SOME DYNAMIC MIXED BOUNDARY VALUE PROBLEMS IN ISOTROPIC AND ORTHOTROPIC ELASTIC MEDIA

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Low frequency scattering of elastic waves by Griffith cracks in orthotropic elastic medium

INTRODUCTION

The theory of wave propagation in elastic solids possesses a distinguished history of nearly hundred years. The eminent mathematician Galileo Galilei paid his attention to the vibrations of pendulums, the resonance phenomenon and the vibrations of strings. The first stage of the investigation on wave propagation associated with the names of **NAVIER, CAUCHY, HOOKE, POISSON, STOKES, RAYLEIGH, KELVIN, GREEN, LAME** and **CLEBSCH** is characterized by development of the extensive theory of elasticity to the problem of wave propagation and vibrating bodies in elastic material.

During the first quarter of this century the subject lost much of its glamour and interest, perhaps because of a gap between the advancement of theoratical and experimental work, as there was no practical methods available in laboratory for observing the passage of stress waves in elastic materials. But in the later part of the century the interest in the study of elastic waves has been growing rapidly because of the application of the theory in Seismology, Geophysics and in Engineering science. During the last three decades there has been a remarkable revival of interest in this subject.

Most of the experimental works carried out on the wave propagation of elasticity are concerned with studying propagation in specimens of comparatively simple geometrical shape, the results of this experiment could be compared directly with the exact or approximate theoratical predictions. With increasing confidence in the experimental techniques and in the interpretation of observations, it is now possible to study more complicated problems of elastodynamics.

All the elastic bodies may be divided roughly into two categories

(i)homogeneous and non-homogeneous

(ii) isotropic and anisotropic

A homogeneous body is the body whose elastic properties are the same at different points and a non-homogeneous body has different elastic properties at different points. If the elastic moduli vary from point to point in a continuous manner, the non-homogeneity may also be termed continuous. If, however, the elastic moduli undergo discontinuties in passing from point to point, for example change abruptly, the non-homogeneity is said to be discontinuous or discrete.

An isotropic body , with regard to its elastic properties , is one in which these properties are the same for all directions drawn through a given point. An anisotropic body has, in general, different elastic properties for different directions drawn through a given point. A body may be isotropic or anisotropic and at the same time homogeneous or non-homogeneous depending on its own structure.

In an unbounded homogeneous isotropic solid, two types of elastic waves may be propagated with two different velocities. These are dilatational wave or longitudinal wave and distortional wave or shear wave. Obviously longitudinal waves arrive earlier than shear waves. In the case of the deformation of elastic body, both longitudinal and distortional waves will normally be produced and when a wave of either type impinges on the boundary of the solid, two types of waves are generated. In addition to the existence of these two types of waves of the body, a third type of wave may exist whose effects are confined closely along the surface of the body; this type of waves are known as Rayleigh-wave. Their effects decrease exponentially with depth and their velocity of propagation is smaller than the other types of elastic waves. They are of great importance in seismic phenomena. Bullein (1963), Ewing et. al. (1957), Cagniard (1962) and Pilant (1979) have discussed about seismic waves in their books.

Some important equations governing the motion of a homogeneous, isotropic, linearly isotropic elastic solid are listed below:
Consider a rectangular cartesian co-ordinate system with reference to the co-ordinate axes x_i (i= 1,2,3) and assume u_i to be the components of the displacement vector field. The system of displacement equations of motion in indicial notation may be expressed as

$$\mu \, u_{i,jj} + (\lambda+\mu) \, u_{j,ji} = \rho \, \ddot{u}_i \qquad (1)$$

where λ , μ are Lame's elastic constants and ρ is the mass density.
A plane displacement wave propagating in an arbitrary direction in

an unbounded medium is represented by

$$u_i = f(x_k p_k - ct)\, d_i \tag{2}$$

where d_i and p_i are the components of unit vectors in the directions of motion and of propagation and x_i are the components of the position vector, $p_i x_i$ = constant represents a plane normal to the unit vector with components p_i. Equation (2) represents a plane wave whose planes of constant phase propagate with velocity c. Substitution of equation (2) into the displacement equation of motion yields

$$(\mu - \rho c^2) d_i + (\lambda+\mu)(p_j d_j) p_i = 0 \tag{3}$$

Since p_i and d_i denote two different unit vectors, equation (3) may be satisfied in only two ways:

(i) $p_i = \pm d_i$, consequently $p_j d_j = \pm 1$ and equation (3) yields

$$c^2 = c_L^2 = (\lambda+2\mu)/\rho \tag{4}$$

(ii) If $p_i \neq d_i$, both terms in equation (4) have to vanish independently yielding

$$c^2 = c_T^2 = \mu/\rho \qquad \text{and} \qquad p_j d_j = 0 \tag{5}$$

The displacement corresponding to transverse wave whose velocity of propagation is given by (5) can have any direction in a plane normal to the direction of propagation but usually the x_1-x_2 plane is chosen to contain the vector $\vec{p}$ and transverse motions are considered in the x_1-x_2 plane or normal to the x_1-x_2 plane. These are called "vertically" and "horizontally" polarised transverse waves, respectively.

A convenient representation of the displacement components is

$$u_i = \phi,_j + e_{ijk}\,\psi_{k,j}\,, \qquad \psi_{k,k} = 0 \tag{6}$$

where e_{ijk} is the alternating tensor. Substitution of equation (6) in equation (1) shows that this representation satisfies the displacement equations of motion, provided that

$$\phi,_{ii} = \frac{1}{c_L^2}\ddot{\phi} \tag{7.a}$$

and

$$\psi_{k,jj} = \frac{1}{c_T^2}\ddot{\psi}_k \tag{7.b}$$

where c_L and c_T are given by equations (4) and (5) respectively and equations (7.a-b) are uncoupled wave equations.

When the field variables are independent of one of the cartesian co-ordinates say x_3, wave motions uncouple into anti-plane and in-plane motions. A displacement distribution defined by $u_3(x_1,x_2,t)$ describes anti-plane strain and $u_1(x_1,x_2,t)$, $u_2(x_1,x_2,t)$ defines a state of plane strain.

Since the x,y,z co-ordinate system is more convenient to refer the motions, so the displacement components are denoted by u(x,y,t), v(x,y,t) and w(x,y,t) respectively and the components of the stress tensor by $\tau_x(x,y,t)$, $\tau_{xy}(x,y,t)$ etc.

The two dimensional anti-plane motion governing the displacement component w(x,y,t) is

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{c_T^2}\frac{\partial^2 w}{\partial t^2} \tag{8}$$

The non-vanishing stress components are

$$\tau_{xz} = \mu \frac{\partial w}{\partial x} \quad \text{and} \quad \tau_{yz} = \mu \frac{\partial w}{\partial y} \tag{9}$$

For the case of plane strain it is expedient to employ the decomposition (6) and reduce the displacement components to

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \tag{10}$$

$$v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \tag{11}$$

The functions ϕ and ψ satisfy the following two dimensional wave equations

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c_L^2} \frac{\partial^2 \phi}{\partial t^2} \tag{12}$$

and

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{1}{c_T^2} \frac{\partial^2 \psi}{\partial t^2} \tag{13}$$

where c_L and c_T are defined by equations (4) and (5) respectively. The corresponding components of the stress tensor are

$$\tau_{xx} = \lambda \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) + 2\mu \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}\right) \tag{14}$$

$$\tau_{yy} = \lambda \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) + 2\mu \left(\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x \partial y}\right) \tag{15}$$

$$\tau_{xy} = \mu \left(2\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) \tag{16}$$

However, natural or artificial materials in our surroundings are generally inhomogeneous and anisotropic.

Since 1950 the theory of elasticity for anisotropic bodies has been continually developed and enriched with new investigations of both serious problems as a general nature and individual aspects of these problems. Thus the general theory has been placed on a rigorous scientific basis and a number of laws have been established with the result that this theory, first worked out by B.de Saint Venant and P.V.Bekhterev (1925),has been revived.

Of great importance is the development and construction of many entirely new anisotropic materials which possesses a number of advantages over those previously known (for example,glass-fibre reinforced plastics). Thus, over two or three decades this branch of science has made great progress, both in a theoratical and a purely practical way, i.e. in constructing new anisotropic materials.

Now we recapitulate the fundamental principles of the theory of elasticity and the general equations which will be used in what follows for the construction of solutions to specific problems of the theory of elasticity for anisotropic bodies.

In studying the states of stress and strain in anisotropic bodies produced by an external load, we make a number of assumptions imposing certain restrictions. The most important of these assumptions reduce to the following:

(1) A body is solid (a continuous medium), the stresses on any plane within the body and on its surface are forces per unit area. In otherwords, the couple stresses are neglected, as is done in the classical theory of elasticity.

(2) The relation between the components of strain and the projections of displacement and their first derivatives with respect to the co-ordinates is linear.

(3) The stress-strain relations are linear, i.e., the material follows the generalized Hooke's law, the co-efficients in these linear relations may be either constant (homogeneous body) or variable, i.e. functions of position, continuous or discontinuous (in the case of a non-homogeneous body).

(4) The initial stresses i.e. those existing without any external load, including the thermal stresses are disgarded; specific problems of dynamics are not considered.

Thus, the theory of anisotropic elastic bodies can be studied from the classical linear theory of homogeneous or non-homogeneous elastic bodies.

The stresses acting on planes normal to the co-ordinate directions are each resolved into three components: one normal(normal stress) and two tangential (shearing stresses).

The deformation of a body in the neighbourhood of a point is characterized by the components of strain, viz three extensions and three shearing strains.

The components of the displacement of a point on the axes of cartesian co-ordinates (x,y,z) are denoted as u,v,w. Let ε_x and ε_y are the extensions of segments of unit length originally parallel to x and y, γ_{xy} is the change in angle between segments whose original directions are x and y.

The strain components ε_i (i = 1,2,3) and $\frac{1}{2}\gamma_{ij}$ (j≠i, j = 1,2,3) constitute a symmetrical tensor of rank two. For a cartesion system x,y,z, it can be written as in the matrix form

$$\begin{vmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{vmatrix}$$

The relation between the components of displacement and strain in cartesian system are given below:

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad (17)$$

If the strains are not small, the extensions and shears,

$\varepsilon_i (i=1,2,3)$, $\frac{1}{2}\ \gamma_{ij} (i \neq j,\ j=1,2,3)$ are related to the displacements by non-linear equation which are given as follows:

$$\varepsilon_x = \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} - 1$$

$$\varepsilon_y = \sqrt{1 + 2\frac{\partial v}{\partial y} + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2} - 1 \qquad (18)$$

$$\sin \gamma_{xy} = \frac{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}}{(1 + \varepsilon_x)(1 + \varepsilon_y)}$$

The other three components ε_z, γ_{yz}, γ_{xz} are found from (18) by cyclic permutation of the subscripts.

The corresponding stress components are in the matrix form

$$\begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix}$$

In the general case of anisotropy each strain component is a linear function of all six components. For a homogeneous body having anisotropy of the most general kind, the equations expressing the generalized Hooke's law for this system are

$$\begin{aligned}
\varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z + a_{14}\tau_{yz} + a_{15}\tau_{xz} + a_{16}\tau_{xy} \\
\varepsilon_y &= a_{21}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z + a_{24}\tau_{yz} + a_{25}\tau_{xz} + a_{26}\tau_{xy} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\gamma_{xy} &= a_{61}\sigma_x + a_{62}\sigma_y + a_{63}\sigma_z + a_{64}\tau_{yz} + a_{65}\tau_{xz} + a_{66}\tau_{xy}
\end{aligned} \qquad (19)$$

In the general case equations (19) contain 36 co-efficients a_{ij}, but actually they are always fewer;

Suppose that the 6th order determinent of the co-efficients a_{ij}, written down successively, is not zero, amd hence equation (19) for σ and τ are solvable. The generalized Hooke's law equations for the general case are thus obtained in an alternative equivalent form:

$$\begin{aligned}
\sigma_x &= A_{11}\varepsilon_x + A_{12}\varepsilon_y + A_{13}\varepsilon_z + A_{14}\gamma_{yz} + A_{15}\gamma_{xz} + A_{16}\gamma_{xy} \\
\sigma_y &= A_{21}\varepsilon_x + A_{22}\varepsilon_y + A_{23}\varepsilon_z + A_{24}\gamma_{yz} + A_{25}\gamma_{xz} + A_{26}\gamma_{xy} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\tau_{xy} &= A_{61}\varepsilon_x + A_{62}\varepsilon_y + A_{63}\varepsilon_z + A_{64}\gamma_{yz} + A_{65}\gamma_{xz} + A_{66}\gamma_{xy}
\end{aligned} \qquad (20)$$

However if strain energy function $\bar{V}$ exists such that

$$\sigma_x = \frac{\partial \bar{V}}{\partial \varepsilon_x}, \quad \sigma_y = \frac{\partial \bar{V}}{\partial \varepsilon_y}, \quad ; \tau_{xy} = \frac{\partial \bar{V}}{\partial \gamma_{xy}}$$

then differentiation of the stress components with respect to the strain components yield

$$\frac{\partial \sigma_x}{\partial \varepsilon_y} = \frac{\partial \sigma_y}{\partial \varepsilon_x}, \quad \frac{\partial \sigma_x}{\partial \gamma_{xy}} = \frac{\partial \tau_{xy}}{\partial \varepsilon_x} \quad \text{etc.} \qquad (21)$$

It follows from the equalities of (21) and (20) that

$$A_{21} = A_{12}, \quad A_{31} = A_{13}, \ldots\ldots\ldots\ldots, A_{65} = A_{56}$$

and, in general,

$$A_{ij} = A_{ji} \quad (i,j = 1,2,..,6) \tag{22}$$

Solving equation (19), we obtain six expressions for ε and γ in which the co-efficients on the right hand sides are also symmetrical:

$$a_{ij} = a_{ji} \quad (i,j = 1,2,...,6)$$

So we can write the generalized Hooke's law equations in the general case as

$$\begin{aligned} \varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + \ldots\ldots\ldots\ldots + a_{16}\tau_{xy} \\ \varepsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + \ldots\ldots\ldots\ldots + a_{26}\tau_{xy} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \gamma_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + \ldots\ldots\ldots\ldots + a_{66}\tau_{xy} \end{aligned} \tag{23}$$

or

$$\begin{aligned} \sigma_x &= A_{11}\varepsilon_x + A_{12}\varepsilon_y + \ldots\ldots\ldots\ldots + A_{16}\gamma_{xy} \\ \sigma_y &= A_{12}\varepsilon_x + A_{22}\varepsilon_y + \ldots\ldots\ldots\ldots + A_{26}\gamma_{xy} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \tau_{xy} &= A_{16}\varepsilon_x + A_{26}\varepsilon_y + \ldots\ldots\ldots\ldots + A_{66}\gamma_{xy} \end{aligned} \tag{24}$$

In general case of anisotropy the number of elastic constants A_{ij}, a_{ij} is 21, but among these the independent constants are fewer.

NOVOZHILOV (1958) states that, all co-ordinate systems are equivalent in Geometry, neverthless as regards the elastic and in general, physical properties symmetry may be observed even in the most general case. Consequently, even in the most general case the number of independent elastic constants is not 21, but fewer, namely 18.

By changing the notation for the elastic constants and stress components, we may write the generalized Hooke's law equations in an extremely simple form. Let the elastic constants be denoted by "a" with four subscripts and setting

(1) $a_{ij} = a_{mnkl}$ if $i, j = 1, 2, 3$ (all possible cases where $j=i$ are included)

(2) $a_{ij} = 2a_{mnkl}$ if either of the two subscripts, i or j is 4,5,6.

(3) $a_{ij} = 4a_{mnkl}$ if both subscripts $i, j = 4, 5, 6$.

The six equations (23) are then written as a single one:

$$\varepsilon_{ij} = a_{ijkl}\sigma_{kl} \qquad (i, j, k, l = 1, 2, 3) \tag{25}$$

The number of all constants a_{ijkl} with four subscripts is 81, but, when grouped, they reduce to 21 (of these 18 constants are independent) elastic constants.

The generalized Hooke's law equations, solved for the stress components, are of the form

$$\sigma_{ij} = A_{ijkl}\varepsilon_{kl} \tag{26}$$

The notation for the elastic constants "a" and "A" with four subscripts has been used by Malmeister, Tamuzh and Terters (1972) in their book.

If the structure of an anisotropic body has some kind of symmetry, the elastic properties also exhibit symmetry. The elastic symmetry is expressed in the fact that at each point there are symmetrical directions equivalent as regards the elastic properties.

F.Neumann (1885) established the relationship between the structural symmetry and the elastic symmetry for crystals, which may be stated as follows:

With respect to its physical properties (including the elastic properties), a material exhibits the same kind of symmetry as its crystallographic form or more perfect symmetry. The principle is also extended to bodies that are not crystals, but have structural symmetry (wood, plywood, glass fibre reinforced plastics).

If there is symmetry of the elastic properties (elasticsymmetry) in an anisotropic body, the generalized Hooke's law equations for it are simplified since some of the co-efficients a_{ij} are zero, while among others there are linear relations.

The following four cases of elastic symmetry are the most important, and these will now be discussed below:

(1) PLANE OF ELASTIC SYMMETRY:-

Suppose a plane passing through each point of a body possesses the following property:

Every two symmetric directions with respect to this plane are equivalent as regards the elastic properties. A direction normal to the plane of elastic symmetry will be termed the principal direction of elasticity. In this case only one principal direction passes through a point of the body.

If the z-axis is taken normal to the plane of elastic symmetry and the other two axes lie in this plane, we conclude that 8 elastic constants must be zero, namely

$$a_{14} = a_{24} = a_{34} = a_{46} = a_{15} = a_{25} = a_{35} = a_{56} = 0$$

and the number of elastic constants a_{ij} reduces to 13 which are given below

$$\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & 0 & 0 & a_{16} \\ & a_{22} & a_{23} & 0 & 0 & a_{26} \\ & & a_{33} & 0 & 0 & a_{36} \\ & & & a_{44} & a_{45} & 0 \\ & & & & a_{55} & 0 \\ & & & & & a_{66} \end{array} \qquad (27)$$

For arbitrary directions of the axes, these equations contain 13 strain co-efficients a_{ij}, not explicitly related in any way.

(2) THREE PLANES OF ELASTIC SYMMETRY (ORTHOGONAL BODY)

If through each point of a body there pass three mutually perpendicular (orthogonal) planes of elastic symmetry and the like planes of elastic symmetry are parallel at each point, then, taking the co-ordinate axes normal to the planes of elastic symmetry (along the principal directions) we find, in addition to 8-elastic constants of the preceeding case, there are 4-more constants equal to zero:

$$a_{16} = a_{26} = a_{36} = a_{45} = 0$$

The generalized Hooke's law equations and the Schematic expression for the elastic potential in terms of the constants a_{ij} take the form

$$\varepsilon_x = a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z$$

$$\varepsilon_y = a_{12}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z$$

$$\varepsilon_z = a_{13}\sigma_x + a_{23}\sigma_y + a_{33}\sigma_z \qquad (28)$$

$$\gamma_{yz} = a_{44}\tau_{yz}, \quad \gamma_{xz} = a_{55}\tau_{xz}, \quad \gamma_{xy} = a_{66}\tau_{xy}$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ & a_{22} & a_{23} & 0 & 0 & 0 \\ & & a_{33} & 0 & 0 & 0 \\ & & & a_{44} & 0 & 0 \\ & & & & a_{55} & 0 \\ & & & & & a_{66} \end{matrix} \qquad (29)$$

Introducing the engineering constants E_i, G_{ij}, ν_{ij} equation (28) can be rewritten in the following form

$$\varepsilon_x = \frac{1}{E_1}\sigma_x - \frac{\nu_{21}}{E_2}\sigma_y - \frac{\nu_{31}}{E_3}\sigma_z$$

$$\varepsilon_y = -\frac{\nu_{12}}{E_1}\sigma_x + \frac{1}{E_2}\sigma_y - \frac{\nu_{32}}{E_3}\sigma_z$$

$$\varepsilon_z = -\frac{\nu_{13}}{E_1}\sigma_x - \frac{\nu_{23}}{E_2}\sigma_y + \frac{1}{E_3}\sigma_z \qquad (30)$$

$$\gamma_{yz} = \frac{1}{G_{23}}\tau_{yz}, \qquad \gamma_{xz} = \frac{1}{G_{13}}\tau_{xz}, \qquad \gamma_{xy} = \frac{1}{G_{12}}\tau_{xy}$$

A body having three orthogonal planes of elastic symmetry at each point is said to be orthogonally anisotropic or in short, orthotropic. The principal directions at a given point may not be equivalent. Of the 12 elastic constants entering into equations (30) only 9 constants are independent. By virtue of the symmetry of the matrix of the right hand side of the equations expressing the generalized Hooke's law, we always have

$$E_1\nu_{21} = E_2\nu_{12}, \quad E_2\nu_{32} = E_3\nu_{23}, \quad E_3\nu_{13} = E_1\nu_{31} \tag{31}$$

It is important to note that no further reduction of the elastic constants is possible here since, in contrast to the case of a plane symmetry, a_{ij} from equations (28) or E_i, G_{ij}, ν_{ij} from equations (30) are invariant constants themselves. They are alternatively called the principal constants.

(3) PLANE OF ISOTROPY (AXIS OF ROTATIONAL SYMMETRY)

(3a) TRANSVERSELY ISOTROPIC BODY.

Suppose a body possesses the properties that through all points there pass parallel planes of elastic symmetry in which all directions are elastically equivalent (planes of isotropy). A body with such properties is said to be transversely isotropic.

Considering the z-axis to be taken normal to a plane of isotropy, with the x and y axes arbitrarily in this plane, the generalized Hooke's law equations with the 5 independent elastic constants are then written as

$$\varepsilon_x = a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z$$

$$\varepsilon_y = a_{12}\sigma_x + a_{11}\sigma_y + a_{13}\sigma_z$$

$$\varepsilon_z = a_{13}(\sigma_x + \sigma_y) + a_{33}\sigma_z \tag{32}$$

$$\gamma_{yz} = a_{44}\tau_{yz}, \quad \gamma_{xz} = a_{44}\tau_{xz}, \quad \gamma_{xy} = 2(a_{11} - a_{12})\tau_{xy}$$

In some cases a transversely isotropic material is called, in short, transtropic.

(4) ISOTROPIC BODY:-

If all directions in a body are elastically equivalent and principal, then the generalized Hooke's law for an isotropic body of Young's modulus E, Poission's ratio ν, and shear modulus G, is

$$\varepsilon_x = \frac{1}{E} [\sigma_x - \nu (\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E} [\sigma_y - \nu (\sigma_x + \sigma_z)]$$

$$\varepsilon_z = \frac{1}{E} [\sigma_z - \nu (\sigma_x + \sigma_y)] \tag{33}$$

$$\gamma_{yz} = \frac{1}{G} \tau_{yz} , \qquad \gamma_{xz} = \frac{1}{G} \tau_{xz} , \qquad \gamma_{xy} = \frac{1}{G} \tau_{xy}$$

In recent years problems of diffraction of elastic waves by cracks or by inclusions have attracted considerable attention in view of their application in Seismology and Geophysics. Cracks or inclusions are present in essentially all structural materials either as natural defects or as a result of fabrication processes. Moreover, in many cases the cracks or inclusions are sufficiently small so that their presence does not significantly reduce the strength of the material. In other cases, however, the imperfections are large enough through fatigue, stress corosion cracking etc., so that they must be taken into account in determining the strength.

From the standpoint of engineering applications it has been the macroscopic theories based on the notions of continuum solid mechanics and classical thermodynamics which have provided the quantitative working tools for dealing with the fracture of structural materials. In the macroscopic-continuum approach to fracture it is implicitly assumed that the material contains some macroscopic flaw which may act as fracture nuclei and that the medium is a homogeneous continuum in the sense that the size of the macroscopic flaws is large in comparison with the characteristic microstructural dimension of the material. The problem is then to study the effects of the applied loads, the flaw geometry and the environmental conditions on the fracture process in the solid.

Fracture mechanics is concerned with the analysis of the stability of cracks. A fracture criterion can subsequently be employed to determine the conditions for crack propagation, both stable and unstable, and for crack arrest.

Fracture mechanics problems that have to be treated as dynamic problem may be classified in two types:

(1) Cracked bodies subjected to rapidly varying loads

(2) Bodies containing rapidly propagating cracks.

In both the cases the crack tip is an environment disturbed by wave motions.

Impact and vibration problems fall into the first type of dynamic problems. It is often found that at inhomogeneities in a body the dynamic stresses are higher than the stresses computed from the corresponding problem of static equillibrium in the analysis of this type of problem.

The second type of problem is equally important. There are several kinds of large engineering structures e.g., gas transmission pipelines, ship-hulls, aircraft fuselages and nuclear reactor components, in which rapid crack growth is a definite possibility. The study of earthquake mechanisms is the another area to which the analysis of rapidly propagating cracks is relevant.

Recently, there have been a number of comprehensive articles in the general area of fracture mechanics. Some references are those of Achenbach (1972,1976) , Freund (1975,1976,1990) and Kanninen (1978).

Engineering structures requiring protection against the possibility of large scale catastrophic crack propagation are, however, generally constructed of ductile, tough materials. Current progress in this area, and a starting point for the development of a dynamic plastic propagating crack tip analysis have recently been presented by Achenbach and Kanninen.

A problem of central importance in dynamic fracture mechanics is that of predicting the way in which a crack will grow in a deformable solid, given the geometrical configuration of the solid, a characterization of the material, the applied load distribution and suitable initial conditions. In the interpretation of laboratory data on rapid crack propagation, a problem of equal importance is that of determining the values of fracture characterizing parameters from measurments of the crack motion and applied load distribution.

In order to determine an equation of motion for a crack tip, two main ingradients are essential. The first of these is a crack propagation criterion which must be stated as a fundamental physical postulate, distinct from the postulates dealing with bulk material behaviour and momentum balance. Generally these later postulates can be satisfied for any motion of the crack tip. It is the role of the fracture criterion to select the motion of the crack tip from the class of all such dynamically admissible motions.

The only geometrical configuration for which exact solutions of the elastodynamic field equations, valid for nonuniform crack motion, have been found is a semi-infinite crack motion in an otherwise unbounded solid or configurations which can be shown to be equivalent to this by linear superposition arguments. The solution for anti -plane deformation was presentedby Kostrov (1964,1966) and Achenbach (1970) and for in-plane deformation by Freund (1973), Burridge (1976) and Kostrov (1975). Although these solutions have been of major importance in addressing certain fundamental questions on rapid crack propagation, they have been found to be inadequate for describing some dynamic fracture processes of practical importance.

The shape of the cracks which have been studied uptil now are as follows:

(i) Semi-infinite plane cracks

(ii) Finite Griffith cracks

(iii) Penny shaped and annular cracks

(iv) Non-planer cracks

A transient problem in which a semi-infinite crack appears suddenly in a stretched elastic sheet was solved by Maue (1954) and was also discussed by Ang (1958) as his dissertation. Baker (1962) solved the problem of a semi-infinite crack suddenly bearing and growing at a constant velocity in a stretched body. A steady state problem in which a semi-infinite crack extends at constant speed through an elastic sheet was solved by Craggs (1960). Using the method of matched asymptotic expansion the problem involving diffraction of plane elastic waves by a semi-infinite boundary of finite width was solved by Viswanathan and Sharma (1978) and by Viswanathan, Sharma and Datta (1982).

The diffraction problem of a semi-infinite crack has been solved by the Wiener-Hopf (1958) technique.

In 1921 Griffith considered the problem of a fracture of a glass containing crack like defects. Griffith's work presented a theory of fracture. Among other workers investigating crack problems are

Orowan (1948), Sack (1946), Irwin (1957) etc. A number of crack problems in the theory of classical elasticity can be found in the literature [e.g. Sneddon and Lowengrub (1969), Sih (1972)]. Yoffe (1951) considered the inplane problem of propagation of a finite Griffith crack of fixed length at a constant speed in an isotropic elastic solid of infinite extent. Other references treating elastodynamic problems involving a single finite Griffith crack are of Sato (1961), Williams (1957,1961), Karp and Karal (1962), Ang and Knopoff (1964), Loeber and Sih (1968), Sih and Loeber (1968,1969,1970), Willis (1967), Atkinson and Eshebly (1968), Mal (1970a,1970b,1972), Hilton and Sih (1971), Chang (1971), Thau and Lu (1971), Sih, Embley and Ravera (1972), Kanninen (1974), Chen (1978), Sih and Chen (1980), Takei, Shindo and Atsumi (1982), Ueda, Shindo and Atsumi (1983), Shindo (1985). Some other references are of Srivastava, Palaiya and Karaulia (1980a, 1980b), Srivastava, Gupta and Palaiya (1981), Erguven (1987).

Carrier (1946) studied the propagation of waves in orthotropic medium. Achenbach and Bazant (1975) considered the problem of elastodynamic near-tip stress and displacement fields for rapidly propagating cracks in orthotropic meterials. Kassir and Tse (1983) also solved the problem of moving a Griffith crack in an orthotropic material.

The problem of diffraction of finite Griffith crack along the interface of two dissimilar elastic media have been solved by Goldshtein (1966,1967), Brock and Achenbach (1973a,1973b,1974), Atkinson (1974), Matczynski (1974), Brock (1975), Srivastava, Gupta and Palaiya (1978), Neerhoff (1979), Srivastava, Palaiya and Karaulia (1980), Bostrom (1987). Bostrom (1987) solved the two dimensional scalar problem of scattering of elastic waves under anti-plane strain from an interfacecrackbetweentwoelastic half-spaces. The problem of interaction of anti-plane shear waves by a Griffith crack at the interface of two bonded dissimilar elastic half-spaces was considered by Srivastava, Palaiya and Karaulia (1980).

The transient stress and displacement fields arround an embedded crack in the shape of a circle were first investigated by Embley and Sih (1971) for extensional impact and by Sih and Embley (1971) for torsional impact. Their method of solution involves isolating the singular portion dynamic stresses in the Laplace transform domain such that the dynamic stress intensity factor can be obtained by direct application of the Laplace inversion theorem. Some other references are Mal (1968, 1970.a), Olesiak and Sneddon (1959), Pal and Sridharan (1980.a, 1980.b), Arwin and Erdogan (1971), Green (1949), Dhawan (1973), Krenk and Schimidt (1982). Robertson (1967) solved the problem of diffraction of a plane longitudinal wave by a penny-shaped crack.

We now discuss a certain type of mixed boundary value problems which are known as contact problem in the theory of elasticity. The contact problem is formulated as a problem about the influence of a rigid body or an elastic body.

Hertz investigated the punch problem in 1882. In his time many researchers followed his work. Chaplygin (1950) collected a number of punch problems worked out during the 19th century. Many authors such as Glagolev (1942), Mushkelishvilli (1953,1963), Mossakovski (1958), Ufliand (1956), Spence (1968,1975) investigated punch problems. In the literature [e.g. Gladwell (1980)] a variety of punch problems can be found.

The problem of diffraction of anti-plane shear wave by one or more finite rigid strip at the interface has been treated by Palaiya and Majumder (1981), Singh and Dhaliwal (1984), Tait and Moodie (1981), Mandal and Ghosh (1992a,1992b). Palaiya and Majumder (1981) considered the problem of diffraction of anti-plane shear wave by a finite rigid strip at the interface of two bonded dissimilar half spaces. The problem of diffraction of anti-plane shear wave by a pair of parallel rigid strips at the interface of two bonded dissimilar elastic media was solved by Mandal and Ghosh (1992a). De Sarkar (1985a,1985b) solved the punch problem on a micropolar elastic solid.

Different techniques have been adopted by many authors to solve these type of crack and inclusion problems. From these standpoint, these problems may be divided into two categories:

(i) One for low frequency oscillation of the source or long wave scattering or transmission and

(ii) the other for high frequency oscillation or short wave scattering or transmission in the medium.

The term long and short are used in comparison to the region of the source of disturbance or the size of the crack or strip etc. inside the medium to the wave length of disturbance. In case of low frequency oscillations Noble's (1963) method of solving dual integral equations, Tranter's (1968) technique for solving dual integral equations, Matched asymtotic expassion, and Variational principle are found to be very useful techniques.

NOBLE'S METHOD :

Suppose that a mixed boundary value problem is formulated by suitable integral transform so as to be governed by a set of dual integral equations of the form

$$\int_0^{\infty} x^{-1}\left[1+K(x)\right]S(x)J_{\nu}(rx)dx = f(r) \quad , \quad 0\le r<a$$

$$\int_0^{\infty} S(x)J_{\nu}(rx)dx = g(r) \quad , \quad r>a$$

where the functions K(x), f(r) and g(r) are known.
According to Noble (1963), when $\nu>-\frac{1}{2}$.

$$S(x) = \sqrt{\frac{2x}{\pi}}\left\{\int_0^{a} t^{1/2}\theta(t)J_{\nu-1/2}(xt)dt + \int_a^{\infty} t^{\nu+1/2}G(t)J_{\nu-1/2}(xt)dt\right\}$$

where $\theta(t)$ satisfies the Fredholm integral equation

$$\theta(t) + \frac{1}{\pi}\int_0^{a} M(\tau,t)\theta(\tau)d\tau = t^{-\nu}F(t) - H(t) \qquad (0\le t<a) \qquad (34)$$

in which $$M(\tau,t) = \pi\sqrt{\tau t}\int_0^\infty xK(x)J_{\nu-1/2}(\tau x)J_{\nu-1/2}(tx)dx$$

$$F(t) = \frac{d}{dt}\int_0^t f(r)r^{\nu+1}(t^2-r^2)^{-1/2}\,dr$$

$$H(t) = t^{1/2}\int_0^\infty xK(x)J_{\nu-1/2}(xt)dx\int_a^\infty \xi^{\nu+1/2}G(\xi)J_{\nu-1/2}(x\xi)d\xi$$

$$G(\xi) = \int_\xi^\infty g(r)r^{-\nu+1}(r^2-\xi^2)^{-1/2}\,dr\ .$$

The integral equation (34) can be solved for $\theta(t)$ iteratively for low frequency and consequently $S(x)$ can be determined.

Singh and Dhaliwal (1984) solved the closed form solutions of dynamic punch problems by integral transform method. The mixed boundary value problem was reduced to a set of dual integral equations with trigonometrical kernels. The solutions were obtained by using Hilbert transform technique [Srivastava and Lowengrub (1968)]. We now discuss the Hilbert transform technique as follows.

HILBERT TRANSFORM TECHNIQUE :

Using the theorem (Tricomi, 1951), if $p \in L_2(a,b)$, then the equation

$$\frac{1}{\pi}\int_a^b \frac{h(x)}{x-y}\,dx = p(y)\quad,\qquad y\in(a,b)$$

has the solution

$$h(x) = -\frac{1}{\pi}\left(\frac{x-a}{b-x}\right)^{1/2}\int_a^b \left(\frac{b-y}{y-a}\right)^{1/2}\frac{p(y)}{y-x}\,dy + \frac{C}{\sqrt{(x-a)(b-x)}}$$

where C is an arbitrary constant and the first term belongs to the class $L_2(a,b)$. Srivastava and Lowengrub (1968) found that the solution of the integral equation

$$\frac{1}{\pi}\int_a^b \frac{2th(t^2)}{t^2-y^2}\,dt = p(y) \quad , \quad y \in (a,b)$$

(provided that p satisfies the conditions of the above theorem) is given by

$$h(t^2) = -\frac{1}{\pi}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}\int_a^b \left(\frac{b^2-y^2}{y^2-a^2}\right)^{1/2}\frac{2yp(y)}{y^2-t^2}\,dy + \frac{C}{\sqrt{(t^2-a^2)(b^2-t^2)}}$$

where C is an arbitrary constant.

Using Hilbert transform technique problems involving pair of cracks or strips can easily be solved. Using Hilbert transform technique and also applying the modified Tricomi (1951) theorem of Singh (1973) Singh and Dhaliwal (1984) obtained a closed form solution of dynamic punch problem involving two moving punches.

All the axisymmetrical contact problems may be solved by using Hankel transforms and they then reduce to the solution of a number of sets (or pairs) of dual integral equations. To solve these dual integral equations there are various methods one of which is

Tranter's method. We discuss briefly the method of Tranter (1968) in solving axisymmetric problems.

TRANTER'S METHOD :

The solution of certain physical problems involving axisymmetric geometry can be reduced to the determination of F(p) from so called dual integral equations of the form

$$\begin{aligned} &\int_0^\infty G(p)F(p)J_\nu(rp)dp = f(r) \quad , \quad 0<r<1 \\ &\int_0^\infty pF(p)J_\nu(rp)dp = 0 \quad , \quad 1<r<\infty \end{aligned} \tag{35}$$

where G(p) and f(r) are known functions.

A solution F(p) of the above integral equations as a series of Bessel functions can be found by setting

$$F(p) = p^{-k} \sum_{m=0}^{\infty} a_m J_{\nu+2m+k}(p) \tag{36}$$

where k is at present an arbitrary parameter, and proceeding as follows.

Substituting from (36) in the second equation of (35) and changing the order of integration and summation, one gets

$$\int_0^\infty pF(p)J_\nu(rp)dp = \sum_{m=0}^{\infty} a_m \int_0^\infty p^{1-k} J_\nu(rp)J_{\nu+2m+k}(p)dp \tag{37}$$

Provided $\nu>-1$ and $k>0$, the formula

$$I(\nu,\mu,\lambda,a,b) = \int_0^\infty \frac{J_\nu(at)J_\mu(bt)}{t^\lambda}\,dt = \frac{b^\mu\Gamma(\frac{\nu}{2}+\frac{\mu}{2}-\frac{\lambda}{2}+\frac{1}{2})}{2^\lambda a^{\mu-\lambda+1}\Gamma(\mu+1)\Gamma(\frac{\lambda}{2}+\frac{\nu}{2}-\frac{\mu}{2}+\frac{1}{2})}$$

$$\times\ {}_2F_1\left(\frac{\nu+\mu-\lambda+1}{2},\ \frac{\mu-\lambda-\nu+1}{2}\ ;\ \mu+1\ ;\ \frac{b^2}{a^2}\right)$$

shows that all the integrals on the right hand side of (37) vanish when $r>1$ (because of the factor $\Gamma(-m)$ in the denominator of the term multiplying the hypergeometric function) and hence the series in (36) automatically satisfies the second of the dual equations (35). The coefficients a_m have now to be chosen so that the series in (36) satisfies the first of the dual equations (35). For this purpose we need the result

$$p^{-k} J_{\nu+2n+k}(p) = \frac{\Gamma(\nu+n+1)}{2^{k-1}\Gamma(\nu+1)\Gamma(n+k)} \int_0^1 r^{\nu+1}(1-r^2)^{k-1}F_n(k+\nu,\nu+1,r^2) \times$$

$$\times J_\nu(pr)dr \qquad (38)$$

where n is a positive integer or zero and

$$F_n(\alpha,\gamma,x) = {}_2F_1(-n,\alpha+n;\gamma;x) \qquad (39)$$

is Jacobi's polynomial.

Substituting from (36) in the first of (35), multiplication by

$$r^{\nu+1}(1-r^2)^{k-1}F_n(k+\nu,\nu+1,r^2),$$

integration with respect to r between 0 and 1, interchange of the order of integrations and use of (38) give

$$\sum_{m=0}^{\infty} a_m \int_0^\infty G(p)p^{-2k}\ J_{\nu+2m+k}(p)J_{\nu+2n+k}(p)dp = E(\nu,n,k) \qquad (40)$$

where

$$E(\nu,n,k) = \frac{\Gamma(\nu+n+1)}{2^{k-1}\Gamma(\nu+1)\Gamma(n+k)} \int_0^1 f(r) r^{\nu+1}(1-r^2)^{k-1} F_n(k+\nu,\nu+1,r^2)\,dr \tag{41}$$

Equation (40) with n=0,1,2,3,....... gives a set of simultaneous equations for the determination of the coefficients a_m. These simultaneous equations can be rewritten in a more convenient form by making use of the formula

$$\int_0^\infty p^{-1} J_{\nu+2m+k}(p) J_{\nu+2n+k}(p)\,dp = \begin{cases} 0 , & m\neq n \\ (2\nu+4n+2k)^{-1} , & m=n \end{cases} \tag{42}$$

this being the form taken by equation

$$\int_0^\infty \frac{J_\nu(at)J_\mu(at)}{t}\,dt = \frac{\Gamma(\frac{\nu}{2}+\frac{\mu}{2})}{2\Gamma(1+\frac{\nu}{2}-\frac{\mu}{2})\Gamma(1+\frac{\nu}{2}+\frac{\mu}{2})\Gamma(1-\frac{\nu}{2}+\frac{\mu}{2})}$$

$$= \frac{2}{\pi}\frac{\sin\frac{1}{2}(\mu-\nu)\pi}{\mu^2-\nu^2} \tag{43}$$

when μ and ν are replaced respectively by $\nu+2n+k$, $\nu+2m+k$ and when 'at' is replaced by p. We find in this way

$$a_n + \sum_{m=0}^{\infty} L_{m,n}\, a_m = (2\nu+4n+2k)E(\nu,n,k) \tag{44}$$

where

$$L_{m,n} = (2\nu+4n+2k)\int_0^\infty \left[G(p)p^{1-2k} - 1\right] p^{-1} J_{\nu+2m+k}(p) J_{\nu+2n+k}(p)\,dp \tag{45}$$

The iterative solution of the simultaneous equations (44) is

$$a_n = E_n - E_n' + E_n'' - \ldots\ldots. \tag{46}$$

where

$$E_n = (2\nu+4n+2k)E(\nu,n,k)$$

$$E'_n = \sum_{m=0}^{\infty} L_{m,n}\, E_m \quad , \quad E''_n \doteq \sum_{m=0}^{\infty} L_{m,n}\, E'_m \qquad (47)$$

and so on.

Equations (36), (46), (47), (45) and (42) provide a theoratical solution of the dual integral equations. For a practical solution it is necessary to be able to choose the parameter k so that the expression $\left[G(p)p^{1-2k} - 1\right]$, which occurs in the formula (45) for $L_{m,n}$, is fairly small.

Low frequency diffraction due to disc, cone and rigid cylinder have been studied by Asvestas and Kleinman (1970), Senior (1971), Datta (1974), Roy (1982a,1982b,1982c), Sleeman (1967), Roy and Sabina (1982). Datta (1970) considered the problem of diffraction of a plane compressional elastic wave by a rigid circular disc.

The problem of diffraction of elastic waves by two or more co-planar Griffith cracks are very few in number. As regards the dynamic crack problem, research has been restricted mainly to the case of a single crack because of the severe mathematical complexity encountered in finding solutions for two or more cracks.

Itou (1978) solved the problem of dynamic stress concentration around two co-planar Griffith cracks in an infinite elastic medium. Itou (1980a,1980b) also considered two different problems involving two finite cracks. The problem of determining the transient stress distribution in an infinite elastic medium weakened by two coplanar Griffith cracks has been reduced to the following equation

$$\sum_{n=1}^{\infty} c_n(s)\left[-\frac{4c_L^3}{k^2 s^2 b}\int_0^{\infty} g(s,\xi)\sin\left(\frac{a+b}{2}\xi - \frac{n\pi}{2}\right)J_n\left(\frac{b-a}{2}\xi\right)\cos(\xi x)d\xi\right]$$

$$= - Pc_L(bs) \quad , \quad a<x<b \tag{48}$$

with

$$g(s,\xi) = \frac{[\xi^2+k^2s^2/(2c_L^2)]^2 - \xi^2\gamma_1\gamma_2}{\xi\gamma_1} \tag{49}$$

where locations of the cracks are $a\le|x|\le b$, $|y|<\infty$, $z = 0$,

$c_L = \left(\frac{\lambda+2\mu}{\rho}\right)^{1/2}$, $c_T = \left(\frac{\mu}{\rho}\right)^{1/2}$, $k = c_L/c_T$ and $c_n(s)$ are the unknown coefficients.

To determine the coefficients $c_n(s)$ by Schmidt's method (1958) equation (48) can be rewritten as

$$\sum_{n=1}^{\infty} c_n(s)E_n(s,x) = - u(s,x) \quad , \quad a<|x|<b \tag{50}$$

where $E_n(s,x)$ and $u(s,x)$ are known functions and the coefficients $c_n(s)$ are unknown.

A set of functions $P_n(s,x)$ which satisfy the orthogonality condition

$$\int_{a}^{b} P_m(s,x)P_n(s,x)dx = N_n\,\delta_{mn} \quad , \quad N_n = \int_a^b P_n^2(s,x)dx \tag{51}$$

can be constructed from the function, $E_n(s,x)$, such that

$$P_n(s,x) = \sum_{n=1}^{\infty} \frac{M_{in}}{M_{nn}} E_i(s,x) \tag{52}$$

where M_{in} is the cofactor of the element d_{in} of D_n, which is defined as

$$D_n = \begin{vmatrix} d_{11} & d_{12} & \cdots\cdots & d_{1n} \\ d_{21} & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ d_{n1} & \cdots\cdots & \cdots\cdots & d_{nn} \end{vmatrix} \tag{53}$$

$$d_{in} = \int_a^b E_i(s,x)E_n(s,x)\ dx\ .$$

Using equations (50) and (51) one can obtain

$$c_n(s) = \sum_{j=n}^{\infty} q_j \frac{M_{nj}}{M_{jj}} \tag{54}$$

with $$q_j = -\frac{1}{N_j}\int_a^b u(s,x)P_j(s,x)\ dx \tag{55}$$

In case of high frequency oscillation Wiener-Hopf (Noble, 1958) technique and Keller's (1958) geometrical theory are found to be most suitable. We now briefly discuss these methods.

WIENER- HOPF METHOD:

The typical problem obtained by applying Fourier transforms to partial differential equations is the following. One shall have to find unknown functions $\Phi_+(\alpha)$, $\Psi_-(\alpha)$ satisfying

$$A(\alpha)\Phi_+(\alpha) + B(\alpha)\Psi_-(\alpha) + C(\alpha) = 0 \qquad (56)$$

where this equation holds in a strip $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$ of the complex α-plane, $\Phi_+(\alpha)$ is regular in the half-plane $\tau > \tau_-$, $\Psi_-(\alpha)$ is regular in $\tau < \tau_+$, and certain information which will be specified later is available regarding the behaviour of these functions as α tends to infinity in appropriate half-planes. The functions $A(\alpha)$, $B(\alpha)$, $C(\alpha)$ are given function of α, regular in the strip. For simplicity let us assume that A, B are also non-zero in the strip.

The fundamental step in the Wiener-Hopf procedure for solution of this equaion is to find $K_+(\alpha)$ regular and non-zero in $\tau > \tau_-$, $K_-(\alpha)$ regular and non-zero in $\tau < \tau_+$, such that

$$A(\alpha)/B(\alpha) = K_+(\alpha)/K_-(\alpha) \qquad (57)$$

Use (57) to rearrange (56) as

$$K_+(\alpha)\Phi_+(\alpha) + K_-(\alpha)\Psi_-(\alpha) + K_-(\alpha)C(\alpha)/B(\alpha) = 0 \qquad (58)$$

Decompose $K_-(\alpha)C(\alpha)/B(\alpha)$ in the form

$$K_-(\alpha)C(\alpha)/B(\alpha) = C_+(\alpha) + C_-(\alpha) \tag{59}$$

where $C_+(\alpha)$ is regular in $\tau>\tau_-$, $C_-(\alpha)$ is regular in $\tau<\tau_+$. With the help of (59) rearrange (58) so as to define a function $J(\alpha)$ by

$$J(\alpha) = K_+(\alpha)\Phi_+(\alpha) + C_+(\alpha) = -K_-(\alpha)\Psi_-(\alpha) - C_-(\alpha) \tag{60}$$

So far this equation defines $J(\alpha)$ only in the strip $\tau_- < \tau < \tau_+$. But the second part of the equation is defined and is regular in $\tau > \tau_-$, and the third part is defined and is regular in $\tau < \tau_+$. Hence by analytic continuation $J(\alpha)$ must be regular in the whole α-plane. Then by the extended form of Liouville's theorem $J(\alpha)$ is a polynomial $p(\alpha)$

$$\begin{aligned} K_+(\alpha)\Phi_+(\alpha) + C_+(\alpha) &= p(\alpha) \\ K_-(\alpha)\Psi_-(\alpha) + C_-(\alpha) &= -p(\alpha) \end{aligned} \tag{61}$$

These equations determine $\Phi_+(\alpha)$, $\Psi_-(\alpha)$ to within the arbitrary polynomial $p(\alpha)$, i.e. to within a finite number of arbitrary constants which must be determined otherwise.

KELLER'S GEOMETRICAL METHOD :

Keller's theory of geometrical diffraction applied to elastodynamics states that the two conical surfaces of diffracted rays are generated when an incident ray strikes an edge. The surface of the inner cone consists of rays of longitudinal motion,

while the surface of the outer cone is composed of rays of transverse motion. The half-angles of the cones are related by Snell's law. Fig.1 shows the cones generated by an incident longitudinal ray. For this case the diffracted longitudinal rays make the same angle ϕ_L with the tangent to the edge as the incident ray, and the diffracted rays of transverse motion are under an angle ϕ_T with the edge, where $C_L \cos\phi_T = C_T \cos\phi_L$.
For a straight diffracting edge, and an incident longitudinal ray, the diffracted displacement fields are related quantitatively to the incident field by

$$\vec{u}_d^L = e^{i\omega s_1/C_L} \left[S_1(1+S_1/R_i)\right]^{-1/2} D_L \hat{i}_L^d A e^{i\omega(s_o/C_L - t)}$$

$$\vec{u}_d^T = e^{i\omega s_2/C_T} \left[S_2(1+S_2/R_d)\right]^{-1/2} D_T \hat{i}_T^d A e^{i\omega(s_o/C_L - t)}$$

Here $A\exp[i\omega(S_o/C_L - t)]$ defines the amplitude and the phase of the incident field at the point of diffraction, and D_L and D_T are diffraction coefficents which relate the diffracted field to the incident field. Also S_1 and S_2 are the smaller of the principal radii of curvature of the diffracted wave front, or equivalently the distances along the diffracted rays from the points of diffraction to the observation point. The unit vectors $\hat{i}_L^d$ and $\hat{i}_T^d$ relate the directions of displacement of the diffracted field to the direction of displacement of the incident field. For a straight diffracting edge R_i is the radius of curvature at the point of diffraction of the curve formed by the intersection of the incident wave front and the plane which contains the incident

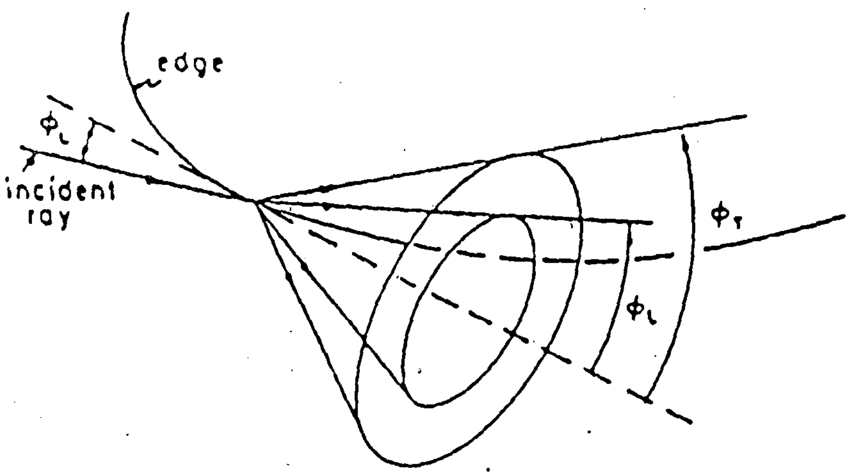


FIG. 1. Cones of diffracted longitudinal and transverse rays for an incident longitudinal ray.

ray and the edge, and

$$R_d = R_i \frac{\sin \phi_T}{\sin \phi_L} \frac{\tan \phi_T}{\tan \phi_L}.$$

In the thesis presented here we have studied some mixed boundary value problems in elastodynamics involving punches, inclusions and cracks. The work has been presented in three chapters. The first chapter deals with the diffraction problems in elastic media by propagating semi-infinite cracks. The second chapter deals with the diffraction problems in isotropic media involving finite width Griffith cracks when the boundaries are present in the medium. The last chapter i.e. chapter III deals with diffraction problem of elastic waves in an infinite orthotropic elastic medium in presence of strips or cracks of finite width. Here we give the summary of the thesis chapterwise.

In the first paper of the chapter-I, we have considered the problem of a series of semi-infinite, parallel and equally spaced cracks subjected to identic loads satisfying the conditions of anti-plane state of strain and steadily propagating in an infinite inhomogeneous medium. Cracks are assumed to move steadily in the direction of modulus variation, it being assumed that the moduli vary exponentially. We further assume that the medium possesses constant elastic wave speeds. These assumptions are necessary for the steady state solution to exist. We assume that the loading is such that Mode III conditions prevail. Mode III is the simplest mode to analyze mathematically. This problem has been solved by the application of Wiener-Hopf technique. We have solved the problem for two types of anti-plane loading. Firstly, the case when the crack edges are loaded at fixed distance from the crack-edge by a concentrated force of constant magnitude has been solved. Secondly, the crack propagation in the case of constant strain on the crack edges has been treated. In both the cases expressions of the stress intensity factor and the crack opening displacement have been derived in closed form and the effect of inhomogeneity of the medium has been shown by means of graphs.

In paper-2, we have considered the problem of the diffraction of a plane harmonic SH-wave by a semi-infinite crack running uniformly along the interface of two dissimilar semi-infinite elastic media. We have applied Fourier transform and Wiener-Hopf technique (1958) to solve the mixed boundary value problem. The resulting integrals have been evaluated asymptotically to obtain the displacement and stress-field near about the crack-tip. The effect of different values of the material parameter, the angle of incidence of incident wave and the crack propagation velocity on the stress intensity factor have been illustrated graphically.

In the second chapter of the thesis, we have considered three problems involving the diffraction of elastic waves by finite width Griffith cracks in a strip and also a problem of diffraction of elastic waves by a series of periodically placed Griffith cracks in an isotropic elastic medium.

First problem of chapter II deals with the diffraction of elastic SH-waves by a Griffith crack in an infinitely long inhomogeneous elastic strip. The shear modulus (μ) and density (ρ) of the material have been assumed to vary in the vertical direction. Applying the Fourier transform, the mixed boundary value problem has been reduced to the solution of the dual integral equations. The dual integral equations have been finally reduced to a Fredholm integral equation of second kind by applying the Abel transform. The numerical values of stress intensity factor and crack opening displacement have been illustrated graphically to show the effect of inhomogeneity of the material.

In the second paper of this chapter we have studied the two dimensional problems of diffraction of longitudinal waves by a series of periodically spaced co-planar Griffith cracks in an infinite, isotropic elastic medium. Due to the periodicity of the geometry, the problem can be reduced to the problem with a single crack in a strip with boundaries such that shear stress and normal displacement are zero on them. On use of Fourier transform the mixed boundary value problem for a typical strip has been reduced to the solution of dual integral equations and finally to that of a Fredholm integral equation of the second kind by applying Abel's transform. Expressions for the stress intensity factor and crack opening displacement have been derived in closed form. Numerical values of stress intensity factor and the crack opening displacement have been plotted graphically.

Paper 3 deals with the dynamic antiplane problem of determining

stress and displacement due to three coplanar Griffith cracks moving steadily at a subsonic speed in an infinite elastic strip. Employing Fourier integral transform, the problem when the rigidly clamped edges on the strip are pulled apart in opposite directions has been reduced to solving a set of four integral equations. These integral equations have been solved using the finite Hilbert transform technique and Cook's result (1970) to obtain the exact form of crack opening displacement and stress intensity factors. Making the length of the inner crack tend to zero, the diffraction problem for two cracks have been obtained.
Again, letting the distance between the edges of the inner and outer cracks tend to zero, the diffraction problem for a single crack has also been derived. Numerical results of stress intensity factors are presented in the form of graphs.

In the last problem, i.e. in paper 4 of chapter II, we investigated the problem of determining the antiplane dynamic stress distributions around four coplanar finite length Griffith cracks moving steadily with constant velocity in an infinitely long finite width strip. The two-dimensional Fourier transform has been used to reduce the mixed boundary value problem to the solution of five integral equations. These integral equations have been solved using the finite Hilbert transform technique to obtain the analytical form of crack opening displacement and stress intensity factors. Numerical results for the stress intensity factors at the crack tips have also been depicted graphically. Letting the distance between the inner cracks tend to zero, the corresponding solution of diffraction problem in the presence of three cracks has been derived.

Chapter III deals with some contact problems and crack problems in elastodynamics in an orthotropic elastic medium.

In the first problem of chapter III, the elastodynamic response of a pair of parallel rigid strips embedded in an infinite orthotropic medium due to elastic waves incident normally on the strips has been investigated. The resulting mixed boundary value problem is reduced to the solution of a triple integral equation which has further been reduced to the solution of an integro-differential equation. Iterative solution valid for low frequency has been obtained. From the solution of the integral equation, we have found out the normal stress and vertical displacement at points in the plane of the strips. Finally, choosing the engineering elastic constants of the orthotropic material suitably the results for isotropic material have been deduced and compared with the results obtained by Jain and Kanwal (1972.b) Numerical values of the stress intensity factors at inner and outer edges of the strips for several orthotropic materials have been calculated and plotted graphically to show the effect of material orthotropy.

Problem 2 of this chapter deals with the problem of diffraction of normally incident elastic waves by two coplanar Griffith cracks situated in an infinite orthotropic medium. Fourier and Hilbert transform techniques have been used to solve this mixed boundary value problem. The resulting triple integral equation has been reduced to the solution of an integro-differential equation and approximate solution has been obtained. These solutions have been used to obtain approximate analytical results for stress intensity factors and crack opening displacements when the wave lengths are large compared to the crack length. Making the distance between two cracks zero, the corresponding results for a single crack has been presented. Finally, choosing the engineering elastic constants of the orthotropic material suitably the results for isotropic material have been deduced and compared with the results obtained by Jain and Kanwal (1972a) To display the influence of the material orthotropy numerical values of stress intensity factors and crack opening displacements have been plotted for several orthotropic materials.

In paper 3, we have considered the problem of dynamic response of three coplanar Griffith cracks in an infinite orthotropic medium due to elastic waves incident normally on the cracks. Fourier transform technique has been used to reduce the elastodynamic problem to the solution of a set of four integral equations. These integral equations have been solved by using finite Hilbert transform technique and Cook's result. The analytical forms of crack opening displacements and stress intensity factors have been derived for low frequency vibration. Making the length of the central crack tend to zero, the corresponding results for two Griffith cracks have been obtained. Numerical results for stress intensity factors and crack opening displacements have been plotted against dimensionless frequency and distance respectively.

for different orthotropic materials which have been shown graphically.

The last problem of this chapter deals with the problem of diffraction of normally incident elastic waves due to four coplanar Griffith cracks in an infinite orthotropic elastic medium. The faces of each of the cracks do not come into contact during small deformation of the solid because a small distance are assumed to be separated. By the use of the Fourier integral, the mixed boundary value problem has been reduced to solving a set of five integral equations which have been solved by finite Hilbert transform technique. This enables us to obtain approximate value of the crack opening displacements and stress intensity factors at the crack tips. The effect of stress intensity factors and crack opening displacements at the edges of the cracks for several orthotropic materials have been calculated and plotted by means of graphs. Also letting the distance of the inner cracks tend to zero, the corresponding results for three cracks have been obtained.

With this much of introduction, we now present the thesis chapterwise. References given in the thesis do not include all the previous workers in this line. But attempt has been made to include most of them.

CHAPTER - I

SEMI-INFINITE CRACKS PROPAGATING IN AN ELASTIC MEDIUM

PAPER 1 : Steady state propagation of a series of parallel cracks in antiplane state of strain in an inhomogeneous elastic medium.

PAPER 2 : Scattering of antiplane shear wave by a propagating crack at the interface of two dissimilar elastic media.

media when the crack moves in the direction of the modulus variation. Steady state crack propagation due to shear waves in a medium of monoclinic type has recently been studied by Chattopadhyay and Bandyopadhyay (1988).

In our paper, we have considered the steady state propagation of a series of semi-infinite, rectilinear parallel and uniformly spaced cracks in an infinite inhomogeneous medium. Cracks are assumed to move steadily in the direction of modulus variation, it being assumed that the moduli vary exponentially. We further assume that the medium possesses constant elastic wave speeds. These assumptions are necessary for the steady state solution to exist. We assume that the loading is such that Mode III conditions prevail. Mode III is the simplest mode to analyze mathematically. Nevertheless, it can be expected that the results for the stress intensity factor obtained here will be qualitatively similar to other modes, even though the specific structure of the stress variation near the crack tip will differ in each case. Following Atkinson and List (1978), we have also assumed in our paper that the edges of the cracks are loaded on their entire length by constant strain.

2. FORMULATION OF THE PROBLEM

Consider an infinite elastic medium with spatially varying density and elastic moduli divided partially by an infinite number of semi-infinite, rectilinear, parallel and uniformly spaced cracks. The semi-infinite cracks are situated parallel to the negative x_1-axis at 2h distances apart and move along positive x_1-direction at a constant velocity $c < c_2$.

The cracks are assumed to propagate steadily in the direction of modulus variation. We assume that the elastic moduli and density both vary exponentially in the same manner; so that the medium may have constant elastic wave speeds.

Owing to the symmetry of the problem, it is reduced to the problem of an infinite elastic strip of thickness 2h weakened in the middle plane $x_2=0$ by a semi-infinite crack $x_1<0$, the surfaces $x_2= \pm h$ of the strip being rigidly clamped.

The displacement $\vec{U}$ in the anti-plane state of strain in a rectangular co-ordinate system (x_1, x_2, x_3) is in the form

$$\vec{U} = [0, 0, w(x_1, x_2, t)] \tag{1}$$

The non-vanishing components of this state of strain are given by the following relations:-

$$e_{13} = \frac{\partial w}{\partial x_1}, \qquad e_{23} = \frac{\partial w}{\partial x_2}$$

$$\begin{aligned} \sigma_{13} &= \mu \frac{\partial w}{\partial x_1}, & \sigma_{23} &= \mu \frac{\partial w}{\partial x_2} \\ &= \mu_0 e^{2\alpha x_1} \frac{\partial w}{\partial x_1} & &= \mu_0 e^{2\alpha x_1} \frac{\partial w}{\partial x_2} \end{aligned} \tag{2}$$

where the shear modulous $\mu(x_1) = \mu_0 e^{2\alpha x_1}$, μ_0 and α are constants.

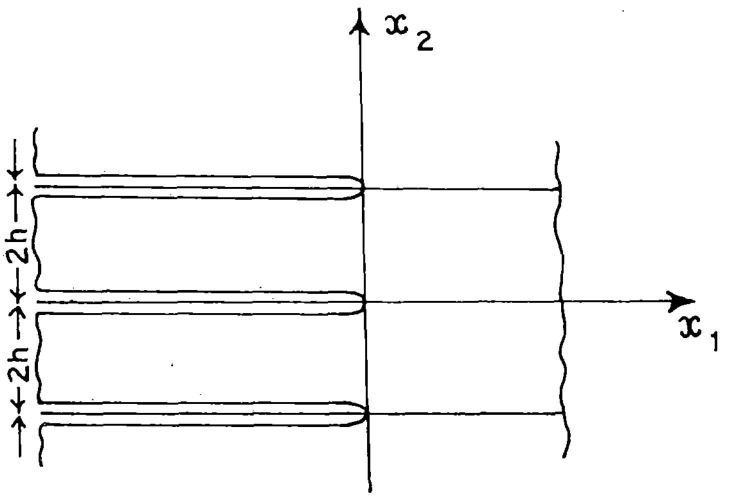


Fig. 1. Geometry of the problem .

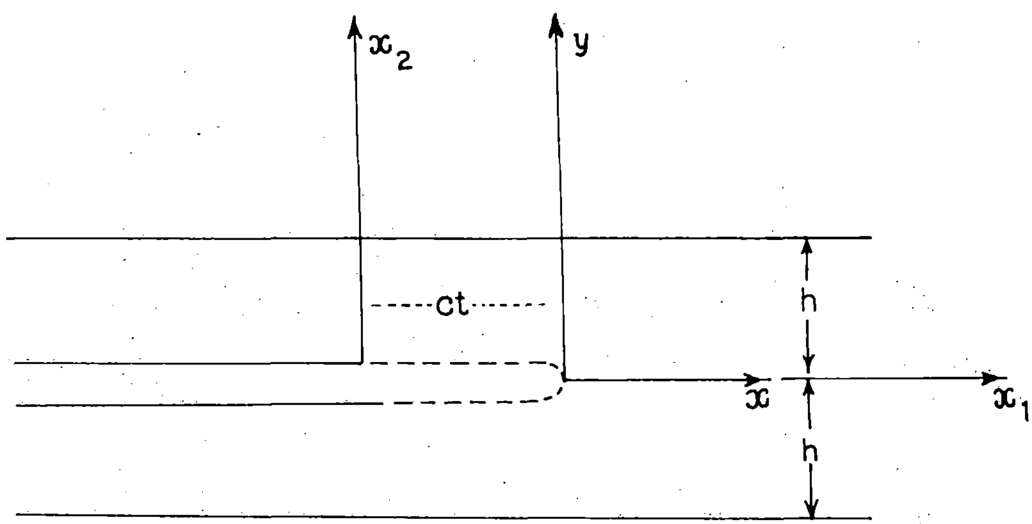


Fig. 2. Crack propagating in the strip.

Using relation (2), the equation of motion of SH-waves is

$$\frac{\partial}{\partial x_1}\left[\mu(x_1)\frac{\partial w}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[\mu(x_1)\frac{\partial w}{\partial x_2}\right] = \rho(x_1)\frac{\partial^2 w}{\partial t^2}$$

or,

$$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + 2\alpha\frac{\partial w}{\partial x_1} = c_2^{-2}\frac{\partial^2 w}{\partial t^2} \tag{3}$$

where $\rho(x_1)=\rho_o e^{2\alpha x_1}$; so $c_2 = \sqrt{\mu(x_1)/\rho(x_1)} = \sqrt{\mu_o/\rho_o}$ is the shear-wave velocity.

The fixed co-ordinate system may be repalced by the conventional system (x,y,z) moving with the crack tip,

$$x_1 = x + ct, \quad x_2 = y, \quad x_3 = z \tag{4}$$

Using relation (4) , equation (3) becomes

$$\left(1-\frac{c^2}{c_2^2}\right)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 2\alpha\frac{\partial w}{\partial x} = 0 \tag{5}$$

Applying complex Fourier transform in x, equation (5) becomes

$$\frac{d^2\overline{W}}{dy^2} - \beta^2\overline{W} = 0 \tag{6}$$

where

$$\beta^2 = \left(1-\frac{c^2}{c_2^2}\right)\zeta^2 + 2i\alpha\zeta \tag{7.1}$$

and

$$\overline{W}(\zeta,y) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} w(x,y)\, e^{i\zeta x}\, dx \tag{7.2}$$

The solution of equation (6) becomes

$$\overline{W}(\zeta,y) = A\sinh(\beta y) + B\cosh(\beta y) \tag{8}$$

where the constants A and B are to be determined.

3. SOLUTION OF THE PROBLEM FOR CONSTANT STRAIN $\frac{\partial w}{\partial y} = P$ OF THE CRACK EDGES $x<0$

We now consider the problem when the constant strain given by

$$\frac{\partial w}{\partial y} = P \tag{9}$$

is applied to the crack faces $y = 0$, $x < 0$.

We shall therefore consider the steady state crack propagation under the boundary conditions

$$\frac{\partial w}{\partial y} = P, \qquad \text{for} \quad x<0, \; y=0 \tag{10.1}$$

$$w(x,y) = 0, \qquad \text{for} \quad x>0, \; y=0 \tag{10.2}$$

$$w(x,y) = 0, \qquad \text{for} \quad |x|<\infty, \; y=h. \tag{10.3}$$

Now we can write

$$\frac{\partial w}{\partial y} = P, \qquad \text{for} \quad x<0, \; y=0$$

$$= e(x) \qquad \text{for} \quad x>0, \; y=0$$

where $e(x)$ is the unknown function which is to be determined.

In our case

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty} \frac{\partial w}{\partial y} e^{i\zeta x} dx = (2\pi)^{-1/2}\int_{-\infty}^{0} \frac{\partial w}{\partial y} e^{i\zeta x} dx + (2\pi)^{-1/2}\int_{0}^{\infty} \frac{\partial w}{\partial y} e^{i\zeta x} dx$$

$$\frac{\partial \overline{W}(\zeta,0)}{\partial y} = (2\pi)^{-1/2}\int_{-\infty}^{0} P \, e^{i\zeta x} dx + (2\pi)^{-1/2}\int_{0}^{\infty} e(x) \, e^{i\zeta x} dx \tag{11}$$

Therefore using (8) and writing $(2\pi)^{-1/2}\int_0^\infty e(x)\, e^{i\zeta x}\, dx = E_+(\zeta)$,

$$\beta A = (2\pi)^{-1/2}\frac{P}{i\zeta} + E_+(\zeta) \quad \text{for } -k < \operatorname{Im}\zeta < 0 \tag{12}$$

$$\text{if } e(x) \sim O(e^{-kx}) \text{ as } x \longrightarrow \infty.$$

Using the conditions (10.2) and(10.3), it can be easily shown that

$$A = -\frac{\overline{W}_-(\zeta,0)}{\tanh(\beta h)} \tag{13}$$

where $\overline{W}_-(\zeta,0) = (2\pi)^{-1/2}\int_{-\infty}^0 w(x,0)\, e^{i\zeta x}\, dx$ is analytic in the lower half-plane $\operatorname{Im}\zeta<k_1$, if we assume $w(x,0)$ $O(e^{k_1 x})$ as $x \longrightarrow -\infty$.

Eliminating A by equations (12) and (13)

$$-\beta\,\frac{\overline{W}_-(\zeta,0)}{\tanh(\beta h)} = -\frac{iP}{\sqrt{2\pi}}\,\frac{1}{\zeta} + E_+(\zeta) \tag{14}$$

Let $K(\zeta) = \beta\,\coth(\beta h) = \frac{1}{h}\,\beta h\,\frac{\cosh(\beta h)}{\sinh(\beta h)} = \frac{1}{h}\prod_{n=1}^{\infty}\left\{\frac{1-\left(\frac{i\beta h}{\pi(n-1/2)}\right)^2}{1-\left(\frac{i\beta h}{n\pi}\right)^2}\right\}$ (15)

[cf. Noble (1958) eqns (3.96a) and (3.96b), p.123]

Now consider

$$1-\left(\frac{i\beta h}{n\pi}\right)^2 = 1+\left(\frac{\beta h}{n\pi}\right)^2 = \left(\frac{h}{n\pi}\right)^2\left[\nu^2\zeta^2 + 2i\alpha\zeta + \left(\frac{n\pi}{h}\right)^2\right]$$

$$= \left(\frac{\nu h}{n\pi}\right)^2\left[\zeta^2 + \frac{2i\alpha\zeta}{\nu^2} + \left(\frac{n\pi}{\nu h}\right)^2\right] \tag{16}$$

where $\nu^2 = 1 - c^2/c_2^2$.

So equation (16) can be written as

$$1 - \left(\frac{i\beta h}{n\pi}\right)^2 = \left(\frac{\nu h}{n\pi}\right)^2 (\zeta + i\eta_n^+)(\zeta + i\eta_n^-)$$

where

$$\eta_n^{\pm} = \frac{\alpha}{\nu^2} \pm \left[\frac{\alpha^2}{\nu^4} + \left(\frac{n\pi}{\nu h}\right)^2\right]^{1/2}$$

Similarly, $$1 - \left[\frac{i\beta h}{(n-1/2)\pi}\right]^2 = \left(\frac{\nu h}{n\pi}\right)^2 (\zeta + i\eta^+_{n-1/2})(\zeta + i\eta^-_{n-1/2})$$

It may be noted that η_n^- and $\eta^-_{n-1/2}$ are negative real quantities. So equation (15) becomes

$$K(\zeta) = \frac{1}{h}\prod_{n=1}^{\infty} \frac{(\zeta + i\eta^-_{n-1/2})(\zeta + i\eta^+_{n-1/2})\, n^2}{(\zeta + i\eta_n^-)(\zeta + i\eta_n^+)\ (n-1/2)^2}$$

$$= \frac{1}{h}\prod_{n=1}^{\infty} \frac{(\zeta + i\eta^-_{n-1/2})}{(\zeta + i\eta_n^-)} \frac{n}{(n-1/2)} \cdot \prod_{n=1}^{\infty} \frac{(\zeta + i\eta^+_{n-1/2})}{(\zeta + i\eta_n^+)} \frac{n}{(n-1/2)}$$

$$= K^-(\zeta).K^+(\zeta) \quad \text{(say)} \tag{17}$$

where $K^-(\zeta)$ is analytic in the lower half-plane given by $\mathrm{Im}\ \zeta < -\eta^-_{1/2}$ whereas $K^+(\zeta)$ is analytic in the upper half plane given by $\mathrm{Im}\ \zeta > -\eta^+_{1/2}$.

Now

$$K^+(\zeta) = \prod_{n=1}^{\infty} \frac{(\zeta + i\eta^+_{n-1/2})}{(\zeta + i\eta_n^+)} \frac{(n-0)}{(n-1/2)}$$

$$= \prod_{n=1}^{\infty} \frac{\left[\zeta + i\left(\frac{\alpha}{\nu^2} + \left(\frac{\alpha^2}{\nu^4} + \frac{(n-1/2)^2\pi^2}{\nu^2 h^2}\right)^{1/2}\right)\right](n-0)}{\left[\zeta + i\left(\frac{\alpha}{\nu^2} + \left(\frac{\alpha^2}{\nu^4} + \frac{n^2\pi^2}{\nu^2 h^2}\right)^{1/2}\right)\right](n-1/2)}$$

$$= \prod_{n=1}^{\infty} \frac{\left[\frac{\zeta\nu h}{\pi} + i\left(\frac{\alpha h}{\nu\pi} + \left\{\frac{\alpha^2 h^2}{\nu^2\pi^2} + (n-1/2)^2\right\}^{1/2}\right)\right](n-0)}{\left[\frac{\zeta\nu h}{\pi} + i\left(\frac{\alpha h}{\nu\pi} + \left\{\frac{\alpha^2 h^2}{\nu^2\pi^2} + n^2\right\}\right)^{1/2}\right](n-1/2)}$$

Now elastic moduli and density are assumed to be varying slowly with x_1 so that αh may be assumed to be small.

So neglecting $\alpha^2 h^2$ we get

$$K^+(\zeta) = \prod_{n=1}^{\infty} \frac{\left[\frac{\zeta\nu h}{\pi} + i\left(\frac{\alpha h}{\nu\pi} + (n-1/2)\right)\right](n-0)}{\left[\frac{\zeta\nu h}{\pi} + i\left(\frac{\alpha h}{\nu\pi} + n\right)\right](n-1/2)}$$

$$= \prod_{n=1}^{\infty} \frac{\left[n - \left(\frac{1}{2} + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right)\right](n-0)}{\left[n - \left(\frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right)\right](n-1/2)} \tag{18}$$

Next using the formula

$$\prod_{n=1}^{\infty} \left\{\frac{(n-a_1)\ldots\ldots\ldots(n-a_k)}{(n-b_1)\ldots\ldots\ldots(n-b_k)}\right\} = \prod_{m=1}^{k} \frac{\Gamma(1-b_m)}{\Gamma(1-a_m)}$$

which expresses the general infinite product in terms of the Gamma function (cf. Whittaker and Watson ,1969, p.239) we obtain from

(18)

$$K^{+}(\zeta) = \frac{\Gamma(\frac{1}{2})\ \Gamma\,[\,1 - (\frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi})\,]}{\Gamma(1)\ \Gamma\,[\frac{1}{2} - (\frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi})\,]} \tag{19}$$

Similarly, for small values of αh, neglecting $\alpha^2 h^2$, it can be easily shown that

$$K^{-}(\zeta) = \frac{\sqrt{\pi}}{h}\,\frac{\Gamma\,[\,1 + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\,]}{\Gamma\,[\frac{1}{2} + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\,]} \tag{20}$$

Now writing $\beta\coth(\beta h) = K(\zeta) = K^{+}(\zeta)K^{-}(\zeta)$, equation (14) becomes

$$-\,K^{+}(\zeta)K^{-}(\zeta)\ \overline{W}_{-}(\zeta,0) = -\,\frac{iP}{\sqrt{2\pi}}\,\frac{1}{\zeta} + E_{+}(\zeta)$$

so,

$$-\,K^{-}(\zeta)\ \overline{W}_{-}(\zeta,0) = -\,\frac{iP}{\sqrt{2\pi}}\,\frac{1}{\zeta}\,\frac{1}{K^{+}(\zeta)} + \frac{E_{+}(\zeta)}{K^{+}(\zeta)}$$

$$= -\,\frac{iP}{\sqrt{2\pi}}\,\frac{1}{\zeta}\left[\frac{1}{K^{+}(\zeta)} - \frac{1}{K^{+}(0)}\right] - \frac{iP}{\sqrt{2\pi}}\,\frac{1}{\zeta}\,\frac{1}{K^{+}(0)} + \frac{E_{+}(\zeta)}{K^{+}(\zeta)}$$

Therefore,

$$-\,K^{-}(\zeta)\ \overline{W}_{-}(\zeta,0) + \frac{iP}{\sqrt{2\pi}}\,\frac{1}{\zeta}\,\frac{1}{K^{+}(0)} = \frac{E_{+}(\zeta)}{K^{+}(\zeta)} - \frac{iP}{\sqrt{2\pi}}\,\frac{1}{\zeta}\left[\frac{1}{K^{+}(\zeta)} - \frac{1}{K^{+}(0)}\right]. \tag{21}$$

The expression on the left hand side of equation (21) is regular in the half-plane $\mathrm{Im}\,\zeta < 0$ whereas R.H.S. is regular in $\mathrm{Im}\,\zeta > -K_1$

where $K_1 = \min(k, \eta^+_{1/2})$. The equation (21) holds in the strip $-K_1 < \mathrm{Im}\ \zeta < 0$ and therefore using analytic continuation and Liouville's theorem we can write

$$\overline{W}_-(\zeta,0) = \frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\frac{1}{K^+(0)K^-(\zeta)} \tag{22}$$

and

$$E_+(\zeta) = \frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\left[1 - \frac{K^+(\zeta)}{K^+(0)}\right] \tag{23}$$

Therefore, by help of (11) and (23), we obtain

$$\frac{\partial\overline{W}(\zeta,0)}{\partial y} = -\frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\frac{K^+(\zeta)}{K^+(0)} .$$

So,

$$\frac{\partial w}{\partial y} = -\frac{iP}{\sqrt{2\pi}}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon}\frac{1}{\zeta}\frac{K^+(\zeta)}{K^+(0)}e^{-i\zeta x}\,d\zeta \quad \text{where } -K_1 < \varepsilon < 0. \tag{24}$$

For $x<0$, considering a semi-circular contour in the upper half ζ-plane it can easily be verified that

$$\frac{\partial w}{\partial y} = P$$

Now for $x>0$, substituting the values of $K^+(\zeta)$ and $K^+(0)$ from (19) and (24) we obtain

$$\frac{\partial w}{\partial y} = -\frac{iP}{2\pi}\frac{\Gamma\left[\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{\alpha h}{\nu\pi}\right]}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon}\frac{1}{\zeta}\frac{\Gamma\left[1-\frac{i\zeta\nu h}{\pi}+\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[\frac{1}{2}-\frac{i\zeta\nu h}{\pi}+\frac{\alpha h}{\nu\pi}\right]}e^{-i\zeta x}\,d\zeta$$

$(x>0)$

$$= -\frac{iP}{2\pi}\frac{\Gamma[\frac{1}{2}+\frac{\alpha h}{\nu\pi}]}{\Gamma[1+\frac{\alpha h}{\nu\pi}]} e^{-\frac{\pi x}{\nu h}(\frac{1}{2}+\frac{\alpha h}{\nu\pi})}\int_{-i\infty+s}^{i\infty+s}\frac{1}{(\frac{1}{2}-p+\frac{\alpha h}{\nu\pi})}\times$$

$$\times\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)} e^{\frac{\pi x}{\nu h}p}\,dp$$

where $s = \frac{1}{2}+\frac{\alpha h}{\nu\pi}-\frac{\nu h\varepsilon}{\pi}$

$$= \frac{iP}{2\pi}\frac{\Gamma[\frac{1}{2}+\frac{\alpha h}{\nu\pi}]}{\Gamma[1+\frac{\alpha h}{\nu\pi}]} e^{-\frac{\pi x}{\nu h}(\frac{1}{2}+\frac{\alpha h}{\nu\pi})}\int_{-i\infty+s}^{i\infty+s}\frac{\Gamma(p-\frac{1}{2}-\frac{\alpha h}{\nu\pi})}{\Gamma(p+\frac{1}{2}-\frac{\alpha h}{\nu\pi})}\times$$

$$\times\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)} e^{\frac{\pi x}{\nu h}p}\,dp$$

$$= -P\frac{\Gamma[\frac{1}{2}+\frac{\alpha h}{\nu\pi}]}{\Gamma[1+\frac{\alpha h}{\nu\pi}]} e^{-\frac{\pi x}{\nu h}(\frac{1}{2}+\frac{\alpha h}{\nu\pi})}\frac{e^{\frac{\pi x}{\nu h}(\frac{1}{2}+\frac{\alpha h}{\nu\pi})}(1-e^{-\frac{\pi x}{\nu h}})^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}$$

$$\times\ {}_2F_1(-\tfrac{1}{2};\ -\tfrac{\alpha h}{\nu\pi};\ \tfrac{1}{2};\ 1-e^{-\frac{\pi x}{\nu h}})$$

[cf. Erdélyi et al. (1954) formula no. 7 . p.262]

$$= -\frac{P}{\sqrt{\pi}}\frac{\Gamma[\frac{1}{2}+\frac{\alpha h}{\nu\pi}]}{\Gamma[1+\frac{\alpha h}{\nu\pi}]}\frac{1}{\sqrt{1-\exp(-\pi x/\nu h)}}\ {}_2F_1(-\tfrac{1}{2};\ -\tfrac{\alpha h}{\nu\pi};\ \tfrac{1}{2};\ 1-e^{-\frac{\pi x}{\nu h}})$$

where ${}_2F_1(-\frac{1}{2};\ -\frac{\alpha h}{\nu\pi};\ \frac{1}{2};\ 1-e^{-\frac{\pi x}{\nu h}})$ is the hypergeometric function.

It is known that the Hypergeometric series

$${}_2F_1(a,b,c,z) = 1+\frac{a.b}{1.c}z+\frac{a(a+1)\ b(b+1)}{1.2.c(c+1)}z^2+$$

$$+ \frac{a(a+1)(a+2)\ b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)} z^3 + \ldots\ldots\ldots\ldots$$

therefore neglecting $\left(\frac{\alpha h}{\nu\pi}\right)^2$ and higher powers of $\frac{\alpha h}{\nu\pi}$,

$${}_2F_1\left(-\tfrac{1}{2};\ -\tfrac{\alpha h}{\nu\pi};\ \tfrac{1}{2};\ 1 - e^{-\frac{\pi x}{\nu h}}\right) = 1 + \frac{\alpha h}{\nu\pi}\left(\frac{z}{1} + \frac{z^2}{2.3} + \frac{z^3}{3.4} + \frac{z^4}{4.7} + \ldots\ldots\right)$$

where $z = 1 - e^{-\frac{\pi x}{\nu h}}$;

After a little algebraic simplification it can be shown that for small $\frac{\alpha h}{\nu\pi}$

$${}_2F_1\left(-\tfrac{1}{2};\ -\tfrac{\alpha h}{\nu\pi};\ \tfrac{1}{2};\ z\right) = 1 + \frac{\alpha h}{\nu\pi}\left[(1+\sqrt{z})\log(1+\sqrt{z}) + (1-\sqrt{z})\log(1-\sqrt{z})\right]$$

Therefore

$$\frac{\partial w}{\partial y} = -\frac{P}{\sqrt{\pi}} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \frac{1}{\sqrt{1 - \exp(-\pi x/\nu h)}} \times$$

$$\times\left\{1 + \frac{\alpha h}{\nu\pi}\left[(1+\sqrt{z})\log(1+\sqrt{z}) + (1-\sqrt{z})\log(1-\sqrt{z})\right]\right\} \quad (x>0) \qquad (25)$$

Next in order to determine the crack opening displacement consider equation (22) viz.

$$\bar{W}_-(\zeta,0) = \frac{iP}{\sqrt{2\pi}} \frac{1}{\zeta} \frac{1}{K^+(0)K^-(\zeta)}$$

which by help of equations (19) and (20) becomes

$$\bar{W}_-(\zeta,0) = \frac{iP}{\sqrt{2\pi}} \frac{h}{\pi} \frac{\Gamma\left[\frac{1}{2} + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \frac{1}{\zeta}$$

Therefore

$$w(x,0) = \frac{ihP}{\pi}\frac{1}{2\pi}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{\Gamma\left[\frac{1}{2} + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]}\frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]}\frac{1}{\zeta}e^{-i\zeta x}\,d\zeta$$

Obviously for $x > 0$, $w(x,0) \equiv 0$. In order to find $w(x,0)$ for $x < 0$, we firstly evaluate $\frac{dw(x,0)}{dx}$ which is given by

$$\frac{dw}{dx} = \frac{hP}{\pi}\frac{1}{2\pi}\frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{\Gamma\left[\frac{1}{2} + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]} e^{-i\zeta x}\,d\zeta$$

$$\text{so,}\quad \frac{dw}{dx} = \frac{P\,e^{\frac{\pi x}{\nu h}\left(\frac{1}{2} - \frac{\alpha h}{\nu\pi}\right)}}{\nu\sqrt{\pi}}\frac{1}{2\pi i}\frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]}\int_{s-i\infty}^{s+i\infty}\frac{\Gamma(\frac{1}{2})\,\Gamma(p)}{\Gamma(p+\frac{1}{2})}e^{-\frac{\pi x}{\nu h}p}\,dp$$

where $p = \frac{1}{2} + \frac{i\nu\zeta h}{\pi} - \frac{\alpha h}{\nu\pi}$ and $s = \frac{1}{2} - \frac{\alpha h}{\nu\pi} + \frac{\nu h\varepsilon}{\pi}$.

Using the table of inverse Laplace transform (1954), we find

$$\frac{dw}{dx} = \frac{P\,e^{\frac{\pi x}{\nu h}\left(\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right)}}{\nu\sqrt{\pi}}\frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]}\frac{1}{\sqrt{1 - \exp(\pi x/\nu h)}}$$

Integrating w.r.t. x we obtain

$$w(x,0) = \frac{P}{\nu\sqrt{\pi}}\frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]}\int_0^x e^{-\frac{\alpha x}{\nu^2}}\frac{e^{\frac{\pi x}{2\nu h}}}{\sqrt{1 - \exp(\pi x/\nu h)}}\,dx \qquad (\text{for } x<0)$$

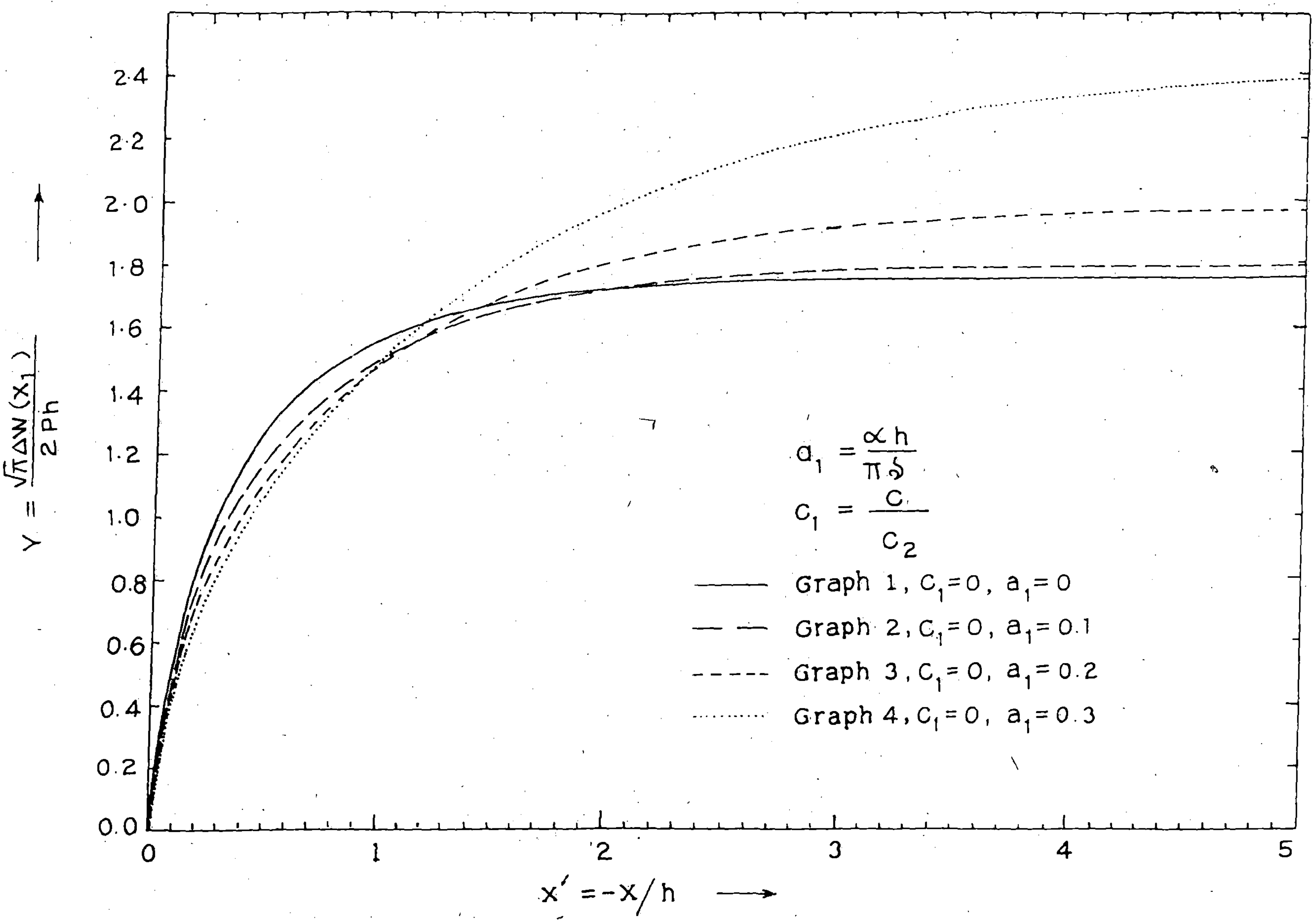
Making $x \longrightarrow -\infty$, it can easily be shown that

$$w(x,0) \longrightarrow -\frac{Ph}{\pi}\,\frac{\Gamma\left[\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right]\,\Gamma\left[\frac{1}{2}-\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{\alpha h}{\nu\pi}\right]\,\Gamma\left[1-\frac{\alpha h}{\nu\pi}\right]} \qquad (26)$$

Putting $\alpha = 0$ in (25) and (26) expressions for $\frac{\partial w(x,0)}{\partial y}$ and $w(x,0)$ for homogeneous medium can be derived and they are found to be identical with the results given by Matczynski (1973).

