e^{\xi z})+\xi(C_1+D_1 z)(2\sinh \xi z - e^{-\xi z})\right. \qquad (48)$$

$$\int_0^\infty r(u_z)_1 J_o(\xi r)dr = \frac{1+\nu_1}{E_1}\xi\Big[2(1-2\nu_1)B_1(2\cosh\xi z - e^{-\xi z}) + 2(1-2\nu_1)D_1$$

$$(2\sinh\xi z - e^{\xi z}) - \xi(A_1+B_1 z)(2\sinh\xi z + e^{-\xi z}) - \xi(C_1+D_1 z)(2\cosh\xi z - e^{\xi z})\tag{49}$$

Stresses and displacements for the underlying mass

$$\int_0^\infty r(\sigma_z)_2 J_o(\xi r)\, dr = \Big[(A_2+B_2 z)e^{-\xi z} - (C_2+D_2 z)e^{\xi z}\Big]\xi^3 +$$

$$+(1-2\nu_2)\xi^2\Big[B_2 e^{-\xi z} + D_2 e^{\xi z}\Big]\tag{50}$$

$$\int_0^\infty r(\tau_{zr})_2 J_1(\xi r)\, dr = \Big[(A_2+B_2 z)e^{-\xi z} + (C_2+D_2 z)e^{\xi z}\Big]\xi^3$$

$$-2\nu_2\xi^2\Big[B_2 e^{-\xi z} - D_2 e^{\xi z}\Big]\tag{51}$$

$$\int_0^\infty r(u_r)_2 J_1(\xi r)\, dr = -\frac{1+\nu_2}{E_2}\Big\{\Big[\xi(A_2+B_2 z)+B_2\Big]e^{-\xi z} - \Big[\xi(C_2+D_2 z)+D_2\Big]e^{\xi z}\Big\}\tag{52}$$

$$\int_0^\infty r(u_z)_3\, J_o(\xi r)\, dr = -\frac{1+\nu_2}{E_2}\left\{\left[(A_2+B_2 z)e^{-\xi z}+(C_2+D_2 z)e^{\xi z}\right]\xi^2 + +2(1-2\nu_2)\xi\left[B_2 e^{-\xi z}-D_2 e^{\xi z}\right]\right\} \tag{53}$$

4.BOUNDARY CONDITION

In order to nullify the stresses on the boundaries the following conditions are to be satisfied:

At $z=0$,

$$-(u_r)_{T_1} = (u_r)_1$$

$$-(u_z)_{T_1} = (u_z)_1$$

$$-(\sigma_z)_{T_1} = (\sigma_z)_1$$

$$-(\tau_{zr})_{T_1} = (\tau_{zr})_1 \tag{54}$$

At $z=h_1$,

$$-(u_r)_{T_2} = (u_r)_2$$

$$-(u_z)_{T_2} = (u_z)_2$$

$$-(\sigma_z)_{T_2} = (\sigma_z)_2$$

$$-(\tau_{zr})_{T_2} = (\tau_{zr})_2 \tag{55}$$

Conditions (54),(55) with the help of (40)-(43) and (46)-(49) give the values of constants as

$$A_1 = A'_1 A_o \ , \ A'_1 = S_1 - S_2 + S_3$$

$$B_1 = B'_1 A_o, \ \ B'_1 = S_2 - S_3$$

$$C_1 = C'_1 A_o \ , \ C'_1 = S_1 - S_2 - S_3$$

$$D_1 = D'_1 A_o \ , \ D'_1 = - S_3$$

where

$$S_1 = \frac{2\xi+1}{\xi} \beta_1 \mu_1$$

$$S_2 = \frac{\alpha_1}{(1-\nu_1)^2 \xi^2} \left[\mu_1 (1+\nu_1) - \xi E_1\right]$$

$$S_3 = \frac{\alpha_1}{(1-\nu_1)\xi^3} \left[\mu_1 (1+\nu_1) + (2\xi+1) E_1\right]$$

$$D_2 = \frac{\beta_2 r_1}{4(1-\nu_2^2)\xi^2 s_1} \left[\mu_2 (1+\nu_2)(\xi h_1 - h_1 - \xi) + (1+2\xi-2\xi h_1) E_2\right] A_o = D'_2 A_o$$

$$B_2 = \frac{\beta_2}{4(1-\nu_2^2)\xi^2} \left[\mu_2 (1+\nu_2)(2+\xi-\xi h_1 - h_1) + E_2\right] A_o = B'_2 A_o$$

$$A_2 = \frac{\beta_2 E_2}{4(1+\nu_2)\ \xi^3} \left\{ \xi^3 (\xi+1)(h_1 - 1) - \xi^2 \mu_2 (1+\nu_2) \right.$$

$$+ \frac{1}{2\xi r_1}\left[(2h_1\xi - 4\nu_2 + 1) r_1 B'_2 + (1-2\nu_2) s_1 D'_2\right]\Bigg\} A_o$$

$$C_2 = \left\{ \frac{\beta_2 E_2}{(1-\nu_2) s_1} (1-h_1) r_1 + \frac{r_1}{s_1}\left[\xi A'_2 + (\xi h_1 - 1) B'_2\right] - (\xi h_1 + 1) D'_2 \right\} A_o$$

Thus the constants A'_i, B'_i, C'_i, D'_i, $i=1,2$ are independent of A_o.

So, on the surface $z=0$, the resultant displacement and stress components are

$$\int_0^\infty r(u_r)_{R_1} J_1(\xi r)\, dr = \int_0^\infty r[(u_r)_1 + (u_r)_{T_1}] J_1(\xi r)\, dr =$$

$$= \left[\frac{1+\nu_1}{E_1} (\xi A'_1 + B'_1 - \xi C'_1 + D'_1) - \beta_1 \right] A_o \qquad (56)$$

$$\int_0^\infty r(u_z)_{R_1} J_o(\xi r)\, dr = \int_0^\infty r[(u_z)_1 + (u_z)_{T_1}] J_o(\xi r)\, dr =$$

$$= - \left[\frac{1+\nu_1}{E_1} \xi \left\{ \xi (A'_1 + C'_1) - 2(1-2\nu_1)(B'_1 + D'_1) \right\} + \frac{\xi+1}{\xi} \beta_1 \right] A_o \qquad (57)$$

$$\int_0^\infty r(\sigma_z)_{R_1}\ J_o(\xi r)\ dr = \int_0^\infty r[(\sigma_z)_1+(\sigma_z)_{T_1}]\ J_o(\xi r)\ dr =$$

$$= \xi\left[\mu_1\beta_1-\xi^2\left\{(A'_1-C'_1)+(1-2\nu_1)(B'_1+D'_1)\right\}\right]A_o \qquad (58)$$

$$\int_0^\infty r(\tau_{zr})_{R_1}\ J_1(\xi r)\ dr = \int_0^\infty r[(\tau_{zr})_1+(\tau_{zr})_{T_1}]\ J_1(\xi r)\ dr =$$

$$= \left[(1+\xi)\mu_1\beta_1+\xi^2\left\{(A'_1+C'_1)+2\nu_1(B'_1-D'_1)\right\}\right]A_o \qquad (59)$$