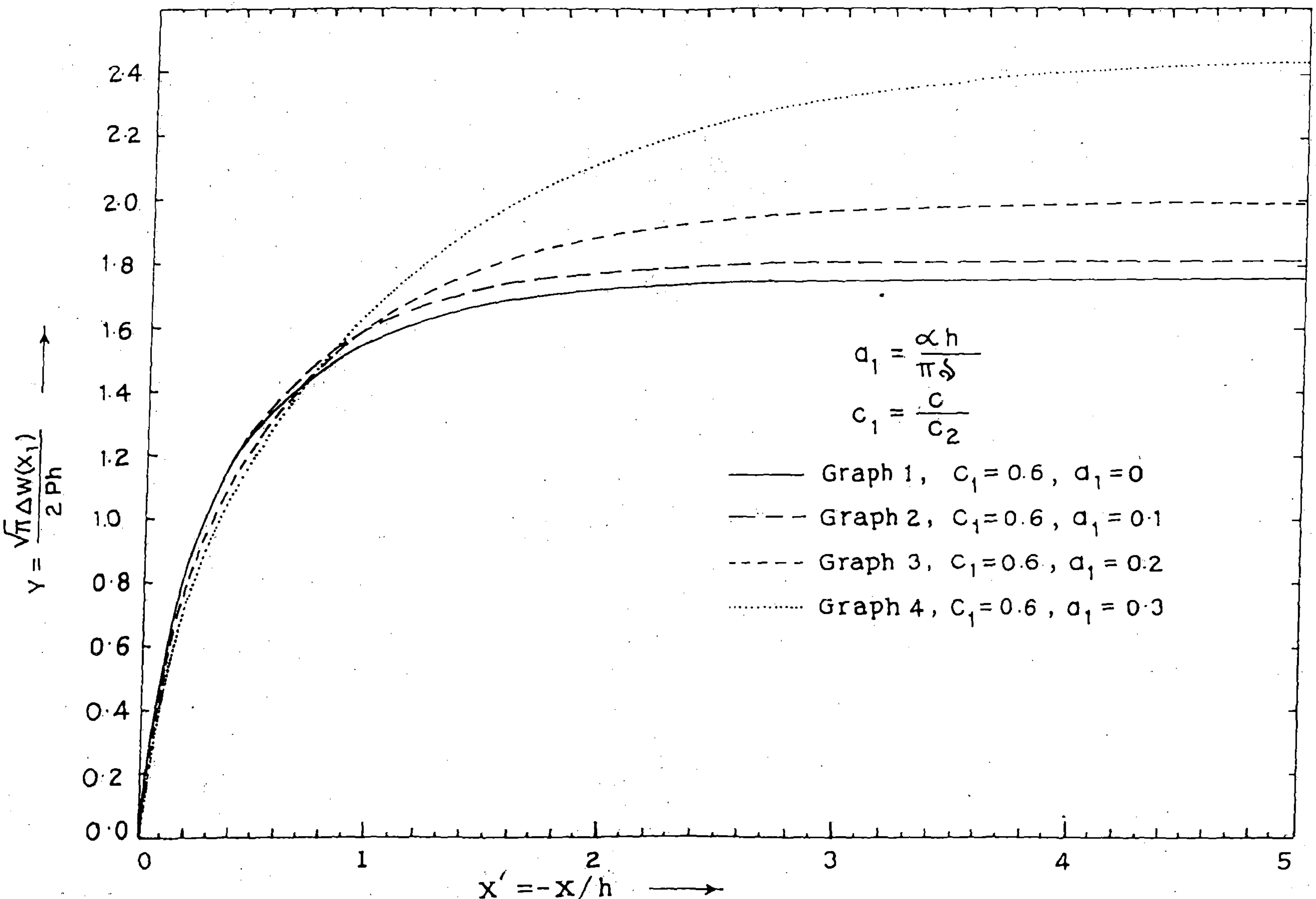
Crack opening displacement is obviously $\Delta w = 2w(x,0)$ where $w(x,0)$ is given by (26). In Figs. 3-5 dimensionless values of the crack opening displacement given by $Y = \frac{\pi\Delta w}{2ph}$ have been plotted against the dimensionless distance $x' = -\frac{x}{h}$ along the length of the crack for different values of $\alpha_1 = \frac{\alpha h}{\nu\pi}$ and $c_1 = c/c_2$.

It is interesting to note that for a fixed value of c_1, crack opening displacement increases with the increase in the values of the inhomogeneity parameter α_1 for large values of x' whereas for small values of $x'(x' \neq 0)$, the result is just the opposite. Further it may be noted that for any given value of the inhomogeneity parameter α_1, crack opening displacement Y at any point x' increases with the increase in the value of the crack propagation velocity.



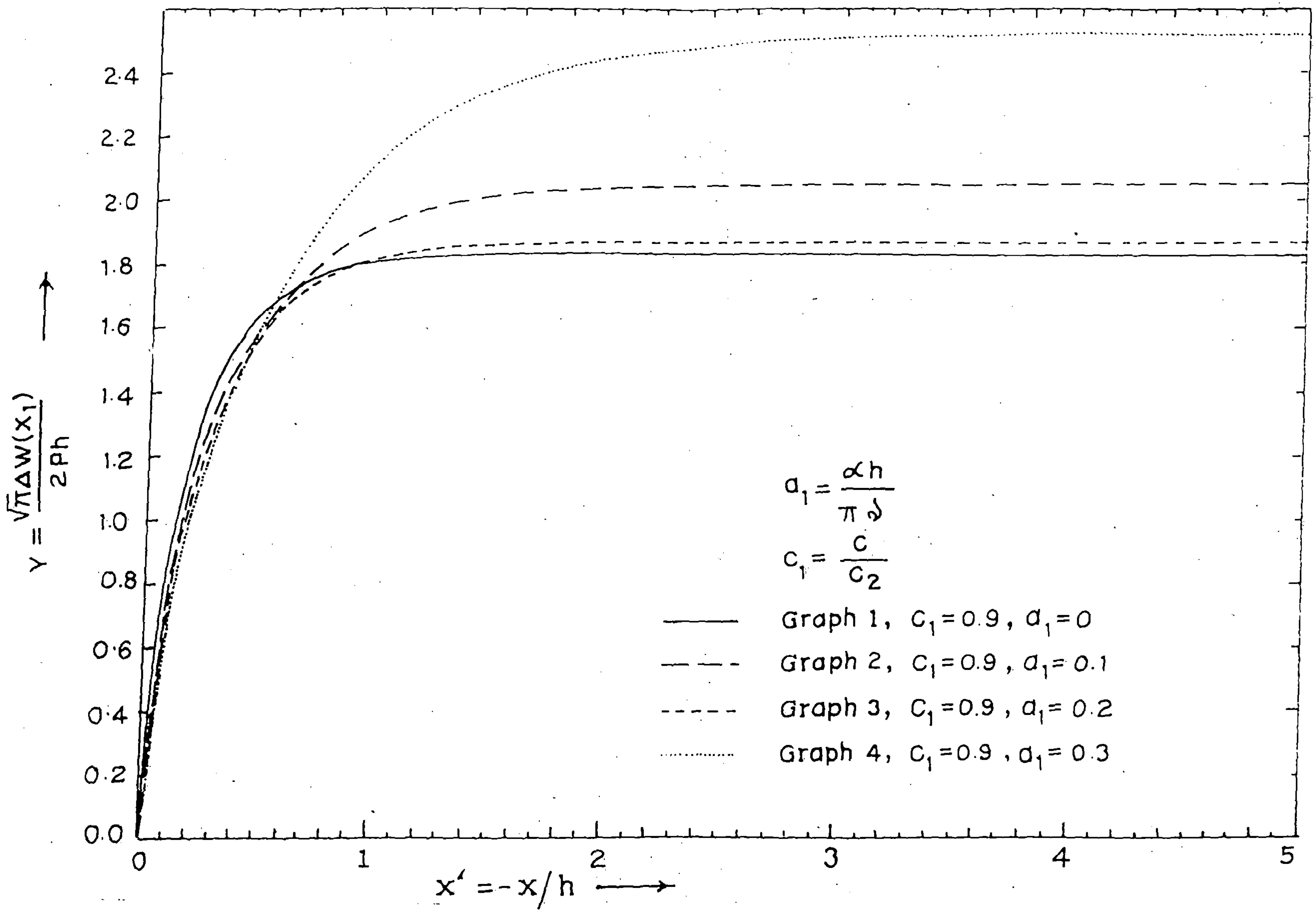
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
Y = √π Δw (x_1) / 2Ph
0
1
2
3
4
5
x' = -x/h
a_1 = αh / πδ
c_1 = c / c_2
Graph 1, C_1 = 0, a_1 = 0
Graph 2, C_1 = 0, a_1 = 0.1
Graph 3, C_1 = 0, a_1 = 0.2
Graph 4, C_1 = 0, a_1 = 0.3

Fig. 3. Y vs x'



2·4
2·2
2·0
1·8
1·6
1·4
1·2
1.0
0·8
0·6
0·4
0·2
0·0
0
1
2
3
4
5
$Y = \frac{\sqrt{\pi}\Delta W(x_1)}{2Ph}$
$x' = -x/h$
$a_1 = \frac{\alpha h}{\pi \delta}$
$c_1 = \frac{c}{c_2}$
Graph 1, $C_1 = 0.6$, $a_1 = 0$
Graph 2, $C_1 = 0.6$, $a_1 = 0.1$
Graph 3, $C_1 = 0.6$, $a_1 = 0.2$
Graph 4, $C_1 = 0.6$, $a_1 = 0.3$

Fig. 4 Y vs. x'



$Y = \frac{\sqrt{\pi}\Delta W(x_1)}{2Ph}$
2·4
2·2
2·0
1·8
1·6
1·4
1·2
1·0
0·8
0·6
0·4
0·2
0.0
0
1
2
3
4
5
$x' = -x/h$
$a_1 = \frac{\alpha h}{\pi \delta}$
$c_1 = \frac{c}{c_2}$
Graph 1, $c_1 = 0.9$, $a_1 = 0$
Graph 2, $c_1 = 0.9$, $a_1 = 0.1$
Graph 3, $c_1 = 0.9$, $a_1 = 0.2$
Graph 4, $c_1 = 0.9$, $a_1 = 0.3$

Fig. 5. Y vs x'

SCATTERING OF ANTIPLANE SHEAR WAVE BY A PROPAGATING CRACK AT THE INTERFACE OF TWO DISSIMILAR ELASTIC MEDIA

1. INTRODUCTION

It is well known that the problems of diffraction of elastic wave by cracks or inclusions are of considerable importance in view of their application in seismology and geophysics. If the cracks or inclusions are located at the interface of layered media, the study becomes more relevant. The extensive use of composite materials in modern technology has also evoked interest in the wave propagation problems in layered media with interfacial discontinuities. Onder et al. (1975) studied the diffraction of monochromatic plane SH-waves obliquely incident on a rigid half-plane between the two different semi-infinite media.

In this paper we have considered the problem of the diffraction of a plane harmonic SH-wave by a semi-infinite crack running uniformly along the interface of two dissimilar semi-infinite elastic media. The problem of scattering of plane harmonic polarized shear wave by a half-plane crack in an infinite isotropic medium extending under antiplane strain was studied

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earlier by Jahanshahi (1967). Chen and Sih (1973, 1975) also solved the in-plane problem of the diffraction of stress waves by a running crack in an incident wave field in an infinite elastic medium. We have applied Fourier transform and Wiener-Hopf technique (1958) to solve the mixed boundary value problem. The resulting integrals have been evaluated asymptotically to obtain the displacement and stress field near about the crack tip. It is found that the stress intensity factor depends sensitively upon the speed of crack propagation, the angle of incidence of the incoming wave and on the material properties of the elastic media. Quantitative assessment of the effect of the aforementioned parameters on the stress intensity factor has been made by displaying the numerical results graphically for two pairs of different materials.

2. FORMULATION OF THE PROBLEM AND ITS SOLUTION

Let a plane crack move at a constant velocity V on the interface of two bonded dissimilar elastic semi-infinite medium due to the incidence of the plane harmonic SH-wave

$$v_1^o = V_1 \exp[-i\{\Lambda_1(X\cos\Theta_1 + Y\sin\Theta_1) + \Omega T\}] \tag{1}$$

in the medium where the co-efficient of rigidity, density and shear wave velocity respectively are given by μ_1, ρ_1 and C_1. The crack lies on the bimaterial interface along Y=0 with respect to the fixed rectangular co-ordinate system (X,Y,Z).

We assume that the displacement and stress due to the scattered field are

$$v_j = v_j(X, Y, T) \tag{2}$$

and $$\left(\tau_{xz}\right)_j = \mu_j \frac{\partial v_j}{\partial X}, \qquad \left(\tau_{yz}\right)_j = \mu_j \frac{\partial v_j}{\partial Y} \tag{3}$$

where the subscript j=1,2 refers to the upper and lower half-planes and T, the time.

The equations of SH-wave motion in either elastic half-space are given by

$$\frac{\partial^2 v_j}{\partial X^2} + \frac{\partial^2 v_j}{\partial Y^2} = \frac{1}{C_j^2}\frac{\partial^2 v_j}{\partial T^2} \quad (j=1,2) \tag{4}$$

where $C_j = (\mu_j/\rho_j)^{1/2}$ is the shear-wave velocity. Without any loss of generality, we further assume that $C_1 > C_2$.

Due to the incident wave given in (1), the reflected and transmitted waves in the absence of the crack may be written in the form

$$v_1^r(X, Y, T) = V_1^r \exp[-i\{\Lambda_1(X\cos\Theta_1 - Y\sin\Theta_1) + \Omega T\}]$$

and (5)

$$v_2^T(X, Y, T) = V_2^T \exp[-i\{\Lambda_2(X\cos\Theta_2 + Y\sin\Theta_2) + \Omega T\}]$$

where

$$V_1^r = \frac{\mu_1\Lambda_1\sin\Theta_1 - \mu_2\Lambda_2\sin\Theta_2}{\mu_1\Lambda_1\sin\Theta_1 + \mu_2\Lambda_2\sin\Theta_2} V_1 = A_1^R V_1 \text{ (say)}$$

and (6)

$$V_2^T = \frac{2\mu_1\Lambda_1\sin\Theta_1}{\mu_1\Lambda_1\sin\Theta_1 + \mu_2\Lambda_2\sin\Theta_2} V_1 = A_2^T V_1 \text{ (say)}$$

with

$$\Lambda_1 \cos\Theta_1 = \Lambda_2 \cos\Theta_2 \ .$$

V_1, V_1^r and V_2^T are the incident, reflected and transmitted wave amplitude respectively, Λ_j the wave number, $\Omega = \Lambda_j C_j$ the circular frequency and Θ_1, Θ_2 the angles of incidence and refraction respectively.

Assume that the crack has been moving in the horizontal direction along the interface for a sufficiently long time and that a steady state has been reached in the neighbourhood of the crack.

A set of moving co-ordinate systems (x,y,z,t) attached to the crack tip moving at a constant velocity V is introduced in accordance with

$$x = X - Vt \ , \quad y_j = s_j Y \ , \quad z = Z \ , \quad t = T \tag{7}$$

where $s_j = (1-M_j^2)^{1/2}$ and $M_j = V/C_j$ is the Mach number.

In terms of the moving co-ordinate system (x,y,t), (4) becomes

$$\frac{\partial^2 v_j}{\partial x^2} + \frac{\partial^2 v_j}{\partial y_j^2} + \frac{1}{C_j^2 \ s_j^2} \frac{\partial}{\partial t}\left[2M_j C_j \frac{\partial v_j}{\partial x} - \frac{\partial v_j}{\partial t}\right] = 0. \tag{8}$$

It is convenient to define an apparent circular frequency $\omega = \alpha\,\Omega$ and the angles of reflection ϕ_1 and refraction ϕ_2 are given by

$$\cos\phi_j = M_j + (\Lambda_j / \lambda_j)\cos\Theta_j \ , \quad \sin\phi_j = (s_j/\alpha)\sin\Theta_j \ ,$$

where (9)

$$\alpha = (1 + M_j \cos\Theta_j) \quad \text{and} \quad \lambda_j = (\Lambda_j / s_j^2)\alpha.$$

Using these relations in a moving system, (1) and (5) take the form

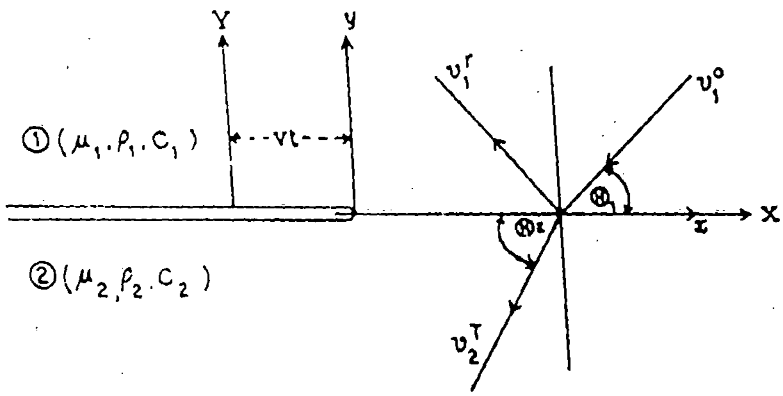

Figure 1. Geometry of the propagating crack.

$$\begin{bmatrix} v_1^O \\ v_1^r \\ v_2^T \end{bmatrix} = \begin{bmatrix} w_1^O(x,y_1) \\ w_1^r(x,y_1) \\ w_2^T(x,y_2) \end{bmatrix} \exp\left\{i(M_1\lambda_1 x-\omega t)\right\} \tag{10}$$

where

$$w_1^O(x,y_1) = V_1\exp\left\{-i\lambda_1(x\cos\phi_1+y_1\sin\phi_1)\right\}$$

$$w_1^r(x,y_1) = A_1^R V_1\exp\left\{-i\lambda_1(x\cos\phi_1-y_1\sin\phi_1)\right\}$$

$$w_2^T(x,y_2) = A_2^T V_1\exp\left[-i\left\{(\beta_2+\lambda_2\cos\phi_2)x+\lambda_2 y_2\sin\phi_2\right\}\right] \tag{11}$$

and

$$\beta_2 = M_1\lambda_1\left[1-\frac{\lambda_2 C_1}{\lambda_1 C_2}\right] < 0 \quad \text{since} \quad C_1>C_2.$$

Using (10), we assume the solution of the governing equation (8) as

$$v_j(x,y_j,t) = w_j(x,y_j)\exp[i(M_j\lambda_j x-\omega t)]. \tag{12}$$

Substitution of (12) in (8) yields the Helmhotz equation

$$\frac{\partial^2 w_j}{\partial x^2} + \frac{\partial^2 w_j}{\partial y_j^2} + \lambda_j^2 w_j = 0 \quad (j=1,2). \tag{13}$$

Applying Fourier transform, (13) can be solved and the result is

$$w_1(x,y_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_1(\xi)\exp\left\{-i\xi x-(\xi^2-\lambda_1^2)^{1/2}y_1\right\} d\xi, \quad (y_1>0)$$

and

$$w_2(x,y_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_2(\xi)\exp\left\{-i\xi x-(\xi^2-\lambda_2^2)^{1/2}y_2\right\} d\xi, \quad (y_2<0)$$

(14)

where $A_1(\xi)$ and $A_2(\xi)$ are the unknown functions to be determined.

From (12) and (14) we obtain the displacement components due to scattered field as

$$v_1 = \exp[i(M_1\lambda_1 x-\omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty} A_1(u)\exp[-iux-\gamma_1 y_1]du , \quad (y_1>0)$$

and

$$v_2 = \exp[i(M_1\lambda_1 x-\omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty} A_2(u)\exp[-iux+\gamma_2 y_2]du , \quad (y_2<0)$$

(15)

where $\gamma_1 = (u^2-\lambda_1^2)^{1/2}$ and $\gamma_2 = [(u-\beta_2)^2-\lambda_2^2]^{1/2}$. (16)

Therefore, the expressions for the stresses are

$$\left(\tau_{xz}\right)_1 = -i\mu_1\exp[i(M_1\lambda_1 x-\omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty}(u-M_1\lambda_1)A_1(u)\exp[-iux-\gamma_1 y_1]du$$

$$\left(\tau_{xz}\right)_2 = -i\mu_2\exp[i(M_1\lambda_1 x-\omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty}(u-M_1\lambda_1)A_2(u)\exp[-iux+\gamma_2 y_2]du$$

and

$$\left(\tau_{y_1z}\right)_1 = -\mu_1 s_1\exp[i(M_1\lambda_1 x-\omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty} \gamma_1 A_1(u)\exp[-iux-\gamma_1 y_1]du$$

$$\left(\tau_{y_2z}\right)_2 = \mu_2 s_2\exp[i(M_1\lambda_1 x-\omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty} \gamma_2 A_2(u)\exp[-iux+\gamma_2 y_2]du.$$

(17)

The unknown functions $A_1(u)$ and $A_2(u)$ are to be determined from the following boundary conditions at the interface y=0

(i) $\quad v_1(x,0) = v_2(x,0) , \quad x>0$

(ii) $\quad \mu_1 s_1 \frac{\partial v_1}{\partial y_1} = \mu_2 s_2 \frac{\partial v_2}{\partial y_2} , \quad -\infty<x<\infty$

and

$$\text{(iii)} \qquad \frac{\partial v_1^o}{\partial y_1} + \frac{\partial v_1^r}{\partial y_1} + \frac{\partial v_1}{\partial y_1} = 0 , \quad x<0, \quad y \to 0+ .$$

From the boundary condition (ii) we obtain

$$-\mu_1 s_1 \gamma_1 A_1(u) + \mu_2 s_2 \gamma_2 A_2(u) = 0 \tag{18}$$

and from other two boundary conditions, we get

$$\int_{-\infty}^{\infty} B_1(u)\exp(-iux)du = 0 \quad (x>0)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} M(u)B_1(u)\exp(-iux)du = N \exp(-i\lambda_1 x\cos\phi_1) , \quad (x<0) \tag{19}$$

where

$$B_1(u) = \frac{\mu_1 s_1 \gamma_1 + \mu_2 s_2 \gamma_2}{\mu_2 s_2 \gamma_2} A_1(u)$$

$$M(u) = \gamma_1 \frac{\mu_2 s_2 \gamma_2}{(\mu_1 s_1 \gamma_1 + \mu_2 s_2 \gamma_2)} \tag{20}$$

and

$$N = -\frac{i\Lambda_1 v_1 \sin\Theta_1}{s_1} (1-A_1^R).$$

The solution of the dual integral equation may be obtained by a method based on the Wiener-Hopf technique. The first part of (19) can be satisfied if we choose

$$B_1(u) = L_-(u) \tag{21}$$

where $L_-(u)$ is a function of u, analytic in the lower half of the complex u-plane. The second part is satisfied if we take

$$M(u)B_1(u) = \frac{N}{i(u-\alpha_1)} \frac{U_+(u)}{U_+(\alpha_1)} \tag{22}$$

where $\alpha_1=\lambda_1\cos\phi_1$ and $U_+(u)$ is a function free from zeros and singularities in the upper half of the complex u-plane. Thus (22) is a solution of the second part of (19) can easily be shown by completing the path from $-\infty$ to ∞ by a semi-circle of infinite radius in the upper u-plane and then applying the residue theorem and Jordan's Lemma. The path of integration is chosen to avoid possible branch points and is indented below the pole $u=\alpha_1$.

Eliminiting $B_1(u)$ from (21) and (22) we obtain

$$\frac{L_-(u)}{U_+(u)} = \frac{N}{i(u-\alpha_1)M(u)} \frac{1}{U_+(\alpha_1)} \tag{23}$$

and

$$M(u) = \frac{\mu_2 s_2}{\mu_1 s_1+\mu_2 s_2} (u^2-\lambda_1^2)^{1/2} F(u) \tag{24}$$

where

$$F(u) = \frac{\gamma_2(\mu_1 s_1+\mu_2 s_2)}{(\mu_1 s_1 \gamma_1+\mu_2 s_2 \gamma_2)}$$

and

$$F(u) \to 1 \quad \text{as} \quad |u| \to \infty .$$

The function F(u) can be expressed as the product of two functions such that

$$F(u) = F_+(u)F_-(u) \tag{25}$$

where $F_+(u)$ and $F_-(u)$ are analytic in the upper and lower half of the complex u-plane respectively. The expressions for $F_+(u)$ and $F_-(u)$ have been derived in the appendix.

In view of (25), (24) assumes the form

$$\frac{U_+(u)}{(u+\lambda_1)^{1/2}F_+(u)} = \frac{L_-(u)}{\dfrac{N}{iU_+(\alpha_1)}\,\dfrac{\mu_1 s_1+\mu_2 s_2}{\mu_2 s_2(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)}} \tag{26}$$

where

$$U_+(u) = (u+\lambda_1)^{1/2}\, F_+(u). \tag{27}$$

So

$$L_-(u) = \frac{N}{i(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}\,\frac{\mu_1 s_1+\mu_2 s_2}{\mu_2 s_2(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)} \tag{28}$$

Hence the functions $A_1(u)$ and $A_2(u)$ are

$$A_1(u) = \frac{N}{i(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}\,\frac{\gamma_2(\mu_1 s_1+\mu_2 s_2)}{(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)}$$

and

$$A_2(u) = \frac{-N}{i(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}\,\frac{\mu_1 s_1\gamma_1(\mu_1 s_1+\mu_2 s_2)}{\mu_2 s_2(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)} \tag{29}$$

The singular behaviour of the stress components for the scattered waves at the crack-tip is due to the divergence of the integrals around $x=y_j=0$ in (17). Making use of (29) and asymptotic expressions of the integrands of (17) for large values of u, we obtain near about the crack-tip,

$$\left(\tau_{xz}\right)_1 = \frac{B(1+i)}{s_1}\int_0^\infty u^{-1/2} \exp[-s_1uY](\cos ux-\sin ux)du$$

$$\left(\tau_{xz}\right)_2 = -\frac{B(1+i)}{s_2}\int_0^\infty u^{-1/2} \exp[-s_2u|Y|](\cos ux-\sin ux)du$$

$$\left(\tau_{yz}\right)_1 = -B(1+i)\int_0^\infty u^{-1/2} \exp[-s_1uY](\cos ux+\sin ux)du$$

$$\left(\tau_{yz}\right)_2 = -B(1+i)\int_0^\infty u^{-1/2} \exp[-s_2u|Y|](\cos ux+\sin ux)du \tag{30}$$

where

$$B = -\frac{N\mu_1 s_1}{2\pi(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)} \; ; \quad y_j=s_jY \quad (j=1,2).$$

Using the results

$$\int_0^\infty u^{-1/2} \exp[-s_1uY]\cos ux \, dx = \sqrt{\frac{\pi}{2}}\left[\frac{(s_1^2y^2+x^2)^{1/2}+s_1y}{s_1^2y^2+x^2}\right]^{1/2}$$

$$\int_0^\infty u^{-1/2} \exp[-s_1uY]\sin ux \, dx = \sqrt{\frac{\pi}{2}}\left[\frac{(s_1^2y^2+x^2)^{1/2}-s_1y}{s_1^2y^2+x^2}\right]^{1/2} \tag{31}$$

the stresses near about the crack tip given by (30) can be evaluated. The displacement near about the crack tip can be obtained from the crack tip stresses by integration.

Now introducing the factor $\exp[i(M_1\lambda_1x-\omega t)]$ and taking the real part, the stresses and displacements near about the moving crack-tip are found to be equal to

$$\begin{bmatrix} (\tau_{yz})_j \\ \\ (\tau_{xz})_j \\ \\ v_j \end{bmatrix} = \mathrm{Re}\begin{bmatrix} K_1\left[\dfrac{(s^2Y^2+x^2)^{1/2}+x}{s_j^2Y^2+x^2}\right]^{1/2} \\ \\ (-1)^j\,\dfrac{K_1}{s_1}\left[\dfrac{(s_j^2Y^2+x^2)^{1/2}-x}{s_j^2Y^2+x^2}\right]^{1/2} \\ \\ (-1)^{j+1}\,\dfrac{2K_1}{\mu_j s_j}\left[(x^2+s_j^2Y^2)^{1/2}-x\right]^{1/2} \end{bmatrix}\exp\left[i(M_1\lambda_1 x-\omega t-\frac{\pi}{4})\right] \tag{32}$$

where

$$K_1 = \sqrt{\frac{2}{\pi}}\,\frac{\mu_1\mu_2\Lambda_1\Lambda_2V_1\sin\Theta_1\sin\Theta_2}{(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)(\mu_1\Lambda_1\sin\Theta_1+\mu_2\Lambda_2\sin\Theta_2)}\,. \tag{33}$$

In the case of crack propagation in an isotropic elastic medium using the result $\mu_1=\mu_2$, $\rho_1=\rho_2$ and $F_+(\alpha_1)=1$, we obtain

$$K_1 = (1/\pi)^{1/2}\mu_1\Lambda_1^{1/2}V_1(1-M_1)^{1/2}\sin(\Theta_1/2). \tag{34}$$

Putting $r=(x^2+y^2)^{1/2}$, $\tan\phi=|Y|/x$, the expression of displacements and stresses given by (32) near about the moving crack-tip is found to be equal to

$$v_1 = \frac{2K_1}{\mu_1 s_1}\,r^{1/2}\left\{(1-M_1^2\sin^2\phi)^{1/2} - \cos\phi\right\}^{1/2}\cos(\omega t+\frac{\pi}{4}) + O(r^{3/2})$$

$$v_2 = -\frac{2K_1}{\mu_2 s_2}\,r^{1/2}\left\{(1-M_2^2\sin^2\phi)^{1/2} - \cos\phi\right\}^{1/2}\cos(\omega t+\frac{\pi}{4}) + O(r^{3/2})$$

$$\left[\tau_{yz}\right]_1 = \frac{K_1}{r^{1/2}} \left\{ \frac{(1-M_1^2\sin^2\phi)^{1/2}-\cos\phi}{1-M_1^2\sin^2\phi} \right\}^{1/2} \cos(\omega t+\frac{\pi}{4}) + O(r^{1/2})$$

$$\left[\tau_{yz}\right]_2 = \frac{K_1}{r^{1/2}} \left\{ \frac{(1-M_2^2\sin^2\phi)^{1/2}-\cos\phi}{1-M_2^2\sin^2\phi} \right\}^{1/2} \cos(\omega t+\frac{\pi}{4}) + O(r^{1/2})$$

$$\left[\tau_{xz}\right]_1 = \frac{-K_1}{s_1 r^{1/2}} \left\{ \frac{(1-M_1^2\sin^2\phi)^{1/2}+\cos\phi}{1-M_1^2\sin^2\phi} \right\}^{1/2} \cos(\omega t+\frac{\pi}{4}) + O(r^{1/2})$$

$$\left[\tau_{xz}\right]_2 = \frac{K_1}{s_2 r^{1/2}} \left\{ \frac{(1-M_2^2\sin^2\phi)^{1/2}+\cos\phi}{1-M_2^2\sin^2\phi} \right\}^{1/2} \cos(\omega t+\frac{\pi}{4}) + O(r^{1/2})$$

(35)

Taking the value of K_1 given by (34), the results given by (35) agree with the results of the crack propagation in an isotropic elastic medium as given by Jahanshahi (1967).

When the crack is stationary, the corresponding results of stresses and displacements near about the crack-tip can be derived by making M_1 and M_2 approach zero and are given by

$$\left[\tau_{yz}\right]_1 = K_1^*(2/r)^{1/2} \cos\tfrac{1}{2}\phi \cos(\Omega t+\frac{\pi}{4}) + O(r^{1/2})$$

$$\left[\tau_{yz}\right]_2 = K_1^*(2/r)^{1/2} \cos\tfrac{1}{2}\phi \cos(\Omega t+\frac{\pi}{4}) + O(r^{1/2})$$

$$\left[\tau_{xz}\right]_1 = -K_1^*(2/r)^{1/2} \sin\tfrac{1}{2}\phi \cos(\Omega t+\frac{\pi}{4}) + O(r^{1/2})$$

$$\left[\tau_{xz}\right]_2 = K_1^*(2/r)^{1/2} \sin\tfrac{1}{2}\phi \cos(\Omega t+\frac{\pi}{4}) + O(r^{1/2}) \qquad (36)$$

and

$$v_1 = \frac{2\sqrt{2}K_1^*}{\mu_1} r^{1/2} \sin\tfrac{1}{2}\phi \cos(\Omega t+\tfrac{\pi}{4}) + O(r^{3/2})$$

$$v_2 = -\frac{2\sqrt{2}K_1^*}{\mu_2} r^{1/2} \sin\tfrac{1}{2}\phi \cos(\Omega t+\tfrac{\pi}{4}) + O(r^{3/2}) \qquad (37)$$

where

$$K_1^* = \sqrt{\frac{2}{\pi}} \frac{\mu_1\mu_2\Lambda_1\Lambda_2 V_1 \sin\Theta_1 \sin\Theta_2}{(\Lambda_1\cos\phi_1+\Lambda_1)^{1/2} F_+^*(\Lambda_1\cos\phi_1)(\mu_1\Lambda_1\sin\phi_1+\mu_2\Lambda_2\sin\phi_2)} \qquad (38)$$

and

$$F_+^*(\Lambda_1\cos\phi_1) = \exp\left[\frac{1}{\pi}\int_{\Lambda_1}^{\Lambda_2} \tan^{-1}\left\{\frac{\mu_1(s^2-\Lambda_1^2)^{1/2}}{\mu_2(\Lambda_2^2-s^2)^{1/2}}\right\} \frac{ds}{s+\Lambda_1\cos\phi_1}\right]. \qquad (39)$$

If we put $\mu_1=\mu_2$, $\rho_1=\rho_2$, the corresponding results of the stationary crack in an isotropic elastic medium are found to be

$$\left(\tau_{yz}\right)_{1,2} = V_1 \sin\tfrac{1}{2}\Theta_1 \cos\tfrac{1}{2}\phi \cos(\Omega t+\tfrac{\pi}{4}) \left[\frac{2\Lambda_1\mu_1^2}{\pi r}\right]^{1/2} + O(r^{1/2})$$

$$\left(\tau_{xz}\right)_{1,2} = \mp V_1 \sin\tfrac{1}{2}\Theta_1 \cos\tfrac{1}{2}\phi \cos(\Omega t+\tfrac{\pi}{4}) \left[\frac{2\Lambda_1\mu_1^2}{\pi r}\right]^{1/2} + O(r^{1/2})$$

and

$$v_{1,2} = \pm V_1 \sin\tfrac{1}{2}\Theta_1 \sin\tfrac{1}{2}\phi \cos(\Omega t+\tfrac{\pi}{4}) \left[\frac{8\Lambda_1 r}{\pi}\right]^{1/2} + O(r^{3/2}) \qquad (40)$$

which are same as given by Jahanshahi (1967).

3. RESULTS AND DISCUSSION

K_1 given by (33) is the dynamic stress intensity factor at the moving crack-tip and K_1^* given by (38) is the value of the corresponding quantity when the crack is stationary. The variation of K_1/K_1^* with the values of V/C_2 where V is the crack speed has been depicted graphically for the following two sets of materials.

First set :

Wrought iron	$\rho_1 = 7.8 g/cm^3$,	$\mu_1 = 7.7\times10^{11} dyn/cm^2$
Copper	$\rho_2 = 8.96 g/cm^3$,	$\mu_2 = 4.5\times10^{11} dyn/cm^2$

Second set :

Steel	$\rho_1 = 7.6 g/cm^3$,	$\mu_1 = 8.33\times10^{11} dyn/cm^2$
Aluminium	$\rho_2 = 2.7 g/cm^3$,	$\mu_2 = 2.63\times10^{11} dyn/cm^2$

It is found that in both the cases the stress intensity factor gradually decreases with an increase in the value of V/C_2 and approaches zero as $V/C_2 \to 1$; the decrease in the value of K_1/K_1^* for the second set being more rapid than for the first set. We also find that in both the cases for any fixed value of C_1/C_2, K_1/K_1^* decreases with decrease in the value of Θ.

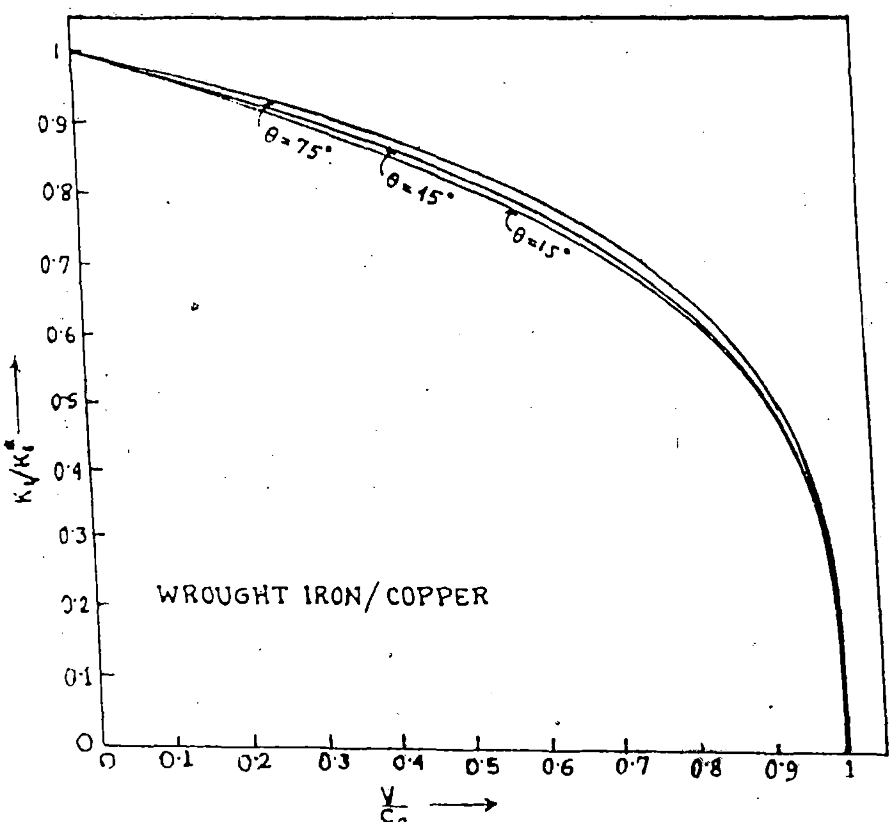

Figure 2. Stress intensity factor vs. dimensionless crack speed (wrought iron/copper).

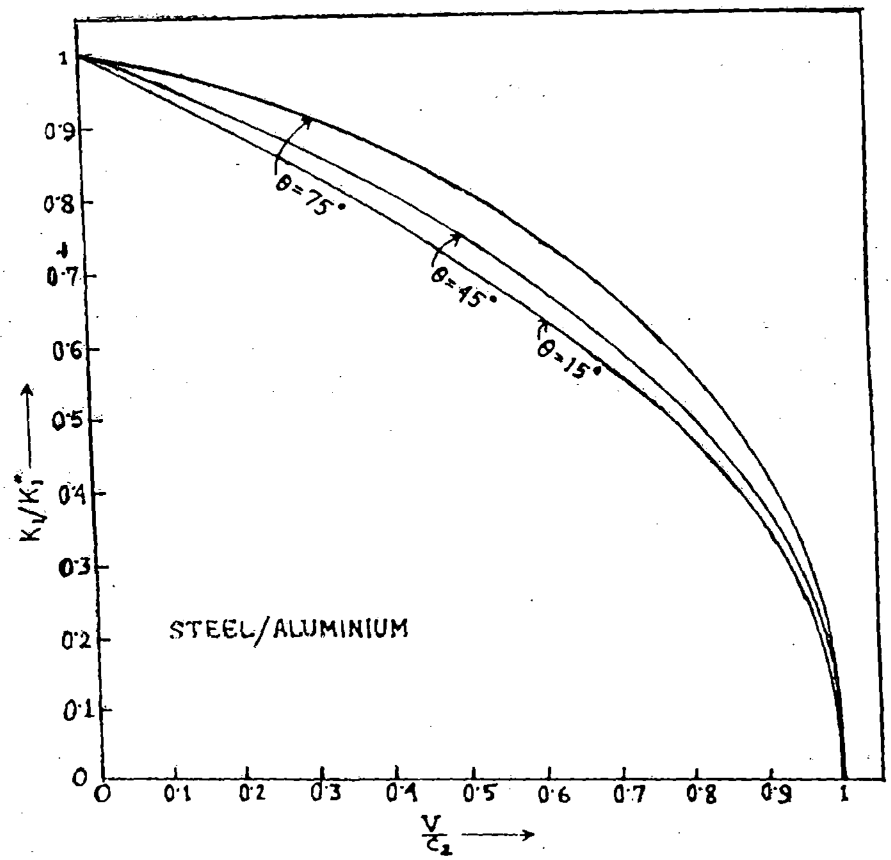

Figure 3. Stress intensity factor vs dimensionless crack speed (steel/aluminium).

APPENDIX

FACTORIZATION OF $F(\xi)$ INTO $F_+(\xi)$ AND $F_-(\xi)$:

Consider

$$F(\xi) = \frac{\gamma_2(\mu_1 s_1+\mu_2 s_2)}{(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)} \tag{A1}$$

The branch points of $F(\xi)$ are at $\xi=\lambda_1,\ -\lambda_1,\ \lambda_2+\beta_2,\ -(\lambda_2-\beta_2)$ where

$$-(\lambda_2-\beta_2)<-\lambda_1<\lambda_1<\lambda_2+\beta_2 \quad \text{since } C_2<C_1.$$

Since $F(\xi)\to1$ as $|\xi|\to\infty$, $F(\xi)$ possesses no singularity within the rectangular contour (shown in fig.4), by Cauchy's residue theorem we can write

$$\log F(\xi) = \frac{1}{2\pi i}\int_{c_++c_-} \frac{\log F(s)}{s-\xi}\,ds \tag{A2}$$

$$= \frac{1}{2\pi i}\int_{c_+} \frac{\log F(s)}{s-\xi}\,ds + \frac{1}{2\pi i}\int_{c_-} \frac{\log F(s)}{s-\xi}\,ds$$

$$\log F(\xi) = \log F_+(\xi) + \log F_-(\xi), \tag{A3}$$

where $F_+(\xi)$ and $F_-(\xi)$ are analytic in the upper and lower half of the complex ξ-plane respectively and can be expressed as

$$F_+(\xi) = \exp\left[\frac{1}{2\pi i}\int_{c_+} \frac{\log F(s)}{s-\xi}\,ds\right]$$

and (A4)

$$F_-(\xi) = \exp\left[\frac{1}{2\pi i}\int_{c_-} \frac{\log F(s)}{s-\xi}\,ds\right]$$

In order to evaluate $F_-(\xi)$ the path of integration C_- can be deformed to the path C_1 round the branch cut through λ_1 and $\lambda_2+\beta_2$ as shown in fig.5.

After a little algebraic manipulation it can be shown that

$$F_-(\xi) = \exp\left[\frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2+\beta_2} \frac{1}{s-\xi}\log\left\{1+i\,\frac{m_1}{m_2}\,\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2-s)^2]^{1/2}}\right\} ds\ -\right.$$

$$\left. -\ \frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2+\beta_2} \frac{1}{s-\xi}\log\left\{1-i\,\frac{m_1}{m_2}\,\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2-s)^2]^{1/2}}\right\} ds\right] \qquad \text{(A5)}$$

which after simplification becomes

$$F_-(\xi) = \exp\left[\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2+\beta_2} \frac{1}{s-\xi}\tan^{-1}\left\{\frac{m_1}{m_2}\,\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2-s)^2]^{1/2}}\right\} ds\right] \qquad \text{(A6)}$$

where

$$m_1 = \frac{\mu_1 s_1}{\mu_1 s_1+\mu_2 s_2} \quad \text{and} \quad m_2 = \frac{\mu_2 s_2}{\mu_1 s_1+\mu_2 s_2}\,. \qquad \text{(A7)}$$

Similarly it can be shown that

$$F_+(\xi) = \exp\left[\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2-\beta_2} \frac{1}{s+\xi}\tan^{-1}\left\{\frac{m_1}{m_2}\,\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2+s)^2]^{1/2}}\right\} ds\right] \qquad \text{(A8)}$$

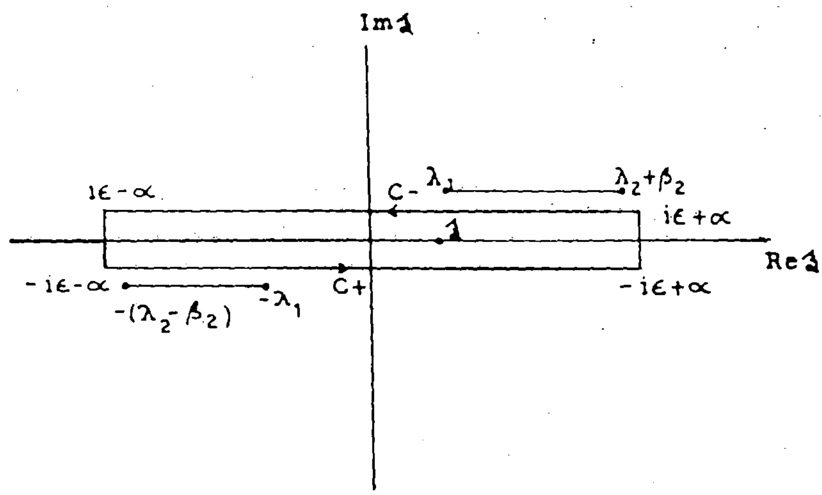

Figure 4. Rectangular contour in the complex ξ-plane.

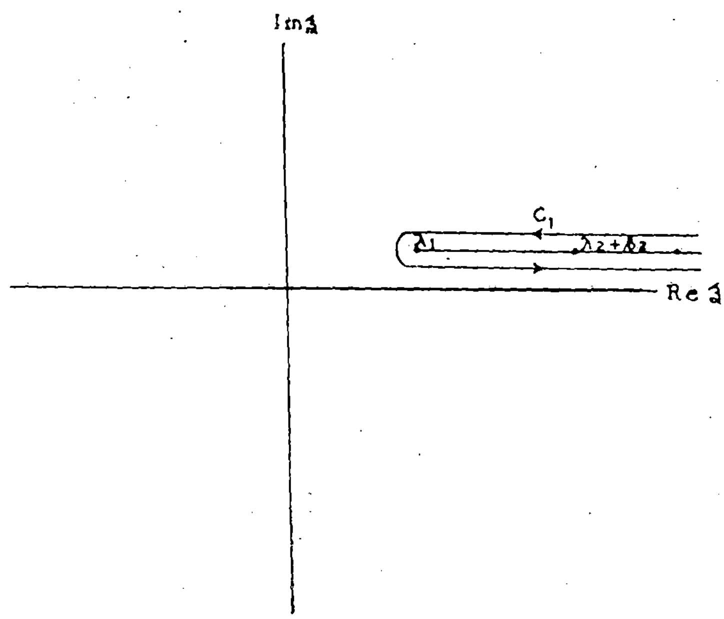

Figure 5. Path of integration C_1 round the branch cut.

CHAPTER - II

SOME ELASTODYNAMIC DIFFRACTION PROBLEMS INVOLVING GRIFFITH
IN ISOTROPIC ELASTIC MEDIUM

PAPER 3 : Diffraction of SH-waves by a griffith crack in no
geneous elastic strip.

PAPER 4 : Inplane problem of diffraction of elastic waves
periodic array of coplanar Griffith cracks.

PAPER 5 ; An elastic strip with three coplanar Griffith crac

PAPER 6 : Four coplanar Griffith cracks moving in an infi
long elastic strip under antiplane shear stress.

DIFFRACTION OF SH-WAVES BY A GRIFFITH CRACK IN NONHOMOGENEOUS ELASTIC STRIP

1. INTRODUCTION

The natural or artificial materials are usually inhomogeneous; so in recent years great attention has been given to the study of diffraction of elastic waves by cracks or obstacles in inhomogeneous media in view of their application in fracture mechanics. Many problems have been solved involving one or more cracks in an infinite homogeneous elastic medium. Loeber and Sih (1960) and Mal (1970b) have studied the problem of diffraction of elastic waves by a Griffith crack in an infinite medium. The problem of finite crack at the interface of two elastic half-spaces has been discussed by Srivastava et al. (1980a) and Bostrom (1987). Singh et al. (1977, 1980) considered the problem of scattering of a SH-wave by cracks or strips in a nonhomogeneous infinite elastic medium. Papers involving cracks located in an infinitely long elastic strip are very few. The problem of an infinite elastic strip containing an arbitrary number of unequal Griffith cracks, located parallel to its surfaces and opened by an arbitrary internal pressure, has been treated by Adams (1980). Finite crack perpendicular to the surface of the infinitely long elastic strip has been studied by Chen (1978) (for an impact load) and by Srivastava et al. (1981) (for normally incident waves).

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Recently Shindo et al. (1986) considered the problem of impact response of a finite crack in an orthotropic strip.

In our paper, the diffraction of normally incident SH-waves by a Griffith crack situated in an infinitely long inhomogeneous elastic strip has been discussed. The shear modulus (μ) and the density (ρ) of the material have been assumed to vary in the vertical direction. Applying the Fourier transform, the mixed boundary value problem has been converted to the solution of dual integral equations. The dual integral equations have been finally reduced to a Fredholm integral equation of second kind by applying the Abel transform. Expressions for the stress intensity factor and crack opening displacement have been derived. The numerical values of stress intensity factor and crack opening displacement have been depicted by means of graphs to show the effect of material inhomogeneity.

2. FORMULATION OF THE PROBLEM

Consider the problem of diffraction of SH-waves by a Griffith crack in an inhomogeneous elastic strip of width $2h_1$. The crack is located in the region $-a\le x_1\le a$, $-\infty<y_1<\infty$, $z_1=0$ (fig.1). Normalizing all the lengths with respect to a and putting $x_1/a=x$, $y_1/a=y$, $z_1/a=z$, $h_1/a=h$ it is found that the location of the crack is $-1\le x\le 1$, $-\infty<y<\infty$, $z=0$ referred to a cartesian co-ordinate system (x,y,z). Let a plane harmonic SH-wave originating at $z=-\infty$ impinge on the crack normally to the x-axis. The variation of the shear

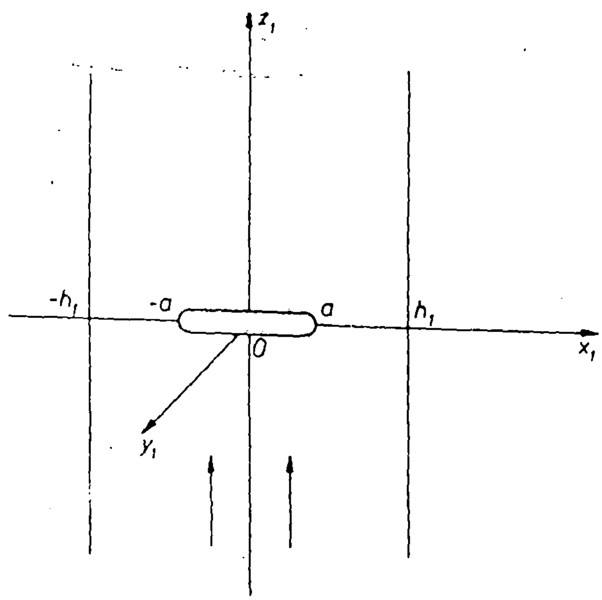

FIG. 1. Crack in the inhomogeneous strip.

modulus μ and the density ρ is taken in the vertical (z) direction in such a manner that the shear velocity $(\mu_o/\rho_o)^{1/2}$ is constant. The only non-vanishing y-component of the displacement which is independent of y is $v = v(x,z,t)$.

The equation of motion is given by

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) = \rho\frac{\partial^2 v}{\partial t^2} \tag{2.1}$$

If we consider $v(x,z,t)$ in the form

$$v(x,z,t) = \frac{W(x,z,t)}{\sqrt{\mu(z)}} \tag{2.2}$$

then (2.1) becomes

$$\mu\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial z^2} \right) + \frac{1}{2}\left[\frac{1}{2\mu}\left(\frac{\partial\mu}{\partial z}\right)^2 - \frac{\partial^2\mu}{\partial z^2} \right]W = \rho\, \frac{\partial^2 W}{\partial t^2} \tag{2.3}$$

Putting $W(x,z,t) = F(x)G(z)e^{-i\omega t}$ and $\mu(z) = \mu_o f(z)$, $\rho(z) = \rho_o f(z)$ in equation (2.3) where μ_o, ρ_o are constants, such that $(\mu_o/\rho_o)^{1/2} = c_2$ is the shear wave velocity, it is found that F(x) and G(z) satisfy the following equations

$$\frac{\partial^2 F}{\partial x^2} + n^2 F = 0 \tag{2.4}$$

$$\frac{\partial^2 G}{\partial z^2} + \left(\frac{a^2\omega^2}{c_2^2} - b^2 - n^2\right)G = 0 \tag{2.5}$$

provided f(z) is of the form

$$-\frac{1}{4}\left(\frac{\partial f}{\partial z}/f\right)^2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial z^2}/f\right) = b^2 \tag{2.6}$$

where n and b are contants.

Let us assume f(z) in the form

$$f(z) = \cosh^2(bz) \tag{2.7}$$

so that equation (2.6) is automatically satisfied.

Now the shear modulus $\mu(z)$ and density of the medium $\rho(z)$ are

$$\mu = \mu_o \cosh^2(bz) , \qquad \rho = \rho_o \cosh^2(bz) \tag{2.8}$$

Using equations (2.8), (2.2) and $W(x,z,t) = W(x,z)e^{-i\omega t}$, equation (2.1) takes the form

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial z^2} + k^2 W = 0 \tag{2.9}$$

where

$$k^2 = (k_z^2 - b^2), \qquad k_z = \frac{a\omega}{c_2} .$$

The displacement component $v^{(i)}(x,z,t)$ and stress $\tau^{(i)}(x,z,t)$ due to incident wave are given by

$$v^{(i)}(x,z,t) = \frac{A_o \; e^{i(kz-\omega t)}}{\sqrt{\mu_o}\; \cosh(bz)} \tag{2.10}$$

and

$$\tau_{yz}^{(i)}(x,z,t) = A_o\sqrt{\mu_o}\; [ik \cosh(bz) - b \sinh(bz)]e^{i(kz-\omega t)}, \tag{2.11}$$

where A_o is a constant.

Henceforth the time factor $e^{-i\omega t}$ will be suppressed in the sequel.

Solution of equation (2.9) is

$$W(x,z) = \int_0^\infty B_1(\xi)e^{-\beta z} \cos(\xi x)d\xi + \int_0^\infty C_1(\zeta)\cosh(\alpha x)\sin(\zeta z)d\zeta , \tag{2.12}$$

where $\alpha = (\zeta^2-k^2)^{1/2}$, $\zeta>k$, $\beta = (\xi^2-k^2)^{1/2}$, $\xi>k$,

$= -i(k^2-\zeta^2)^{1/2}$, $\zeta<k$, $= -i(k^2-\xi^2)^{1/2}$, $\xi<k$.

Now displacement $v(x,z)$ and stresses $\tau_{yz}(x,z)$, $\tau_{xy}(x,z)$ due to scattered field are

$$v(x,z) = \frac{1}{\cosh(bz)}\left[\int_0^\infty B(\xi)e^{-\beta z}\cos(\xi x)d\xi + \int_0^\infty C(\zeta)\cosh(\alpha x)\sin(\zeta z)d\zeta\right] \tag{2.13}$$

$$\tau_{yz}(x,z) = -\mu_o b\sinh(bz)\left[\int_0^\infty B(\xi)e^{-\beta z}\cos(\xi x)d\xi + \right.$$
$$\left. + \int_0^\infty C(\zeta)\cosh(\alpha x)\sin(\zeta z)d\zeta\right] + \mu_o\cosh(bz) \times$$
$$\times\left[-\int_0^\infty \beta B(\xi)e^{-\beta z}\cos(\xi x)d\xi + \int_0^\infty \zeta C(\zeta)\cosh(\alpha x)\cos(\zeta z)d\zeta\right] \tag{2.14}$$

$$\tau_{xy}(x,z) = \mu_o\cosh(bz)\left[-\int_0^\infty \xi B(\xi)e^{-\beta z}\sin(\xi x)d\xi + \right.$$
$$\left. + \int_0^\infty \alpha C(\zeta)\sinh(\alpha x)\sin(\zeta z)d\zeta\right] \tag{2.15}$$

where

$$B(\xi) = \frac{1}{\sqrt{\mu_o}}B_1(\xi), \qquad C(\zeta) = \frac{1}{\sqrt{\mu_o}}C_1(\zeta).$$

The boundary conditions are

$$\tau_{yz}(x,0) = -\tau_o, \qquad |x|\le 1, \tag{2.16}$$

$$v(x,0) = 0, \qquad 1\le|x|\le h, \tag{2.17}$$

$$\tau_{xy}(\pm h,z) = 0, \qquad |z|<\infty, \tag{2.18}$$

where $\tau_o = ikA_o\sqrt{\mu_o}$.

From the boundary condition (2.18) $C(\zeta)$ is found to be expressible in terms of $B(\xi)$ as follows :

$$C(\zeta) = \frac{2\zeta}{\pi\alpha\sinh(\alpha h)}\int_0^\infty \frac{\xi B(\xi)\sin(\xi h)}{\xi^2+\alpha^2}d\xi \tag{2.19}$$

Next, the use of equation (2.19) in the boundary condition (2.16) and (2.17) yields the following dual integral equations from which the unknown function $B(\xi)$ is to be determined :

$$\int_0^\infty \xi[1+M(\xi)]B(\xi)\cos(\xi x)d\xi = p(x) , \qquad |x|\le 1 \tag{2.20}$$

and

$$\int_0^\infty B(\xi)\cos(\xi x)d\xi = 0 , \qquad 1\le |x| \le h \tag{2.21}$$

where

$$M(\xi) = \left[\frac{\beta}{\xi} - 1 \right], \tag{2.22}$$

$$p(x) = \frac{\tau_o}{\mu_o} + \frac{2}{\pi}\int_0^\infty \frac{\zeta^2\cosh(\alpha x)}{\alpha\sinh(\alpha h)} d\zeta \int_0^\infty \frac{\xi B(\xi)\sin(\xi h)}{\xi^2+\alpha^2} d\xi \tag{2.23}$$

3. METHOD OF SOLUTION

In order to solve the dual integral equations (2.20) and (2.21), $B(\xi)$ is taken in the form

$$B(\xi) = \frac{\tau_o}{\mu_o}\int_0^1 t\phi(t)J_o(\xi t)dt , \tag{3.1}$$

so that equation (2.21) is automatically satisfied.

Substitution of the value of $B(\xi)$ from equation (3.1) in equation (2.20), yields a Fredholm integral equation of second kind

$$\phi(t) + \int_0^1 u\left[L_1(u,t) + L_2(u,t)\right]\phi(u)\, du = 1 , \tag{3.2}$$

where

$$L_1(u,t) = \int_0^\infty \xi M(\xi)J_o(\xi u)J_o(\xi t)\, d\xi , \tag{3.3}$$

$$L_2(u,t) = -\int_0^\infty \frac{\zeta^2 I_0(\alpha t) I_0(\alpha u) e^{-\alpha h}}{\alpha \sinh(\alpha h)} d\zeta . \qquad (3.4)$$

Using contour integration technique (Srivastava et al., 1980a) the infinite integral arising in the kernel $L_1(u,t)$ can be converted to a finite integral and is given by

$$L_1(u,t) = -ik^2 \int_0^1 (1-\eta^2)^{1/2} J_0(k\eta t) H_0^{(1)}(k\eta u)\, d\eta, \quad u>t,$$

$$= -ik^2 \int_0^1 (1-\eta^2)^{1/2} J_0(k\eta u) H_0^{(1)}(k\eta t)\, d\eta, \quad u<t \qquad (3.5)$$

Now

$$L_2(u,t) = \int_0^k \frac{\zeta^2 J_0(\alpha_1 t) J_0(\alpha_1 u)\, e^{i\alpha_1 h}}{\alpha_1 \sin(\alpha_1 h)} d\zeta - \int_k^\infty \frac{\zeta^2 I_0(\alpha t) I_0(\alpha u) e^{-\alpha h}}{\alpha \sinh(\alpha h)} d\zeta$$

$$= \int_0^k \frac{\zeta^2}{\alpha_1} J_0(\alpha_1 t) J_0(\alpha_1 u) \cot(\alpha_1 h) d\zeta + i\int_0^k \frac{\zeta^2}{\alpha_1} J_0(\alpha_1 t) J_0(\alpha_1 u) d\zeta$$

$$- \int_k^\infty \frac{\zeta^2 I_0(\alpha t) I_0(\alpha u) e^{-\alpha h}}{\alpha \sinh(\alpha h)} d\zeta ,$$

where $\alpha_1 = (k^2-\zeta^2)^{1/2}$.

Putting $\zeta^2 = k^2(1-y^2)$ in the first and second integrals and $\zeta^2 = k^2(1+y^2)$ in the third integral, it is found that

$$L_2(u,t) = k^2 \Big[\int_0^1 (1-y^2)^{1/2} J_0(kyt) J_0(kyu) \cot(kyh) dy +$$

$$+ i\int_0^1 (1-y^2)^{1/2} J_0(kyt) J_0(kyu)dy -$$

$$- \int_0^\infty (1+y^2)^{1/2} I_0(kyt) I_0(kyu) e^{-kyh} \operatorname{cosech}(kyh)dy\Big] \qquad (3.6)$$

4. STRESS INTENSITY FACTOR AND CRACK OPENING DISPLACEMENT

From equation (2.14) the stress τ_{yz} on the plane z=0 can be written as

$$\tau_{yz}(x,0) = \mu_0\left[-\int_0^\infty \beta B(\xi)\cos(\xi x)d\xi + \int_0^\infty \zeta C(\zeta)\cosh(\alpha x)d\zeta \right]. \qquad (4.1)$$

Substituting the value of $C(\zeta)$ and $B(\xi)$ from equations (2.19) and (3.1), the expression for the stress can finally be presented as

$$\tau_{yz}(x,0) = \frac{\tau_0 x}{(x^2-1)^{1/2}}\phi(1) + O(1), \qquad |x|>1.$$

Defining the stress intensity factor N by

$$N = \lim_{x\to 1+}\left|\frac{(x-1)^{1/2}\tau_{yz}(x,0)}{\tau_0}\right|,$$

we obtain

$$N = \frac{1}{\sqrt{2}}\,|\phi(1)|. \qquad (4.2)$$

Now the crack opening displacement $\Delta v(x,0) = v(x,0+) - v(x,0-)$ can be obtained from equation (2.13) as

$$\Delta v(x,0) = 2\int_0^\infty B(\xi)\cos(\xi x)\, d\xi, \qquad |x|\le 1,$$

which, on substitution of the value of $B(\xi)$ from equation (3.1), takes the form

$$\Delta v(x,0) = \frac{2\tau_0}{\mu_0}\int_x^1 \frac{t\phi(t)}{(t^2-x^2)^{1/2}}\,dt\ , \qquad |x|\le 1 \tag{4.3}$$

5. NUMERICAL RESULTS AND DISCUSSION

Using the method of Fox and Goodwin (1953), the Fredholm integral equation given by equation (3.2) has been solved numerically for different values of the material inhomogeneity parameters. In this method the integral in equation (3.2) has been represented at first by a quadrature formula involving the values of the desired function $\phi(t)$ at the pivotal points inside the specified range of integration, and then converted to a set of simultaneous linear algebraic equations; their solutions yield the first approximations to the required pivotal values of $\phi(t)$. Applying the difference-correction technique, the first approximations have been improved. After solving the integral equation (3.2) numerically, the stress intensity factor N and the crack opening displacement $\mu_0\Delta v(x,0)/\tau_0$ have been calculated numerically and plotted separately against the dimensional frequency $k_2(0.5\le k_2\le 1)$ and dimensionless distance $x(0\le x\le 1)$, respectively, for different values of the material inhomogeneity parameter b and strip width 2h.

In fig.2, the effect of the width of the strip on the stress

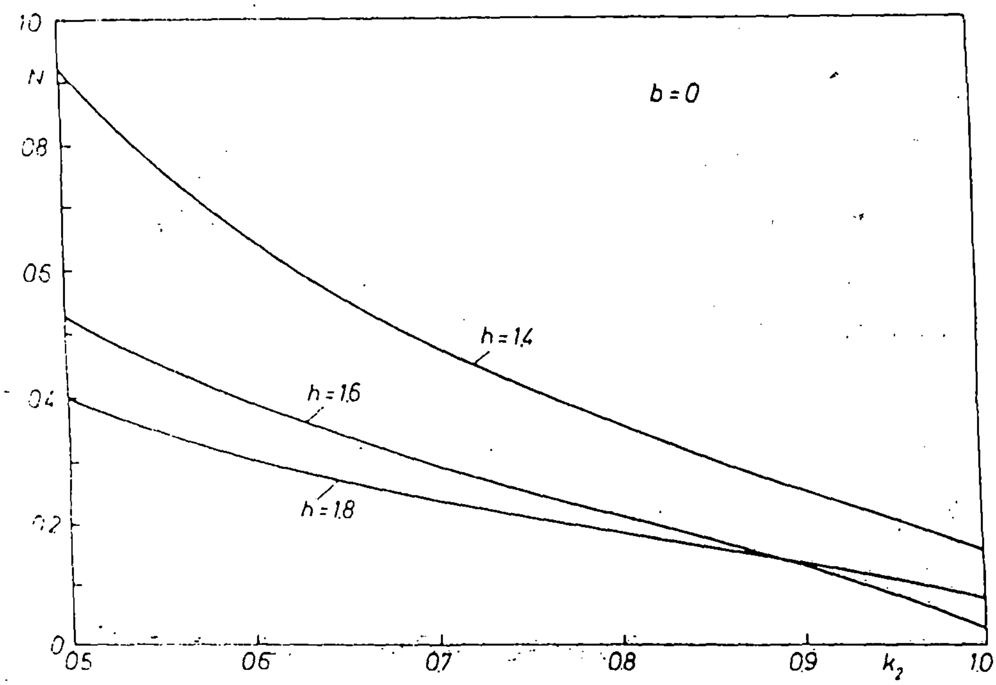

Fig. 2. Stress intensity factor N vs. dimensionless frequency k_2 for homogeneous medium ($b = 0$).

intensity factor for a homogeneous material has been shown; the effect of inhomogeneity of the material on the stress intensity factor for different widths of the strip has been depicted in figs.3-5.

It is found that in both the homogeneous and nonhomogeneous cases, the effect of the strip width decreases with the increase of the frequency, and the graphs of the stress intensity factor N become flat with the increase of strip width 2h. From fig.3 it is clear that the effect of inhomogeneity parameter b is prominent for low frequency k_2 and stress intensity factor is greater for higher values of the inhomogeneity parameter b.

In figs.4-8 the crack opening displacements against dimensionless distance x for different values of the material inhomogeneity parameter b and the strip width 2h have been illustrated by means of graphs. Case b=0 corresponds to the homogeneous case(fig.4). From figs.4-6 it is seen that for a fixed value of inhomogeneity parameter b, the crack opening displacement is greater for lower values of h when the frequencies are small, but the reverse effect is found for higher frequencies.

Next, in figs.7-8 we see that for a fixed value of h, the crack opening displacement is greater for higher values of the inhomogeneity parameter b when the frequencies are small, but for higher frequencies the effect is just reverse.

Finally it is found in all cases that the crack opening displacement reaches its maximum at about x=0, and then it gradually decreases and becomes zero at x=1.

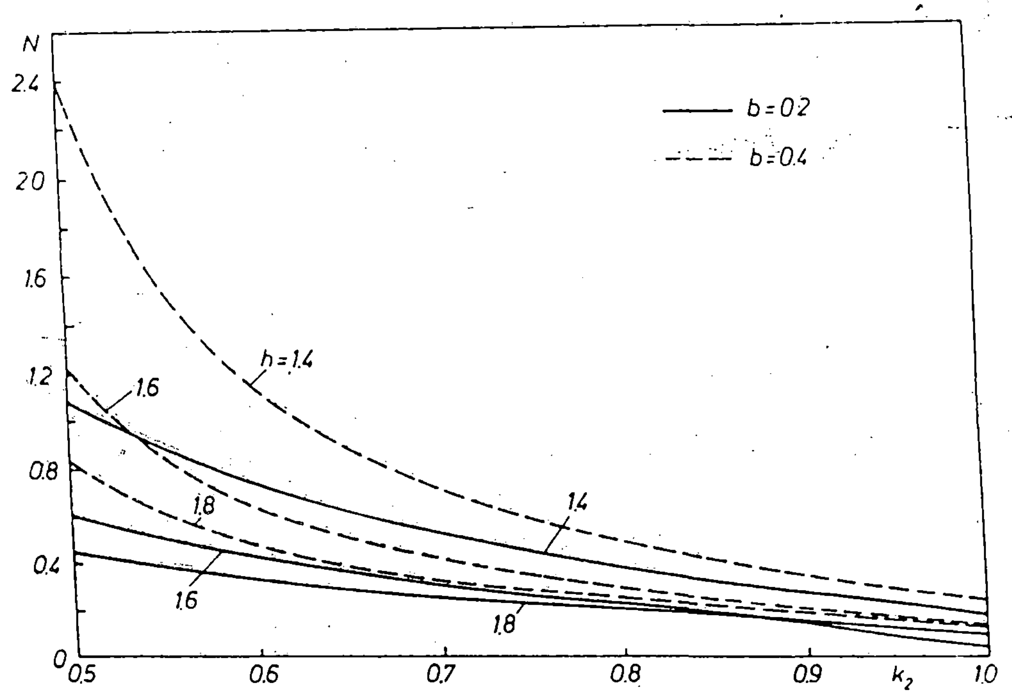

FIG. 3. Stress intensity factor N vs. dimensionless frequency k_2.

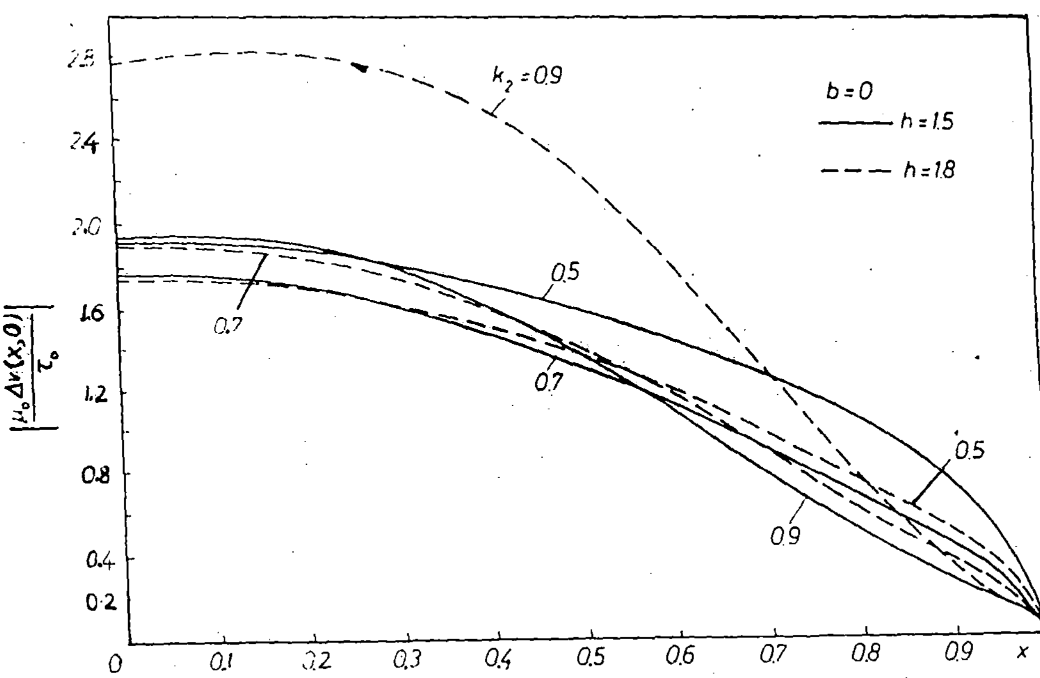

FIG. 4. Crack opening displacement vs. dimensionless distance $x(b = 0)$.

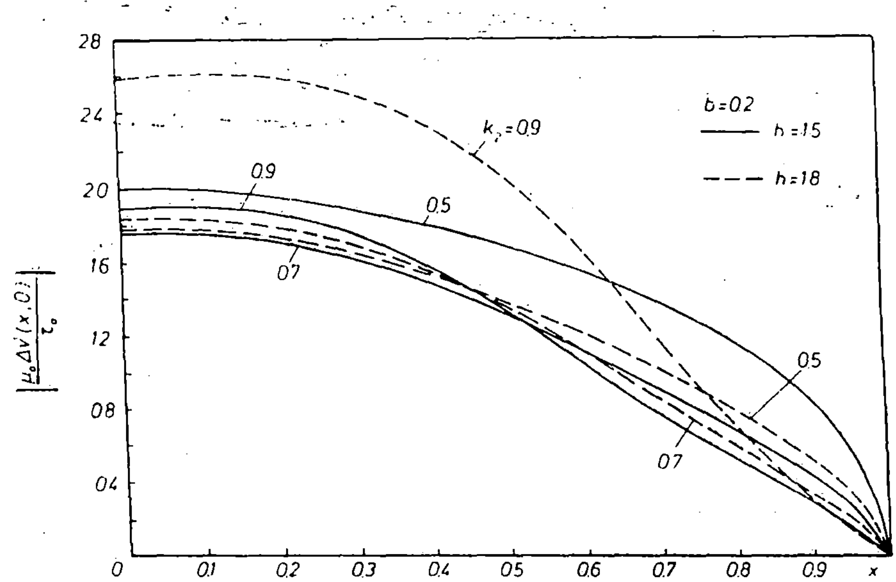

FIG 5. Crack opening displacement vs. dimensionless distance $x(b = 0.2)$.

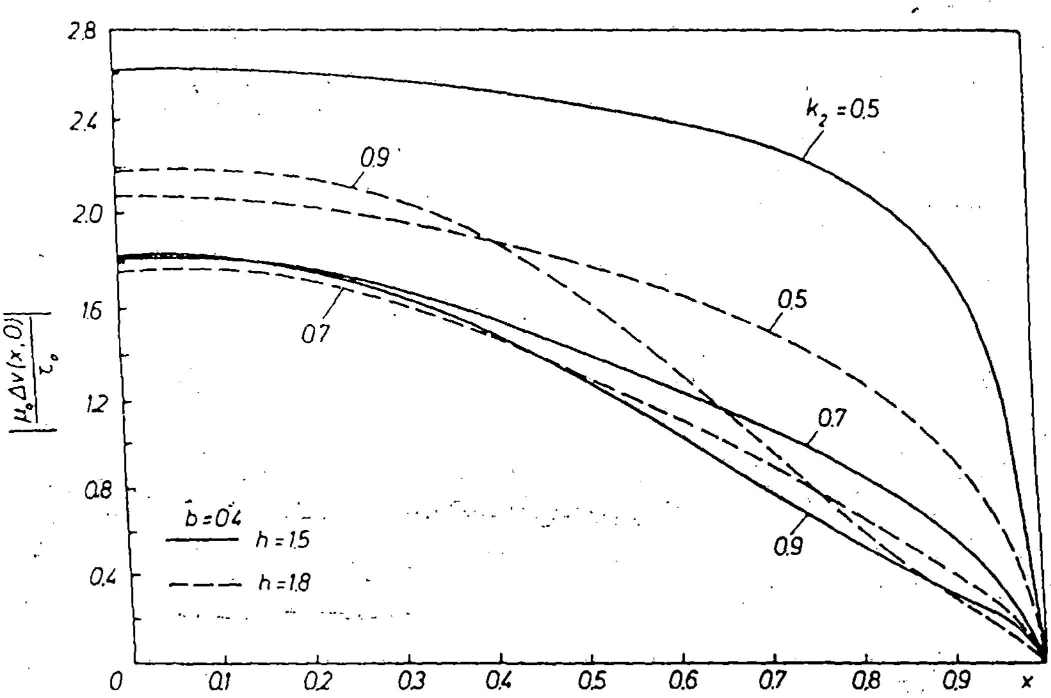

FIG. 6. Crack opening displacement vs. dimensionless distance $x(b = 0.4)$.

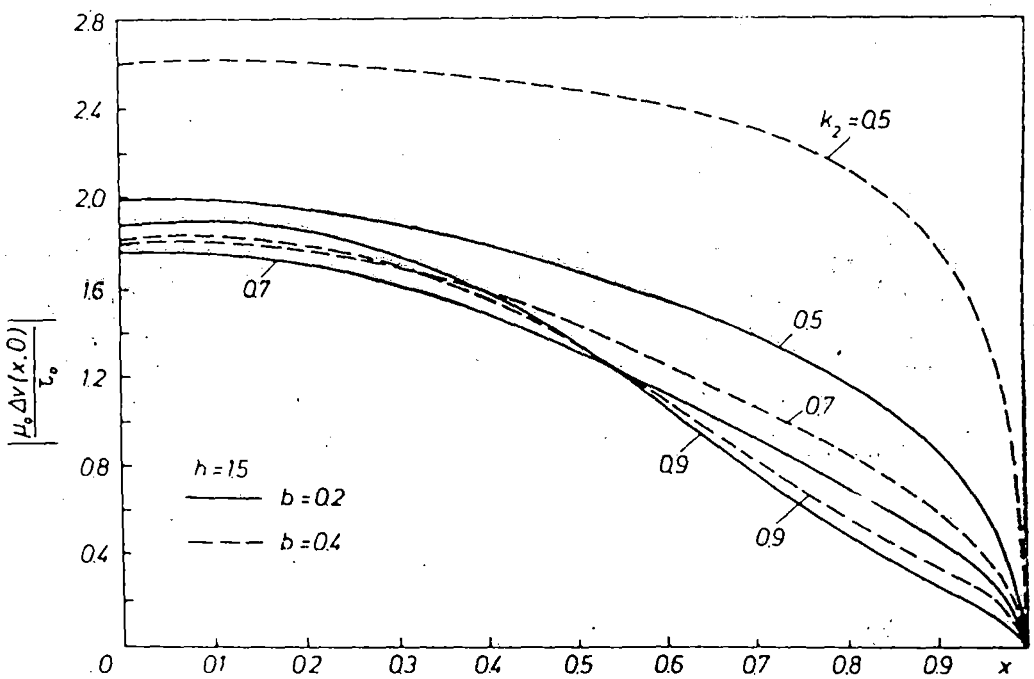

FIG. 7. Crack opening displacement vs. dimensionless distance x (h = 1.5).

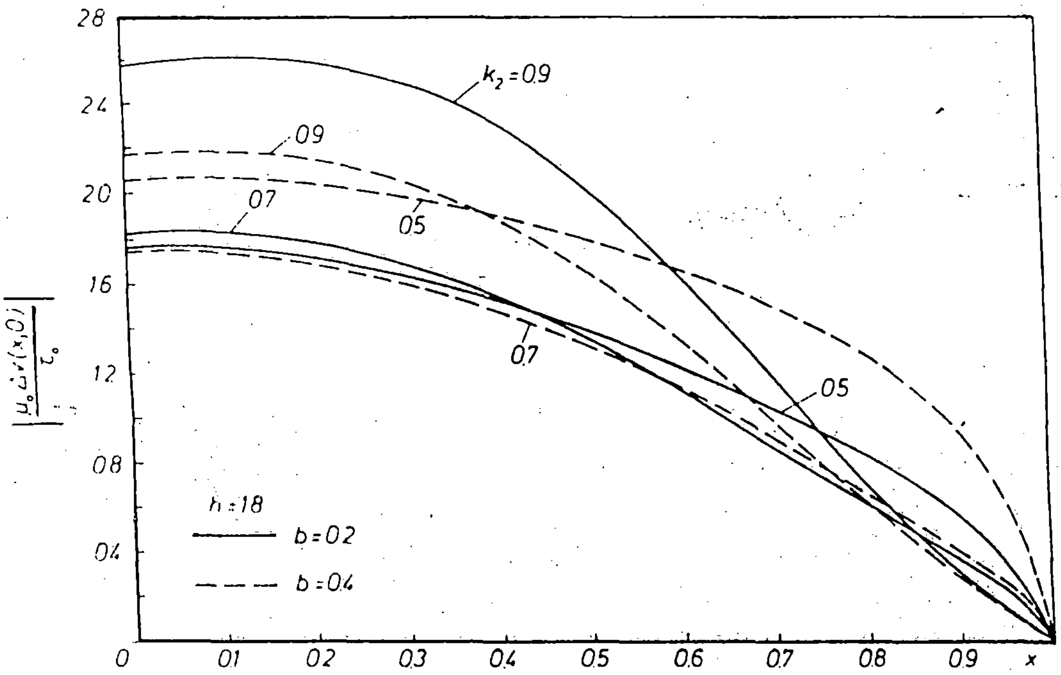

FIG. 8. Crack opening displacement vs. dimensionless distance x(h = 1.8).

INPLANE PROBLEM OF DIFFRACTION OF ELASTIC WAVES BY A PERIODIC ARRAY OF COPLANAR GRIFFITH CRACKS

1. INTRODUCTION

The problems involving cracks or inclusions in elastodynamics are of much importance in view of their application in geophysics and earthquake engineering. Uptil now many problems have been solved involving one or two cracks in an infinite homogeneous elastic medium. Loeber and Sih (1968) and Mal (1970b) have studied the problem of diffraction of elastic waves by a Griffith crack in an infinite medium. The problem of a finite crack at the interface of two elastic half spaces has been discussed by Srivastava et al (1980a) and Bostrom (1987). Finite crack perpendicular to the surface of the infinitely long elastic strip has been studied by Chen (1978) for impact load and by Srivastava et al. (1981) for normally incident waves. But elastodynamic problems involving two or more Griffith cracks have not yet received much attention. Jain and Kanwal (1972a) have studied the problem of scattering of elastic waves by two Griffith cracks for normally incident waves and the same problem has been considered by Itou (1980b) for impact load. Angel and Achenbach (1985) have studied the problem of reflection and transmission of elastic waves by a periodic array of cracks in an infinite isotropic medium. The problem of diffraction of

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SH-waves by a series of cuts in nonhomogeneous solid was investigated by De Sarkar (1983). The steady state vibration of an infinite isotropic medium with a periodic system of coplanar cracks has been discussed by Parton and Morozov (1978) using the method of the finite Fourier transforms to reduce the relevant mixed relations.

In our paper, the diffraction of normally incident time harmonic elastic waves by a periodic array of coplanar Griffith cracks in infinite elastic medium has been analyzed. Due to geometrical symmetry the problem has been reduced to the solution of the problem of a single crack in a strip whose boundaries are shear free and constrained in a way not to permit normal displacement. Applying Fourier transform the problem has been converted to the solution of dual integral equations. The dual integral equations finally have been reduced to a Fredholm integral equation of second kind by applying Abel's transform. Expressions for stress intensity factor and crack opening displacement have been derived in closed form. The numerical values of stress intensity factor and crack opening displacement have been presented graphically to bring out the salient features of the problem.

2. FORMULATION OF THE PROBLEM

We consider a homogeneous, isotropic, linearly elastic, unbounded solid weakened by a infinite number of collinear cracks of equal length which are equally spaced on a line taken as the x_1-axis.

The length of each crack is 2a and the period of the crack-array is $2h_1$ as shown in fig.1. The cracks lie in the plane $x_2=0$ and extend to infinity in the x_3-direction which is perpendicular to the plane of the figure.

For convenience we make all the lengths dimensionless by writting

$$x_1/a=x, \quad x_2/a=y, \quad x_3/a=z, \quad h_1/a=h.$$

Let an incident time-harmonic body wave travel in the direction of the positive y-axis. The steady state term $e^{-i\omega t}$, which is common to all field variables, has been omitted in the sequel.

By simple symmetry considerations, the displacement and stress distribution due to the scattered field in the entire xy-plane can be derived by considering only the isotropic elastic strip $|x|\le h$ with a central crack $|x|\le 1$, $y=0$; the boundaries of the strip $x=\pm h$ being shear free and constrained in a way not to permit normal displacement.

The displacement components are

$$u = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y}$$

and

$$v = \frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial x} \tag{1}$$

where ϕ and ψ are scalar and vector potentials satisfying the following equations :

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{a^2}{c_1^2}\frac{\partial^2\phi}{\partial t^2}$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{a^2}{c_2^2}\frac{\partial^2\psi}{\partial t^2} \tag{2}$$

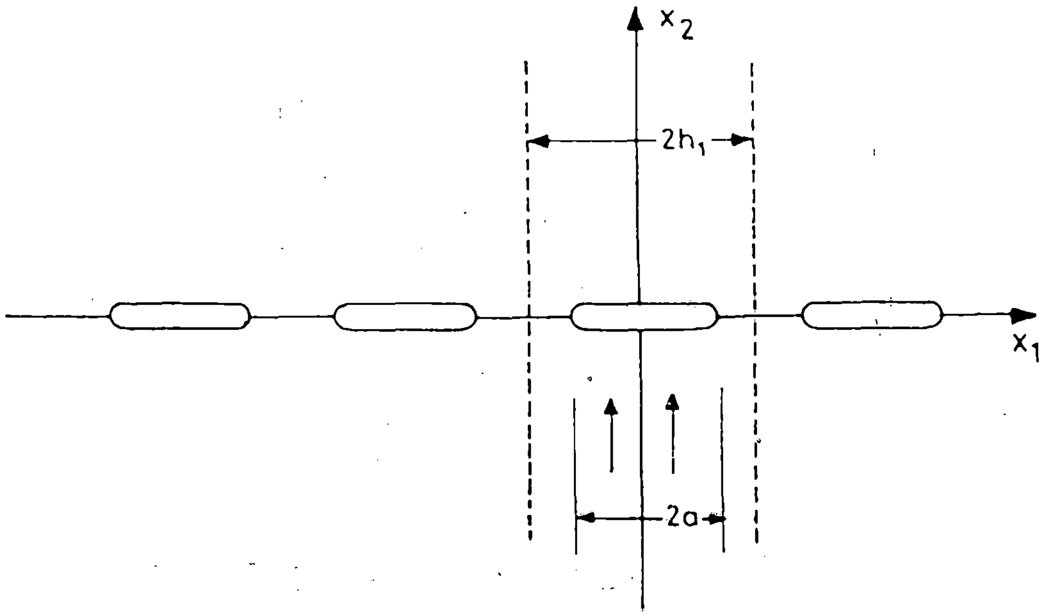

Fig. 1 Incidence of plane time-harmonic wave on a periodic array of cracks.

where $c_1 = \left(\frac{\lambda+2\mu}{\rho}\right)^{1/2}$ and $c_2 = \left(\frac{\mu}{\rho}\right)^{1/2}$ are the dilatational and shear wave velocities, λ, μ are the Lame's constant, ρ is the density of the material.

Therefore, substituting $\phi(x,y,t) = \phi(x,y)e^{-i\omega t}$ and $\psi(x,y,t) = \psi(x,y)e^{-i\omega t}$, our problem reduces to the solution of the equations

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + k_1^2\phi = 0$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + k_2^2\psi = 0 \tag{3}$$

subject to the boundary conditions

$$\tau_{yy}(x,0) = -p(x) \quad , \quad |x|<1 \tag{4}$$

$$\tau_{xy}(x,0) = 0 \quad , \quad |x|\le h \tag{5}$$

$$v(x,0) = 0 \quad , \quad 1\le|x|\le h \tag{6}$$

$$\tau_{xy}(\pm h,y) = 0 \quad , \quad |y|<\infty \tag{7}$$

$$u(\pm h,y) = 0 \quad , \quad |y|<\infty \tag{8}$$

where $k_i = a\omega/c_i$ $(i=1,2)$.

Solutions of the equations (3) are

$$\phi(x,y) = \sqrt{\frac{2}{\pi}}\left[\int_0^\infty A_1(\zeta)e^{-\alpha y}\cos\zeta x\, d\zeta + \int_0^\infty A_2(\xi)\cosh(\alpha_1 x)\cos\xi y\, d\xi\right]$$

and (9)

$$\psi(x,y) = \sqrt{\frac{2}{\pi}}\left[\int_0^\infty B_1(\zeta)e^{-\beta y}\sin\zeta x\, d\zeta + \int_0^\infty B_2(\xi)\sinh(\beta_1 x)\sin\xi y\, d\xi\right]$$

where $A_1(\zeta)$, $A_2(\xi)$, $B_1(\zeta)$, $B_2(\xi)$ are constants and

$$\alpha = (\zeta^2 - k_1^2)^{1/2} \quad , \quad \zeta > k_1 \qquad\qquad \beta = (\zeta^2 - k_2^2)^{1/2} \quad , \quad \zeta > k_2$$

$$= -i(k_1^2 - \zeta^2)^{1/2} \quad , \quad \zeta < k_1 \qquad\qquad = -i(k_2^2 - \zeta^2)^{1/2} \quad , \quad \zeta < k_2$$

$$\alpha_1 = (\xi^2 - k_1^2)^{1/2} \quad , \quad \xi > k_1 \qquad\qquad \beta_1 = (\xi^2 - k_2^2)^{1/2} \quad , \quad \xi > k_2$$

$$= -i(k_1^2 - \xi^2)^{1/2} \quad , \quad \xi < k_1 \qquad\qquad = -i(k_2^2 - \xi^2)^{1/2} \quad , \quad \xi < k_2 .$$

Now the stress τ_{xy} can be expressed as

$$a\tau_{xy}(x,y) = \sqrt{\frac{2}{\pi}}\left[-\mu\int_0^\infty \left\{-2\zeta\alpha A_1(\zeta)e^{-\alpha y} + (\zeta^2+\beta^2)B_1(\zeta)e^{-\beta y}\right\}\sin\zeta x \; d\zeta \; + \right.$$

$$\left. +\mu\int_0^\infty \left\{-2\xi\alpha_1 A_2(\xi)\sinh(\alpha_1 x) + (\xi^2+\beta_1^2)B_2(\xi)\sinh(\beta_1 x)\right\}\sin\xi y \; d\xi\right] \tag{10}$$

The boundary condition (5) yields

$$B_1(\zeta) = \frac{2\zeta\alpha}{\zeta^2+\beta^2} A_1(\zeta) \tag{11}$$

Assuming $\quad -\zeta A_1(\zeta) = A(\zeta) \quad , \quad \alpha_1 A_2(\xi) = C(\xi) \quad , \quad -\xi B_2(\xi) = D(\xi)$

and using the relation (11), expressions for displacements and stresses finally can be written as

$$u = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[e^{-\alpha y} - \frac{2\alpha\beta}{2\zeta^2 - k_2^2} e^{-\beta y}\right]A(\zeta) \; \sin\zeta x \; d\zeta \; +$$

$$+ \sqrt{\frac{2}{\pi}}\int_0^\infty \left[C(\xi)\sinh(\alpha_1 x) + D(\xi)\sinh(\beta_1 x)\right]\cos\xi y \; d\xi \tag{12}$$

$$v = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[e^{-\alpha y} - \frac{2\zeta^2}{2\zeta^2 - k_2^2} e^{-\beta y}\right]\alpha\zeta^{-1} \; A(\zeta) \; \cos\zeta x \; d\zeta \; -$$

$$- \sqrt{\frac{2}{\pi}} \int_0^\infty \left[\xi \alpha_1^{-1} C(\xi) \cosh(\alpha_1 x) + \beta_1 \xi^{-1} D(\xi) \cosh(\beta_1 x)\right] \sin\xi y \, d\xi \tag{13}$$

$$a\tau_{yy} = -\mu\sqrt{\frac{2}{\pi}} \int_0^\infty \left[(2\zeta^2 - k_2^2)e^{-\alpha y} - \frac{4\alpha\beta\zeta^2}{2\zeta^2 - k_2^2} e^{-\beta y}\right]\zeta^{-1} A(\zeta) \cos\zeta x \, d\zeta \; -$$

$$- \mu\sqrt{\frac{2}{\pi}}\int_0^\infty \left[\frac{(2\alpha_1^2 + k_2^2)}{\alpha_1} C(\xi)\cosh(\alpha_1 x) + 2\beta_1 D(\xi)\cosh(\beta_1 x)\right]\cos\xi y \, d\xi \tag{14}$$

$$a\tau_{xy} = -\mu\sqrt{\frac{2}{\pi}} \int_0^\infty \left[e^{-\alpha y} - e^{-\beta y} \right] 2\alpha A(\zeta) \sin\zeta x \, d\zeta \; -$$

$$- \mu\sqrt{\frac{2}{\pi}}\int_0^\infty \left[2\xi C(\xi)\sinh(\alpha_1 x) + \xi^{-1}(2\xi^2 - k_2^2) D(\xi)\sinh(\beta_1 x)\right]\sin\xi y \, d\xi \tag{15}$$

3. SOLUTION OF THE PROBLEM

The boundary conditions (4) and (6) yield the following two integral equations :

$$\int_0^\infty \frac{1}{\zeta}\left[1 + H(\zeta)\right] B(\zeta) \sin\zeta x \, d\zeta = R(x) \quad , \quad 0 \le |x| \le 1 \tag{16}$$

$$\int_0^\infty \frac{1}{\zeta} B(\zeta) \cos\zeta x \, d\zeta = 0 \quad , \quad 1 \le |x| \le h \tag{17}$$

where,

$$B(\zeta) = \frac{2\alpha(k_1^2 - k_2^2) A(\zeta)}{2\zeta^2 - k_2^2} \tag{18}$$

$$H(\zeta) = \frac{(2\zeta^2 - k_2^2)^2 - 4\alpha\beta\zeta^2}{2\alpha\zeta(k_1^2 - k_2^2)} - 1 \tag{19}$$

$$H(\zeta) \to 0 \quad \text{as} \quad \zeta \to \infty.$$

$$R(x) = \sqrt{\frac{2}{\pi}}\,\mu^{-1} a \int_0^x p(x)dx -$$

$$- \int_0^\infty \left[\frac{(2\alpha_1^2 + k_2^2)}{\alpha_1^2} C(\xi)\sinh(\alpha_1 x) + 2D(\xi)\sinh(\beta_1 x) \right] d\xi \tag{20}$$

Let us consider the solution of integral equations (16) and (17) in the form

$$B(\zeta) = \sqrt{\frac{\pi}{2}}\,\zeta \int_0^1 t f(t) J_0(\zeta t)dt \tag{21}$$

so that the integral equation (17) is automatically satiesfied. Now, substituting the value of $B(\zeta)$ from (21) in (16) and using Abel's transform we obtain the following Fredholm integral equation of second kind :

$$f(t) + \int_0^1 u f(u) L_1(t,u)\, du = Q(t) \tag{22}$$

where,

$$Q(t) = \frac{2a}{\mu\pi t}\frac{d}{dt}\int_0^t (t^2 - z^2)^{1/2} p(z)dz -$$

$$- \sqrt{\frac{2}{\pi}} \int_0^\infty \left[\alpha_1^{-1}(2\alpha_1^2 + k_2^2) I_0(\alpha_1 t) C(\xi) + 2\beta_1 I_0(\beta_1 t) D(\xi) \right] d\xi \tag{23}$$

and

$$L_1(t,u) = \int_0^\infty \zeta H(\zeta) J_o(\zeta u) J_o(\zeta t) d\zeta \qquad (24)$$

From the boundary conditions (7) and (8), the unknown functions $C(\xi)$ and $D(\xi)$ can be found to be related to $B(\zeta)$ as :

$$C(\xi) = \frac{2}{\pi k_2^2 (k_1^2 - k_2^2) \sinh(\alpha_1 h)} \left[-\xi^2 \int_0^\infty g_1(\xi,\zeta) B(\zeta) d\zeta + \right.$$

$$\left. + \frac{(2\xi^2 - k_2^2)}{2} \int_0^\infty g_2(\xi,\zeta) B(\zeta) d\zeta \right] \qquad (25)$$

$$D(\xi) = \frac{2}{\pi k_2^2 (k_1^2 - k_2^2) \sinh(\beta_1 h)} \left[\xi^2 \int_0^\infty g_1(\xi,\zeta) B(\zeta) d\zeta - \xi^2 \int_0^\infty g_2(\xi,\zeta) B(\zeta) d\zeta \right] \qquad (26)$$

where,

$$g_1(\xi,\zeta) = \left\{ \frac{2\beta_1^2 + k_2^2}{\zeta^2 + \beta_1^2} - \frac{2\alpha_1^2 + k_2^2}{\zeta^2 + \alpha_1^2} \right\} \sin(\zeta h)$$

$$(27)$$

$$g_2(\xi,\zeta) = \left\{ \frac{2(\beta_1^2 + k_2^2)}{\zeta^2 + \beta_1^2} - \frac{2\alpha_1^2 + k_2^2}{\zeta^2 + \alpha_1^2} \right\} \sin(\zeta h)$$

Next, substituting the value of $B(\zeta)$ from (21) in the expressions of $C(\xi)$ and $D(\xi)$ given by (25) and (26) and using the result (Gradshteyn et al., 1965)

$$\int_0^\infty \frac{\zeta \sin(\zeta h) J_o(\zeta u)}{\zeta^2 + \alpha_1^2} d\zeta = \frac{\pi}{2} I_o(\alpha_1 u) e^{-\alpha_1 u}$$

$C(\xi)$ and $D(\xi)$ can be written in terms of $f(t)$ as

$$C(\xi) = \sqrt{\frac{\pi}{2}}\,\frac{1}{2(k_1^2-k_2^2)}\int_0^1 \Big[\,(2\alpha_1^2+k_2^2)\,I_0(\alpha_1 u)\,e^{-\alpha_1 h}\Big]\frac{uf(u)du}{\sinh(\alpha_1 h)}$$

$$D(\xi) = -\sqrt{\frac{\pi}{2}}\,\frac{\xi^2}{(k_1^2-k_2^2)}\int_0^1 \Big[\,I_0(\beta_1 u)\,e^{-\beta_1 h}\Big]\frac{uf(u)du}{\sinh(\beta_1 h)} \tag{28}$$

Using the above relations (28) in (23) we obtain

$$Q(t) = \frac{2a}{\mu\pi t}\,\frac{d}{dt}\int_0^t \sqrt{t^2-z^2}\;p(z)dz + \int_0^1 u\big[L_2(t,u)+L_3(t,u)\big]f(u)du \tag{29}$$

where,

$$L_2(t,u) = -\frac{1}{2(k_1^2-k_2^2)}\int_0^\infty\Big[\alpha_1^{-1}(2\alpha_1^2+k_2^2)^2\,I_0(\alpha_1 t)\,I_0(\alpha_1 u)\,e^{-\alpha_1 h}\Big]\frac{d\xi}{\sinh(\alpha_1 h)} \tag{30}$$

$$L_3(t,u) = \frac{2}{(k_1^2-k_2^2)}\int_0^\infty\Big[\beta_1(\beta_1^2+k_2^2)\,I_0(\beta_1 t)\,I_0(\beta_1 u)\,e^{-\beta_1 h}\Big]\frac{d\xi}{\sinh(\beta_1 h)} \tag{31}$$

Next substituting $Q(t)$ from (29) in (22) and assuming $p(x)=p_0$ and $f(t)= ap_0 g(t)/\mu$ we finally obtain the following Fredholm integral equation of second kind for the determination of $g(t)$:

$$g(t) + \int_0^1 ug(u)L(t,u)\,du = 1 \tag{32}$$

where $L(t,u) = L_1(t,u) - L_2(t,u) - L_3(t,u)$ (33)

and $L_1(t,u)$, $L_2(t,u)$ and $L_3(t,u)$ are given by (24),(30) and (31) respectively.