At $z=h_1$,

$$\int_0^\infty r(u_r)_{R_2}\ J_1(\xi r)dr = \int_0^\infty r[(u_r)_2+(u_r)_{T_2}]\ J_1(\xi r)dr =$$

$$=\left[\beta_2(h_1-1)\xi r_1+\frac{1+\nu_2}{E_2}\left\{\left[-\xi A'_2+(1-h_1)B'_2\right]r_1+\left[\xi C'_2+(1+h_1)D'_2\right]s_1\right\}\right]A_o \qquad (60)$$

$$\int_0^\infty r(u_z)_{R_2}\ J_o(\xi r)dr = \int_0^\infty r[(u_z)_2+(u_z)_{T_2}]\ J_o(\xi r)dr =$$

$$= -\left[\frac{\xi h_1 - \xi - 1}{\xi}\beta_2 r_1 + \frac{1+\nu_2}{E_2}\left\{(4\nu_2 - \xi h_1 - 2)(r_1 B'_2 + s_1 D'_2) - (A'_2 - C'_2)\right\}\right] A_o \quad (61)$$

$$\int_0^\infty r(\sigma_z)_{R_2} J_o(\xi r)\, dr = \int_0^\infty r[(\sigma_z)_2 + (\sigma_z)_{T_2}] J_o(\xi r)\, dr = \Big[\mu_2\beta_2(1-h_1)\xi r_1 +$$

$$\xi^2(A'_2 r_1 - C'_2 s_1) + r_1\xi(\xi h_1 - 2\nu_2 + 1)B'_2 - s_1\xi(\xi h_1 + 2\nu_2 - 1)D'_2\Big] A_o \quad (62)$$

$$\int_0^\infty r(\tau_{zr})_{R_2} J_1(\xi r)dr = \int_0^\infty r[(\tau_{zr})_2 + (\tau_{zr})_{T_2}] J_1(\xi r)dr = \Big[\mu_2\beta_2(\xi h_1 - \xi - 1)r_1 +$$

$$\xi^3(A'_2 r_1 + C'_2 s_1) + r_1\xi^2(\xi h_1 - 2\nu_2)B'_2 + s_1\xi(\xi h_1 + 2\nu_2)D'_2\Big] A_o \quad (63)$$

5. Flux Of Heat On The Boundaries

Let the flux of heat in a region of the surface $z=0$, distributed through layers in the underlying mass be

$$\frac{\partial T}{\partial z} = f(r/a), \quad 0<r<a \quad (64)$$

$$= 0, \quad r>a$$

using dimensionless variables

$$\xi A_o(\xi) = aX(\xi a), \quad \eta=\xi a, \quad \rho=r/a, \quad \zeta=z/a$$

where a is some length and η, a new variable of integration, we get from (24) and (39), on z=0,

$$\frac{\partial T}{\partial z} = -a^{-1} \int_0^{\infty} \eta X(\eta) J_o(\rho\eta)\ d\eta \tag{65}$$

By Hankel inversion theorem

$$X(\eta) = -a^{-1} \int_0^{1} \rho f(\rho) J_o(\rho\eta)\ d\rho \tag{66}$$

For a simple physical situation we consider a linear temperature distribution $f(\rho) = K\rho$, K = constant, then

$$X(\eta) = -\frac{K}{a\eta^2} J_1(\eta) \tag{67}$$

So, the unknown $A_o(\xi)$ being known the problem is completely solved.

6·Numerical Results

If the upper layer be concrete pavement, and the underlying mass be natural soil, then elastic constants for those materials are [8]

$E_1 = 2.18\times10^8$ gms/cm^2 , $\alpha_1 = -5\times10^{-6}/0^\circ C$

$E_2 = 1.1\times10^8$ gms/cm^2 , $\alpha_2 = 7.5\times10^{-6}/0^\circ C$

$\nu_1 = 0.15$, $\mu_1 = 0.94\times10^8$ gms/cm^2

$\nu_2 = 0.25$, $\mu_2 = 0.43\times10^8$ gms/cm^2

$K_1 = 6.4\times10^{-3}$, $K_2 = 6.7\times10^{-3}$

So, evaluating constants for a given value of η when a=1 and h_1=2, we have

$$S_1 = 10.8\times10^2; \quad S_2 = 7.605\times10^2; \quad S_3 = -50.6\times10^2$$

Thus

$$A'_1 = -32.1845\times10^2, B'_1 = -42.9945\times10^2, C'_1 = 53.8045\times10^2, D'_1 = 50.6\times10^2,$$

$$A'_2 = 0.9794\times10^2, B'_2 = 1.8723\times10^2, C'_2 = 1.2163\times10^2, D'_2 = -7.559\times10^2,$$

So, applying dimensionless variables on (56)-(63) substituting the value of $X(\eta)$ from (67) and the values of constants , we get

$$\int_0^\infty r(u_r)_R\, J_1(\eta r)dr = \int_0^\infty r[(u_r)_{R_1} + (u_r)_{R_2}]\, J_1(\eta r)dr$$

$$= -0.19593\times10^{-3} K\eta^{-2} J_1(\eta) \tag{68}$$

$$\int_0^\infty r(u_z)_R \, J_o(\eta r)dr = 0.00324\times10^{-3} \, K\eta^{-2} J_1(\eta) \tag{69}$$

$$\int_0^\infty r(\sigma_z)_R \, J_o(\eta r)dr = -99.2487\times10^{2} \, K\eta^{-2} J_1(\eta) \tag{70}$$

Inversions gives

$$|(u_r)_R| = 0.19593\times10^{-3} \, K\int_0^\infty \frac{J_1(\eta)J_1(\rho\eta)}{\eta} d\eta \tag{71}$$

$$|(u_z)_R| = 0.00324\times10^{-3} \, K\int_0^\infty \frac{J_1(\eta)J_o(\rho\eta)}{\eta} d\eta \tag{72}$$

$$|(\sigma_z)_R| = 99.2483\times10^{2} \, K\int_0^\infty \frac{J_1(\eta)J_o(\rho\eta)}{\eta} d\eta \tag{73}$$

Thus

$$|(u_r)_R| = 0.19593\times10^{-3} K \left\{ \begin{array}{ll} (2\rho)^{-1} F(1,0;2;\rho^{-2}), & \rho>1 \\ \frac{1}{2}, & \rho=1 \\ .5\rho \; F(1,0;2;\rho^{2}), & \rho<1 \end{array} \right\} \tag{74}$$

$$|(u_z)_R| = 0.00324\times10^{-3}K \left\{ \begin{array}{ll} F(.5,.5;1;\rho^2) & ,\ \rho<1 \\ \frac{2}{\pi}\ , & ,\ \rho=1 \\ .5F(.5,.5;2;\rho^{-2}), & \rho>1 \end{array} \right\} \quad (75)$$

$$|(\sigma_z)_R| = 99.2483\times10^{2}\ K \left\{ \begin{array}{ll} F(.5,.5;1;\rho^2) & ,\ \rho<1 \\ \frac{2}{\pi}\ , & ,\ \rho=1 \\ .5F(.5,.5;1;\rho^{-2}), & \rho>1 \end{array} \right\} \quad (76)$$

F denotes hypergeometric function.

DISCUSSIONS

Solution of the system of equations (40)-(43) and (46)-(49) for the unknowns A_i, B_i, i=1,2,3,4, depend on A_o, where A_o is a function of ξ. So existence of uniqueness of the solutions depends on the uniqueness of A_o. A_o is unique for a definite type of distribution of heat flux on the boundary. So, the solutions A_i, B_i are unique subject to the condition that A_o is unique.

Here the distribution of heat flux is taken to be linear but it would have been physically more interesting and suitable if it is considered other that linear.

Figures (1) and (2) show how the radial component of displacement and component of stress in the z- direction vary with ρ.

Stress Concentration In Elastic Layered Media Due To Thermal Effect (Three Layered Case)

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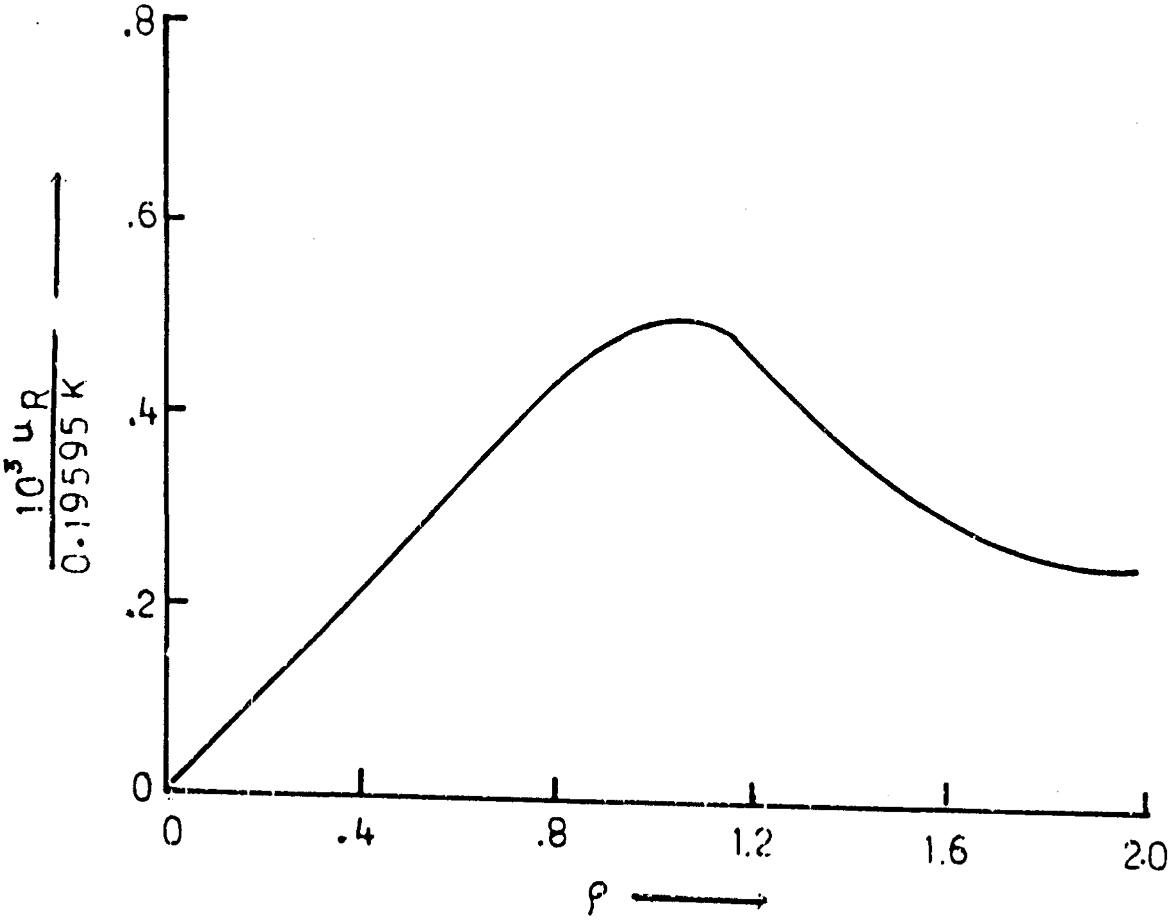