It is to be noted that the kernel $L_1(t,u)$ represented by the semi-infinite integral given by equation (24) has a slow rate of convergence. In order to make the numerical analysis easier, the semi-infinite integral has therefore been converted to finite integrals by using simple contour integration technique (Srivastava et al. 1980a) and is given by

$$L_1(t,u) = -\frac{ik_2^4}{2(k_2-k_1)}\left[\int_0^{\gamma}\frac{(2\eta^2-1)^2}{(\gamma^2-\eta^2)^{1/2}}J_0(k_2\eta u)H_0^{(1)}(k_2\eta t)d\eta + \right.$$

$$\left. + \int_0^1 4\eta^2(1-\eta^2)^{1/2}J_0(k_2\eta u)H_0^{(1)}(k_2\eta t)d\eta\right], \quad t>u \qquad (34)$$

where $\gamma=k_1/k_2$. The corresponding expression of $L_1(t,u)$ for $t<u$ can be obtained by interchanging t and u in (34).

4. STRESS INTENSITY FACTOR AND DISPLACEMENT

The normal stress $\tau_{yy}(x,y)$ in the plane y=0 in the vicinity of the crack tip can be found from equation (14) and is given by

$$\tau_{yy}(x,0) = -\mu\sqrt{\frac{2}{\pi}}\int_0^{\infty}B(\zeta)\cos\zeta x\, d\zeta + O(1), \qquad x>1$$

$$= -\frac{p_0 x}{\sqrt{x^2-1}}g(1) + O(1), \qquad x>1$$

Defining the stress intensity factor by

$$K = \underset{x \to 1+}{Lt} \left| \frac{\tau_{yy}(x,0)\sqrt{x-1}}{p_o} \right|$$

it is found that

$$K = \frac{|g(1)|}{\sqrt{2}} \tag{35}$$

Now the crack opening displacement $\Delta v(x,0) = v(x,0+)-v(x,0-)$ can be obtained from (13) as

$$\Delta v(x,0) = -\frac{k^2}{\sqrt{2\pi}(k_1^2-k_2^2)} \int_0^\infty \frac{1}{\zeta} B(\zeta)\cos(\zeta x)d\zeta \quad , \quad |x| \le 1$$

which on substitution of the value of $B(\zeta)$ from (21) takes the form

$$\Delta v(x,0) = \frac{ap_o}{\mu(1-\gamma^2)} \int_x^1 \frac{tg(t)\ dt}{(t^2-x^2)^{1/2}} \quad , \quad |x| \le 1 \tag{36}$$

5. NUMERICAL RESULTS AND DISCUSSION

Using the method of Fox and Goodwin (1953) the Fredholm integral equation given by equation (32) has been solved numerically for different values of dimensionless frequency k_2 and h, the separating distance of the cracks. At first the integral in (32) has been presented by a quadrature formula involving values of the desired function g(t) at pivotal points inside the specified range of integration and then converted to a set of linear algebraic simultaneous equations, solving which the first approximation to the required pivotal values of g(t) has been obtained. Applying difference-correction technique the first approximations has been improved. Standard numerical integration technique has been used to evaluate the kernals $L_1(t,u)$, $L_2(t,u)$ and $L_3(t,u)$ given by (34), (30) and (31). After solving the integral equation (32) numerically, the stress intensity factor K and the crack opening displacement $\mu\Delta v(x,0)/ap_0$ have been calculated numerically and plotted separately against dimensionless frequency k_2 $(0<k_2\leq 1)$ and dimensionless distance $x(0\leq x\leq 1)$ respectively for different values of h. The value of γ is taken to be $1/\sqrt{3}$. From fig.2 it is interesting to note that the number of oscillations in stress intensity factor K increases with the increase in the values of h. The crack opening displacement(fig.3) is greater for higher values of h and also for higher values of dimensionless frequency k_2.

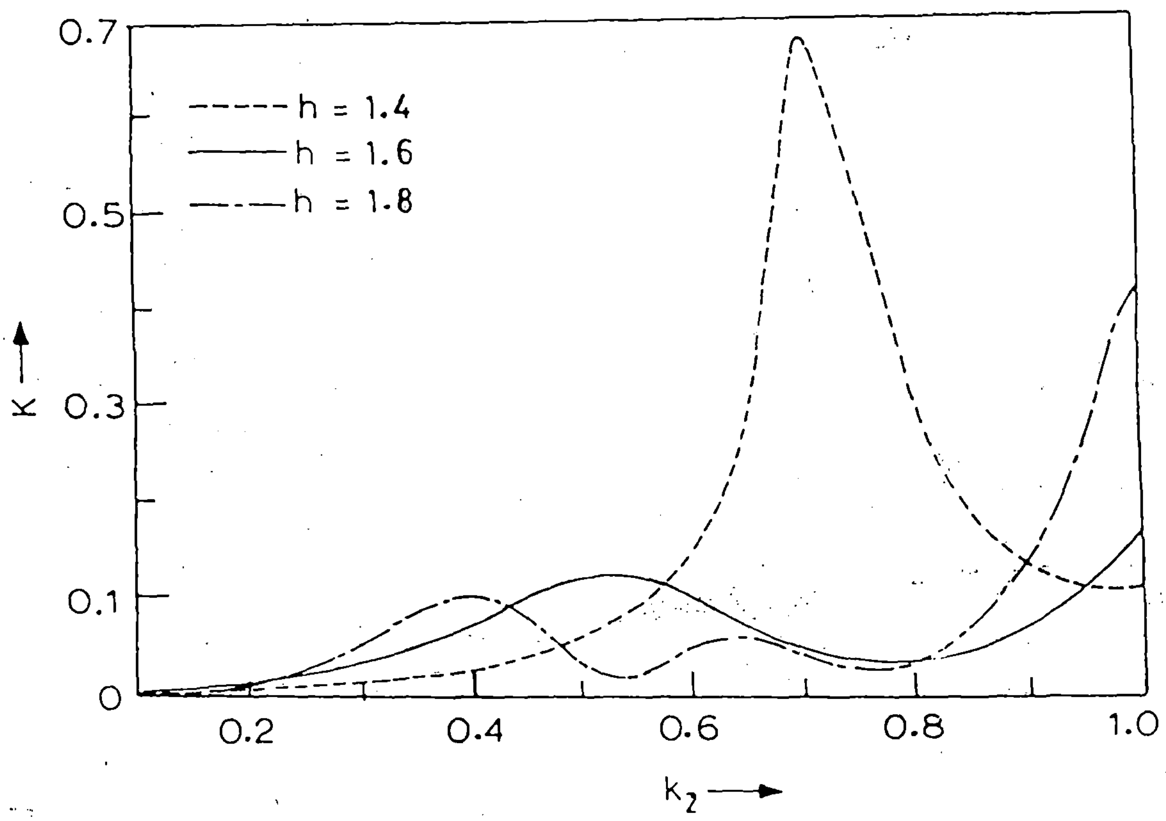

Fig. 2 Stress intensity factor K vs dimensionless frequency k_2

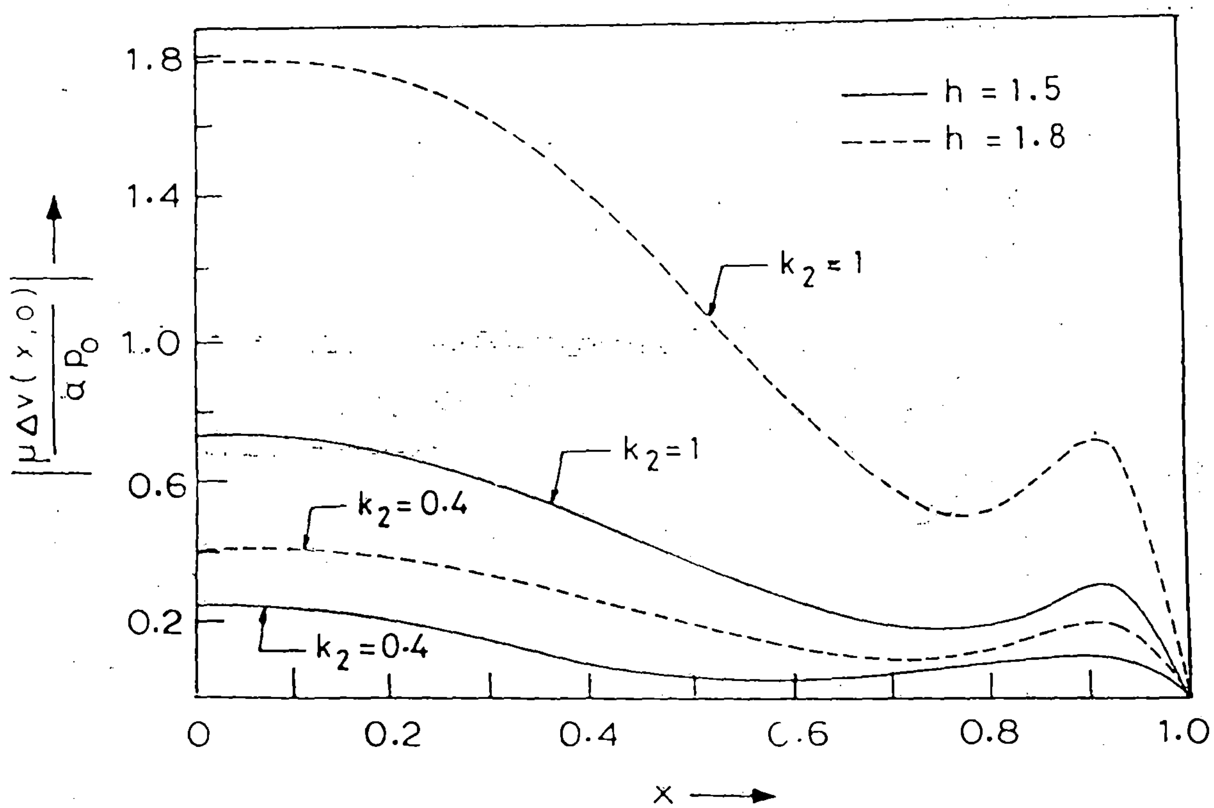

Fig. 3 Crack opening displacement vs distance

AN ELASTIC STRIP WITH THREE CO-PLANAR MOVING GRIFFITH CRACKS

1. INTRODUCTION

In fracture mechanics, the problem of diffraction of elastic waves by cracks of finite dimension in a strip of elastic material has been examined by several investigators. Sih and Chen (1972) investigated the problem of propagation of a crack of finite length in a strip under plane extension. Closed-form solutions for a finite length crack moving in a strip under anti-plane shear stress were obtained by Singh et al. (1981). Using a finite Hilbert transform technique developed by Srivastava and Lowengrub (1968), Lowengrub and Srivastava (1968·b)solved the static problem of distribution of stress and displacement in an infinitely long elastic strip containing two co-planar Griffith cracks. Recently, several dynamic problems of determining stress and displacement due to moving Griffith cracks have been solved by Das and Ghosh (1991, 1992a, 1992b, 1992c) and by Das (1993, 1992). Dhawan and Dhaliwal (1978) also solved the static problem of determining the stress distribution in an infinite transversely isotropic medium containing three co-planar Griffith cracks.

In this paper, the problem of propagation of three co-planar

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Griffith cracks in a fixed direction with constant velocity V in an infinitely long but finite width elastic strip is considered. Employing the Fourier integral transform, the problem when the lateral boundaries are assumed to be clamped and displaced by an equal amount has been reduced to solving a set of four integral equations which are solved using the finite Hilbert transform technique and Cook's result (1970) to derive the exact form of stress intensity factors and crack opening displacement. Numerical results for stress intensity factors are presented graphically to show their variations with crack speed, crack length and the separating distance between the cracks.

2. STATEMENT OF THE PROBLEM

Consider an infinitely long elastic strip occupying the region $-h \le y \le h$, weakened by three co-planar Griffith cracks moving steadily at a constant velocity V in the X-direction, referred to a fixed co-ordinate system (X, Y, Z) as shown in Fig.1.

In dynamic problems of anti-plane shear, the non-vanishing component of displacement W directed in the Z-direction satisfies the equation of motion :

$$W_{,XX} + W_{,YY} = \frac{1}{C_2^2} W_{,TT} \tag{1}$$

where $C_2 = (\mu/\rho)^{1/2}$ is the shear wave velocity, ρ is the material

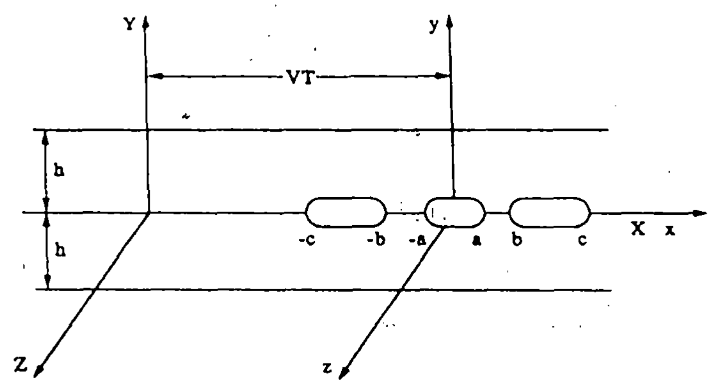

Fig. 1. Geometry and coordinate system

density and $W_{,x}$ represents partial derivatives of W with respect to X.

For cracks moving at a constant velocity V in the X-direction, it is convenient to introduce the Galilean transformation :

$$x=X-VT, \quad y=Y, \quad z=Z, \quad t=T \tag{2}$$

where (x,y,z) represents the translating co-ordinate system shown in Fig.1.

Let three co-planar Griffith cracks of finite length located along the X-axis be moving steadily with velocity V in the direction of the X-axis so that their positions referred to translating co-ordinates (x,y,z) are $-c<x<-b$, $-a<x<a$ and $b<x<c$ on $y=0$. The edges of the strip $y=\pm h$ are assumed to be clamped and displaced by an equal amount W_o, where W_o is a constant.

The boundary conditions of the proposed problem are

$$\sigma_{yz}(x,0) = 0, \quad |x|<a, \; b<|x|<c \tag{3}$$

$$W(x,\pm h) = \pm W_o , \quad -\infty<x<\infty \tag{4}$$

$$W(x,0) = 0 , \quad a<|x|<b , \; |x|>c. \tag{5}$$

In order to apply the integral transform technique it is required to solve a different but equivalent problem which can be obtained from the clamped strip problem (without any cracks) while the uniform strain is applied. The equivalent stress conditions on the cracks are

$$\sigma_{yz}(x,0) = -\frac{\mu W_o}{h} , \quad |x|<a , \; b<|x|<c \tag{6}$$

and the boundary conditions for the displacement are

$$W(x,\pm h) = 0\ , \quad -\infty<x<\infty \tag{7}$$

$$W(x,0) = 0\ , \quad a<|x|<b\ , \quad |x|>c \tag{8}$$

In the moving co-ordinate system, the equation of motion becomes independent of time and takes the form

$$s^2\, W_{,xx} + W_{,yy} = 0, \tag{9}$$

with $$s = \surd(1-V^2/C_2^2). \tag{10}$$

Introducing

$$\overline{W}_c(\xi,y) = \int_0^\infty W(x,y)\cos(\xi x)\,dx$$

$$W(x,y) = \frac{2}{\pi}\int_0^\infty \overline{W}_c(\xi,y)\cos(\xi x)\,d\xi \tag{11}$$

in equation (3), the solution of equation (3) is obtained as

$$W(x,y) = \frac{2}{\pi}\int_0^\infty \left[C_1(\xi)e^{-\xi ys} + C_3(\xi)e^{\xi ys}\right]\cos(\xi x)\,d\xi, \tag{12}$$

with

$$\sigma_{yz}(x,y) = -\frac{2\mu s}{\pi}\int_0^\infty \xi\left[C_1(\xi)e^{-\xi ys} - C_3(\xi)e^{\xi ys}\right]\cos(\xi x)\,d\xi. \tag{13}$$

Using the expression for W(x,y) given in (6) in equation (9), it has been found that

$$C_1(\xi) = \frac{C(\xi)}{1-e^{-2\xi hs}}$$

$$C_3(\xi) = -\frac{C(\xi)e^{-2\xi hs}}{1-e^{-2\xi hs}}, \tag{14}$$

where the unknown function $C(\xi)$ is to be determined.
From conditions (8) and (10) it is determined that $C(\xi)$ satisfies the following quadruple integral equations

$$\int_0^{\infty} \xi C(\xi)\coth(\xi hs)\cos(\xi x)d\xi = \frac{\pi W_0}{2hs}, \quad x \in I_1, I_3 \qquad (15a,b)$$

and

$$\int_0^{\infty} C(\xi)\cos(\xi x)d\xi = 0, \quad x \in I_2, I_4, \qquad (16)$$

where

$$I_1=(0,a), \quad I_2=(a,b), \quad I_3=(b,c), \quad I_4=(c,\infty).$$

3. METHOD OF SOLUTION

In order to solve the quadruple integral equations given by equations (15) and (16), let us take

$$C(\xi) = \frac{1}{\xi}\int_0^{a} h(u)\sin(\xi u)du + \frac{1}{\xi}\int_b^{c} g(v^2)\operatorname{sech}^2(ev)\sin(\xi v)dv, \qquad (17)$$

where $h(u)$ and $g(v^2)$ are the unknown functions to be determined from the boundary conditions of the proposed problem. Substituting the value of $C(\xi)$ given by (17) in (16) and using the following result :

$$\int_0^{\infty} \frac{\sin(\xi u)\cos(\xi x)}{\xi} d\xi = \begin{cases} \frac{\pi}{2}, & u>x>0 \\ \frac{\pi}{4}, & u=x>0 \\ 0, & x>u>0, \end{cases}$$

it is found that this choice of $C(\xi)$ leads to the condition

$$\int_b^c g(v^2)\operatorname{sech}^2(ev)dv = 0. \tag{18}$$

Rewriting equation (15a) as

$$\frac{d}{dx}\int_0^\infty C(\xi)\coth(\xi hs)\sin(\xi x)d\xi = \frac{\pi W_o}{2hs}, \quad x \in I_1 \tag{19}$$

and inserting the value of $C(\xi)$ from equation (17) in (19), it is found that h(u) is the solution of the following singular integral equation :

$$\int_0^a h(u)\log\left|\frac{\tanh(ex)+\tanh(eu)}{\tanh(ex)-\tanh(eu)}\right|du = \pi f(x), \qquad x\in I_1. \tag{20}$$

with

$$f(x) = \int_0^x \left[\frac{W_o}{hs} - \frac{2}{\pi}\int_b^c \frac{eg(v^2)\operatorname{sech}^2(ex')\operatorname{sech}^2(ev)\tanh(ev)}{\tanh^2(ev) - \tanh^2(ex')}dv\right]dx' ,$$

where the following result (Gradshteyn et al., 1965) has been used:

$$\int_0^\infty \coth(\xi hs)\frac{\sin(\xi u)\sin(\xi x)}{\xi}d\xi = \frac{1}{2}\log\left|\frac{\tanh(ex)+\tanh(eu)}{\tanh(ex)-\tanh(eu)}\right|, \quad e=\frac{\pi}{2hs}. \tag{21}$$

Now using Cook's result (1970), the solution of (20) has been obtained with the aid of the following result :

$$\int_0^a \frac{\sqrt{[\tanh^2(ea)-\tanh^2(ex)]}\, e\operatorname{sech}^2(ex)}{[\tanh^2(ex)-\tanh^2(eu)][\tanh^2(ev)-\tanh^2(ex)]}dx$$

$$= - \frac{\pi}{2\tanh(ev)} \frac{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}{\tanh^2(ev)-\tanh^2(eu)} \quad \text{for} \quad u \in I_1 \text{ and } v \in I_3,$$

$$h(u) = \frac{-2e \tanh(eu)\operatorname{sech}^2(eu)}{\pi[\tanh^2(ea)-\tanh^2(eu)]} \left[\frac{W_o}{hs}\int_0^a \frac{\sqrt{[\tanh^2(ea)-\tanh^2(ex)]}}{\tanh^2(ex)-\tanh^2(eu)} dx + \right.$$

$$\left. + \int_b^c \frac{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}{\tanh^2(ev)-\tanh^2(eu)} g(v^2)\operatorname{sech}^2(ev) \, dv \right]. \qquad (22)$$

Substituting the resulting value of $C(\xi)$, obtained using equation (22) in equation (17), in condition (15b) and making use of the following results :

$$\int_0^a \frac{e \operatorname{sech}^2(eu)\tanh^2(eu) \, du}{[\tanh^2(eu)-\tanh^2(ex)][\tanh^2(ev)-\tanh^2(eu)]\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}}$$

$$= \frac{\pi}{2[\tanh^2(ev)-\tanh^2(ex)]} \left[\frac{\tanh(ev)}{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}} - \right.$$

$$\left. - \frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}} \right],$$

$$\int_0^a \frac{e \operatorname{sech}^2(eu)\tanh^2(eu) \, du}{[\tanh^2(eu)-\tanh^2(ex)][\tanh^2(ey')-\tanh^2(eu)]\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}}$$

$$= \frac{\pi}{2[\tanh^2(ex)-\tanh^2(ey')]} \frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}},$$

for $x, v \in I_3$ and $y' \in I_1$,

it can be shown that $g(v^2)$ is the solution of the following singular integral equation :

$$\int_b^c \frac{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}{\tanh^2(ev)-\tanh^2(ex)}\, eg(v^2)\operatorname{sech}^2(ev)dv$$

$$= \frac{\pi W_o}{2hs}\left[\frac{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}{\operatorname{sech}^2(ex)\tanh(ex)} + \frac{e}{\pi}\int_o^a \frac{\sqrt{[\tanh^2(ea)-\tanh^2(ey')]}}{\tanh^2(ex)-\tanh^2(ey')}\,dy'\right]$$

$$, \quad \text{for } x \in I_3. \qquad (23)$$

Using the finite Hilbert transform technique (Srivastava et al., 1968) and the following result :

$$\int_b^c \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ex)}{\tanh^2(ex)-\tanh^2(eb)}\right]} \times$$

$$\times \frac{2\operatorname{sech}^2(ex)\tanh(ex)\ dx}{[\tanh^2(ex)-\tanh^2(ey')][\tanh^2(ex)-\tanh^2(ev)]}$$

$$= -\frac{\pi}{e[\tanh^2(ev)-\tanh^2(ey')]}\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ey')}{\tanh^2(eb)-\tanh^2(ey')}\right]},$$

the solution of equation (23) is found as

$$g(v^2) = -\frac{2eW_o}{\pi hs}\,\frac{\tanh^2(ev)\sqrt{[\tanh^2(ev)-\tanh^2(eb)]}}{\sqrt{\{[\tanh^2(ev)-\tanh^2(ea)][\tanh^2(ec)-\tanh^2(ev)]\}}}$$

$$\times\left[\int_b^c \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ex)}{\tanh^2(ex)-\tanh^2(eb)}\right]}\frac{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}{\tanh^2(ex)-\tanh^2(ev)}\,dx\ -\right.$$

$$\left.-\int_o^a \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ey')}{\tanh^2(eb)-\tanh^2(ey')}\right]}\frac{\sqrt{[\tanh^2(ea)-\tanh^2(ey')]}}{\tanh^2(ev)-\tanh^2(ey')}\,dy'\right] +$$

$$\frac{C_1 \tanh(ev)}{\sqrt{\{[\tanh^2(ev)-\tanh^2(ea)][\tanh^2(ev)-\tanh^2(eb)][\tanh^2(ec)-\tanh^2(ev)]\}}} \tag{24}$$

Next substituting the value of $g(v^2)$ from equation (24) in equation (22) and finally using the following result :

$$\int_b^c \sqrt{\left[\frac{\tanh^2(ev)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ev)}\right]} \times \times \frac{2\operatorname{sech}^2(ev)\tanh(ev)\,dv}{[\tanh^2(ev)-\tanh^2(eu)][\tanh^2(ex')-\tanh^2(ev)]}$$

$$= -\frac{\pi}{e[\tanh^2(eu)-\tanh^2(ex')]}\left[\sqrt{\left[\frac{\tanh^2(eb)-\tanh^2(eu)}{\tanh^2(ec)-\tanh^2(eu)}\right]} - \sqrt{\left[\frac{\tanh^2(eb)-\tanh^2(ex')}{\tanh^2(ec)-\tanh^2(ex')}\right]}\right],$$

for $u, x' \in I_1$,

$h(u)$ is derived in the form :

$$h(u) = -\frac{2eW_o}{h\pi s}\frac{\operatorname{sech}^2(eu)\tanh(eu)\sqrt{[\tanh^2(eb)-\tanh^2(eu)]}}{\sqrt{\{[\tanh^2(ea)-\tanh^2(eu)][\tanh^2(ec)-\tanh^2(eu)]\}}} \times$$

$$\times \left[\int_0^a \sqrt{\left[\frac{\tanh^2(ea)-\tanh^2(ey')}{\tanh^2(eb)-\tanh^2(ey')}\right]}\frac{\sqrt{[\tanh^2(ec)-\tanh^2(ey')]}}{\tanh^2(ey')-\tanh^2(eu)}\,dy' + \right.$$

$$\left. + \int_b^c \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ex)}{\tanh^2(ex)-\tanh^2(eb)}\right]}\frac{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}{\tanh^2(ex)-\tanh^2(eu)}\,dx\right] -$$

$$\frac{C_1 \tanh(eu)\operatorname{sech}^2(eu)}{\sqrt{\{[\tanh^2(ea)-\tanh^2(eu)][\tanh^2(eb)-\tanh^2(eu)][\tanh^2(ec)-\tanh^2(eu)]\}}} \tag{25}$$

Substitution of the value of $g(v^2)$ from equation (24) in the condition (18) yields

$$C_1 = -\frac{2eW_o}{\pi hs}\left[\int_b^c \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ex)}{\tanh^2(ex)-\tanh^2(eb)}\right]}\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}\ \times\right.$$

$$\times\left\{\frac{\tanh^2(ex)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ex)} \times \Pi\left[\frac{\pi}{2},\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ex)},q\right]/F(\frac{\pi}{2},q)+1\right\}dx\ +$$

$$+\int_o^a \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(es)}{\tanh^2(eb)-\tanh^2(es)}\right]}\sqrt{[\tanh^2(ea)-\tanh^2(es)]}\ \times$$

$$\left.\times\left\{1-\frac{\tanh^2(eb)-\tanh^2(es)}{\tanh^2(ec)-\tanh^2(es)}\ \Pi\left[\frac{\pi}{2},\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(es)},q\right]/F(\frac{\pi}{2},q)\right\}ds,\right. \tag{26}$$

where $F(\phi,q)$ and $\Pi(\phi,n,q)$ are elliptic integrals of the first and third kinds respectively and

$$q = \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ea)}\right]}.$$

The relevant displacement and stress components in the plane of the crack can now be shown to be given by

$$W(x,0) = \int_x^a h(u)du\ , \qquad 0\le x\le a$$

$$= \int_x^c g(v^2)\cosh(ev)dv\ , \qquad b\le x\le c \tag{27}$$

and

$$\left[\sigma_{yz}(x,0)\right]_{a<x<b} = \frac{2\mu s}{\pi}\left[\int_0^a \frac{eh(u)\tanh(eu)du}{\tanh^2(ex)-\tanh^2(eu)} - \right.$$

$$\left. - \int_b^c \frac{eg(v^2)\tanh(ev)\mathrm{sech}^2(ev)}{\tanh^2(ev)-\tanh^2(ex)}\, dv\right]\mathrm{sech}^2(ex)$$

$$\left[\sigma_{yz}(x,0)\right]_{x>c} = \frac{2\mu s}{\pi}\left[\int_0^a \frac{eh(u)\tanh(eu)du}{\tanh^2(ex)-\tanh^2(eu)} + \right.$$

$$\left. + \int_b^c \frac{eg(v^2)\tanh(ev)\mathrm{sech}^2(ev)}{\tanh^2(ex)-\tanh^2(ev)}\, dv\right]\mathrm{sech}^2(ex). \qquad (28)$$

Now insertion of the values of $h(u)$ and $g(v^2)$ as given by equations (25) and (24) in the expressions (28) yields, after some algebraic manipulations,

$$\left[\sigma_{yz}(x,0)\right]_{a<x<b} = \frac{2\mu eW_o}{\pi hs}\left[-\sqrt{\left[\frac{\tanh^2(eb)-\tanh^2(ea)}{\tanh^2(ec)-\tanh^2(ea)}\right]} \times \right.$$

$$\times \frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}\left\{\int_0^a F_2(u,x)du + \int_b^c F_2(v,x)dv\right\} -$$

$$- \frac{2e[\tanh^2(ec)-\tanh^2(eb)]}{\pi}\left\{\int_0^a F_2(u',x)du'\int_0^a F_4(c,u)\; F_3(0,x,u)du + \right.$$

$$\left. + \int_b^c F_2(v,x)dv\int_0^a F_4(c,u)\; F_3(v,x,u)du\right\} +$$

$$+ \frac{\mu sh}{eW_o}\, C_1\left\{\frac{\pi}{2}\, \frac{1 - \tanh(ex)/\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}{\sqrt{\{[\tanh^2(eb)-\tanh^2(ea)][\tanh^2(ec)-\tanh^2(ea)]\}}} + \right.$$

$$+ e\int_0^a F_4(c,u)\ F_5(x,u)du\Big\} + \frac{e[\tanh^2(eb)-\tanh^2(ea)]}{\pi} \times$$

$$\times \left\{\int_b^c F_2(v',x)dv' \int_b^c F_4(a,v)\ F_6(v',x,v)dv + \int_0^a F_2(u,x)du\right.$$

$$\times\int_c^b F_4(a,v)\ F_6(u,x,v)dv - \frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(eb)-\tanh^2(ea)} \times$$

$$\left.\times\int_0^a F_1(u,x)du\int_0^a F_4(c,u')\ F_9(u,u')du'\right\} - \frac{\mu sh}{eW_0}\frac{C_1}{X_1} \times$$

$$\times \left\{\frac{\pi}{2}\frac{\tanh(ec)}{\sqrt{[\tanh^2(ec)-\tanh^2(ea)]}} + e\tanh^2(ea)\int_b^c F_7(x,v)dv\right\}\Bigg]\mathrm{sech}^2(ex)$$

and

$$[\sigma_{yz}(x,0)]_{x>c} = \frac{2\mu eW_0}{\pi hs}\left[-\sqrt{\left[\frac{\tanh^2(eb)-\tanh^2(ea)}{\tanh^2(ec)-\tanh^2(ea)}\right]} \times\right.$$

$$\times \frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}\left\{\int_0^a F_2(u,x)du + \int_b^c F_2(v,x)dv\right\} -$$

$$- \frac{2e[\tanh^2(ec)-\tanh^2(eb)]}{\pi}\left\{\int_0^a F_2(u',x)du' \int_0^a F_4(c,u)\ F_3(0,x,u)du +\right.$$

$$\left.+ \int_b^c F_2(v,x)dv\int_0^a F_4(c,u)\ F_3(v,x,u)du\right\} +$$

$$+ \frac{\mu sh}{eW_0} C_1\left\{\frac{\pi}{2}\frac{1 - \tanh(ex)/\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}{\sqrt{\{[\tanh^2(eb)-\tanh^2(ea)][\tanh^2(ec)-\tanh^2(ea)]\}}} +\right.$$

$$\left.+ e\int_0^a F_4(c,u)\ F_5(u,x)du\right\} - \frac{e[\tanh^2(eb)-\tanh^2(ea)]}{\pi} \times$$

$$\times \left\{ \int_b^c F_2(v',x)dv' \int_b^c F_4(a,v)\; F_8(v',v,x)dv + \int_0^a F_2(u,x)du \right.$$

$$\times \int_b^c F_4(a,v)\; F_8(u,v,x)dv + \frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(eb)-\tanh^2(ea)} \times$$

$$\left. \times \int_0^a F_1(u,x)du \int_0^a F_4(c,u')\; F_9(u,u')du' \right\} + \frac{\mu sh}{eW_0} \frac{C_1}{X_1} \times$$

$$\times \left\{ \frac{\pi}{2} \frac{\tanh(ec)}{\sqrt{[\tanh^2(ec)-\tanh^2(ea)]}} + e\tanh^2(ea) \int_b^c F_7(x,v)dv \right\} -$$

$$-\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ea)} \right]} \frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ec)]}} \times$$

$$\times \left. \left\{ \int_0^a F_2(u,x)du + \int_b^c F_2(v,x)dv \right\} \right] \operatorname{sech}^2(ex) , \qquad (29)$$

where

$$F_1(u,x) = \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eu)}{\tanh^2(eb)-\tanh^2(eu)} \right]} \frac{\tanh(eu)}{\tanh^2(ex)-\tanh^2(eu)}$$

$$F_2(v,x) = \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ev)}{\tanh^2(ev)-\tanh^2(eb)} \right]} \frac{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}{\tanh^2(ev)-\tanh^2(ex)}$$

$$F_3(v,x,u) = \frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}} \tan^{-1}\left\{ \frac{\tanh(eu)}{\tanh(ex)} \times \right.$$

$$\left. \times \sqrt{\left[\frac{\tanh^2(ex)-\tanh^2(ea)}{\tanh^2(ea)-\tanh^2(eu)} \right]} \right\} - \frac{\tanh(ev)}{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}} \times$$

$$\times \tan^{-1}\left\{\frac{\tanh(eu)}{\tanh(ev)}\sqrt{\left[\frac{\tanh^2(ev)-\tanh^2(ea)}{\tanh^2(ea)-\tanh^2(eu)}\right]}\right\}$$

$$F_4(w,u) = \frac{\operatorname{sech}^2(eu)\tanh(eu)}{\sqrt{\{[\tanh^2(ew)-\tanh^2(eu)]^3[\tanh^2(eb)-\tanh^2(eu)]\}}}$$

$$F_5(u,x) = [2\tanh^2(eu)-\tanh^2(ec)-\tanh^2(eb)]\left\{\sin^{-1}\left(\frac{\tanh(eu)}{\tanh(ea)}\right) - \right.$$

$$\left. - F_3(0,x,u)\right\}$$

$$F_6(u,x,v) = \frac{\tanh(ex)}{\sqrt{[\tanh^2(ec)-\tanh^2(ex)]}} \times$$

$$\log\left|\frac{\tanh(ex)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}+\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(ex)]}}{\tanh(ex)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}-\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(ex)]}}\right|$$

$$- \frac{\tanh(eu)}{\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}} \times$$

$$\log\left|\frac{\tanh(eu)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}+\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}}{\tanh(eu)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}-\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}}\right|$$

$$F_7(x,v) = \tan^{-1}\left\{\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ex)}{\tanh^2(ec)-\tanh^2(ev)}\right]}\sqrt{\left[\frac{\tanh^2(ev)-\tanh^2(eb)}{\tanh^2(eb)-\tanh^2(ex)}\right]}\right\}$$

$$\times \frac{\operatorname{sech}^2(ev)}{\sqrt{\{[\tanh^2(ev)-\tanh^2(ea)]^3\}}}$$

$$F_8(u,v,x) = -\frac{2\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ec)]}}\tan^{-1}\left\{\frac{\tanh(ev)}{\tanh(ex)}\right. \times$$

$$\times \sqrt{\left[\frac{\tanh^2(ex)-\tanh^2(ec)}{\tanh^2(ec)-\tanh^2(ev)} \right]} \Bigg\} + \frac{\tanh(eu)}{\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}} \times$$

$$\log\left| \frac{\tanh(eu)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}+\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}}{\tanh(eu)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}-\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}} \right|$$

$F_{\rho}(u,u') =$

$$\log\left| \frac{\tanh(eu)\sqrt{[\tanh^2(ea)-\tanh^2(eu')]}+\tanh(eu')\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}}{\tanh(eu)\sqrt{[\tanh^2(ea)-\tanh^2(eu')]}-\tanh(eu')\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}} \right|$$

and $\quad X_1 = \sqrt{\{[\tanh^2(eb)-\tanh^2(ex)][\tanh^2(ec)-\tanh^2(ex)]\}}.$

(30)

The dynamic stress intensity factors are defined by

$$N_a = \lim_{x \to a+} \sqrt{[2(x-a)]}[\sigma_{yz}(x,0)]_{a<x<b}$$

$$N_b = \lim_{x \to b-} \sqrt{[2(b-x)]}[\sigma_{yz}(x,0)]_{a<x<b}$$

$$N_c = \lim_{x \to c+} \sqrt{[2(x-c)]}[\sigma_{yz}(x,0)]_{x>c} \qquad (31)$$

Substitution of the results given by equations (29) in expressions (31) yields

$$N_a = \sqrt{\left[\frac{\tanh(ea)}{e}\right]}\left[- \sqrt{\left[\frac{\tanh^2(eb)-\tanh^2(ea)}{\tanh^2(ec)-\tanh^2(ea)} \right]}\frac{2eW_0}{\pi h} \left\{\int_0^a F_2(u,a)du + \right.\right.$$

$$\left. + \int_b^c F_2(v,a)dv \right\} -$$

$$- \frac{\mu s C_1}{\sqrt{\{[\tanh^2(eb)-\tanh^2(ea)][\tanh^2(ec)-\tanh^2(ea)]\}}}\Bigg] \mathrm{sech}(ea)$$

$$N_b = - \frac{\mu s C_1}{\sqrt{\{[\tanh^2(eb)-\tanh^2(ea)][\tanh^2(ec)-\tanh^2(eb)]\}}} \times$$

$$\sqrt{\left[\frac{\tanh(eb)}{e}\right]} \mathrm{sech}(eb)$$

$$N_c = \sqrt{\left[\frac{\tanh(ec)}{e}\right]}\Bigg[- \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ea)}\right]}\frac{2eW_o}{\pi h}\left\{\int_o^a F_2(u,c)du + \right.$$

$$\left. + \int_b^c F_2(v,c)dv\right\} +$$

$$+ \frac{\mu s C_1}{\sqrt{\{[\tanh^2(ec)-\tanh^2(ea)][\tanh^2(ec)-\tanh^2(eb)]\}}}\Bigg] \mathrm{sech}(ec)$$

(32a-c)

Again insertion of the values of $h(u)$ and $g(v^2)$, given by equations (24) and (25), in the expressions for displacements given by equations (27) yields

$$[W(x,0)]_{o\le x\le a} = - \frac{W_o}{h\mu\pi s}\left[\frac{2[\tanh^2(eb)-\tanh^2(ea)]}{\sqrt{[\tanh^2(ec)-\tanh^2(ea)]}} \times\right.$$

$$\times \left\{\int_b^c \Pi\left\{\lambda, \frac{\tanh^2(ev)-\tanh^2(eb)}{\tanh^2(ev)-\tanh^2(ea)}, q\right\} \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ev)}{\tanh^2(ev)-\tanh^2(eb)}\right]} \times\right.$$

$$\times \frac{dv}{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}} - \int_o^a \Pi\left\{\lambda, \frac{\tanh^2(eb)-\tanh^2(eu)}{\tanh^2(ea)-\tanh^2(eu)}, q\right\} \times$$

$$\times \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eu)}{\tanh^2(eb)-\tanh^2(eu)}\right]} \frac{du}{\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}}\Bigg\}\Bigg] -$$

$$- \frac{C_1 F(\lambda, q)}{e\sqrt{[\tanh^2(ec)-\tanh^2(ea)]}}$$

and

$$[W(x,0)]_{b\le x\le c} = \left[\frac{2W_o}{h\mu\pi s} \left(\int_b^c \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ev)}{\tanh^2(ev)-\tanh^2(eb)}\right]} \times \right.\right.$$

$$\times \sqrt{[\tanh^2(ev)-\tanh^2(ea)]}\left\{F(\lambda',q)+ \frac{\tanh^2(ev)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ev)} \times\right.$$

$$\times \Pi\left\{\lambda', \frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ev)}, q\right\}\Bigg\}dv + \int_o^a \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eu)}{\tanh^2(eb)-\tanh^2(eu)}\right]}\times$$

$$\times \sqrt{[\tanh^2(ea)-\tanh^2(eu)]}\left\{F(\lambda',q)- \frac{\tanh^2(eb)-\tanh^2(eu)}{\tanh^2(ec)-\tanh^2(eu)} \times\right.$$

$$\left.\left.\times \Pi\left\{\lambda', \frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(eu)}, q\right\}\right\}du\right) + \frac{C_1}{e} F(\lambda',q)\Bigg] \times$$

$$\times \frac{1}{\sqrt{[\tanh^2(ec)-\tanh^2(ea)]}}, \qquad (33a,b)$$

where

$$\sin\lambda = \sqrt{\left[\frac{\tanh^2(ea)-\tanh^2(ex)}{\tanh^2(eb)-\tanh^2(ex)}\right]}, \quad \sin\lambda' = \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ex)}{\tanh^2(ec)-\tanh^2(eb)}\right]}$$

and $F(\phi,q)$, $\Pi(\phi,n,q)$ and q have been defined earlier.

On putting b=c and simplifying, it may be noted that the results (33a) and (32a) become those given by equations (3.18) and (3.21) of Singh et al (1981) and for a=0 the results given by (32b), (32c) and (33b) coincide with those given by equations (4.21), (4.22) and (4.17) of Das and Ghosh (1991).

4. NUMERICAL RESULTS AND DISCUSSION

Numerical results for stress intensity factor at the tips of the cracks for different values of crack speed, crack length and the separating distance between the cracks are presented in this section. The crack length dependence of the stress intensity factors and its variations with V/C_2 are shown in Figs.2-5. It is shown in Figs.2 and 3 that stress intensity factors at the edges of the cracks decrease with an increase in the values of V/C_2 and have a prominent variation when $V/C_2 \to 1$. Variations of stress intensity factors at the edge x=a become more prominent than those at the tips x=b and x=c when the length of the inner crack increases.

Variations of stress intensity factors at the edges of the cracks with a/b for different values of c/b and those with b/a for different values of c/a are plotted in Figs.4 and 5, respectively. It is found that when the separating distance between the inner crack and outer pair of cracks decreases the stress intensity factors at the tips x=a and x=b become more prominent than that at the edge x=c.

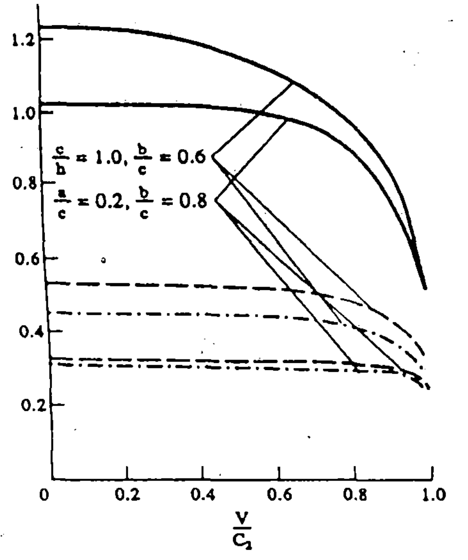

Fig. 2. Variations of stress intensity factors with V/C_2: (——) $hN_a/\mu W_0\sqrt{a}$; (– – –) $hN_b/\mu W_0\sqrt{b}$; (– · – · –) $hN_c/\mu W_0\sqrt{c}$.

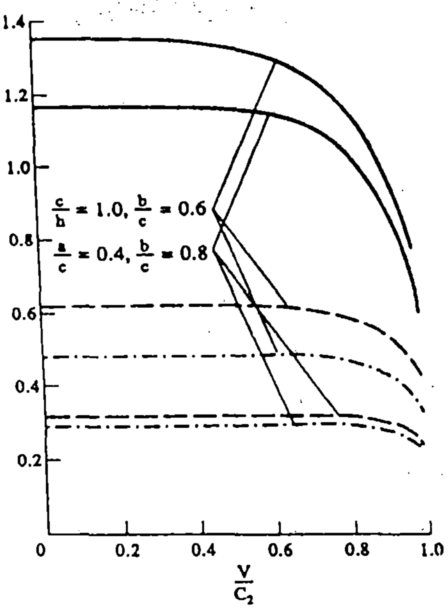

Fig. 3. Variations of stress intensity factors with V/C_2: (——) $hN_a/\mu W_0\sqrt{a}$; (– – –) $hN_b/\mu W_0\sqrt{b}$; (– · – · –) $hN_c/\mu W_0\sqrt{c}$.

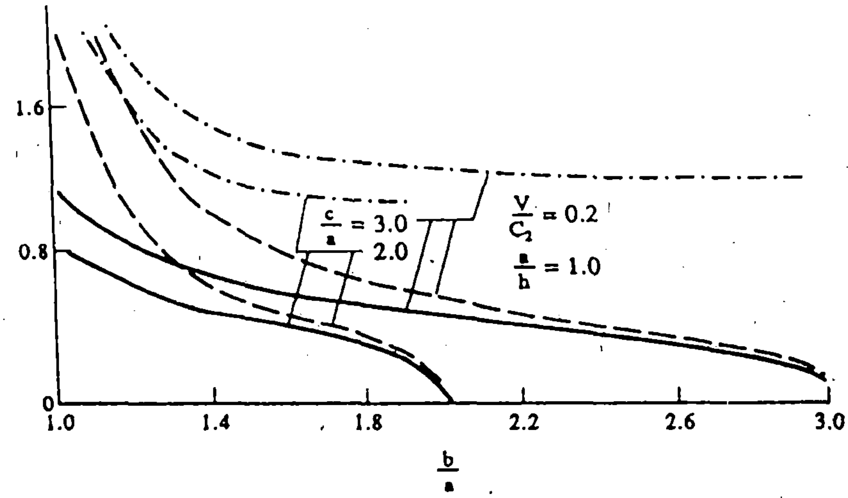

Fig. 4. Stress intensity factors vs b/a: (–·–·–) $hN_a/\mu W_0\sqrt{a}$; (– – –) $hN_b/\mu W_0\sqrt{b}$; (——) $hN_c/\mu W_0\sqrt{c}$.

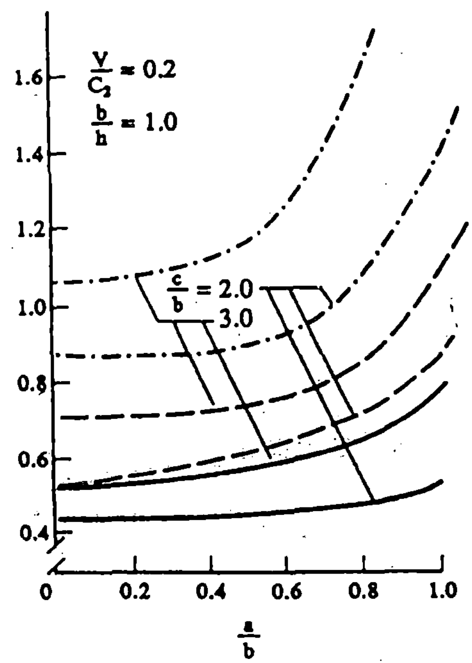

Fig. 5. Stress intensity factors vs a/b: (—·—·—) $hN_a/\mu W_0\sqrt{a}$; (– – –) $hN_b/\mu W_0\sqrt{b}$; (——) $hN_c/\mu W_0\sqrt{c}$.

FOUR COPLANAR GRIFFITH CRACKS MOVING IN AN INFINITELY LONG ELASTIC STRIP UNDER ANTIPLANE SHEAR STRESS

1. INTRODUCTION

In recent years, scattering of elastic waves by cracks of finite dimension in a strip of elastic material has been investigated by several investigators. The theory of cracks in 2-dimensional medium was first developed by Griffith (1920). Sih and Chen (1972) solved the problem of a uniformly propagating finite crack in a strip of isotropic material under plane extension. Singh et al. (1981) also studied the problem of propagation for a finite length crack moving in a strip under anti-plane shear stress and gave the closed form solution. In the above analysis, the usual method of solving mixed boundary value problems by integral transforms is to reduce the problem to a Fredholm integral equation of second kind and then proceed to its numerical solution.

As regards the crack problem research has been restricted mainly to the case of a single crack or a pair of cracks because of the severe mathematical complexity encountered in solving the problems of three or more cracks. Jain and Kanwal (1972a) solved the low frequency solution of diffraction of normally incident longitudinal waves by two co-planar Griffith cracks in an infinite isotropic elastic medium. Using a completely different technique Itou (1980b) solved the diffraction problem of elastic waves by two co-planar Griffith cracks in an infinite elastic medium. Problems on three coplanar Griffith cracks moving steadily in an elastic strip has been solved by Das and Sarkar (1993).

To the best knowledge of the authors, the problem of stress distribution around four co-planar Griffith cracks in a strip has not been investigated so far. In this paper we have considered the problem of propagation of four co-planar Griffith cracks moving steadily in an infinitely long finite width strip under antiplane shear stress. Cracks are assumed to be moving steadily along a fixed direction with a constant speed V less than the shear wave velocity in the medium. The application of two-dimensional Fourier transforms reduced this problem to that of solving a set of five integral equations with cosine kernel and weight function. Employing finite Hilbert transform technique (Srivastava et al., 1968), the closed form solutions are obtained when the lateral boundaries are subjected to shearing stresses. The dynamic stress intensity factors and the crack opening displacement have been evaluated numerically for various values of crack velocity and distance between the cracks and the results have been presented by means of graphs.

2. FORMULATION OF THE PROBLEM

We first consider a strip of elastic material occupying the region $-h' \le Y \le h'$ referred to a fixed co-ordinate system (X', Y', Z') as shown in fig.1. The strip extends from $-\infty$ to ∞ in X'-direction and contains four coplanar Griffith cracks such that these cracks are located in the region $-d' \le X' \le c'$, $-b' \le X' \le -a'$, $a' \le X' \le b'$, $c' \le X' \le d'$, $|Z'| < \infty$, $Y' = 0$ moving at a constant speed v in the X'-direction.

In dynamic problem of antiplane shear, there exists a single non-vanishing component of displacement $W = W(X', Y', t)$ in the Z'-direction. The corresponding stress components are

$$\sigma_{X'Z'} = \mu\frac{\partial W}{\partial X'} \qquad , \qquad \sigma_{Y'Z'} = \mu\frac{\partial W}{\partial Y'} \tag{2.1}$$

where μ is the shear-modulus of elastic material.

The two dimensional wave equation for $W(X', Y', t)$ is given by

$$\frac{\partial^2 W}{\partial X'^2} + \frac{\partial^2 W}{\partial Y'^2} = \frac{1}{c_2^2}\frac{\partial^2 W}{\partial t^2} \tag{2.2}$$

where $c_2 = (\mu/\rho)^{1/2}$ is the shear wave velocity and ρ is the density of the material.

Using Galilean transformation, $x' = X' - Vt$, $y' = Y'$, $z' = Z'$, $t' = t$ where (x', y', z') represents the translating co-ordinate system as shown in fig.1 and also normalizing all the lengths with respect to 'd'' so that $x' = d'x$, $y' = d'y$, $a' = ad'$, $b' = bd'$, $c' = cd'$, $h' = d'h$, $W = d'w$, equation (2.2) reduces to

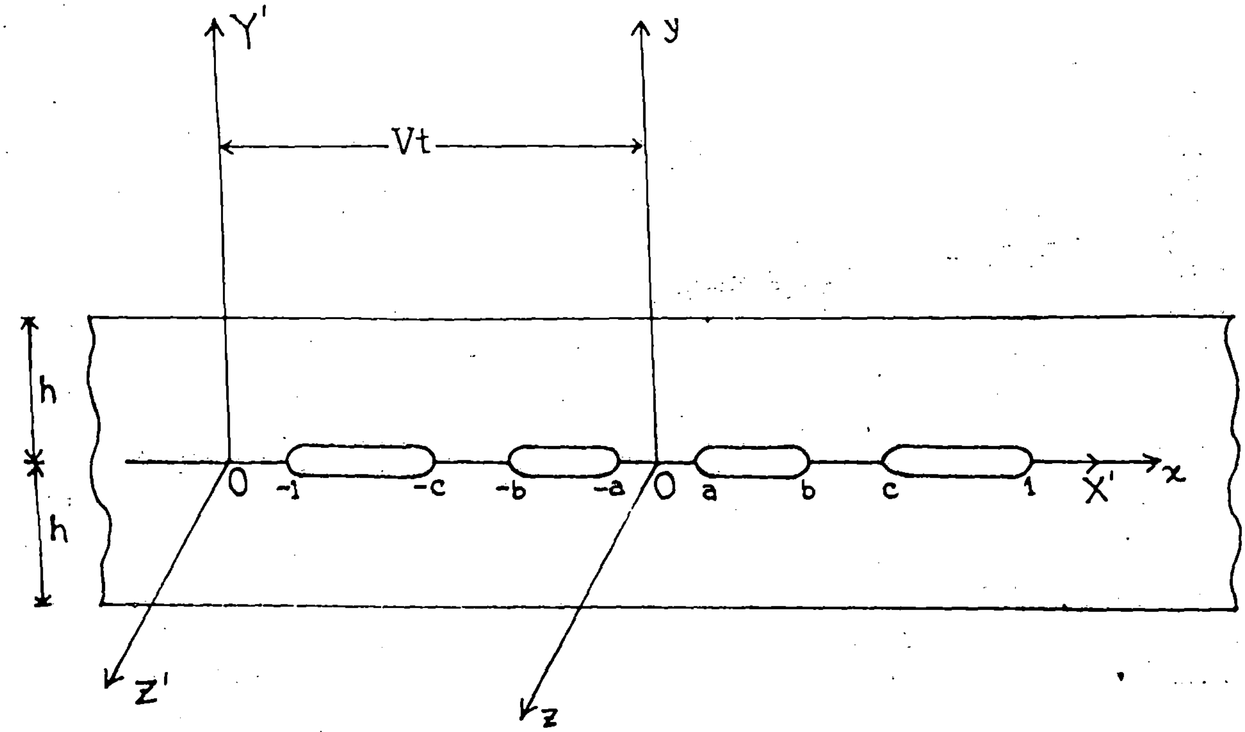

Fig.1. Geometry of the cracks.

$$s^2 \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \tag{2.3}$$

with

$$s^2 = 1 - v^2/c_2^2. \tag{2.4}$$

Since the geometry of the problem is symmetric about the y-axis, so introducing Fourier cosine transform

$$A_1(\xi) = \int_0^{\infty} A(x)\cos(\xi x)\, dx$$

and

$$A(x) = \frac{2}{\pi}\int_0^{\infty} A_1(\xi)\cos(\xi x)\, d\xi$$

we obtain the solution of equation (2.3) as

$$w(x,y) = \pm\frac{2}{\pi}\int_0^{\infty} \Big[A_1(\xi)\exp(-\xi|y|s) + A_2(\xi)\exp(\xi|y|s)\Big]\cos(\xi x)\,d\xi \quad (y \gtrless 0) \tag{2.5}$$

with

$$\sigma_{yz}(x,y) = -\frac{2\mu s}{\pi}\int_0^{\infty}\Big[A_1(\xi)\exp(-\xi|y|s) - A_2(\xi)\exp(\xi|y|s)\Big]\xi\cos(\xi x)\,d\xi \quad (y \gtrless 0) \tag{2.6}$$

where s is the positive root of equation (2.4) and $A_1(\xi)$, $A_2(\xi)$ are the unknown functions to be determined.

In our case uniform shearing stress p is applied to the upper and lower boundaries $y=\pm h$ of the strip. The equivalent problem in our case involves the application of the shear stress $-p$ to the crack faces at $y=0$. Accordingly, the boundary conditions are

$$\sigma_{yz}(x,\pm h) = 0 , \qquad 0<x<\infty \tag{2.7}$$

$$w(x,0) = 0 , \qquad x \in I_1,\ I_3,\ I_5 \tag{2.8a-c}$$

$$\sigma_{yz}(x,0) = -p , \qquad x \in I_2,\ I_4 \tag{2.9a-b}$$

where $I_1=(0,a)$, $I_2=(a,b)$, $I_3=(b,c)$, $I_4=(c,1)$, $I_5=(1,\infty)$.

3. SOLUTION OF THE PROBLEM

Using the expression for $w(x,y)$ from (2.5) in (2.7) it has been found that

$$A_1(\xi) = \frac{A(\xi)}{1 + \exp(-2\xi hs)}$$

and

$$A_2(\xi) = \frac{A(\xi)\exp(-2\xi hs)}{1 + \exp(-2\xi hs)}$$

where $A(\xi)$ is to be determined from the boundary conditions. With the help of boundary conditions (2.8) and (2.9) $A(\xi)$ is found to satisfy the following set of five integral equations

$$\int_0^\infty A(\xi)\cos(\xi x)d\xi = 0 , \qquad x \in I_1,\ I_3,\ I_5 \tag{3.1a-c}$$

and

$$\int_0^\infty \xi H_1(\xi hs)A(\xi)\cos(\xi x)d\xi = \frac{\pi p}{2\mu s} , \qquad x \in I_2,\ I_4 \tag{3.2a-b}$$

where

$$H_1(\xi hs) = \frac{1 - \exp(-2\xi hs)}{1 + \exp(-2\xi hs)} = \tanh(\xi hs) \qquad (3.3)$$

In order to solve the set of five integral equations given by equations (3.1) and (3.2), let us take

$$A(\xi) = \frac{1}{\xi}\int_a^b g(u^2)\cosh(eu)\sin(\xi u)du +$$

$$+ \frac{1}{\xi}\int_c^1 h(v^2)\cosh(ev)\sin(\xi v)dv. \qquad (3.4)$$

In equation (3.4), $g(u^2)$ and $h(v^2)$ are unknown functions to be determined from the boundary conditions and $e = \frac{\pi}{2hs}$.

Using the following result (Gradshteyn et al., 1965)

$$\int_0^\infty \frac{\sin(\xi u)\cos(\xi x)}{\xi} d\xi = \begin{cases} \frac{\pi}{2}, & u>x>0 \\ 0, & x>u>0 \end{cases}$$

it is found that the choice of $A(\xi)$ satisfies equations (3.1a,c) if $g(u^2)$ and $h(v^2)$ satisfy

$$\int_a^b g(u^2)\cosh(eu)du = 0 \qquad (3.5a)$$

and

$$\int_c^1 h(v^2)\cosh(ev)dv = 0 \qquad (3.5b)$$

Now equations (3.2a-b) may be written in the form

$$\frac{d}{dx}\int_0^\infty \tanh(\xi hs)A(\xi)\sin(\xi x)d\xi = \frac{\pi p}{2\mu s}, \quad x \in I_2, I_4 \qquad (3.6a\text{-}b)$$

Substitution of equation (3.4) in (3.6a) and use of the following result (Das, 1992)

$$\int_0^{\infty} \xi^{-1}\tanh(\xi hs)\sin(\xi u)\sin(\xi x)d\xi = \frac{1}{2}\log\left|\frac{\sinh(ex)+\sinh(eu)}{\sinh(ex)-\sinh(eu)}\right|$$

yields

$$\int_a^b \frac{eg(u^2)\sinh(2eu)}{\sinh^2(eu)-\sinh^2(ex)}\,du + \int_c^1 \frac{eh(v^2)\sinh(2ev)}{\sinh^2(ev)-\sinh^2(ex)}\,dv \tag{3.7}$$

$$= \frac{\pi p}{\mu s\cosh(ex)}, \qquad x \in l_2 .$$

Substituting cosh(eu) = U, cosh(ev) = S equation (3.7) is found to reduce to the form

$$\int_A^B \frac{UG(U^2)}{U^2-X^2}\,dU = \frac{\pi}{2}F(X), \qquad (A<X<B) \tag{3.8}$$

where $X = \cosh(ex)$, $A = \cosh(ea)$, $B = \cosh(eb)$, $C = \cosh(ec)$, $D = \cosh(e)$, $g(u^2) = G(U^2)$, $h(v^2) = H(S^2)$ and

$$F(X) = \frac{p}{\mu sX} - \frac{2}{\pi}\int_C^D \frac{SH(S^2)}{S^2-X^2}\,dS \tag{3.9}$$

Using the finite Hilbert transform technique (Srivastava et al., 1968) the solution of equation (3.8) is

$$G(U^2) = -\frac{2}{\pi}\sqrt{\frac{U^2-A^2}{B^2-U^2}}\int_A^B \sqrt{\frac{B^2-X^2}{X^2-A^2}}\left[\frac{p}{\mu sX} - \frac{2}{\pi}\int_C^D \frac{SH(S^2)}{S^2-X^2}\,dS\right] \times$$

$$\times \frac{X\,dX}{X^2-U^2} + \frac{B_1}{\sqrt{(U^2-A^2)(B^2-U^2)}}, \qquad (A<U<B) \tag{3.10}$$

The constant B_1 is to be determined from equation (3.5a). Using the result

$$\int_A^B \sqrt{\frac{B^2-X^2}{X^2-A^2}} \frac{X \, dX}{(U^2-X^2)(S^2-X^2)} = \frac{\pi}{2} \sqrt{\frac{S^2-B^2}{S^2-A^2}} \frac{1}{S^2-U^2}$$

equation (3.10) can be rewritten as

$$G(U^2) = \frac{2p}{\pi\mu s} \sqrt{\frac{U^2-A^2}{B^2-U^2}} \int_A^B \sqrt{\frac{B^2-X^2}{X^2-A^2}} \frac{dX}{U^2-X^2} - \frac{2}{\pi} \sqrt{\frac{U^2-A^2}{B^2-U^2}} \times$$

$$\times \int_C^D \sqrt{\frac{S^2-B^2}{S^2-A^2}} \frac{SH(S^2)}{S^2-U^2} dS + \frac{B_1}{\sqrt{(U^2-A^2)(B^2-U^2)}}, \quad (A<U<B) \qquad (3.11)$$

Substitution of expression for $A(\xi)$ from (3.4) in (3.66) yields with aid of (3.11) the following singular integral equation involving $H(S^2)$

$$\int_C^D \sqrt{\frac{S^2-B^2}{S^2-A^2}} \frac{SH(S^2)}{S^2-U^2} dS = \frac{\pi}{2}\left[\frac{p}{\mu s X} \sqrt{\frac{X^2-B^2}{X^2-A^2}} - \frac{2pA^2}{\pi\mu s B X^2} \times \right.$$

$$\left. \times \left\{ \left(\frac{X^2-B^2}{X^2-A^2}\right) \Pi\left(\frac{\pi}{2}, \frac{X^2(B^2-A^2)}{B^2(X^2-A^2)}, q\right) - \frac{B^2}{A^2} F\left(\frac{\pi}{2}, q\right) \right\} + \frac{B_1}{X^2-A^2} \right]$$

(3.12)

where $q = \frac{\sqrt{B^2-A^2}}{B}$ and $F(\phi,k)$, $\Pi(\phi,n,k)$ are elliptic integrals of first and third kind respectively.

While deriving equation (3.12), the following results have been made use of.

$$\int_A^B \sqrt{\frac{U^2-A^2}{B^2-U^2}} \frac{U\,dU}{(X^2-U^2)(S^2-U^2)} = \frac{\pi}{2(V^2-X^2)} \left[\sqrt{\frac{X^2-A^2}{X^2-B^2}} - \sqrt{\frac{S^2-A^2}{S^2-U^2}} \right]$$

$$\int_A^B \sqrt{\frac{U^2-A^2}{B^2-U^2}} \frac{U\,dU}{(X^2-U^2)} \int_A^B \sqrt{\frac{B^2-Z^2}{Z^2-A^2}} \frac{dZ}{(U^2-Z^2)} = -\frac{\pi}{2BX^2} \sqrt{\frac{X^2-A^2}{X^2-B^2}} \left[A^2 \left(\frac{X^2-B^2}{X^2-A^2}\right) \Pi - B^2 F \right]$$

and

$$\int_A^B \frac{U\,dU}{(X^2-U^2)\sqrt{(U^2-A^2)(B^2-U^2)}} = \frac{\pi}{2\sqrt{(X^2-A^2)(X^2-B^2)}} \qquad (C<X<D).$$

Again, using finite Hilbert transform technique (Srivastava et al., 1968) it is found that

$$H(S^2) = -\frac{2}{\pi} \sqrt{\frac{(S^2-A^2)(S^2-C^2)}{(S^2-B^2)(D^2-S^2)}} \left[\frac{p}{\mu s} \left\{ \int_C^D \sqrt{\frac{(D^2-X^2)(X^2-B^2)}{(X^2-C^2)(X^2-A^2)}} \times \right.\right.$$

$$\left.\left. \times \frac{dX}{(X^2-S^2)} - \int_C^D \sqrt{\frac{(D^2-Y^2)(B^2-Y^2)}{(Y^2-A^2)(C^2-Y^2)}} \frac{dY}{(S^2-Y^2)} \right\} - \frac{\pi}{2} \sqrt{\frac{D^2-A^2}{C^2-A^2}} \frac{B_1}{(S^2-A^2)} \right] +$$

$$+ \frac{B_2 \sqrt{S^2-A^2}}{\sqrt{(S^2-B^2)(S^2-C^2)(D^2-S^2)}} \qquad (C<S<D) \qquad (3.13)$$

where we have used

$$\int_C^D \sqrt{\frac{D^2-X^2}{X^2-C^2}} \frac{X\,dX}{(X^2-A^2)(X^2-S^2)} = -\frac{\pi}{2}\sqrt{\frac{D^2-A^2}{C^2-A^2}} \frac{1}{(S^2-A^2)}$$

the constant B_z occuring in (3.13) is to be determined using the condition given by equation (3.5b).

Next, substituting the value of $H(S^2)$ from equation (3.13) in equation (3.11) and using the following results

$$\int_C^D \sqrt{\frac{S^2-C^2}{D^2-S^2}} \frac{S\,dS}{(S^2-U^2)(X^2-S^2)} = -\frac{\pi}{2}\sqrt{\frac{C^2-U^2}{D^2-U^2}} \frac{1}{(X^2-U^2)}$$

$$\int_C^D \sqrt{\frac{S^2-C^2}{D^2-S^2}} \frac{S\,dS}{(S^2-A^2)(S^2-U^2)} = -\frac{\pi}{2(U^2-A^2)}\left[\sqrt{\frac{C^2-A^2}{D^2-A^2}} - \sqrt{\frac{C^2-U^2}{D^2-U^2}}\right]$$

$$\int_C^D \frac{S\,dS}{(S^2-U^2)\sqrt{(S^2-C^2)(D^2-S^2)}} = \frac{\pi}{2\sqrt{(C^2-U^2)(D^2-U^2)}} \qquad (A<U<B).$$

$G(U^2)$ may be written in the following form

$$G(U^2) = \frac{2}{\pi}\sqrt{\frac{U^2-A^2}{B^2-U^2}}\,\frac{p}{\mu s}\left[\frac{(B^2-U^2)}{BU^2(U^2-A^2)}\left\{A^2\Pi\left(\frac{\pi}{2}, \frac{X^2(B^2-A^2)}{B^2(X^2-A^2)}, q\right)+\right.\right.$$

$$\left. + (U^2-A^2)F(\frac{\pi}{2}, q)\right\} + \frac{1}{B}F(\frac{\pi}{2}, q) - \sqrt{\frac{C^2-U^2}{D^2-U^2}}\left\{\int_C^D\sqrt{\frac{(D^2-X^2)(X^2-B^2)}{(X^2-C^2)(X^2-A^2)}}\times\right.$$

$$\times \frac{dX}{(X^2-U^2)} + \int_A^B \sqrt{\frac{(D^2-Y^2)(B^2-Y^2)}{(Y^2-A^2)(C^2-Y^2)}} \frac{dY}{(Y^2-U^2)} \Bigg\} + \int_A^B \sqrt{\frac{(B^2-Y^2)}{(Y^2-A^2)}} \frac{dY}{(Y^2-U^2)} \Bigg] +$$

$$+ \sqrt{\frac{(D^2-A^2)(C^2-U^2)}{(C^2-A^2)(D^2-U^2)}} \frac{B_1}{\sqrt{(U^2-A^2)(B^2-U^2)}} - \frac{B_2 \sqrt{U^2-A^2}}{\sqrt{(B^2-U^2)(C^2-U^2)(D^2-U^2)}}$$

$$(A<U<B) \tag{3.14}$$

To determine the values of the unknown constants B_1 and B_2 , we substitute $H(S^2)$ and $G(U^2)$ given by (3.13) and (3.14) in (3.5a,b) and obtain

$$B_1 = \frac{p}{\mu s} \left\{ \frac{K_3(K_{1,2} - K_{1,1}) - K_6(K_{1,3} + K_{2,3})}{RK_4K_6 + K_3K_5} \right\} \tag{3.15a}$$

$$B_2 = \frac{p}{\mu s} \left\{ \frac{RK_4(K_{1,1} - K_{1,2}) - K_5(K_{1,3} + K_{2,3})}{RK_4K_6 + K_3K_5} \right\} \tag{3.15b}$$

where

$$K_{1,1} = \int_C^D M_1(X)dX \int_C^D \frac{M_2(S)}{X^2-S^2} dS \tag{3.16}$$

$$K_{1,2} = \int_A^B M_1(Y)dY \int_C^D \frac{M_2(S)}{S^2-Y^2} dS \tag{3.17}$$

$$K_{1,3} = \int_C^D M_1(X)dX \int_A^B \frac{M_2(U)}{X^2-U^2} dU \tag{3.18}$$

$$K_{2,3} = \int_A^B M_1(Y)dY \int_A^B \frac{M_2(U)}{Y^2-U^2} dU \tag{3.19}$$

$$K_3 = \frac{\pi}{2} \int_A^B \frac{M_2(U)}{C^2 - U^2}\, dU \quad , \quad K_4 = \int_A^B \frac{M_2(U)}{U^2 - A^2}\, dU \tag{3.20}$$

$$K_5 = R \int_C^D \frac{M_2(S)}{S^2 - A^2}\, dS \quad , \quad K_6 = \frac{\pi}{2} \int_C^D \frac{M_2(S)}{S^2 - C^2}\, dS \tag{3.21}$$

$$M_1(T) = \sqrt{\frac{(D^2-T^2)(T^2-B^2)}{(T^2-C^2)(T^2-A^2)}} \quad , \quad M_2(T) = \sqrt{\frac{(T^2-A^2)(T^2-C^2)}{(T^2-B^2)(D^2-T^2)}}\ \frac{T}{\sqrt{T^2-1}} \tag{3.22}$$

and

$$R = -\frac{\pi}{2} \sqrt{\frac{D^2-A^2}{C^2-A^2}} \tag{3.23}$$

4. STRESS INTENSITY FACTORS

The corresponding displacement and stress components in the plane of the cracks may be written as

$$w(x,0) = \frac{1}{e} \int_x^B \frac{UG(U^2)}{\sqrt{U^2-1}}\, dU \quad , \qquad x \in l_2$$

$$= \frac{1}{e} \int_x^D \frac{SH(S^2)}{\sqrt{S^2-1}}\, dS \quad , \qquad x \in l_4 \tag{4.1a,b}$$

and

$$\left[\sigma_{yz}(x,0)\right]_{0<x<a} = -\frac{2\mu s X}{\pi} \left[\int_A^B \frac{UG(U^2)}{U^2-X^2}\, dU + \int_C^D \frac{SH(S^2)}{S^2-X^2}\, dS \right] \tag{4.2a}$$

$$\left[\sigma_{yz}(x,0)\right]_{b<x<c} = \frac{2\mu s X}{\pi}\left[\int_A^B \frac{UG(U^2)}{X^2-U^2}\,dU - \int_C^D \frac{SH(S^2)}{S^2-X^2}\,dS\right] \qquad (4.2b)$$

$$\left[\sigma_{yz}(x,0)\right]_{x>1} = \frac{2\mu s X}{\pi}\left[\int_A^B \frac{UG(U^2)}{X^2-U^2}\,dU + \int_C^D \frac{SH(S^2)}{X^2-S^2}\,dS\right] \qquad (4.2c)$$

With the aid of the results given by equations (3.13) and (3.14) the expressions (4.2a-c) yield after some algebraic manipulation, the results

$$\left[\sigma_{yz}(x,0)\right]_{0<x<a} = \frac{2\mu s}{\pi}\left[F_1(X) - F_2(X) - F_3(X) + F_4(X) - \right.$$
$$\left. - F_5(X) - F_6(X) - F_7(X) - F_8(X)\right] \qquad (4.3a)$$

$$\left[\sigma_{yz}(x,0)\right]_{b<x<c} = \frac{2\mu s}{\pi}\left[F_1(X) - F_2(X) + F_3(X) + F_4(X) - \right.$$
$$\left. - F_5(X) - F_6(X) - F_7(X) - F_8(X)\right] \qquad (4.3b)$$

$$\left[\sigma_{yz}(x,0)\right]_{x>1} = \frac{2\mu s}{\pi}\left[F_1(X) - F_2(X) + F_3(X) + F_4(X) - \right.$$
$$\left. - F_5(X) - F_6(X) - F_7(X) + F_8(X)\right] \qquad (4.3c)$$

where

$$F_1(X) = \frac{2pX}{\pi\mu s}\int_C^D \sqrt{\frac{(D^2-Y^2)(Y^2-B^2)}{(Y^2-C^2)(Y^2-A^2)}}\left[\frac{\pi}{2(Y^2-X^2)}\left\{\sqrt{\frac{Y^2-A^2}{Y^2-B^2}} - \right.\right.$$
$$\left.\left. - \sqrt{\frac{A^2-X^2}{B^2-X^2}}\right\}\sqrt{\frac{C^2-B^2}{D^2-B^2}} + I_{A,C}^{B,D}(X,Y)\right] dY \qquad (4.4a)$$

$$F_2(X) = \frac{2pX}{\pi\mu s}\int_A^B \sqrt{\frac{(D^2-Y^2)(B^2-Y^2)}{(C^2-Y^2)(Y^2-A^2)}}\left[\frac{\pi}{2(Y^2-X^2)}\times\right.$$

$$\left.\times\sqrt{\frac{(C^2-B^2)(A^2-X^2)}{(D^2-B^2)(B^2-X^2)}} + L_{A,C}^{B,D}(X,Y)\right]dY \qquad (4.4b)$$

$$F_3(X) = \frac{B_1X}{X_1}\left[\frac{\pi}{2}\sqrt{\frac{(C^2-B^2)}{(D^2-B^2)}} + J_{A,C}^{B,D}(X)\right]\sqrt{\frac{(D^2-A^2)}{(C^2-A^2)}} \qquad (4.4c)$$

$$F_4(X) = B_2X\left[\frac{\pi}{2\sqrt{(C^2-B^2)(D^2-B^2)}}\left\{1-\sqrt{\frac{(A^2-X^2)}{(B^2-X^2)}}\right\} + K_{A,C}^{B,D}(X)\right] \qquad (4.4d)$$

$$F_5(X) = \frac{2pX}{\pi\mu s}\int_C^D \sqrt{\frac{(D^2-Y^2)(Y^2-B^2)}{(Y^2-C^2)(Y^2-A^2)}}\left[\frac{\pi}{2(Y^2-X^2)}\times\right.$$

$$\left.\times\sqrt{\frac{(D^2-A^2)(C^2-X^2)}{(D^2-B^2)(D^2-X^2)}} - L_{C,A}^{D,B}(X,Y)\right]dY \qquad (4.4e)$$

$$F_6(X) = \frac{2pX}{\pi\mu s}\int_A^B \sqrt{\frac{(D^2-Y^2)(B^2-Y^2)}{(C^2-Y^2)(Y^2-A^2)}}\left[\frac{\pi}{2(Y^2-X^2)}\left\{\sqrt{\frac{C^2-X^2}{D^2-X^2}}-\right.\right.$$

$$\left.\left.-\sqrt{\frac{C^2-Y^2}{D^2-Y^2}}\right\}\sqrt{\frac{D^2-A^2}{D^2-B^2}} + I_{C,A}^{D,B}(X,Y)\right]dY \qquad (4.4f)$$

$$F_7(X) = \frac{B_1 X}{(A^2-X^2)} \left[\frac{\pi}{2} \sqrt{\frac{(D^2-A^2)}{(D^2-B^2)}} \left\{ \sqrt{\frac{C^2-X^2}{D^2-X^2}} - \sqrt{\frac{C^2-A^2}{D^2-A^2}} \right\} + \right.$$

$$\left. + I_{C,A}^{D,B}(X, A) \right] \sqrt{\frac{(D^2-A^2)}{(C^2-A^2)}} \qquad (4.4g)$$

$$F_8(X) = \frac{B_1 X}{X_2} \left[\frac{\pi}{2} \sqrt{\frac{(D^2-A^2)}{(D^2-B^2)}} - J_{C,A}^{D,B}(X) \right] \qquad (4.4h)$$

$$I_{P,R}^{Q,S}(X, Y) = \int_P^Q \left(\frac{S^2-R^2}{Y^2-X^2}\right) \left\{ \sqrt{\frac{(Y^2-P^2)}{(Y^2-Q^2)}} \tan^{-1} \sqrt{\frac{(U^2-P^2)(Y^2-Q^2)}{(Q^2-U^2)(Y^2-P^2)}} - \right.$$

$$\left. - \sqrt{\frac{(P^2-X^2)}{(Q^2-X^2)}} \tan^{-1} \sqrt{\frac{(U^2-P^2)(Q^2-X^2)}{(Q^2-U^2)(P^2-X^2)}} \right\} \frac{U\, dU}{\sqrt{(R^2-U^2)(S^2-U^2)^3}} \qquad (4.4i)$$

$$L_{P,R}^{Q,S}(X, Y) = \int_P^Q \left(\frac{S^2-R^2}{Y^2-X^2}\right) \left\{ \sqrt{\frac{(P^2-X^2)}{(Q^2-X^2)}} \tan^{-1} \sqrt{\frac{(U^2-P^2)(Q^2-X^2)}{(Q^2-U^2)(P^2-X^2)}} + \right.$$

$$\left. + \frac{1}{2} \sqrt{\frac{(Y^2-P^2)}{(Q^2-Y^2)}} \log \left| \frac{\sqrt{(U^2-P^2)(Q^2-Y^2)} - \sqrt{(Q^2-U^2)(Y^2-P^2)}}{\sqrt{(U^2-P^2)(Q^2-Y^2)} + \sqrt{(Q^2-U^2)(Y^2-P^2)}} \right| \right\} \times$$

$$\times \frac{U\, dU}{\sqrt{(R^2-U^2)(S^2-U^2)^3}} \qquad (4.4j)$$

$$J^{Q,S}_{P,R}(X) = \int_P^Q \left\{ \tan^{-1}\sqrt{\frac{(U^2-P^2)(Q^2-X^2)}{(Q^2-U^2)(P^2-X^2)}} \right\} \frac{U(S^2-R^2)\ dU}{\sqrt{(R^2-U^2)(S^2-U^2)^3}} \tag{4.4k}$$

$$K^{Q,S}_{P,R}(X) = \int_P^Q \left\{ \tan^{-1}\sqrt{\frac{(U^2-P^2)}{(Q^2-U^2)}} - \sqrt{\frac{(P^2-X^2)}{(Q^2-X^2)}} \times \right.$$

$$\left. \times \tan^{-1}\sqrt{\frac{(U^2-P^2)(Q^2-X^2)}{(Q^2-U^2)(P^2-X^2)}} \right\} \frac{U(2U^2-R^2-S^2)\ dU}{\sqrt{(R^2-U^2)^3(S^2-U^2)^3}} \tag{4.4l}$$

$$X_1 = \sqrt{(A^2-X^2)(B^2-X^2)} \quad , \quad X_2 = \sqrt{(C^2-X^2)(D^2-X^2)} . \tag{4.4m}$$

The dynamic stress intensity factors are given by

$$N_a = \underset{x \to a-}{\mathrm{Lt}} \sqrt{2(a-x)} \left| \frac{\sigma_{yz}(x,0)}{p} \right|_{0<x<a} \tag{4.5a}$$

$$N_b = \underset{x \to b+}{\mathrm{Lt}} \sqrt{2(x-b)} \left| \frac{\sigma_{yz}(x,0)}{p} \right|_{b<x<c} \tag{4.5b}$$

$$N_c = \underset{x \to c-}{\mathrm{Lt}} \sqrt{2(c-x)} \left| \frac{\sigma_{yz}(x,0)}{p} \right|_{b<x<c} \tag{4.5c}$$

$$N_1 = \underset{x \to 1+}{\mathrm{Lt}} \sqrt{2(x-1)} \left| \frac{\sigma_{yz}(x,0)}{p} \right|_{x>1} \tag{4.5d}$$

With the aid of the results given by (4.3) in (4.5) it follows that

$$N_a = - \frac{\mu s \sqrt{A}}{\sqrt{e(A^2-1)^{1/2}(B^2-A^2)}} B_1 \tag{4.6a}$$

$$N_b = -\frac{\mu s\sqrt{B}}{\sqrt{e(B^2-1)^{1/2}}}\left[-\frac{2p}{\pi\mu s}\sqrt{\frac{(B^2-A^2)(C^2-B^2)}{(D^2-B^2)}}\left\{\int_A^B G_1(Y)\,dY + \right.\right.$$

$$\left.\left.+\int_C^D G_1(Y)\,dY\right\} + \sqrt{\frac{(C^2-B^2)(D^2-A^2)}{(B^2-A^2)(C^2-A^2)(D^2-B^2)}}\,B_1 - \sqrt{\frac{(B^2-A^2)}{(C^2-B^2)(D^2-B^2)}}\,B_2\right] \tag{4.6b}$$

$$N_c = -\frac{\mu s\sqrt{C(C^2-A^2)}}{\sqrt{e(C^2-1)^{1/2}(C^2-B^2)(D^2-C^2)}}\,B_2 \tag{4.6c}$$

$$N_1 = -\frac{\mu s\sqrt{D}}{\sqrt{e(D^2-1)^{1/2}}}\left[-\frac{2p}{\pi\mu s}\sqrt{\frac{(D^2-A^2)(D^2-C^2)}{(D^2-B^2)}}\left\{\int_A^B G_2(Y)\,dY + \right.\right.$$

$$\left.\left.+\int_C^D G_2(Y)\,dY\right\} + \sqrt{\frac{(D^2-C^2)}{(D^2-B^2)(C^2-A^2)}}\,B_1 + \sqrt{\frac{(D^2-A^2)}{(D^2-C^2)(D^2-B^2)}}\,B_2\right] \tag{4.6d}$$

where

$$G_1(Y) = \frac{\sqrt{(D^2-Y^2)}}{\sqrt{(Y^2-A^2)(Y^2-B^2)(Y^2-C^2)}} \tag{4.7a}$$

$$G_2(Y) = \frac{\sqrt{(B^2-Y^2)}}{\sqrt{(Y^2-A^2)(C^2-Y^2)(D^2-Y^2)}} \tag{4.7b}$$

The crack opening displacements are obtained by using the expressions for $G(U^2)$ and $H(S^2)$ from equations (3.14) and (3.13) in equations (4.1a,b).

Again letting $a \to 0$ and simplifying, it may be noted that the results (4.6b), (4.6c) and (4.6d) become those given by equations (3.16) of Das (1993).

5. NUMERICAL RESULTS

The numerical values of stress intensity factors (SIF) N_a, N_b, N_c and N_1 given by (4.6a-d) at the tips of the crack have been plotted against crack speed (V/c_2) for different values of crack lengths, separating distances of the cracks and strip width(h).

Keeping the length of the outer cracks and distance between inner and outer cracks fixed (b=0.6, c=0.8) SIFs at the tips of the cracks have been plotted against crack speed ($0.1 \le V/c_2 < 1$) for different lengths of the inner cracks (a=0.2, 0.4) and strip width (h=1,3,5). It is found from the graphs (fig.2-5) that SIFs increase rapidly as $V/c_2 \to 1$ and with the decrease in the value of inner crack length i.e. with the increase in the value of the distance between inner cracks the value of SIF decreases.

When lengths of the outer cracks and the distance between inner cracks are kept fixed (a=0.2, c=0.8) it is noted from the graphs (fig.6-9) that with the increase in the value of b (0.4, 0.6) i.e. with the decrease in the value of the distance between inner and

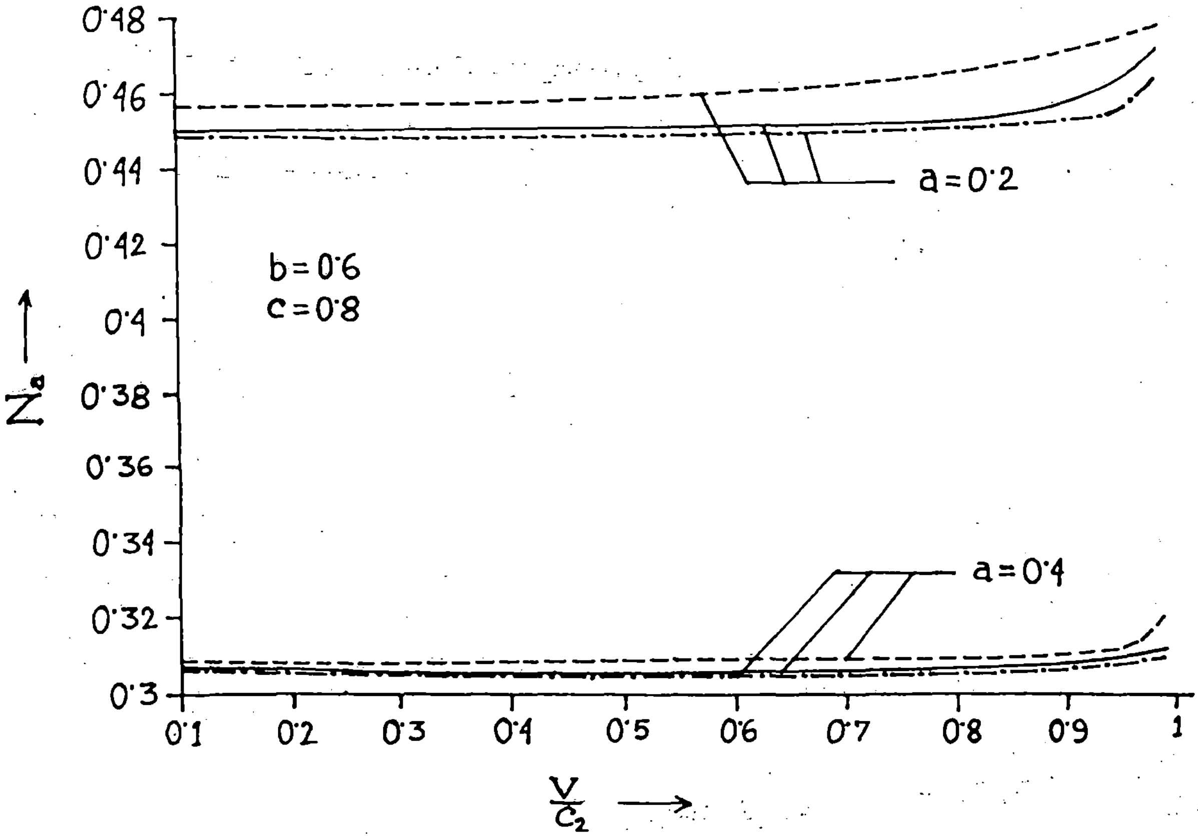

Fig. 2. Stress intensity factor N_a vs. V/c_2.
(---- h=1, —— h=2, —.—.— h=5).