Fig. 1

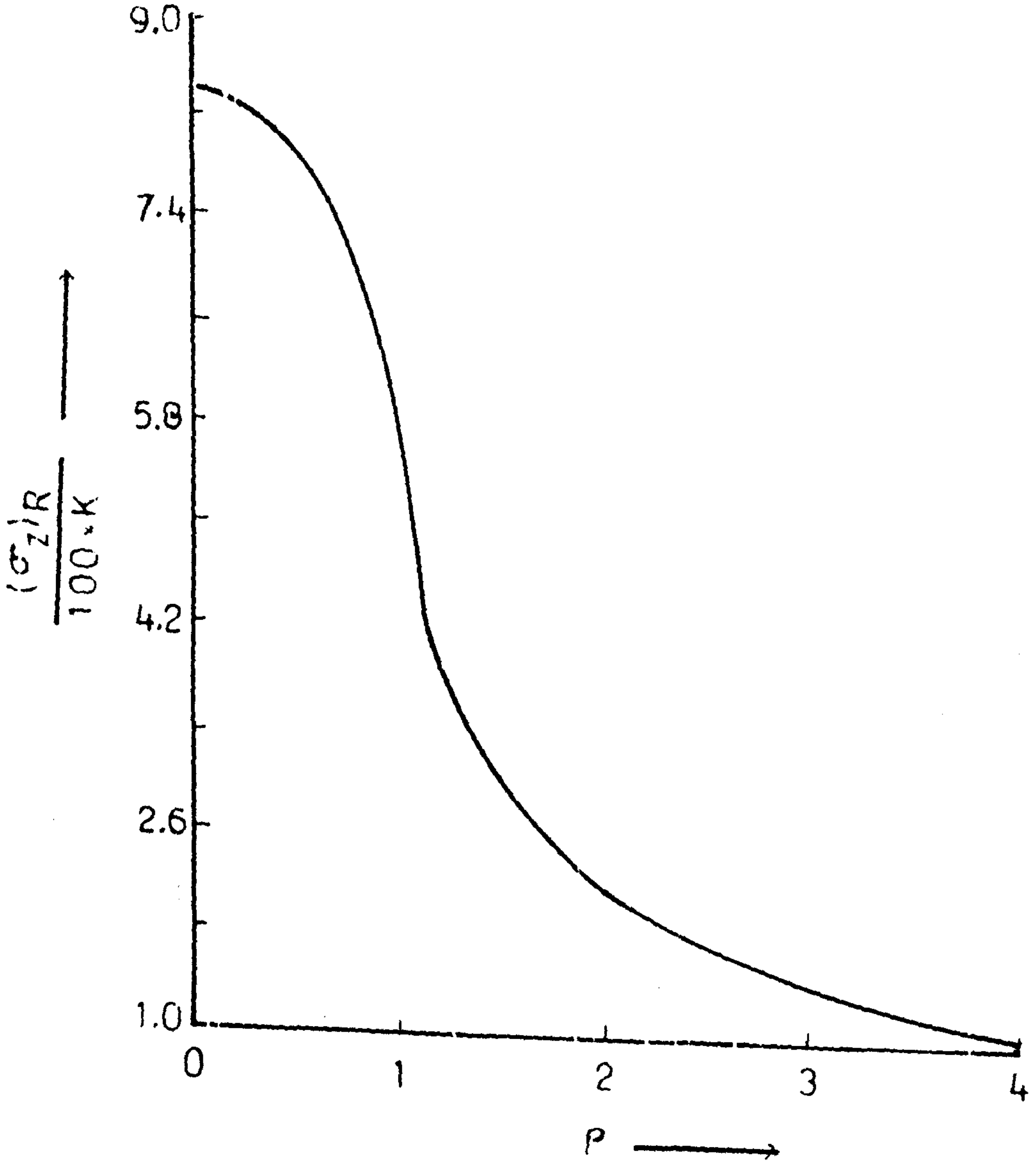

FIG.2 : DISTRIBUTION OF THERMOELASTIC STRESS $(\sigma_z)_R$ IN THE UNDERLYING MASS FOR A TWO LAYERED SYSTEM.

1. Statement Of The Problem

Material under consideration occupy the lower half of the plane z=0 and the axis of z being taken positive when drawn into the material. Physical quantities involved here are all symmetrical about z-axis. Lower boundaries of the first and second layers are given by the planes $z=h_1$ and $z=h_2$ and the underlying mass extended to infinity. Interfaces are supposed to be perfectly rough so that stresses and displacements are continuous across it.

2. Method Of Solution

Physical quantities involved in the problem are all symmetrical about z-axis, so four non-zero stress components $\sigma_r, \sigma_\theta, \sigma_z, \tau_{zr}$ and two non-zero displacements u_r, u_z are retained in terms of the stress function ϕ satisfying the differential equation [9]

$$\nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right)\phi = 0 \tag{1}$$

The stress and displacement components are given by [17]

$$\sigma_r = \frac{\partial}{\partial z}\left[\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}\right] \tag{2}$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left[\nu\nabla^2\phi - r^{-1}\frac{\partial\phi}{\partial r}\right] \tag{3}$$

$$\sigma_z = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \tag{4}$$

$$\tau_{zr} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \tag{5}$$

$$u_r = - \frac{1 + \nu\partial^2\phi}{E \quad \partial r\partial z} \tag{6}$$

$$u_z = \frac{1 + \nu}{E}\left[(1-2\nu)\nabla^2\phi + \frac{\partial^2\phi}{\partial r^2} + r^{-1}\frac{\partial\phi}{\partial r}\right] \tag{7}$$

Here the stress function ϕ due to Hankel transform [14] will take form G, defined as

$$G(\xi, z) = \int_0^\infty r\phi(r, z)\ J_o(\xi r)\ dr, \quad (\xi > 0, z > 0) \tag{8}$$

where $J_o(\xi r)$ denotes the Bessel function [19] of order zero, whose inverse is given by

$$\phi(r, z) = \int_0^\infty \xi G(\xi, z)\ J_o(\xi r)\ d\xi \tag{9}$$

Applying Hankel transform [14] on (1) and (4)-(7) we get

$$\left(\frac{d^2}{dz^2} - \xi^2\right)^2 G(\xi, z) = 0 \tag{10}$$

$$\int_0^\infty r\sigma_z J_o(\xi r)\, dr = (1-\nu)\frac{d^3G}{dz^3} - (2-\nu)\xi^2 \frac{dG}{dz} \tag{11}$$

$$\int_0^\infty r\tau_{zr} J_1(\xi r)\, dr = \left[\nu \frac{d^2G}{dz^2} + (1-\nu)\xi^2\, G\right] \tag{12}$$

$$\int_0^\infty r u_r J_1(\xi r)\, dr = \frac{1+\nu}{E}\,\xi\,\frac{dG}{dz} \tag{13}$$

$$\int_0^\infty r u_z J_o(\xi r)\, dr = \frac{1+\nu}{E}\left[(1-2\nu)\frac{d^2G}{dz^2} - 2(1-\nu)\xi^2\, G\right] \tag{14}$$

Computing inverses of (11)-(14) we get

$$\sigma_z = \int_0^\infty \xi\left[(1-\nu)\frac{d^3G}{dz^3} - (2-\nu)\xi^2\frac{dG}{dz}\right] J_o(\xi r)\, d\xi \tag{15}$$

$$\tau_{zr} = \int_0^\infty \xi^2\left[\nu\frac{d^2G}{dz^2} + (1-\nu)\xi^2\, G\right] J_1(\xi r)\, d\xi \tag{16}$$

$$u_r = \frac{1+\nu}{E}\int_0^\infty \xi^2 \frac{dG}{dz} J_1(\xi r)\, d\xi \tag{17}$$

$$u_r = \frac{1+\nu}{E}\int_0^\infty \left[\xi(1-2\nu)\frac{d^2G}{dz^2} - 2(1-\nu)\xi^2 G\right] J_0(\xi r)\, d\xi \tag{18}$$

Let the solution of (10) be given by

$$G(\xi, z) = (A+Bz)\cosh \xi z + (C+Dz) \sinh \xi z$$

where A,B,C and D are functions of ξ to be determined from the suitable boundary conditions.