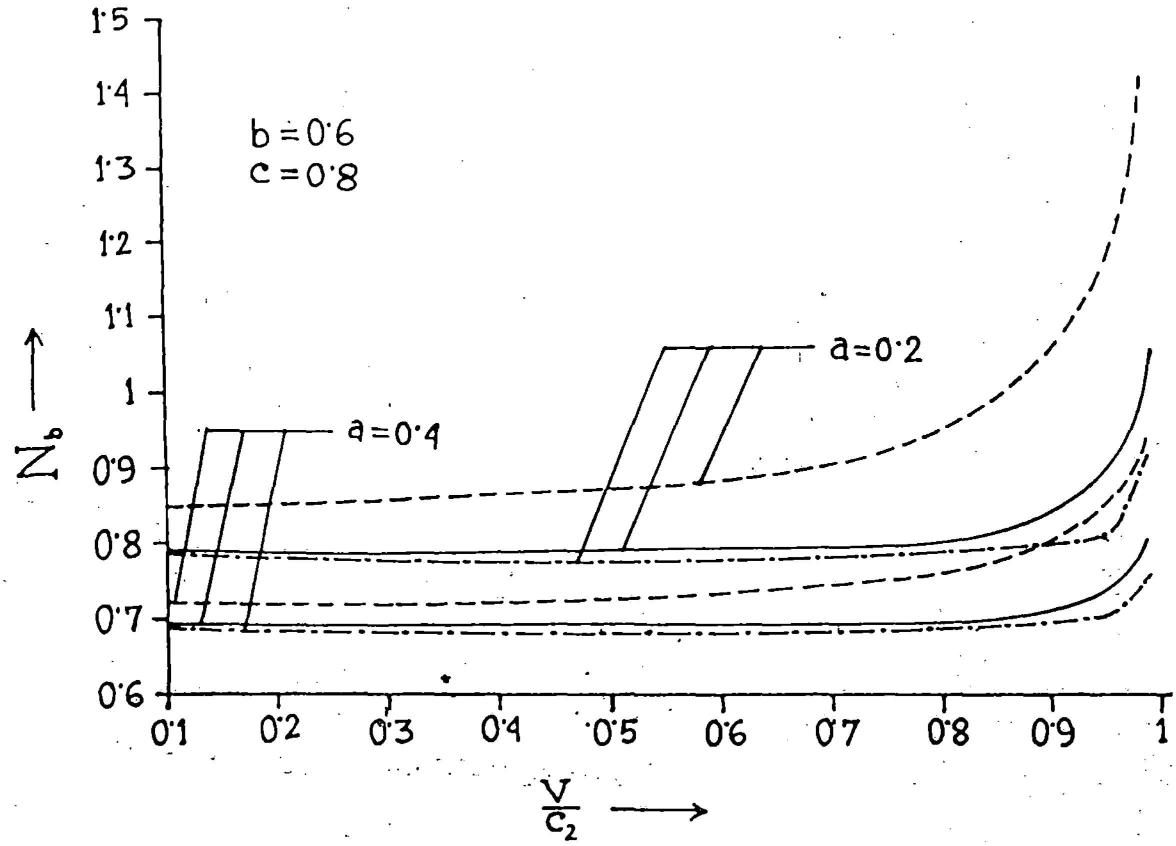

Fig. 3. Stress intensity factor N_b vs. V/c_2.
(---- h=1, —— h=2, —.—.— h=5).

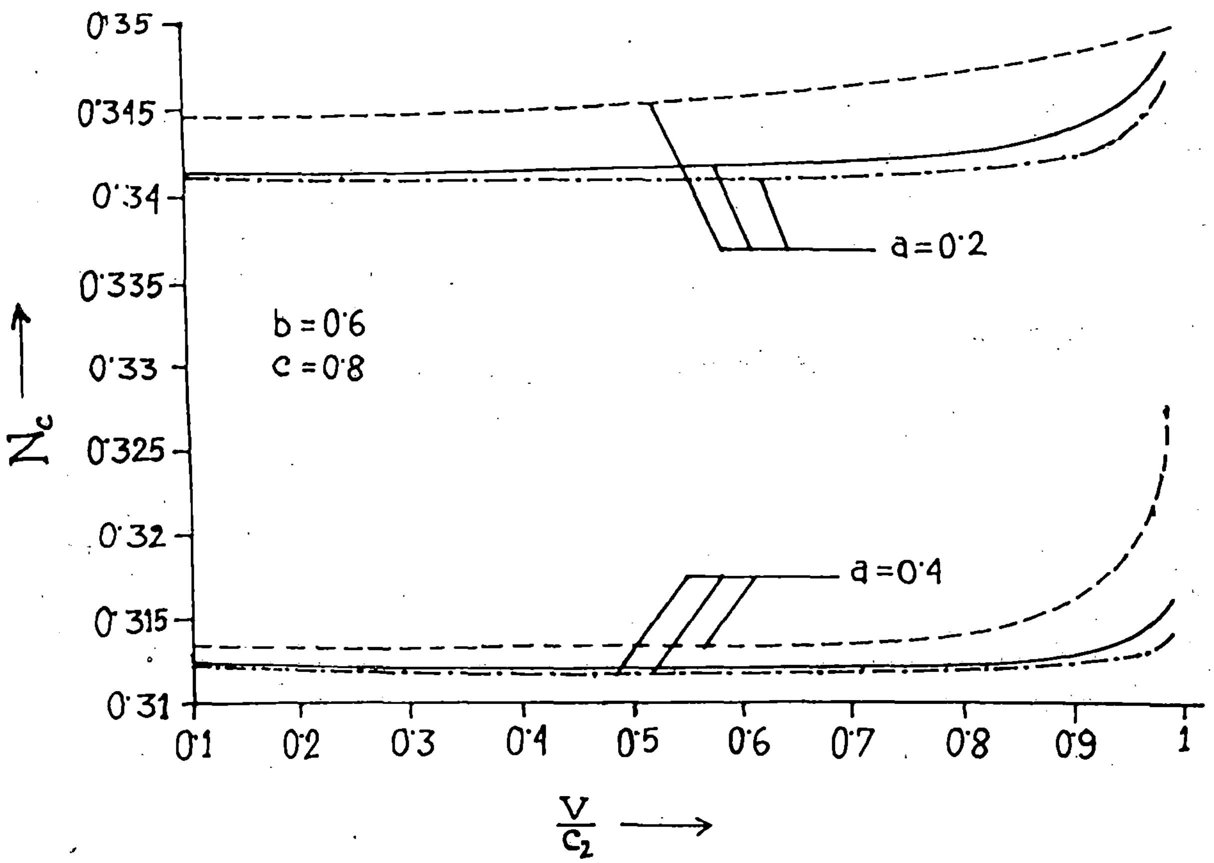

Fig. 4. Stress intensity factor N_c vs. V/c_2.
(---- h=1, —— h=2, —.—.— h=5).

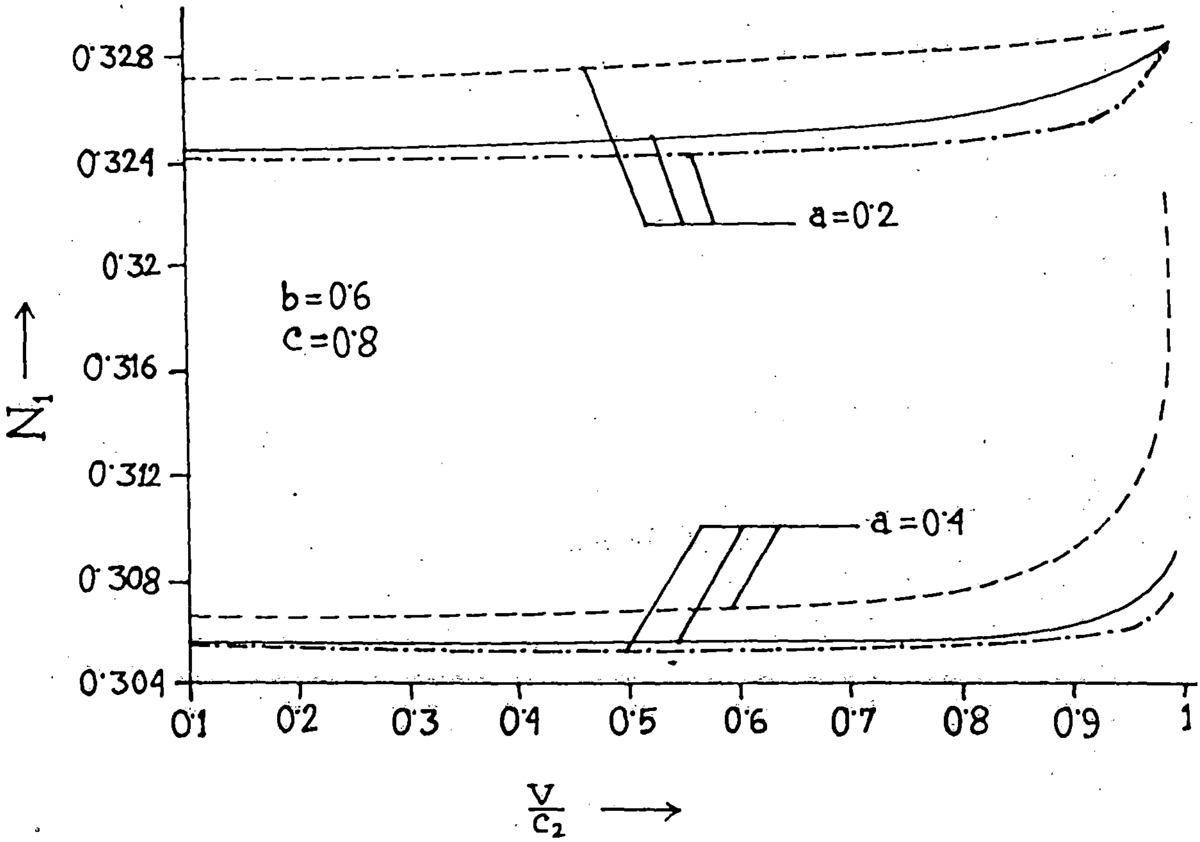

Fig. 5. Stress intensity factor N_1 vs. V/c_2.
(---- h=1, —— h=2, —.—.— h=5).

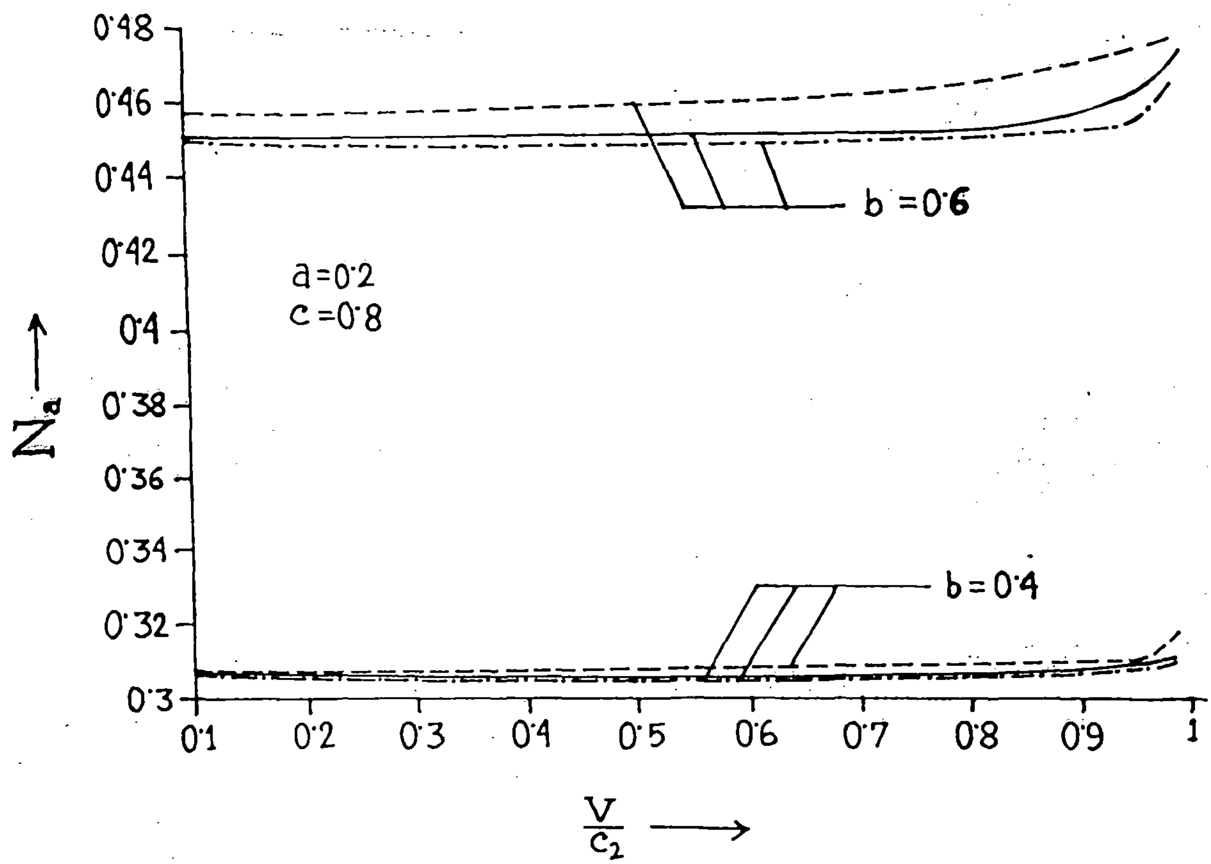

Fig.6. Stress intensity factor N_a vs. V/c_2.
(---- h=1, —— h=2, —.—.— h=5).

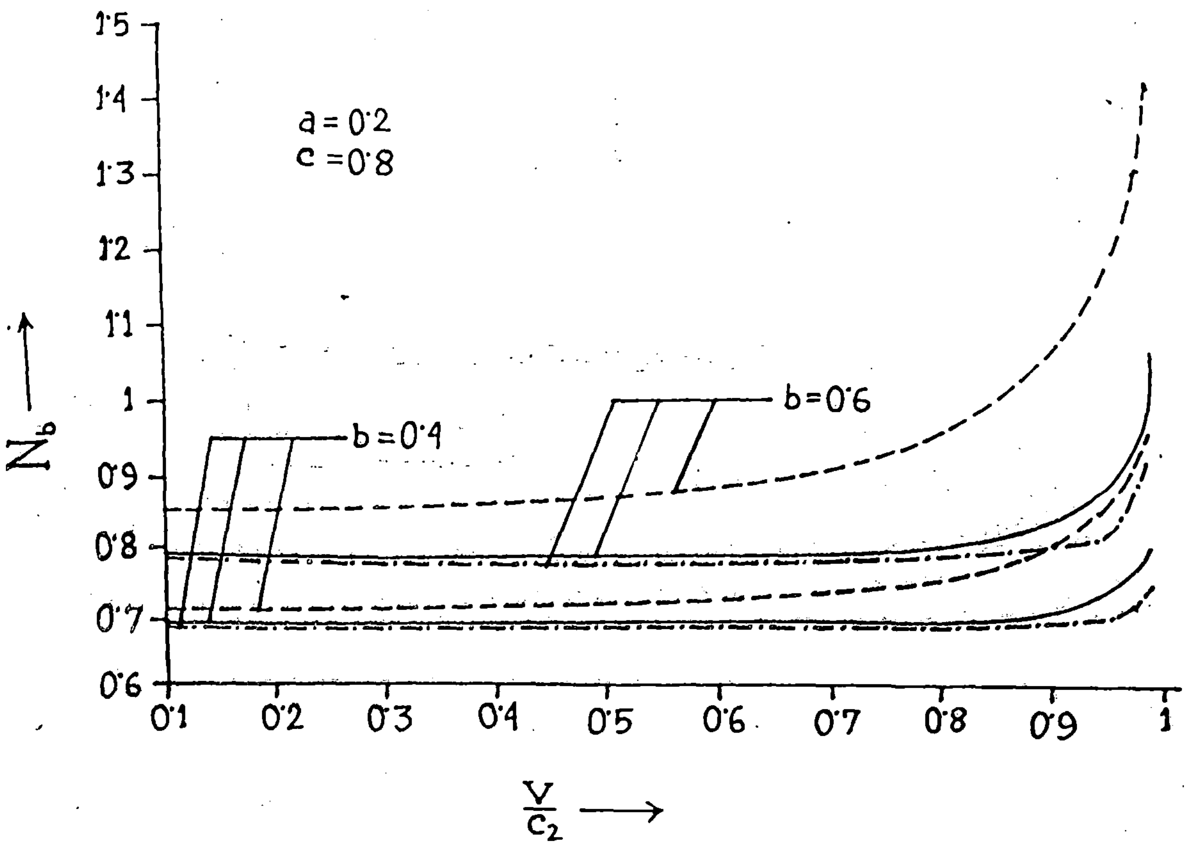

Fig.7. Stress intensity factor N_b vs. V/c_2.
(---- h=1, —— h=2, —.—.— h=5).

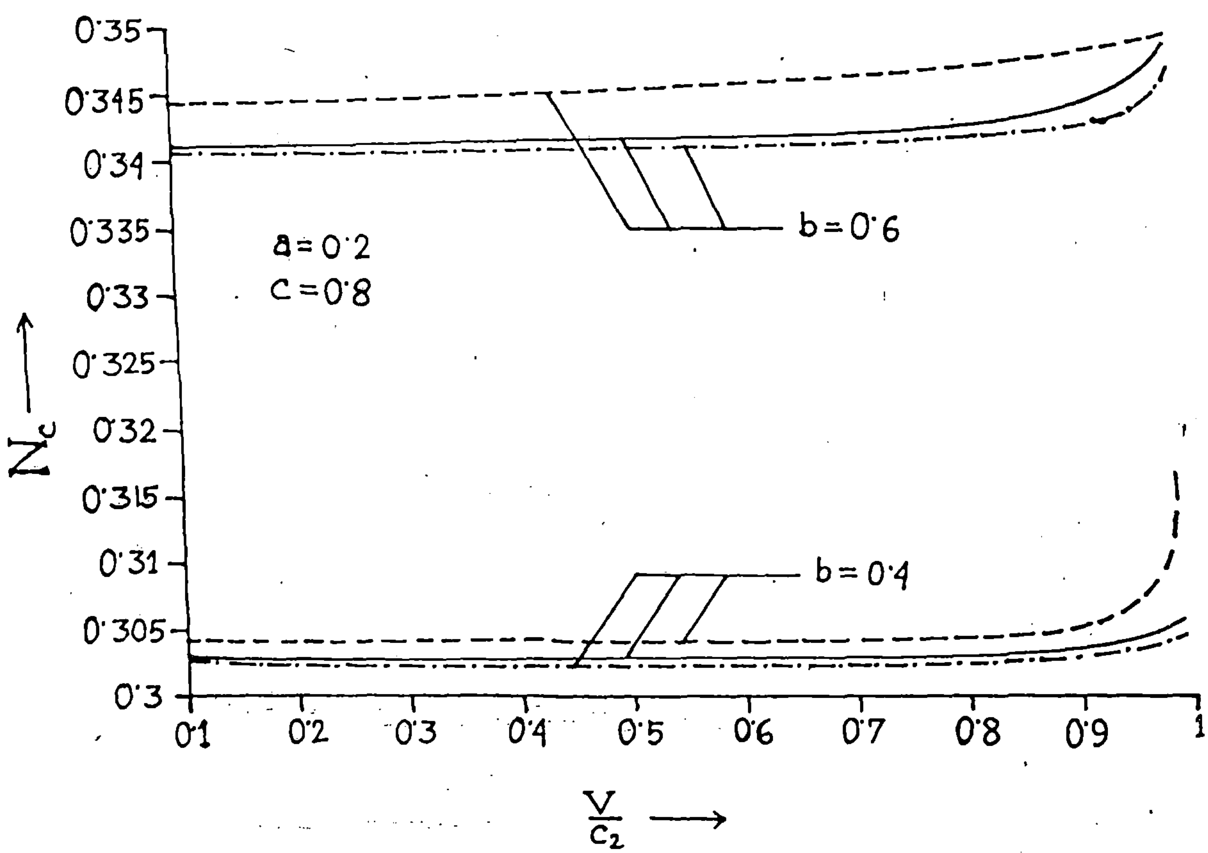

Fig. 8. Stress intensity factor N_c vs. V/c_2.
(---- h=1, —— h=2, —.—.— h=5).

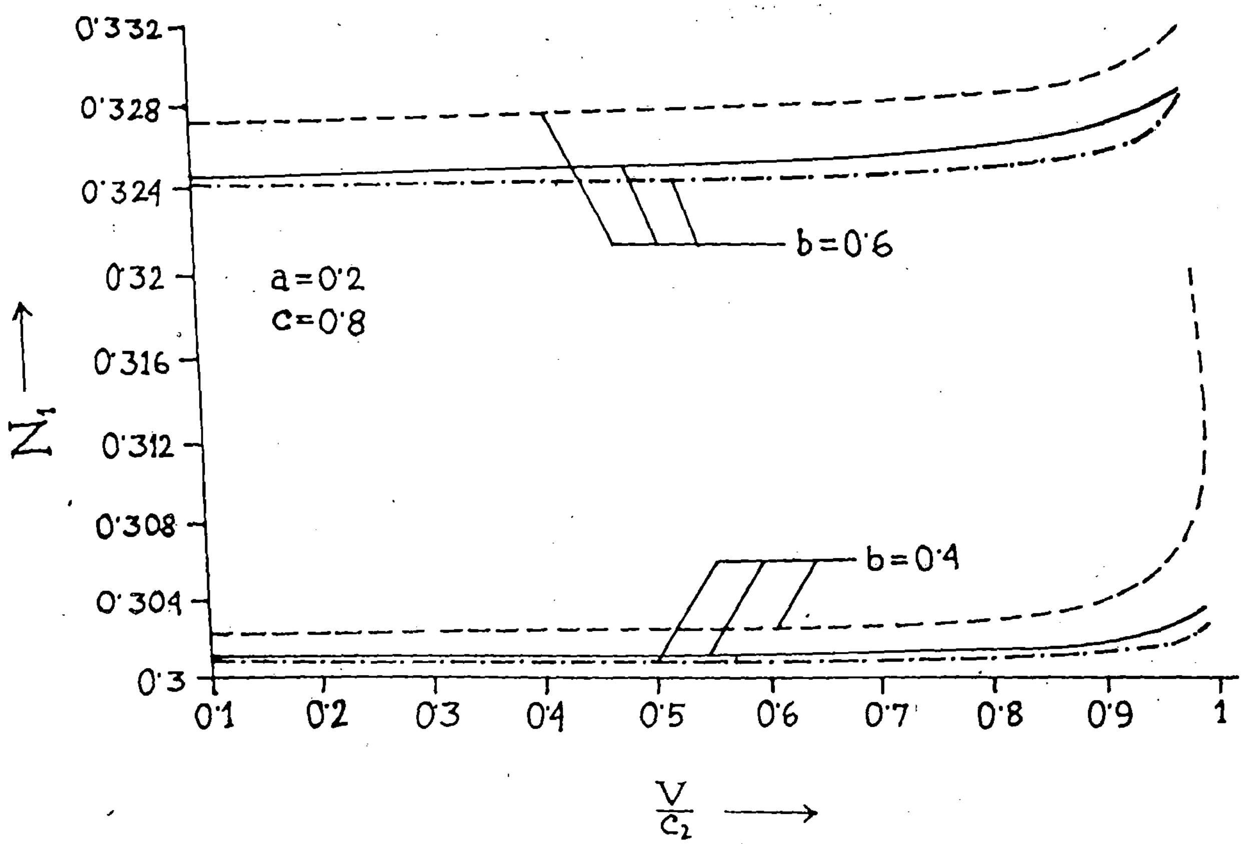

Fig.9. Stress intensity factor N_1 vs. V/c_2.
(---- h=1, —— h=2, —.—.— h=5).

outer cracks SIF increases.

Next, keeping the lengths of the inner cracks fixed (a=0.2, b=0.4), it is seen from the graphs (fig.10-13) that the value of SIF N_b is higher for higher values of c(0.6, 0.8). But the nature is opposite in case of N_a, N_b and N_1.

In all the cases mentioned above the SIFs increase with the increase in the value of V/c_2 gradually at a slow rate in the begining but increase rapidly as $V/c_2 \to 1$. Also the value of SIFs are higher for lower values of h in these cases.

The nature of SIFs, when plotted against 'a' are exhibited in fig.14-17. In fig.14 for fixed strip width (h=2) SIFs have been plotted against 'a' for different values of V/c_2(0.1,0.8). From the graph it is found that SIF N_b firstly increases with the increase in the value of 'a', attains a maximum and then decreases rapidly whereas SIFs N_b and N_1 decrease gradually with the increase in the value of 'a'. Further it is found that SIFs are higher for higher values of V/c_2.

In fig.15-17, SIFs have been plotted against 'a' for different values of h(1,2,3,4) when V/c_2 is kept fixed. With the increase in the value of 'a', N_b first increases and then decreases sharply. But N_c and N_1 decrease gradually (fig.16-17) with the increase in the value of 'a'. In all the cases (fig.15-17) values of SIFs are found to be higher for lower values of h as expected from the physical standpoint.

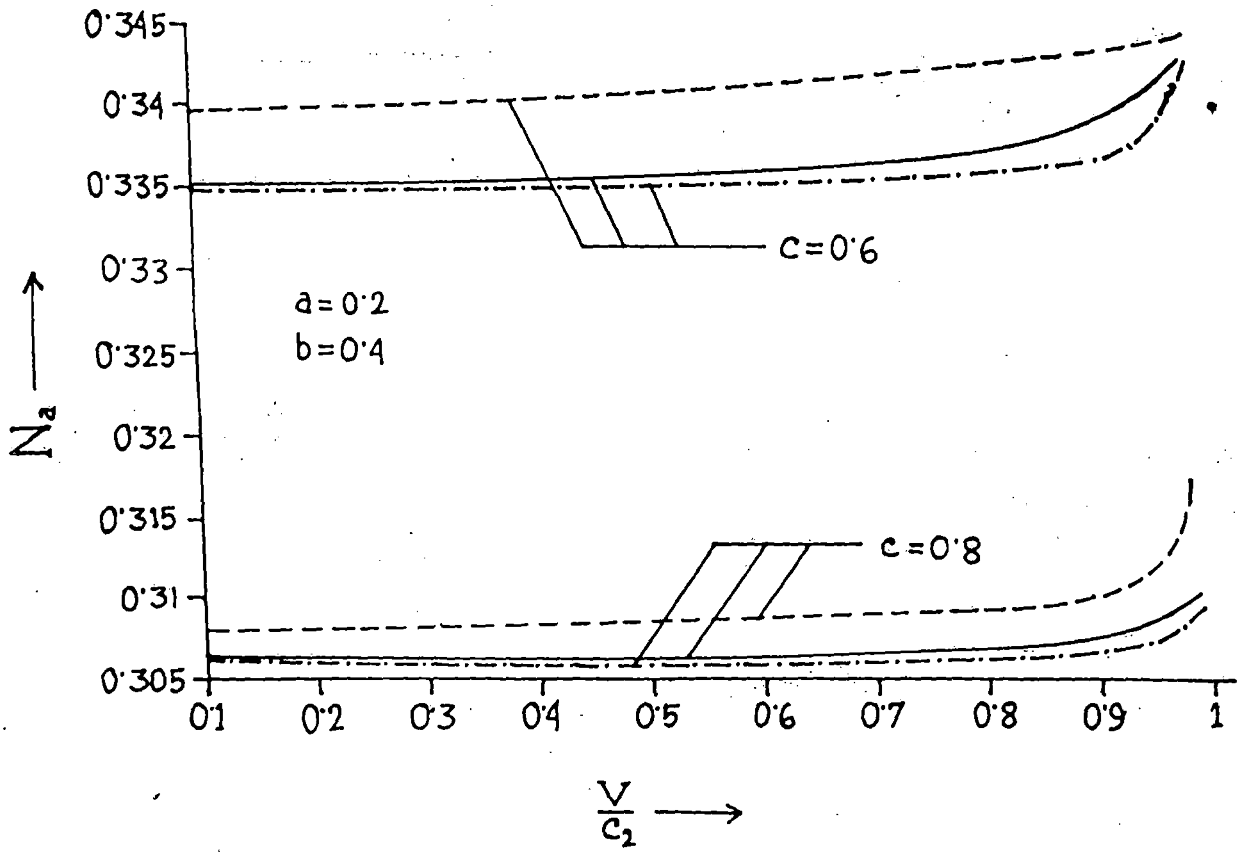

Fig.10. Stress intensity factor N_a vs. V/c_2.
(---- h=1, —— h=2, —.—.— h=5).

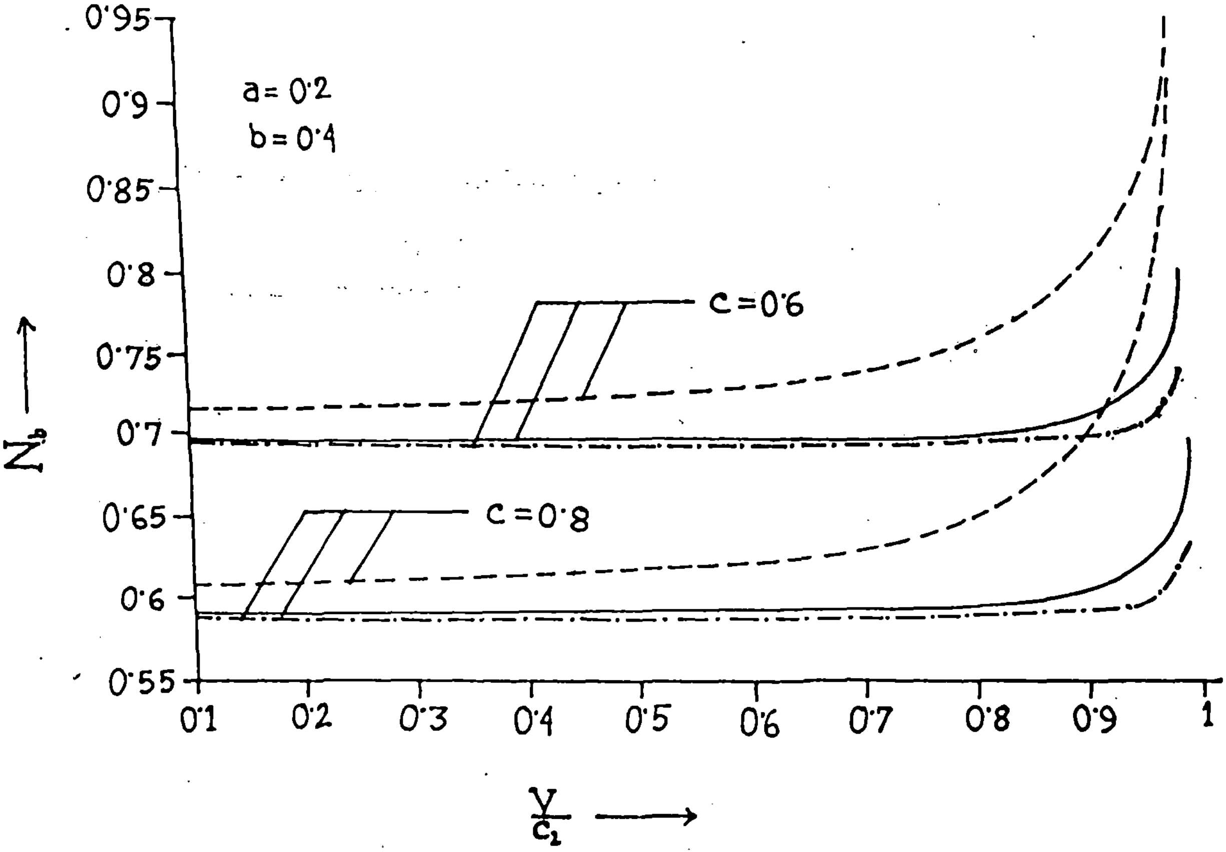

Fig.11. Stress intensity factor N_b vs. V/c_2.
(---- h=1, —— h=2, —.—.— h=5).

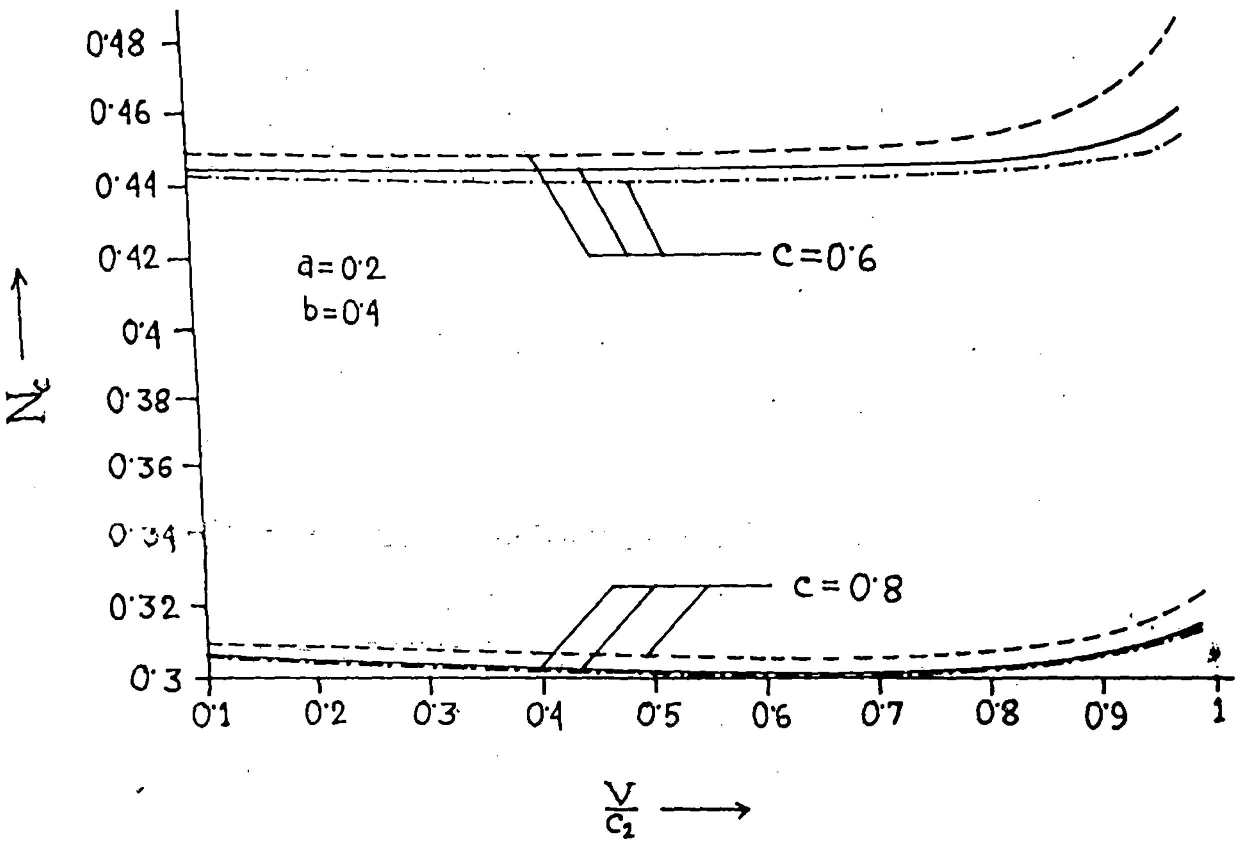

Fig.12. Stress intensity factor N_c vs. V/c_2.
(---- h=1, —— h=2, —·—·— h=5).

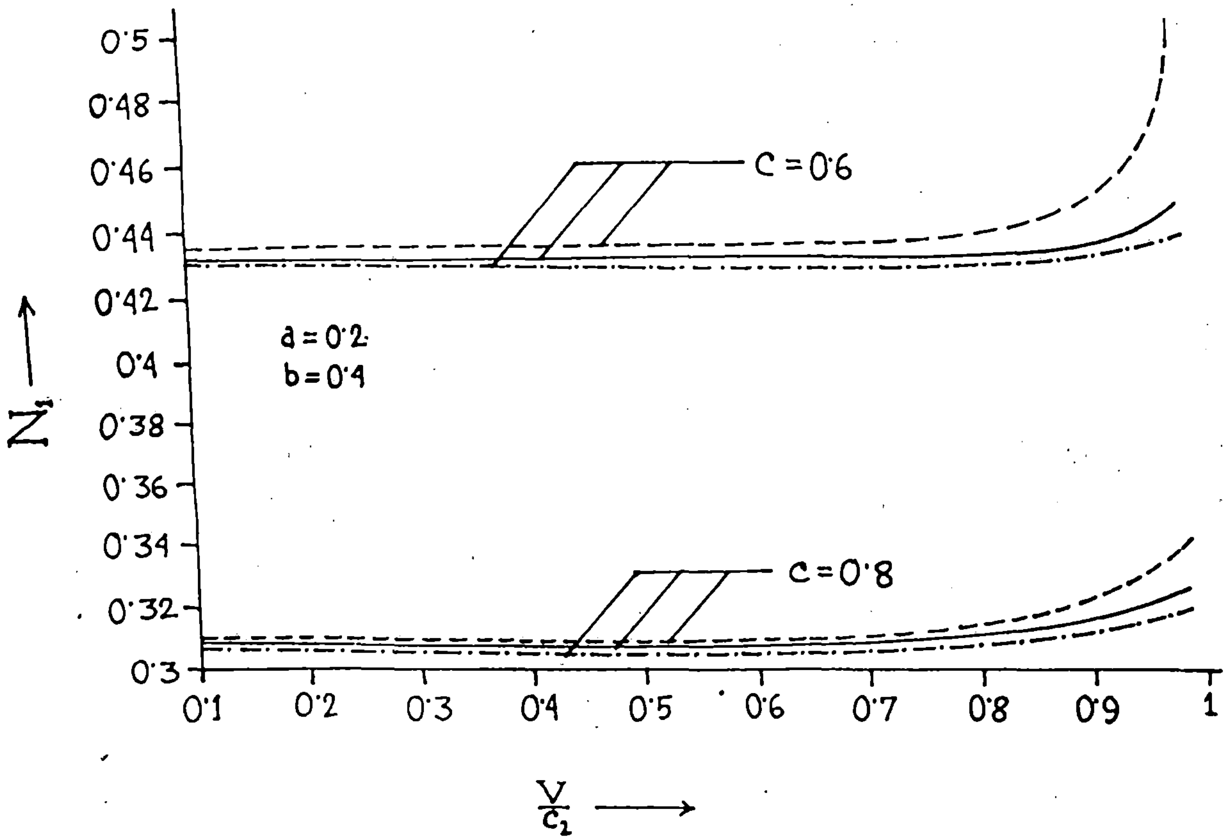

Fig.13. Stress intensity factor N_1 vs. V/c_2.
(---- h=1, —— h=2, —.—.— h=5).

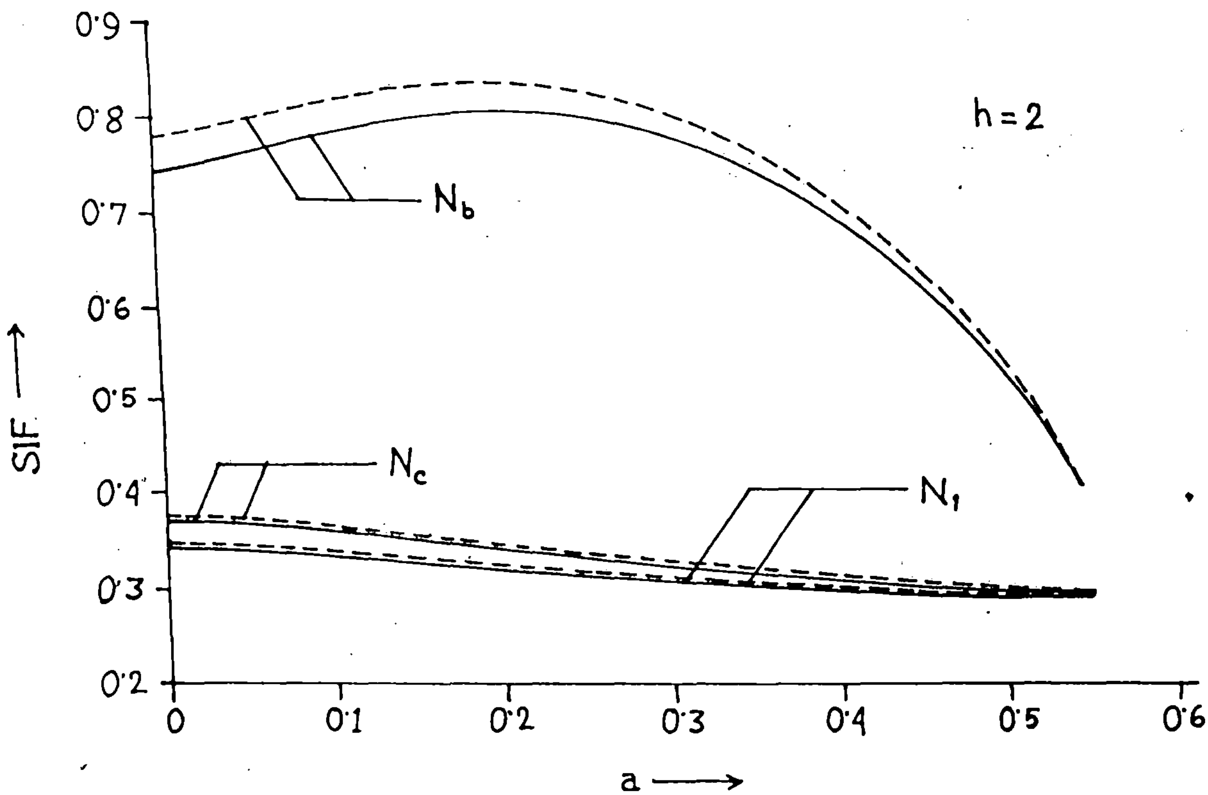

Fig.14. Stress intensity factors vs. a.
(—— V/c_2=0.1, ----- V/c_2=0.8).

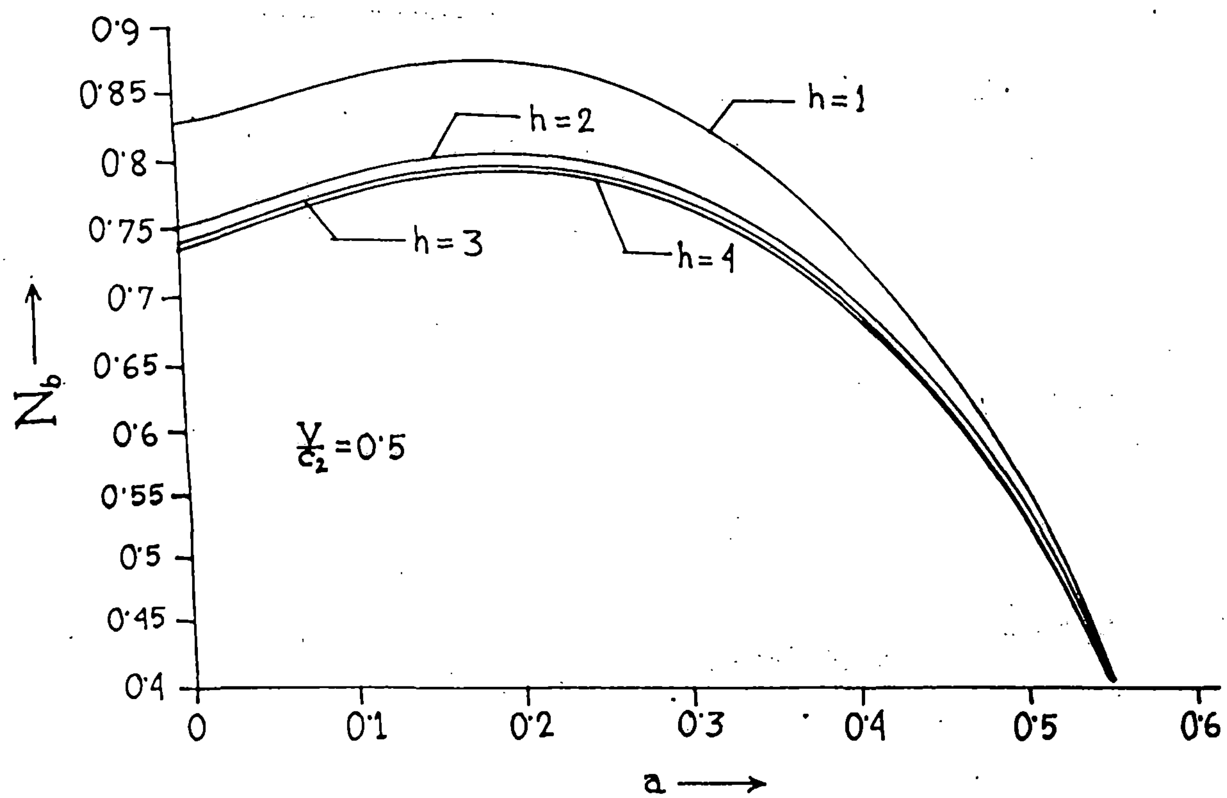

Fig.15. Stress intensity factor N_b vs. a.
(V/c_2=0.5).

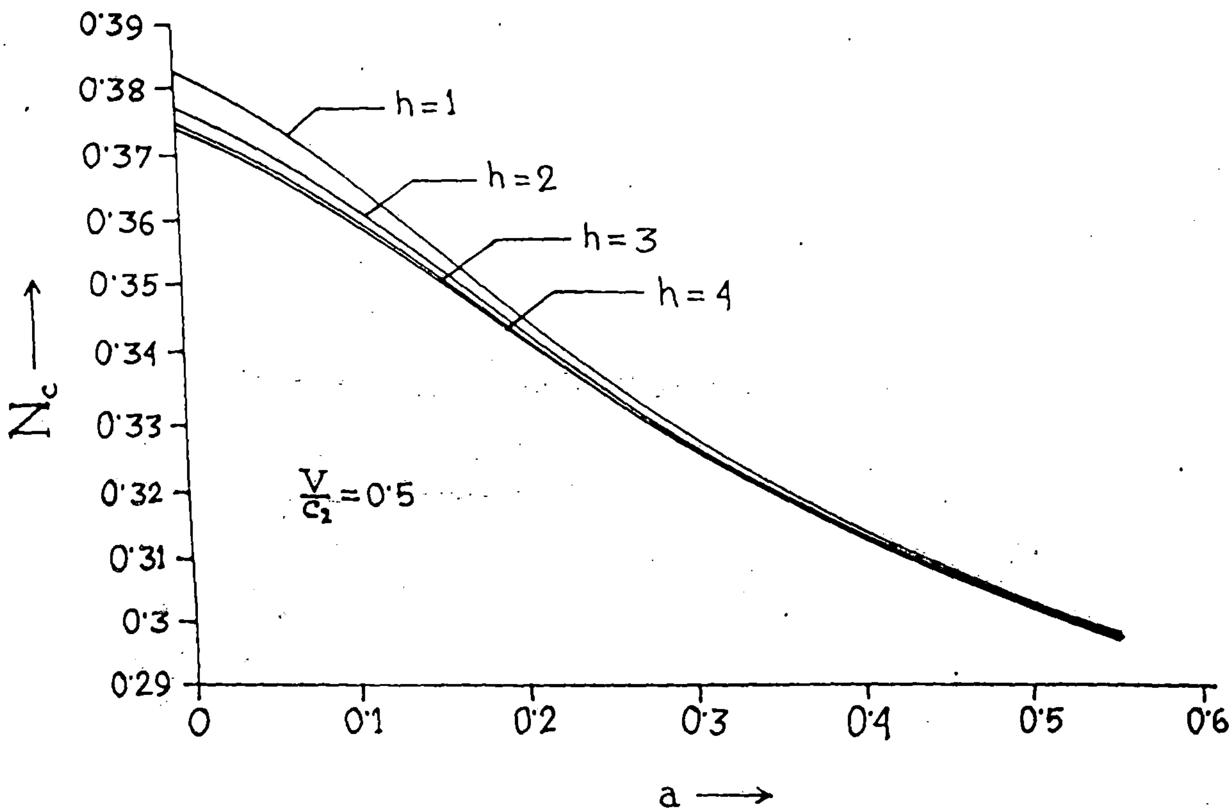

Fig.16. Stress intensity factor N_c vs. a. (V/c_2=0.5).

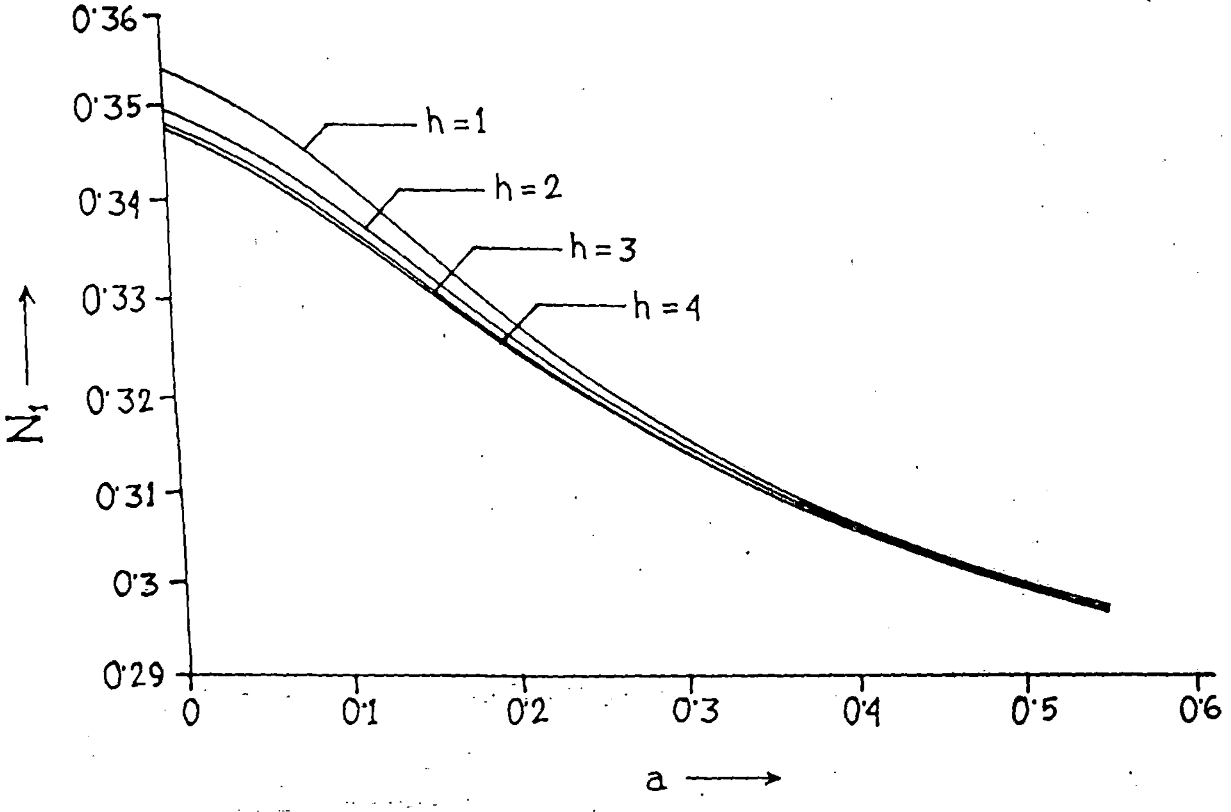

Fig.17. Stress intensity factor N_1 vs. a. (V/c_2=0.5).

CHAPTER - III

LOW FREQUENCY SCATTERING OF ELASTIC WAVES BY GRIFFITH CRACKS IN ORTHOTROPIC ELASTIC MEDIUM

PAPER 7 : Diffraction of elastic waves by two parallel rigid strips embedded in an orthotropic medium.

PAPER 8 : Interaction of elastic waves with two coplanar Griffith cracks in an orthotropic medium.

PAPER 9 : Diffraction of elastic waves by three coplanar Griffith cracks in an orthotropic medium.

PAPER 10 : Elastic wave scattering from four coplanar Griffith cracks in an orthotropic medium.

DIFFRACTION OF ELASTIC WAVES BY TWO PARALLEL RIGID STRIPS EMBEDDED IN AN INFINITE ORTHOTROPIC MEDIUM

1. INTRODUCTION

Recently, with the increased usage of macroscopically anisotropic construction materials such as fibre-reinforced composites, the study of interaction of elastic waves with cracks or inclusions in composite materials has gained much importance. The problems involving inclusions in isotropic medium have been studied by many authors. Palaiya and Majumder (1981) considered the problem of a single strip at a bimaterial interface. Forced vertical vibration of a single strip was treated by Wickham (1977). Jain and Kanwal (1972b)have solved the problem of two rigid strips embedded in an isotropic elastic medium. Recently Mandal and Ghosh (1992b) have treated the problem of vertical vibration of two rigid strips on the surface of a semi-infinite medium. The problem involving single Griffith crack in orthotropic medium was investigated by Kassir and Bandyopadhyay (1983), Shindo et al. (1986), De and Patra (1990). Shindo et al. (1991) have investigated the impact response of symmetric edge cracks in an orthotropic strip. But perhaps, due to mathematical complexity, elastodynamic problems involving two or more Griffith cracks or strips in anisotropic materials have not yet received much attention.

In our problem, the interaction of normally incident time harmonic elastic waves with two rigid strips embedded in an infinite orthotropic medium has been considered. The resulting mixed boundary value problem is reduced to the solution of a triple integral equation which has further been reduced to the solution of an integro-differential equation. Iterative solution valid for low frequency has been obtained. From the solution of the integral equation we have found out the normal stress and vertical displacement at points in the plane of the strips. Finally choosing the engineering elastic constants of the orthotropic material suitably the results for isotropic material have been deduced and compared with the results obtained by Jain and Kanwal (1972b). To display the influence of the material orthotropy numerical values of stress intensity factors and vertical displacement have been plotted against dimensionless frequency and distance respectively for several orthotropic materials.

2. FORMULATION OF THE PROBLEM

Let us consider the diffraction of normally incident longitudinal wave by two symmetric coplanar and parallel rigid strips embedded in an infinite orthotropic elastic medium and the strips occupy the region $b\le|x_1|\le a$, $x_2=0$, $-\infty<x_3<\infty$. Let E_i, μ_{ij} and ν_{ij} $(i,j=1,2,3)$ denote the engineering elastic constants of the material where the subscripts 1,2,3 correspond to the x_1, x_2, x_3 directions which coincide with the axes of material orthotropy. Normalizing all lengths with respect to 'a' and putting $x_1/a=x$, $x_2/a=y$, $x_3/a=z$, $b/a=c$, the rigid strips are defined by $c\leq|x|\leq 1$, $y=0$, $|z|<\infty$(Fig.1). Let a time harmonic wave given by $u=0$ and $v=v_0\exp[i(ky-\omega t)]$ where $k=a\omega/c_s\sqrt{c_{22}}$, $c_s=(\mu_{12}/\rho)^{1/2}$ and v_0 is a constant, travelling in the direction of positive y-axis be incident normally on the strips.

The non-zero stress components τ_{yy} and τ_{xy} are given by

$$\begin{aligned} \tau_{yy}/\mu_{12} &= c_{12}\frac{\partial u}{\partial x} + c_{22}\frac{\partial v}{\partial y} \\ \tau_{xy}/\mu_{12} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned} \tag{1}$$

where $c_{ij}(i,j=1,2)$ are nondimensional parameters related to the elastic constants by the relations

$$\begin{aligned} c_{11} &= E_1/\mu_{12}(1-\nu_{12}^2E_2/E_1) \\ c_{22} &= E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1) = c_{11}E_2/E_1 \\ c_{12} &= \nu_{12}E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1) = \nu_{12}c_{22} = \nu_{21}c_{11} \end{aligned} \tag{2}$$

for generalized plane stress, and by

$$c_{11}=(E_1/\Delta\mu_{12})(1-\nu_{23}\nu_{32})$$

$$c_{22}=(E_2/\Delta\mu_{12})(1-\nu_{13}\nu_{31})$$

$$c_{12}=E_1(\nu_{21}+\nu_{13}\nu_{32}E_2/E_1)/\Delta\mu_{12}$$

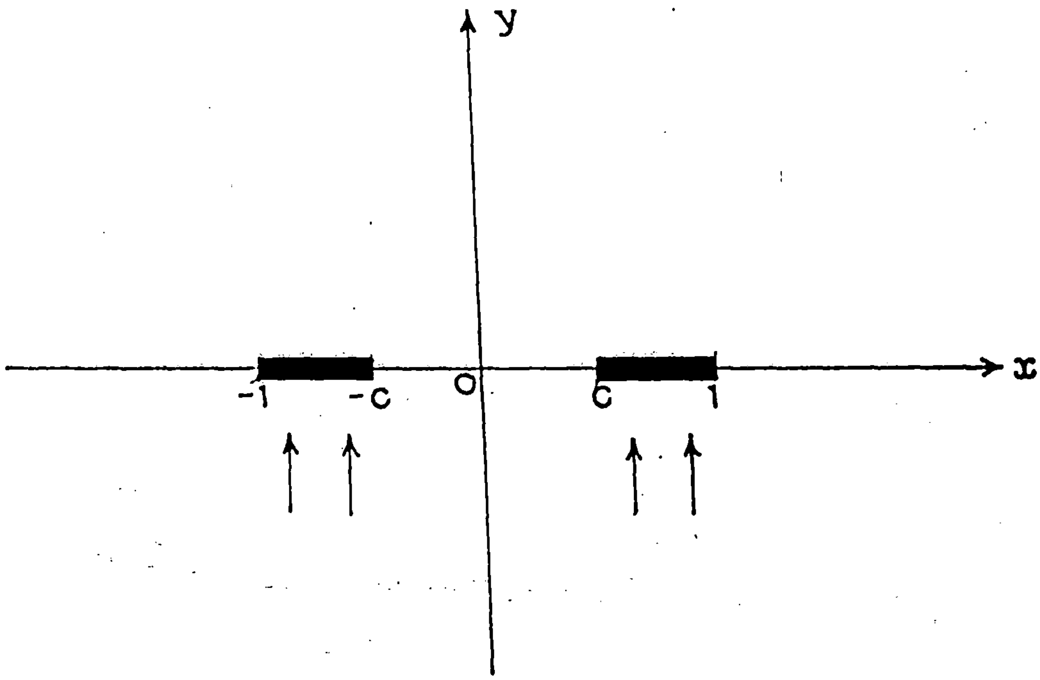

Fig. 1. Geometry of the strips.

$$=E_2(\nu_{12}+\nu_{23}\nu_{31}E_1/E_2)/\Delta\mu_{12}$$

$$\Delta = 1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{13}\nu_{21}\nu_{32} \tag{3}$$

for plane strain. The constants E_i and ν_{ij} satisfy the Maxwell's relation $\nu_{ij}/E_i = \nu_{ji}/E_j$.

The equations of motion for orthotropic material, in terms of displacements are

$$c_{11}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+(1+c_{12})\frac{\partial^2 v}{\partial x\partial y} = \frac{a^2}{c_s^2}\frac{\partial^2 u}{\partial t^2}$$

$$c_{22}\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}+(1+c_{12})\frac{\partial^2 u}{\partial x\partial y} = \frac{a^2}{c_s^2}\frac{\partial^2 v}{\partial t^2} \tag{4}$$

Therefore, substituting $u(x,y,t) = u(x,y)\exp(-i\omega t)$ and $v(x,y,t) = v(x,y)\exp(-i\omega t)$ our problem reduces to the solution of the equations

$$c_{11}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+(1+c_{12})\frac{\partial^2 v}{\partial x\partial y}+\frac{a^2\omega^2}{c_s^2}u = 0$$

and

$$c_{22}\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}+(1+c_{12})\frac{\partial^2 u}{\partial x\partial y}+\frac{a^2\omega^2}{c_s^2}v = 0 \tag{5}$$

subject to the boundary conditions

$$v(x,0) = -v_0 \quad , \quad c\le|x|\le 1 \tag{6}$$

$$\tau_{yy}(x,0) = 0 \quad , \quad |x|<c,\ |x|>1 \tag{7}$$

$$u(x,0) = 0 \quad , \quad |x|<\infty. \tag{8}$$

Henceforth the time factor exp(-iωt) which is common to all field variables would be omitted in the sequel.

The solutions of equations (5) are taken as

$$u(x,y) = \frac{2}{\pi}\int_0^\infty [A_1(\xi)\exp(-\gamma_1|y|) + A_2(\xi)\exp(-\gamma_2|y|)]\sin\xi x \, d\xi \;, \quad y>0 \tag{9}$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}[\alpha_1 A_1(\xi)\exp(-\gamma_1|y|) + \alpha_2 A_2(\xi)\exp(-\gamma_2|y|)]\cos\xi x \, d\xi \tag{10}$$

$$\text{where} \quad \alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1+c_{12})\gamma_i} \;, \quad i=1,2 \;, \quad k_s^2 = \frac{a^2\omega^2}{c_s^2} \tag{11}$$

and $A_i(\xi)(i=1,2)$ are the unknowns to be solved, γ_1^2 ,γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \left\{(c_{12}^2+2c_{12}-c_{11}c_{22})\xi^2 + (1+c_{22})k_s^2\right\}\gamma^2 + (c_{11}\xi^2-k_s^2)(\xi^2-k_s^2) = 0 \tag{12}$$

From the boundary condition (8) it is found that

$$A_2(\xi) = -A_1(\xi).$$

Therefore displacements u, v and stresses τ_{yy}, τ_{xy} finally can be written as

$$u(x,y) = \frac{2}{\pi}\int_0^\infty [\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)]A_1(\xi)\sin\xi x \, d\xi \;, \quad y>0 \tag{13}$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}[\alpha_1\exp(-\gamma_1|y|) - \alpha_2\exp(-\gamma_2|y|)]A_1(\xi)\cos\xi x \, d\xi \tag{14}$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty \left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \right.$$

$$\left. - \left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\right]A_1(\xi)\cos\xi x \; d\xi \quad , \quad y>0 \tag{15}$$

$$\tau_{xy}/\mu_{12} = -\frac{2}{\pi}\int_0^\infty \left[(\gamma_1+\alpha_1)\exp(-\gamma_1|y|) - (\gamma_2+\alpha_2)\exp(-\gamma_2|y|)\right]A_1(\xi) \times$$

$$\times \sin\xi x \; d\xi \tag{16}$$

Next putting

$$A(\xi) = \frac{\alpha_1\gamma_1 - \alpha_2\gamma_2}{\xi} A_1(\xi)$$

the boundary conditions (6) and (7) lead to the following dual integral equations in $A(\xi)$:

$$\int_0^\infty \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2}\right) A(\xi)\cos\xi x \; d\xi = -\frac{\pi}{2} v_o \quad , \qquad c\le|x|\le 1 \tag{17}$$

and

$$\int_0^\infty A(\xi)\cos\xi x \; d\xi = 0 \quad , \qquad |x|<c \; , \; |x|>1 \tag{18}$$

3. SOLUTION OF THE PROBLEM

We consider the solution of the integral equations (17) and (18) in the form

$$A(\xi) = \int_c^1 t f(t^2)\cos\xi t \; dt \tag{19}$$

where $f(t^2)$ is an unknown function to be determined.

By the choice of $A(\xi)$ given by (19) the relation (18) is satisfied automatically and the equation (17) becomes

$$\int_c^1 tf(t^2)dt\int_0^\infty\left(\frac{\alpha_1}{\alpha_1\gamma_1} - \frac{\alpha_2}{\alpha_2\gamma_2}\right)\cos\xi x\ \cos\xi t\ d\xi = -\frac{\pi}{2}v_o\ ,\quad c\le|x|\le 1 \qquad (20)$$

Using the relation

$$\frac{\sin\xi x\ \sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_o(\xi w)J_o(\xi v)dvdw}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}$$

the above equation is converted to the form

$$\frac{d}{dx}\int_c^1 tf(t^2)dt\ \frac{\partial}{\partial t}\int_0^x\int_0^t \frac{vwL_1(v,w)\ dwdv}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} = -\frac{\pi}{2}v_o\ ,\qquad c\le|x|\le 1 \qquad (21)$$

where

$$L_1(v,w) = \int_0^\infty\left(\frac{\alpha_1}{\alpha_1\gamma_1} - \frac{\alpha_2}{\alpha_2\gamma_2}\right)J_o(\xi w)J_o(\xi v)\ d\xi \qquad (22)$$

By a contour integration technique, the infinite integral in $L_1(v,w)$ can be converted to the following finite integrals (details have been given in the appendix)

$$L_1(v,w) = -i\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{11}\eta^2-1-\bar\gamma_1\bar\gamma_2}{\bar\gamma_1\bar\gamma_2(\bar\gamma_1+\bar\gamma_2)}\ J_o(k_s\eta v)H_o^{(1)}(k_s\eta w)\ d\eta\ - \right.$$

$$\left. - \int_{1/\sqrt{c_{11}}}^1 \frac{c_{11}\eta^2-1+\bar\gamma_2'^2}{\bar\gamma_2'(\bar\gamma_1'^2+\bar\gamma_2'^2)}\ J_o(k_s\eta v)H_o^{(1)}(k_s\eta w)\ d\eta\right],\quad w>v \qquad (23)$$

where
$$\bar{\gamma}_1 = \left[\frac{1}{2}\left\{X_1-(X_1^2-4X_2)^{1/2}\right\}\right]^{1/2}$$

$$\bar{\gamma}_2 = \left[\frac{1}{2}\left\{X_1+(X_1^2-4X_2)^{1/2}\right\}\right]^{1/2}$$

$$\bar{\gamma}'_1 = \left[\frac{1}{2}\left\{-X_1+(X_1^2+4X_3)^{1/2}\right\}\right]^{1/2} \qquad (24)$$

$$\bar{\gamma}'_2 = \left[\frac{1}{2}\left\{X_1+(X_1^2+4X_3)^{1/2}\right\}\right]^{1/2}$$

$$X_1 = \frac{1}{c_{22}}\left\{(c_{12}^2+2c_{12}-c_{11}c_{22})\eta^2 + (1+c_{22})\right\}$$

$$X_2 = \frac{c_{11}}{c_{22}}\left(1-\eta^2\right)\left(\frac{1}{c_{11}} - \eta^2\right)$$

$$X_3 = \frac{c_{11}}{c_{22}}\left(1-\eta^2\right)\left(\eta^2 - \frac{1}{c_{11}}\right)$$

The corresponding expression of $L_1(v,w)$ for $w<v$ follows from (23) by interchanging w and v.

Substituting the series expansion of J_o and $H_o^{(1)}$ in (23) we find after some algebraic manipulation

$$L_1(v,w) = \frac{2}{\pi}\left[\left(\gamma+\log(k_s w/2)-\frac{\pi i}{2}\right)M + N - \frac{(w^2+v^2)}{4}Rk_s^2\log k_s\right] + O(k_s^2) \quad , \; w>v$$

$$= \frac{2}{\pi}\left[\left(\gamma+\log(k_s v/2)-\frac{\pi i}{2}\right)M + N - \frac{(w^2+v^2)}{4}Rk_s^2\log k_s\right] + O(k_s^2) \quad , \; v>w \qquad (25)$$

where γ = 0.5772157............. is Euler's constant,

$$M = \int_0^{1/\sqrt{c_{11}}} \frac{c_{11}\eta^2 - 1 - \bar{\gamma}_1\bar{\gamma}_2}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1 + \bar{\gamma}_2)} d\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_2'^2}{\bar{\gamma}_2'(\bar{\gamma}_1'^2 + \bar{\gamma}_2'^2)} d\eta \tag{26}$$

$$N = \int_0^{1/\sqrt{c_{11}}} \frac{c_{11}\eta^2 - 1 - \bar{\gamma}_1\bar{\gamma}_2}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1 + \bar{\gamma}_2)} \log\eta \, d\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_2'^2}{\bar{\gamma}_2'(\bar{\gamma}_1'^2 + \bar{\gamma}_2'^2)} \log\eta \, d\eta \tag{27}$$

and
$$R = \int_0^{1/\sqrt{c_{11}}} \frac{\eta^2(c_{11}\eta^2 - 1 - \bar{\gamma}_1\bar{\gamma}_2)}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1 + \bar{\gamma}_2)} d\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{\eta^2(c_{11}\eta^2 - 1 + \bar{\gamma}_2'^2)}{\bar{\gamma}_2'(\bar{\gamma}_1'^2 + \bar{\gamma}_2'^2)} d\eta \tag{28}$$

Now differentiating both sides of the relation (20) with respect to x we obtain

$$\int_c^1 tf(t^2)dt \int_0^\infty \xi \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2}\right) \cos\xi x \, \cos\xi t \, d\xi = 0 , \qquad c \le |x| \le 1$$

Following similar procedure as done for deriving equation (21), we obtain

$$\int_c^1 \frac{tf(t^2)}{x^2 - t^2} dt = \int_c^1 tf(t^2)dt \, \frac{\partial}{\partial t}\int_0^x \int_0^t \frac{vwL_2(v,w) \, dwdv}{(x^2 - w^2)^{1/2}(t^2 - v^2)^{1/2}} , \quad c \le |x| \le 1 \tag{29}$$

where

$$L_2(v,w) = \int_0^\infty \left[\xi - \frac{\xi^2}{\theta}\left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2}\right)\right] J_0(\xi w) J_0(\xi v) \, d\xi \tag{30}$$

$$\theta = \frac{c_{11} + N_1 N_2}{N_1 + N_2}$$

$$N_1^2 = \frac{1}{2c_{22}}\left[-(c_{12}^2+2c_{12}-c_{11}c_{22})+\sqrt{(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}}\right]$$

and $$N_2^2 = \frac{1}{2c_{22}}\left[-(c_{12}^2+2c_{12}-c_{11}c_{22})-\sqrt{(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}}\right].$$

We use the contour integration technique mentioned earlier and get from (30)

$$L_2(v,w) = \frac{ik_s^2}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{\eta^2(c_{11}\eta^2-1-\bar{\gamma}_1\bar{\gamma}_2)}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1+\bar{\gamma}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,d\eta\right. -$$

$$\left. - \int_{1/\sqrt{c_{11}}}^{1} \frac{\eta^2(c_{11}\eta^2-1+\bar{\gamma}_2'^2)}{\bar{\gamma}_2'(\bar{\gamma}_1'^2+\bar{\gamma}_2'^2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,d\eta\right], \quad w>v$$

(31)

By the process similar to the one which led to the equation (25), (30) for small values of k_s can be written as

$$L_2(v,w) = -\frac{2}{\pi}Pk_s^2\log k_s + O(k_s^2) \tag{32}$$

where

$P = \frac{1}{\theta}R$ and R is given by (28).

Now, let us consider

$$f(t^2) = f_0(t^2) + k_s^2\log k_s f_1(t^2) + O(k_s^2) \tag{33}$$

Putting the above expansion of $f(t^2)$ and the value of $L_2(v,w)$ given by (32) in the equation (29) and equating the coefficients of like powers of k_s we obtain,

$$\int_c^1 \frac{tf_o(t^2)}{x^2-t^2}\,dt = 0 \qquad , \qquad c\le|x|\le 1 \tag{34}$$

and $$\int_c^1 \frac{tf_1(t^2)}{x^2-t^2}\,dt = -\frac{2P}{\pi}\int_c^1 tf_o(t^2)\,dt \ , \qquad c\le|x|\le 1 \tag{35}$$

Following Srivastava and Lowengrub (1968) the solutions of the above integral equations can be obtained as

$$f_o(t^2) = \frac{D_1}{\sqrt{(1-t^2)(t^2-c^2)}} \tag{36}$$

$$f_1(t^2) = \frac{2}{\pi}PD_1\left(\frac{t^2-c^2}{1-t^2}\right)^{1/2} + \frac{D_2}{\sqrt{(1-t^2)(t^2-c^2)}} \tag{37}$$

where D_1 and D_2 are constants which can be calculated as follows. We substitute the value of $L_1(v,w)$ from (25) as well as the expansion of $f(t^2)$ obtained from (33), (36) and (37) in the equation (21). When the coefficients of like powers of k_s from both sides of the resulting equation are equated we get the following results :

$$D_1 = -\frac{\pi v_o}{2\left[\left(\gamma + \log(k_s/2) - \frac{\pi i}{2} + \log(1-c^2)^{1/2}\right)M + N\right]} \tag{38}$$

and

$$D_2 = -\frac{2D_1^2}{\pi v_o}\left[\frac{R}{4}(2x^2+c^2+1) - \frac{MP}{2\pi}(1-2x^2+c^2) + \frac{Pv_o(1-c^2)}{2D_1}\right] \tag{39}$$

4. DISPLACEMENT AND STRESS

The vertical displacement $v(x,y)$ on the plane $y=0$ can be obtained from equations (15) and (19) as

$$
\begin{aligned}
v(x,0) &= -v_o + \frac{2M}{\pi}\left[D_1 + k_s^2 \log k_s \left\{ D_2 + \frac{(1-c^2)PD_1}{\pi} \right\} \right] \sinh^{-1}\sqrt{\frac{(x^2-1)}{(1-c^2)}} + \\
&\quad + \frac{2PD_1M}{\pi^2} k_s^2 \log k_s \sqrt{(x^2-1)(x^2-c^2)} + O(k_s^2) , \qquad |x|>1 \\
&= -v_o , \qquad c\le |x| \le 1 \\
&= -v_o + \frac{2M}{\pi}\left[D_1 + k_s^2 \log k_s \left\{ D_2 + \frac{(1-c^2)PD_1}{\pi} \right\} \right] \sinh^{-1}\sqrt{\frac{(c^2-x^2)}{(1-c^2)}} - \\
&\quad - \frac{2PD_1M}{\pi^2} k_s^2 \log k_s \sqrt{(1-x^2)(c^2-x^2)} + O(k_s^2) , \qquad |x|<c
\end{aligned}
\tag{40}
$$

The normal stress $\tau_{yy}(x,y)$ in the plane $y=0$ can be found from the relation (15) as

$$
\begin{aligned}
\tau_{yy}(x,\pm 0) &= \mp c_{22}\mu_{12}|x| \left[\frac{D_1}{\sqrt{(1-x^2)(x^2-c^2)}} + k_s^2 \log k_s \left\{ \frac{2}{\pi} PD_1 \left(\frac{x^2-c^2}{1-x^2} \right)^{1/2} + \right.\right. \\
&\quad \left.\left. + \frac{D_2}{\sqrt{(1-x^2)(x^2-c^2)}} \right\} \right] + O(k_s^2) , \qquad c\le |x| \le 1 \\
&= 0 , \qquad 0\le |x| <c, \quad |x|>1
\end{aligned}
\tag{41}
$$

For isotropic medium, putting

$$c_{11} = c_{22} = \frac{\lambda+2\mu}{\mu} , \qquad \mu_{12} = \mu$$

so that $\alpha_1 = \gamma_1$, $\alpha_2 = \xi^2/\gamma_2$, $k_2 = k_s$, $k_1 = k_s/\sqrt{c_{11}}$, $\tau^2 = \dfrac{1}{c_{11}}$

where $\gamma_i = (\xi^2 - k_i^2)^{1/2}$, $i = 1, 2,$

the expressions for displacement and stress are found to be

$$v(x,0) = -v_0 - \frac{(1+\tau^2)}{2\tau^2}\left[D_1' + k_2^2 \log k_2 \left\{D_2' - \frac{(3+\tau^4)}{8(1+\tau^2)}(1-c^2)D_1'\right\}\right] \times$$

$$\times \sinh^{-1}\sqrt{\frac{(x^2-1)}{(1-c^2)}} + \frac{(3+\tau^4)}{16\tau^2} D_1' k_2^2 \log k_2 \sqrt{(x^2-1)(x^2-c^2)} + O(k_2^2)$$

$$; \qquad |x|>1$$

$$= -v_0 \quad , \qquad c \le |x| \le 1$$

$$= -v_0 - \frac{(1+\tau^2)}{2\tau^2}\left[D_1' + k_2^2 \log k_2 \left\{D_2' - \frac{(3+\tau^4)}{8(1+\tau^2)}(1-c^2)D_1'\right\}\right] \times$$

$$\times \sinh^{-1}\sqrt{\frac{(c^2-x^2)}{(1-c^2)}} - \frac{(3+\tau^4)}{16\tau^2} D_1' k_2^2 \log k_2 \sqrt{(1-x^2)(c^2-x^2)} + O(k_2^2)$$

$$, \qquad |x|<c \tag{42}$$

$$\tau_{yy}(x,\pm 0) = \mp \frac{\mu}{\tau^2}|x|\left[\frac{D_1'}{\sqrt{(1-x^2)(x^2-c^2)}} + k_2^2 \log k_2 \left\{- \frac{(3+\tau^4)}{4(1+\tau^2)} D_1' \left(\frac{x^2-c^2}{1-x^2}\right)^{1/2}\right.\right.$$

$$\left.\left. + \frac{D_2'}{\sqrt{(1-x^2)(x^2-c^2)}}\right\}\right] + O(k_2^2) \quad , \qquad c \le |x| \le 1$$

$$= 0 \quad , \qquad 0 \le |x| < c \quad , \qquad |x|>1 \tag{43}$$

where

$$D_1' = \frac{2\tau^2 v_o}{\left[q_2+\tau^2 q_1+(1+\tau^2)\log(1-c^2)^{1/2}+\frac{1}{2}(1-\tau^2)\right]} \tag{44}$$

$$D_2' = \frac{D_1'^{\,2}(3+\tau^4)}{8v_o(1+\tau^2)}\left[\frac{(1+\tau^2)(1+c^2)}{2\tau^2}+\frac{v_o(1-c^2)}{D_1}\right] \tag{45}$$

$$q_i = \gamma + \log(k_i/4) - \pi i/2 \quad , \quad i=1,2 \tag{46}$$

Now, substituting $v_o=1$, $m_i=k_i$, $i=1,2$,

$$C = \frac{-2}{\pi\left[(q_1\tau^2+q_2)+\frac{1}{2}(1-\tau^2)+(1+\tau^2)\log(1-c^2)^{1/2}\right]}$$

and dropping term involving $k_2^2\log k_2$ the displacement and stress can be written as

$$\begin{aligned} v(x,0) &= -1 + \frac{\pi C}{2}(1+\tau^2)\sinh^{-1}\sqrt{\frac{(c^2-x^2)}{(1-c^2)}} + O(m_2^2) \quad , \quad |x|<c \\ &= -1 \quad , \quad c\le|x|\le 1 \\ &= -1 + \frac{\pi C}{2}(1+\tau^2)\sinh^{-1}\sqrt{\frac{(x^2-1)}{(1-c^2)}} + O(m_2^2) \quad , \quad |x|>1 \end{aligned} \tag{47}$$

$$\tau_{yy}(x,\pm 0) = \pm\frac{\mu\pi C|x|}{\sqrt{(1-x^2)(x^2-c^2)}} + O(m_2^2) \quad , \quad c\le|x|\le 1 \tag{48}$$

which coincide with the results obtained by Jain and Kanwal (1972b).

4. NUMERICAL RESULTS

The vertical displacement field for points near about the rigid strips has been plotted against dimensionless distance for two different types of orthotropic materials whose engineering constants have been listed in table-1. Type I-a, II-a and Type I-b, II-b correspond to the cases of x and y-directional fibre-reinforced composites respectively.

It is interesting to note that in both the cases (c=0.5 and c=0.8) the real part of the displacement viz. $Re(v/v_0)$ increases with the increase in the values of nondimensional frequency k_s [(Fig.2) - (Fig.9)].

The stress intensity factors T_c and T_1 at inner and outer edges of the strips defined by

$$T_c = \left| \underset{x \to c+}{Lt} \; Re\left[\frac{\tau_{yy}(x,0)(x-c)^{1/2}}{c_{22}\mu_{12}} \right] \right|$$

and

$$T_1 = \left| \underset{x \to 1-}{Lt} \; Re\left[\frac{\tau_{yy}(x,0)(1-x)^{1/2}}{c_{22}\mu_{12}} \right] \right|$$

have been plotted against frequency k_s.

It is found from the graphs that for low frequency, stress intensity factors (Fig.10-Fig.13) at both the edges increasegradually, attain maximum values and then go on decreasing. It may be noted further that at the inner edge, stress intensity factor increases with the increase in the values of the

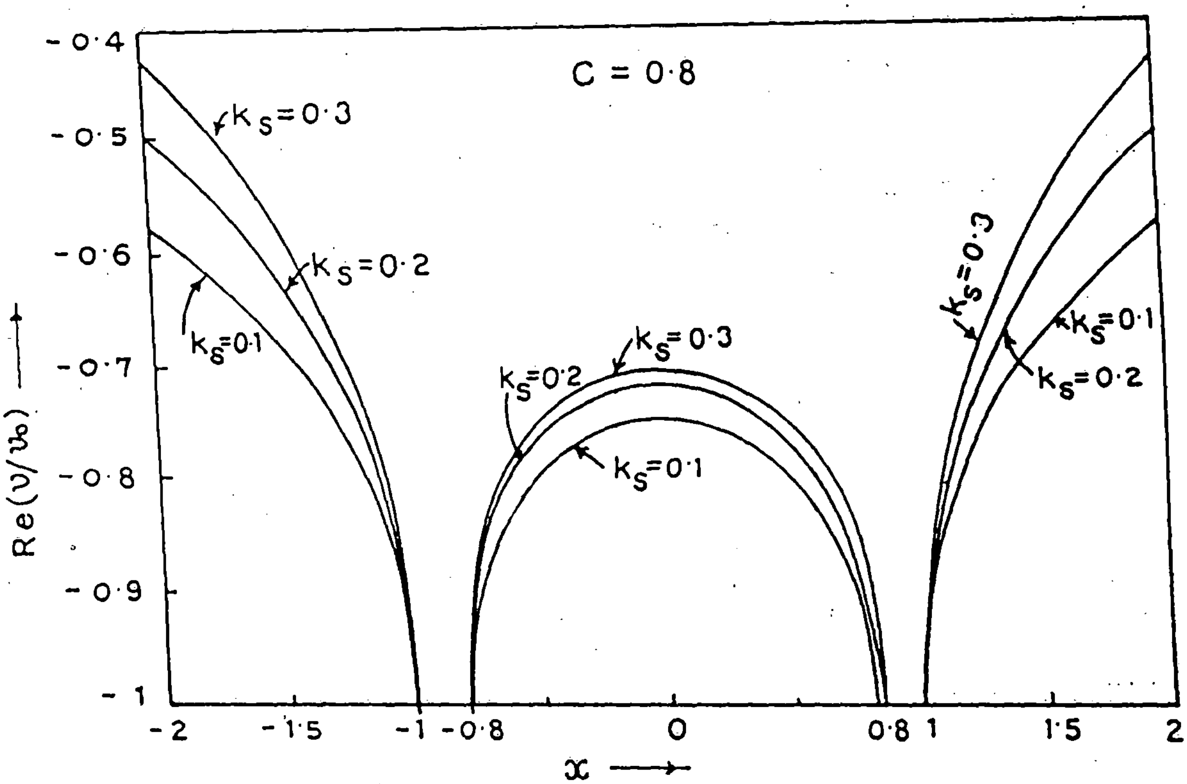

Fig.2. Displacement vs. distance for generalized plane stress (Type Ia, c=0.8).

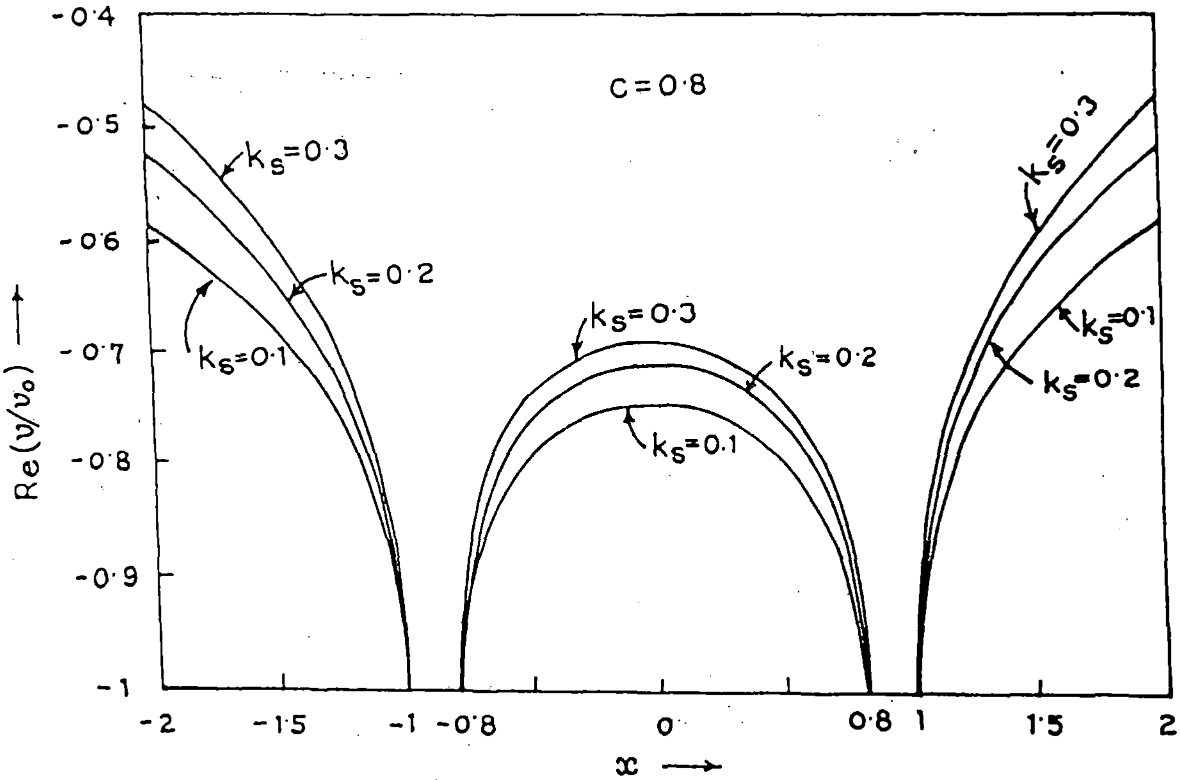

Fig. 3. Displacement vs. distance for generalized plane stress (Type Ib, c=0.8).

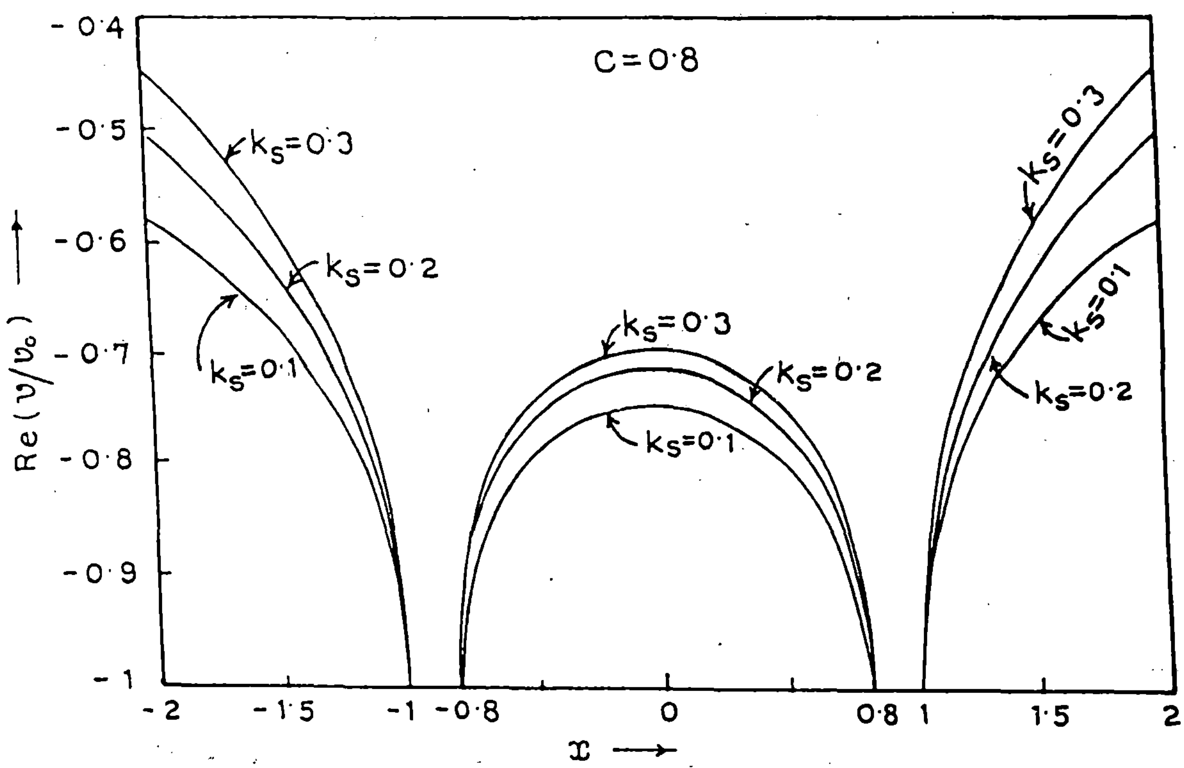

Fig. 4. Displacement vs. distance for generalized plane stress (Type IIa, c=0.8).

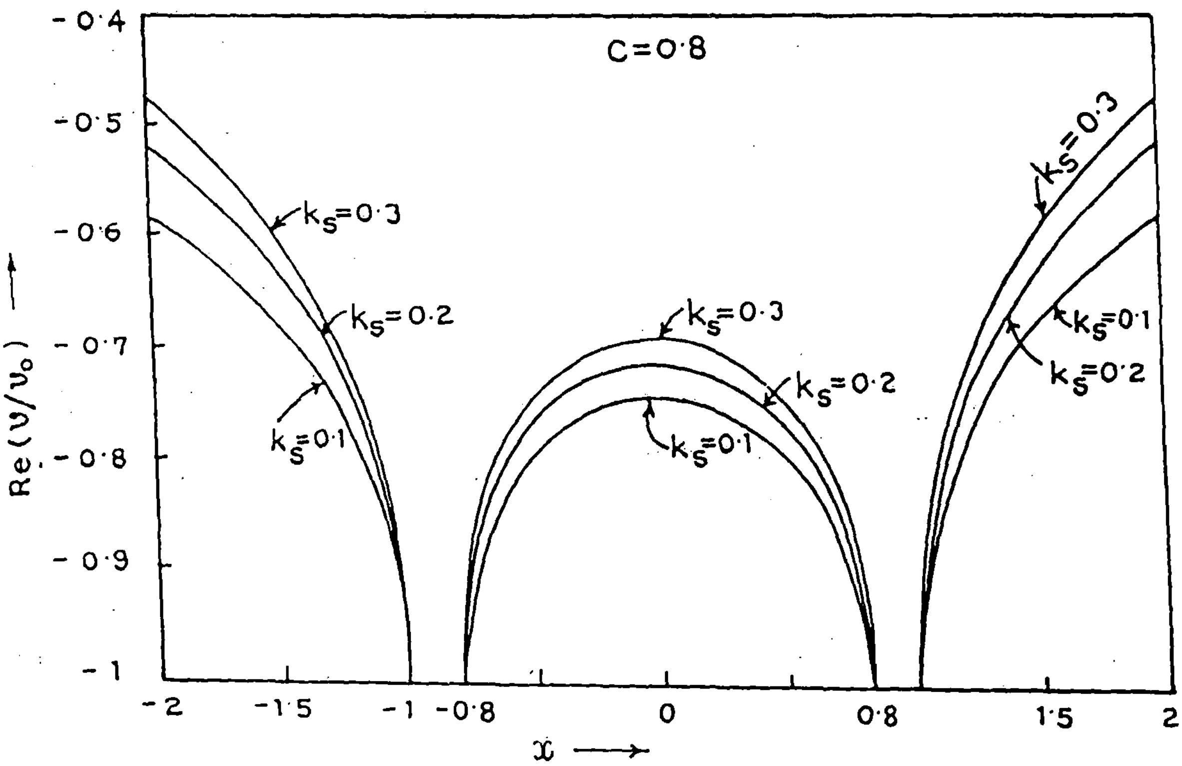

Fig.5. Displacement vs. distance for generalized plane stress (Type IIb, c=0.8).

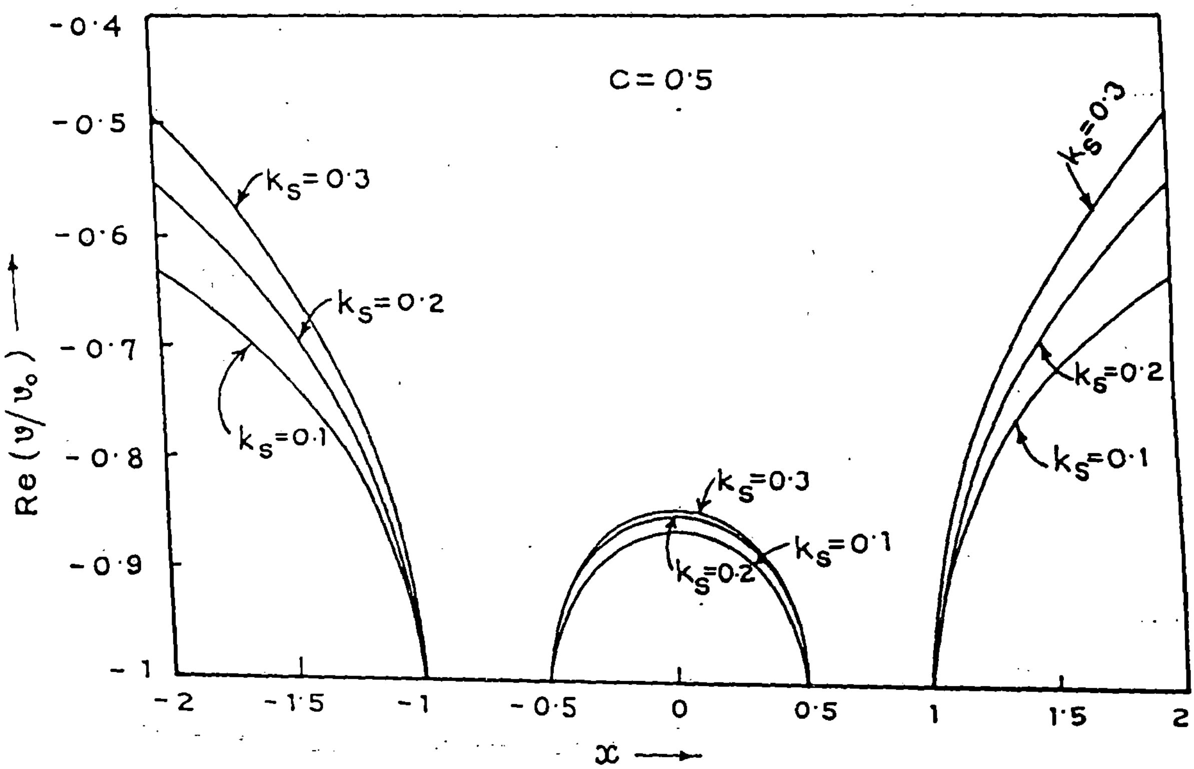

Fig.6. Displacement vs. distance for generalized plane stress (Type Ia, c=0.5).

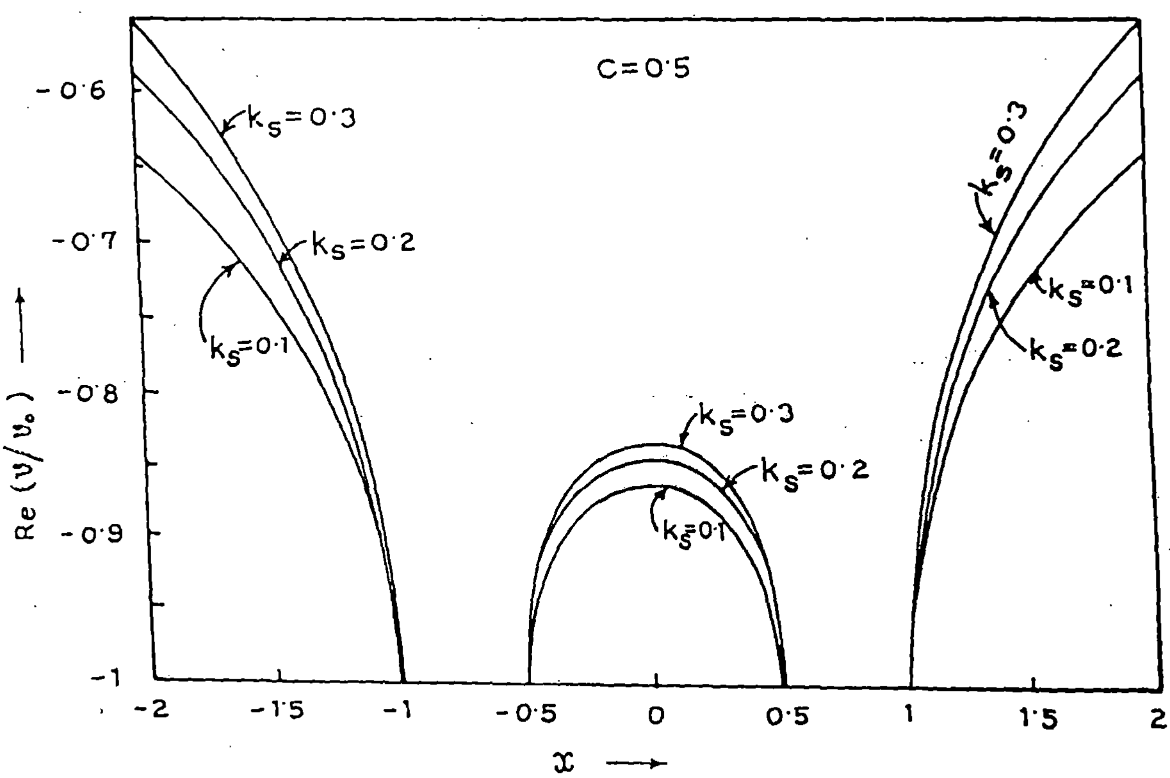

Fig. 7. Displacement vs. distance for generalized plane stress (Type Ib, c=0.5).

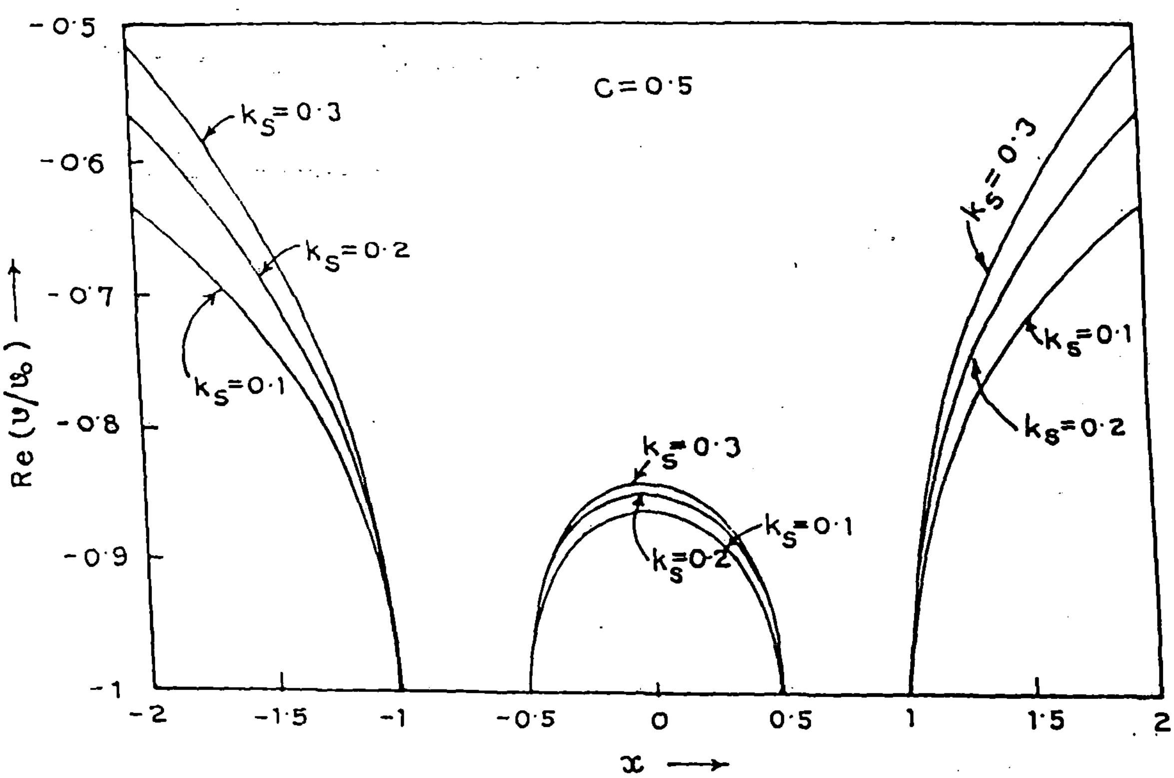

Fig.8. Displacement vs. distance for generalized plane stress (Type IIa, c=0.5).

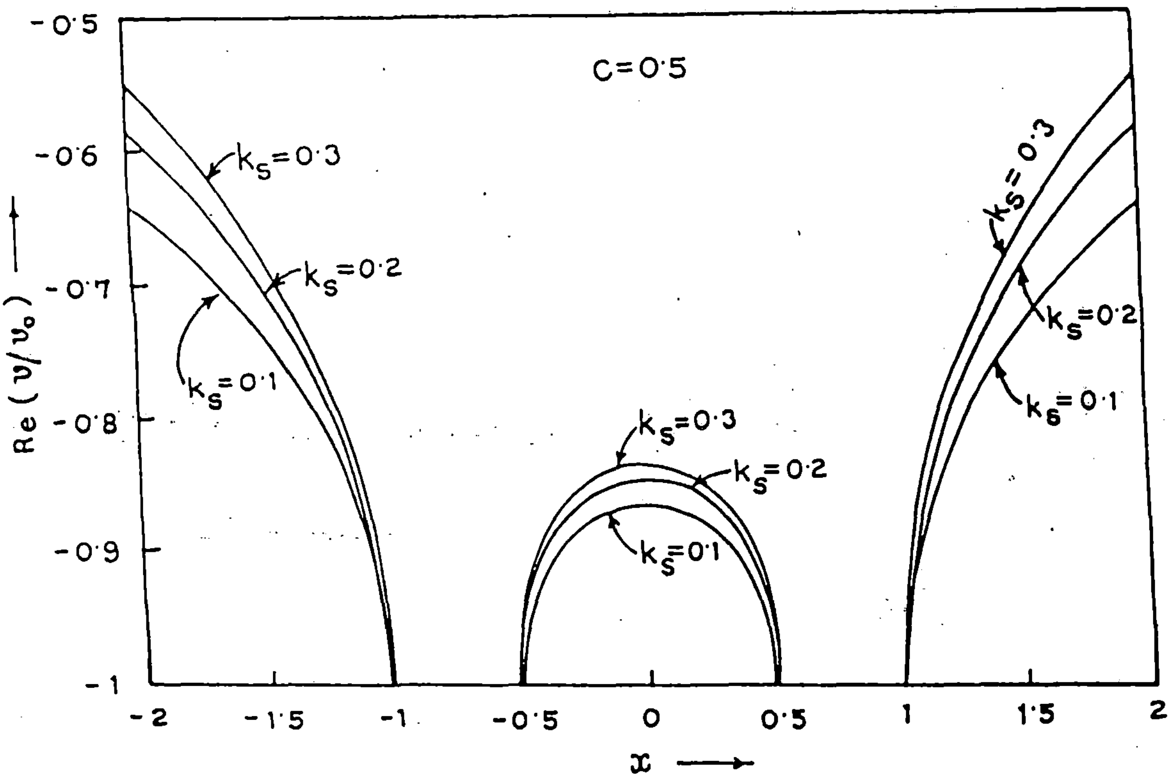

Fig. 9. Displacement vs. distance for generalized plane stress (Type IIb, c=0.5).

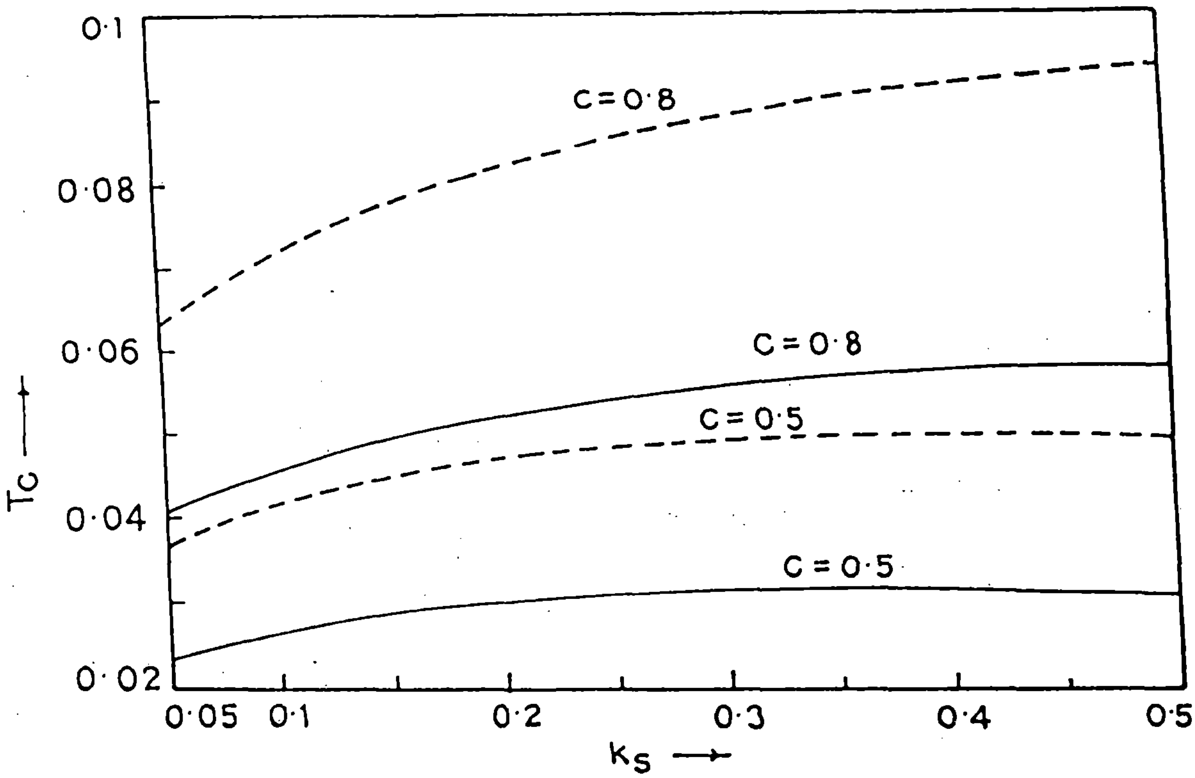

Fig.10. Stress intensity factor T_c vs. frequency k_s for generalized plane stress.
(——— Type Ia, ----- Type IIa).

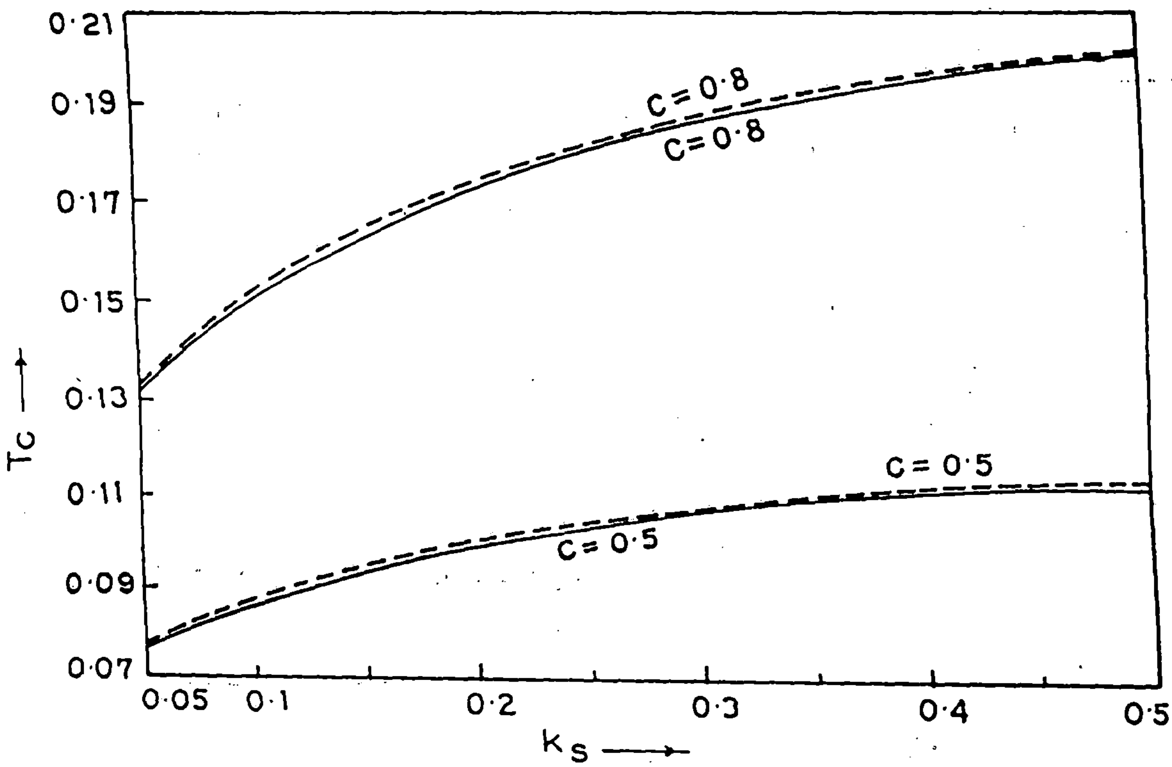

Fig.11. Stress intensity factor T_c vs. frequency k_s for generalized plane stress. (——— Type Ib, ----- Type IIb).

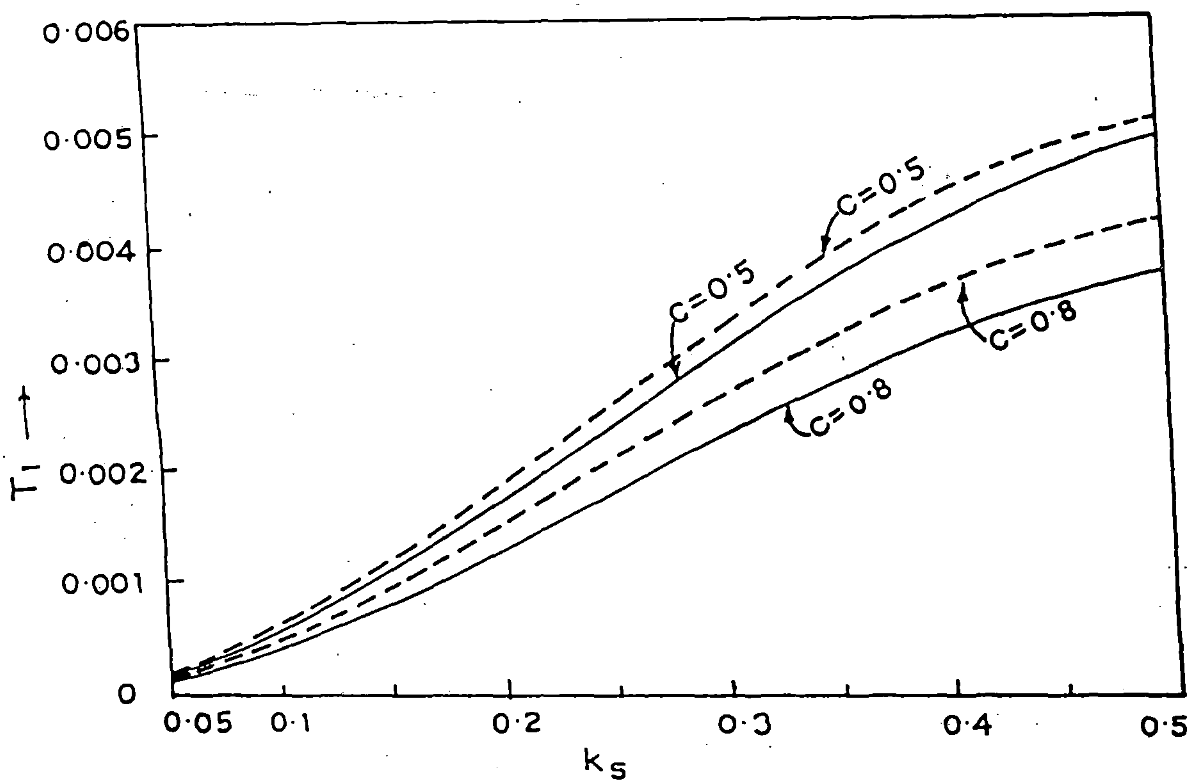

Fig. 12. Stress intensity factor T_1 vs. frequency k_s for generalized plane stress. (——— Type Ia, ----- Type IIa).

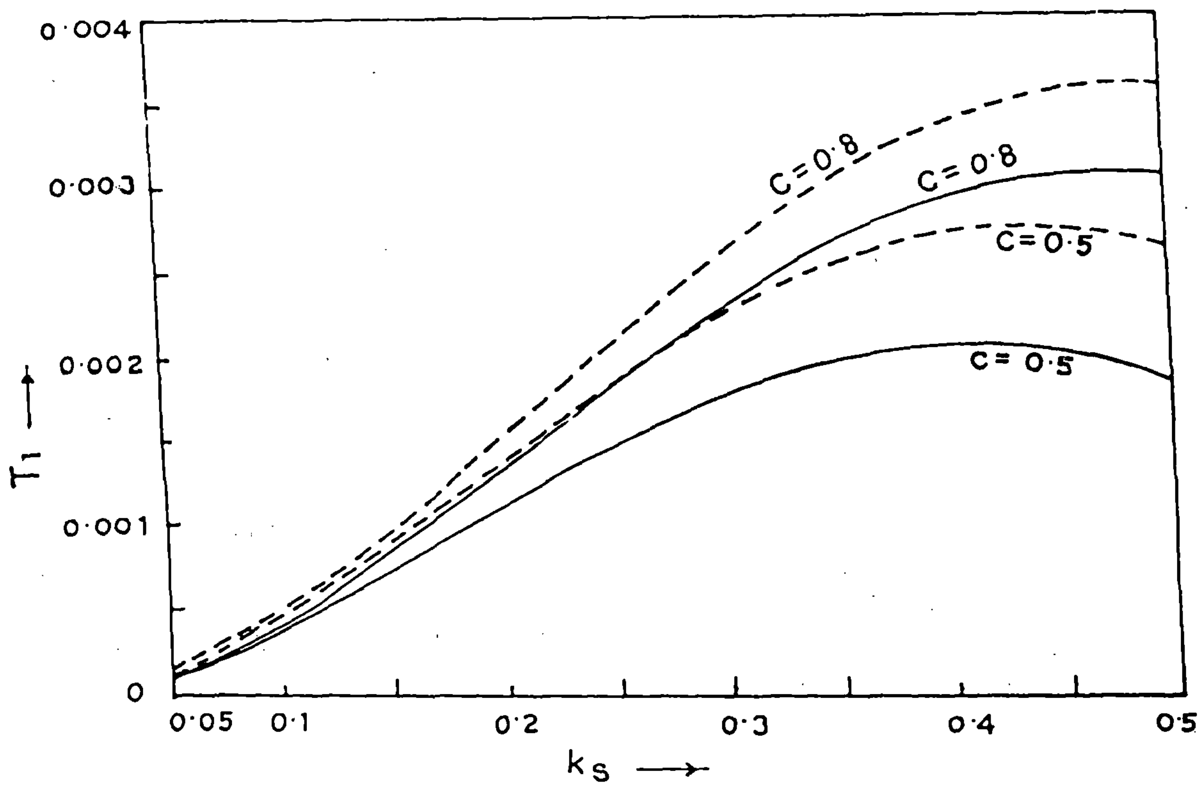

Fig.13. Stress intensity factor T_1 vs. frequency k_s for generalized plane stress.
(——— Type Ib, ----- Type IIb).

strip length whereas at the outer edge the stress intensity factor exhibits similar behaviour where the fibres are perpendicular to the strip but in case the fibres are parallel to the strips, the behaviour is just the opposite.

It may also be noted from the graphs that in case the fibres are perpendicular to the strips, the variation of the stress intensity factors at the inner edge do not vary significantly with the material though their variations at the outer edge are prominant.

APPENDIX

EVALUATION OF $L_1(v,w)$:

The integral $L_1(v,w)$ given by (22) is

$$L_1(v,w) = \int_0^\infty K(\xi,\gamma_1,\gamma_2) J_0(\xi w) J_0(\xi v)\, d\xi \tag{A1}$$

where

$$K(\xi,\gamma_1,\gamma_2) = \frac{\alpha_1-\alpha_2}{\alpha_1\gamma_1-\alpha_2\gamma_2} = \frac{c_{11}\xi^2-k_s+\gamma_1\gamma_2}{\gamma_1\gamma_2(\gamma_1+\gamma_2)} \tag{A2}$$

$$\begin{aligned} \gamma_1 &= \left[\frac{1}{2}\left\{-B_1+(B_1^2-4B_2)^{1/2}\right\}\right]^{1/2} \\ \gamma_2 &= \left[\frac{1}{2}\left\{-B_1-(B_1^2-4B_2)^{1/2}\right\}\right]^{1/2} \\ B_1 &= \frac{1}{c_{22}}\left\{(c_{12}^2+2c_{12}-c_{11}c_{22})\xi^2 + (1+c_{22})k_s^2\right\} \end{aligned} \tag{A3}$$

$$B_2 = \frac{1}{c_{22}}\left(\xi^2 - k_s^2\right)\left(c_{11}\xi^2 - k_s^2\right)$$

To evaluate the integral (A1) we consider two contour integrals :

$$I_1 = \int_{\Gamma_1} K(\xi,\gamma_1,\gamma_2) J_0(\xi v) H_0^{(1)}(\xi w)\, d\xi \quad , \quad w>v$$

(A4)

$$I_2 = \int_{\Gamma_2} K(\xi,\gamma_1,\gamma_2) J_0(\xi v) H_0^{(2)}(\xi w)\, d\xi \quad , \quad w>v$$

where Γ_1 and Γ_2 are the closed contours defined in fig.14.

Assuming the relation

$$\left\{\frac{(c_{12}^2+2c_{12}-c_{11}c_{22})(1+c_{22})}{c_{22}^2} + \frac{2(1+c_{11})}{c_{22}}\right\}^2 - \left\{\frac{(c_{12}^2+2c_{12}-c_{11}c_{22})^2}{c_{22}^2} - \frac{4c_{11}}{c_{22}}\right\}$$

$$\times \left\{\frac{(1+c_{22})^2}{c_{22}^2} + \frac{4}{c_{22}}\right\} < 0 \qquad \text{(A5)}$$

it is noted the branch points $\xi = \lambda_i (i=1-4)$ corresponding to the roots of the equation $B_1^2 - 4B_2 = 0$ are always complex.

Now, the branch points corresponding to the roots of the equations

$$- B_1 + (B_1^2 - 4B_2)^{1/2} = 0 \quad \text{and} \quad - B_1 - (B_1^2 - 4B_2)^{1/2} = 0$$

are $\xi=\pm k_s$ and $\xi=\pm k_s/\sqrt{c_{11}}$ respectively, where it is assumed that

$$c_{11}c_{22} - c_{12}^2 - 2c_{12} > 1 + c_{22}$$

(A6)

and $$c_{12}^2 + 2c_{12} + c_{11} > 0$$

Most of the orthotropic materials satisfy the relations (A5) and (A6). Therefore under the above condition, $\xi=\pm k_s/\sqrt{c_{11}}$ and $\xi=\pm k_s$

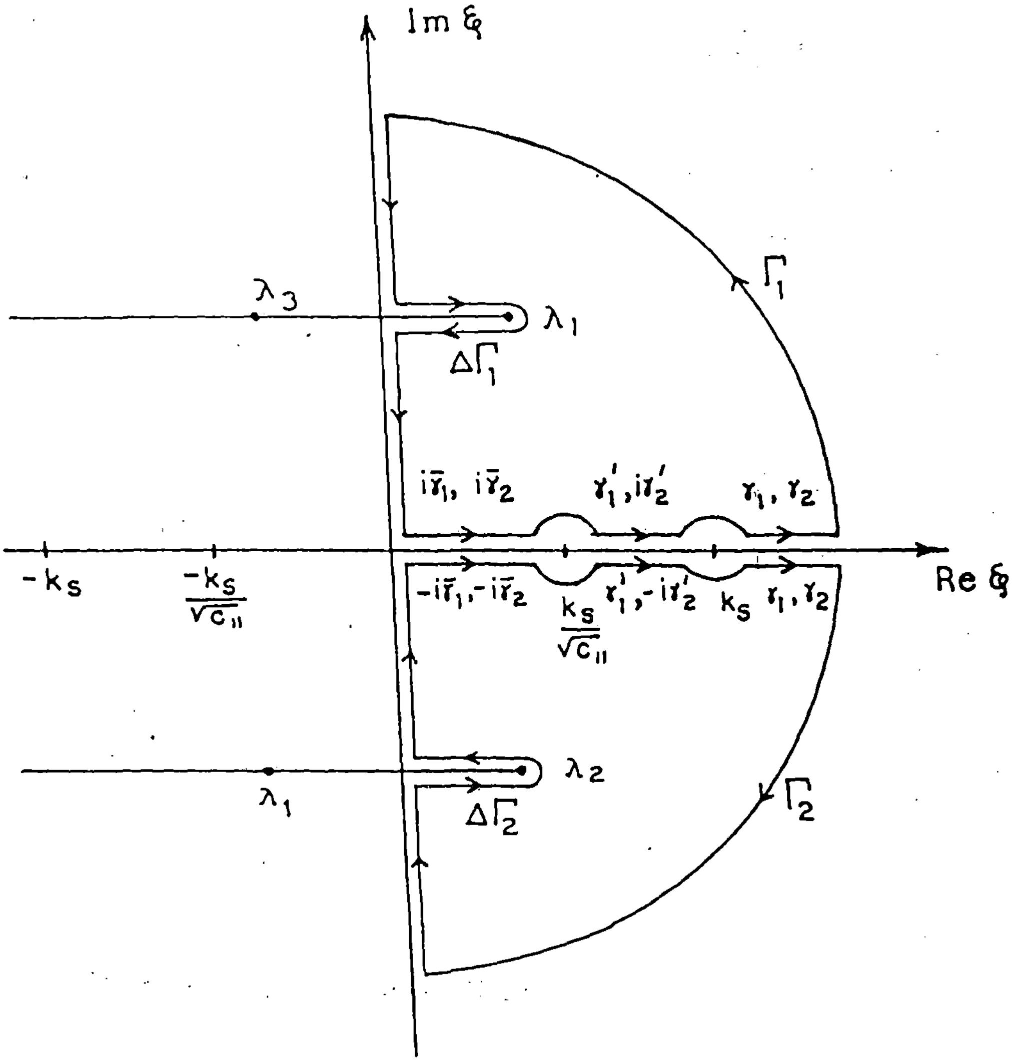

Fig. 14. Contours of integration for integral in equation (A1).

are the branch points of γ_1 and γ_2 respectively.

The integrals in equation (A4) are found to be zero on the contours $\Delta\Gamma_1$ and $\Delta\Gamma_2$ (fig.14) around the branch cuts from λ_1 and λ_2. Thus integrating along the contours Γ_1 and Γ_2 the integral $L_1(v,w)$ for w>v can be finally written as

$$L_1(v,w) = -i\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{11}\eta^2-1-\bar{\gamma}_1\bar{\gamma}_2}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1+\bar{\gamma}_2)} J_o(k_s\eta v)H_o^{(1)}(k_s\eta w)\, d\eta\; - \right.$$

$$\left. - \int_{1/\sqrt{c_{11}}}^{1} \frac{c_{11}\eta^2-1+\bar{\gamma}_2'^{\,2}}{\bar{\gamma}_2'(\bar{\gamma}_1'^{\,2}+\bar{\gamma}_2'^{\,2})} J_o(k_s\eta v)H_o^{(1)}(k_s\eta w)\, d\eta\right],\quad w>v \tag{A7}$$

where $\bar{\gamma}_1$, $\bar{\gamma}_2$, $\bar{\gamma}_1'$ and $\bar{\gamma}_2'$ are given by (24).

TABLE - 1. ENGINEERING ELASTIC CONSTANTS

	E_1 (Pa)	E_2 (Pa)	μ_{12} (Pa)	ν_{12}
Type I	Modulite II Graphite-Epoxy Composite			
a	15.3×10^9	158.0×10^9	5.52×10^9	0.033
b	158.0×10^9	15.3×10^9		0.34
Type II	E-Type Glass-Epoxy Composite			
a	9.79×10^9	42.3×10^9	3.66×10^9	0.063
b	42.3×10^9	9.79×10^9		0.27

INTERACTION OF ELASTIC WAVES WITH TWO COPLANAR GRIFFITH CRACKS IN AN ORTHOTROPIC MEDIUM

1. INTRODUCTION

Dynamic fracture problems involving anisotropic materials weakened by crack-like imperfections have drawn much attention to the investigators because of the increased usage of macroscopically anisotropic construction materials such as fibre reinforced composites. The different possible location of cracks with respect to the planes of material symmetry introduce great modifications in the strain and stress distribution. The problems are also of considerable interest in seismology and exploration geophysics. The problems involving single or two Griffith cracks in isotropic elastic medium have been studied by many authors (Loeber and Sih 1960, Mal 1978, Srivastava et al. 1981, Jain and Kanwal 1972a, Itou 1980b). Mathematical difficulties encountered in solving the governing equations of the anisotropic elasticity theory are responsible for the availability of few results only for special classes of materials. Kassir and Bandyopadhyay (1983) have studied the elastodynamic response of an infinite orthotropic solid containing a crack under the action of impact loading and the elastodynamic problem of a finite Griffith crack in an orthotropic strip under normal impact was investigated by Shindo et al.(1986).

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PP. 411-423 (1994)

Problem involving a moving Griffith crack in an orthotropic strip has also been studied by De and Patra (1990). Recently, Kundu and Bostrom (1991) solved the problem of scattering of elastic waves by a circular crack situated in a transversely isotropic solid.

In our paper, the diffraction of normally incident time harmonic elastic waves by two coplanar Griffith cracks in an infinite orthotropic medium has been investigated. The faces of each of the cracks are assumed to be separated by a small distance so that, during small deformations of the solid, the crack faces do not come into contact. The resulting mixed boundary value problem is reduced to the solution of a triple integral equation which has further been reduced to the solution of an integro-differential equation. Iterative solution valid for low frequency has been obtained. Analytical formulae for stress intensity factor and crack opening displacement have been derived. Making the distance between two crack zero the corresponding results for single crack have been presented. Finally, choosing the engineering elastic constants of the orthotropic material suitably the results for isotropic material have been deduced and compared with the results obtained by Jain and Kanwal (1972a). To display the influence of the material orthotropy numerical values of stress intensity factors and crack opening displacement have been plotted for several orthotropic materials.

2. STATEMENT AND FORMULATION OF THE PROBLEM

Consider the plane problem of diffraction of normally incident longitudinal wave by two symmetrical coplanar Griffith cracks situated in an infinite orthotropic elastic medium. The cracks are assumed to occupy the region $b \le |X| \le a$, $Y=0$, $|Z|<\infty$. It is convenient to normalize all lengths with respect to 'a' and so setting $X/a=x$, $Y/a=y$, $Z/a=z$, $b/a=c$, the new position of the cracks are defined by $c \le |x| \le 1$, $y=0$, $|z|<\infty$ (Fig.1).

Let a plane time harmonic elastic wave originating at $y=-\infty$ be incident normally on the two cracks is defined by $v_0=\exp[i(ky-\omega t)]$ where $k=a\omega/c_s\sqrt{c_{22}}$, $c_s=(\mu_{12}/\rho)^{1/2}$ with ρ being the density of the material. In the isotropic solid, c_s, represents the velocity of the shear wave.

The non-zero stress components τ_{yy} and τ_{xy} are given by

$$\tau_{yy}/\mu_{12} = c_{12}\, u_{,x} + c_{22}\, v_{,y}$$

$$\tau_{xy}/\mu_{12} = u_{,y} + v_{,x}\,, \qquad (1)$$

where u, v denote the component of the displacement in the x, y directions respectively and comma denotes partial differentiation with respect to the co-ordinates or time; $c_{ij}\,(i,j=1,2)$ are nondimensional parameters related to the elastic constants by the relations :

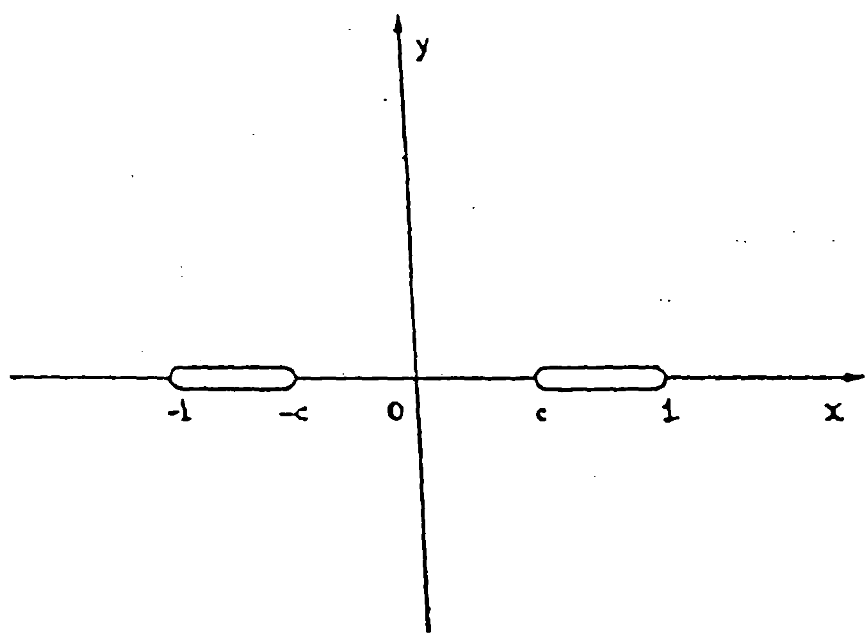

Fig. 1 Geometry of the cracks

$$c_{11}=E_1/\mu_{12}(1-\nu_{12}^2E_2/E_1)$$

$$c_{22}=E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1)=c_{11}E_2/E_1 \tag{2}$$

$$c_{12}=\nu_{12}E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1)=\nu_{12}c_{22}=\nu_{21}c_{11}$$

for generalized plane stress, and by

$$c_{11}=(E_1/\Delta\mu_{12})(1-\nu_{23}\nu_{32})$$

$$c_{22}=(E_2/\Delta\mu_{12})(1-\nu_{13}\nu_{31})$$

$$c_{12}=E_1(\nu_{21}+\nu_{13}\nu_{32}E_2/E_1)/\Delta\mu_{12} \tag{3}$$

$$=E_2(\nu_{12}+\nu_{23}\nu_{31}E_1/E_2)/\Delta\mu_{12}$$

$$\Delta = 1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{13}\nu_{21}\nu_{32}$$

for plane strain. In the above equations E_i, μ_{ij} and ν_{ij} (i, j=1, 2, 3) denote the engineering ealstic constants of the material where the subscripts 1,2,3 correspond to the x, y, z directions which coincide with the axes of material orthotropy and the constants E_i and ν_{ij} satisfy the Maxwell's relation :

$$\nu_{ij}/E_i = \nu_{ji}/E_j \tag{4}$$

The equations of motion for orthotropic material, in terms of displacements are

$$c_{11}\, u_{,xx} + u_{,yy} + (1+c_{12})v_{,xy} = \frac{a^2}{c_s^2}\, u_{,tt}$$

$$c_{22}\, v_{,yy} + v_{,xx} + (1+c_{12})u_{,xy} = \frac{a^2}{c_s^2}\, v_{,tt}. \tag{5}$$

Therefore, substituting $u(x,y,t) = u(x,y)\exp(-i\omega t)$ and $v(x,y,t) = v(x,y)\exp(-i\omega t)$ in equation (5) we obtain

$$c_{11}\, u_{,xx} + u_{,yy} + (1+c_{12})v_{,xy} + k_s^2 u = 0$$

and (6)

$$c_{22}\, v_{,yy} + v_{,xx} + (1+c_{12})u_{,xy} + k_s^2 v = 0$$

with $k_s^2 = a^2\omega^2/c_s^2$.

The boundary conditions of the problem are

$$\tau_{xy}(x,0) = 0 \quad , \qquad |x|<\infty \tag{7}$$

$$\tau_{yy}(x,0) + \tau_{yy}^{(0)}(x,0) = 0 \quad , \quad c\le|x|\le 1 \tag{8}$$

$$v(x,0) = 0 \quad , \qquad |x|<c \;,\; |x|>1. \tag{9}$$

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables would be omitted in the sequel.

The solutions of equations (6) can be taken as

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[A_1(\xi)\exp(-\gamma_1|y|) + A_2(\xi)\exp(-\gamma_2|y|)\right]\sin(\xi x)d\xi$$

(10)

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\left[\alpha_1 A_1(\xi)\exp(-\gamma_1|y|) + \alpha_2 A_2(\xi)\exp(-\gamma_2|y|)\right]\cos(\xi x)d\xi$$

, $y>0$ (11)

where
$$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1+c_{12})\gamma_i} \quad , \quad i=1,2 \tag{12}$$

and $A_i(\xi)$ $(i=1,2)$ are the unknown function to be determined, γ_1^2 , γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \left\{(c_{12}^2+2c_{12}-c_{11}c_{22})\xi^2+(1+c_{22})k_s^2\right\}\gamma^2 + (c_{11}\xi^2-k_s^2)(\xi^2-k_s^2) = 0. \tag{13}$$

From the boundary condition (7), it is found that

$$A_2(\xi) = -\beta A_1(\xi) \tag{14}$$

where

$$\beta = \frac{\gamma_1+\alpha_1}{\gamma_2+\alpha_2}. \tag{15}$$

Employing equation (14) the expressions for displacements and stresses reduce to

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[\exp(-\gamma_1|y|) - \beta\ \exp(-\gamma_2|y|)\right]A_1(\xi)\sin(\xi x)d\xi, \tag{16}$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\left[\alpha_1\exp(-\gamma_1|y|) - \beta\alpha_2\ \exp(-\gamma_2|y|)\right]A_1(\xi)\cos(\xi x)d\xi \quad , \quad y>0 \tag{17}$$

$$\tau_{xy}/\mu_{12} = -\frac{2}{\pi}\int_0^\infty (\gamma_1+\alpha_1)\left[\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)\right]A_1(\xi)\sin(\xi x)d\xi \quad , \quad y>0 \tag{18}$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty\left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \beta\left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\right]A_1(\xi)\cos(\xi x)d\xi. \tag{19}$$

We further substitute

$$A(\xi) = \frac{\alpha_1-\beta\alpha_2}{\xi}A_1(\xi)$$

so that the boundary conditions (9) and (8) yield the following integral equations in $A(\xi)$:

$$\int_0^\infty A(\xi)\cos(\xi x)d\xi = 0 \quad , \qquad |x|<c, \ |x|>1 \tag{20}$$

and

$$\int_0^\infty H(\xi)A(\xi)\cos(\xi x)d\xi = -\frac{\pi p_o}{2\mu_{12}} \quad , \qquad c\le|x|\le 1 \tag{21}$$

where $p_o = ik\mu_{12}c_{22}$

and $$H(\xi) = \frac{c_{12}\xi^2 - c_{22}\alpha_1\gamma_1 - \beta(c_{12}\xi^2 - c_{22}\alpha_2\gamma_2)}{(\alpha_1 - \beta\alpha_2)} \tag{22}$$

3. METHOD OF SOLUTION

In order to solve the set of integral equations (20) and (21), assume

$$A(\xi) = \frac{1}{\xi}\int_c^1 h(t^2)\sin(\xi t)dt \tag{23}$$

where $h(t^2)$ is an unknown function to be determined from the boundary conditions.

Inserting the value of $A(\xi)$ from equation (23) in equation (20) and using the following result (Gradshteyn and Ryzhik, 1965)

$$\int_0^\infty \frac{\sin(\xi t)\cos(\xi x)}{\xi}d\xi = \begin{cases} \frac{\pi}{2}, & t>x \\ 0, & t<x \end{cases}$$

it is found that the choice of $A(\xi)$ leads to the equation

$$\int_c^1 h(t^2)\,dt = 0. \tag{24}$$

Further substitution of $A(\xi)$ from equation (23) in equation (21) leads to

$$\int_c^1 h(t^2)dt\int_0^\infty \sin(\xi t)\cos(\xi x)d\xi$$

$$= q_o - \frac{d}{dx}\int_c^1 h(t^2)dt\int_0^\infty \xi H_1(\xi)\frac{\sin(\xi t)\sin(\xi x)}{\xi^2}\,d\xi \quad , \quad c\le|x|\le 1 \tag{25}$$

where

$$q_o = -\frac{\pi p_o}{2\theta\mu_{12}} \tag{26}$$

$$H_1(\xi) = \frac{H(\xi)}{\xi\theta} - 1 \to 0 \text{ as } \xi\to\infty \tag{27}$$

$$\theta = \frac{(c_{12}^2+c_{12}-c_{11}c_{22})(c_{12}N_1N_2-c_{11})-c_{22}\left[c_{12}N_1^2N_2^2+c_{11}(N_1^2+N_1N_2+N_2^2)\right]}{c_{11}(1+c_{12})(N_1+N_2)} \tag{28}$$

$$N_1^2 = \frac{1}{2c_{22}}\left\{c_{11}c_{22}-c_{12}^2-2c_{12}+\left[(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}\right]^{1/2}\right\}$$

(29)

$$N_2^2 = \frac{1}{2c_{22}}\left\{c_{11}c_{22}-c_{12}^2-2c_{12}-\left[(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}\right]^{1/2}\right\}$$

Using the relation

$$\frac{\sin\xi x\ \sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_o(\xi w)J_o(\xi v)dvdw}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} \tag{30}$$

equation (25) can be rewritten in the following form

$$\int_c^1 \frac{th(t^2)}{t^2-x^2}\,dt = q_o - \frac{d}{dx}\int_c^1 h(t^2)dt \int_o^x \int_o^t \frac{vwL(v,w)\ dw\ dv}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}\ ,\quad c\le|x|\le 1 \tag{31}$$

where

$$L(v,w) = \int_o^\infty \xi H_1(\xi)J_o(\xi w)J_o(\xi v)\ d\xi . \tag{32}$$

Applying a contour integration technique, (Mandal and Ghosh, 1994) the infinite integral in L(v,w) can be converted to the following finite integrals

$$L(v,w) = -ik_s^2\Bigg[\int_o^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2-c_{22}\bar{\alpha}_1\bar{\gamma}_1 - \bar{\beta}(c_{12}\eta^2-c_{22}\bar{\alpha}_2\bar{\gamma}_2)}{\theta(\bar{\alpha}_1-\bar{\beta}\,\bar{\alpha}_2)}\times$$

$$\times J_o(k_s\eta v)H_o^{(1)}(k_s\eta w)\ d\eta -$$

$$-\int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2-c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta(\hat{\alpha}_1-\hat{\beta}\,\hat{\alpha}_2)} J_o(k_s\eta v)H_o^{(1)}(k_s\eta w)\ d\eta\Bigg],\quad w>v \tag{33}$$

where

$$\bar{\gamma}_1 = \left[\frac{1}{2}\left\{R_1-(R_1^2-4\bar{R}_2)^{1/2}\right\}\right]^{1/2}$$

$$\bar{\gamma}_2 = \left[\frac{1}{2}\left\{R_1+(R_1^2-4\bar{R}_2)^{1/2}\right\}\right]^{1/2}$$

$$\hat{\gamma}_1 = \left[\frac{1}{2}\left\{-R_1+(R_1^2+4R_2')^{1/2}\right\}\right]^{1/2}$$

$$\hat{\gamma}_2 = \left[\frac{1}{2}\left\{R_1+(R_1^2+4R_2')^{1/2}\right\}\right]^{1/2}$$

$$R_1 = \frac{1}{c_{22}} \left\{ (c_{12}^2 + 2c_{12} - c_{11}c_{22})\eta^2 + (1+c_{22}) \right\}$$

$$\overline{R}_2 = \frac{c_{11}}{c_{22}} \left[1-\eta^2\right] \left[\frac{1}{c_{11}} - \eta^2\right]$$

$$R_2' = \frac{c_{11}}{c_{22}} \left[1-\eta^2\right] \left[\eta^2 - \frac{1}{c_{11}}\right]$$

$$\overline{\alpha}_i = \frac{c_{11}\eta^2 - 1 + \overline{\gamma}_i^{\,2}}{(1+c_{12})\overline{\gamma}_i} \qquad (i=1,2)$$

$$\hat{\alpha}_i = \frac{c_{11}\eta^2 - 1 + (-1)^i \hat{\gamma}_i^{\,2}}{(1+c_{12})\hat{\gamma}_i} \qquad (i=1,2)$$

$$\overline{\beta} = \frac{\overline{\alpha}_1 - \overline{\gamma}_1}{\overline{\alpha}_2 - \overline{\gamma}_2} \quad \text{and} \quad \hat{\beta} = \frac{\hat{\alpha}_1 + \hat{\gamma}_1}{\hat{\alpha}_2 - \hat{\gamma}_2} \qquad (34)$$

The corresponding expression of $L(v,w)$ for $w<v$ follows from (33) by interchanging w and v.

Employing the series expansions for the Bessel function J_o and the Hankel function $H_o^{(1)}$ in equation (33), it is found that

$$L(v,w) = \frac{2}{\pi} P\, k_s^2 \log k_s + O(k_s^2) \qquad (35)$$

where

$$P = \frac{1}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - c_{22}\overline{\alpha}_1\overline{\gamma}_1 - \overline{\beta}(c_{12}\eta^2 - c_{22}\overline{\alpha}_2\overline{\gamma}_2)}{(\overline{\alpha}_1 - \overline{\beta}\,\overline{\alpha}_2)}\, d\eta \; - \right.$$

$$- \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{(\hat{\alpha}_1 - \hat{\beta}\,\hat{\alpha}_2)}\, d\eta \Bigg].$$

Now, let us expand $h(t^2)$ in the form

$$h(t^2) = h_o(t^2) + k_s^2 \log k_s \; h_1(t^2) + O(k_s^2). \tag{36}$$

Inserting the above expansion of $h(t^2)$ and the value of $L(v,w)$ given by equation (35) into equation (31) and equating the coefficients of like powers of k_s, we obain the equations

$$\int_c^1 \frac{t h_o(t^2)}{t^2 - x^2}\, dt = q_o \quad , \quad c \le |x| \le 1 \tag{37}$$

and
$$\int_c^1 \frac{t h_1(t^2)}{t^2 - x^2}\, dt = - \frac{2P}{\pi} \int_c^1 t h_o(t^2) dt \; , \quad c \le |x| \le 1. \tag{38}$$

Using the finite Hilbert transform technique (Srivastava and Lowengrub, 1968), the solutions of the above integral equations can be obtained as

$$h_o(t^2) = \frac{2}{\pi} q_o \sqrt{\frac{t^2 - c^2}{1 - t^2}} + \frac{D_1}{\sqrt{(t^2 - c^2)(1 - t^2)}} \tag{39}$$

$$h_1(t^2) = - \frac{2}{\pi} P \left[\frac{q_o(1 - c^2)}{\pi} + D_1 \right] \sqrt{\frac{t^2 - c^2}{1 - t^2}} + \frac{D_2}{\sqrt{(t^2 - c^2)(1 - t^2)}} \tag{40}$$

where D_1 and D_2 are consants to be determined using the condition given by equation (24) so that

$$\int_c^1 h_o(t^2)\,dt = 0 \quad \text{and} \quad \int_c^1 h_1(t^2)\,dt = 0. \tag{41}$$

Substitution of the values of $h_o(t^2)$ and $h_1(t^2)$ given by equations (39) and (40) in (41), yields

$$D_1 = \frac{2}{\pi} q_o \left[c^2 - \frac{E}{F}\right] \tag{42}$$

$$D_2 = \frac{2}{\pi^2} q_o \left[1 + c^2 - \frac{2E}{F}\right]\left[\frac{E}{F} - c^2\right], \tag{43}$$

where

$$F = F\left[\frac{\pi}{2}, \sqrt{1-c^2}\,\right] \quad \text{and} \quad E = E\left[\frac{\pi}{2}, \sqrt{1-c^2}\,\right]$$

are the elliptic integrals of first and second kind, respecively. Substitituting the value of D_1 and D_2 given by equations (42) and (43) into equations (39-40), we obtain

$$h_o(t^2) = -\frac{p_o}{\mu_{12}\theta} \frac{\left[t^2 - \frac{E}{F}\right]}{\sqrt{(t^2-c^2)(1-t^2)}} \tag{44}$$

$$h_1(t^2) = -\frac{P\, p_o}{\pi\mu_{12}\theta} \frac{\left[t^2 - \frac{E}{F}\right]\left[1+c^2 - \frac{2E}{F}\right]}{\sqrt{(t^2-c^2)(1-t^2)}} \tag{45}$$

4. CRACK OPENING DISPLACEMENT AND STRESS INTENSITY FACTORS

The crack opening displacement and the normal stress component in the plane of the crack can be written as

$$\Delta v(x,0) = v(x,0+) - v(x,0-) = 2\int_x^1 h(t^2)\,dt \quad , \quad c \leq x \leq 1 \tag{46}$$

and

$$\tau_{yy}(x,0) = \frac{2\mu_{12}\theta}{\pi} \int_c^1 \frac{th(t^2)}{t^2 - x^2}\, dt \quad , \quad 0<x<c \qquad (47)$$

$$= -\frac{2\mu_{12}\theta}{\pi} \int_c^1 \frac{th(t^2)}{x^2 - t^2}\, dt \quad , \quad x>1 \qquad (48)$$

Expressions (47) and (48) with the aid of the equations (36), (44) and (45) yield

$$\tau_{yy}(x,0) = -p_o\left[1+ \frac{\left[x^2 - \frac{E}{F}\right]}{\sqrt{(c^2-x^2)(1-x^2)}}\right]\left[1-\frac{P}{\pi}\left\{1+c^2-\frac{2E}{F}\right\} k_s^2 \log k_s\right] + O(k_s^2)$$

$$, \quad 0<x<c \qquad (49)$$

$$\tau_{yy}(x,0) = -p_o\left[1- \frac{\left[x^2 - \frac{E}{F}\right]}{\sqrt{(x^2-c^2)(x^2-1)}}\right]\left[1-\frac{P}{\pi}\left\{1+c^2-\frac{2E}{F}\right\} k_s^2 \log k_s\right] + O(k_s^2)$$

$$, \quad x>1 \qquad (50)$$

The stress intensity factors are defined as (in physical units)

$$K_c = \lim_{x\to c-} \left[\frac{\sqrt{(c-x)}\;\tau_{yy}(x,0)}{p_o}\right]_{0<x<c} \qquad (51)$$

$$K_1 = \lim_{x\to 1+} \left[\frac{\sqrt{(x-1)}\;\tau_{yy}(x,0)}{p_o}\right]_{x>1} \qquad (52)$$

Substituting equations (49-50) into equations (51-52) it can be shown that

$$K_c = -\frac{\left[c^2 - \frac{E}{F}\right]}{\sqrt{2c(1-c^2)}}\left[1-\frac{P}{\pi}\left\{1+c^2-\frac{2E}{F}\right\}k_s^2\log k_s\right] + O(k_s^2) \qquad (53)$$

$$K_1 = \frac{\left[1 - \frac{E}{F}\right]}{\sqrt{2(1-c^2)}}\left[1-\frac{P}{\pi}\left\{1+c^2-\frac{2E}{F}\right\}k_s^2\log k_s\right] + O(k_s^2) \qquad (54)$$

Further substituting equations (36), (44-45) in the expression given by equation (46), the crack opening displacement is obtained as

$$\Delta v(x,0) = \frac{2p_o}{\mu_{12}\theta}\left[1-\frac{P}{\pi}\left\{1+c^2-\frac{2E}{F}\right\}k_s^2\log k_s\right]\left[\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)}F(\lambda,q) - E(\lambda,q)\right] +$$

$$+ O(k_s^2) \quad , \quad c \le x \le 1 \qquad (55)$$

where

$$\sin\lambda = \sqrt{\frac{1-x^2}{1-c^2}} \quad \text{and} \quad q = \sqrt{1-c^2} \; .$$

Letting $c \to 0$ in the expression for stress intensity factor and crack opening displacement, the results for a single crack occupying the region $|x| \le 1$, $y=0$, $|z| < \infty$ are found to be

$$K_1 = \frac{1}{\sqrt{2}}\left[1-\frac{P}{\pi}k_s^2\log k_s\right] + O(k_s^2) \qquad (56)$$

$$\Delta v(x,0) = -\frac{2p_o}{\mu_{12}\theta}\sqrt{1-x^2}\left[1-\frac{P}{\pi}k_s^2\log k_s\right] + O(k_s^2) \quad , \quad 0 \le x \le 1 \qquad (57)$$

For isotropic medium, putting

$$c_{11}=c_{22}=\frac{\lambda+2\mu}{\mu}\ , \quad \mu_{12}=\mu\ , \quad c_{12}=c_{11}-2=\frac{\lambda}{\mu}$$

so that $\alpha_1=\gamma_1$, $\alpha_2=\xi^2/\gamma_2$, $k_s=m_2$, $k_s/\sqrt{c_{11}}=m_1$, $\tau=\dfrac{1}{c_{11}}$

$N_1=1=N_2$, $\theta=-2(1-\tau^2)$ and $P=\dfrac{\pi}{2}c_1$,

where

$$c_1=\frac{3\tau^4-4\tau^2-3}{4(1-\tau^2)},\quad \gamma_i=(\xi^2-m_i^2)^{1/2}\quad \text{and}\quad m_i=\frac{a\omega}{c_i}\quad (i=1,2)$$

the expressions for displacemet and stress are found to be

$$\Delta v(x,\pm 0) = \mp\frac{p_o}{2\mu(1-\tau^2)}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]\times$$

$$\times\left[\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)}F(\lambda,q)-E(\lambda,q)\right]+O(m_2^2)\quad,\quad c\le x\le 1$$

$$=0\quad,\quad |x|<c\ ,\quad |x|>1$$

and

$$\tau_{yy}(x,0) = -p_o\left[1+\frac{[x^2-\frac{E}{F}]}{\sqrt{(c^2-x^2)(1-x^2)}}\right]\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]+O(m_2^2)$$

$$,\quad 0<x<c$$

$$=-p_o\quad,\quad c\le|x|\le 1$$

$$=-p_o\left[1-\frac{[x^2-\frac{E}{F}]}{\sqrt{(x^2-c^2)(x^2-1)}}\right]\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]+O(m_2^2)$$

$$,\quad |x|>1.$$

Now, the crack opening displacement and stress intensity factors are found to be

$$\Delta v(x,0) = -\frac{p_o}{\mu(1-\tau^2)}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]\times$$

$$\times\left[\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)}F(\lambda,q) - E(\lambda,q)\right] + O(m_2^2)\quad,\quad c\le x\le 1$$

and

$$K_c = -\frac{\left[c^2-\frac{E}{F}\right]}{\sqrt{2c(1-c^2)}}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right] + O(m_2^2)$$

$$K_1 = \frac{\left[1-\frac{E}{F}\right]}{\sqrt{2(1-c^2)}}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right] + O(m_2^2)$$

which coincide with the results obtained by Jain and Kanwal (1972a) up to the order of $m_2^2\log m_2$ in the isotropic case.

When $c\to 0$, we recover the stress intensity factor and crack opening displacement for a single crack

$$K_1 = \frac{1}{\sqrt{2}}\left[1-\frac{c_1}{2}m_2^2\log m_2\right] + O(m_2^2)$$

$$\Delta v(x,0) = \frac{p_o}{\mu(1-\tau^2)}\sqrt{1-x^2}\left[1-\frac{c_1}{2}m_2^2\log m_2\right] + O(m_2^2)\quad,\quad 0\le x\le 1$$

which agrees with the result of Mal (1978)

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) K_c and K_1 given by (53) and (54) at the inner and outer tips of the cracks and crack opening displacements (COD) given by (55) have been plotted against

TABLE - 1. ENGINEERING ELASTIC CONSTANTS.

	E_1 (Pa)	E_2 (Pa)	μ_{12} (Pa)	ν_{12}
Type I	Modulite II Graphite-Epoxy Composite :			
	15.3×10^9	158.0×10^9	5.52×10^9	0.033
Type II	E-Type Glass-Epoxy Composite :			
	9.79×10^9	42.3×10^9	3.66×10^9	0.063
Type III	Stainless Steel-Aluminium Composite :			
	79.76×10^9	85.91×10^9	30.02×10^9	0.31

dimensionless frequency k_s and distance, respectively for three different types of orthotropic materials whose engineering constants have been listed in table 1.

From Fig.2 it is found that SIF K_c at the inner tip of the crack increases at a slow rate with the increase in the value of frequency k_s($0.1 \le k_s \le 0.6$). On the other hand the rate of increase of the SIF K_1 (Fig.3) with frequency k_s at the outer tip of the crack is found to be higher than that of K_c.

In both the cases the value of SIF is higher for small values of c, i.e., for greater crack length SIF is higher. But it is interesting to note that for different materials the variation of SIFs in both the cases are not significant. In the case of single crack (c=0) the variation of SIF with material properties has been shown in Fig.4.

The COD has been plotted for different crack length. In each case COD increases gradually from zero, attains maximum value and then decreases to zero. It is found that with the increase in the values of c (i.e., for small crack length) the values of COD decreases (Figs.5-6). For a fixed material the variation of COD with frequency is found to be insignificant, but it is noticed that for smaller values of c (Fig.5, Fig.7) the variation of COD with frequency is palpable. c=0 (Fig.7) correspond to the case of single crack.

In all the cases where different values of c has been considered the variation of COD is found to be prominent for different orthotropic materials.

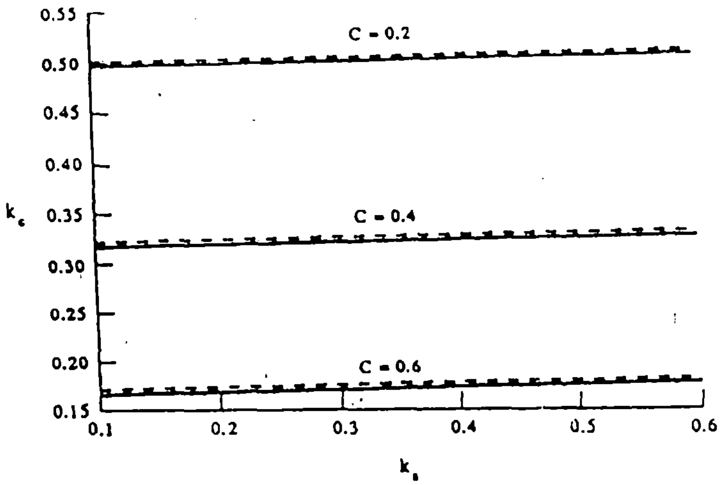

Fig. 2. Stress intensity factor K_c vs frequency k_s for generalized plane stress. (—, Type I; - - - -, Type II).

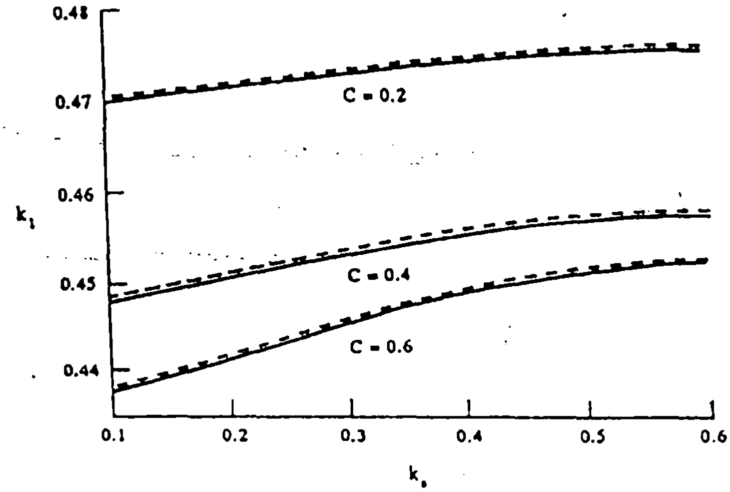

Fig. 3. Stress intensity factor K_1 vs frequency k_s for generalized plane stress. (—, Type I; - - - -, Type II).

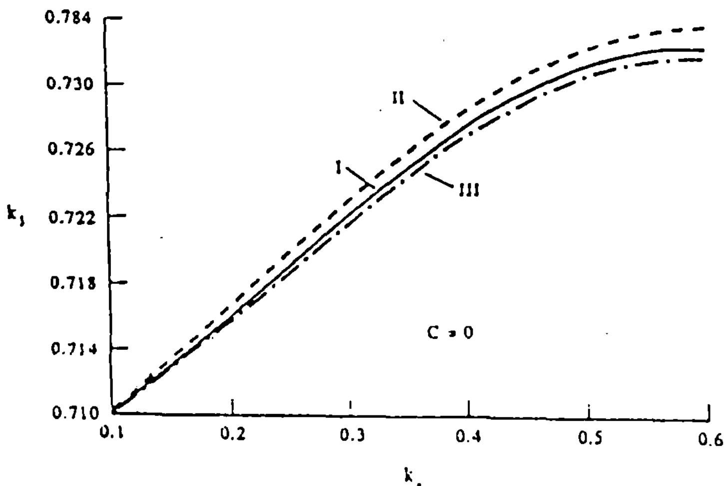

Fig. 4. Stress intensity factor K_1 vs frequency k_s for generalized plane stress. (Single crack, $c = 0$).

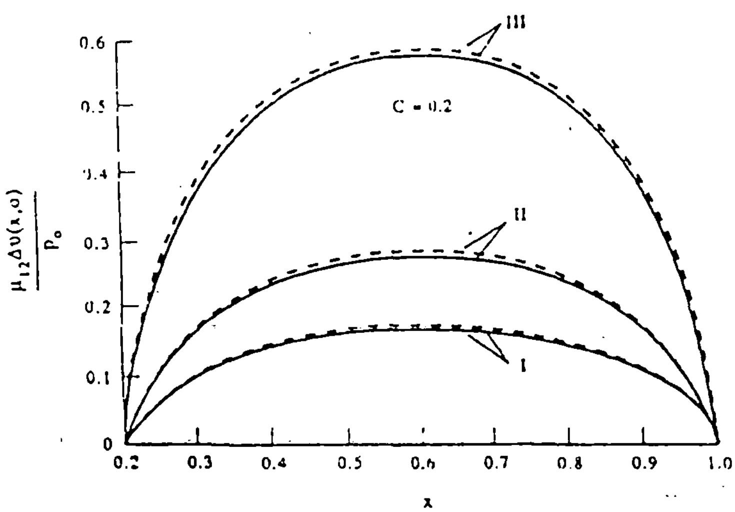

Fig. 5. Crack opening displacement (COD) vs distance ($c = 0.2$) for generalized plane stress. (—, $k_s = 0.2$; - - - -, $k_s = 0.6$).

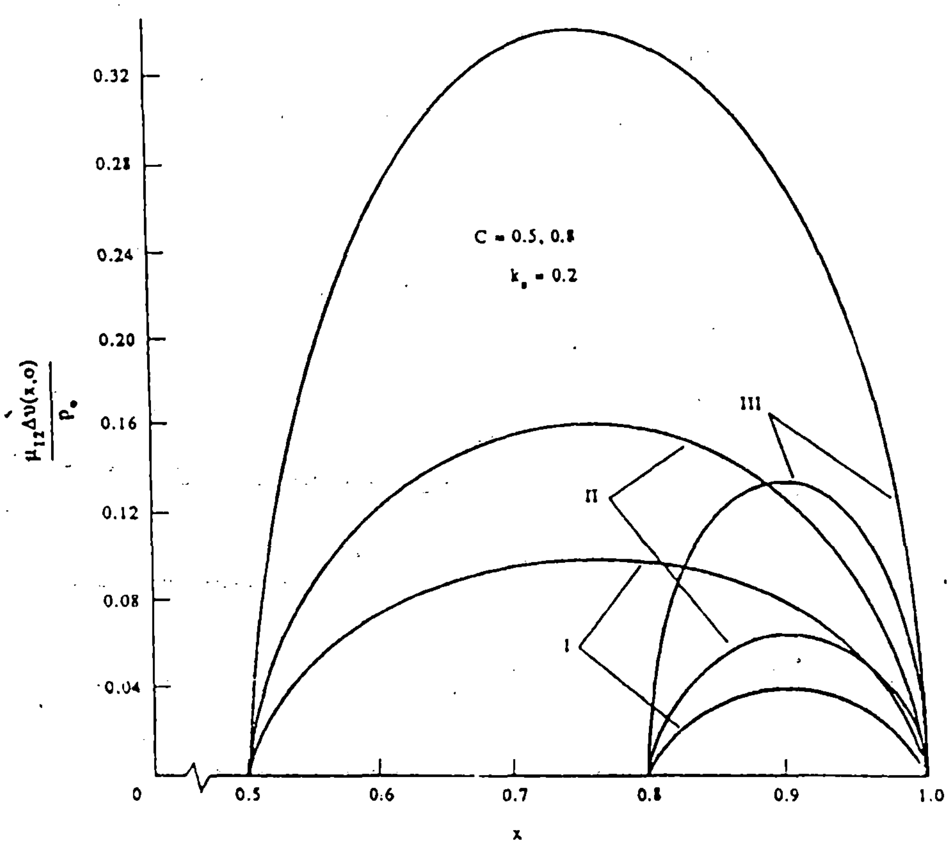

Fig. 6. Crack opening displacement (COD) vs distance ($c = 0.5$ and $c = 0.8$) for generalized plane stress.

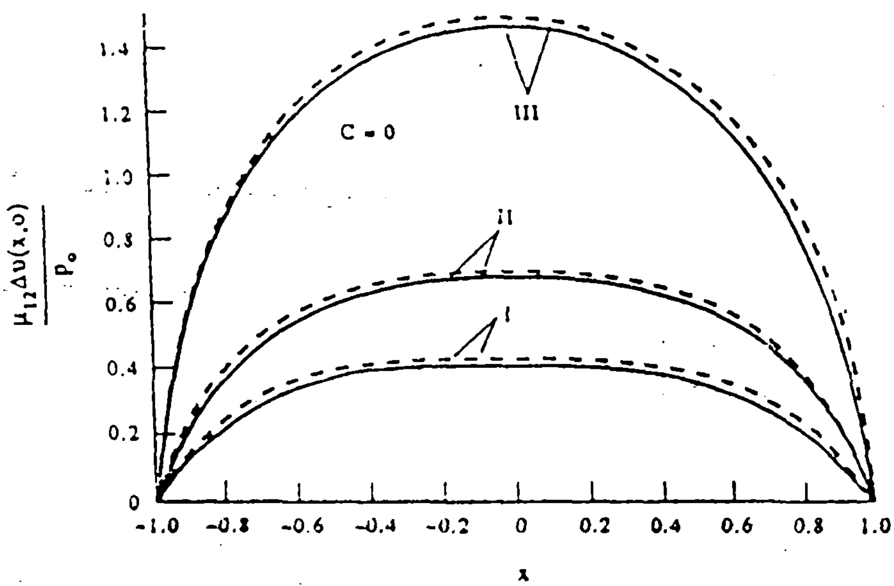

Fig. 7. Crack opening displacement (COD) vs distance (single crack, $c = 0$) for generalized plane stress. (—, $k_r = 0.2$; - - - -, $k_r = 0.6$).

DIFFRACTION OF ELASTIC WAVES BY THREE COPLANAR GRIFFITH CRACKS IN AN ORTHOTROPIC MEDIUM

1. INTRODUCTION

Recently, with the increased usage of macroscopically anisotropic construction materials such as fibre-reinforced materials, the study of diffraction of elastic waves with cracks or inclusions has attracted the attention of scientists. The different possible location of cracks with respect to the planes of material symmetry is of great interest in Seismology and Exploration Geophysics. The problem of scattering of elastic waves by cracks of finite dimension in isotropic medium has been investigated by several investigators. Many investigators (Mal 1970b, Lowengrub et al. 1968a, Itou 1980b, Jain and Kanwal 1972a, Srivastava et al. 1981, Das and Ghosh 1992a) have solved the diffraction problem involving single or two cracks in isotropic medium. Dhawan and Dhaliwal (1978) solved the statical problem involving three coplanar cracks in an infinite transversely isotropic medium. The dynamic problem of singular stresses around cracks in orthotropic medium are few in number. Kassir and Bandyopadhyay (1983) solved the problem of

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elastodynamic response of an infinite orthotropic solid containing a crack under the action of impact loading. The problem of normal impact response of a finite Griffith crack in an orthotropic strip has been solved by Shindo (1986). De and Patra (1990) have also solved the problem involving a moving Griffith crack in an orthotropic strip. Recently Kundu and Bostrom (1991) treated the diffraction problem of a circular crack in orthotropic medium.

To the best knowledge of the authors, the problem of diffraction of elastic waves by three coplanar Griffith cracks in an orthotropic material has not been considered. In our paper, the interaction of normally incident time harmonic elastic waves with three coplanar Griffith cracks in an orthotropic medium has been investigated. It is assumed that the faces of each of the cracks do not come into contact during small deformation of the solid. The resulting mixed boundary value problem is reduced to the solution of a set of four integral equations which has been reduced to the solution of an integro-differential equation. Iteration method has been used to obtain the low frequency solution of the problem. This enables us to obtain approximate value of the crack opening displacements and stress intensity factors. Making the length of the central crack tend to zero, the corresponding results for two Griffith cracks have been obtained. Numerical results for stress intensity factors and crack opening displacements have been plotted against dimensionless frequency and distance respectively for different orthotropic materials which have been shown graphically.

2. STATEMENT AND FORMULATION OF THE PROBLEM

Consider the interaction of normally incident longitudinal wave with three coplanar Griffith cracks situated in an infinite orthotropic elastic medium. The cracks are assumed to occupy the position $|X|\le d_1$, $d_2\le|X|\le d$, $Y=0$, $|Z|<\infty$. Let E_i, μ_{ij} and ν_{ij} $(i,j= 1,2,3)$ denote the engineering elastic constants of the material where the subscripts 1,2,3 correspond to the X, Y, Z directions chosen to coincide with the axes of material orthotropy. Normalizing all the lengths with respect to 'd' and setting $X/d=x$, $Y/d=y$, $Z/d=z$, $d_1/d=b$, $d_2/d=c$, the cracks are defined by $|x|\le b$, $c\le|x|\le 1$, $y=0$, $|z|<\infty$ (fig.1).

Displacement components are also made dimensionless with respect to 'd' so that dimensionless components of displacement in x, y directions are assumed to be u, v respectively, where

$$u = u(x,y,t) \text{ and } v = v(x,y,t).$$

Let a time harmonic plane elastic wave originating at $y=-\infty$ and incident normally on the three cracks be given by $v=v_o\exp[i(ky-\omega t)]/d$ where $k=d\omega/c_s\sqrt{c_{22}}$, $c_s=(\mu_{12}/\rho)^{1/2}$, v_o is a constant, ω and v_o/d are the frequency and dimensionless amplitude of the incident wave respectively, ρ being the density of the material. In the isotropic solid, c_s represents the velocity of the shear-wave.

The non-zero stress components τ_{yy} and τ_{xy} are given by

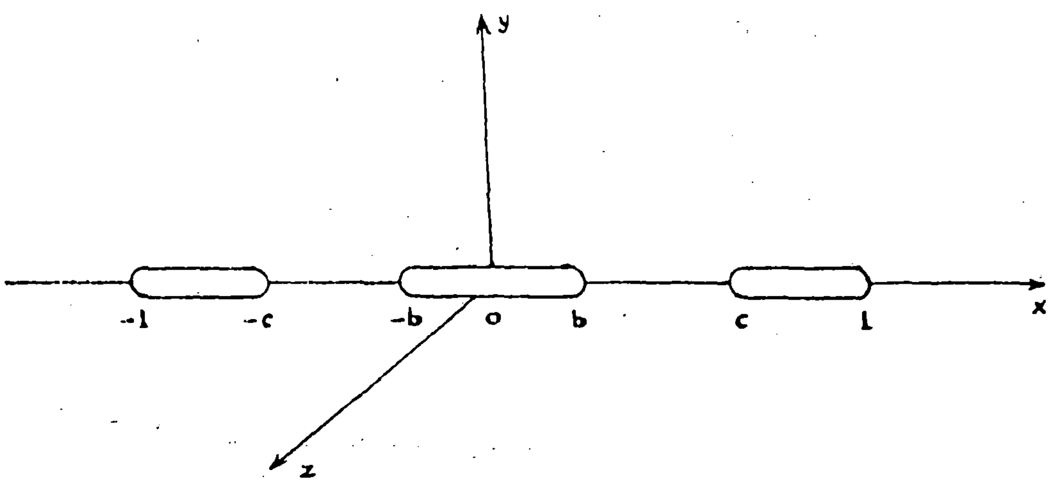

Fig. 1. Geometry of the cracks.

$$\tau_{yy}/\mu_{12} = c_{12}\, u_{,x} + c_{22}\, v_{,y}$$
$$\tau_{xy}/\mu_{12} = u_{,y} + v_{,x} \qquad (2.1)$$

where u, v denote the component of the displacement in the x, y directions respectively and comma denotes partial differentiation with respect to the co-ordinates or time ; $c_{ij}(i,j=1,2)$ are nondimensional parameters related to the elastic constant by the relations :

$$c_{11}=E_1/\mu_{12}(1-\nu_{12}^2E_2/E_1)$$
$$c_{22}=E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1)=c_{11}E_2/E_1 \qquad (2.2)$$
$$c_{12}=\nu_{12}E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1)=\nu_{12}c_{22}=\nu_{21}c_{11}$$

for generalized plane stress, and by

$$c_{11}=(E_1/\Delta\mu_{12})(1-\nu_{23}\nu_{32})$$
$$c_{22}=(E_2/\Delta\mu_{12})(1-\nu_{13}\nu_{31})$$
$$c_{12}=E_1(\nu_{21}+\nu_{13}\nu_{32}E_2/E_1)/\Delta\mu_{12} \qquad (2.3)$$
$$=E_2(\nu_{12}+\nu_{23}\nu_{31}E_1/E_2)/\Delta\mu_{12}$$

$$\Delta = 1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{13}\nu_{21}\nu_{32}$$

for plane strain. The constants E_i and ν_{ij} satisfy Maxwell's relation :

$$\nu_{ij}/E_i = \nu_{ji}/E_j\,. \qquad (2.4)$$

The displacement equations of motion for orthotropic material are

$$c_{11}\, u_{,xx} + u_{,yy} + (1+c_{12})v_{,xy} = \frac{d^2}{c_s^2}\, u_{,tt}$$

$$c_{22}\, v_{,yy} + v_{,xx} + (1+c_{12})u_{,xy} = \frac{d^2}{c_s^2}\, v_{,tt} \qquad (2.5)$$

Substitution of $u(x,y,t) = u(x,y)\exp(-i\omega t)$ and $v(x,y,t) = v(x,y)\exp(-i\omega t)$ in equations (2.5) reduces them to

$$c_{11}\, u_{,xx} + u_{,yy} + (1+c_{12})v_{,xy} + k_s^2 u = 0$$

$$c_{22}\, v_{,yy} + v_{,xx} + (1+c_{12})u_{,xy} + k_s^2 v = 0 \qquad (2.6)$$

with $k_s^2 = d^2\omega^2/c_s^2$, which are to be solved subject to the boundary conditions

$$v(x,0) = 0 \quad , \quad b\le|x|\le c \;,\; |x|\ge 1 \qquad (2.7)$$

$$\tau_{xy}(x,0) = 0 \quad , \quad |x|<\infty \qquad (2.8)$$

$$\tau_{yy}(x,0) + \tau_{yy}^{(0)}(x,0) = 0 \quad , \quad |x|<b,\; c<|x|<1 \qquad (2.9)$$

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables would be omitted in the sequel.

Using the condition (2.8), the solutions of equations (2.6) may be written as

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[\exp(-\gamma_1|y|) - \beta \exp(-\gamma_2|y|)\right]A_1(\xi)\sin(\xi x)d\xi \qquad (2.10)$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\left[\alpha_1 \exp(-\gamma_1|y|) - \beta\alpha_2 \exp(-\gamma_2|y|)\right]A_1(\xi)\cos(\xi x)d\xi \quad , \quad y>0 \qquad (2.11)$$

and the stress components are given by

$$\tau_{xy}/\mu_{12} = -\frac{2}{\pi}\int_0^\infty (\gamma_1+\alpha_1)\left[\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)\right]A_1(\xi)\sin(\xi x)d\xi \quad , \quad y>0 \qquad (2.12)$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty \left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \beta\left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\right]A_1(\xi)\cos(\xi x) \qquad (2.13)$$

where $$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1+c_{12})\gamma_i} \quad , \quad i=1,2 \qquad (2.14)$$

$$\beta = \frac{\gamma_1+\alpha_1}{\gamma_2+\alpha_2} \qquad (2.15)$$

$A_1(\xi)$ is the unknown function to be determined, and γ_1^2 , γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \left\{(c_{12}^2+2c_{12}-c_{11}c_{22})\xi^2+(1+c_{22})k_s^2\right\}\gamma^2 + (c_{11}\xi^2-k_s^2)(\xi^2-k_s^2) = 0 \qquad (2.16)$$

With the aid of the boundary conditions, (2.7) and (2.9) $A(\xi)$ is found to satisfy the integral equations

$$\int_0^\infty A(\xi)\cos(\xi x)d\xi = 0 \quad , \qquad x \in I_2 \, , \, I_4 \tag{2.17}$$

and $$\int_0^\infty H(\xi)A(\xi)\cos(\xi x)d\xi = -\frac{\pi p_0}{2\mu_{12}} \quad , \qquad x \in I_1 \, , \, I_3 \tag{2.18a,b}$$

where $I_1 = (0,b)$, $I_2 = (b,c)$, $I_3 = (c,1)$, $I_4 = (1,\infty)$

and

$$p_0 = ik\mu_{12}c_{22}v_0/d \tag{2.19}$$

$$A(\xi) = \frac{\alpha_1 - \beta\alpha_2}{\xi} A_1(\xi) \tag{2.20}$$

$$H(\xi) = \frac{c_{12}\xi^2 - c_{22}\alpha_1\gamma_1 - \beta(c_{12}\xi^2 - c_{22}\alpha_2\gamma_2)}{(\alpha_1 - \beta\alpha_2)} \tag{2.21}$$

3. METHOD OF SOLUTION

The solution of the integral equations (2.17) and (2.18) is taken in the form

$$A(\xi) = \frac{1}{\xi}\int_0^b h(t)\sin(\xi t)dt + \frac{1}{\xi}\int_c^1 g(u^2)\sin(\xi u)du \tag{3.1}$$

where $h(t)$ and $g(u^2)$ are the unknown functions to be determined. Substituting the value of $A(\xi)$ from (3.1) in (2.17) and using the following result (Gradshteyn et al, 1965)

$$\int_0^\infty \frac{\sin(\xi t)\cos(\xi x)}{\xi}\, d\xi = \begin{cases} \frac{\pi}{2}, & t>x \\ 0, & t<x \end{cases}$$

it is found that the choice of $A(\xi)$ leads to the equation

$$\int_c^1 g(u^2)\, du = 0. \tag{3.2}$$

Further substituting $A(\xi)$ from (3.1) in (2.18a) and using the result (Srivastava et al., 1968)

$$\int_0^\infty \xi^{-1}\sin(\xi u)\sin(\xi x)\, d\xi = \frac{1}{2}\log\left|\frac{u+x}{u-x}\right|$$

we obtain

$$\frac{d}{dx}\int_0^b h(t)\log\left|\frac{t+x}{t-x}\right| dt + \frac{d}{dx}\int_c^1 g(u^2)\log\left|\frac{u+x}{u-x}\right| du$$

$$= 2\Big[q_0 - \frac{d}{dx}\int_0^b h(t)dt\int_0^\infty H_1(\xi)\xi^{-1}\sin(\xi t)\sin(\xi x)d\xi -$$

$$- \frac{d}{dx}\int_c^1 g(u^2)du\int_0^\infty H_1(\xi)\xi^{-1}\sin(\xi u)\sin(\xi x)d\xi\Big], \quad x\in I_1 \tag{3.3}$$

where

$$q_0 = -\frac{\pi p_0}{2\theta\mu_{12}} \tag{3.4}$$

$$H_1(\xi) = \frac{H(\xi)}{\xi\theta} - 1 \to 0 \text{ as } \xi\to\infty \tag{3.5}$$

$$\theta = \frac{(c_{12}^2+c_{12}-c_{11}c_{22})(c_{12}N_1N_2-c_{11})-c_{22}\left[c_{12}N_1^2N_2^2+c_{11}(N_1^2+N_1N_2+N_2^2)\right]}{c_{11}(1+c_{12})(N_1+N_2)}$$

(3.6)

$$N_1^2 = \frac{1}{2c_{22}}\left\{c_{11}c_{22}-c_{12}^2-2c_{12}+\left[(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}\right]^{1/2}\right\}$$

(3.7)

$$N_2^2 = \frac{1}{2c_{22}}\left\{c_{11}c_{22}-c_{12}^2-2c_{12}-\left[(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}\right]^{1/2}\right\}$$

Using the relation

$$\frac{\sin\xi x\ \sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_0(\xi w)J_0(\xi v)dvdw}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}$$

equation (3.3) can now be rewritten in the form

$$\frac{d}{dx}\int_0^b h(t)\ \log\left|\frac{t+x}{t-x}\right|\ dt + \frac{d}{dx}\int_c^1 g(u^2)\ \log\left|\frac{u+x}{u-x}\right|\ du$$

$$= 2\left[\ q_0 - \frac{d}{dx}\int_0^b h(t)dt\int_0^x\int_0^t \frac{vwL(v,w)\ dwdv}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}\right. -$$

$$\left. - \frac{d}{dx}\int_c^1 g(u^2)du\int_0^x\int_0^u \frac{vwL(v,w)\ dwdv}{(x^2-w^2)^{1/2}(u^2-v^2)^{1/2}}\right], \quad x\in I_1 \qquad (3.8)$$

where

$$L(v,w) = \int_0^\infty \xi H_1(\xi)J_0(\xi w)J_0(\xi v)\ d\xi \qquad (3.9)$$

and $J_0(\ )$ is the Bessel function of order zero.

Applying a contour integration technique (Mandal and Ghosh, 1994) the infinite integral in L(v,w) can be converted to the following finite integrals

$$L(v,w) = -ik_s^2\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2-\bar{\alpha}_1\bar{\gamma}_1 c_{22} - \bar{\beta}(c_{12}\eta^2-\bar{\alpha}_2\bar{\gamma}_2 c_{22})}{\theta(\bar{\alpha}_1 - \bar{\beta}\,\bar{\alpha}_2)} \times \right.$$

$$\times\ J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\ d\eta\ -$$

$$\left. - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta(\hat{\alpha}_1-\hat{\beta}\,\hat{\alpha}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\ d\eta\right],\quad w>v \tag{3.10}$$

where

$$\bar{\gamma}_1 = \left[\frac{1}{2}\left\{R_1-(R_1^2-4\bar{R}_2)^{1/2}\right\}\right]^{1/2}$$

$$\bar{\gamma}_2 = \left[\frac{1}{2}\left\{R_1+(R_1^2-4\bar{R}_2)^{1/2}\right\}\right]^{1/2}$$

$$\hat{\gamma}_1 = \left[\frac{1}{2}\left\{-R_1+(R_1^2+4R_2')^{1/2}\right\}\right]^{1/2}$$

$$\hat{\gamma}_2 = \left[\frac{1}{2}\left\{R_1+(R_1^2+4R_2')^{1/2}\right\}\right]^{1/2}$$

$$R_1 = \frac{1}{c_{22}}\left\{(c_{12}^2+2c_{12}-c_{11}c_{22})\eta^2 + (1+c_{22})\right\}$$

$$\bar{R}_2 = \frac{c_{11}}{c_{22}}\left[1-\eta^2\right]\left[\frac{1}{c_{11}} - \eta^2\right] \tag{3.11}$$

$$R_2' = \frac{c_{11}}{c_{22}}\left(1-\eta^2\right)\left(\eta^2 - \frac{1}{c_{11}}\right)$$

$$\bar{\alpha}_i = \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_i^2}{(1+c_{12})\bar{\gamma}_i} \quad , \quad i=1,2$$

$$\hat{\alpha}_i = \frac{c_{11}\eta^2 - 1 + (-1)^i\hat{\gamma}_i^2}{(1+c_{12})\hat{\gamma}_i} \quad , \quad i=1,2$$

$$\bar{\beta} = \frac{\bar{\alpha}_1 - \bar{\gamma}_1}{\bar{\alpha}_2 - \bar{\gamma}_2}$$

$$\hat{\beta} = \frac{\hat{\alpha}_1 + \hat{\gamma}_1}{\hat{\alpha}_2 - \hat{\gamma}_2}$$

The corresponding expression of $L(v,w)$ for $w<v$ is obtained by interchanging v and w in (3.10).

Employing the series expansions for the Bessel function J_o and the Hankel function $H_o^{(1)}$ in equation (3.10), it is found that

$$L(v,w) = \frac{2}{\pi} P \; k_s^2 \log k_s + O(k_s^2) \tag{3.12}$$

where

$$P = \frac{1}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - \bar{\alpha}_1\bar{\gamma}_1 c_{22} - \bar{\beta}(c_{12}\eta^2 - \bar{\alpha}_2\bar{\gamma}_2 c_{22})}{(\bar{\alpha}_1 - \bar{\beta}\,\bar{\alpha}_2)}\, d\eta \; - \right.$$

$$\left. - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{(\hat{\alpha}_1 - \hat{\beta}\,\hat{\alpha}_2)}\, d\eta\right]. \tag{3.13}$$

Let us now expand $h(t)$ and $g(u^2)$ in the form

$$h(t) = h_o(t) + k_s^2 \log k_s \; h_1(t) + O(k_s^2)$$

and
$$g(u^2) = g_o(u^2) + k_s^2 \log k_s \; g_1(u^2) + O(k_s^2). \qquad (3.14)$$

Substituting the above equations (3.14) and the value of $L(v,w)$ given by (3.10) in equations (3.8) and (3.2) and equating the coefficients of like powers of k_s, the following equations are derived.

$$\frac{d}{dx}\int_0^b h_o(t) \log\left|\frac{t+x}{t-x}\right| dt + 2\int_c^1 \frac{ug_o(u^2)}{u^2-x^2} du = 2q_o \quad , \qquad x \in I_1 \; , \; I_3$$

(3.15a,b)

$$\frac{d}{dx}\int_0^b h_1(t) \log\left|\frac{t+x}{t-x}\right| dt + 2\int_c^1 \frac{ug_1(u^2)}{u^2-x^2} du =$$

$$= -\frac{4P}{\pi}\left[\int_0^b th_o(t)dt + \int_c^1 ug_o(u^2)du\right] , \quad x \in I_1 \; , \; I_3 \qquad (3.16a,b)$$

and
$$\int_c^1 g_i(u^2)\, du = 0 \qquad (i=0,1) \qquad (3.17a,b)$$

Rewriting equation (3.15a) as

$$\int_0^b h_o(t) \log\left|\frac{t+x}{t-x}\right| dt = \pi F_1(x) \quad , \qquad x \in I_1 \qquad (3.18)$$

where

$$F_1(x) = -\int_0^x \left[\frac{p_o}{\mu_{12}\theta} + \frac{2}{\pi} \int_c^1 \frac{u g_o(u^2)}{u^2-y^2} du \right] dy.$$

The solution of the integral equation (3.18) with the help of Cook's result (1970) is found to be

$$h_o(t) = -\frac{p_o}{\mu_{12}\theta} \frac{t}{(b^2-t^2)^{1/2}} - \frac{2}{\pi} \frac{t}{(b^2-t^2)^{1/2}} \int_c^1 \frac{\sqrt{u^2-b^2}\, g_o(u^2)}{u^2-t^2} du. \tag{3.19}$$

Substitution of the value of $h_o(t)$ from (3.19) in (3.15b) with the aid of the result

$$\int_0^b \frac{1}{(b^2-t^2)^{1/2}} \frac{t^2\, dt}{(x^2-t^2)(u^2-t^2)} = \frac{\pi}{2}\left[\frac{x}{(x^2-b^2)^{1/2}} - \frac{u}{(u^2-b^2)^{1/2}} \right], \quad x \in I_3$$

yields the singular integral equation

$$\int_c^1 \frac{\sqrt{u^2-b^2}\, g_o(u^2)}{u^2-x^2} du = -\frac{\pi}{2} \frac{p_o}{\mu_{12}\theta}, \quad x \in I_3 \tag{3.20}$$

Next using the finite Hilbert transform technique (Srivastava et al, 1968) the solution of the integral equation is found to be

$$g_o(u^2) = -\frac{p_o}{\mu_{12}\theta} \sqrt{\frac{u^2(u^2-c^2)}{(u^2-b^2)(1-u^2)}} + \frac{uD_1}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}} \tag{3.21}$$

where D_1 is unknown constant to be determined from equation (3.17a).

Now substituting the value of $g_o(u^2)$ from (3.21) in (3.19) and performing the integrations, $h_o(t)$ is obtained in the following form

$$h_o(t) = -\frac{p_o}{\mu_{12}\theta}\left[\sqrt{\frac{t^2(c^2-t^2)}{(b^2-t^2)(1-t^2)}} + \frac{tD_1}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}}\right]. \qquad (3.22)$$

By the procedure similar to one which led to the derivations of the solutions of (3.15) as given by (3.21) and (3.22), the solutions of equation (3.16a,b) can also be obtained and they are found to be

$$h_1(t) = -\frac{4PR}{\pi^2}\left[\sqrt{\frac{t^2(c^2-t^2)}{(b^2-t^2)(1-t^2)}} - \frac{tD_2}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}}\right] \qquad (3.23)$$

$$g_1(u^2) = -\frac{4PR}{\pi^2}\left[\sqrt{\frac{u^2(u^2-c^2)}{(u^2-b^2)(1-u^2)}} + \frac{uD_2}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}}\right] \qquad (3.24)$$

where

$$R = -\frac{p_o}{\mu_{12}\theta}\left[I_o^b + I_c^1\right] - D_1\left[J_o^b - J_c^1\right].$$

$$I_m^n = \int_m^n \frac{t^2\sqrt{(c^2-t^2)}}{\sqrt{(b^2-t^2)(1-t^2)}}\,dt \qquad (3.25)$$

$$J_m^n = \int_m^n \frac{t^2\, dt}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}} .$$

The constant D_2 is to be determined from equation (3.17b).