3.SOLUTION OF THE PROBLEM

To determine the stresses, the potential of thermoelastic displacement ψ, related by the equations

$$\frac{\partial\psi}{\partial r} = (u_r)_T$$

$$\frac{\partial\psi}{\partial z} = (u_z)_T \tag{19}$$

is considered in the steady state of the temperature field given by [3]

$$\nabla^2 T = 0 \tag{20}$$

Since different layers have different thermal properties, different potential displacement functions are chosen. From the stress-strain relations of thermo-elasticity and the equation of equilibrium [17], we have

$$\nabla^2 \psi_i = \beta_i T , \quad i=1,2,3 \tag{21}$$

where

$$\beta_i = \frac{1+\nu_i}{1-\nu_i}\, \alpha_i$$

Applying Hankel transform on (20) and (21) we obtain

$$\left(\frac{d^2}{dz^2} - \xi^2 \right) M(\xi, z) = 0 \tag{22}$$

and

$$\left(\frac{d^2}{dz^2} - \xi^2 \right) L_i(\xi, z) = \beta_i M(\xi, z) \tag{23}$$

where $$M(\xi, z) = \int_0^\infty rTJ_o(\xi r)\, dr \tag{24}$$

$$L_i(\xi, z) = \int_0^\infty r\psi_i J_o(\xi r)\, dr \tag{25}$$

The corresponding thermal stress components $(\sigma_z)_T, (\tau_{zr})_T$ and displacement components $(u_r)_T, (u_z)_T$ are given by

$$(\sigma_z)_{T_i} = 2\mu_i \left\{ \frac{\partial^2 \psi_i}{\partial z^2} - \nabla^2 \phi_i \right\} \tag{26}$$

$$(\tau_{zr})_{T_i} = 2\mu_i \frac{\partial^2 \psi_i}{\partial r \partial z} \tag{27}$$

$$(u_r)_{T_i} = \frac{\partial \psi_i}{\partial r} \tag{28}$$

$$(u_z)_{T_i} = \frac{\partial \psi_i}{\partial z} \tag{29}$$

Now we consider the transformations of (26)-(29) into relation involving $L_i(\xi, z)$ and $M(\xi, z)$. The results of computations are

$$\int_0^\infty r(\sigma_z)_{T_i} J_o(\xi r)\, dr = 2\mu_i \xi^2 L_i(\xi, z) \tag{30}$$

$$\int_0^\infty r(\tau_{zr})_{T_i} J_1(\xi r)\, dr = -2\xi\mu_i \frac{d}{dz} L_i(\xi, z) \tag{31}$$

$$\int_0^\infty r(u_r)_{T_i} J_1(\xi r)\, dr = -\xi\, L_i(\xi, z) \tag{32}$$

$$\int_0^\infty r(u_z)_{T_i} J_o(\xi r)\, dr = \frac{d}{dz} L_i(\xi, z) \tag{33}$$

Applying Hankel inverse transform, we get

$$(\sigma_z)_{T_i} = 2\mu_i \int_0^{\infty} \xi^3 L_i(\xi, z) J_o(\xi r) \, d\xi \tag{34}$$

$$(\tau_{zr})_{T_i} = -2\mu_i \int_0^{\infty} \xi^2 \frac{d}{dz} L_i(\xi, z) J_1(\xi r) \, d\xi \tag{35}$$

$$(u_r)_{T_i} = -\int_0^{\infty} \xi^2 L_i(\xi, z) J_1(\xi r) \, d\xi \tag{36}$$

$$(u_z)_{T_i} = \int_0^{\infty} \xi \frac{d}{dz} L_i(\xi, z) J_o(\xi r) \, d\xi \tag{37}$$

Let the solution of (23) be

$$L_i(\xi, z) = -\frac{\beta_i}{2\xi^2} A_o (1+\xi z) e^{-\xi z} \tag{38}$$

then the solution of (22) is

$$M(\xi, z) = A_o e^{-\xi z} \tag{39}$$

where A_o is a function of ξ.

Applying (38) we have from (30)-(33)

$$\int_0^{\infty} r (\sigma_z)_{T_i} J_o(\xi r) \, dr = \mu_i \beta_i A_o (1+\xi z) e^{-\xi z} \tag{40}$$

$$\int_0^\infty r(\tau_{zr})_{T_i} J_1(\xi r)\ dr = -\mu_i \beta_i A_o \xi z e^{-\xi z} \tag{41}$$

$$\int_0^\infty r(u_r)_{T_i} J_1(\xi r)\ dr = \frac{1}{2\xi} \beta_i A_o (1+\xi z) e^{-\xi z} \tag{42}$$

$$\int_0^\infty r(u_z)_{T_i} J_o(\xi r)\ dr = \frac{1}{2} \beta_i z A_o e^{-\xi z} \tag{43}$$

Taking solution of (10) for different layers [59] as:

For the upper layer

$$G_1(\xi, z) = (A_1+B_1 z)\cosh \xi z + (C_1+D_1 z) \sinh \xi z \tag{44}$$

For the middle layer

$$G_2(\xi, z) = (A_2+B_2 z)\cosh \xi z + (C_2+D_2 z) \sinh \xi z \tag{45}$$

For the underlying layer

$$G_3(\xi, z) = (A_3+B_3 z) e^{-\xi z}, \quad \xi>0, \quad z>h_2 \tag{46}$$

It is to be noted that stress and displacements in the underlying mass vanish as z tends to infinity. So, the components of stress and displacement for the upper and middle layers obtained from (11)-(14) when j=1,2 are

$$\int_0^{\infty} r(\sigma_z)_j \, J_o(\xi r) \, dr = (1-2\nu_j)\xi^2 B_j \cosh \xi z + (1-2\nu_j)\xi^2 D_j \sinh \xi z$$

$$-(A_j + B_j z)\xi^2 \sinh \xi z - (C_j + D_j z)\xi^2 \cosh \xi z \qquad (47)$$

$$\int_0^{\infty} r(\tau_{zr})_j \, J_1(\xi r) \, dr = \xi^2 \left[2\nu_j B_j \sinh \xi z + 2\nu_j D_j \cosh \xi z \right]$$

$$+ \xi^3 \left[(A_j + B_j z)\cosh \xi z + (C_j + D_j z) \sinh \xi z \right] \qquad (48)$$

$$\int_0^{\infty} r(u_r)_j \, J_1(\xi r) \, dr = \frac{1+\nu_j}{E_j} \xi^2 \left[(A_j + B_j z + \frac{D_j}{\xi})\sinh \xi z \right.$$

$$\left. + (C_j + D_j z + \frac{B_j}{\xi})\cosh \xi z \right] \qquad (49)$$

$$\int_0^{\infty} r(u_z)_j \, J_o(\xi r) \, dr = \frac{1+\nu_j}{E_j} \xi^2 \left[\left(\frac{2D_j}{\xi}(1-2\nu_j) - (A_j + B_j z) \right) \cosh \xi z \right.$$

$$\left. + \left(\frac{2B_j}{\xi} (1-2\nu_j) - (C_j + D_j z) \right) \sinh \xi z \right] \qquad (50)$$