In order to determine the values of the unknown constants D_1 and D_2, $g_0(u^2)$ and $g_1(u^2)$ as given by (3.21) and (3.24) respectively are substituted in (3.17a,b) and it is found that

$$D_j = A_j\left[(1-b^2)\frac{E}{F} - (c^2-b^2)\right] \quad , \quad (j=1,2) \tag{3.26}$$

and

$$A_1 = \frac{p_0}{\mu_{12}\theta} \quad , \quad A_2 = \frac{4PR}{\pi^2} \tag{3.27}$$

where $F = F(\frac{\pi}{2}, q)$ and $E = E(\frac{\pi}{2}, q)$ are the elliptic integrals of first and second kind respectively and $q = \sqrt{\frac{1-c^2}{1-b^2}}$.

Substitution of the values of $D_j (j=1,2)$ given by equations (3.26) in equations (3.21) - (3.24) yields

$$h_{j-1}(t) = -A_j\left[(1-b^2)\frac{E}{F} + (b^2-t^2)\right] \frac{t}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}}$$

$$(j=1,2) \tag{3.28}$$

$$g_{j-1}(u^2) = -A_j\left[(1-b^2)\frac{E}{F} - (u^2-b^2)\right] \frac{u}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}}$$

$$(j=1,2) \tag{3.29}$$

4. STRESS INTENSITY FACTORS AND CRACK OPENING DISPLACEMENTS

The stress intensity factors are defined as (in physical units)

$$N_b = \lim_{x \to b+} \left[\frac{\sqrt{(x-b)}\ \tau_{yy}(x,0)}{p_o} \right]_{b<x<c} \tag{4.1}$$

$$N_c = \lim_{x \to c-} \left[\frac{\sqrt{(c-x)}\ \tau_{yy}(x,0)}{p_o} \right]_{b<x<c} \tag{4.2}$$

$$N_1 = \lim_{x \to 1+} \left[\frac{\sqrt{(x-1)}\ \tau_{yy}(x,0)}{p_o} \right]_{x>1} \tag{4.3}$$

and the crack opening displacement can now be shown to be given by

$$\Delta v(x,0) = v(x,0+) - v(x,0-) = 2\int_x^b h(t)\, dt \quad , \qquad 0 \le x \le b \tag{4.4}$$

$$= 2\int_x^1 g(u^2)\, du \quad , \qquad c \le x \le 1. \tag{4.5}$$

Substituting the values of the function $h(t)$ and $g(u^2)$, the stress component τ_{yy} can be evaluated from the expressions (2.13), (2.21) and (3.1). After evaluation of the value of τ_{yy} and putting it in relations (4.1) - (4.3) it is found that

$$N_b = \sqrt{\frac{b(1-b^2)}{2(c^2-b^2)}}\ \frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} \left[1 - \frac{4P}{\pi^2} M_2 k_s^2 \log k_s \right] + O(k_s^2) \tag{4.6}$$

$$N_c = \sqrt{\frac{c}{2(c^2-b^2)(1-c^2)}}\left[(1-b^2)\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} - (c^2-b^2)\right]\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2) \qquad (4.7)$$

$$N_1 = \sqrt{\frac{(1-b^2)}{2(1-c^2)}}\left[1 - \frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)}\right]\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2) \qquad (4.8)$$

where

$$M_2 = \left[I_0^b + I_c^1 + \left\{(1-b^2)\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} - (c^2-b^2)\right\}\left\{J_0^b - J_c^1\right\}\right].$$

Expressions (4.4) - (4.5) with the aid of the equations (3.28) - (3.29) yield

$$\Delta v(x,0) = \frac{2p_0}{\mu_{12}\theta}\left[\sqrt{(1-b^2)}\,F(\beta,q)\left\{\frac{E(\beta,q)}{F(\beta,q)} - \frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)}\right\} - \right.$$

$$\left. - \sqrt{\frac{(1-x^2)(b^2-x^2)}{(c^2-x^2)}}\right]\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2)$$

$$, \quad 0 \le x \le b \qquad (4.9)$$

and

$$\Delta v(x,0) = \frac{2p_0}{\mu_{12}\theta}\left[\sqrt{(1-b^2)}\,F(\lambda,q)\left\{\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} - \frac{E(\lambda,q)}{F(\lambda,q)}\right\}\right] \times$$

$$\times\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2) \quad , \quad c \le x \le 1 \qquad (4.10)$$

where

$$\sin\beta = \sqrt{\frac{b^2-x^2}{c^2-x^2}} \qquad \text{and} \qquad \sin\lambda = \sqrt{\frac{1-x^2}{1-b^2}} \, .$$

When $b \to 0$, we recover the stress intensity factor and the crack opening displacement for two Griffith cracks occupying the region $c \le |x| \le 1$, $y=0$, $|z| < \infty$:

$$N_c = - \frac{[c^2 - \frac{E}{F}]}{\sqrt{2c(1-c^2)}}\left[1 - \frac{P}{\pi}\left\{1+c^2 - \frac{2E}{F}\right\} k_s^2 \log k_s\right] + O(k_s^2)$$

$$N_1 = - \frac{[1 - \frac{E}{F}]}{\sqrt{2(1-c^2)}}\left[1 - \frac{P}{\pi}\left\{1+c^2 - \frac{2E}{F}\right\} k_s^2 \log k_s\right] + O(k_s^2) \tag{4.11}$$

and

$$\Delta v(x,0) = \frac{2p_o}{\mu_{12}\theta}\left[1 - \frac{P}{\pi}\left\{1+c^2 - \frac{2E}{F}\right\} k_s^2 \log k_s\right]\left[\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} F(\lambda,q) - E(\lambda,q)\right] + O(k_s^2) \,, \quad c \le x \le 1 \tag{4.12}$$

where $M_2 = \frac{\pi}{4}(1+c^2-2E/F)$ has been used.

It is noted that if further $c \to 0$, the cracks merge into a single crack of width two units. In this case $F \to \infty$ and $M_2 \to \pi/4$; so the results for stress intensity factor and crack opening displacements corresponding to the single crack are found to be

$$N_1 = \frac{1}{\sqrt{2}}\left[1 - \frac{P}{\pi} k_s^2 \log k_s\right] + O(k_s^2) \tag{4.13}$$

and

$$\Delta v(x,0) = -\frac{2p_o}{\mu_{12}\theta}\sqrt{(1-x^2)}\left[1 - \frac{P}{\pi}k_s^2\log k_s\right] + O(k_s^2) \quad , \quad 0\le x\le 1. \quad (4.14)$$

The results given by (4.11) - (4.14) are found to be in agreement with the results of Sarkar, Mandal and Ghosh (1994a).

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) N_b, N_c and N_1 given by (4.6), (4.7) and (4.8) at the tips of the cracks and crack opening displacements (COD) given by (4.9) and (4.10) have been plotted against dimensionless frequency k_s and distance respectively for three different types of orthotropic materials whose engineering constants have been listed in table 1.

Keeping the length of the central crack fixed (b=0.2) SIFs at the tips of the central and outer cracks have been plotted against frequency $k_s(0.1\le k_s\le 0.6)$ for different lengths (c=0.5, 0.6, 0.7) of the outer crack (fig.2-fig.4). It is noted from the graphs (fig.2-fig.4) that with the decrease in the value of outer crack length, i.e. with the increase in the value of the distance between inner and outer cracks the rate of increase in the SIF is higher with the increase in the value of the frequency k_s.

The same nature of SIFs are seen (fig.5-fig.7) in the case when the length of the outer cracks are fixed (c=0.7) and the length of

TABLE - 1. ENGINEERING ELASTIC CONSTANTS.

	E_1 (Pa)	E_2 (Pa)	μ_{12} (Pa)	ν_{12}
Type I	Modulite II Graphite-Epoxy Composite :			
	15.3×10^{9}	158.0×10^{9}	5.52×10^{9}	0.033
Type II	E-Type Glass-Epoxy Composite :			
	9.79×10^{9}	42.3×10^{9}	3.66×10^{9}	0.063
Type III	Stainless Steel-Aluminium Composite :			
	79.76×10^{9}	85.91×10^{9}	30.02×10^{9}	0.31

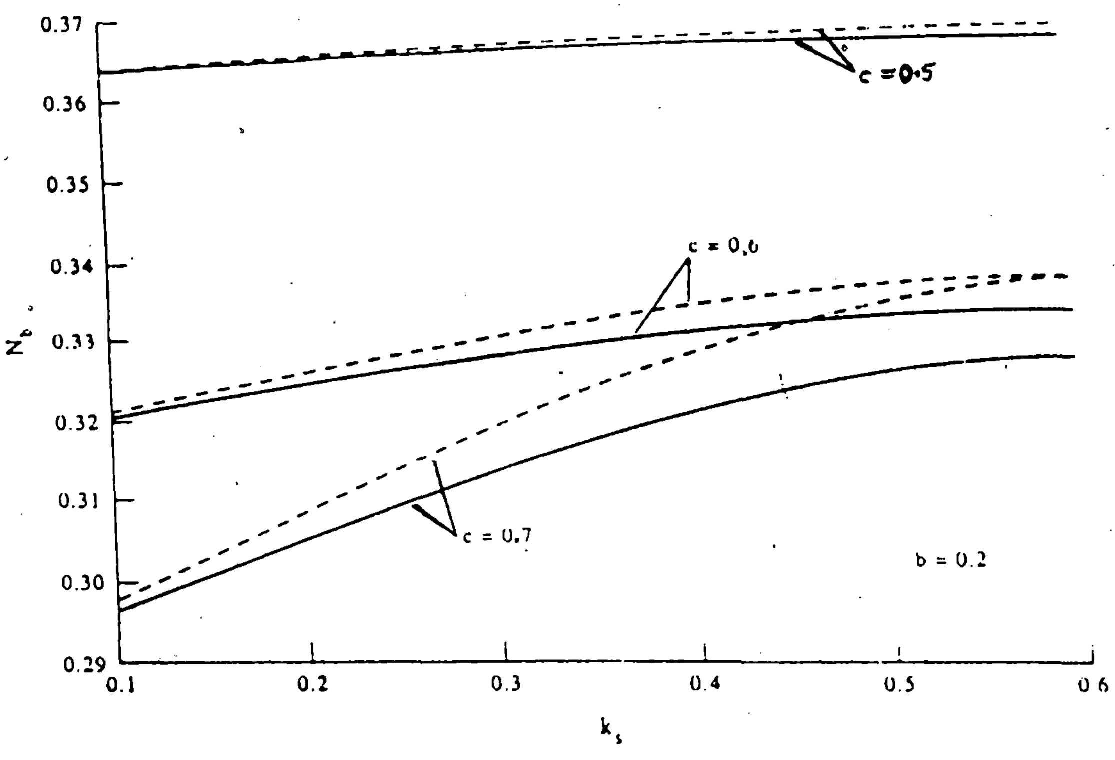

Fig. 2. Stress intensity factor N_b vs frequency k_s for generalized plane stress. (——) type I; (- - - - -) type III.

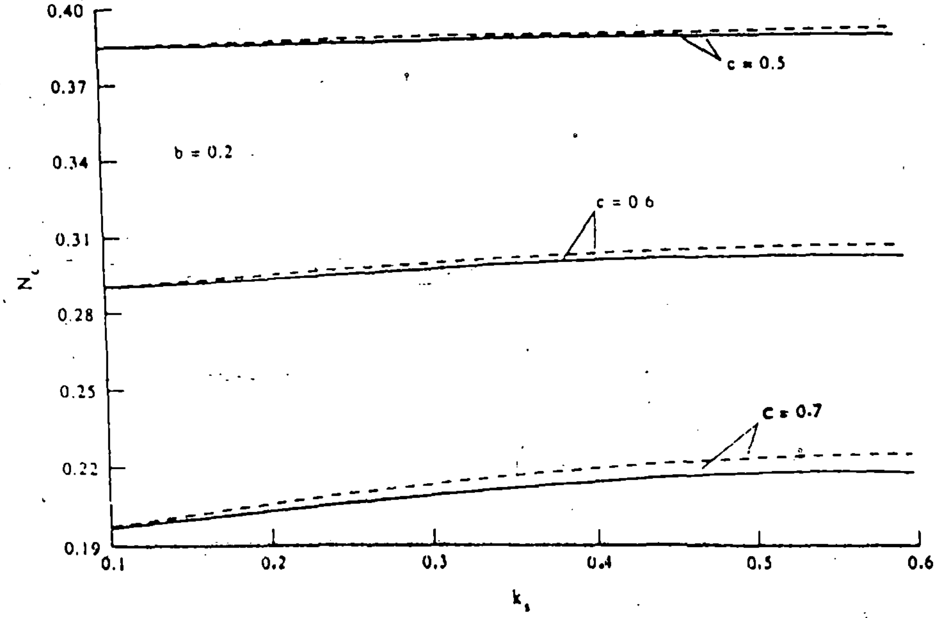

Fig. 3. Stress intensity factor N_c vs frequency k_s for generalized plane stress. (——) type I; (- - - -) type III.

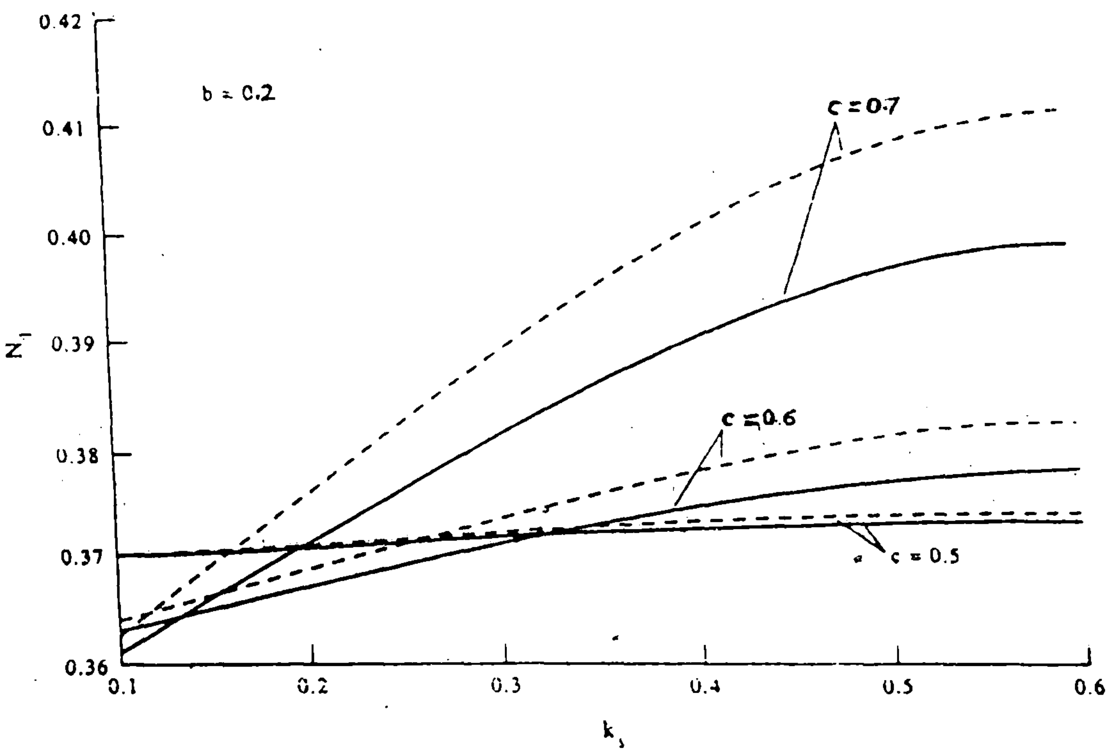

Fig. 4. Stress intensity factor N_1 vs frequency k_s for generalized plane stress. (——) type I; (- - - -) type III.

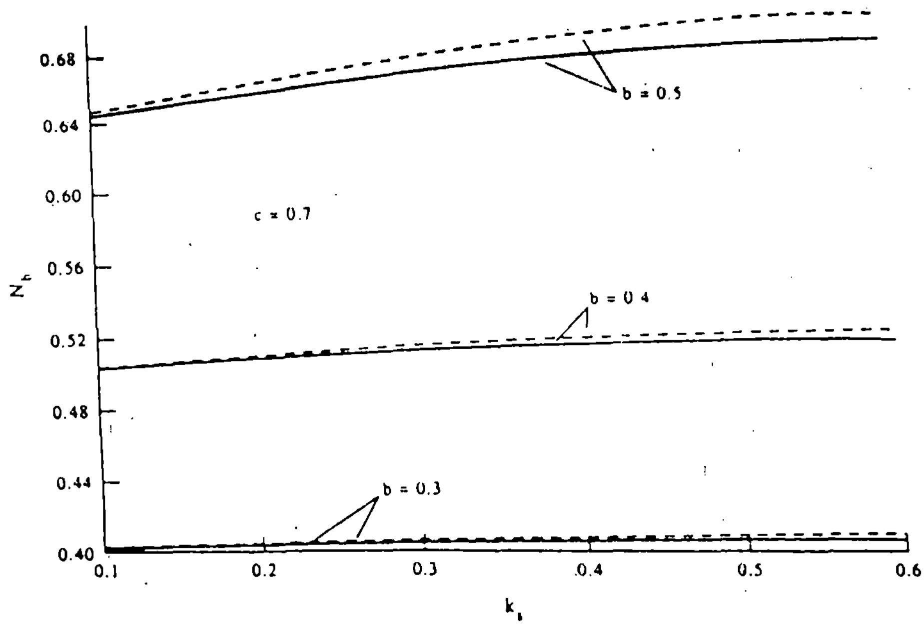

Fig. 5. Stress intensity factor N_b vs frequency k_s for generalized plane stress. (——) type I; (- - - - -) type III.

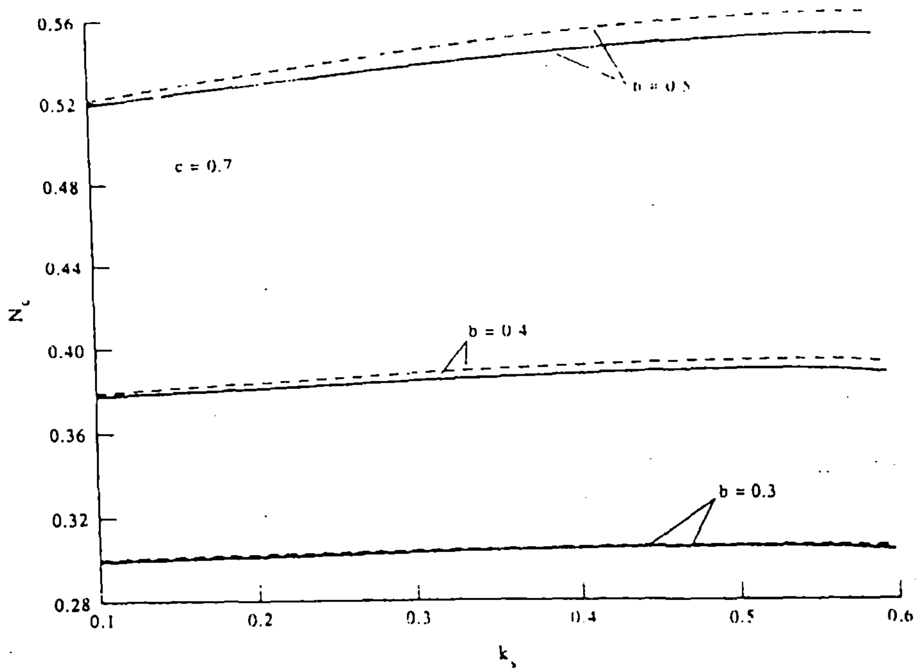

Fig. 6. Stress intensity factor N_c vs frequency k_s for generalized plane stress. (——) type I, (-----) type III.

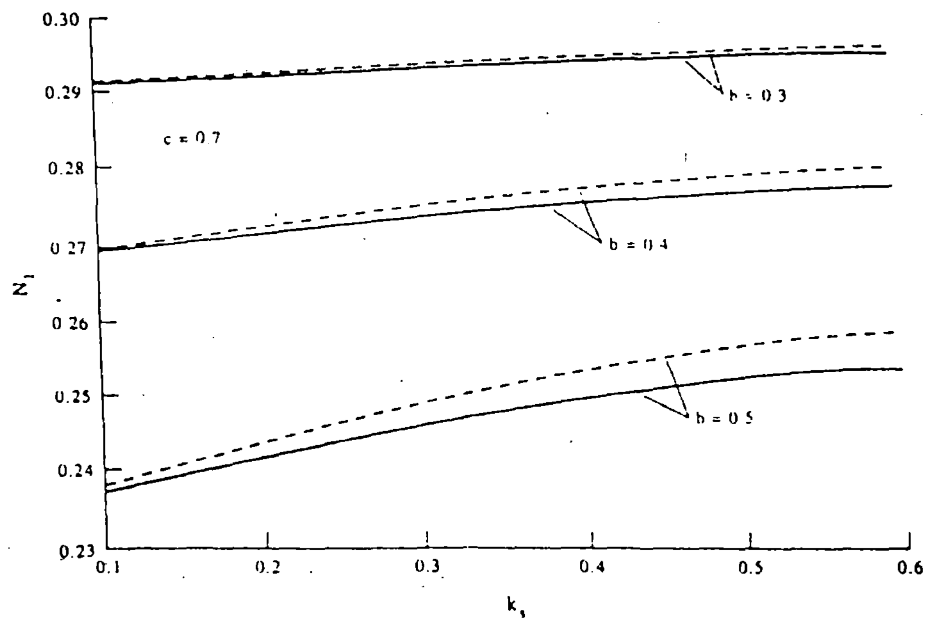

Fig. 7. Stress intensity factor N_1 vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

the central crack increases (b=0.3, 0.4, 0.5). It is interesting to note that for fixed c(=0.7) the SIFs N_b and N_c increase with the increase in the value of b, but the effect is just reverse in case of N_1.

The COD $\mu_{12}\Delta v(x,0)/p_0$ has been plotted for different crack lengths. It is found from fig.8 and fig.9 that with the increase in the value of crack length the value of COD increases. For a fixed material the variation of COD with frequency is found to be insignificant.

In all the cases where different values of crack length have been considered the variation of COD is found to be prominant for different orthotropic materials.

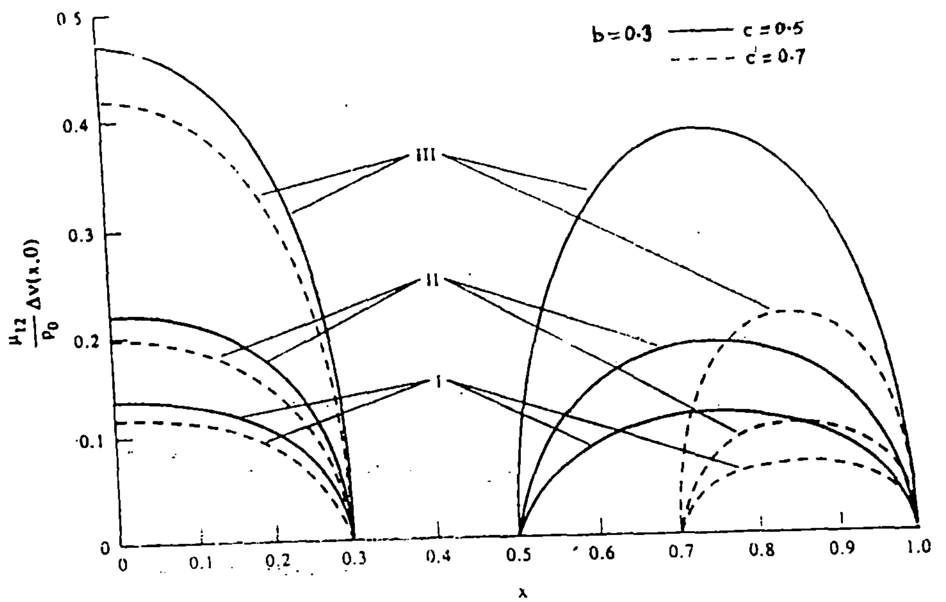

Fig. 8. Crack opening displacement vs distance for generalized plane stress ($k_s = 0.5$, b = 0.3, c = 0.5, 0.7).

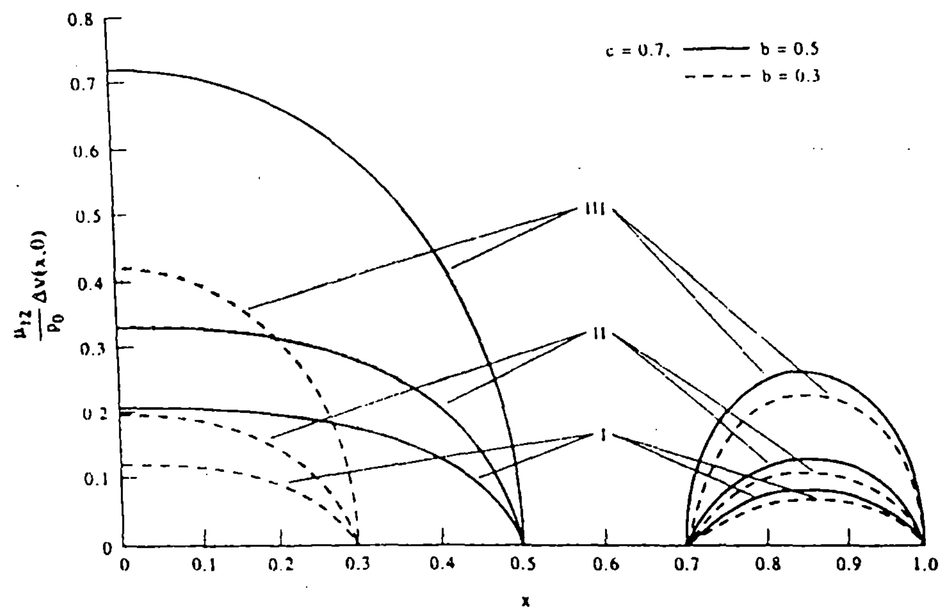

Fig. 9. Crack opening displacement vs distance for generalized plane stress ($k_s = 0.5$, b = 0.3, 0.5, c = 0·7) 0.7).

ELASTIC WAVE SCATTERING FROM FOUR COPLANAR GRIFFITH CRACKS IN AN ORTHOTROPIC MEDIUM

1. INTRODUCTION

With the increased usage of macroscopically anisotropic construction materials such as fibre-reinforced composite, the study of an anisotropic material with crack-like imperfections has become a matter of great importance in fracture analysis of composites (G.C.Sih et al. 1975). The different possible location of cracks with respect to the plane of symmetry is of great importance in seismology and exploration Geophysics. The problems involving the diffraction of elastic waves by cracks in an isotropic medium have been investigated by several investigators (Mal 1970b,Lowengrub et al. 1968a,Itou 1980b,Jain and Kanwal 1972a, Srivastava et al. 1981, Das and Ghosh 1992a,Dhawan et al. 1978), but perhaps, due to mathematical complexity, elastodynamic problems involving two or more Griffith cracks in an anisotropic medium for low frequency have not been treated earlier. Kassir and Tse (1983) have studied the plane stress problem of a moving Griffith crack in an infinite orthotropic stresses medium by using integral transform technique and the same technique has also been employed by De and Patra (1990) to solve the Yoffe's problem in a prestressed orthotropic strip of finite thickness. Kassir and Bandyopadhyay (1983) solved the elastodynamic response of an infinite orthotropic solid containing a crack under the action of impact loading and the problem of normal impact response of an orthotropic strip with a central crack have also been studied by Shindo et al. (1986).

In the present paper, we investigate the problem of diffraction of normally incident time harmonic elastic waves by four coplanar Griffith cracks in an infinite orthotropic medium. The faces of each of the cracks are assumed to be separated by a small distance so that during small deformation of the solid, the crack faces do not come into contact. The resulting mixed boundary value problem has been reduced to solving a set of five integral equations. Iterative method has been used to obtain the low frequency solution of the problem. Employing finite Hilbert transform technique (Srivastava and Lowengrub 1968) the integral equations have been solved to derive crack opening displacement and stress intensity factors. Finally, making the distance between two inner cracks tend to zero, the corresponding results for three cracks have been derived. To display the influence of the material orthotropy, numerical results of stress intensity factors and crack opening displacements have been plotted graphically against the dimensionless frequency and distance respectively for several orthotropic materials.

2. STATEMENT AND FORMULATION OF THE PROBLEM

Consider the problem of diffraction of normally incident elastic waves by four coplanar Griffith cracks situated in an infinite orthotropic elastic medium. The position of the cracks referred to a set of cartesian co-ordinate system (X,Y,Z) are assumed to be $d_1 \le |X| \le d_2$, $d_3 \le |X| \le d$, $Y=0$, $|Z|<\infty$. Let E_i, μ_{ij} and ν_{ij} (i,j= 1,2,3) denote the engineering elastic constants of the material where the subscripts 1,2,3 correspond to the X, Y, Z directions chosen to coincide with the axes of material orthotropy. Normalizing all the lengths with respect to 'd' and setting X/d=x, Y/d=y, Z/d=z, $d_1/d=a$, $d_2/d=b$, $d_3/d=c$ the cracks are defined by $a \le |x| \le b$, $c \le |x| \le 1$, $y=0$, $|z|<\infty$ (Fig.1).

Components of the displacement are also made dimensionless with respect to 'd' so that dimensionless components of displacement in x, y directions are assumed to be u, v respectively, where

$$u = u(x,y,t) \quad \text{and} \quad v = v(x,y,t).$$

Let a time harmonic plane elastic wave given by u=0 and $v=v_o\exp[i(ky-\omega t)]/d$ where $k=\omega d/c_s\sqrt{c_{22}}$, $c_s=(\mu_{12}/\rho)^{1/2}$, ρ the density of the material and v_o a constant, travelling in the direction of positive y-axis be incident normally on the four cracks.

The non-zero stress components τ_{yy} and τ_{xy} are given by

$$\tau_{yy}/\mu_{12} = c_{12}\, u_{,x} + c_{22}\, v_{,y}$$
$$\tau_{xy}/\mu_{12} = u_{,y} + v_{,x} \qquad (2.1a,b)$$

in which a comma denotes partial differentiation with respect to the co-ordinates or the time and c_{ij}(i,j=1,2) are non-dimensional

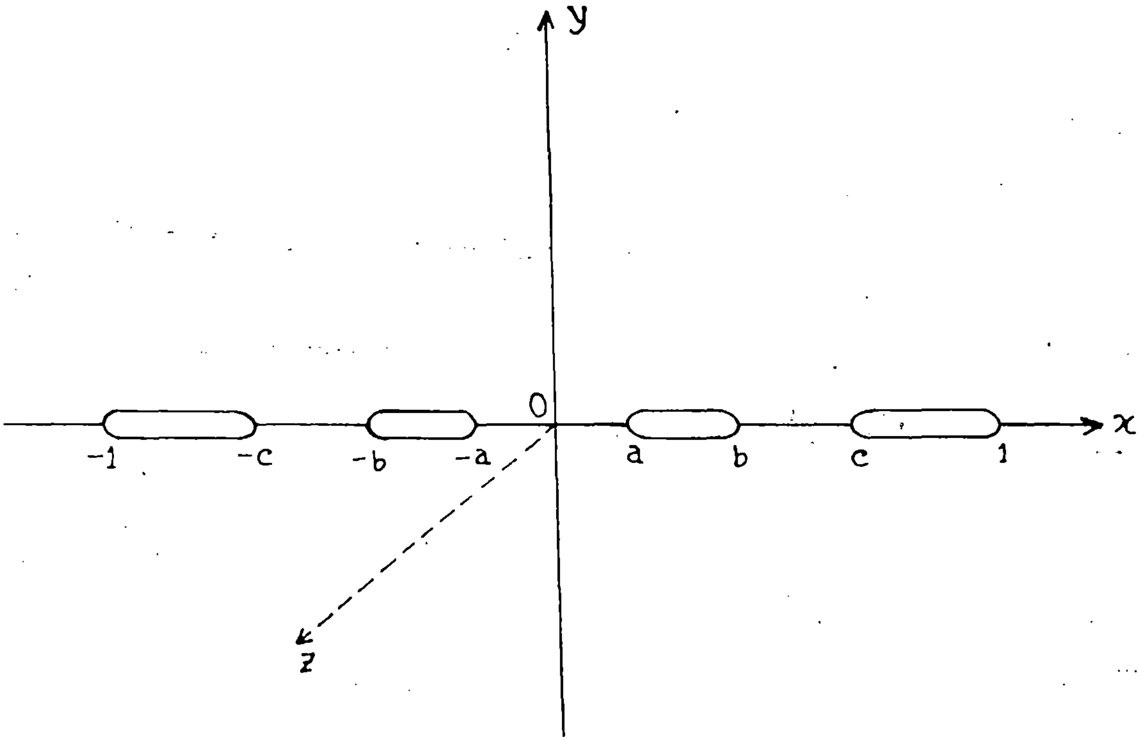

Fig.1. Geometry of the cracks.

parameters related to the elastic constants by the relations :

$$c_{11}=E_1/\mu_{12}(1-\nu_{12}^2E_2/E_1)$$

$$c_{22}=E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1)=c_{11}E_2/E_1 \tag{2.2}$$

$$c_{12}=\nu_{12}E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1)=\nu_{12}c_{22}=\nu_{21}c_{11}$$

for generalized plane stress, and by

$$c_{11}=(E_1/\Delta\mu_{12})(1-\nu_{23}\nu_{32})$$

$$c_{22}=(E_2/\Delta\mu_{12})(1-\nu_{13}\nu_{31})$$

$$c_{12}=E_1(\nu_{21}+\nu_{13}\nu_{32}E_2/E_1)/\Delta\mu_{12} \tag{2.3}$$

$$=E_2(\nu_{12}+\nu_{23}\nu_{31}E_1/E_2)/\Delta\mu_{12}$$

$$\Delta = 1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{13}\nu_{21}\nu_{32}$$

for plane strain. The constants E_i and ν_{ij} satisfy Maxwell's relation :

$$\nu_{ij}/E_i = \nu_{ji}/E_j \tag{2.4}$$

The displacement equations of motion for orthotropic material are

$$c_{11}\,u_{,xx} + u_{,yy} + (1+c_{12})v_{,xy} = \frac{d^2}{c_s^2}\,u_{,tt}$$

$$\tag{2.5}$$

$$c_{22}\,v_{,yy} + v_{,xx} + (1+c_{12})u_{,xy} = \frac{d^2}{c_s^2}\,v_{,tt}$$

Substitution of $u(x,y,t) = u(x,y)\exp(-i\omega t)$ and $v(x,y,t) = v(x,y)\exp(-i\omega t)$ in equations (2.5) reduces them to

$$c_{11}\, u_{,xx} + u_{,yy} + (1+c_{12}) v_{,xy} + k_s^2 u = 0$$

and (2.6)

$$c_{22}\, v_{,yy} + v_{,xx} + (1+c_{12}) u_{,xy} + k_s^2 v = 0$$

with $k_s^2 = d^2\omega^2/c_s^2$.

The boundary conditions of the problem on account of the symmetry with respect to the y-axis are

$$\tau_{xy}(x,0) = 0 \quad , \qquad |x|<\infty \tag{2.7}$$

$$\tau_{yy}(x,0) + \tau_{yy}^{(0)}(x,0) = 0 \quad , \qquad x \in I_2,\ I_4 \tag{2.8}$$

$$v(x,0) = 0 \quad , \qquad x \in I_1,\ I_3,\ I_5 \tag{2.9}$$

where $I_1=(0,a)$, $I_2=(a,b)$, $I_3=(b,c)$, $I_4=(c,1)$, $I_5=(1,\infty)$.

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables is suppressed throughout the analysis.

The solution of equations (2.6) are taken as

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \Big[A_1(\xi)\exp(-\gamma_1|y|) + A_2(\xi)\exp(-\gamma_2|y|)\Big]\sin(\xi x)d\xi ,$$

(2.10)

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\Big[\alpha_1 A_1(\xi)\exp(-\gamma_1|y|) + \alpha_2 A_2(\xi)\exp(-\gamma_2|y|)\Big]\cos(\xi x)d\xi$$

$$, \quad y>0 \tag{2.11}$$

where
$$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1+c_{12})\gamma_i} \quad , \quad i=1,2 \tag{2.12}$$

$A_i(\xi)$ $(i=1,2)$ are the unknown functions to be determined and γ_1^2 , γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \left\{(c_{12}^2+2c_{12}-c_{11}c_{22})\xi^2+(1+c_{22})k_s^2\right\}\gamma^2 + (c_{11}\xi^2-k_s^2)(\xi^2-k_s^2) = 0 \tag{2.13}$$

Using the condition (2.7), it is found that

$$A_2(\xi) = -\frac{\gamma_1+\alpha_1}{\gamma_2+\alpha_2}A_1(\xi) \tag{2.14}$$

By the help of the relation (2.14), the expressions for displacements and stresses reduce to

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[\exp(-\gamma_1|y|) - \beta\exp(-\gamma_2|y|)\right]A_1(\xi)\sin(\xi x)d\xi, \tag{2.15}$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\left[\alpha_1\exp(-\gamma_1|y|) - \beta\alpha_2\exp(-\gamma_2|y|)\right]A_1(\xi)\cos(\xi x)d\xi \quad , \quad y>0 \tag{2.16}$$

$$\tau_{xy}/\mu_{12} = -\frac{2}{\pi}\int_0^\infty (\gamma_1+\alpha_1)\left[\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)\right]A_1(\xi)\sin(\xi x)d\xi \quad , \quad y>0 \tag{2.17}$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty\left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \right.$$

$$\left. - \beta\left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\right]A_1(\xi)\cos(\xi x)d\xi. \tag{2.18}$$

where $\beta = \dfrac{\gamma_1+\alpha_1}{\gamma_2+\alpha_2}$.

Finally, with the aid of the boundary conditions (2.9) and (2.8) the following set of five integral equations

$$\int_0^\infty A(\xi)\cos(\xi x)d\xi = 0 \quad , \qquad x \in I_1 \,,\ I_3 \,,\ I_5 \qquad (2.19a\text{-}c)$$

and
$$\int_0^\infty H(\xi)A(\xi)\cos(\xi x)d\xi = -\frac{\pi p_0}{2\mu_{12}} \quad , \qquad x \in I_2 \,,\ I_4 \qquad (2.20a,b)$$

are obtained for the determination of the unknown function $A(\xi)$

where
$$p_0 = ik\mu_{12}c_{22}v_0/d$$

$$A(\xi) = \frac{\alpha_1-\beta\alpha_2}{\xi} A_1(\xi) \qquad (2.21)$$

$$H(\xi) = \frac{c_{12}\xi^2-c_{22}\alpha_1\gamma_1-\beta(c_{12}\xi^2-c_{22}\alpha_2\gamma_2)}{(\alpha_1-\beta\alpha_2)} .$$

3. METHOD OF SOLUTION

The solution of the integral equations (2.19) and (2.20) is taken in the form

$$A(\xi) = \frac{1}{\xi}\int_a^b h(t^2)\sin(\xi t)dt + \frac{1}{\xi}\int_c^1 g(u^2)\sin(\xi u)du \qquad (3.1)$$

where $h(t^2)$ and $g(u^2)$ are the unknown functions to be determined. Substituting the value of $A(\xi)$ from (3.1) in (2.19) and using the following result (Gradshteyn and Ryzhik, 1965)

$$\int_0^\infty \frac{\sin(\xi t)\cos(\xi x)}{\xi} d\xi = \begin{cases} \frac{\pi}{2} \ , & t>x \\ 0 \ , & t<x \end{cases}$$

it is found that the choice of $A(\xi)$ leads to the equations

$$\int_a^b h(t^2)\,dt = 0 \quad \text{and} \quad \int_c^1 g(u^2)\,du = 0 \tag{3.2a,b}$$

Further substitution of $A(\xi)$ from (3.1) in (2.20) leads to

$$\int_a^b \frac{th(t^2)}{t^2-x^2}\,dt + \int_c^1 \frac{ug(u^2)}{u^2-x^2}\,du =$$

$$= q_o - \frac{d}{dx}\int_a^b h(t^2)dt\int_0^\infty H_1(\xi)\xi^{-1}\sin(\xi t)\sin(\xi x)d\xi -$$

$$- \frac{d}{dx}\int_c^1 g(u^2)du\int_0^\infty H_1(\xi)\xi^{-1}\sin(\xi u)\sin(\xi x)d\xi \quad , \ x\in I_2,\ I_4 \tag{3.3}$$

where

$$q_o = -\frac{\pi p_o}{2\theta\mu_{12}} \tag{3.4}$$

$$H_1(\xi) = \frac{H(\xi)}{\xi\theta} - 1 \to 0 \text{ as } \xi\to\infty \tag{3.5}$$

$$\theta = \frac{(c_{12}^2+c_{12}-c_{11}c_{22})(c_{12}N_1N_2-c_{11})-c_{22}\left[c_{12}N_1^2N_2^2+c_{11}(N_1^2+N_1N_2+N_2^2)\right]}{c_{11}(1+c_{12})(N_1+N_2)} \tag{3.6}$$

$$N_1^2 = \frac{1}{2c_{22}}\left\{c_{11}c_{22}-c_{12}^2-2c_{12}+\left[(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}\right]^{1/2}\right\}$$

$$N_2^2 = \frac{1}{2c_{22}}\left\{c_{11}c_{22}-c_{12}^2-2c_{12}-\left[(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}\right]^{1/2}\right\} \tag{3.7}$$

Using the relation

$$\frac{\sin\xi x\ \sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_o(\xi w)J_o(\xi v)dvdw}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}$$

equation (3.3) can now be rewritten in the form

$$\int_a^b \frac{th(t^2)}{t^2-x^2}\,dt + \int_c^1 \frac{ug(u^2)}{u^2-x^2}\,du =$$

$$= q_o - \frac{d}{dx}\int_a^b h(t^2)dt \int_0^x \int_0^t \frac{vwL(v,w)\ dwdv}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} -$$

$$- \frac{d}{dx}\int_c^1 g(u^2)du \int_0^x \int_0^u \frac{vwL(v,w)\ dwdv}{(x^2-w^2)^{1/2}(u^2-v^2)^{1/2}} \quad , \ x \in I_2 \ , \ I_4 \tag{3.8}$$

where

$$L(v,w) = \int_0^\infty \xi H_1(\xi) J_o(\xi w) J_o(\xi v)\ d\xi \tag{3.9}$$

and $J_o(\ )$ is the Bessel function of order zero.

Applying a contour integration technique (Mandal and Ghosh, 1994) the infinite integral in L(v,w) can be converted to the following finite integrals

$$L(v,w) = -ik_s^2\Bigg[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2-\bar{\alpha}_1\bar{\gamma}_1 c_{22} - \bar{\beta}(c_{12}\eta^2-\bar{\alpha}_2\bar{\gamma}_2 c_{22})}{\theta(\bar{\alpha}_1 - \bar{\beta}\ \bar{\alpha}_2)} \times$$

$$\times\ J_o(k_s\eta v)H_o^{(1)}(k_s\eta w)\ d\eta\ -$$

$$- \int_{1/\sqrt{c_{11}}}^1 \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta(\hat{\alpha}_1-\hat{\beta}\ \hat{\alpha}_2)}\ J_o(k_s\eta v)H_o^{(1)}(k_s\eta w)\ d\eta\Bigg], \quad w>v \tag{3.10}$$

where

$$\bar{\gamma}_1 = \left[\frac{1}{2}\left\{R_1 - (R_1^2 - 4\bar{R}_2)^{1/2}\right\}\right]^{1/2}$$

$$\bar{\gamma}_2 = \left[\frac{1}{2}\left\{R_1 + (R_1^2 - 4\bar{R}_2)^{1/2}\right\}\right]^{1/2}$$

$$\hat{\gamma}_1 = \left[\frac{1}{2}\left\{-R_1 + (R_1^2 + 4R_2')^{1/2}\right\}\right]^{1/2}$$

$$\hat{\gamma}_2 = \left[\frac{1}{2}\left\{R_1 + (R_1^2 + 4R_2')^{1/2}\right\}\right]^{1/2}$$

$$R_1 = \frac{1}{c_{22}}\left\{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\eta^2 + (1 + c_{22})\right\}$$

$$\bar{R}_2 = \frac{c_{11}}{c_{22}}\left[1 - \eta^2\right]\left[\frac{1}{c_{11}} - \eta^2\right]$$

$$R_2' = \frac{c_{11}}{c_{22}}\left[1 - \eta^2\right]\left[\eta^2 - \frac{1}{c_{11}}\right]$$

$$\bar{\alpha}_i = \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_i^2}{(1 + c_{12})\bar{\gamma}_i} \quad , \quad i = 1, 2$$

$$\hat{\alpha}_i = \frac{c_{11}\eta^2 - 1 + (-1)^i\hat{\gamma}_i^2}{(1 + c_{12})\hat{\gamma}_i} \quad , \quad i = 1, 2$$

$$\bar{\beta} = \frac{\bar{\alpha}_1 - \bar{\gamma}_1}{\bar{\alpha}_2 - \bar{\gamma}_2} \quad \text{and} \quad \hat{\beta} = \frac{\hat{\alpha}_1 + \hat{\gamma}_1}{\hat{\alpha}_2 - \hat{\gamma}_2}\,. \tag{3.11}$$

The corresponding expression of $L(v,w)$ for $w<v$ is obtained by interchanging v and w in (3.10).

Employing the series expansions for the Bessel function J_o and the Hankel function $H_o^{(1)}$ in equation (3.10), it is found that

$$L(v,w) = \frac{2}{\pi} P\, k_s^2 \log k_s + O(k_s^2) \tag{3.12}$$

where

$$P = \frac{1}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - \bar{\alpha}_1\bar{\gamma}_1 c_{22} - \bar{\beta}(c_{12}\eta^2 - \bar{\alpha}_2\bar{\gamma}_2 c_{22})}{(\bar{\alpha}_1 - \bar{\beta}\,\bar{\alpha}_2)}\, d\eta \right. -$$

$$- \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{(\hat{\alpha}_1 - \hat{\beta}\,\hat{\alpha}_2)}\, d\eta \Bigg] \qquad (3.13)$$

Next expanding $h(t^2)$ and $g(u^2)$ in the form

$$h(t^2) = h_o(t^2) + k_s^2 \log k_s\, h_1(t^2) + O(k_s^2)$$

$$(3.14)$$

and

$$g(u^2) = g_o(u^2) + k_s^2 \log k_s\, g_1(u^2) + O(k_s^2)$$

and substituting this expansion as well as the result (3.12) in equation (3.8) and finally equating the coefficients of like powers of k_s, the following equations are derived.

$$\int_a^b \frac{t h_o(t^2)}{t^2 - x^2}\, dt + \int_c^1 \frac{u g_o(u^2)}{u^2 - x^2}\, du = q_o \quad , \qquad x \in I_2 \; , \; I_4$$

$$(3.15a,b)$$

$$\int_a^b \frac{t h_1(t^2)}{t^2 - x^2}\, dt + \int_c^1 \frac{u g_1(u^2)}{u^2 - x^2}\, du =$$

$$= - \frac{4P}{\pi}\Big[\int_a^b t h_o(t^2) dt + \int_c^1 u g_o(u^2) du \Big] \; , \qquad x \in I_2 \; , \; I_4$$

$$(3.16a,b)$$

and also equation (3.2) with the aid of equation (3.14) yields

$$\int_a^b h_i(t^2)\, dt = 0 \qquad (i=0,1)$$

$$(3.17a\text{-}d)$$

$$\int_c^1 g_i(u^2)\, du = 0 \qquad (i=0,1)$$

Rewriting equation (3.15a) as

$$\int_a^b \frac{t h_o(t^2)}{t^2-x^2}\, dt = \frac{\pi}{2} F_1(x) \quad , \quad x \in I_2 \tag{3.18}$$

where $$F_1(x) = -\left[\frac{p_o}{\mu_{12}\theta} + \frac{2}{\pi}\int_c^1 \frac{u g_o(u^2)}{u^2-x^2}\, du \right]$$

Applying finite Hilbert transform technique (Srivastava and Lowengrub, 1968), the solution of the integral equation (3.18) is found to be

$$h_o(t^2) = -\frac{p_o}{\mu_{12}\theta}\sqrt{\frac{t^2-a^2}{b^2-t^2}} - \frac{2}{\pi}\sqrt{\frac{t^2-a^2}{b^2-t^2}} \int_c^1 \sqrt{\frac{u^2-b^2}{u^2-a^2}}\, \frac{u\, g_o(u^2)}{u^2-t^2}\, du + \frac{C_1}{\sqrt{(t^2-a^2)(b^2-t^2)}} \tag{3.19}$$

where C_1 is the unknown constant to be determined from equation (3.17a).

Substitution of the value of $h_o(t^2)$ from (3.19) in (3.15b) with the aid of the results

$$\int_a^b \sqrt{\frac{t^2-a^2}{b^2-t^2}}\, \frac{t\, dt}{(x^2-t^2)(u^2-t^2)} = \frac{\pi}{2(u^2-x^2)}\left[\sqrt{\frac{x^2-a^2}{x^2-b^2}} - \sqrt{\frac{u^2-a^2}{u^2-b^2}} \right]$$

$$\int_a^b \frac{t\, dt}{(x^2-t^2)\sqrt{(t^2-a^2)(b^2-t^2)}} = \frac{\pi}{2\sqrt{(t^2-a^2)(b^2-t^2)}} \quad \text{for} \quad x \in I_4$$

yields the singular integral equation

$$\int_c^1 \sqrt{\frac{u^2-b^2}{u^2-a^2}}\, \frac{u\, g_o(u^2)}{u^2-x^2}\, du = \frac{\pi}{2} F_2(x) \quad , \quad x \in I_4 \tag{3.20}$$

where

$$F_2(x) = -\frac{p_o}{\mu_{12}\theta} + \frac{C_1}{x^2-a^2} .$$

Next using the finite Hilbert transform technique (Srivastava and Lowengrub, 1968) the solution of the integral equation (3.20) is found to be

$$g_o(u^2) = -\frac{p_o}{\mu_{12}\theta}\sqrt{\frac{(u^2-a^2)(u^2-c^2)}{(u^2-b^2)(1-u^2)}} + \sqrt{\frac{1-a^2}{c^2-a^2}}\,\frac{C_1\sqrt{(u^2-c^2)}}{\sqrt{(u^2-a^2)(u^2-b^2)(1-u^2)}} +$$

$$+ \frac{C_2\sqrt{(u^2-a^2)}}{\sqrt{(u^2-b^2)(u^2-a^2)(1-u^2)}} \qquad (3.21)$$

where C_2 is unknown constant to be determined from equation (3.17c).

Further substituting the value of $g_o(u^2)$ from (3.21) in (3.19) and performing the resulting integrations, $h_o(t^2)$ is obtained in the following form

$$h_o(t^2) = -\frac{p_o}{\mu_{12}\theta}\sqrt{\frac{(t^2-a^2)(c^2-t^2)}{(b^2-t^2)(1-t^2)}} + \sqrt{\frac{1-a^2}{c^2-a^2}}\,\frac{C_1\sqrt{(c^2-t^2)}}{\sqrt{(t^2-a^2)(b^2-t^2)(1-t^2)}} -$$

$$- \frac{C_2\sqrt{(t^2-a^2)}}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}} \qquad (3.22)$$

By the procedure similar to one which led to the derivations of the solutions of (3.15) as given by (3.21) and (3.22), the solutions of equation (3.16a,b) can also be obtained and they are found to be

$$h_1(t^2) = -\frac{4PR}{\pi^2}\sqrt{\frac{(t^2-a^2)(c^2-t^2)}{(b^2-t^2)(1-t^2)}} + \sqrt{\frac{1-a^2}{c^2-a^2}}\,\frac{D_1\sqrt{(c^2-t^2)}}{\sqrt{(t^2-a^2)(b^2-t^2)(1-t^2)}} -$$

$$- \frac{D_2\sqrt{(t^2-a^2)}}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}} \qquad (3.23)$$

$$g_1(u^2) = -\frac{4PR}{\pi^2}\sqrt{\frac{(u^2-a^2)(u^2-c^2)}{(u^2-b^2)(1-u^2)}} + \sqrt{\frac{1-a^2}{c^2-a^2}}\,\frac{D_1\sqrt{(u^2-c^2)}}{\sqrt{(u^2-a^2)(u^2-b^2)(1-u^2)}} +$$

$$+ \frac{D_2\sqrt{(u^2-a^2)}}{\sqrt{(u^2-b^2)(u^2-a^2)(1-u^2)}} \qquad (3.24)$$

where

$$R = -\frac{p_o}{\mu_{12}\theta}\left[R_a^b + R_c^1 \right] + \left[\sqrt{\frac{1-a^2}{c^2-a^2}}\, C_1 + C_2 \right] J_2$$

$$R_m^n = \int_m^n \sqrt{\frac{(t^2-a^2)(c^2-t^2)}{(b^2-t^2)(1-t^2)}}\, dt \qquad (3.25)$$

$$J_2 = \frac{(c^2-b^2)}{\sqrt{(c^2-a^2)(1-b^2)}}\left[\Pi\left(\frac{\pi}{2}, \frac{b^2-a^2}{c^2-a^2}, r\right) + \Pi\left(\frac{\pi}{2}, \frac{1-c^2}{1-b^2}, r\right) - F\left(\frac{\pi}{2}, r\right)\right]$$

$$r = \sqrt{\frac{(1-c^2)(b^2-a^2)}{(1-b^2)(c^2-a^2)}} .$$

The constants D_1 and D_2 are to be determined from (3.17b) and (3.17d). In equations (3.25), F() is the elliptic integral of the first kind and Π() is the elliptic integral of the third kind. Substitution of the values of $h(t^2)$ and $g(u^2)$ given by equations (3.21-3.24) in equations (3.17a-d)yield

$$C_i = \frac{p_o}{\mu_{12}\theta} Q_i \qquad (i=1,2) \qquad (3.26)$$

$$D_i = \frac{4PR}{\pi^2} Q_i \qquad (i=1,2) \qquad (3.27)$$

where

$$Q_1 = \left[\frac{K_a^b \, I_c^1 + K_c^1 \, I_a^b}{K_a^b \, J_c^1 + K_c^1 \, J_a^b} \right] \sqrt{\frac{(c^2-a^2)}{(1-a^2)}}$$

$$Q_2 = \left[\frac{J_a^b \, I_c^1 - J_c^1 \, I_a^b}{K_a^b \, J_c^1 + K_c^1 \, J_a^b} \right]$$

$$I_m^n = \int_m^n \sqrt{\frac{(u^2-a^2)(u^2-c^2)}{(u^2-b^2)(1-u^2)}} \, du \qquad (3.28)$$

$$J_m^n = \int_m^n \frac{\sqrt{(u^2-c^2)}}{\sqrt{(u^2-a^2)(u^2-b^2)(1-u^2)}} \, du$$

$$K_m^n = \int_m^n \frac{\sqrt{(u^2-a^2)}}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}} \, du$$

Substitution of the values of C_i and D_i given by equations (3.26) and (3.27) in equations (3.21-3.24) yields

$$h_{i-1}(t^2) = -A_i\left[1 - \frac{Q_1}{t^2-a^2}\sqrt{\frac{(1-a^2)}{(c^2-a^2)}} + \frac{Q_2}{c^2-t^2}\right]\sqrt{\frac{(t^2-a^2)(c^2-t^2)}{(b^2-t^2)(1-t^2)}} \quad (i=1,2)$$

(3.29)

$$g_{i-1}(u^2) = -A_i\left[1 - \frac{Q_1}{u^2-a^2}\sqrt{\frac{(1-a^2)}{(c^2-a^2)}} - \frac{Q_2}{u^2-c^2}\right]\sqrt{\frac{(u^2-a^2)(u^2-c^2)}{(u^2-b^2)(1-u^2)}} \quad (i=1,2)$$

where

$$A_1 = \frac{p_o}{\mu_{12}\theta} \quad , \quad A_2 = \frac{4PR}{\pi^2} \; .$$

4. STRESS INTENSITY FACTORS AND CRACK OPENING DISPLACEMENTS

The stress intensity factors are defined as (in physical units)

$$N_a = \lim_{x\to a-}\left[\frac{\sqrt{(a-x)}\;\tau_{yy}(x,0)}{p_o}\right]_{0<x<a}$$

$$N_b = \lim_{x\to b+}\left[\frac{\sqrt{(x-b)}\;\tau_{yy}(x,0)}{p_o}\right]_{b<x<c}$$

$$N_c = \lim_{x\to c-}\left[\frac{\sqrt{(c-x)}\;\tau_{yy}(x,0)}{p_o}\right]_{b<x<c}$$

$$N_1 = \lim_{x\to 1+}\left[\frac{\sqrt{(x-1)}\;\tau_{yy}(x,0)}{p_o}\right]_{x>1} \qquad (4.1a\text{-}d)$$

and the crack opening displacement can now be shown to be given by

$$\Delta v(x,0) = v(x,0+)-v(x,0-) = 2\int_x^b h(t^2)\,dt \quad , \quad a\le x\le b$$

$$= 2\int_x^1 g(u^2)\,du \quad , \quad c\le x\le 1 \qquad (4.2a\text{-}b)$$

The stress component τ_{yy} can be evaluated from the equations (2.18), (2.21) and (3.1) when the values of the functions $h(t^2)$ and $g(u^2)$ as obtained above from (3.29) are substituted. Next substitution of the value of τ_{yy} in the relations (4.1a-d) yields finally,

$$N_a = \sqrt{\frac{1}{2a(b^2-a^2)}}\, Q_1\left[1 - \frac{4P}{\pi^2} M_2 k_s^2 \log k_s \right] + O(k_s^2)$$

$$N_b = \left[\sqrt{\frac{(b^2-a^2)(c^2-b^2)}{2b(1-b^2)}} - Q_1 \sqrt{\frac{(c^2-b^2)(1-a^2)}{2b(b^2-a^2)(1-b^2)(c^2-a^2)}} + \right.$$

$$\left. + Q_2 \sqrt{\frac{(b^2-a^2)}{2b(c^2-b^2)(1-b^2)}} \right]\left[1 - \frac{4P}{\pi^2} M_2 k_s^2 \log k_s \right] + O(k_s^2)$$

$$N_c = \sqrt{\frac{(c^2-a^2)}{2c(c^2-b^2)(1-c^2)}}\, Q_2\left[1 - \frac{4P}{\pi^2} M_2 k_s^2 \log k_s \right] + O(k_s^2)$$

$$N_1 = \left[\sqrt{\frac{(1-a^2)(1-c^2)}{2(1-b^2)}} - Q_1 \sqrt{\frac{(1-c^2)}{2(1-b^2)(c^2-a^2)}} - Q_2 \sqrt{\frac{(1-a^2)}{2(1-b^2)(1-c^2)}} \right]\times$$

$$\times\left[1 - \frac{4P}{\pi^2} M_2 k_s^2 \log k_s \right] + O(k_s^2) \qquad (4.3a\text{-}d)$$

where $$M_2 = [\, R_a^b + R_c^1 \,] - \left[\sqrt{\frac{1-a^2}{c^2-a^2}}\, Q_1 + Q_2 \right] J_2 \,.$$

Expressions (4.2a-b) with the aid of the equations (3.14) and

(3.29) yield

$$\Delta v(x,0) = -2\left[A_1 + A_2\, k_s^2 \log k_s \right]\int_x^b \sqrt{\frac{(t^2-a^2)(c^2-t^2)}{(b^2-t^2)(1-t^2)}}\, \times$$

$$\times \left[1- \frac{Q_1}{t^2-a^2}\sqrt{\frac{(1-a^2)}{(c^2-a^2)}} + \frac{Q_2}{c^2-t^2}\right]dt + O(k_s^2) \;, \quad a\le x\le b$$

$$= -2\left[A_1 + A_2\, k_s^2 \log k_s \right]\int_x^1 \sqrt{\frac{(u^2-a^2)(u^2-c^2)}{(u^2-b^2)(1-u^2)}}\, \times$$

$$\times \left[1- \frac{Q_1}{u^2-a^2}\sqrt{\frac{(1-a^2)}{(c^2-a^2)}} - \frac{Q_2}{u^2-c^2}\right]du + O(k_s^2) \;, \quad c\le x\le 1$$

(4.4a-b)

When $a=d_1/d \to 0$, the stress intensity factor and the crack opening displacement for three Griffith cracks occupying the region $|x|\le b$, $c\le|x|\le 1$, $y=0$, $|z|<\infty$ are recovered (Sarkar et al. , 1994b)

$$N_b = \sqrt{\frac{b(1-b^2)}{2(c^2-b^2)}}\;\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)}\left[1 - \frac{4P}{\pi^2} M_2 k_s^2 \log k_s \right] + O(k_s^2)$$

$$N_c = \sqrt{\frac{c}{2(c^2-b^2)(1-c^2)}}\left[(1-b^2)\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} - (c^2-b^2)\right]\left[1 - \frac{4P}{\pi^2} M_2 k_s^2 \log k_s \right] + O(k_s^2)$$

$$N_1 = \sqrt{\frac{(1-b^2)}{2(1-c^2)}}\left[1 - \frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)}\right]\left[1 - \frac{4P}{\pi^2} M_2 k_s^2 \log k_s \right] + O(k_s^2)$$

$$\Delta v(x,0) = \frac{2p_o}{\mu_{12}\theta}\left[\sqrt{(1-b^2)}\;F(\beta,q)\left\{\frac{E(\beta,q)}{F(\beta,q)} - \frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)}\right\} -\right.$$

$$\left. - \sqrt{\frac{(1-x^2)(b^2-x^2)}{(c^2-x^2)}}\;\right]\left[\,1 - \frac{4P}{\pi^2} M_2 k_s^2 \log k_s\,\right] + O(k_s^2)\ , \qquad 0 \le x \le b$$

and

$$\Delta v(x,0) = \frac{2p_o}{\mu_{12}\theta}\left[\sqrt{(1-b^2)}\;F(\lambda,q)\left\{\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} - \frac{E(\lambda,q)}{F(\lambda,q)}\right\}\right]\times$$

$$\times\left[\,1 - \frac{4P}{\pi^2} M_2 k_s^2 \log k_s\,\right] + O(k_s^2)\ , \qquad c \le x \le 1$$

where

$$M_2 = \left[\,R_o^b + R_c^1 + \left\{(1-b^2)\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} - (c^2-b^2)\right\}\left(\,L_o^b - L_c^1\,\right)\right]$$

$$L_m^n = \int_m^n \frac{t^2\,dt}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}}$$

$$\sin\beta = \sqrt{\frac{b^2-x^2}{c^2-x^2}} \qquad \text{and} \qquad \sin\lambda = \sqrt{\frac{1-x^2}{1-b^2}}\ .$$

and $E(\frac{\pi}{2},q)$ is the elliptic integral of the second kind with

$$q = \sqrt{\frac{1-c^2}{1-b^2}}\ .$$

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) N_a, N_b, N_c and N_1 given by (4.3a-d) at the tips of the cracks and crack opening displacements (COD) given by (4.4a-b) have been plotted against dimensionless frequency k_s and distance respectively for three different types of orthotropic materials whose engineering constants have been listed in table 1.

Keeping the length of the outer cracks and distance between inner and outer cracks fixed (b=0.6, c=0.8) SIFs at the tips of the cracks have been plotted against frequency k_s ($0.1\le k_s \le 0.6$) for different lengths of the inner cracks (a=0.2,0.3,0.4). It is noted from the graphs (Fig.2-Fig.5) that with the decrease in the value of inner crack length i.e. with the increase in the value of the distance between inner cracks the rate of increase in the SIF is higher with the increase in the value of the frequency k_s.

It is also found that the value of SIF is higher for lower value of a. When lengths of the outer cracks and the distance between inner cracks are kept fixed (a=0.2, c=0.8) it is noted from the graphs (Fig.6-Fig.9) that with the increase in the value of b(0.4, 0.5,0.6) i.e. with the decrease in the value of the distance between inner and outer cracks the rate of decrease of SIFs are higher. It is interesting to note that the value of SIF N_a is lower for higher values of b but in case of the SIFs N_b, N_c and N_1 the effect is just reverse.

Next, keeping the lengths of the inner cracks fixed (a=0.2, b=0.4)

TABLE - 1. ENGINEERING ELASTIC CONSTANTS.

	E_1 (Pa)	E_2 (Pa)	μ_{12} (Pa)	ν_{12}
Type I	Modulite II Graphite-Epoxy Composite :			
	15.3×10^9	158.0×10^9	5.52×10^9	0.033
Type II	E-Type Glass-Epoxy Composite :			
	9.79×10^9	42.3×10^9	3.66×10^9	0.063
Type III	Stainless Steel-Aluminium Composite :			
	79.76×10^9	85.91×10^9	30.02×10^9	0.31

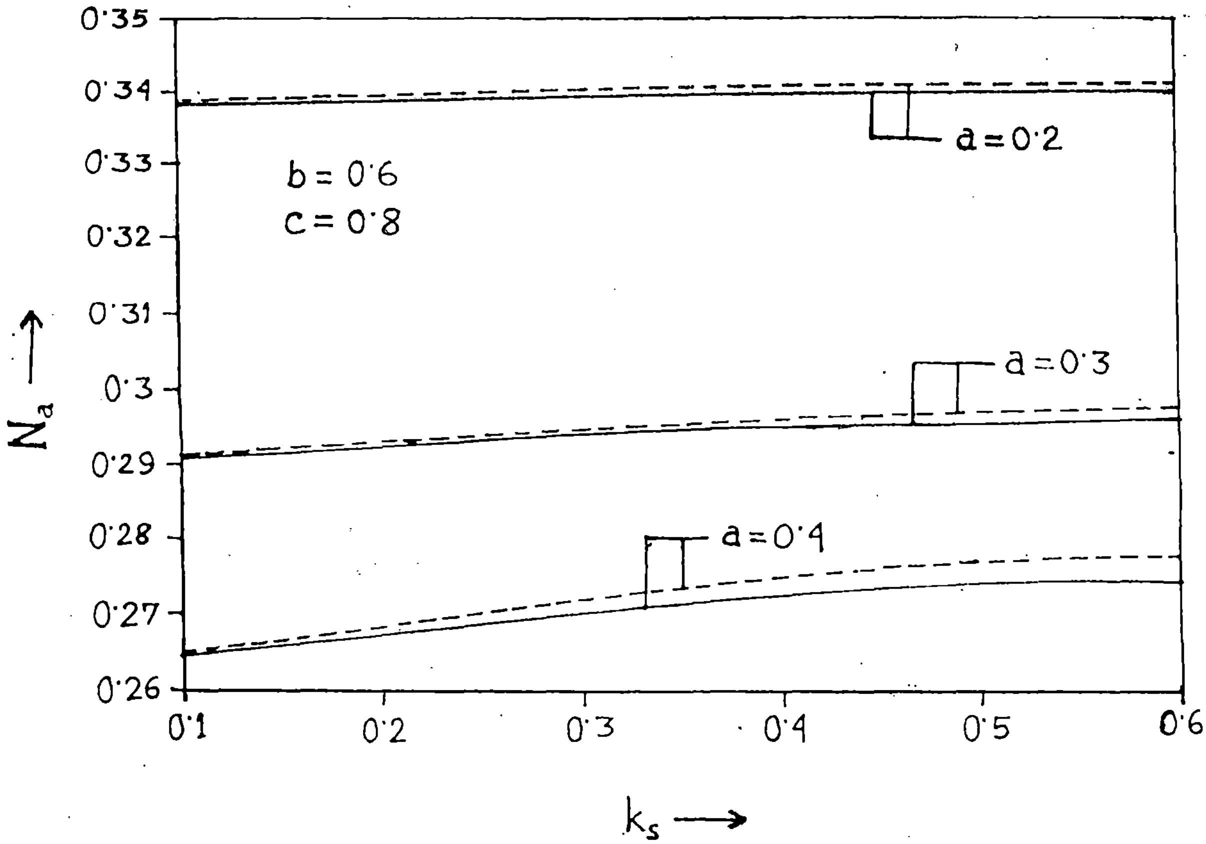

Fig. 2. Stress intensity factor N_a vs. frequency k_s for generalized plane stress. (——— Type I, ------ Type III).

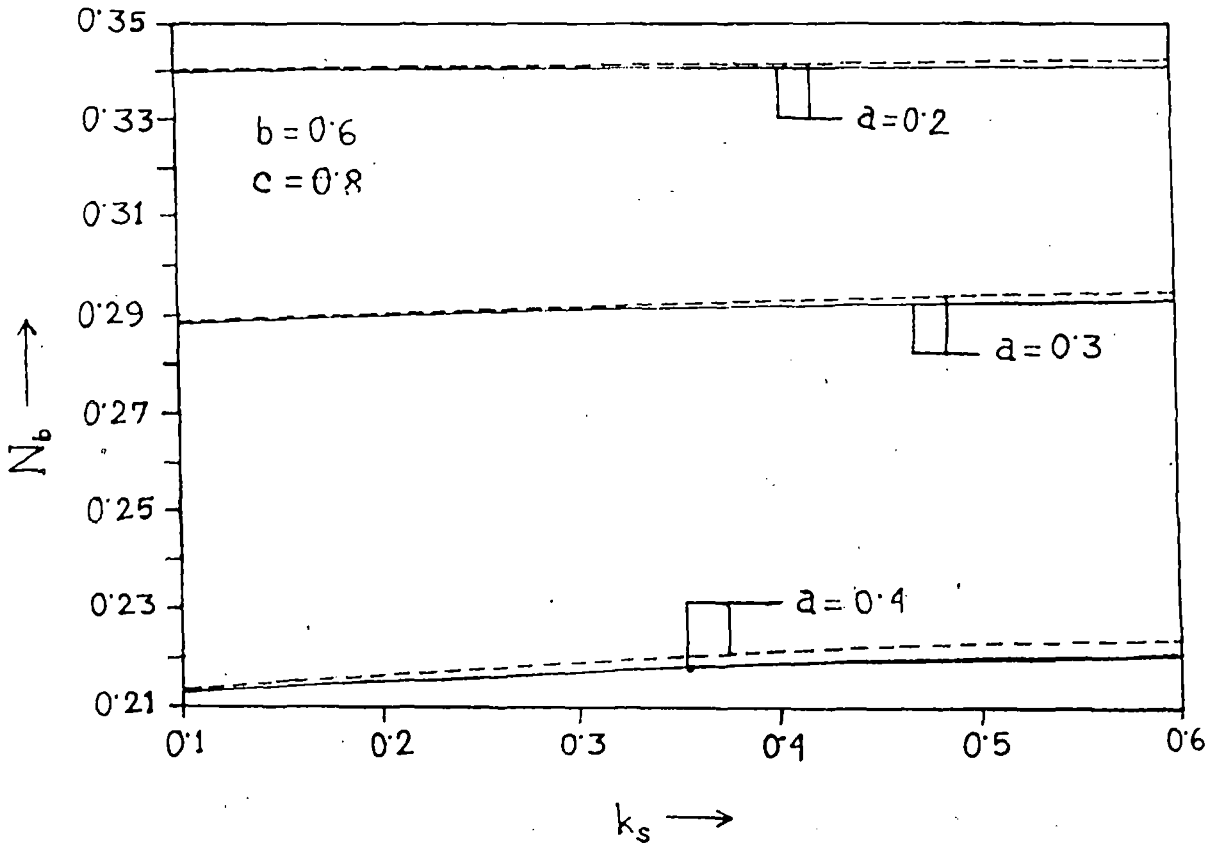

Fig.3. Stress intensity factor N_b vs. frequency k_s for generalized plane stress.
(——— Type I, ------ Type III).

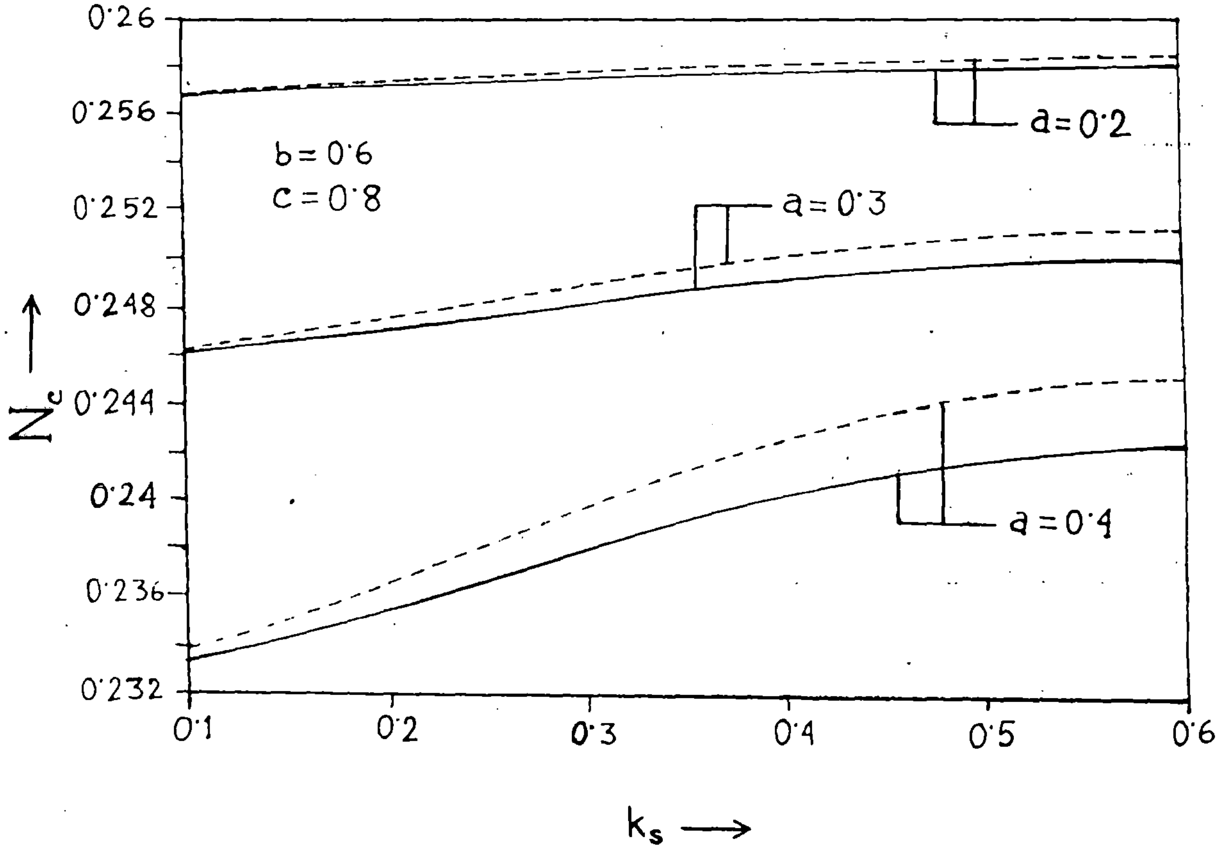

Fig. 4. Stress intensity factor N_c vs. frequency k_s for generalized plane stress.
(——— Type I, ------ Type III).

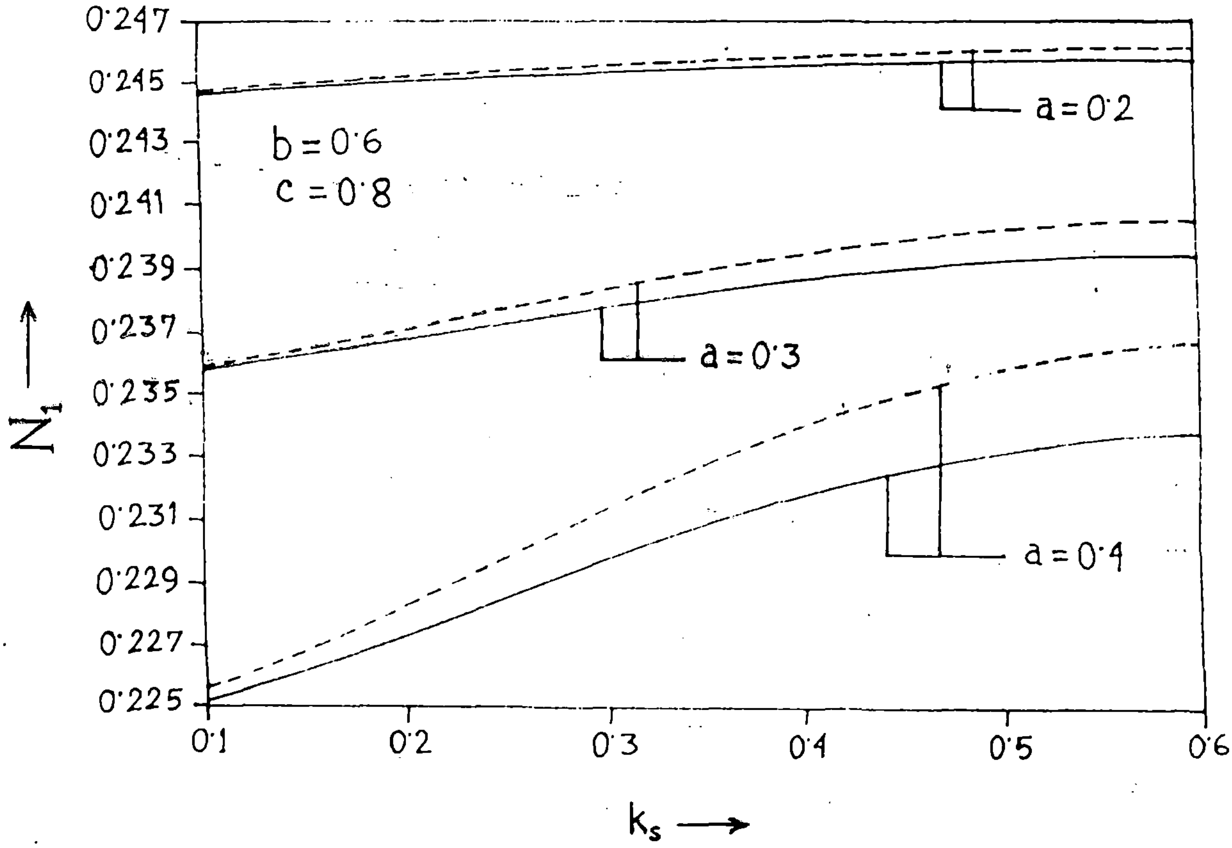

Fig. 5. Stress intensity factor N_1 vs. frequency k_s for generalized plane stress.
(——— Type I, ------ Type III).

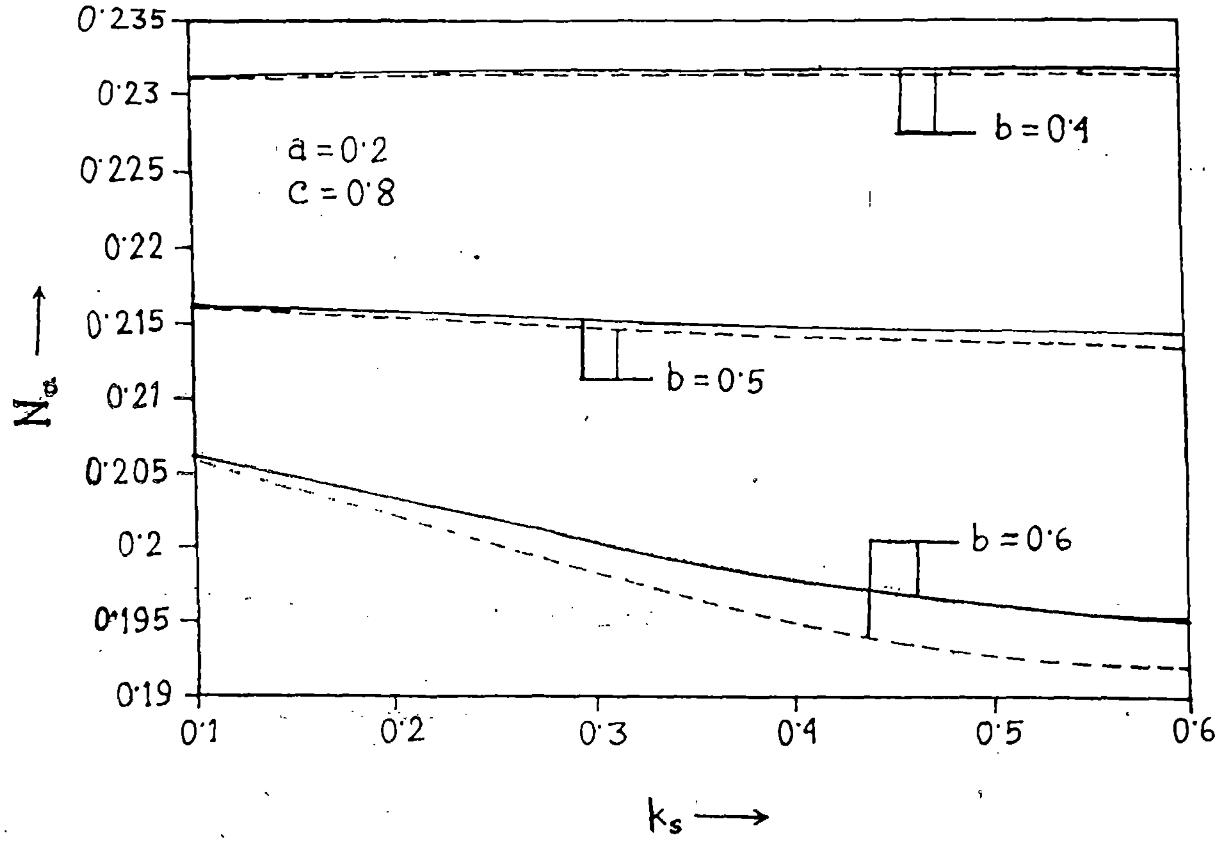

Fig. 6. Stress intensity factor N_a vs. frequency k_s for generalized plane stress.
(——— Type I, ------ Type III).

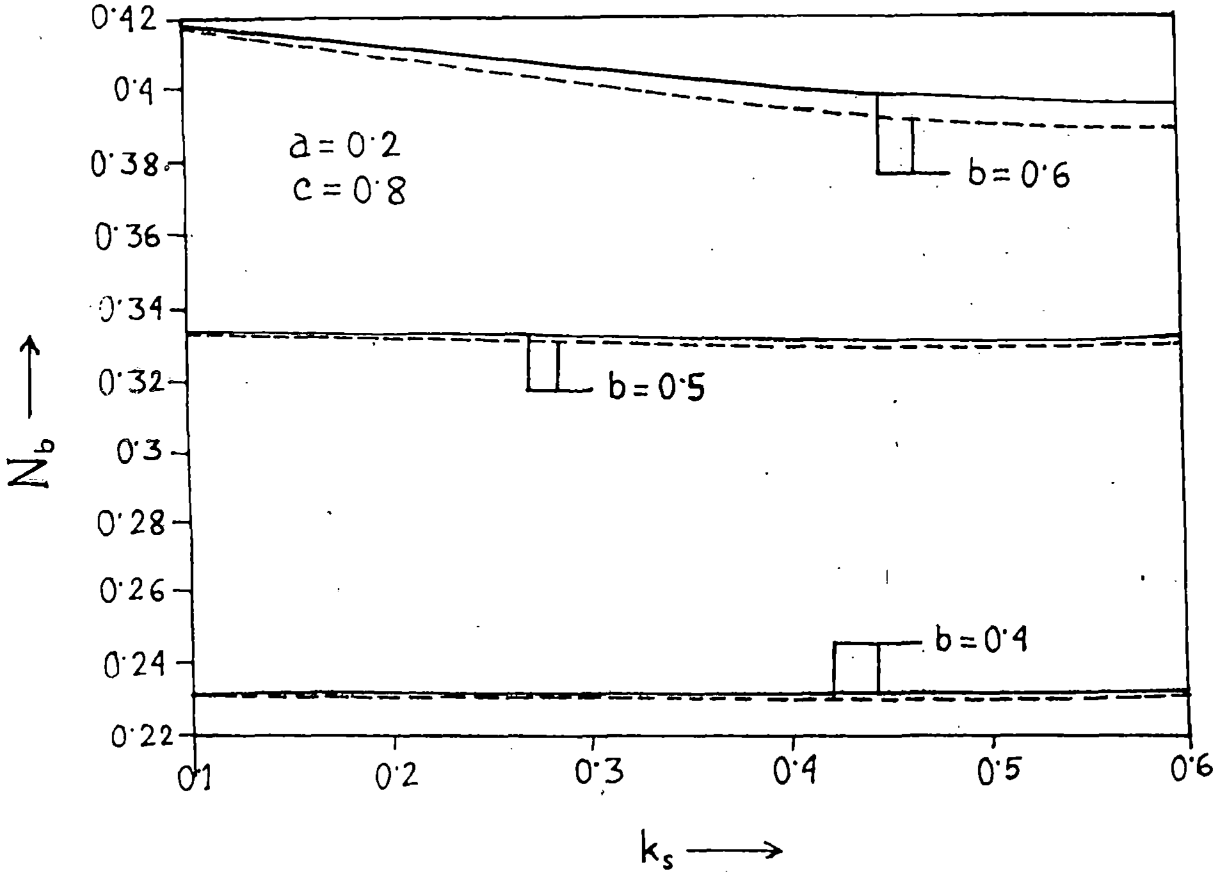

Fig. 7. Stress intensity factor N_b vs. frequency k_s for generalized plane stress. (——— Type I, - - - - - - Type III).

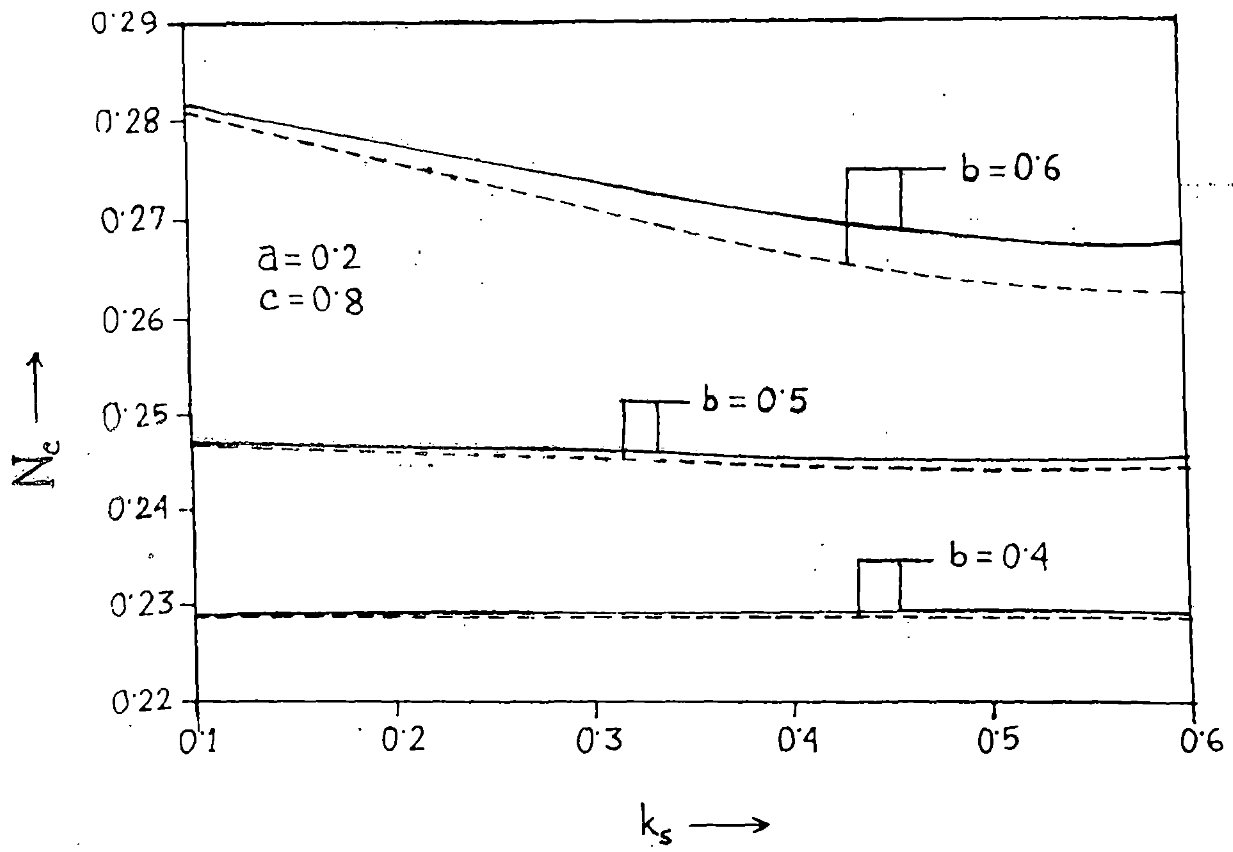

Fig. 8. Stress intensity factor N_c vs. frequency k_s for generalized plane stress. (——— Type I, ------ Type III).

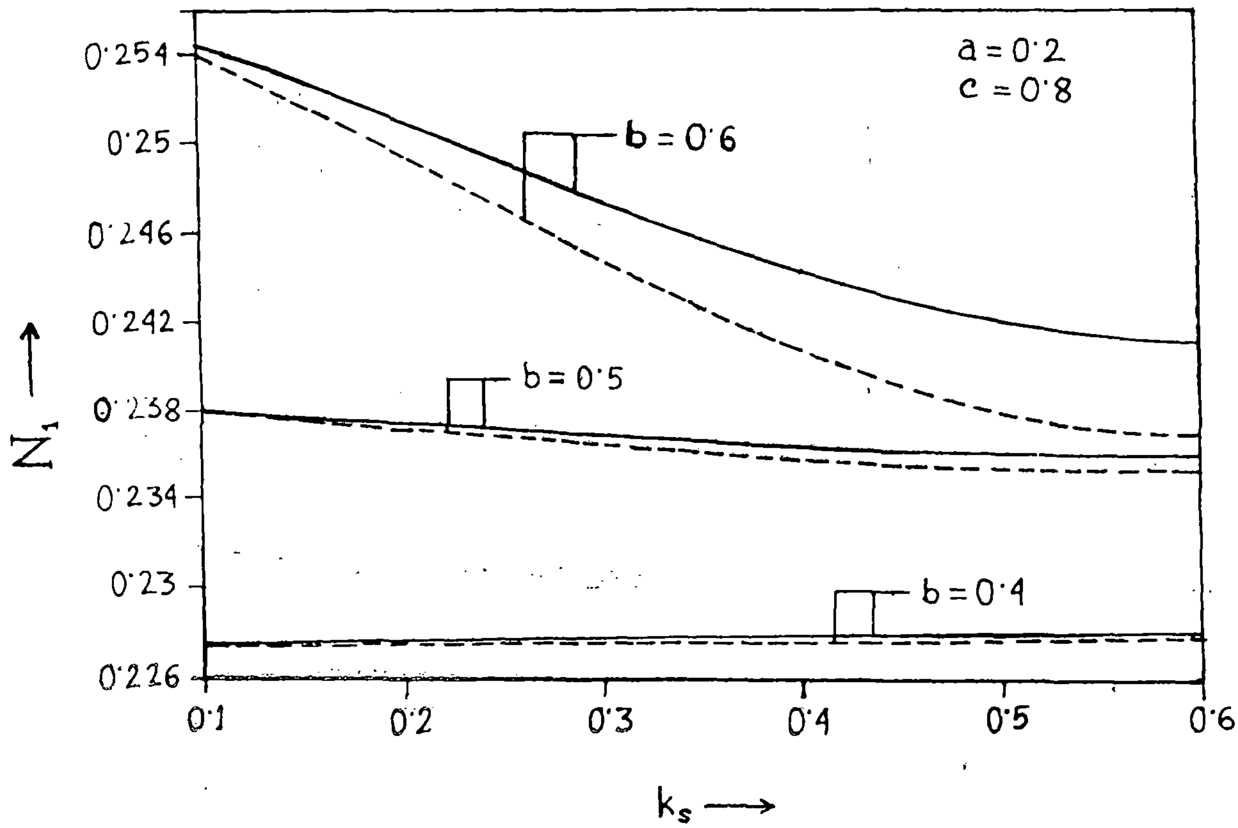

Fig. 9. Stress intensity factor N_1 vs. frequency k_s for generalized plane stress. (——— Type I, ------ Type III).

it is seen from the graphs (Fig.10-Fig.13) that SIFs increase with the increase in the value of k_s for lower values of c(0.6,0.7) but decrease for higher values of c(0.8). The value of SIF N_a is higher for higher values of c. But the nature is opposite in case of N_b, N_c and N_1.

The COD $\mu_{12}\Delta v(x,0)/p_0$ has been plotted for different crack lengths. It is found from Fig.14-Fig.16 that with the increase in the value of crack length the value of COD increases. For a fixed material the variation of COD with frequency is found to be insignificant.

In all the cases where different values of crack length have been considered the variation of COD shows marked difference for different orthotropic materials.