For the underlying mass

$$\int_0^\infty r(\sigma_z)_s J_o(\xi r)\, dr = \left[A_s + B_s z + \frac{B_s}{\xi}(1-2\nu_s)\right]\xi^s\ e^{-\xi z} \tag{51}$$

$$\int_0^\infty r(\tau_{zr})_s J_1(\xi r)\, dr = \left[A_s + B_s z + \frac{B_s}{\xi}\ 2\nu_s\right]\xi^s\ e^{-\xi z} \tag{52}$$

$$\int_0^\infty r(u_r)_s J_1(\xi r)\, dr = \frac{1+\nu_s}{E_s}\ \xi^2\ \left[\ \frac{B_s}{\xi}\ -A_s - B_s z\ \right]\ e^{-\xi z} \tag{53}$$

$$\int_0^\infty r(u_z)_s J_o(\xi r)\, dr = \frac{1+\nu_s}{E_s}\ \xi^s \left[A_s + B_s z + \frac{2B_s}{\xi}\ (1-2\nu_s)\right]e^{-\xi z} \tag{54}$$

4. BOUNDARY CONDITIONS

In order to nullify the stresses on the boundaries the following conditions are to be satisfied:

At z=0, $-(\sigma_z)_{T_i} = (\sigma_z)_1$

$$-(\tau_{zr})_{T_i} = (\tau_{zr})_1 \tag{55}$$

At the interfaces,

$z=h_1$, $|(\sigma_z)_2| = |(\sigma_z)_1|$, $|(\tau_{zr})_2| = |(\tau_{zr})_1|$.

So at $z=h_1$,

$$-(\sigma_z)_{T_2} = (\sigma_z)_1$$

$$-(\tau_{zr})_{T_2} = (\tau_{zr})_1$$

and

$$-(\sigma_z)_{T_2} = (\sigma_z)_2$$

$$-(\tau_{zr})_{T_2} = (\tau_{zr})_2 \qquad (56)$$

Since at the interfaces

$z=h_2$, $|(\sigma_z)_2| = |(\sigma_z)_3|$ and $|(\tau_{zr})_2| = |(\tau_{zr})_3|$.

So at $z=h_2$,

$$-(\sigma_z)_{T_3} = (\sigma_z)_3$$

$$-(\tau_{zr})_{T_3} = (\tau_{zr})_3$$

and

$$-(\sigma_z)_{T_3} = (\sigma_z)_2$$

$$-(\tau_{zr})_{T_3} = (\tau_{zr})_2 \qquad (57)$$

Boundary conditions relating to the continuities of displacement are assumed to be identically satisfied at the interfaces.

Conditions (55) with the help of (40),(41) and(47),(48) we have

$$A_1 = -\frac{2\nu_1 D_1}{\xi} \qquad (58)$$

$$B_1 = \frac{\beta_1\mu_1 A_o + \xi^2 C_1}{(1-2\nu_1)\xi^2} \tag{59}$$

Using (40),(41),(51) and (52) on the first two conditions of (57) we get

$$A_3 = \frac{2\nu_3 A_o}{\xi^3} \beta_3\mu_3 \tag{60}$$

$$B_3 = A_o\beta_3\mu_3 \xi^{-2} \tag{61}$$

From the boundary conditions (56) and last two of (57)

$$(q_1 - \xi h_1 p_1)D_1 - \frac{h_1 q_1 \xi^2}{1-2\nu_1} C_1 = A_o\xi^{-2}\left[\beta_2\mu_2(1+\xi h_1)e^{-\xi h_1} - \mu_1\beta_1\left\{p_1 - \frac{h_1 q_1 \xi}{1-2\nu_1}\right\}\right] = S_5 \tag{62}$$

$$\xi q_1 h_1 D_1 + \frac{(q_1+\xi h_1 p_1)\xi}{1-2\nu_1} C_1 = -A_o\xi^{-2}\beta_1\mu_1 \frac{2\mu_1 q_1 + \xi h_1 p_1}{1-2\nu_1} + S_5 = S_6 \tag{63}$$

$$\left[(1-2\nu_2)p_1 - \xi h_1 q_1\right]B_2 + \left[(1-2\nu_2)q_1 - \xi h_1 p_1\right]D_2 - \xi q_1 A_2 - \xi p_1 C_2 - S_1 = 0 \tag{64}$$

$$\left[2\nu_2 q_1 + \xi h_1 p_1\right]B_2 + \left[2\nu_2 p_1 + \xi h_1 q_1\right]D_2 + \xi p_1 A_2 + \xi q_1 C_2 - S_3 = 0 \tag{65}$$

$$\left[(1-2\nu_2)p_2-\xi h_2 q_2\right]B_2+\left[(1-2\nu_2)q_2-\xi h_2 p_2\right]D_2-\xi q_2 A_2-\xi p_2 C_2-S_2=0 \quad (66)$$

$$\left[2\nu_2 q_2+\xi h_2 p_2\right]B_2+\left[2\nu_2 p_2+\xi h_2 q_2\right]D_2+\xi p_2 A_2+\xi q_2 C_2-S_4=0 \quad (67)$$

where

$$S_1 = S_1' A_o ; \quad S_1' = \frac{\mu_2\beta_2}{\xi^2}(1+\xi h_1)e^{-\xi h_1}$$

$$S_2 = S_2' A_o ; \quad S_2' = \frac{\mu_3\beta_3}{\xi^2}(1+\xi h_2)e^{-\xi h_2}$$

$$S_3 = S_3' A_o ; \quad S_3' = \frac{\mu_2\beta_2}{\xi^2} h_1 e^{-\xi h_1}$$

$$S_4 = S_4' A_o ; \quad S_4' = \frac{\mu_3\beta_3}{\xi^2} h_2 e^{-\xi h_2}$$

$$S_5 = S_5' A_o ; \quad S_5' = S_1' - \frac{\mu_1\beta_1}{\xi^2}\left\{p_1 - \frac{h_1 q_1 \xi}{1-2\nu_1}\right\}$$

$$S_6 = S_6' A_o ; \quad S_6' = S_3' - \xi^{-2}\beta_1\mu_1\frac{2\nu_1 q_1+\xi h_1 p_1}{1-2\nu_1}$$