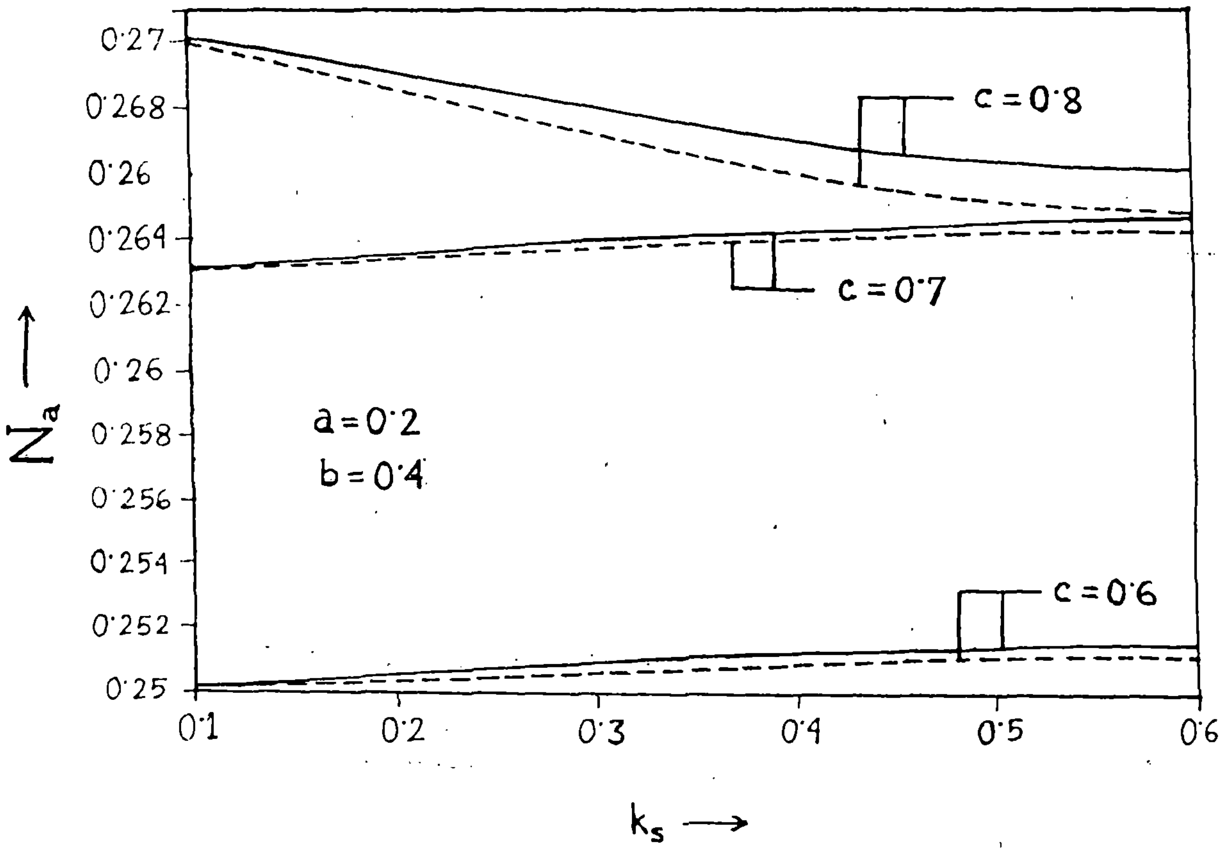

Fig.10. Stress intensity factor N_a vs. frequency k_s for generalized plane stress.
(——— Type I, ------ Type III).

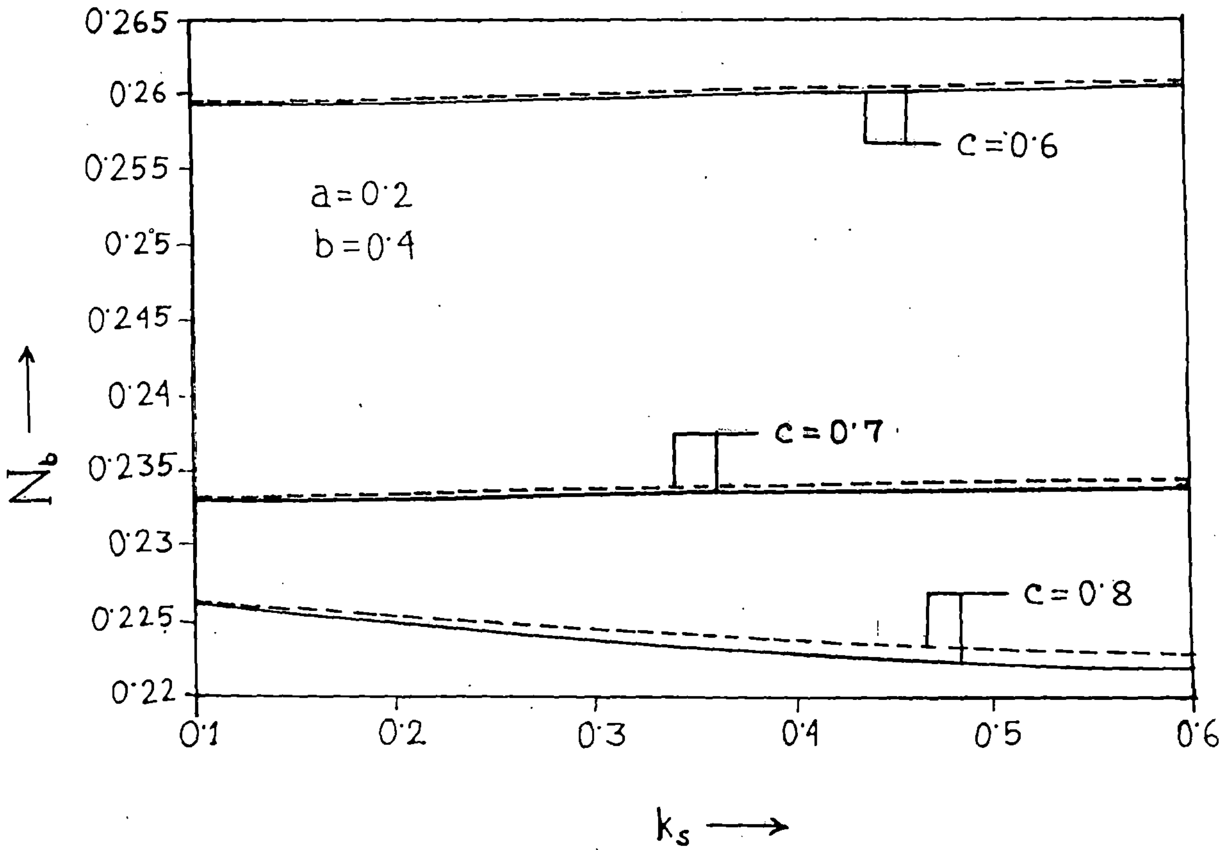

Fig.11. Stress intensity factor N_b vs. frequency k_s for generalized plane stress.
(——— Type I, ------ Type III).

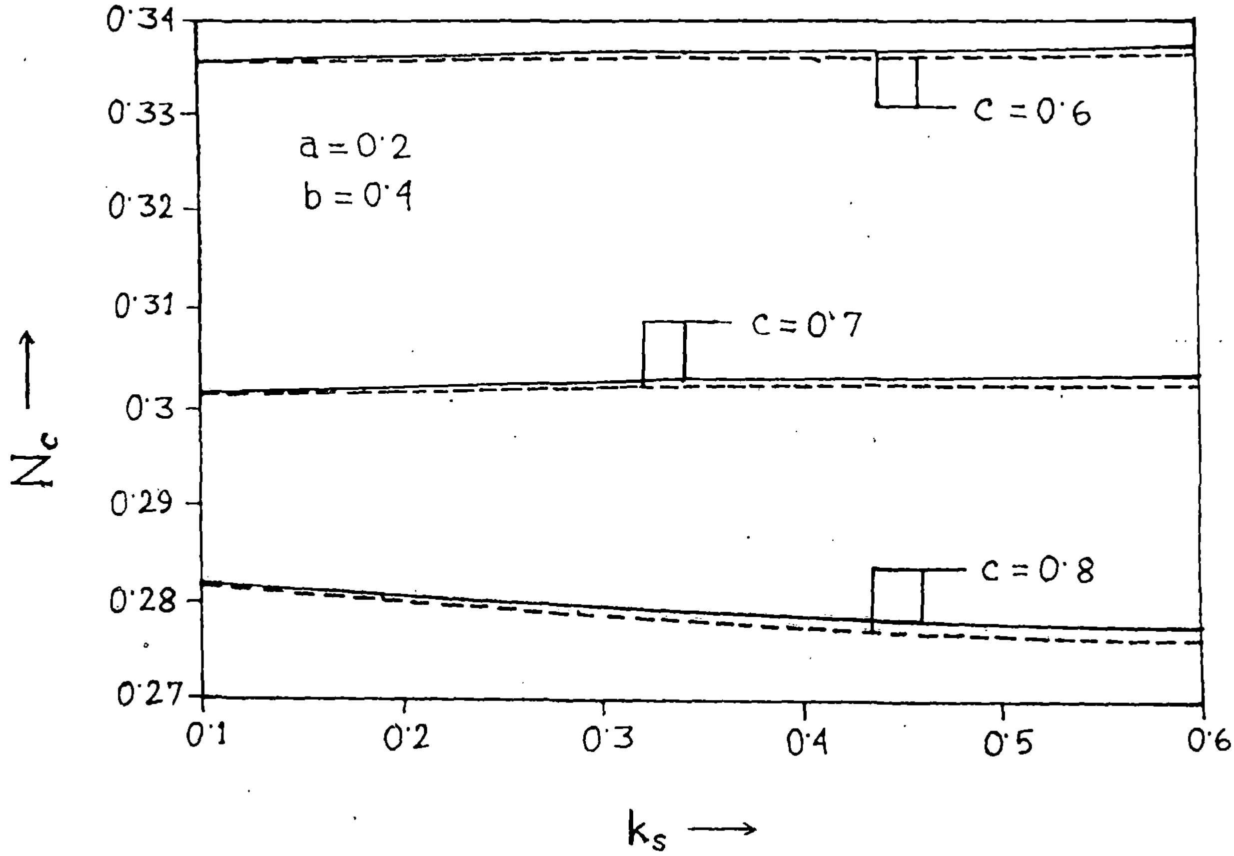

Fig.12. Stress intensity factor N_c vs. frequency k_s for generalized plane stress. (——— Type I, ------ Type III).

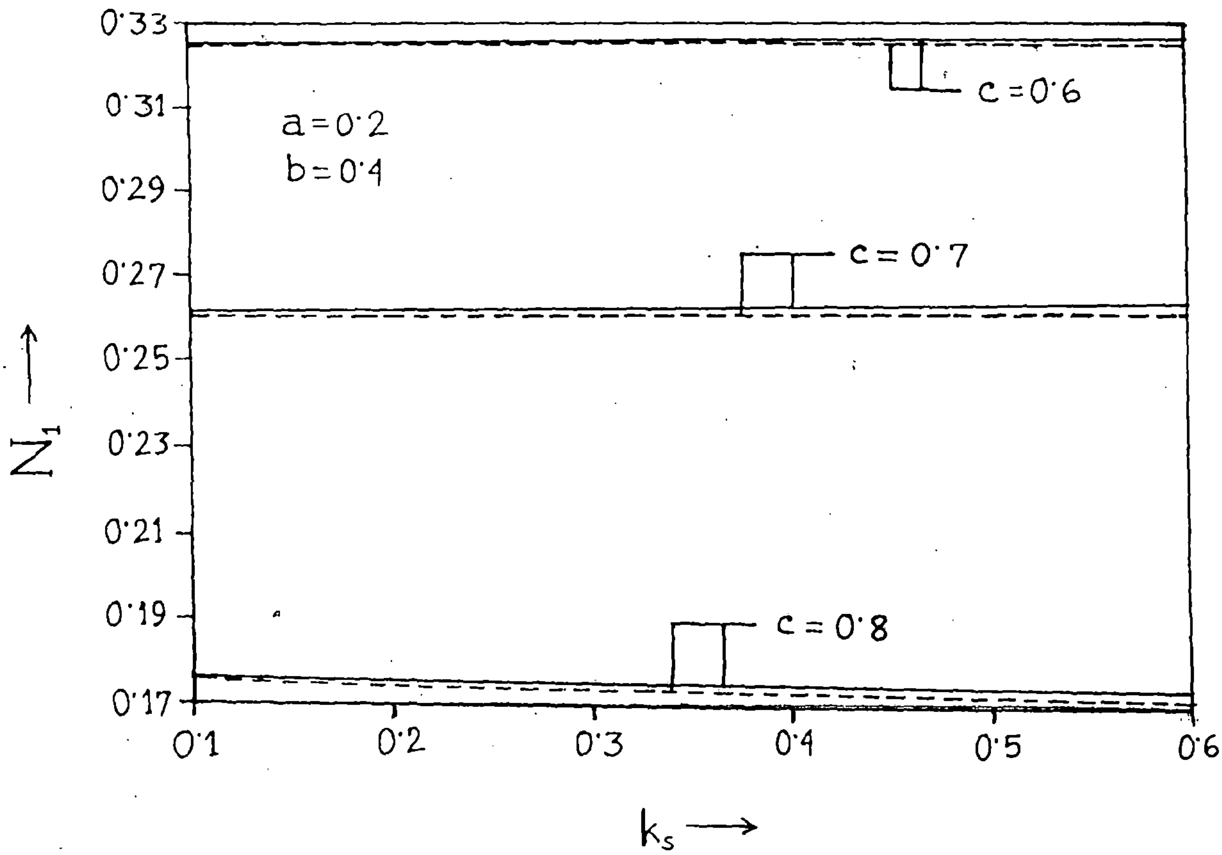

Fig. 13. Stress intensity factor N_1 vs. frequency k_s for generalized plane stress. (——— Type I, ------ Type III).

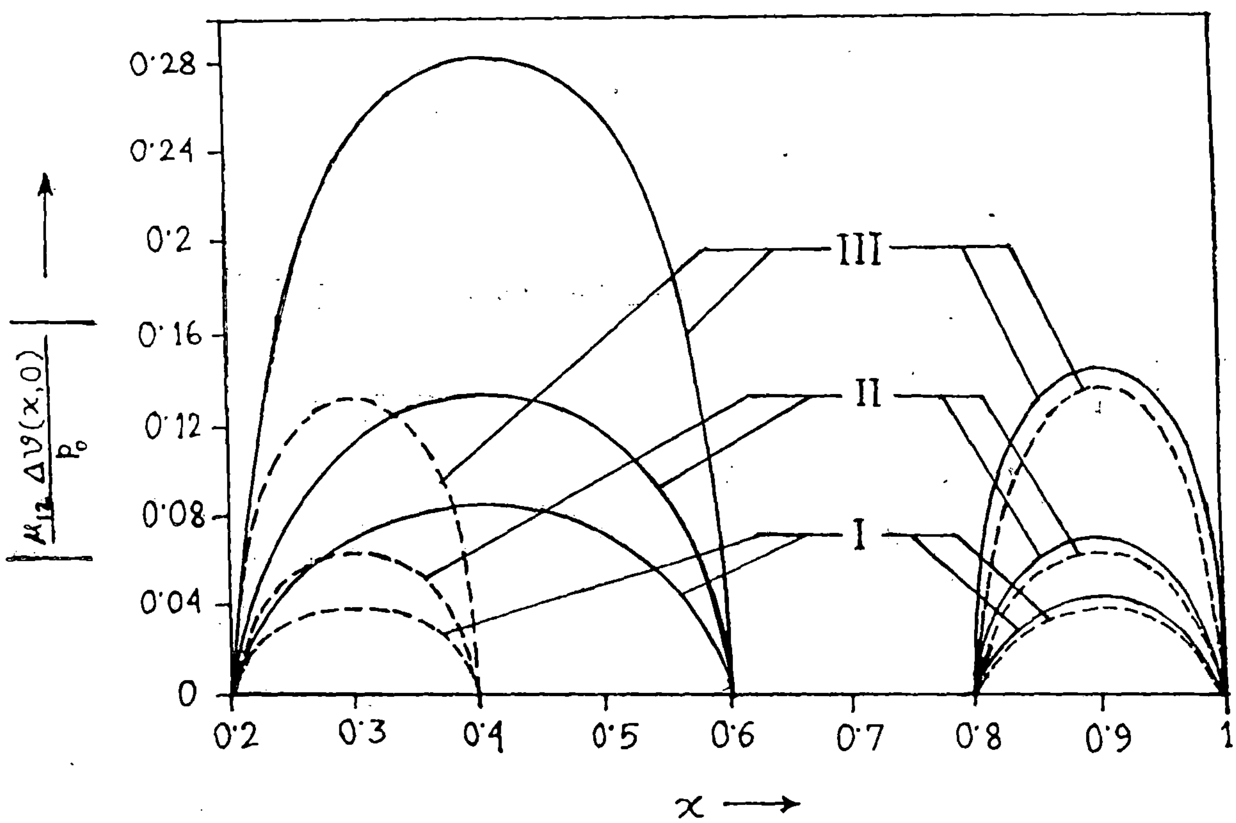

Fig.14. Crack opening displacement vs. distance for generalized plane stress.
(k_s=0.5, a=0.2, b=0.4,0.6, c=0.8).

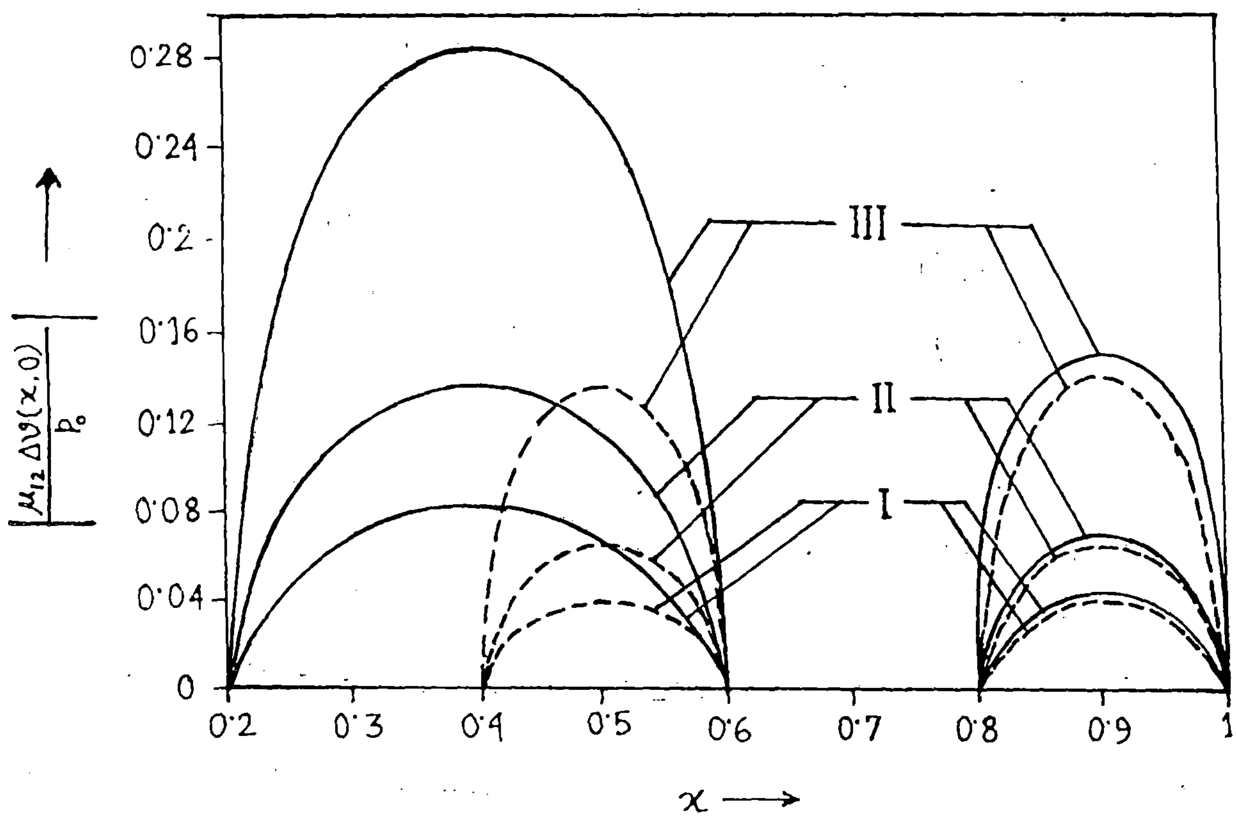

Fig. 15. Crack opening displacement vs. distance for generalized plane stress. (k_s=0.5, a=0.2, 0.4, b=0.6, c=0.8).

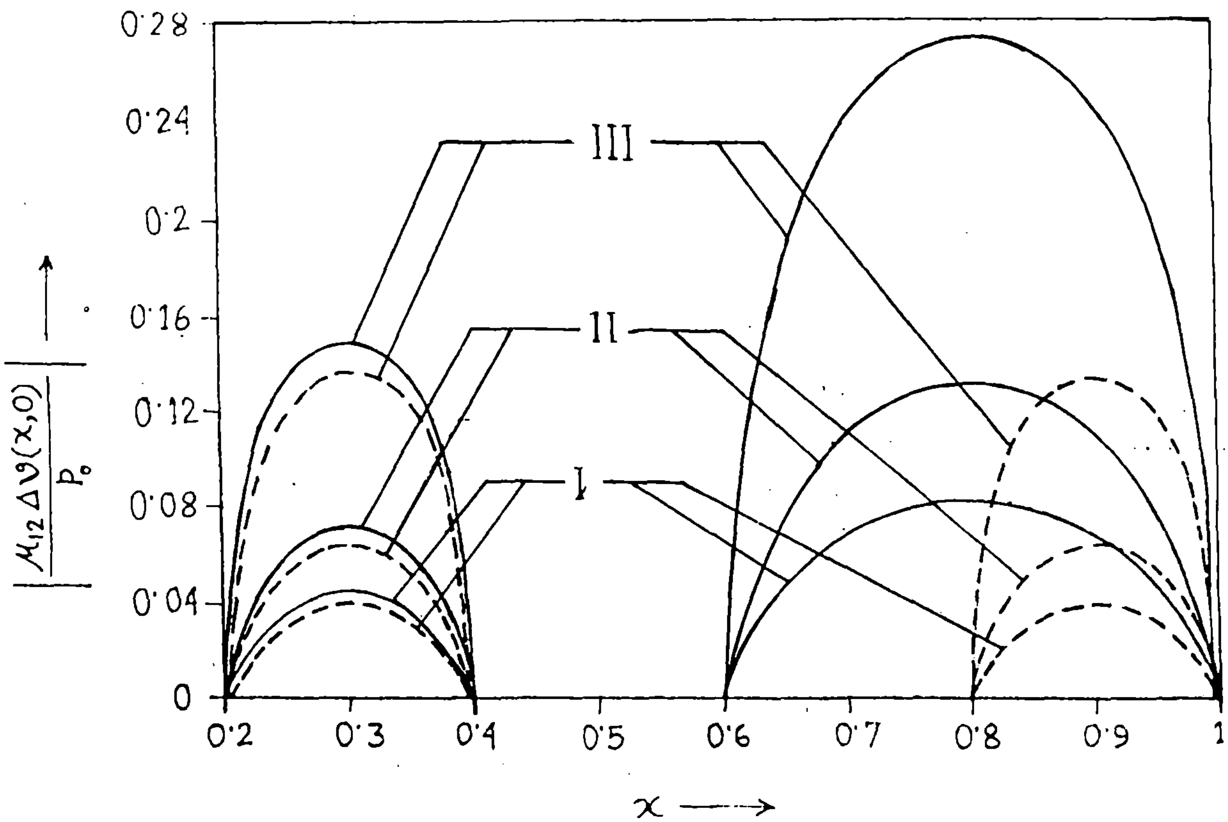

Fig.16. Crack opening displacement vs. distance for generalized plane stress. (k_s=0.5, a=0.2, b=0.4, c=0.6,0.8).

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Scattering of antiplane shear wave by a propagating crack at the interface of two dissimilar elastic media

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Abstract. An analysis of the scattering of horizontally polarized shear wave by a semi-infinite crack running with uniform velocity along the interface of two dissimilar semi-infinite elastic media has been carried out. The mixed boundary value problem has been solved completely by the Wiener–Hopf technique. The effect of different values of the material parameter, the angle of incidence of incident wave and the crack propagation velocity on the stress intensity factor have been illustrated graphically.

Keywords. Diffraction of elastic waves; propagating crack; SH-wave; stress intensity factor.

1. Introduction

It is well known that the problems of diffraction of elastic wave by cracks or inclusions are of considerable importance in view of their application in seismology and geophysics. If the cracks or inclusions are located at the interface of layered media, the study becomes more relevant. The extensive use of composite materials in modern technology has also evoked interest in the wave propagation problems in layered media with interfacial discontinuities. Onder *et al* [5] studied the diffraction of monochromatic plane SH-waves obliquely incident on a rigid half plane between the two different semi-infinite media.

In this paper we have considered the problem of the diffraction of a plane harmonic SH-wave by a semi-infinite crack running uniformly along the interface of two dissimilar semi-infinite elastic media. The problem of scattering of plane harmonic polarized shear wave by a half plane crack in an infinite isotropic medium extending under antiplane strain was studied earlier by Jahanshahi [3]. Chen and Sih [1,2] also solved the in plane problem of the diffraction of stress waves by a running crack in an incident wave field in an infinite elastic medium. We have applied Fourier transform and Wiener–Hopf technique [4] to solve the mixed boundary value problem. The resulting integrals have been evaluated asymptotically to obtain the displacement and stress field near about the crack tip. It is found that the stress intensity factor depends sensitively upon the speed of crack propagation, the angle of incidence of the incoming wave and on the material properties of the elastic media. Quantitative assessment of the effect of the aforementioned parameters on the stress intensity factor has been made by displaying the numerical results graphically for two pairs of different materials.

2. Formulation of the problem and its solution

Let a plane crack move at a constant velocity V on the interface of two bonded dissimilar elastic semi-infinite medium due to the incidence of the plane harmonic SH-wave

$$v_1^0 = V_1 \exp[-i\{\Lambda_1(X\cos\Theta_1 + Y\sin\Theta_1) + \Omega T\}] \tag{1}$$

in the medium where the co-efficient of rigidity, density and shear-wave velocity respectively are given by μ_1, ρ_1 and C_1. The crack lies on the bimaterial interface along $Y = 0$ with respect to the fixed rectangular co-ordinate system (X, Y, Z).

We assume that the displacement and stress due to the scattered field are

$$v_j = v_j(X, Y, T) \tag{2}$$

and

$$(\tau_{xz})_j = \mu_j \frac{\partial v_j}{\partial X}, \quad (\tau_{yz})_j = \mu_j \frac{\partial v_j}{\partial Y} \tag{3}$$

where the subscript $j = 1, 2$ refers to the upper and lower half-planes and T the time.

The equations of SH-wave motion in either elastic half-space are given by

$$\frac{\partial^2 v_j}{\partial X^2} + \frac{\partial^2 v_j}{\partial Y^2} = \frac{1}{C_j^2}\frac{\partial^2 v_j}{\partial T^2} \quad (j = 1, 2) \tag{4}$$

where $C_j = (\mu_j/\rho_j)^{1/2}$ is the shear-wave velocity. Without any loss of generality, we further assume that $C_1 > C_2$.

Due to the incident wave given in (1), the reflected and transmitted waves in the absence of the crack may be written in the form

$$v_1^r(X, Y, T) = V_1^r \exp[-i\{\Lambda_1(X\cos\Theta_1 - Y\sin\Theta_1) + \Omega T)\}]$$

and

$$v_2^T(X, Y, T) = V_2^T \exp[-i\{\Lambda_2(X\cos\Theta_2 + Y\sin\Theta_2) + \Omega T)\}] \tag{5}$$

where

$$V_1^r = \frac{\mu_1\Lambda_1\sin\Theta_1 - \mu_2\Lambda_2\sin\Theta_2}{\mu_1\Lambda_1\sin\Theta_1 + \mu_2\Lambda_2\sin\Theta_2} V_1 = A_1^R V_1 \quad \text{(say)}$$

and

$$V_2^T = \frac{2\mu_1\Lambda_1\sin\Theta_1}{\mu_1\Lambda_1\sin\Theta_1 + \mu_2\Lambda_2\sin\Theta_2} V_1 = A_2^T V_1 \quad \text{(say)} \tag{6}$$

with

$$\Lambda_1\cos\Theta_1 = \Lambda_2\cos\Theta_2.$$

V_1, V_1^r and V_2^T are the incident, reflected and transmitted wave amplitude respectively, Λ_j the wave number, $\Omega = \Lambda_j C_j$ the circular frequency and Θ_1, Θ_2 the angles of incidence and refraction respectively.

Assume that the crack has been moving in the horizontal direction along the interface for a sufficiently long time and that a steady state has been reached in the neighbourhood of the crack.

A set of moving co-ordinate systems (x, y, z, t) attached to the crack tip moving at

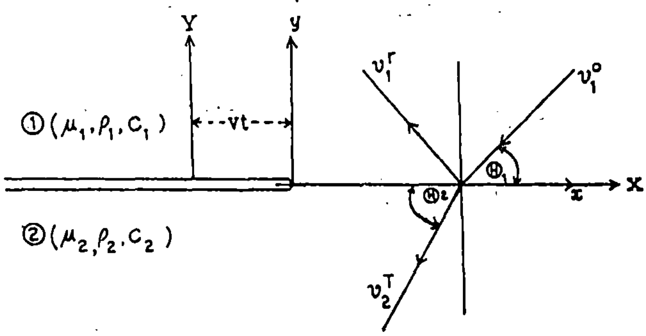

Figure 1. Geometry of the propagating crack.

a constant velocity V is introduced in accordance with

$$x = X - Vt, \quad y_j = s_j Y, \quad z = Z, \quad t = T \tag{7}$$

where $s_j = (1 - M_j^2)^{1/2}$ and $M_j = V/C_j$ is the Mach number.

In terms of the moving co-ordinate system (x, y, t), (4) becomes

$$\frac{\partial^2 v_j}{\partial x^2} + \frac{\partial^2 v_j}{\partial y_j^2} + \frac{1}{C_j^2 s_j^2}\frac{\partial}{\partial t}\left(2M_j C_j \frac{\partial v_j}{\partial x} - \frac{\partial v_j}{\partial t}\right) = 0. \tag{8}$$

It is convenient to define an apparent circular frequency $\omega = \alpha\Omega$ and the angles of reflection ϕ_1 and refraction ϕ_2 are given by

$$\cos\phi_j = M_j + (\Lambda_j/\lambda_j)\cos\Theta_j, \quad \sin\phi_j = (s_j/\alpha)\sin\Theta_j,$$

where

$$\alpha = (1 + M_j \cos\Theta_j) \quad \text{and} \quad \lambda_j = (\Lambda_j/s_j^2)\alpha. \tag{9}$$

Using these relations in a moving system, (1) and (5) take the form

$$\begin{bmatrix} v_1^0 \\ v_1^r \\ v_2^T \end{bmatrix} = \begin{bmatrix} w_1^0(x, y_1) \\ w_1^r(x, y_1) \\ w_2^T(x, y_2) \end{bmatrix} \exp\{i(M_1\lambda_1 x - \omega t)\} \tag{10}$$

where

$$w_1^0(x, y_1) = V_1 \exp\{-i\lambda_1(x\cos\phi_1 + y_1\sin\phi_1)\}$$
$$w_1^r(x, y_1) = A_1^R V_1 \exp\{-i\lambda_1(x\cos\phi_1 - y_1\sin\phi_1)\}$$
$$w_2^T(x, y_2) = A_2^T V_1 \exp[-i\{(\beta_2 + \lambda_2\cos\phi_2)x + \lambda_2 y_2 \sin\phi_2\}] \tag{11}$$

and

$$\beta_2 = M_1\lambda_1\left(1 - \frac{\lambda_2 C_1}{\lambda_1 C_2}\right) < 0 \quad \text{since} \quad C_1 > C_2.$$

Using (10), we assume the solution of the governing equation (8) as

$$v_j(x, y_j, t) = w_j(x, y_j)\exp[i(M_j\lambda_j x - \omega t)]. \tag{12}$$

Substitution of (12) in (8) yields the Helmholtz equation

$$\frac{\partial^2 w_j}{\partial x^2}+\frac{\partial^2 w_j}{\partial y_j^2}+\lambda_j^2 w_j=0 \quad (j=1,2). \tag{13}$$

Applying Fourier transform, (13) can be solved and the result is

$$w_1(x,y_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty} A_1(\xi)\exp\{-i\xi x-(\xi^2-\lambda_1^2)^{1/2}y_1\}\mathrm{d}\xi, \quad (y_1>0)$$

and

$$w_2(x,y_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty} A_2(\xi)\exp\{-i\xi x-(\xi^2-\lambda_2^2)^{1/2}y_2\}\mathrm{d}\xi, \quad (y_2<0) \tag{14}$$

where $A_1(\xi)$ and $A_2(\xi)$ are the unknown functions to be determined. From (12) and (14) we obtain the displacement components due to scattered field as

$$v_1=\exp[i(M_1\lambda_1 x-\omega t]\frac{1}{2\pi}\int_{-\infty}^{\infty} A_1(u)\exp[-iux-\gamma_1 y_1]du, \quad (y_1>0)$$

and

$$v_2=\exp[i(M_1\lambda_1 x-\omega t]\frac{1}{2\pi}\int_{-\infty}^{\infty} A_2(u)\exp[-iux+\gamma_2 y_2]du, \quad (y_2<0) \tag{15}$$

where

$$\gamma_1=(u^2-\lambda_1^2)^{1/2} \quad \text{and} \quad \gamma_2=[(u-\beta_2)^2-\lambda_2^2]^{1/2}. \tag{16}$$

Therefore, the expressions for the stresses are

$$(\tau_{xz})_1=-i\mu_1\exp[i(M_1\lambda_1 x-\omega t)] \times\frac{1}{2\pi}\int_{-\infty}^{\infty}(u-M_1\lambda_1)A_1(u)\exp[-iux-\gamma_1 y_1]du$$

$$(\tau_{xz})_2=-i\mu_2\exp[i(M_1\lambda_1 x-\omega t)] \times\frac{1}{2\pi}\int_{-\infty}^{\infty}(u-M_1\lambda_1)A_2(u)\exp[-iux+\gamma_2 y_2]du$$

and

$$(\tau_{y_1z})_1=-\mu_1 s_1\exp[i(M_1\lambda_1 x-\omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty}\gamma_1 A_1(u)\exp[-iux-\gamma_1 y_1]du$$

$$(\tau_{y_2z})_2=\mu_2 s_2\exp[i(M_1\lambda_1 x-\omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty}\gamma_2 A_2(u)\exp[-iux+\gamma_2 y_2]du. \tag{17}$$

The unknown functions $A_1(u)$ and $A_2(u)$ are to be determined from the following boundary conditions at the interface $y=0$

(i) $v_1(x,0)=v_2(x,0), \quad x>0$

(ii) $\mu_1 s_1\frac{\partial v_1}{\partial y_1}=\mu_2 s_2\frac{\partial v_2}{\partial y_2}, \quad -\infty<x<\infty$

and

(iii) $$\frac{\partial v_1^0}{\partial y_1}+\frac{\partial v_1^r}{\partial y_1}+\frac{\partial v_1}{\partial y_1}=0, \quad x<0, \quad y\to 0+.$$

From the boundary condition (ii) we obtain

$$\mu_1 s_1 \gamma_1 A_1(u)+\mu_2 s_2 \gamma_2 A_2(u)=0 \tag{18}$$

and from other two boundary conditions, we get

$$\int_{-\infty}^{\infty} B_1(u)\exp(-iux)du=0 \quad (x>0)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} M(u)B_1(u)\exp(-iux)du=N\exp[-i\lambda_1 x\cos\phi_1], \quad (x<0) \tag{19}$$

where

$$B_1(u)=\frac{\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2}{\mu_2 s_2\gamma_2}A_1(u)$$

$$M(u)=\gamma_1\frac{\mu_2 s_2\gamma_2}{(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)} \tag{20}$$

and

$$N=-\frac{i\Lambda_1 v_1\sin\Theta_1}{s_1}(1-A_1^R).$$

The solution of the dual integral equation may be obtained by a method based on the Wiener–Hopf technique. The first part of (19) can be satisfied if we choose

$$B_1(u)=L_-(u) \tag{21}$$

where $L_-(u)$ is a function of u, analytic in the lower half of the complex u-plane. The second part is satisfied if we take

$$M(u)B_1(u)=\frac{N}{i(u-\alpha_1)}\frac{U_+(u)}{U_+(\alpha_1)} \tag{22}$$

where $\alpha_1=\lambda_1\cos\phi_1$ and $U_+(u)$ is a function free from zeros and singularities in the upper half of the complex u-plane. Thus (22) is a solution of the second part of (19) can easily be shown by completing the path from $-\infty$ to ∞ by a semi-circle of infinite radius in the upper u-plane and then applying the residue theorem and Jordan's Lemma. The path of integration is chosen to avoid possible branch points and is indented below the pole $u=\alpha_1$.

Eliminating $B_1(u)$ from (21) and (22) we obtain

$$\frac{L_-(u)}{U_+(u)}=\frac{N}{i(u-\alpha_1)M(u)}\frac{1}{U_+(\alpha_1)} \tag{23}$$

and

$$M(u)=\frac{\mu_2 s_2}{\mu_1 s_1+\mu_2 s_2}(u^2-\lambda_1^2)^{1/2}F(u) \tag{24}$$

where

$$F(u)=\frac{(\mu_1 s_1+\mu_2 s_2)\gamma_2}{(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)}$$

and

$$F(u)\to 1 \quad \text{as} \quad |u|\to\infty.$$

The function $F(u)$ can be expressed as the product of two functions such that

$$F(u)=F_+(u)\cdot F_-(u) \tag{25}$$

where $F_+(u)$ and $F_-(u)$ are analytic in the upper and lower half of the complex u-plane respectively. The expressions for $F_+(u)$ and $F_-(u)$ have been derived in the appendix.

In view of (25), (24) assumes the form

$$\frac{U_+(u)}{(u+\lambda_1)^{1/2}F_+(u)}=\frac{L_-(u)}{\frac{N}{iU_+(\alpha_1)}\frac{\mu_1 s_1+\mu_2 s_2}{\mu_2 s_2(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)}} \tag{26}$$

where

$$U_+(u)=(u+\lambda_1)^{1/2}F_+(u). \tag{27}$$

So

$$L_-(u)=\frac{N}{i(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}\frac{\mu_1 s_1+\mu_2 s_2}{\mu_2 s_2(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)}. \tag{28}$$

Hence the functions $A_1(u)$ and $A_2(u)$ are

$$A_1(u)=\frac{N}{i(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}\frac{\gamma_2(\mu_1 s_1+\mu_2 s_2)}{(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)}$$

and

$$A_2(u)=\frac{-N}{i(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}\frac{\mu_1 s_1\gamma_1(\mu_1 s_1+\mu_2 s_2)}{\mu_2 s_2(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)}. \tag{29}$$

The singular behaviour of the stress components for the scattered waves at the crack-tip is due to the divergence of the integrals around $x=y_j=0$ in (17). Making use of (29) and asymptotic expressions of the integrands of (17) for large values of u, we obtain near about the crack-tip,

$$(\tau_{xz})_1=\frac{B(1+i)}{s_1}\int_0^\infty u^{-1/2}\exp[-s_1 uY](\cos ux-\sin ux)\,du$$

$$(\tau_{xz})_2=\frac{-B(1+i)}{s_2}\int_0^\infty u^{-1/2}\exp[-s_2 u|Y|](\cos ux-\sin ux)\,du$$

$$(\tau_{yz})_1=-B(1+i)\int_0^\infty u^{-1/2}\exp[-s_1 uY](\cos ux+\sin ux)\,du$$

$$(\tau_{yz})_2=-B(1+i)\int_0^\infty u^{-1/2}\exp[-s_2 u|Y|](\cos ux+\sin ux)\,du \tag{30}$$

where

$$B=-\frac{N\mu_1 s_1}{2\pi(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)};\quad y_j=s_j Y\quad (j=1,2).$$

Using the results

$$\int_0^\infty u^{-1/2}\exp[-s_1 uY]\cos ux\,dx=(\pi/2)^{1/2}\left[\frac{(s_1^2y^2+x^2)^{1/2}+s_1y}{s_1^2y^2+x^2}\right]^{1/2}$$

$$\int_0^\infty u^{-1/2}\exp[-s_1 uY]\sin ux\,dx=(\pi/2)^{1/2}\left[\frac{(s_1^2y^2+x^2)^{1/2}-s_1y}{s_1^2y^2+x^2}\right]^{1/2} \tag{31}$$

the stresses near about the crack tip given by (30) can be evaluated. The displacement near about the crack tip can be obtained from the crack tip stresses by integration.

Now introducing the factor $\exp[i(M_1\lambda_1x-\omega t)]$ and taking the real part, the stresses and displacements near about the moving crack-tip are found to be equal to

$$\begin{bmatrix}(\tau_{yz})_j\\ (\tau_{xz})_j\\ v_j\end{bmatrix}=\mathrm{Re}\begin{bmatrix}K_1\left[\dfrac{(s_j^2Y^2+x^2)^{1/2}+x}{s_j^2Y^2+x^2}\right]^{1/2}\\ (-1)^j\dfrac{K_1}{s_1}\left[\dfrac{(s_j^2Y^2+x^2)^{1/2}-x}{s_j^2Y^2+x^2}\right]^{1/2}\\ (-1)^{j+1}\dfrac{2K_1}{\mu_js_j}[(x^2+s_j^2Y^2)^{1/2}-x]^{1/2}\end{bmatrix}\exp\left[i\left(M_1\lambda_1x-\omega t-\frac{\pi}{4}\right)\right] \tag{32}$$

where

$$K_1=(2/\pi)^{1/2}\frac{\mu_1\mu_2\Lambda_1\Lambda_2V_1\sin\Theta_1\sin\Theta_2}{(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)(\mu_1\Lambda_1\sin\Theta_1+\mu_2\Lambda_2\sin\Theta_2)}. \tag{33}$$

In the case of crack propagation in an isotropic elastic medium using the result $\mu_1=\mu_2$, $\rho_1=\rho_2$ and $F_+(\alpha_1)=1$, we obtain

$$K_1=(1/\pi)^{1/2}\mu_1\Lambda_1^{1/2}V_1(1-M_1)^{1/2}\sin(\Theta_1/2). \tag{34}$$

Putting $r=(x^2+y^2)^{1/2}$, $\tan\phi=|Y|/x$, the expression of displacements and stresses given by (32) near about the moving crack-tip is found to be equal to

$$v_1=\frac{2K_1}{\mu_1s_1}r^{1/2}\{(1-M_1^2\sin^2\phi)^{1/2}-\cos\phi\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{3/2})$$

$$v_2=-\frac{2K_1}{\mu_2s_2}r^{1/2}\{(1-M_2^2\sin^2\phi)^{1/2}-\cos\phi\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{3/2})$$

$$(\tau_{yz})_1=\frac{K_1}{r^{1/2}}\left\{\frac{(1-M_1^2\sin^2\phi)^{1/2}+\cos\phi}{1-M_1^2\sin^2\phi}\right\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{1/2})$$

$$(\tau_{yz})_2 = \frac{K_1}{r^{1/2}}\left\{\frac{(1-M_2^2\sin^2\phi)^{1/2}+\cos\phi}{1-M_2^2\sin^2\phi}\right\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{1/2})$$

$$(\tau_{xz})_1 = -\frac{K_1}{s_1}\frac{1}{r^{1/2}}\left\{\frac{(1-M_1^2\sin^2\phi)^{1/2}-\cos\phi}{1-M_1^2\sin^2\phi}\right\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{1/2})$$

$$(\tau_{xz})_2 = \frac{K_1}{s_2}\frac{1}{r^{1/2}}\left\{\frac{(1-M_2^2\sin^2\phi)^{1/2}-\cos\phi}{1-M_2^2\sin^2\phi}\right\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{1/2}). \tag{35}$$

Taking the value of K_1 given by (34), the results given by (35) agree with the results of the crack propagation in an isotropic elastic medium as given by Jahanshahi [3].

When the crack is stationary, the corresponding results of stresses and displacements near about the crack-tip can be derived by making M_1 and M_2 approach zero and are given by

$$(\tau_{yz})_1 = K_1^*(2/r)^{1/2}\cos\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{1/2})$$

$$(\tau_{yz})_2 = K_1^*(2/r)^{1/2}\cos\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{1/2})$$

$$(\tau_{xz})_1 = -K_1^*(2/r)^{1/2}\sin\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{1/2})$$

$$(\tau_{xz})_2 = K_1^*(2/r)^{1/2}\sin\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{1/2}) \tag{36}$$

and

$$v_1 = \frac{2\sqrt{2}K_1^*}{\mu_1}(r)^{1/2}\sin\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{3/2})$$

$$v_2 = \frac{-2\sqrt{2}K_1^*}{\mu_2}(r)^{1/2}\sin\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{3/2}) \tag{37}$$

where

$$K_1^* = \sqrt{2/\pi}\,\frac{\mu_1\mu_2\Lambda_1\Lambda_2 V_1\sin\Theta_1\sin\Theta_2}{(\Lambda_1\cos\phi_1+\Lambda_1)^{1/2}F_+^*(\Lambda_1\cos\phi_1)(\mu_1\Lambda_1\sin\phi_1+\mu_2\Lambda_2\sin\phi_2)} \tag{38}$$

and

$$F_+^*(\Lambda_1\cos\phi_1) = \exp\left[\frac{1}{\pi}\int_{\Lambda_1}^{\Lambda_2}\tan^{-1}\left\{\frac{\mu_1(s^2-\Lambda_1^2)^{1/2}}{\mu_2(\Lambda_2^2-s^2)^{1/2}}\right\}\frac{ds}{s+\Lambda_1\cos\phi_1}\right]. \tag{39}$$

If we put $\mu_1=\mu_2$, $\rho_1=\rho_2$, the corresponding results of the stationary crack in an isotropic elastic medium are found to be

$$(\tau_{yz})_{1,2} = V_1(\sin\tfrac{1}{2}\Theta_1)(\cos\tfrac{1}{2}\phi)\cos(\Omega t+\tfrac{1}{4}\pi)\left[\frac{2\Lambda_1\mu_1^2}{\pi r}\right]^{1/2}+0(r^{1/2})$$

$$(\tau_{xz})_{1,2} = \mp V_1(\sin\tfrac{1}{2}\Theta_1)(\cos\tfrac{1}{2}\phi)\cos(\Omega t+\tfrac{1}{4}\pi)\left[\frac{2\Lambda_1\mu_1^2}{\pi r}\right]^{1/2}+0(r^{1/2})$$

and

$$v_{1,2} = \pm V_1(\sin\tfrac{1}{2}\Theta_1)(\sin\tfrac{1}{2}\phi)\cos(\Omega t+\tfrac{1}{4}\pi)\left[\frac{8\Lambda_1 r}{\pi}\right]^{1/2}+0(r^{3/2}) \tag{40}$$

which are same as given by Jahanshahi [3].

3. Results and discussion

K_1 given by (33) is the dynamic stress intensity factor at the moving crack-tip and K_1^* given by (38) is the value of the corresponding quantity when the crack is stationary. The variation of K_1/K_1^* with the values of V/C_2 where V is the crack speed has been depicted graphically for the following two sets of materials.

First set:

Wrought iron $\rho_1 = 7{\cdot}8\,\mathrm{g/cm^3}$, $\mu_1 = 7{\cdot}7 \times 10^{11}\,\mathrm{dyn/cm^2}$
Copper $\rho_2 = 8{\cdot}96\,\mathrm{g/cm^3}$, $\mu_2 = 4{\cdot}5 \times 10^{11}\,\mathrm{dyn/cm^2}$

Second set:

Steel $\rho_1 = 7{\cdot}6\,\mathrm{g/cm^3}$, $\mu_1 = 8{\cdot}32 \times 10^{11}\,\mathrm{dyn/cm^2}$
Aluminium $\rho_2 = 2{\cdot}7\,\mathrm{g/cm^3}$, $\mu_2 = 2{\cdot}63 \times 10^{11}\,\mathrm{dyn/cm^2}$.

It is found that in both the cases the stress intensity factor gradually decreases with an increase in the value of V/C_2 and approaches zero as $V/C_2 \to 1$; the decrease in the value of K_1/K_1^* for the second set being more rapid than for the first set. We also find that in both the cases for any fixed value of C_1/C_2, K_1/K_1^* decreases with decrease in the value of Θ.

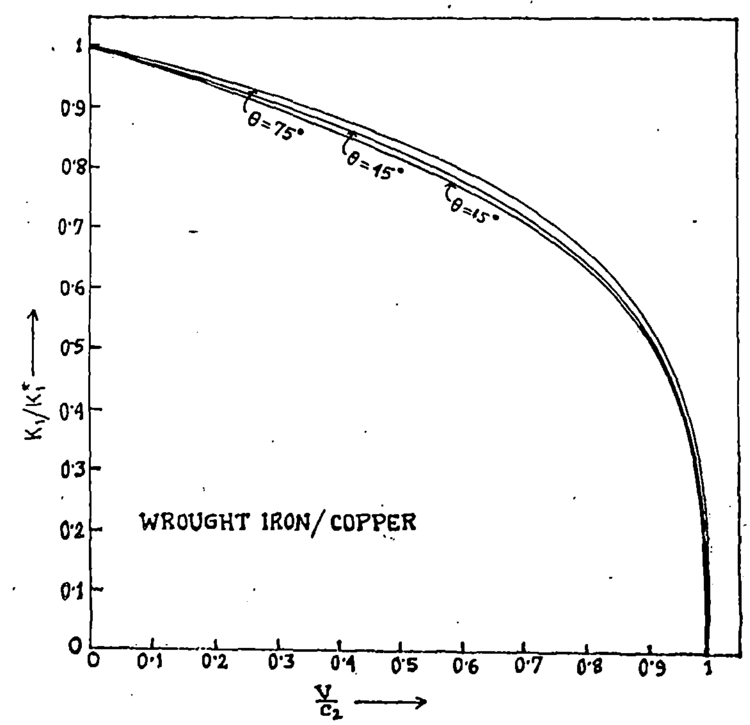

Figure 2. Stress intensity factor vs dimensionless crack speed (wrought iron/copper).

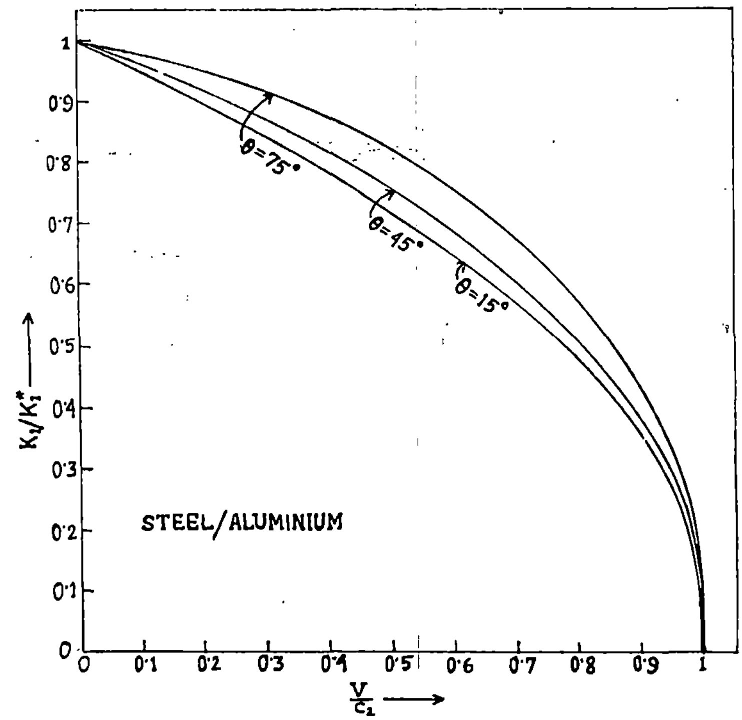

Figure 3. Stress intensity factor vs dimensionless crack speed (steel/aluminium).

Appendix

Factorization of $F(\xi)$ into $F_+(\xi)$ and $F_-(\xi)$

Consider

$$F(\xi)=\frac{(\mu_1 s_1+\mu_2 s_2)\gamma_2}{(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)} \tag{A1}$$

The branch points of $F(\xi)$ are at $\xi=\lambda_1,\ -\lambda_1,\ \lambda_2+\beta_2,\ -(\lambda_2-\beta_2)$ where

$$-(\lambda_2-\beta_2)<-\lambda_1<\lambda_1<\lambda_2+\beta_2 \text{ since } C_2<C_1.$$

Since $F(\xi)\to 1$ as $|\xi|\to\infty$, $F(\xi)$ possesses no singularity within the rectangular contour (shown in figure 4), by Cauchy's residue theorem we can write

$$\log F(\xi)=\frac{1}{2\pi i}\int_{c_+ + c_-}\frac{\log F(s)}{s-\xi}\mathrm{d}s \tag{A2}$$

$$=\frac{1}{2\pi i}\int_{c_+}\frac{\log F(s)}{s-\xi}\mathrm{d}s+\frac{1}{2\pi i}\int_{c_-}\frac{\log F(s)}{s-\xi}\mathrm{d}s$$

$$\log F(\xi)=\log F_+(\xi)+\log F_-(\xi), \tag{A3}$$

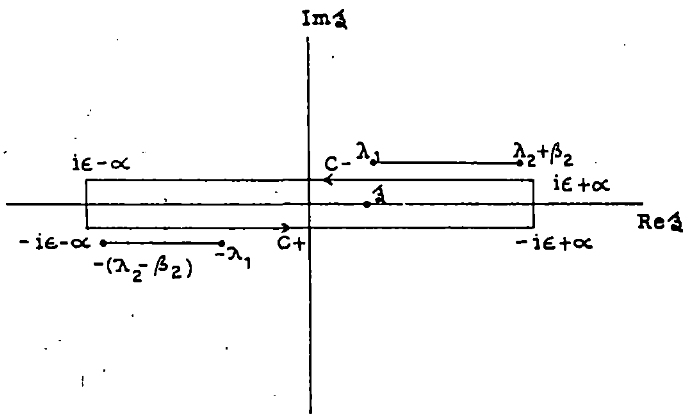

Figure 4. Rectangular contour in the complex ξ-plane.

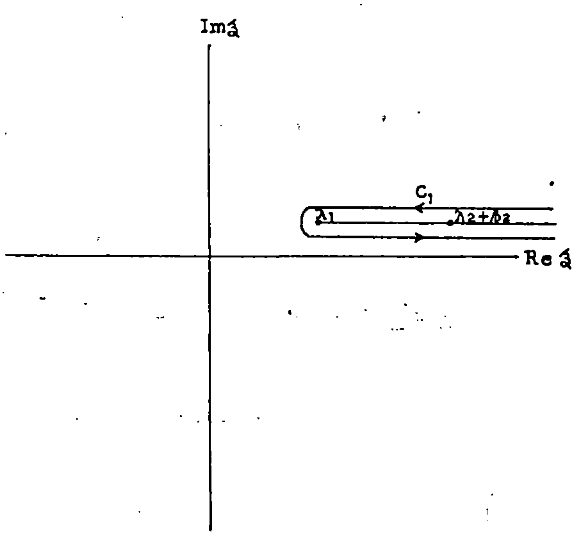

Figure 5. Path of integration C_1 round the branch cut.

where $F_+(\xi)$ and $F_-(\xi)$ are analytic in the upper and lower half of the complex ξ-plane respectively and can be expressed as

$$F_+(\xi) = \exp\left[\frac{1}{2\pi i}\int_{c+}\frac{\log F(s)}{s-\xi}\mathrm{d}s\right]$$

and

$$F_-(\xi) = \exp\left[\frac{1}{2\pi i}\int_{c-}\frac{\log F(s)}{s-\xi}\mathrm{d}s\right]. \tag{A4}$$

In order to evaluate $F_-(\xi)$ the path of integration C_- can be deformed to the path C_1 round the branch cut through λ_1 and $\lambda_2+\beta_2$ as shown in figure 5.

After a little algebraic manipulation it can be shown that

$$F_-(\xi) = \exp\left[\frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2+\beta_2}\frac{1}{s-\xi}\log\left\{1 + i\frac{m_1}{m_2}\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2-s)^2]^{1/2}}\right\}ds\right.$$

$$\left. -\frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2+\beta_2}\frac{1}{s-\xi}\log\left\{1 - i\frac{m_1}{m_2}\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2-s)^2]^{1/2}}\right\}ds\right] \tag{A5}$$

which after simplification becomes

$$F_-(\xi) = \exp\left[\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2+\beta_2}\frac{1}{s-\xi}\tan^{-1}\left\{\frac{m_1}{m_2}\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2-s)^2]^{1/2}}\right\}ds\right] \tag{A6}$$

where

$$m_1 = \frac{\mu_1 s_1}{\mu_1 s_1 + \mu_2 s_2} \quad \text{and} \quad m = \frac{\mu_2 s_2}{\mu_1 s_1 + \mu_2 s_2}. \tag{A7}$$

Similarly it can be shown that

$$F_+(\xi) = \exp\left[\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2-\beta_2}\frac{1}{s+\xi}\tan^{-1}\left\{\frac{m_1}{m_2}\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2+s)^2]^{1/2}}\right\}ds\right]. \tag{A8}$$

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Diffraction of SH-waves by a Griffith crack in nonhomogeneous elastic strip

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IN THIS PAPER the scattering of elastic SH-waves by a Griffith crack situated in an infinitely long inhomogeneous strip has been analyzed. Assuming that the shear modulus (μ) and density (ϱ) of the material vary in the vertical direction and applying Fourier transform, the mixed boundary value problem has been reduced to the solution of dual integral equations which finally has been reduced to the solution of a Fredholm integral equation of second kind. The numerical values of stress intensity factor and crack opening displacement have been illustrated graphically to show the effect of inhomogeneity of the material.

1. Introduction

THE NATURAL or artificial materials are usually inhomogeneous; so in recent years great attention has been given to the study of diffraction of elastic waves by cracks or obstacles in inhomogeneous media in view of their application in fracture mechanics. Many problems have been solved involving one or more cracks in an infinite homogeneous elastic medium. LOEBER and SIH [1] and MAL [2] have studied the problem of diffraction of elastic waves by a Griffith crack in an infinite medium. The problem of finite crack at the interface of two elastic half-spaces has been discussed by SRIVASTAVA *et al.* [3] and BOSTROM [4]. SINGH *et al.* [5, 6] considered the problem of scattering of a SH-wave by cracks or strips in a nonhomogeneous infinite elastic medium. Papers involving cracks located in an infinitely long elastic strip are very few. The problem of an infinite elastic strip containing an arbitrary number of unequal Griffith cracks, located parallel to its surfaces and opened by an arbitrary internal pressure, has been treated by ADAMS [7]. Finite crack perpendicular to the surface of the infinitely long elastic strip has been studied by CHEN [8] (for an impact load) and by SRIVASTAVA *et al.* [9] (for normally incident waves). Recently SHINDO *et al.* [10] considered the problem of impact response of a finite crack in an orthotropic strip. In our paper, the diffraction of normally incident SH-waves by a Griffith crack situated in an infinitely long inhomogeneous elastic strip has been discussed. The shear modulus (μ) and the density (ϱ) of the material have been assumed to vary in the vertical direction. Applying the Fourier transform, the mixed boundary value problem has been converted to the solution of dual integral equations. The dual integral equations have been finally reduced to a Fredholm integral equation of second kind by applying the Abel transform. Expressions for the stress intensity factor and crack opening displacement have been derived. The numerical values of stress intensity factor and crack opening displacement have been depicted by means of graphs to show the effect of material inhomogeneity.

2. Formulation of the problem

Consider the problem of diffraction of SH-waves by a Griffith crack in an inhomogeneous elastic strip of width $2h_1$. The crack is located in the region $-a \le x_1 \le a$, $-\infty < y_1 < \infty$, $z_1 = 0$ (Fig. 1). Normalizing all the lengths with respect to a and putting $x_1/a = x$, $y_1/a = y$, $z_1/a = z$, $h_1/a = h$ it is found that the location of the crack is $-1 \leq x \leq 1$, $-\infty < y < \infty$, $z = 0$ referred to a Cartesian coordinate system (x, y, z). Let a plane harmonic SH-wave originating at $z = -\infty$ impinge on the crack normally to the x-axis. The variation of the shear modulus μ and the density ϱ is taken in the vertical (z) direction in such a manner that the shear velocity $(\mu_0/\varrho_0)^{1/2}$ is constant. The only non-vanishing y-component of the displacement which is independent of y is $v = v(x, z, t)$.

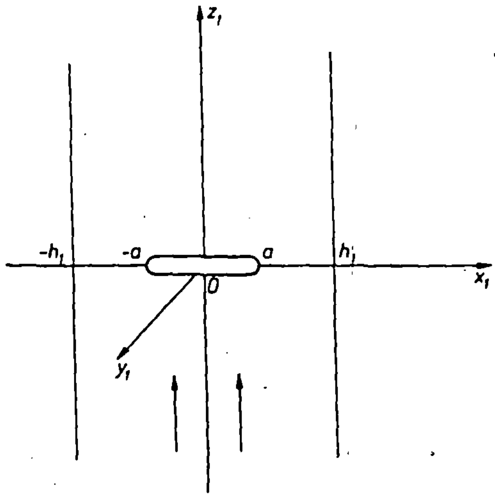

FIG. 1. Crack in the inhomogeneous strip.

The equation of motion is given by

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) = \varrho\frac{\partial^2 v}{\partial t^2}. \tag{2.1}$$

If we consider $v(x, z, t)$ in the form

$$v(x, z, t) = \frac{W(x, z, t)}{\sqrt{\mu(z)}}, \tag{2.2}$$

then

$$\mu\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial z^2}\right) + \frac{1}{2}\left[\frac{1}{2\mu}\left(\frac{\partial \mu}{\partial z}\right)^2 - \frac{\partial^2 \mu}{\partial z^2}\right]W = \varrho\frac{\partial^2 W}{\partial t^2}. \tag{2.3}$$

Putting $W(x, z, t) = F(x)G(z)e^{-i\omega t}$ and $\mu(z) = \mu_0 f(z)$, $\varrho(z) = \varrho_0 f(z)$ in Eq. (2.3) where μ_0, ϱ_0 are constants, such that $(\mu_0/\varrho_0)^{1/2} = c_2$ is the shear wave velocity, it is found that $F(x)$ and $G(z)$ satisfy the following equations

$$\frac{\partial^2 F}{\partial x^2} + n^2 F = 0, \tag{2.4}$$

$$\frac{\partial^2 G}{\partial z^2} + \left(\frac{a^2\omega^2}{c_2^2} - b^2 - n^2\right)G = 0, \tag{2.5}$$

provided $f(z)$ is of the form

$$-\frac{1}{4}\left(\frac{\partial f}{\partial z}/f\right)^2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial z^2}/f\right) = b^2, \tag{2.6}$$

where a and b are constants.

Let us assume $f(z)$ in the form

$$f(z) = \cosh^2(bz) \tag{2.7}$$

so that Eq. (2.6) is automatically satisfied.

Now the shear modulus $\mu(z)$ and density of the medium $\varrho(z)$ are

$$\mu = \mu_0 \cosh^2(bz), \qquad \varrho = \varrho_0 \cosh^2(bz). \tag{2.8}$$

Using Eqs. (2.8), (2.2) and $W(x,z,t) = W(x,z)e^{-i\omega t}$, Eq. (2.1) takes the form

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial z^2} + k^2 W = 0, \quad k^2 = (k_2^2 - b^2), \quad k_2 = \frac{a\omega}{c_2}. \tag{2.9}$$

The displacement component $v^{(i)}(x,z,t)$ and stress $\tau^{(i)}(x,z,t)$ due to incident waves are given by

$$v^{(i)}(x,z,t) = \frac{A_0 e^{i(kz-\omega t)}}{\sqrt{\mu_0}\cosh(bz)} \tag{2.10}$$

and

$$\tau^{(i)}_{yz}(x,z,t) = A_0\sqrt{\mu_0}[ik\cosh(bz) - b\sinh(bz)]e^{i(kz-\omega t)}, \tag{2.11}$$

where A_0 is a constant.

Henceforth the time factor $e^{-i\omega t}$ will be suppressed in the sequel.

Solution of Eq. (2.9) is

$$W(x,z) = \int_0^\infty B_1(\xi)e^{-\beta z}\cos(\xi x)d\xi + \int_0^\infty C_1(\zeta)\cosh(\alpha x)\sin(\zeta z)d\zeta, \tag{2.12}$$

where

$$\alpha = (\zeta^2 - k^2)^{1/2}, \quad \zeta > k, \quad \beta = (\xi^2 - k^2)^{1/2}, \quad \xi > k,$$
$$= i(k^2 - \zeta^2)^{1/2}, \quad \zeta < k, \quad = -i(k^2 - \xi^2)^{1/2}, \quad \xi < k.$$

Now displacement $v(x,z)$ and stresses $\tau_{yz}(x,z), \tau_{xy}(x,z)$ due to the scattered field are

$$v(x,z) = \frac{1}{\cosh(bz)}\left[\int_0^\infty B(\xi)e^{-\beta z}\cos\xi x\,d\xi + \int_0^\infty C(\zeta)\cosh(\alpha x)\sin\zeta z\,d\zeta\right], \tag{2.13}$$

$$\begin{aligned}\tau_{yz}(x,z) = -\mu_0 b\sinh(bz)\Big[\int_0^\infty B(\xi)e^{-\beta z}\cos\xi x\,d\xi \\ + \int_0^\infty C(\zeta)\cosh(\alpha x)\sin\zeta z\,d\zeta\Big] + \mu_0\cosh(bz) \\ \Big[-\int_0^\infty \beta B(\xi)e^{-\beta z}\cos\xi x\,d\xi + \int_0^\infty \zeta C(\zeta)\cosh(\alpha x)\cos\zeta z\,d\zeta\Big],\end{aligned} \tag{2.14}$$

$$\tau_{xy}(x,z) = \mu_0 \cosh(bz)\Big[- \int_0^\infty \xi B(\xi) e^{-\beta z} \sin \xi x \, d\xi + \int_0^\infty \alpha C(\zeta) \sinh(\alpha x) \sin \zeta z \, d\zeta \Big], \tag{2.15}$$

where

$$B(\xi) = \frac{1}{\sqrt{\mu_0}} B_1(\xi), \quad C(\zeta) = \frac{1}{\sqrt{\mu_0}} C_1(\zeta).$$

The boundary conditions are

$$\tau_{yz}(x,0) = -\tau_0, \qquad |x| \le 1, \tag{2.16}$$

$$v(x,0) = 0, \qquad 1 \le |x| \le h, \tag{2.17}$$

$$\tau_{xy}(\pm h, z) = 0, \qquad |z| < \infty, \tag{2.18}$$

where $\tau_0 = ikA_0\sqrt{\mu_0}$.

From the boundary condition (2.18) $C(\zeta)$ is found to be expressible in terms of $B(\xi)$ as follows:

$$C(\zeta) = \frac{2\zeta}{\pi \alpha \sinh(\alpha h)} \int_0^\infty \frac{\xi B(\xi) \sin(\xi h)}{\xi^2 + \alpha^2} d\xi. \tag{2.19}$$

Next, the use of Eq. (2.19) in the boundary condition (2.16) and (2.17) yields the following dual integral equations from which the unknown function $B(\xi)$ is to be determined:

$$\int_0^\infty \xi [1 + M(\xi)] B(\xi) \cos(\xi x) \, d\xi = p(x), \quad |x| \le 1 \tag{2.20}$$

and

$$\int_0^\infty B(\xi) \cos(\xi x) \, d\xi = 0, \quad 1 \le |x| \le h \tag{2.21}$$

where

$$M(\xi) = \left(\frac{\beta}{\xi} - 1\right), \tag{2.22}$$

$$p(x) = \frac{\tau_0}{\mu_0} + \frac{2}{\pi} \int_0^\infty \frac{\zeta^2 \cosh(\alpha x)}{\alpha \sinh(\alpha h)} d\zeta \int_0^\infty \frac{\xi B(\xi) \sin(\xi h)}{\xi^2 + \alpha^2} d\xi. \tag{2.23}$$

3. Method of solution

In order to solve the dual integral equations (2.20) and (2.21), $B(\xi)$ is taken in the form

$$B(\xi) = \frac{\tau_0}{\mu_0} \int_0^1 t \phi(t) J_0(\xi t) \, dt, \tag{3.1}$$

so that Eq. (2.21) is automatically satisfied.

Substitution of the value of $B(\xi)$ from Eq. (3.1) in Eq. (2.20), yields a Fredholm integral equation of second kind

$$\phi(t) + \int_0^1 u[L_1(u,t) + L_2(u,t)]\phi(u)\,du = 1, \tag{3.2}$$

where

$$L_1(u,t) = \int_0^\infty \xi M(\xi) J_0(\xi u) J_0(\xi t)\,d\xi, \tag{3.3}$$

$$L_2(u,t) = -\int_0^\infty \frac{\zeta^2 I_0(\alpha t) I_0(\alpha u) e^{-\alpha h}}{\alpha \sinh(\alpha h)}\,d\zeta. \tag{3.4}$$

Using contour integration technique [3], the infinite integral arising in the kernel $L_1(u,t)$ can be converted to a finite integral and is given by

$$\begin{aligned} L_1(u,t) &= -ik^2 \int_0^1 (1-\eta^2)^{1/2} J_0(k\eta t) H_0^{(1)}(k\eta u)\,d\eta, \qquad u > t,\\ &= -ik^2 \int_0^1 (1-\eta^2)^{1/2} J_0(k\eta u) H_0^{(1)}(k\eta t)\,d\eta, \qquad u < t. \end{aligned} \tag{3.5}$$

Now

$$\begin{aligned} L_2(u,t) &= \int_0^k \frac{\zeta^2 J_0(\alpha_1 t) J_0(\alpha_1 u) e^{i\alpha_1 h}}{\alpha_1 \sin(\alpha_1 h)}\,d\zeta - \int_k^\infty \frac{\zeta^2 I_0(\alpha t) I_0(\alpha u) e^{-\alpha h}}{\alpha \sinh(\alpha h)}\,d\zeta \\ &= \int_0^k \frac{\zeta^2}{\alpha_1} J_0(\alpha_1 t) J_0(\alpha_1 u) \operatorname{ctg}(\alpha_1 h)\,d\zeta + i \int_0^k \frac{\zeta^2}{\alpha_1} J_0(\alpha_1 t) J_0(\alpha_1 u)\,d\zeta \\ &\qquad - \int_k^\infty \frac{\zeta^2 I_0(\alpha t) I_0(\alpha u) e^{-\alpha h}}{\alpha \sinh(\alpha h)}\,d\zeta, \end{aligned}$$

where

$$\alpha_1 = (k^2 - \zeta^2)^{1/2}.$$

Putting $\zeta^2 = k^2(1-y^2)$ in the first and second integrals and $\zeta^2 = k^2(1+y^2)$ in the third integral, it is found that

$$\begin{aligned} L_2(u,t) = k^2 \Big[&\int_0^1 (1-y^2)^{1/2} J_0(kyt) J_0(kyu) \operatorname{ctg}(kyh)\,dy \\ &+ i \int_0^1 (1-y^2)^{1/2} J_0(kyt) J_0(kyu)\,dy \\ &- \int_0^\infty (1+y^2)^{1/2} I_0(kyt) I_0(kyu) e^{-kyh} \operatorname{cosech}(kyh)\,dy \Big]. \end{aligned} \tag{3.6}$$

4. Stress intensity factor and crack opening displacement

From Eq. (2.14) the stress τ_{yz} on the plane $z = 0$ can be written as

$$(4.1) \qquad \tau_{yz}(x,0) = \mu_0\Big[-\int_0^\infty \beta B(\xi)\cos\xi x\,d\xi + \int_0^\infty \zeta C(\zeta)\cosh(\alpha x)\,d\zeta\Big].$$

Substituting the value of $C(\zeta)$ and $B(\xi)$ from Eqs. (2.19) and (3.1), the expression for the stress can finally be presented as

$$\tau_{yz}(x,0) = \frac{\tau_0 x}{(x^2-1)^{1/2}}\phi(1) + O(1), \qquad |x| > 1.$$

Defining the stress intensity factor N by

$$N = \operatorname*{Lt}_{x\to 1^+}\left|\frac{(x-1)^{1/2}\tau_{yz}(x,0)}{\tau_0}\right|,$$

we obtain

$$(4.2) \qquad N = \frac{1}{\sqrt{2}}|\phi(1)|.$$

Now the crack opening displacement $\Delta v(x,0) = v(x,0^+) - v(x,0^-)$ can be obtained from Eq. (2.13) as

$$\Delta v(x,0) = 2\int_0^\infty B(\xi)\cos(\xi x)\,d\xi, \qquad |x| \le 1,$$

which, on substitution of the value of $B(\xi)$ from Eq. (3.1), takes the form

$$(4.3) \qquad \Delta v(x,0) = \frac{2\tau_0}{\mu_0}\int_x^1 \frac{t\phi(t)}{(t^2-x^2)^{1/2}}\,dt, \qquad |x| \le 1.$$

5. Numerical results and discussion

Using the method of FOX and GOODWIN [11], the Fredholm integral equation given by Eq. (3.2) has been solved numerically for different values of the material inhomogeneity parameters. In this method the integral in Eq. (3.2) has been represented at first by a quadrature formula involving the values of the desired function $\phi(t)$ at the pivotal points inside the specified range of integration, and then converted to a set of simultaneous linear algebraic equations; their solutions yield the first approximations to the required pivotal values of $\phi(t)$. Applying the difference-correction technique, the first approximations have been improved. After solving the integral equation (3.2) numerically, the stress intensity factor N and the crack opening displacement $\mu_0\Delta v(x,0)/\tau_0$ have been calculated numerically and plotted separately against the dimensional frequency k_2 ($0.5 \le k_2 \le 1$) and dimensionless distance x ($0 \le x \le 1$), respectively, for different values of the material inhomogeneity parameter b and strip width $2h$.

In Fig. 2, the effect of the width of the strip on the stress intensity factor for a homogeneous material has been shown; the effect of inhomogeneity of the material on the stress intensity factor for different widths of the strip has been depicted in Figs. 3–5.

It is found that in both the homogeneous and nonhomogeneous cases, the effect of the strip width decreases with the increase of the frequency, and the graphs of the stress

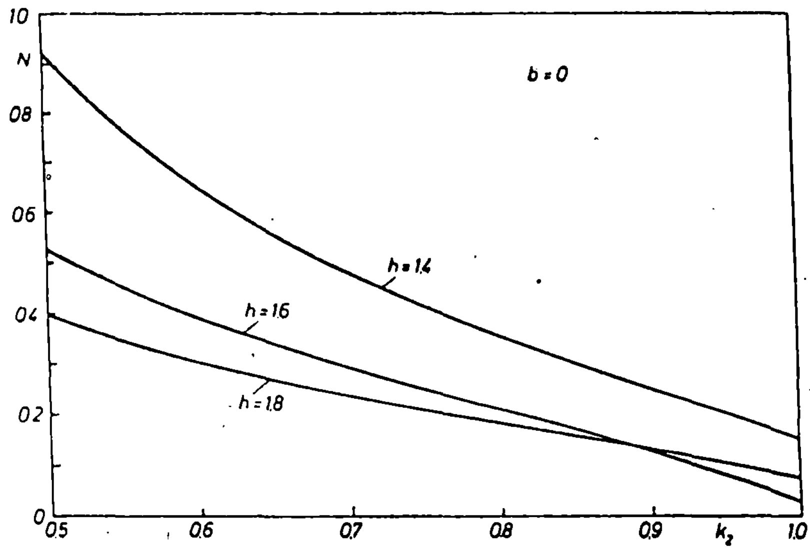

FIG. 2. Stress intensity factor N vs. dimensionless frequency k_2 for homogeneous medium ($b = 0$).

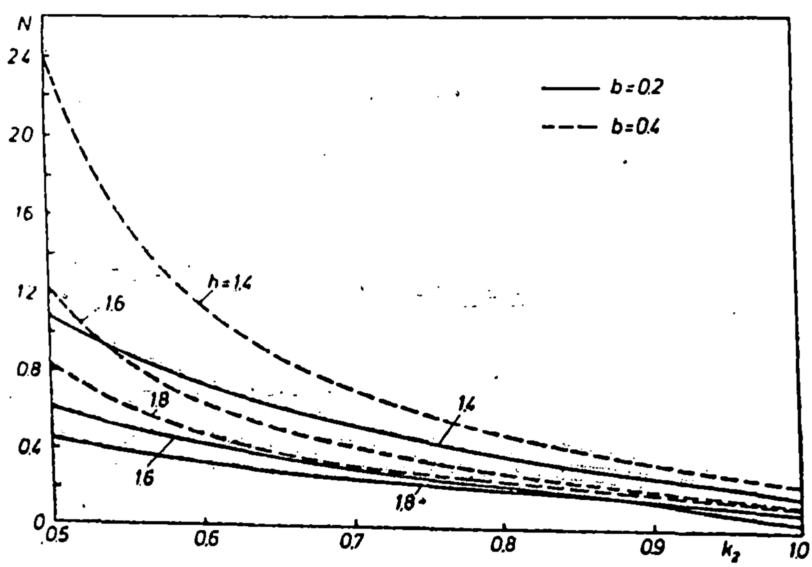

FIG. 3. Stress intensity factor N vs. dimensionless frequency k_2.

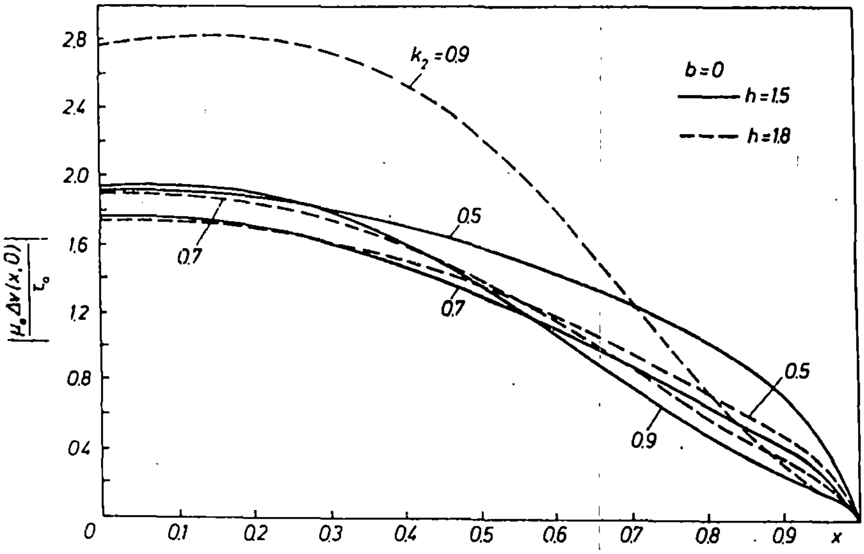

FIG. 4. Crack opening displacement vs. dimensionless distance $x(b = 0)$.

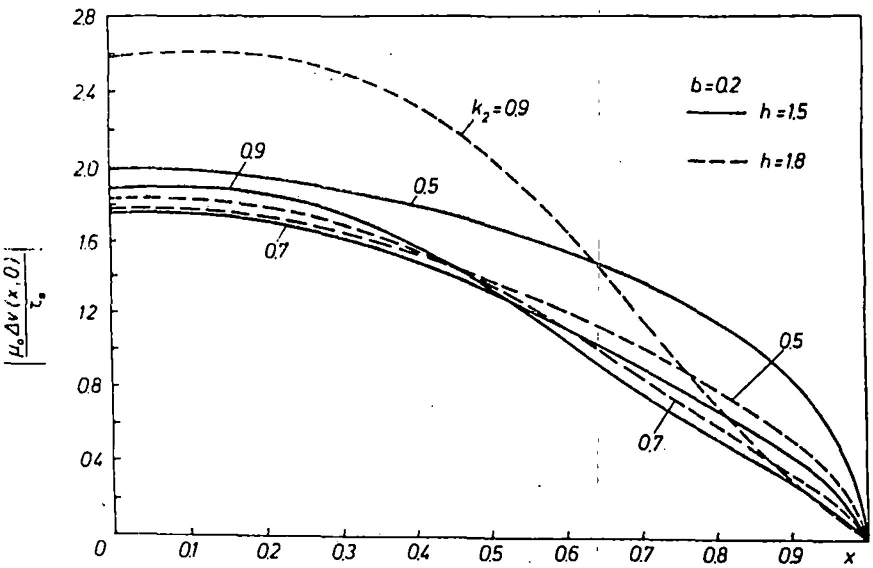

FIG. 5. Crack opening displacement vs. dimensionless distance $x(b = 0.2)$.

intensity factor N become flat with the increase of strip width $2h$. From Fig. 3 it is clear that the effect of inhomogeneity parameter b is prominent for low frequency k_2 and stress intensity factor is greater for higher values of the inhomogeneity parameter b.

In Figs. 4–8 the crack opening displacements against dimensionless distance x for different values of the material inhomogeneity parameter b and the strip width $2h$ have been illustrated by means of graphs. Case $b = 0$ corresponds to the homogeneous case (Fig. 4). From Figs. 4–6 it is seen that for a fixed value of inhomogeneity parameter b,

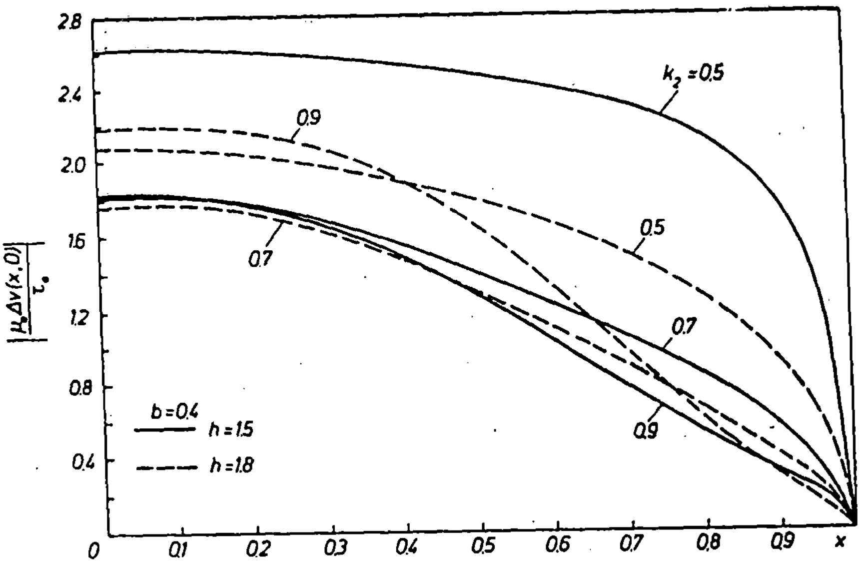

FIG. 6. Crack opening displacement vs. dimensionless distance $x (b = 0.4)$.

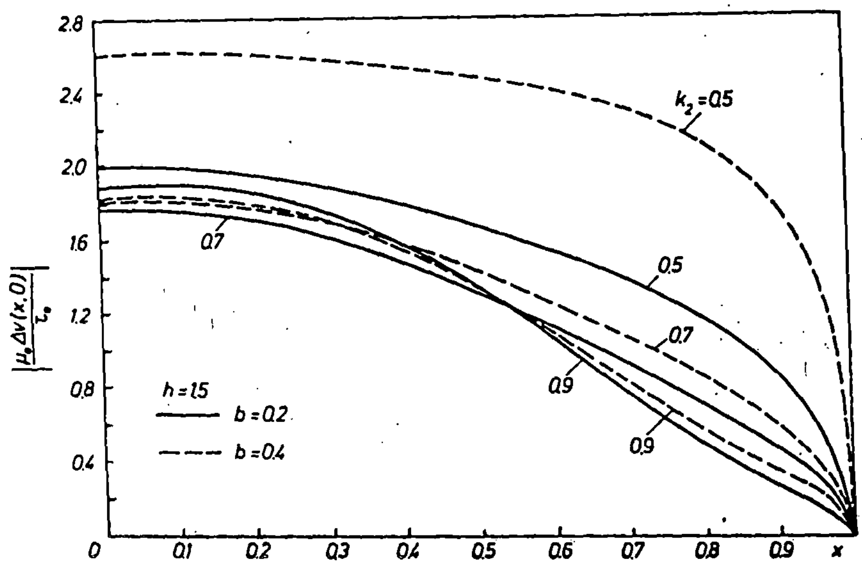

FIG. 7. Crack opening displacement vs. dimensionless distance $x (h = 1.5)$.

the crack opening displacement is greater for lower values of h when the frequencies are small, but the reverse effect is found for higher frequencies.

Next, in Figs. 7 and 8 we see that for a fixed value of h, the crack opening displacement is greater for higher values of the inhomogeneity parameter b when the frequencies are small, but for higher frequencies the effect is just reverse.

Finally it is found in all the cases that the crack opening displacement reaches its maximum at about $x = 0$, and then it gradually decreases and becomes zero at $x = 1$.

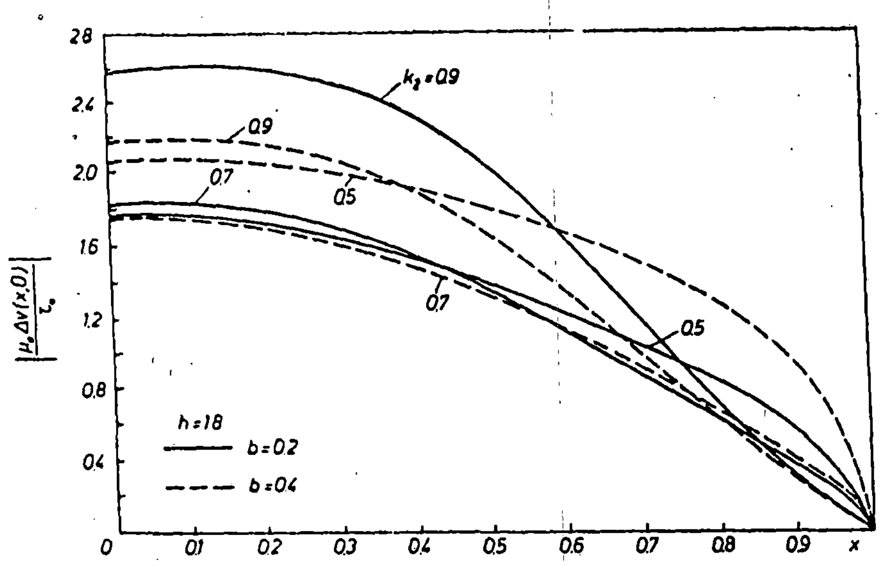

FIG. 8. Crack opening displacement vs. dimensionless distance x(h = 1.8).

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AN ELASTIC STRIP WITH THREE CO-PLANAR MOVING GRIFFITH CRACKS

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Abstract—The dynamic anti-plane problem of determining stress and displacement due to three co-planar Griffith cracks moving steadily at a subsonic speed in an infinite elastic strip has been considered. Employing Fourier integral transform, the problem when the rigidly clamped edges of the strip are pulled apart in opposite directions has been reduced to solving a set of four integral equations. These integral equations have been solved using the finite Hilbert transform technique and Cook's result [*Glas. Math. J.* 11, 9 (1970)] to obtain the exact form of crack opening displacement and stress intensity factors. Numerical results for stress intensity factors are presented in the form of graphs.

1. INTRODUCTION

IN FRACTURE MECHANICS, the problem of diffraction of elastic waves by cracks of finite dimension in a strip of elastic material has been examined by several investigators. Sih and Chen [1] investigated the problem of propagation of a crack of finite length in a strip under plane extension. Closed-form solutions for a finite length crack moving in a strip under anti-plane shear stress were obtained by Singh *et al.* [2]. Using a finite Hilbert transform technique developed by Srivastava and Lowengrub [3], Lowengrub and Srivastava [4] solved the static problem of distribution of stress and displacement in an infinitely long elastic strip containing two co-planar Griffith cracks. Recently, several dynamic problems of determining stress and displacement due to moving Griffith cracks have been solved by Das and Ghosh [5–8] and by Das [9, 10]. Dhawan and Dhaliwal [11] also solved the static problem of determining the stress distribution in an infinite transversely isotropic medium containing three co-planar Griffith cracks.

In this paper, the problem of propagation of three co-planar Griffith cracks in a fixed direction with constant velocity V in an infinitely long but finite width elastic strip is considered. Employing the Fourier integral transform, the problem when the lateral boundaries are assumed to be clamped and displaced by an equal amount has been reduced to solving a set of four integral equations which are solved using the finite Hilbert transform technique and Cook's result [12] to derive the exact form of stress intensity factors and crack opening displacement. Numerical results for stress intensity factors are presented graphically to show their variations with crack speed, crack length and the separating distance between the cracks.

2. STATEMENT OF THE PROBLEM

Consider an infinitely long elastic strip occupying the region $-h \leqslant y \leqslant h$, weakened by three co-planar Griffith cracks moving steadily at a constant velocity V in the X-direction, referred to a fixed coordinate system (X, Y, Z) as shown in Fig. 1.

In dynamic problems of anti-plane shear, the non-vanishing component of displacement W directed in the Z-direction satisfies the equation of motion:

$$W_{,XX} + W_{,YY} = \frac{1}{C_2^2} W_{,TT}, \tag{1}$$

where $C_2 = (\mu/\rho)^{1/2}$ is the shear wave velocity, ρ is the material density and $W_{,X}$ represents partial derivatives of W with respect to X.

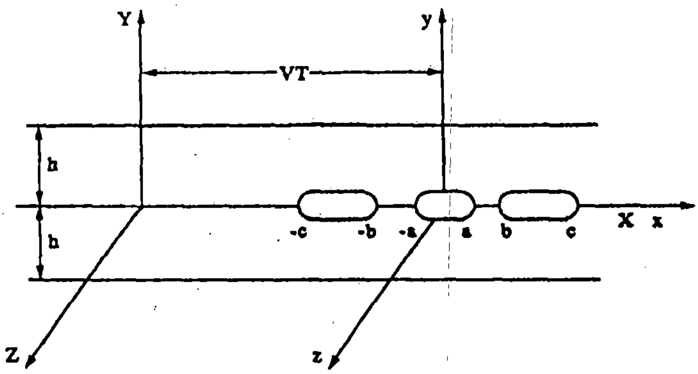

Fig. 1. Geometry and coordinate system.

For cracks moving at a constant velocity V in the X-direction, it is convenient to introduce the Galilean transformation:

$$x = X - VT, \quad y = Y, \quad z = Z, \quad t = T, \tag{2}$$

where (x, y, z) represents the translating coordinate system shown in Fig. 1.

Let three co-planar Griffith cracks of finite length located along the X-axis be moving steadily with velocity V in the direction of the X-axis so that their positions referred to translating coordinates (x, y, z) are $-c < x < -b$, $-a < x < a$ and $b < x < c$ on $y = 0$. The edges of the strip $y = \pm h$ are assumed to be clamped and displaced by an equal amount W_0, where W_0 is a constant. The boundary conditions of the proposed problem are

$$\sigma_{yz}(x, 0) = 0, \quad |x| < a, \quad b < |x| < c \tag{3}$$

$$W(x, \pm h) = \pm W_0, \quad -\infty < x < \infty \tag{4}$$

$$W(x, 0) = 0, \quad a < |x| < b, \quad |x| > c. \tag{5}$$

In order to apply the integral transform technique it is required to solve a different but equivalent problem which can be obtained from the clamped strip problem (without any cracks) while the uniform strain is applied. The equivalent stress conditions on the cracks are

$$\sigma_{yz}(x, 0) = \frac{\mu W_0}{h}, \quad |x| < a, \quad b < |x| < c \tag{6}$$

and the boundary conditions for the displacement are

$$W(x, \pm h) = 0, \quad -\infty < x < \infty \tag{7}$$

$$W(x, 0) = 0, \quad a < |x| < b, \quad |x| > c. \tag{8}$$

In the moving coordinate system, the equation of motion becomes independent of time and takes the form

$$s^2 W_{,xx} + W_{,yy} = 0, \tag{9}$$

with

$$s = \sqrt{(1 - V^2/C_2^2)}. \tag{10}$$

Introducing

$$\bar{W}_c(\xi, y) = \int_0^\infty W(x, y)\cos(\xi x)\, dx$$

$$W(x, y) = \frac{2}{\pi}\int_0^\infty \bar{W}_c(\xi, y)\cos(\xi x)\, d\xi \tag{11}$$

in eq. (3), the solution of eq. (3) is obtained as

$$W(x,y)=\frac{2}{\pi}\int_0^\infty [C_1(\xi)e^{-\xi ys}+C_3(\xi)e^{\xi ys}]\cos(\xi x)\,d\xi, \tag{12}$$

with

$$\sigma_{yz}(x,y)=-\frac{2\mu s}{\pi}\int_0^\infty \xi[C_1(\xi)e^{-\xi ys}-C_3(\xi)e^{\xi ys}]\cos(\xi x)\,d\xi. \tag{13}$$

Using the expression for $W(x,y)$ given in (6) in eq. (9), it has been found that

$$C_1(\xi)=\frac{C(\xi)}{1-e^{-2\xi hs}}$$

$$C_3(\xi)=-\frac{C(\xi)e^{-2\xi hs}}{1-e^{-2\xi hs}}, \tag{14}$$

where the unknown function $C(\xi)$ is to be determined.