Solving (62) and (63)

$$C_1 = C'_1 A_0 \quad \text{and} \quad D_1 = D'_1 A_0$$

where

$$C'_1 = \frac{1-2\nu_1}{\xi} \frac{\xi h_1 q_1 S'_5 - (q_1 - \xi h_1 p_1) S'_6}{q_1^2 - \xi^2 h_1^2 p_1^2 - \xi^2 h_1^2 q_1^2}$$

and

$$D'_1 = \frac{\xi h_1 q_1 S'_6 + (q_1 + \xi h_1 p_1) S'_5}{q_1^2 - \xi^2 h_1^2 p_1^2 - \xi^2 h_1^2 q_1^2}$$

Putting the values of C_1 and D_1 in (58) and (59),

$$A_1 = A'_1 A_0 \quad \text{and} \quad B_1 = B'_1 A_0$$

where

$$A'_1 = -2\nu_1 \xi^{-1} \frac{\xi h_1 q_1 S'_6 + (q_1 + \xi h_1 p_1) S'_5}{q_1^2 - \xi^2 h_1^2 p_1^2 - \xi^2 h_1^2 q_1^2}$$

$$B'_1 = \frac{1}{(1-2\nu_1)\xi^2} (\beta_1 \mu_1 + \xi^3 C'_1)$$

Equations (64)-(67) are then solved to find A_2, B_2, C_2 and D_2 and are written in the form

$$A_2 = A'_2 A_0 \ , \ B_2 = B'_2 A_0 , \ C_2 = C'_2 A_0 \ , \ D_2 = D'_2 A_0$$

Subsequently numerical values of A'_i, B'_i, $i=1,2,3$ and C'_i, D'_i, $i=1,2$ are obtained. Here $A_3 = A_3', B_3 = B_3'$. So the relations (58)-(67) give 10 constants $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, A_3$ and B_3 in terms of A_o. So the formal solution of the problem is complete.

The resultant stresses in the direction of z-axis are

On z=0,

$$\int_0^\infty r(\sigma_z)_{R_1} J_o(\xi r)dr = \int_0^\infty r[(\sigma_z)_1 + (\sigma_z)_{T_1}] J_o(\xi r)dr =$$

$$\left[(1-2\nu_1)\xi^2 B_1' - C_1'\xi^3 - \beta_1\mu_1\right] A_o \tag{68}$$

At $z=h_1$,

$$\int_0^\infty r(\sigma_z)_{R_2} J_o(\xi r)dr = \int_0^\infty r[(\sigma_z)_2 + (\sigma_z)_{T_2}] J_o(\xi r)dr =$$

$$= \Big[(1-2\nu_1)\xi^2 B_2' \cosh \xi h_1 + (1-2\nu_1)\xi^2 D_2' \sinh \xi h_1 - (A_2' + B_2' h_1)\xi^3 \sinh \xi h_1$$

$$- (C_2' + D_2' h_1)\xi^3 \cosh \xi h_1 - \beta_2\mu_2(1+\xi h_1)e^{-\xi h_1}\Big] A_o \tag{69}$$

At $z=h_2$

$$\int_0^\infty r(\sigma_z)_{R_3} J_o(\xi r)dr = \int_0^\infty r[(\sigma_z)_3 + (\sigma_z)_{T_3}] J_o(\xi r)dr =$$

$$=\left[(1-2\nu_3)\xi^2 B_3' \; e^{-\xi h_2}+(A_3'+B_3' h_2)\xi^3 \; e^{-\xi h_2}-\beta_3\mu_3(1+\xi h_2)e^{-\xi h_2}\right]A_o \qquad (70)$$

Constants $A_1', B_1', C_1', D_1', A_2', B_2', C_2', D_2', A_3'$ and B_3' are independent of A_o. Thus the total thermoelastic stress $(\sigma_z)_R$ in the underlying mass is

$$\int_0^\infty r(\sigma_z)_R \; J_o(\xi r)dr = \int_0^\infty r[(\sigma_z)_{R_1}+(\sigma_z)_{R_2}+(\sigma_z)_{R_3}] \; J_o(\xi r)dr \qquad (71)$$

5. Flux Of Heat On The Boundaries

Let the flux of heat in a region of the surface $z=0$, distributed through layers in the underlying mass be

$$\frac{\partial T}{\partial z} = f(r/a), \quad 0<r<a \qquad (72)$$

$$= 0, \quad r>a$$

using dimensionless variables

$$\xi A_o(\xi) = aX(\xi a), \; \eta=\xi a, \; \rho=r/a, \; \zeta=z/a$$

where a is some length and η, a new variable of integration, we get from (24) and (39), on $z=0$,

$$\frac{\partial T}{\partial z} = -a^{-1} \int_0^{\infty} \eta X(\eta) J_o(\rho\eta) \, d\eta \tag{73}$$

By Hankel inversion theorem

$$X(\eta) = -a^{-1} \int_0^1 \rho f(\rho) J_o(\rho\eta) \, d\rho \tag{74}$$

For a simple physical situation we consider a linear temperature distribution $f(\rho) = K\rho$, K = constant, then from (74)

$$X(\eta) = -\frac{K}{a\eta} J_1(\eta) \tag{75}$$

With this value of $X(\eta)$, the problem is completely solved since only unknown $A_o(\xi)$ is now known.

6. Numerical Results

If the upper layer be concrete pavement, the middle layer be gravel base and the underlying mass be natural soil, then elastic constants for those materials are [8]

$E_1 = 2.18 \times 10^8$ gms/cm^2 , $\alpha_1 = -5 \times 10^{-6}/0^\circ C$

$E_2 = 1.1 \times 10^8$ gms/cm^2 , $\alpha_2 = 7.5 \times 10^{-6}/0^\circ C$

$E_3 = 0.4\times10^{8}$ gms/cm^2 , $\alpha_3 = 2.3\times10^{-6}/0^\circ C$

$\nu_1 = 0.15$, $\mu_1 = 0.94\times10^{8}$ gms/cm^2

$\nu_2 = 0.25$, $\mu_2 = 0.43\times10^{8}$ gms/cm^2

$\nu_3 = 0.50$, $\mu_3 = 0.14\times10^{3}$ gms/cm^2

$K_1 = 6.4\times10^{-8}$, $K_2 = 6.7\times10^{-3}$

$K_3 = 2.9\times10^{-3}$

So, evaluating constants for a given value of η when $a=1$ and $h_1=2, h_2=4$ we have

$$A'_1 = 3.504\times10^{2}\ ,\ B'_1 = 77.2671\times10^{2},\ C'_1 = 47.73\times10^{2}\ ,D'_1 = -166.8\times10^{2},$$

$$A'_2 = 0.005829\times10^{2},\ B'_2 = 0.01207\times10^{-2},\ C'_2 = 0.007071\times10^{2},$$

$$D'_2 = -0.02578\times10^{2},\ A'_3 = -14.49\times10^{2},\ B'_3 = 0.966\times10^{2}$$

So, applying dimensionless variables on (71) substituting the value of $X(\eta)$ from (75) and the values of constants in (71), we get

$$\int_0^\infty r(\sigma_z)_R\, J_o(\rho\eta)\ d\rho = 4.94423\times10^{2}\,\frac{K}{\eta}\,J_1(\eta)$$

whose Hankel transform is

$$(\sigma_z)_R = 4.94423\times10^2\ K\int_0^\infty \frac{J_1(\eta)J_o(\rho\eta)}{\eta}\,d\eta$$

$$= 4.94423\times10^2\ K\ F\left[1/2,\ 1/2;1;\rho^2\right] \qquad ,\ \rho<1$$

$$= 4.94423\times10^2\ 2K/\pi, \qquad \rho=1$$

$$= 4.94423\times10^2\ \frac{1}{2}\ K\ F\left[1/2,\ 1/2;1;\rho^{-2}\right], \qquad \rho>1$$

F denotes hypergeometric function.

7·Discussion

It is important to study the existence of uniqueness of the solutions of the system of equations (62)-(67) for the unknowns A'_i, B'_i, $i=1,2,3$ and C'_i, D'_i, $i=1,2$. Here, it is found that all the above solutions depend on the quantities A_o. But A_o is unique for a particular type of distribution of heat flux on the boundary. So the solutions A'_i, B'_i, $i=1,2,3$ and C'_i, D'_i, $i=1,2$ are unique subject to the condition that A_o is unique for a definite kind of heat flux used in this problem.

For simplicity, linear heat flux has been applied on the boundary z=0 in this problem. It will be more interesting and physically suitable if the heat flux is considered in the form other than linear one.

The numerical results of this paper have been compared with the works of Paria [59] who solved a problem of this nature in absence of temperature and the results tally completely with the results of Paria.

Fig.3 displays the nature of distribution of thermoelastic stress $(\sigma_z)_a$ against ρ.

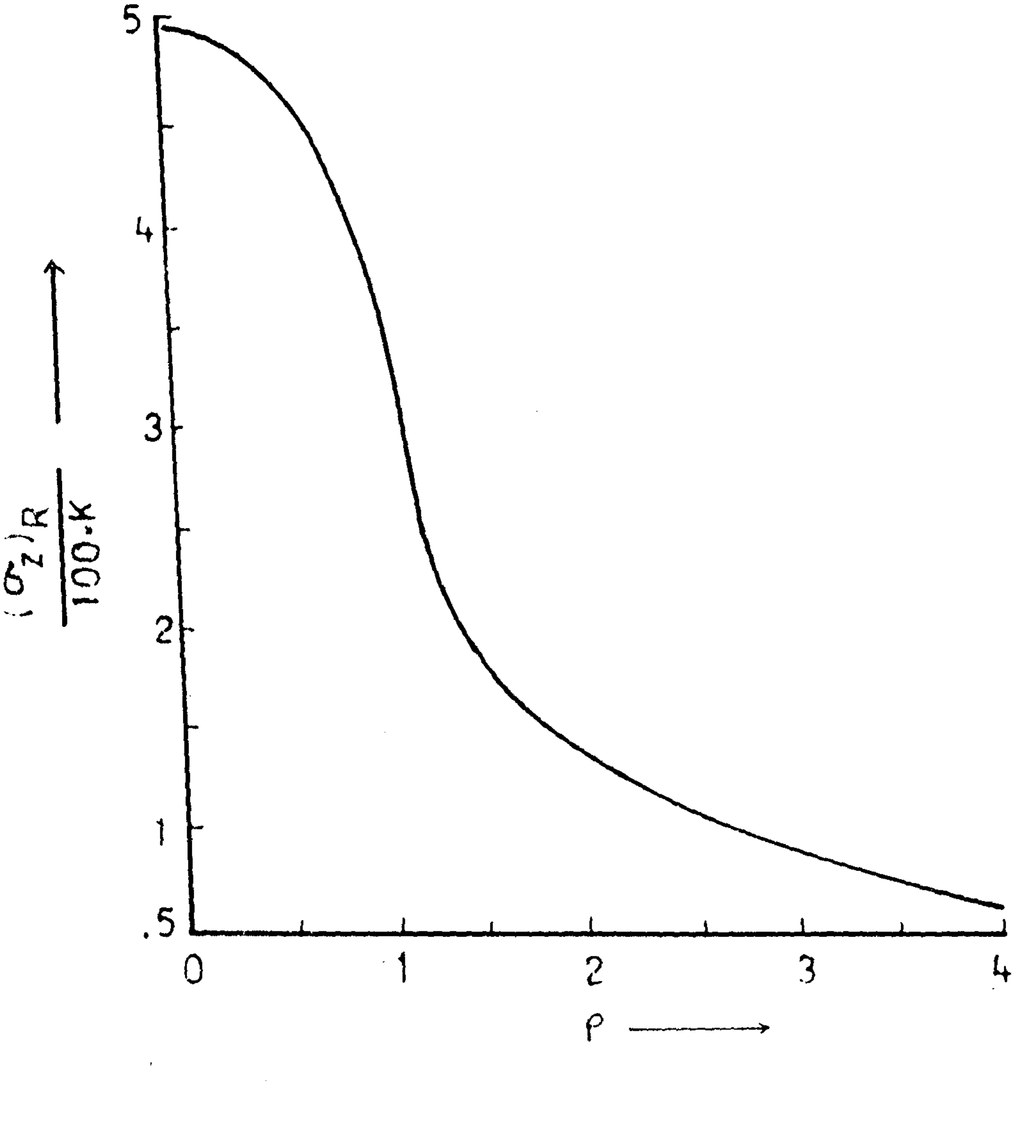

FIG.3 : DISTRIBUTION OF THERMOELASTIC STRESS $(\sigma_z)_R$ IN THE UNDERLYING MASS FOR A THREE LAYERED SYSTEM.