From conditions (8) and (10) it is determined that $C(\xi)$ satisfies the following quadruple integral equations

$$\int_0^\infty \xi C(\xi)\coth(\xi hs)\cos(\xi x)\,d\xi=\frac{\pi W_0}{2hs},\quad x\in I_1,I_3 \tag{15a, b}$$

and

$$\int_0^\infty C(\xi)\cos(\xi x)\,d\xi=0,\quad x\in I_2,I_4, \tag{16}$$

where

$$I_1=(0,a),\quad I_2=(a,b),\quad I_3=(b,c),\quad I_4=(c,\infty).$$

3. METHOD OF SOLUTION

In order to solve the quadruple integral equations given by eqs (15) and (16), let us take

$$C(\xi)=\frac{1}{\xi}\int_0^a h(u)\sin(\xi u)\,du+\frac{1}{\xi}\int_b^c g(v^2)\operatorname{sech}^2(ev)\sin(\xi v)\,dv, \tag{17}$$

where $h(u)$ and $g(v^2)$ are the unknown functions to be determined from the boundary conditions of the proposed problem. Substituting the value of $C(\xi)$ given by (17) in (16) and using the following result:

$$\int_0^\infty \frac{\sin(\xi u)\cos(\xi x)}{\xi}\,d\xi=\begin{cases}\pi/2, & u>x>0\\ \pi/4, & u=x>0\\ 0, & x>u>0,\end{cases}$$

it is found that this choice of $C(\xi)$ leads to the condition

$$\int_b^c g(v^2)\operatorname{sech}^2(ev)\,dv=0. \tag{18}$$

Rewriting eq. (15a) as

$$\frac{d}{dx}\int_0^\infty C(\xi)\coth(\xi hs)\sin(\xi x)\,d\xi=\frac{\pi W_0}{2hs},\quad x\in I_1 \tag{19}$$

and inserting the value of $C(\xi)$ from eq. (17) in (19), it is found that $h(u)$ is the solution of the following singular integral equation:

$$\int_0^a h(u)\log\left|\frac{\tanh(ex)+\tanh(eu)}{\tanh(ex)-\tanh(eu)}\right|du=\pi f(x),\quad x\in I_1, \tag{20}$$

with

$$f(x)=\int_0^x\left[\frac{W_0}{hs}-\frac{2}{\pi}\int_b^c\frac{eg(v^2)\operatorname{sech}^2(ex')\operatorname{sech}^2(ev)\tanh(ev)}{\tanh^2(ev)-\tanh^2(ex')}\,dv\right]dx',$$

where the following result [13] has been used:

$$\int_0^\infty \coth(\xi hs)\frac{\sin(\xi x)\sin(\xi u)}{\xi}\,d\xi = \tfrac{1}{2}\log\left|\frac{\tan(ex)+\tanh(eu)}{\tanh(ex)-\tanh(eu)}\right|, \quad e=\frac{\pi}{2hs}. \tag{21}$$

Now using Cook's result [12], the solution of (20) has been obtained with the aid of the following result:

$$\int_0^u \frac{\sqrt{[\tanh^2(ea)-\tanh^2(ex)]}\,e\,\mathrm{sech}^2(ex)\,dx}{[\tanh^2(ex)-\tanh^2(eu)][\tanh^2(ev)-\tanh^2(ex)]}$$
$$= -\frac{\pi}{2\tanh(ev)}\frac{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}{\tanh^2(ev)-\tanh^2(eu)} \quad \text{for } u\in I_1 \text{ and } v\in I_3,$$

$$h(u)=\frac{-2e\tanh(eu)\mathrm{sech}^2(eu)}{\pi\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}}\left[\frac{W_0}{hs}\int_0^a\frac{\sqrt{[\tanh^2(ea)-\tanh^2(ex)]}}{\tanh^2(ex)-\tanh^2(eu)}dx\right.$$
$$\left.+\int_b^c\frac{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}{\tanh^2(ev)-\tanh^2(eu)}\times g(v^2)\mathrm{sech}^2(ev)\,dv\right]. \tag{22}$$

Substituting the resulting value of $C(\xi)$, obtained using eq. (22) in eq. (17), in condition (15b) and making use of the following results:

$$\int_0^a \frac{e\,\mathrm{sech}^2(eu)\tanh^2(eu)\,du}{[\tanh^2(eu)-\tanh^2(ex)][\tanh^2(ev)-\tanh^2(eu)]\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}}$$
$$=\frac{\pi}{2[\tanh^2(ev)-\tanh^2(ex)]}\left[\frac{\tanh(ev)}{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}-\frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}\right],$$

$$\int_0^a \frac{e\,\mathrm{sech}^2(eu)\tanh^2(eu)\,du}{[\tanh^2(eu)-\tanh^2(ex)][\tanh^2(ey')-\tanh^2(eu)]\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}}$$
$$=\frac{\pi}{2[\tanh^2(ex)-\tanh^2(ey')]}\frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}, \quad \text{for } x,v\in I_3 \text{ and } y'\in I_1,$$

it can be shown that $g(v^2)$ is the solution of the following singular integral equation:

$$\int_b^c\frac{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}{\tanh^2(ev)-\tanh^2(ex)}eg(v^2)\mathrm{sech}^2(ev)\,dv=\frac{\pi W_0}{2hs}\left[\frac{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}{\mathrm{sech}^2(ex)\tanh(ex)}\right.$$
$$\left.+\frac{e}{\pi}\int_0^a\frac{\sqrt{[\tanh^2(ea)-\tanh^2(ey')]}}{\tanh^2(ex)-\tanh^2(ey')}dy'\right], \quad \text{for } x\in I_3. \tag{23}$$

Using the finite Hilbert transform technique [3], and the following result:

$$\int_b^c\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ex)}{\tanh^2(ex)-\tanh^2(eb)}\right]}\frac{2\,\mathrm{sech}^2(ex)\tanh(ex)\,dx}{[\tanh^2(ex)-\tanh^2(ey')][\tanh^2(ex)-\tanh^2(ev)]}$$
$$=-\frac{\pi}{e[\tanh^2(ev)-\tanh^2(ey')]}\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ey')}{\tanh^2(eb)-\tanh^2(ey')}\right]},$$

the solution of eq. (23) is found as

$$g(v^2)=-\frac{2eW_0}{h\pi s}\frac{\tanh^2(ev)\sqrt{[\tanh^2(ev)-\tanh^2(eb)]}}{\sqrt{\{[\tanh^2(ev)-\tanh^2(ea)][\tanh^2(ec)-\tanh^2(ev)]\}}}$$
$$\times\left[\int_b^c\sqrt{\left[\frac{\tan^2(ec)-\tanh^2(ex)}{\tan^2(ex)-\tanh^2(eb)}\right]}\frac{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}{\tanh^2(ex)-\tanh^2(ev)}dx\right.$$
$$\left.-\int_0^a\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ey')}{\tanh^2(eb)-\tanh^2(ey')}\right]}\frac{\sqrt{[\tan^2(ea)-\tanh^2(ey')]}}{\tanh^2(ev)-\tanh^2(ey')}dy'\right]$$
$$+\frac{C_1\tanh(ev)}{\sqrt{\{[\tanh^2(ev)-\tanh^2(ea)][\tanh^2(ev)-\tanh^2(eb)][\tanh^2(ec)-\tanh^2(ev)]\}}}. \tag{24}$$

Next substituting the value of $g(v^2)$ from eq. (24) in eq. (22) and finally using the following result:

$$\int_b^c \sqrt{\left[\frac{\tanh^2(ev)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ev)}\right]}\frac{2\,\mathrm{sech}^2(ev)\tanh(ev)\,dv}{[\tanh^2(ev)-\tanh^2(eu)][\tanh^2(ex')-\tanh^2(ev)]}$$

$$=\frac{\pi}{e[\tan^2(eu)-\tanh^2(ex')]}\left[\sqrt{\left[\frac{\tanh^2(eb)-\tanh^2(eu)}{\tanh^2(ec)-\tanh^2(eu)}\right]}-\sqrt{\left[\frac{\tanh^2(eb)-\tanh^2(ex')}{\tanh^2(ec)-\tanh^2(ex')}\right]}\right],$$

for $u, x' \in I_1$,

$h(u)$ is derived in the form:

$$h(u)=-\frac{2eW_0}{\mu\pi s}\frac{\mathrm{sech}^2(eu)\tanh(eu)\sqrt{[\tanh^2(eb)-\tanh^2(eu)]}}{\sqrt{\{[\tanh^2(ea)-\tanh^2(eu)][\tanh^2(ec)-\tanh^2(eu)]\}}}$$

$$\times\left[\int_0^a \sqrt{\left[\frac{\tanh^2(ea)-\tanh^2(ey')}{\tanh^2(eb)-\tanh^2(ey')}\right]}\frac{\sqrt{[\tanh^2(ec)-\tanh^2(ey')]}}{\tanh^2(ey')-\tanh^2(eu)}\,dy'\right.$$

$$\left.+\int_b^c \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ex)}{\tanh^2(ex)-\tanh^2(eb)}\right]}\frac{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}{\tanh^2(ex)-\tanh^2(eu)}\,dx\right]$$

$$-\frac{C_1\tan(eu)\mathrm{sech}^2(eu)}{\sqrt{\{[\tanh^2(ea)-\tanh^2(eu)][\tanh^2(eb)-\tanh^2(eu)][\tanh^2(ec)-\tanh^2(eu)]\}}}. \tag{25}$$

Substitution of the value of $g(v^2)$ from eq. (24) in the condition (18) yields

$$C_1=-\frac{2eW_0}{\pi hs}\left[\int_b^c \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ex)}{\tanh^2(ex)-\tanh^2(eb)}\right]}\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}\right.$$

$$\times\left\{\frac{\tanh^2(ex)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ex)}\times\Pi\left\{\frac{\pi}{2},\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ex)},q\right\}\Big/F\left(\frac{\pi}{2},q\right)+1\right\}dx$$

$$+\int_0^a \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(es)}{\tanh^2(eb)-\tanh^2(es)}\right]}\sqrt{[\tanh^2(ea)-\tanh^2(es)]}$$

$$\left.\times\left\{1-\frac{\tanh^2(eb)-\tanh^2(es)}{\tanh^2(ec)-\tanh^2(es)}\Pi\left\{\frac{\pi}{2},\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(es)},q\right\}\Big/F\left(\frac{\pi}{2},q\right)\right\}ds\right], \tag{26}$$

where $F(\phi, q)$ and $\Pi(\phi, n, q)$ are elliptic integrals of the first and third kinds respectively and

$$q=\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ea)}\right]}.$$

The relevant displacement and stress components in the plane of the crack can now be shown to be given by

$$W(x,0)=\int_x^a h(u)\,du, \qquad 0\leqslant x\leqslant a$$

$$=\int_x^c g(v^2)\cosh(ev)\,dv, \qquad b\leqslant x\leqslant c \tag{27}$$

and

$$[\sigma_{yz}(x,0)]_{a<x<b}=\frac{2\mu s}{\pi}\left[\int_0^a \frac{eh(u)\tanh(eu)\,du}{\tanh^2(ex)-\tanh^2(eu)}-\int_b^c \frac{eg(v^2)\tanh(ev)\mathrm{sech}^2(ev)}{\tanh^2(ev)-\tanh^2(ex)}\,dv\right]\mathrm{sech}^2(ex)$$

$$[\sigma_{yz}(x,0)]_{x>c}=\frac{2\mu s}{\pi}\left[\int_0^a \frac{eh(u)\tanh(eu)\,du}{\tanh^2(ex)-\tanh^2(eu)}+\int_b^c \frac{eg(v^2)\tanh(ev)\mathrm{sech}^2(ev)}{\tanh^2(ex)-\tanh^2(ev)}\,dv\right]\mathrm{sech}^2(ex). \tag{28}$$

Now insertion of the values of $h(u)$ and $g(v^2)$ as given by eqs (25) and (24) in the expressions (28) yields, after some algebraic manipulations,

$$
\begin{aligned}
[\sigma_{yz}(x,0)]_{a<x<b} = \frac{2\mu e W_0}{\pi h s}\Bigg[& -\sqrt{\left[\frac{\tanh^2(eb)-\tanh^2(ea)}{\tanh^2(ec)-\tanh^2(ea)}\right]}\frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}} \\
& \times\left\{\int_0^a F_2(u,x)\,du + \int_b^c F_2(v,x)\,dv\right\} - \frac{2e[\tanh^2(ec)-\tanh^2(eb)]}{\pi} \\
& \times\left\{\int_0^a F_2(u',x)\,du' \int_0^a F_4(c,u)\times F_3(0,x,u)\,du\right. \\
& \left. + \int_b^c F_2(v,x)\,dv \int_0^a F_4(c,u)F_3(v,x,u)\,du\right\} \\
& + \frac{\mu s h}{e W_0} C_1\left\{\frac{\pi}{2}\frac{1-\tanh(ex)/\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}{\sqrt{\{[\tanh^2(eb)-\tanh^2(ea)][\tanh^2(ec)-\tanh^2(ea)]\}}}\right. \\
& \left. + e\int_0^a F_4(c,u)F_5(u,x)\,du\right\} + \frac{e[\tanh^2(eb)-\tanh^2(ea)]}{\pi} \\
& \times\left\{\int_b^c F_2(v',x)\,dv' \int_b^c F_4(a,v)F_6(v',x,v)\,dv + \int_0^a F_2(u,x)\,du\right. \\
& \times\int_b^c F_4(a,v)F_6(u,x,v)\,dv - \frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(eb)-\tanh^2(ea)} \\
& \left.\times\int_0^a F_1(u,x)\,du \int_0^a F_4(c,u')F_9(u,u')\,du'\right\} - \frac{\mu s h}{e W_0}\frac{C_1}{X_1} \\
& \times\left\{\frac{\pi}{2}\frac{\tanh(ec)}{\sqrt{[\tanh^2(ec)-\tanh^2(ea)]}} + e\tanh^2(ea)\int_b^c F_7(x,v)\,dv\right\}\Bigg]\operatorname{sech}^2(ex)
\end{aligned}
$$

and

$$
\begin{aligned}
[\sigma_{yz}(x,0)]_{x>c} = \frac{2W_0 e\mu}{\pi h s}\Bigg[& -\sqrt{\left[\frac{\tanh^2(eb)-\tanh^2(ea)}{\tanh^2(ec)-\tanh^2(ea)}\right]}\frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}} \\
& \times\left\{\int_0^a F_2(u,x)\,du + \int_b^c F_2(v,x)\,dv\right\} - \frac{2e[\tanh^2(ec)-\tanh^2(eb)]}{\pi} \\
& \times\left\{\int_0^a F_2(u',x)\,du' \int_0^a F_4(c,u)F_3(0,x,u)\,du\right. \\
& \left. + \int_b^c F_2(v,x)\,dv \int_0^a F_4(c,u)F_3(v,x,u)\,du\right\} + \frac{\mu s h}{e W_0} C_1 \\
& \times\left\{\frac{\pi}{2}\frac{1-\tanh(ex)/\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}{\sqrt{\{[\tanh^2(ec)-\tanh^2(ea)][\tanh^2(eb)-\tanh^2(ea)]\}}} + e\right. \\
& \left.\times\int_0^a F_4(c,u)F_5(u,x)\,du\right\} - \frac{e[\tanh^2(eb)-\tanh^2(ea)]}{\pi} \\
& \times\left\{\int_b^c F_2(v',x)\,dv' \int_b^c F_4(a,v)F_8(v',v,x)\,dv + \int_0^a F_2(u,x)\,du\right. \\
& \times\int_b^c F_4(a,v)F_8(u,v,x)\,dv + \frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(eb)-\tanh^2(ea)} \\
& \left.\times\int_0^a F_1(u,x)\,du \int_0^a F_4(c,u')F_9(u,u')\,du'\right\}
\end{aligned}
$$

$$+\frac{\mu s h}{e W_0}\frac{C_1}{X_1}\left\{\frac{\pi}{2}\frac{\tanh(ec)}{\sqrt{[\tanh^2(ec)-\tanh^2(ea)]}}+e\tanh^2(ea)\int_b^c F_7(x,v)\,dv\right\}$$

$$-\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ea)}\right]}\frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ec)]}}$$

$$\times\left\{\int_0^a F_2(u,x)\,du+\int_b^c F_2(v,x)\,dv\right\}\Bigg]\operatorname{sech}^2(ex), \tag{29}$$

where

$$F_1(u,x)=\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eu)}{\tanh^2(eb)-\tanh^2(eu)}\right]}\frac{\tanh(eu)}{\tanh^2(ex)-\tanh^2(eu)}$$

$$F_2(v,x)=\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ev)}{\tanh^2(ev)-\tanh^2(eb)}\right]}\frac{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}{\tanh^2(ev)-\tanh^2(ex)}$$

$$F_3(v,x,u)=\frac{\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ea)]}}\tan^{-1}\left\{\frac{\tanh(eu)}{\tanh(ex)}\sqrt{\left[\frac{\tanh^2(ex)-\tanh^2(ea)}{\tanh^2(ea)-\tanh^2(eu)}\right]}\right\}$$

$$-\frac{\tanh(ev)}{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}\tan^{-1}\left\{\frac{\tanh(eu)}{\tanh(ev)}\sqrt{\left[\frac{\tanh^2(ev)-\tanh^2(ea)}{\tanh^2(ea)-\tanh^2(eu)}\right]}\right\}$$

$$F_4(\omega,u)=\frac{\operatorname{sech}^2(eu)\tanh(eu)}{\sqrt{\{[\tanh^2(e\omega)-\tanh^2(eu)]^3[\tanh^2(eb)-\tanh^2(eu)]\}}}$$

$$F_5(u,x)=[2\tanh^2(eu)-\tanh^2(ec)-\tanh^2(eb)]\left\{\sin^{-1}\left(\frac{\tanh(eu)}{\tanh(ea)}\right)-F_3(0,x,u)\right\}$$

$$F_6(u,x,v)=\frac{\tanh(ex)}{\sqrt{[\tanh^2(ec)-\tanh^2(ex)]}}$$

$$\times\log\left|\frac{\tanh(ex)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}+\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(ex)]}}{\tanh(ex)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}-\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(ex)]}}\right|$$

$$-\frac{\tanh(eu)}{\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}}$$

$$\times\log\left|\frac{\tanh(eu)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}+\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}}{\tanh(eu)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}-\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}}\right|$$

$$F_7(x,v)=\tan^{-1}\left(\frac{\sqrt{[\tanh^2(ec)-\tanh^2(ex)]}}{\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}}\frac{\sqrt{[\tanh^2(ev)-\tanh^2(eb)]}}{\sqrt{[\tanh^2(eb)-\tanh^2(ex)]}}\right)$$

$$\times\frac{\operatorname{sech}^2(ev)}{\sqrt{\{[\tanh^2(ev)-\tanh^2(ea)]^3\}}}$$

$$F_8(u,v,x)=-\frac{2\tanh(ex)}{\sqrt{[\tanh^2(ex)-\tanh^2(ec)]}}\tan^{-1}\left\{\frac{\tanh(ev)}{\tanh(ex)}\sqrt{\left[\frac{\tanh^2(ex)-\tanh^2(ec)}{\tanh^2(ec)-\tanh^2(ev)}\right]}\right\}$$

$$+\frac{\tanh(eu)}{\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}}$$

$$\times\log\left|\frac{\tanh(eu)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}+\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}}{\tanh(eu)\sqrt{[\tanh^2(ec)-\tanh^2(ev)]}-\tanh(ev)\sqrt{[\tanh^2(ec)-\tanh^2(eu)]}}\right|$$

$$F_9(u,u')=\log\left|\frac{\tanh(eu)\sqrt{[\tanh^2(ea)-\tanh^2(eu')]}+\tanh(eu')\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}}{\tanh(eu)\sqrt{[\tanh^2(ea)-\tanh^2(eu')]}-\tanh(eu')\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}}\right|$$

and

$$X_1=\sqrt{\{[\tanh^2(eb)-\tanh^2(ex)][\tanh^2(ec)-\tanh^2(ex)]\}}. \tag{30}$$

The dynamic stress intensity factors are defined by

$$N_a = \lim_{x\to a^+} \sqrt{[2(x-a)]}[\sigma_{yz}(x,0)]_{a<x<b}$$

$$N_b = \lim_{x\to b^-} \sqrt{[2(b-x)]}[\sigma_{yz}(x,0)]_{a<x<b}$$

$$N_c = \lim_{x\to c^+} \sqrt{[2(x-c)]}[\sigma_{yz}(x,0)]_{x>c}. \tag{31}$$

Substitution of the results given by eqs (29) in expressions (31) yields

$$N_a = \sqrt{\left[\frac{\tanh(ea)}{e}\right]}\left[-\sqrt{\left[\frac{\tanh^2(eb)-\tanh^2(ea)}{\tanh^2(ec)-\tanh^2(ea)}\right]}\frac{2W_0 e}{\pi h}\left\{\int_0^a F_2(u,a)\,du + \int_b^c F_2(v,a)\,dv\right\} - \frac{\mu s C_1}{\sqrt{\{[\tanh^2(eb)-\tanh^2(ea)][\tanh^2(ec)-\tanh^2(ea)]\}}}\right]\operatorname{sech}(ea)$$

$$N_b = -\frac{\mu s C_1}{\sqrt{\{[\tanh^2(eb)-\tanh^2(ea)][\tanh^2(ec)-\tanh^2(eb)]\}}}\sqrt{\left[\frac{\tanh(eb)}{e}\right]}\operatorname{sech}(eb)$$

$$N_c = \sqrt{\left[\frac{\tanh(ec)}{e}\right]}\left[-\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ea)}\right]}\frac{2W_0 e}{\pi h}\left\{\int_0^a F_2(u,c)\,du + \int_b^c F_2(v,c)\,dv\right\} + \frac{\mu s C_1}{\sqrt{\{[\tanh^2(ec)-\tanh^2(ea)][\tanh^2(ec)-\tanh^2(eb)]\}}}\right]\sec(ec). \tag{32a–c}$$

Again insertion of the values of $h(u)$ and $g(v^2)$, given by eqs (24) and (25), in the expressions for displacements given by eqs (27) yields

$$[W(x,0)]_{0<x<a} = -\frac{W_0}{h\mu\pi s}\left[\frac{2[\tanh^2(eb)-\tanh^2(ea)]}{\sqrt{[\tanh^2(ec)-\tanh^2(ea)]}}\left\{\int_b^c \Pi\left\{\lambda, \frac{\tanh^2(ev)-\tanh^2(eb)}{\tanh^2(ev)-\tanh^2(ea)}, q\right\}\right.\right.$$
$$\times\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ev)}{\tanh(ev)-\tanh^2(eb)}\right]}\frac{dv}{\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}}$$
$$-\int_0^a \Pi\left\{\lambda, \frac{\tanh^2(eb)-\tanh^2(eu)}{\tanh^2(ea)-\tanh^2(eu)}, q\right\}\sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eu)}{\tanh^2(eb)-\tanh^2(eu)}\right]}$$
$$\left.\left.\times\frac{du}{\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}}\right\}\right] - \frac{C_1 F(\lambda,q)}{e\sqrt{[\tanh^2(ec)-\tanh^2(ea)]}}$$

and

$$[W(x,0)]_{b<x<c} = \left[\frac{2W_0}{h\mu\pi s}\left(\int_b^c \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(ev)}{\tanh^2(ev)-\tanh^2(eb)}\right]}\sqrt{[\tanh^2(ev)-\tanh^2(ea)]}\right.\right.$$
$$\times\left\{F(\lambda',q) + \frac{\tanh^2(ev)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ev)}\Pi\left\{\lambda', \frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(ev)}, q\right\}\right\}dv$$
$$+\int_0^a \sqrt{\left[\frac{\tanh^2(ec)-\tanh^2(eu)}{\tanh^2(eb)-\tanh^2(eu)}\right]}\sqrt{[\tanh^2(ea)-\tanh^2(eu)]}$$
$$\left.\times\left\{F(\lambda,q) - \frac{\tanh^2(eb)-\tanh^2(eu)}{\tanh^2(ec)-\tanh^2(eu)}\Pi\left\{\lambda', \frac{\tanh^2(ec)-\tanh^2(eb)}{\tanh^2(ec)-\tanh^2(eu)}, q\right\}\right\}du\right)$$
$$\left.+\frac{C_1}{e}F(\lambda',q)\right]\frac{1}{\sqrt{[\tanh^2(ec)-\tanh^2(ea)]}}, \tag{33a, b}$$

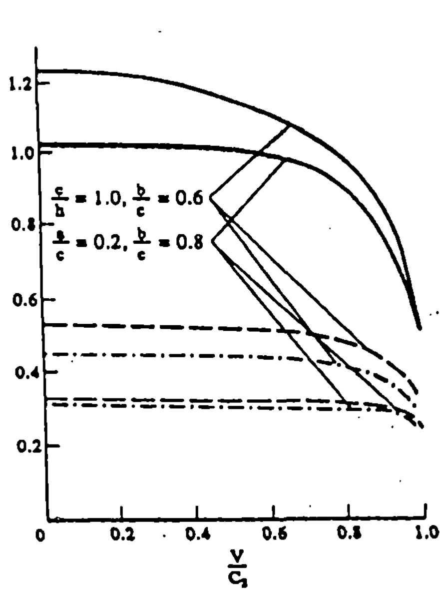

Fig. 2. Variations of stress intensity factors with V/C_2: (——) $hN_a/\mu W_0\sqrt{a}$, (- - -) $hN_b/\mu W_0\sqrt{b}$; (-·-·-) $hN_c/\mu W_0\sqrt{c}$.

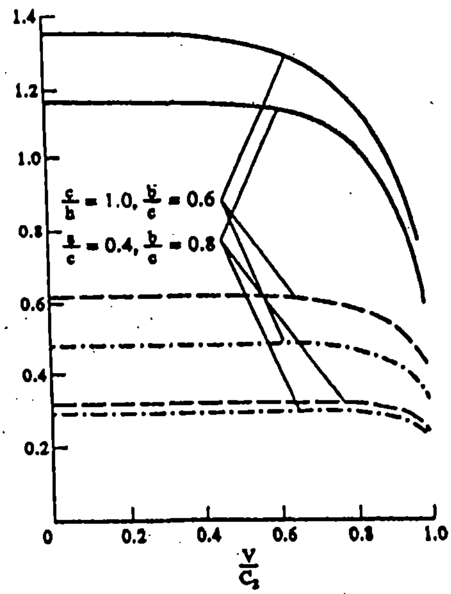

Fig. 3. Variations of stress intensity factors with V/C_2: (——) $hN_a/\mu W_0\sqrt{a}$; (- - -) $hN_b/\mu W_0\sqrt{b}$; (-·-·-) $hN_c/\mu W_0\sqrt{c}$.

where

$$\sin\lambda = \sqrt{\left[\frac{\tanh^2(ea) - \tanh^2(ex)}{\tanh^2(eb) - \tanh^2(ex)}\right]}, \quad \sin\lambda' = \sqrt{\left[\frac{\tanh^2(ec) - \tanh^2(ex)}{\tanh^2(ec) - \tanh^2(eb)}\right]}$$

and $F(\phi, q), \Pi(\phi, n, q)$

and q have been defined earlier.

On putting $b = c$ and simplifying, it may be noted that the results (33a) and (32a) become those given by eqs (3.18) and (3.21) of Singh *et al.* [2] and for $a = 0$ the results given by (32b), (32c) and (33b) coincide with those given by eqs (4.21), (4.22) and (4.17) of Das and Gosh [5].

4. NUMERICAL RESULTS AND DISCUSSION

Numerical results for stress intensity factors at the tips of the cracks for different values of crack speed, crack length and the separating distance between the cracks are presented in this

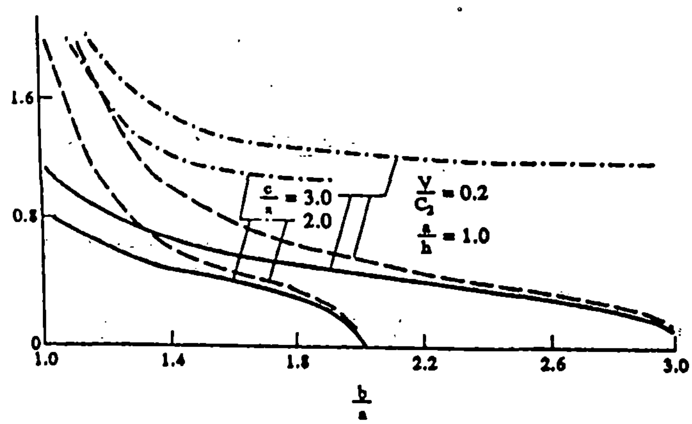

Fig. 4. Stress intensity factors vs b/a: (-·-·-) $hN_a/\mu W_0\sqrt{a}$; (- - -) $hN_b/\mu W_0\sqrt{b}$; (——) $hN_c/\mu W_0\sqrt{c}$.

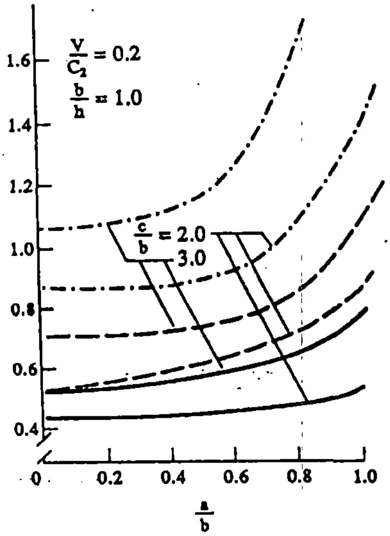

Fig. 5. Stress intensity factors vs a/b: (–·–·–) $hN_a/\mu W_0\sqrt{a}$; (– – –) $hN_b/\mu W_0\sqrt{b}$; (——) $hN_c/\mu W_0\sqrt{c}$.

section. The crack length dependence of the stress intensity factors and its variations with V/C_2 are shown in Figs 2–5. It is shown in Figs 2 and 3 that stress intensity factors at the edges of the cracks decrease with an increase in the values of V/C_2 and have a prominent variation when $V/C_2 \to 1$. Variations of stress intensity factors at the edge $x = a$ become more prominent than those at the tips $x = b$ and $x = c$ when the length of the inner crack increases.

Variations of stress intensity factors at the edges of the cracks with a/b for different values of c/b and those with b/a for different values of c/a are plotted in Figs 4 and 5, respectively. It is found that when the separating distance between the inner crack and outer pair of cracks decreases the stress intensity factors at the tips $x = a$ and $x = b$ become more prominent than that at the edge $x = c$.

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INTERACTION OF ELASTIC WAVES WITH TWO COPLANAR GRIFFITH CRACKS IN AN ORTHOTROPIC MEDIUM

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Abstract—The problem of diffraction of normally incident elastic waves by two coplanar Griffith cracks situated in an infinite orthotropic medium has been analyzed. Fourier and Hilbert transforms have been used to solve this mixed boundary value problem. Approximate analytical results for stress intensity factors and crack opening displacement have been derived when the wave lengths are large compared to the crack length. Numerical values of stress intensity factors and the crack opening displacement for several orthotropic materials have been calculated and plotted graphically to show the effect of material orthotropy.

INTRODUCTION

DYNAMIC fracture problems involving anisotropic materials weakened by crack-like imperfections have drawn much attention by investigators because of the increased usage of macroscopically anisotropic construction materials such as fibre reinforced composites. The different possible location of cracks with respect to the planes of material symmetry introduce great modifications in the strain and stress distribution. The problems are also of considerable interest in seismology and exploration geophysics. The problems involving single or two Griffith cracks in isotropic elastic medium have been studied by many authors [1–6]. Mathematical difficulties encountered in solving the governing equations of the anisotropic elasticity theory are responsible for the availability of few results only for special classes of materials. Kassir and Bandyopadhyay [7] have studied the elastodynamic response of an infinite orthotropic solid containing a crack under the action of impact loading and the elastodynamic problem of a finite Griffith crack in an orthotropic strip under normal impact was investigated by Shindo [8]. Problem involving a moving Griffith crack in an orthotropic strip has also been studied by De and Patra [9]. Recently, Kundu and Bostrom [10] solved the problem of scattering of elastic waves by a circular crack situated in a transversely isotropic solid.

In our paper, the diffraction of normally incident time harmonic elastic waves by two coplanar Griffith cracks in an infinite orthotropic medium has been investigated. The faces of each of the cracks are assumed to be separated by a small distance so that, during small deformations of the solid, the crack faces do not come into contact. The resulting mixed boundary value problem is reduced to the solution of a triple integral equation which has further been reduced to the solution of an integro-differential equation. Iterative solution valid for low frequency has been obtained. Analytical formulae for stress intensity factor and crack opening displacement have been derived. Making the distance between two crack zero the corresponding results for single crack have been presented. Finally, choosing the engineering elastic constants of the orthotropic material suitably the results for isotropic material have been deduced and compared with the results obtained by Jain and Kanwal [5]. To display the influence of the material orthotropy numerical values of stress intensity factors and crack opening displacement have been plotted for several orthotropic materials.

STATEMENT AND FORMULATION OF THE PROBLEM

Consider the plane problem of diffraction of normally incident longitudinal wave by two symmetrical co-planar Griffith cracks situated in an infinite orthotropic elastic medium. The cracks are assumed to occupy the region $b \leqslant |X| \leqslant a$, $Y = 0$, $|Z| < \infty$. It is convenient to normalize all lengths with respect to "a" and so setting $X/a = x$, $Y/a = y$, $Z/a = z$, $b/a = c$, the new position of the cracks are defined by $c \leqslant |x| \leqslant 1$, $y = 0$, $|z| < \infty$ (Fig. 1).

Let a plane time harmonic elastic wave originating at $y = -\infty$ be incident normally on the two cracks is defined by $v_0 = \exp[i(ky - \omega t)]$ where $k = a\omega/c_s\sqrt{c_{22}}$, $c_s = (\mu_{12}/\rho)^{1/2}$ with ρ being the density of the material. In the isotropic solid, c_s, represents the velocity of the shear wave.

The non-zero stress components τ_{yy} and τ_{xy} are given by

$$\tau_{yy}/\mu_{12} = c_{12}u_{,x} + c_{22}v_{,y}$$
$$\tau_{xy}/\mu_{12} = u_{,y} + v_{,x}, \tag{1}$$

where u,v denote the component of the displacement in the x,y directions, respectively and comma denotes partial differentiation with respect to the co-ordinates or time; $c_{ij}(i, j = 1, 2)$ are nondimensional parameters related to the elastic constants by the relations

$$c_{11} = E_1/\mu_{12}(1 - \nu_{12}^2 E_2/E_1)$$
$$c_{22} = E_2/\mu_{12}(1 - \nu_{12}^2 E_2/E_1) = c_{11}E_2/E_1 \tag{2}$$
$$c_{12} = \nu_{12}E_2/\mu_{12}(1 - \nu_{12}^2 E_2/E_1) = \nu_{12}c_{22} = \nu_{21}c_{11}$$

for generalized plane stress, and by

$$c_{11} = (E_1/\Delta\mu_{12})(1 - \nu_{23}\nu_{32})$$
$$c_{22} = (E_2/\Delta\mu_{12})(1 - \nu_{13}\nu_{31})$$
$$c_{12} = E_1(\nu_{21} + \nu_{13}\nu_{32}E_2/E_1)/\Delta\mu_{12} \tag{3}$$
$$= E_2(\nu_{12} + \nu_{23}\nu_{31}E_1/E_2)/\Delta\mu_{12}$$
$$\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{21}\nu_{32}$$

for plane strain. In the above equations E_i, μ_{ij} and ν_{ij} $(i, j = 1, 2, 3)$ denote the engineering elastic constants of the material where the subscripts 1, 2, 3 correspond to the x, y, z directions which coincide with the axes of material orthotropy and the constants E_i and ν_{ij} satisfy the Maxwell's relation

$$\nu_{ij}/E_i = \nu_{ji}/E_j. \tag{4}$$

The equations of motion for orthotropic material, in terms of displacements are

$$c_{11}u_{,xx} + u_{,yy} + (1 + c_{12})v_{,xy} = \frac{a^2}{c_s^2}u_{,tt}$$
$$c_{22}v_{,yy} + v_{,xx} + (1 + c_{12})u_{,xy} = \frac{a^2}{c_s^2}v_{,tt}. \tag{5}$$

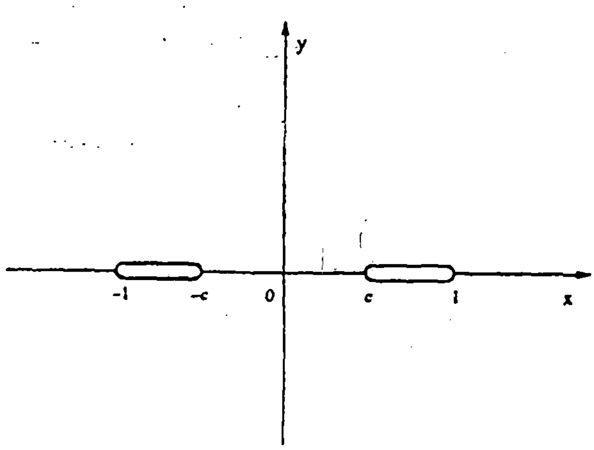

Fig. 1. Geometry of the cracks.

Therefore, substituting $u(x, y, t) = u(x, y)\exp(-i\omega t)$ and $v(x, y, t) = v(x, y)\exp(-i\omega t)$ in eq. (5) we obtain

$$c_{11}u_{,xx} + u_{,yy} + (1 + c_{12})v_{,xy} + k_s^2 u = 0 \tag{6}$$

and

$$c_{22}v_{,yy} + v_{,xx} + (1 + c_{12})u_{,xy} + k_s^2 v = 0$$

where $k_s^2 = a^2\omega^2/c_s^2$.

The boundary conditions of the problem are

$$\tau_{xy}(x, 0) = 0, \quad |x| < \infty \tag{7}$$

$$\tau_{yy}(x, 0) + \tau_{yy}^{(0)}(x, 0) = 0, \quad c \leqslant |x| \leqslant 1 \tag{8}$$

$$v(x, 0) = 0, \quad |x| < c, \quad |x| > 1. \tag{9}$$

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables would be omitted in the sequel.

The solutions of eqs (6) can be taken as

$$u(x, y) = \frac{2}{\pi}\int_0^\infty [A_1(\xi)\exp(-\gamma_1|y|) + A_2(\xi)\exp(-\gamma_2|y|)]\sin\xi x \, d\xi \tag{10}$$

$$v(x, y) = \pm\frac{2}{\pi}\int_0^\infty \frac{1}{\xi}[\alpha_1 A_1(\xi)\exp(-\gamma_1|y|) + \alpha_2 A_2(\xi)\exp(-\gamma_2|y|)]\cos\xi x \, d\xi, \quad y \gtrless 0 \tag{11}$$

where

$$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1 + c_{12})\gamma_i}, \quad i = 1, 2 \tag{12}$$

and $A_i(\xi)(i = 1, 2)$ are the unknown functions to be determined, γ_1^2, γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1 + c_{22})k_s^2\}\gamma^2 + (c_{11}\xi^2 - k_s^2)(\xi^2 - k_s^2) = 0. \tag{13}$$

From the boundary condition (7) it is found that

$$A_2(\xi) = -\beta A_1(\xi) \tag{14}$$

where

$$\beta = \frac{\gamma_1 + \alpha_1}{\gamma_2 + \alpha_2}. \tag{15}$$

Employing eq. (14) the expressions for displacements and stresses reduce to

$$u(x, y) = \frac{2}{\pi}\int_0^\infty [\exp(-\gamma_1|y|) - \beta\exp(-\gamma_2|y|)]A_1(\xi)\sin\xi x \, d\xi, \tag{16}$$

$$v(x, y) = \pm\frac{2}{\pi}\int_0^\infty \frac{1}{\xi}[\alpha_1\exp(-\gamma_1|y|) - \beta\alpha_2\exp(-\gamma_2|y|)]A_1(\xi)\cos\xi x \, d\xi, \quad y \gtrless 0 \tag{17}$$

$$\tau_{xy}/\mu_{12} = \mp\frac{2}{\pi}\int_0^\infty (\gamma_1 + \alpha_1)[\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)]A_1(\xi)\sin\xi x \, d\xi, \quad y \gtrless 0 \tag{18}$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty \left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \right.$$
$$\left. - \beta\left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\right]A_1(\xi)\cos\xi x \, d\xi. \tag{19}$$

We further substitute

$$A(\xi) = \frac{\alpha_1 - \beta\alpha_2}{\xi}A_1(\xi)$$

so that the boundary conditions (9) and (8) yield the following integral equations in $A(\xi)$

$$\int_0^\infty A(\xi)\cos\xi x \, d\xi = 0, \quad |x| < c, \quad |x| > 1 \tag{20}$$

and

$$\int_0^\infty H(\xi)A(\xi)\cos\xi x\,\mathrm{d}\xi = -\frac{\pi p_0}{2\mu_{12}}, \quad c \leqslant |x| \leqslant 1 \tag{21}$$

where $p_0 = ik\mu_{12}c_{22}$
and

$$H(\xi) = \frac{c_{12}\xi^2 - c_{22}\alpha_1\gamma_1 - \beta(c_{12}\xi^2 - c_{22}\alpha_2\gamma_2)}{(\alpha_1 - \beta\alpha_2)}. \tag{22}$$

METHOD OF SOLUTION

In order to solve the set of integral eqs (20) and (21), assume

$$A(\xi) = \frac{1}{\xi}\int_c^1 h(t^2)\sin(\xi t)\mathrm{d}t \tag{23}$$

where $h(t^2)$ is an unknown function to be determined from the boundary conditions.

Inserting the value of $A(\xi)$ from eq. (23) in eq. (20) and using the following result [11]

$$\int_0^\infty \frac{\sin(\xi t)\cos(\xi x)}{\xi}\mathrm{d}\xi = \begin{cases} \frac{\pi}{2}, & t > x \\ 0, & t < x \end{cases}$$

it is found that the choice of $A(\xi)$ leads to the equation

$$\int_c^1 h(t^2)\mathrm{d}t = 0. \tag{24}$$

Further substitution of $A(\xi)$ from eq. (23) in eq. (21) leads to

$$\int_c^1 h(t^2)\mathrm{d}t\int_0^\infty \sin(\xi t)\cos(\xi x)\mathrm{d}\xi = q_0 -$$

$$-\frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 h(t^2)\mathrm{d}t\int_0^\infty \xi H_1(\xi)\frac{\sin(\xi t)\sin(\xi x)}{\xi^2}\mathrm{d}\xi, \quad c \leqslant |x| \leqslant 1 \tag{25}$$

where

$$q_0 = -\frac{\pi p_0}{2\theta\mu_{12}} \tag{26}$$

$$H_1(\xi) = \frac{H(\xi)}{\xi\theta} - 1 \qquad \to 0 \text{ as } \xi \to \infty \tag{27}$$

$$\theta = \frac{(c_{12}^2 + c_{12} - c_{11}c_{22})(c_{12}N_1N_2 - c_{11}) - c_{22}[c_{12}N_1^2N_2^2 + c_{11}(N_1^2 + N_1N_2 + N_2^2)]}{c_{11}(1 + c_{12})(N_1 + N_2)} \tag{28}$$

$$N_1^2 = \frac{1}{2c_{22}}[-(c_{12}^2 + 2c_{12} - c_{11}c_{22}) + \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}}]$$

$$N_2^2 = \frac{1}{2c_{22}}[-(c_{12}^2 + 2c_{12} - c_{11}c_{22}) - \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}}]. \tag{29}$$

Using the relation

$$\frac{\sin\xi x\sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_0(\xi w)J_0(\xi v)\mathrm{d}v\,\mathrm{d}w}{(x^2 - w^2)^{1/2}(t^2 - v^2)^{1/2}} \tag{30}$$

eq. (25) can be rewritten in the following form

$$\int_c^1 \frac{th(t^2)}{t^2 - x^2}\mathrm{d}t = q_0 - \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 h(t^2)\mathrm{d}t\int_0^x\int_0^t \frac{vwL(v, w)\mathrm{d}w\,\mathrm{d}v}{(x^2 - w^2)^{1/2}(t^2 - v^2)^{1/2}}, \quad c \leqslant |x| \leqslant 1 \tag{31}$$

where

$$L(v, w) = \int_0^\infty \xi H_1(\xi)J_0(\xi w)J_0(\xi v)\mathrm{d}\xi. \tag{32}$$

Applying a contour integration technique, the infinite integral in $L(v, w)$ can be converted to the following finite integrals (details given in the appendix)

$$L(v,w) = -ik_s^2\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - c_{22}\bar{\alpha}_1\bar{\gamma}_1 - \bar{\beta}(c_{12}\eta^2 - c_{22}\bar{\alpha}_2\bar{\gamma}_2)}{\theta(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)} \times J_0(k_s\eta v) H_0^{(1)}(k_s\eta w)\mathrm{d}\eta \right.$$
$$\left. - \int_{1/\sqrt{c_{11}}}^1 \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)} J_0(k_s\eta v) H_0^{(1)}(k_s\eta w)\mathrm{d}\eta\right], \quad w > v \quad (33)$$

where

$$\bar{\gamma}_1 = [\tfrac{1}{2}\{R_1 - (R_1^2 - 4\bar{R}_2)^{1/2}\}]^{1/2}$$
$$\bar{\gamma}_2 = [\tfrac{1}{2}\{R_1 + (R_1^2 - r\bar{R}_2)^{1/2}\}]^{1/2}$$
$$\hat{\gamma}_1 = [\tfrac{1}{2}\{-R_1 + (R_1^2 + 4R_2')^{1/2}\}]^{1/2}$$
$$\hat{\gamma}_2 = [\tfrac{1}{2}\{R_1 + (R_1^2 + 4R_2')^{1/2}\}]^{1/2}$$
$$R_1 = \frac{1}{c_{22}}\{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\eta^2 + (1 + c_{22})\}$$
$$\bar{R}_2 = \frac{c_{11}}{c_{22}}(1-\eta^2)\left(\frac{1}{c_{11}} - \eta^2\right) \quad (34)$$
$$R_2' = \frac{c_{11}}{c_{22}}(1-\eta^2)\left(\eta^2 - \frac{1}{c_{11}}\right)$$
$$\bar{\alpha}_i = \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_i^2}{(1 + c_{12})\bar{\gamma}_i}\,(i = 1, 2)$$
$$\hat{\alpha}_i = \frac{c_{11}\eta^2 - 1 + (-1)^i\hat{\gamma}_i^2}{(1 + c_{12})\hat{\gamma}_i}\,(i = 1, 2)$$
$$\bar{\beta} = \frac{\bar{\alpha}_1 - \bar{\gamma}_1}{\bar{\alpha}_2 - \bar{\gamma}_2}$$
$$\hat{\beta} = \frac{\hat{\alpha}_1 + \hat{\gamma}_1}{\hat{\alpha}_2 - \hat{\gamma}_2}.$$

The corresponding expression of $L(v, w)$ for $w < v$ follows from (33) by interchanging w and v.

Employing the series expansions for the Bessel function J_0 and the Hankel function $H_0^{(1)}$ in eq. (33), it is found that

$$L(v, w) = \frac{2}{\pi} P k_s^2 \log k_s + O(k_s^2) \quad (35)$$

where

$$P = \frac{1}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - c_{22}\bar{\alpha}_1\bar{\gamma}_1 - \bar{\beta}(c_{12}\eta^2 - c_{22}\bar{\alpha}_2\bar{\gamma}_2)}{(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)}\mathrm{d}\eta - \int_{1/\sqrt{c_{11}}}^1 \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)}\mathrm{d}\eta\right].$$

Now, let us expand $h(t^2)$ in the form

$$h(t^2) = h_0(t^2) + k_s^2 \log k_s h_1(t^2) + O(k_s^2). \quad (36)$$

Inserting the above expansion of $h(t^2)$ and the value of $L(v, w)$ given by eq. (35) into eq. (31) and equating the coefficients of like powers of k_s, we obtain the equations

$$\int_c^1 \frac{t h_0(t^2)}{t^2 - x^2}\mathrm{d}t = q_0, \quad c \leq |x| \leq 1 \quad (37)$$

and

$$\int_c^1 \frac{t h_1(t^2)}{t^2 - x^2}\mathrm{d}t = -\frac{2P}{\pi}\int_c^1 t h_0(t^2)\mathrm{d}t, \quad c \leq |x| \leq 1. \quad (38)$$

Using the finite Hilbert transform technique [12], the solutions of the above integral equations can be obtained as

$$h_0(t^2) = \frac{2}{\pi} q_0 \left(\frac{t^2-c^2}{1-t^2}\right)^{1/2} + \frac{D_1}{\sqrt{(1-t^2)(t^2-c^2)}} \tag{39}$$

$$h_1(t^2) = -\frac{2}{\pi} P\left[\frac{q_0(1-c^2)}{\pi} + D_1\right]\left(\frac{t^2-c^2}{1-t^2}\right)^{1/2} + \frac{D_2}{\sqrt{(1-t^2)(t^2-c^2)}} \tag{40}$$

where D_1 and D_2 are constants to be determined using the condition given by eq. (24) so that

$$\int_c^1 h_0(t^2)\,dt = 0 \tag{41}$$

and

$$\int_c^1 h_1(t^2)\,dt = 0.$$

Substitution of the values of $h_0(t^2)$ and $h_1(t^2)$ given by eqs (39) and (40) in (41), yields

$$D_1 = \frac{2}{\pi} q_0 \left[c^2 - \frac{E}{F}\right] \tag{42}$$

$$D_2 = \frac{2}{\pi^2} q_0 \left[1 + c^2 - \frac{2E}{F}\right]\left[\frac{E}{F} - c^2\right], \tag{43}$$

where $F = F[\frac{\pi}{2}, \sqrt{1-c^2}]$ and $E = E[\pi/2, \sqrt{1-c^2}]$ are the elliptic integrals of first and second kind, respectively. Substituting the value of D_1 and D_2 given by eqs (42) and (43) into eqs (39) and (40), we obtain

$$h_0(t^2) = -\frac{p_0}{\mu_{12}\theta}\,\frac{\left[t^2 - \frac{E}{F}\right]}{\sqrt{(1-t^2)(t^2-c^2)}} \tag{44}$$

$$h_1(t^2) = -\frac{P p_0}{\pi \mu_{12}\theta}\,\frac{\left[1 + c^2 - \frac{2E}{F}\right]\left[t^2 - \frac{E}{F}\right]}{\sqrt{(1-t^2)(t^2-c^2)}} \tag{45}$$

CRACK OPENING DISPLACEMENT AND STRESS INTENSITY FACTORS

The crack opening displacement and the normal stress component in the plane of the crack can be written as

$$\Delta v(x,0) = v(x,0+) - v(x,0-) = 2\int_x^1 h(t^2)\,dt, \quad c \leq x \leq 1 \tag{46}$$

and

$$\tau_{yy}(x,0) = \frac{2\mu_{12}\theta}{\pi}\int_c^1 \frac{t\,h(t^2)}{t^2-x^2}\,dt, \quad 0 < x < c \tag{47}$$

$$= -\frac{2\mu_{12}\theta}{\pi}\int_c^1 \frac{t\,h(t^2)}{x^2-t^2}\,dt, \quad x > 1. \tag{48}$$

Expressions (47) and (48) with the aid of the eqs (36), (44) and (45) yield

$$\tau_{yy}(x,0) = -p_0\left[1 + \frac{\left[x^2 - \frac{E}{F}\right]}{\sqrt{(1-x^2)(c^2-x^2)}}\right]\left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\}k_s^2 \log k_s\right] + O(k_s^2), \quad 0 < x < c \tag{49}$$

$$\tau_{yy}(x,0) = -p_0\left[1 - \frac{\left[x^2 - \frac{E}{F}\right]}{\sqrt{(x^2-1)(x^2-c^2)}}\right]\left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\}k_s^2 \log k_s\right] + O(k_s^2), \quad x > 1, \tag{50}$$

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The stress intensity factors are defined as (in physical units)

$$K_c = \lim_{x \to c-} \left[\frac{\sqrt{(c-x)}\,\tau_{yy}(x,0)}{p_0} \right]_{0<x<c} \tag{51}$$

$$K_1 = \lim_{x \to 1+} \left[\frac{\sqrt{(x-1)}\,\tau_{yy}(x,0)}{p_0} \right]_{x>1} \tag{52}$$

Substituting eqs (49) and (50) into (51) and (52) it can be shown that

$$K_c = -\frac{\left[c^2 - \frac{E}{F}\right]}{\sqrt{2c(1-c^2)}} \left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\} k_s^2 \log k_s \right] + O(k_s^2) \tag{53}$$

$$K_1 = \frac{\left[1 - \frac{E}{F}\right]}{\sqrt{2(1-c^2)}} \left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\} k_s^2 \log k_s \right] + O(k_s^2) \tag{54}$$

Further substituting eqs (36), (44) and (45) in the expression given by eq. (46), the crack opening displacement is obtained as

$$\Delta v(x,0) = \frac{2p_0}{\mu_{12}\theta} \left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\} k_s^2 \log k_s \right] \left[\frac{E\left(\frac{\pi}{2}, q\right)}{F\left(\frac{\pi}{2}, q\right)} F(\lambda, q) - E(\lambda, q) \right] + O(k_s^2), \quad c \leq x \leq 1 \tag{55}$$

where

$$\sin\lambda = \sqrt{\frac{1-x^2}{1-c^2}} \text{ and } q = \sqrt{(1-c^2)}.$$

Letting $c \to 0$ in the expression for stress intensity factor and crack opening displacement, the results for a single crack occupying the region $|x| \leq 1, y = 0, |z| < \infty$ are found to be

$$K_1 = \frac{1}{\sqrt{2}} \left[1 - \frac{P}{\pi} k_s^2 \log k_s \right] + O(k_s^2) \tag{56}$$

$$\Delta v(x,0) = -\frac{2p_0}{\mu_{12}\theta} \sqrt{(1-x^2)} \left[1 - \frac{P}{\pi} k_s^2 \log k_s \right] + O(k_s^2), \quad 0 \leq x \leq 1. \tag{57}$$

For isotropic medium, putting

$$c_{11} = c_{22} = \frac{\lambda + 2\mu}{\mu}, \quad \mu_{12} = \mu, \quad c_{12} = c_{11} - 2 = \frac{\lambda}{\mu}$$

so that

$$\alpha_1 = \gamma_1, \quad \alpha_1 = \xi^2/\gamma_2, \quad k_s = m_2, \quad k_s/\sqrt{c_{11}} = m_1, \quad \tau^2 = \frac{1}{c_{11}}$$

$$N_1 = 1 = N_2, \quad \theta = -2(1-\tau^2) \text{ and } P = \frac{\pi}{2} c_1,$$

where

$$c_1 = \frac{3\tau^4 - 4\tau^2 - 3}{4(1-\tau^2)}, \quad \gamma_i = (\xi^2 - m_i^2)^{1/2} \text{ and } m_i = \frac{a\omega}{c_i} \, (i = 1, 2)$$

the expressions for displacement and stress are found to be

$$v(x, \pm 0) = \mp \frac{p_0}{2\mu(1-\tau^2)} \left[1 - \frac{c_1}{2}\left\{1 + c^2 - \frac{2E}{F}\right\} m_2^2 \log m_2 \right] \left[\frac{E\left(\frac{\pi}{2}, q\right)}{F\left(\frac{\pi}{2}, q\right)} F(\lambda, q) - E(\lambda, q) \right] + O(m_2^2), \quad c \leq x \leq 1 = 0, \quad |x| < c, \quad |x| > 1$$

$0 \quad |x| < c \quad |x| > 1$

and

$$\tau_{yy}(x,0) = -p_0\left[1 + \frac{\left[x^2 - \frac{E}{F}\right]}{\sqrt{(1-x^2)(c^2-x^2)}}\right]\left[1 - \frac{c_1}{2}\left\{1 + c^2 - \frac{2E}{F}\right\}m_2^{\frac{3}{2}}\log m_2\right] + O(m_2^{\frac{3}{2}})\quad 0 < x < c$$

$$= -p_0,\quad c \leqslant |x| \leqslant 1$$

$$= -p_0\left[1 - \frac{\left[x^2 - \frac{E}{F}\right]}{\sqrt{(x^2-1)(x^2-c^2)}}\right]\left[1 - \frac{c_1}{2}\left\{1 + c^2 - \frac{2E}{F}\right\}m_2^{\frac{3}{2}}\log m_2\right] + O(m_2^{\frac{3}{2}}),\quad |x| > 1.$$

Now, the crack opening displacement and stress intensity factors are found to be

$$\Delta v(x,0) = -\frac{p_0}{\mu(1-\tau^2)}\left[1 - \frac{c_1}{2}\left\{1 + c^2 - \frac{2E}{F}\right\}m_2^{\frac{3}{2}}\log m_2\right]$$
$$\times\left[\frac{E\left(\frac{\pi}{2}, q\right)}{F\left(\frac{\pi}{2}, q\right)}F(\lambda, q) - E(\lambda, q)\right] + O(m_2^{\frac{3}{2}}),\quad c \leqslant x \leqslant 1$$

and

$$K_c = -\frac{\left[c^2 - \frac{E}{F}\right]}{\sqrt{2c(1-c^2)}}\left[1 - \frac{c_1}{2}\left\{1 + c^2 - \frac{2E}{F}\right\}m_2^{\frac{3}{2}}\log m_2\right] + O(m_2^{\frac{3}{2}})$$

$$K_1 = \frac{\left[1 - \frac{E}{F}\right]}{\sqrt{2(1-c^2)}}\left[1 - \frac{c_1}{2}\left\{1 + c^2 - \frac{2E}{F}\right\}m_2^{\frac{3}{2}}\log m_2\right] + O(m_2^{\frac{3}{2}})$$

which coincide with the results obtained by Jain and Kanwal [5] up to the order of $m_2^{\frac{3}{2}}\log m_2$ in the isotropic case.

When $c \to 0$, we recover the stress intensity factor and crack opening displacement for a single crack

$$K_1 = \frac{1}{\sqrt{2}}\left[1 - \frac{c_1}{2}m_2^{\frac{3}{2}}\log m_2\right] + O(m_2^{\frac{3}{2}})$$

$$\Delta v(x,0) = \frac{p_0}{\mu(1-\tau^2)}\sqrt{(1-x^2)}\left[1 - \frac{c_1}{2}m_2^{\frac{3}{2}}\log m_2\right] + O(m_2^{\frac{3}{2}}),\quad 0 \leqslant x \leqslant 1$$

which agrees with the result of Mal [2].

NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) K_c and K_1 given by (53) and (54) at the inner and outer tips of the cracks and crack opening displacement (COD) given by (55) have been plotted against dimensionless frequency k, and distance, respectively for three different types of orthotropic materials whose engineering constants have been listed in Table 1.

From Fig. 2 it is found that SIF K_c at the inner tip of the crack increases at a slow rate with the increase in the value of frequency k, $(0.1 \leqslant k, \leqslant 0.6)$. On the other hand the rate of increase of

Table 1. Engineering elastic constants

	E_1(Pa)	E_2(Pa)	μ_{12}(Pa)	ν_{12}
Type I	Modulite II graphite–epoxy composite:			
	15.3×10^9	158.0×10^9	5.52×10^9	0.033
Type II	E-Type glass–epoxy composite:			
	9.79×10^9	42.3×10^9	3.66×10^9	0.063
Type III	Stainless steel–aluminium composite:			
	79.76×10^9	85.91×10^9	30.02×10^9	0.31

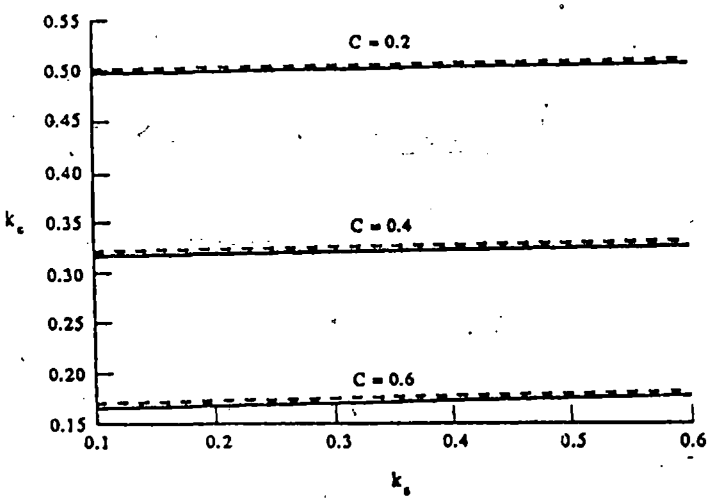

Fig. 2. Stress intensity factor K_c vs frequency k_s for generalized plane stress. (—, Type I; ----, Type II).

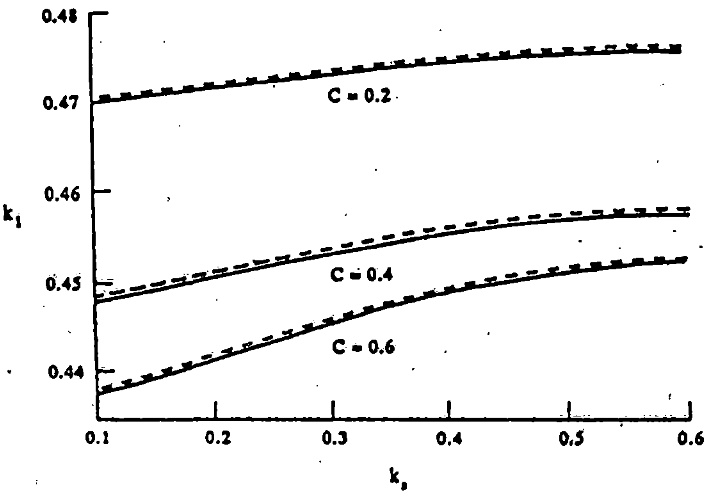

Fig. 3. Stress intensity factor K_1 vs frequency k_s for generalized plane stress. (—, Type I; ----, Type II).

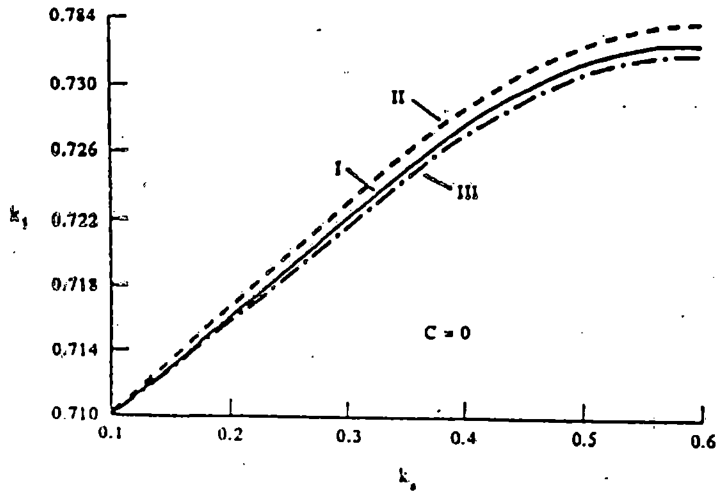

Fig. 4. Stress intensity factor K_1 vs frequency k_s for generalized plane stress. (Single crack, $c = 0$).

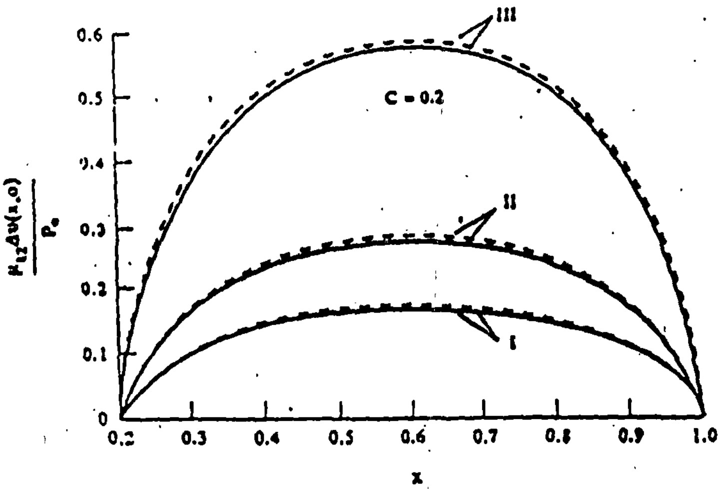

Fig. 5. Crack opening displacement (COD) vs distance ($c = 0.2$) for generalized plane stress. (—, $k_s = 0.2$; -----, $k_s = 0.6$).

the SIF K_i (Fig. 3) with frequency k_s at the outer tip of the crack is found to be higher than that of K_e.

In both the cases the value of SIF is higher for small values of c, i.e. for greater crack length SIF is higher. But it is interesting to note that for different materials the variation of SIFs in both the cases are not significant. In the case of single crack ($c = 0$) the variation of SIF with material properties has been shown in Fig. 4.

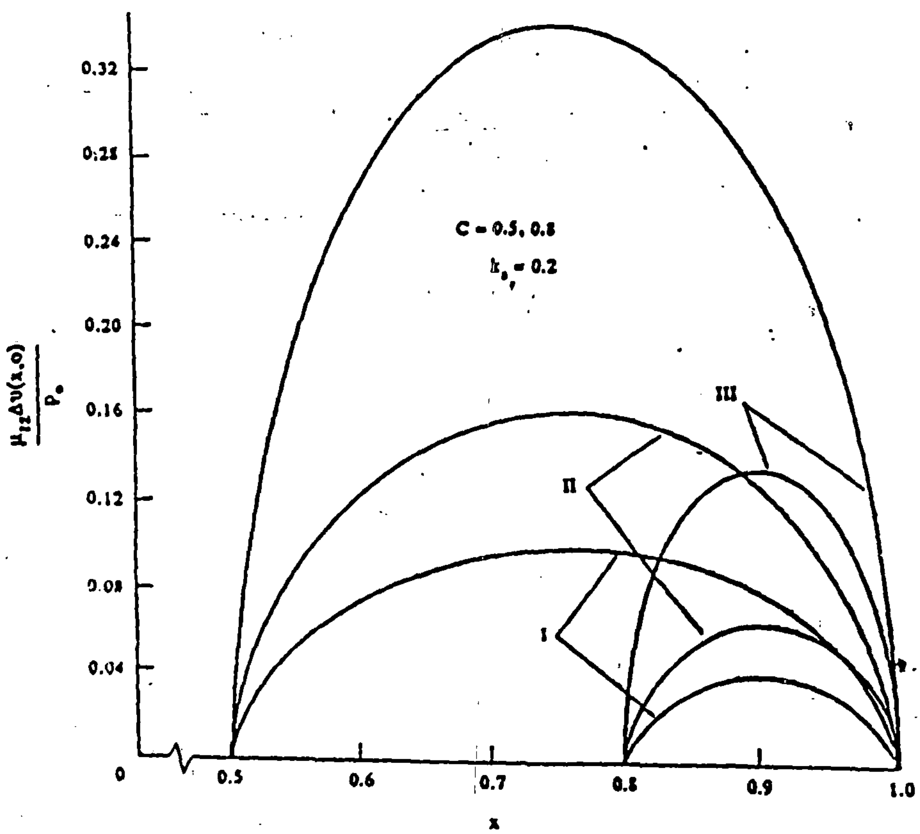

Fig. 6 Crack opening displacement (COD) vs distance ($c = 0.5$ and $c = 0.8$) for generalized plane stress.

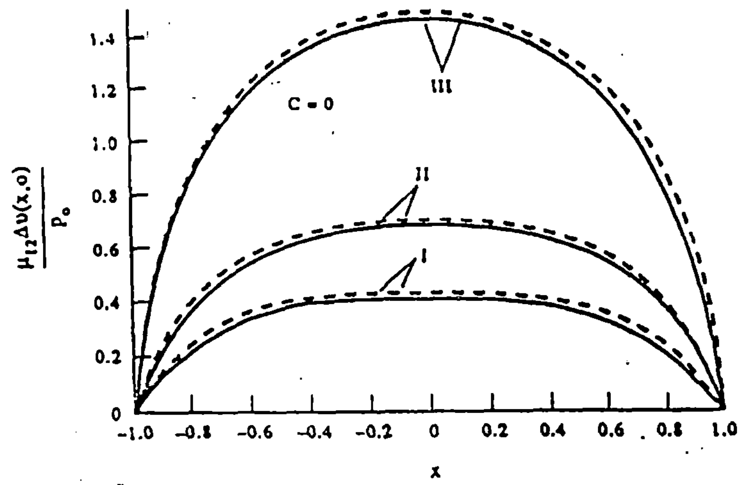

Fig. 7. Crack opening displacement (COD) vs distance (single crack, $c = 0$) for generalized plane stress. (—, $k_t = 0.2$; ----, $k_t = 0.6$).

The COD has been plotted for different crack length. In each case COD increases gradually from zero, attains maximum value and then decreases to zero. It is found that with the increase in the values of c (i.e. for small crack length) the values of COD decreases (Figs 5 and 6). For a fixed material the variation of COD with frequency is found to be insignificant, but it is noticed that for smaller values of c (Figs 5 and 7) the variation of COD with frequency is palpable. $c = 0$ (Fig. 7) correspond to the case of single crack.

In all the cases where different values of c has been considered the variation of COD is found to be prominent for different orthotropic materials.

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APPENDIX

Evaluation of $L(v, w)$

The integral $L(v, w)$, given by eq. (32) is

$$L(v,w) = \int_0^\infty M(\xi,\gamma_1,\gamma_2) J_0(\xi w) J_0(\xi v)\,d\xi \tag{A1}$$

where

$$M(\xi,\gamma_1,\gamma_2) = \xi H_1(\xi) = \frac{c_{12}\xi^2 - c_{22}\alpha_1\gamma_1 - \beta(c_{12}\xi^2 - c_{22}\alpha_2\gamma_2)}{\eta(\alpha_1 - \beta\alpha_2)} - \xi \tag{A2}$$

$$\gamma_1 = [\tfrac{1}{2}\{-B_1 + (B_1^2 - 4B_2)^{1/2}\}]^{1/2}$$
$$\gamma_2 = [\tfrac{1}{2}\{-B_1 - (B_1^2 - 4B_2)^{1/2}\}]^{1/2}$$
$$B_1 = \frac{1}{c_{22}}\{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1 + c_{22})k_s^2\} \tag{A3}$$
$$B_2 = \frac{1}{c_{22}}(\xi^2 - k_s^2)(c_{11}\xi^2 - k_s^2).$$

To evaluate the integral (A1) we consider two contour integrals

$$I_1 = \int_{\Gamma_1} M(\xi, \gamma_1, \gamma_2)J_0(\xi v)H_0^{(1)}(\xi w)\mathrm{d}\xi, \quad w > v$$
$$I_2 = \int_{\Gamma_2} M(\xi, \gamma_1, \gamma_2)J_0(\xi v)H_0^{(2)}(\xi w)\mathrm{d}\xi, \quad w > v, \tag{A4}$$

where Γ_1 and Γ_2 are the closed contours defined in Fig. 8, and $H_0^{(1)}, H_0^{(2)}$ are the zero order Hankel functions of the first and second kind, respectively.

Assuming the relation

$$\left\{\frac{(c_{12}^2 + 2c_{12} - c_{11}c_{22})(1 + c_{22})}{c_{22}^2} + \frac{2(1 + c_{11})}{c_{22}}\right\}^2 - \left\{\frac{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2}{c_{22}^2} - \frac{4c_{11}}{c_{22}}\right\} \times \left\{\frac{(1 + c_{22})^2}{c_{22}^2} - \frac{4}{c_{22}}\right\} < 0 \tag{A5}$$

it is noted the branch points $\xi = \lambda_i (i = 1-4)$ corresponding to the roots of the equation $B_1^2 - 4B_2 = 0$ are always complex. Now, the branch points corresponding to the roots of the equations

$$-B_1 + (B_1^2 - 4B_2)^{1/2} = 0 \text{ and } -B_1 - (B_1^2 - 4B_2)^{1/2} = 0$$

are $\xi = \pm k_s$ and $\xi = \pm k_s/\sqrt{c_{11}}$, respectively where it has been assumed that

$$(c_{11}c_{22} - c_{12}^2 - 2c_{12}) > (1 + c_{22}) \tag{A6}$$

and

$$c_{12}^2 + 2c_{12} + c_{11} > 0.$$

Therefore under the above conditions, $\xi = \pm k_s/\sqrt{c_{11}}$ and $\xi = \pm k_s$ are the branch points of γ_1 and γ_2, respectively. Equations (A5) and (A6) are true for most of the orthotropic materials. The integrals in eq. (A4) can be shown to be zero

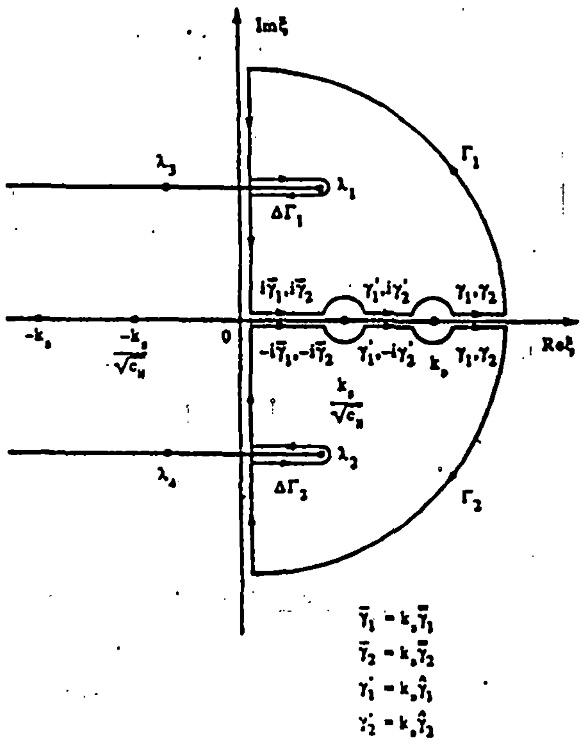

Fig. 8. Contours of integration for integral in eq. (A1).

on the contours $\Delta\Gamma_1$ and $\Delta\Gamma_2$ (Fig. 8) around the branch cuts from λ_1 and λ_2. Thus integrating along the contours Γ_1 and Γ_2 the integral $L(v, w)$ for $w > v$ can be finally written as

$$L(v, w) = -ik\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - c_{22}\bar{\alpha}_1\bar{\gamma}_1 - \beta(c_{12}\eta^2 - c_{22}\bar{\alpha}_2\bar{\gamma}_2)}{\theta(\bar{\alpha}_1 - \beta\bar{\alpha}_2)} \times J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,d\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta\,[illegible]\,(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,d\eta\right], \quad w > v$$

where $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\alpha}_1, \bar{\alpha}_2, \beta, \hat{\beta}, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\gamma}_2$ are given by eq. (34).

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DIFFRACTION OF ELASTIC WAVES BY THREE COPLANAR GRIFFITH CRACKS IN AN ORTHOTROPIC MEDIUM

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Abstract—The dynamic response of three co-planar Griffith cracks situated in an infinite orthotropic medium due to elastic waves incident normally on the cracks has been treated. The Fourier transform technique has been used to reduce the elastodynamic problem to the solution of a set of four integral equations. These integral equations have been solved by using the finite Hilbert transform technique and Cook's result. The analytical forms of crack opening displacement and stress intensity factors have been derived for low frequency vibration. Numerical results of crack opening displacement and stress intensity factors for several orthotropic materials have been calculated and plotted graphically to display the influence of the material orthotropy.

1. INTRODUCTION

Recently, with the increased usage of macroscopically anisotropic construction materials such as fibre-reinforced materials, the study of diffraction of elastic waves with cracks or inclusions has attracted the attention of scientists. The different possible location of cracks with respect to the planes of material symmetry is of great interest in Seismology and Exploration Geophysics. The problem of scattering of elastic waves by cracks of finite dimension in isotropic medium has been investigated by several investigators. Many investigators [1–6] have solved the diffraction problem involving single or two cracks in an isotropic medium. Dhawan and Dhaliwal [7] solved the statical problem involving three coplanar cracks in an infinite transversely isotropic medium. The dynamic problem of singular stresses around cracks in orthotropic medium are few in number. Kassir and Bandyopadhyay [8] solved the problem of elastodynamic response of an infinite orthotropic solid containing a crack under the action of impact loading. The problem of normal impact response of a finite Griffith crack in an orthotropic strip has been solved by Shindo [9]. De and Patra [10] have also solved the problem involving a moving Griffith crack in an orthotropic strip. Recently Kundu and Bostrom [11] treated the diffraction problem of a circular crack in orthotropic medium.

To the best knowledge of the authors, the problem of diffraction of elastic waves by three coplanar Griffith cracks in an orthotropic material has not been considered. In our paper, the interaction of normally incident time harmonic elastic waves with three coplanar Griffith cracks in an orthotropic medium has been investigated. It is assumed that the faces of each of the cracks do not come into contact during small deformation of the solid. The resulting mixed boundary value problem is reduced to the solution of a set of four integral equations which has been reduced to the solution of an integro-differential equation. Iteration method has been used to obtain the low frequency solution of the problem. This enables us to obtain approximate value of the crack opening displacements and stress intensity factors. Making the length of the central crack tend to zero, the corresponding results for two Griffith cracks have been obtained. Numerical results for stress intensity factors and crack opening displacements have been plotted against dimensionless frequency and distance respectively for different orthotropic materials which have been shown graphically.

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2. STATEMENT AND FORMULATION OF THE PROBLEM

Consider the interaction of normally incident longitudinal wave with three coplanar Griffith cracks situated in an infinite orthotropic elastic medium. The cracks are assumed to occupy the position $|X| \le d_1$, $d_2 \le |X| \le d$, $Y = 0$, $|Z| < \infty$. Let E_i, μ_{ij} and ν_{ij} $(i, j = 1, 2, 3)$ denote the engineering elastic constants of the material where the subscripts 1, 2, 3 correspond to the X, Y, Z directions chosen to coincide with the axes of material orthotropy. Normalizing all the lengths with respect to 'd' and setting $X/d = x$, $Y/d = y$, $Z/d = z$, $d_1/d = b$, $d_2/d = c$, the cracks are defined by $|x| \le b$, $c \le |x| \le 1$, $y = 0$, $|z| < \infty$ (Fig. 1).

Displacement components are also made dimensionless with respect to 'd' so that dimensionless components of displacement in x, y directions are assumed to be u, v respectively, where

$$u = u(x, y, t) \quad \text{and} \quad v = v(x, y, t).$$

Let a time harmonic plane elastic wave originating at $y = -\infty$ and incident normally on the three cracks be given by $v = v_0 \exp[i(ky - \omega t)]/d$ where $k = d\omega/c_s\sqrt{c_{22}}$, $c_s = (\mu_{12}/\rho)^{1/2}$, v_0 is a constant, ω and v_0/d are the frequency and dimensionless amplitude of the incident wave respectively, ρ being the density of the material. In the isotropic solid, c_s represents the velocity of the shear wave.

The non-zero stress components τ_{yy} and τ_{xy} are given by

$$\tau_{yy}/\mu_{12} = c_{12}u_{,x} + c_{22}v_{,y}$$

$$\tau_{xy}/\mu_{12} = u_{,y} + v_{,x} \tag{2.1}$$

where u, v denote the component of the displacement in the x, y directions respectively and comma denotes partial differentiation with respect to the coordinates or time ; $c_{ij}(i, j = 1, 2)$ are

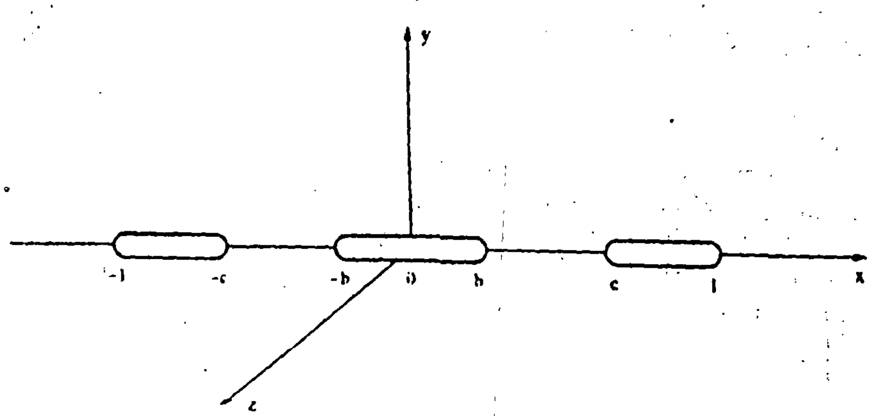

Fig. 1. Geometry of the cracks.

nondimensional parameters related to the elastic constant by the relations:

$$c_{11} = E_1/\mu_{12}(1 - \nu_{12}^2 E_2/E_1)$$
$$c_{22} = E_2/\mu_{12}(1 - \nu_{12}^2 E_2/E_1) = c_{11}E_2/E_1$$
$$c_{12} = \nu_{12}E_2/\mu_{12}(1 - \nu_{12}^2 E_2/E_1) = \nu_{12}c_{22} = \nu_{21}c_{11} \tag{2.2}$$

for generalized plane stress, and by

$$c_{11} = (E_1/\Delta\mu_{12})(1 - \nu_{23}\nu_{32})$$
$$c_{22} = (E_2/\Delta\mu_{12})(1 - \nu_{13}\nu_{31})$$
$$c_{12} = E_1(\nu_{21} + \nu_{13}\nu_{32}E_2/E_1)/\Delta\mu_{12} = E_2(\nu_{12} + \nu_{23}\nu_{31}E_1/E_2)/\Delta\mu_{12}$$
$$\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{21}\nu_{32} \tag{2.3}$$

for plane strain. The constants E_i and ν_{ij} satisfy Maxwell's relation:

$$\nu_{ij}/E_i = \nu_{ji}/E_j. \tag{2.4}$$

The displacement equations of motion for orthotropic material are

$$c_{11}u_{,xx} + u_{,yy} + (1 + c_{12})v_{,xy} = \frac{d^2}{c_s^2}u_{,tt}$$
$$c_{22}v_{,yy} + v_{,xx} + (1 + c_{12})u_{,xy} = \frac{d^2}{c_s^2}v_{,tt} \tag{2.5}$$

Substitution of $u(x, y, t) = u(x, y)\exp(-i\omega t)$ and $v(x, y, t) = v(x, y)\exp(-i\omega t)$ in equations (2.5) reduces them to

$$c_{11}u_{,xx} + u_{,yy} + (1 + c_{12})v_{,xy} + k_s^2 u = 0$$
$$c_{22}v_{,yy} + v_{,xx} + (1 + c_{12})u_{,xy} + k_s^2 v = 0 \tag{2.6}$$

with $k_s^2 = d^2\omega^2/c_s^2$, which are to be solved subject to the boundary conditions

$$v(x, 0) = 0, \quad b \le |x| \le c, \qquad |x| \le 1 \tag{2.7}$$
$$\tau_{xy}(x, 0) = 0, \qquad |x| < \infty \tag{2.8}$$
$$\tau_{yy}(x, 0) + \tau_{yy}^{(in)}(x, 0) = 0, \quad |x| < b, \qquad c < |x| < 1. \tag{2.9}$$

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables would be omitted in the sequel.

Using the condition (2.8), the solutions of equations (2.6) may be written as

$$u(x, y) = \frac{2}{\pi}\int_0^\infty [\exp(-\gamma_1|y|) - \beta\exp(-\gamma_2|y|)]A_1(\xi)\sin(\xi x)\,d\xi \tag{2.10}$$

$$v(x, y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}[\alpha_1\exp(-\gamma_1|y|) - \beta\alpha_2\exp(-\gamma_2|y|)]A_1(\xi)\cos(\xi x)\,d\xi, \qquad y > 0 \tag{2.11}$$

and the stress components are given by

$$\tau_{xy}/\mu_{12} = -\frac{2}{\pi}\int_0^\infty (\gamma_1 + \alpha_1)[\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)]A_1(\xi)\sin(\xi x)\,d\xi, \qquad y > 0 \tag{2.12}$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty \left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \beta\left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\right]A_1(\xi)\cos(\xi x) \tag{2.13}$$

where

$$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1 + c_{12})\gamma_i}, \qquad i = 1, 2 \tag{2.14}$$

$$\beta = \frac{\gamma_1 + \alpha_1}{\gamma_2 + \alpha_2} \tag{2.15}$$

$A_1(\xi)$ is the unknown function to be determined, and γ_1^2, γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1 + c_{22})k_s^2\}\gamma^2 + (c_{11}\xi^2 - k_s^2)(\xi^2 - k_s^2) = 0. \tag{2.16}$$

With the aid of the boundary conditions, (2.7) and (2.9) $A(\xi)$ is found to satisfy the integral equations

$$\int_0^\infty A(\xi)\cos(\xi x)\,d\xi = 0, \qquad x \in I_2, I_4 \tag{2.17}$$

and

$$\int_0^\infty H(\xi)A(\xi)\cos(\xi x)\,d\xi = -\frac{\pi p_0}{2\mu_{12}}, \qquad x \in I_1, I_3 \tag{2.18a, b}$$

where $I_1 = (0, b)$, $I_2 = (b, c)$, $I_3 = (c, 1)$, $I_4 = (1, \infty)$ and

$$p_0 = ik\mu_{12}c_{22}v_0/d \tag{2.19}$$

$$A(\xi) = \frac{\alpha_1 - \beta\alpha_2}{\xi}A_1(\xi) \tag{2.20}$$

$$H(\xi) = \frac{c_{12}\xi^2 - c_{22}\alpha_1\gamma_1 - \beta(c_{12}\xi^2 - c_{22}\alpha_2\gamma_2)}{(\alpha_1 - \beta\alpha_2)}. \tag{2.21}$$

3. METHOD OF SOLUTION

The solution of the integral equations (2.17) and (2.18) is taken in the form

$$A(\xi) = \frac{1}{\xi}\int_0^b h(t)\sin(\xi t)\,dt + \frac{1}{\xi}\int_c^1 g(u^2)\sin(\xi u)\,du \tag{3.1}$$

where $h(t)$ and $g(u^2)$ are the unknown functions to be determined. Substituting the value of $A(\xi)$ from (3.1) in (2.17) and using the following result [12]

$$\int_0^\infty \frac{\sin(\xi t)\cos(\xi x)}{\xi}\,d\xi = \begin{cases} \frac{\pi}{2}, & t > x \\ 0, & t < x \end{cases}$$

it is found that the choice of $A(\xi)$ leads to the equation

$$\int_c^1 g(u^2)\,\mathrm{d}u = 0. \tag{3.2}$$

Further substituting $A(\xi)$ from (3.1) in (2.18a) and using the result [13]

$$\int_0^\infty \xi^{-1}\sin(\xi u)\sin(\xi x)\,\mathrm{d}\xi = \frac{1}{2}\log\left|\frac{u+x}{u-x}\right|$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h(t)\log\left|\frac{t+x}{t-x}\right|\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(u^2)\log\left|\frac{u+x}{u-x}\right|\mathrm{d}u$$
$$= 2\Big[q_0 - \frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h(t)\,\mathrm{d}t\int_0^\infty H_1(\xi)\xi^{-1}\sin(\xi t)\sin(\xi x)\,\mathrm{d}\xi$$
$$-\frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(u^2)\,\mathrm{d}u\int_0^\infty H_1(\xi)\xi^{-1}\sin(\xi u)\sin(\xi x)\,\mathrm{d}\xi\Big], \qquad x \in I_1 \tag{3.3}$$

where

$$q_0 = -\frac{\pi p_0}{2\theta\mu_{12}} \tag{3.4}$$

$$H_1(\xi) = \frac{H(\xi)}{\xi\theta} - 1 \to 0 \quad \text{as} \quad \xi \to \infty \tag{3.5}$$

$$\theta = \frac{(c_{12}^2 + c_{12} - c_{11}c_{22})(c_{12}N_1N_2 - c_{11}) - c_{22}[c_{12}N_1^2N_2^2 + c_{11}(N_1^2 + N_1N_2 + N_2^2)]}{c_{11}(1 + c_{12})(N_1 + N_2)} \tag{3.6}$$

$$N_1^2 = \frac{1}{2c_{22}}\{c_{11}c_{22} - c_{12}^2 - 2c_{12} + [(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}]^{1/2}\}$$

$$N_2^2 = \frac{1}{2c_{22}}\{c_{11}c_{22} - c_{12}^2 - 2c_{12} - [(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}]^{1/2}\}. \tag{3.7}$$

Using the relation

$$\frac{\sin\xi x\,\sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_0(\xi w)J_0(\xi v)\,\mathrm{d}v\,\mathrm{d}w}{(x^2 - w^2)^{1/2}(t^2 - v^2)^{1/2}}$$

equation (3.3) can now be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h(t)\log\left|\frac{t+x}{t-x}\right|\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(u^2)\log\left|\frac{u+x}{u-x}\right|\mathrm{d}u$$
$$= 2\Big[q_0 - \frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h(t)\,\mathrm{d}t\int_0^x\int_0^t \frac{vwL(v,w)\,\mathrm{d}w\,\mathrm{d}v}{(x^2 - w^2)^{1/2}(t^2 - v^2)^{1/2}}$$
$$-\frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(u^2)\,\mathrm{d}u\int_0^x\int_0^u \frac{vwL(v,w)\,\mathrm{d}w\,\mathrm{d}v}{(x^2 - w^2)^{1/2}(u^2 - v^2)^{1/2}}\Big], \qquad x \in I_1 \tag{3.8}$$

where

$$L(v,w) = \int_0^\infty \xi H_1(\xi)J_0(\xi w)J_0(\xi v)\,\mathrm{d}\xi \tag{3.9}$$

and $J_0(\ )$ is the Bessel function of order zero.

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Applying a contour integration technique [14], the infinite integral in $L(v, w)$ can be converted to the following finite integrals

$$L(v,w) = -ik_s^2\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - \bar{\alpha}_1\bar{\gamma}_1 c_{22} - \bar{\beta}(c_{12}\eta^2 - \bar{\alpha}_2\bar{\gamma}_2 c_{22})}{\theta(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,\mathrm{d}\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,\mathrm{d}\eta\right], \qquad w > v \tag{3.10}$$

where

$$\bar{\gamma}_1 = \left[\frac{1}{2}\{R_1 - (R_1^2 - 4\bar{R}_2)^{1/2}\}\right]^{1/2}$$

$$\bar{\gamma}_2 = \left[\frac{1}{2}\{R_1 + (R_1^2 - 4\bar{R}_2)^{1/2}\}\right]^{1/2}$$

$$\hat{\gamma}_1 = \left[\frac{1}{2}\{-R_1 + (R_1^2 + 4R_2')^{1/2}\}\right]^{1/2}$$

$$\hat{\gamma}_2 = \left[\frac{1}{2}\{R_1 + (R_1^2 + 4R_2')^{1/2}\}\right]^{1/2}$$

$$R_1 = \frac{1}{c_{22}}\{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\eta^2 + (1 + c_{22})\}$$

$$\bar{R}_2 = \frac{c_{11}}{c_{22}}(1 - \eta^2)\left(\frac{1}{c_{11}} - \eta^2\right)$$

$$R_2' = \frac{c_{11}}{c_{22}}(1 - \eta^2)\left(\eta^2 - \frac{1}{c_{11}}\right)$$

$$\bar{\alpha}_i = \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_i^2}{(1 + c_{12})\bar{\gamma}_i}, \qquad i = 1, 2$$

$$\hat{\alpha}_i = \frac{c_{11}\eta^2 - 1 + (-1)^i\hat{\gamma}_i^2}{(1 + c_{12})\hat{\gamma}_i}, \qquad i = 1, 2$$

$$\bar{\beta} = \frac{\bar{\alpha}_1 - \bar{\gamma}_1}{\bar{\alpha}_2 - \bar{\gamma}_2}$$

$$\hat{\beta} = \frac{\hat{\alpha}_1 + \hat{\gamma}_1}{\hat{\alpha}_2 - \hat{\gamma}_2}. \tag{3.11}$$

The corresponding expression of $L(v, w)$ for $w < v$ is obtained by interchanging v and w in (3.10).

Employing the series expansions for the Bessel function J_0 and the Hankel function $H_0^{(1)}$ in equation (3.10), it is found that

$$L(v, w) = \frac{2}{\pi}Pk_s^2 \log k_s + O(k_s^2) \tag{3.12}$$

where

$$P = \frac{1}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - \bar{\alpha}_1\bar{\gamma}_1 c_{22} - \bar{\beta}(c_{12}\eta^2 - \bar{\alpha}_2\bar{\gamma}_2 c_{22})}{(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)}\,\mathrm{d}\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)}\,\mathrm{d}\eta\right]. \tag{3.13}$$

Let us now expand $h(t)$ and $g(u^2)$ in the form

$$h(t) = h_0(t) + k_s^2 \log k_s\, h_1(t) + O(k_s^2)$$

and

$$g(u^2) = g_0(u^2) + k_s^2 \log k_s\, g_1(u^2) + O(k_s^2). \tag{3.14}$$

Substituting the above equations (3.14) and the value of $L(v, w)$ given by (3.10) in equations (3.8) and (3.2) and equating the coefficients of like powers of k_s, the following equations are derived.

$$\frac{d}{dx}\int_0^b h_0(t)\log\left|\frac{t+x}{t-x}\right| dt + 2\int_c^1 \frac{ug_0(u^2)}{u^2-x^2}\,du = 2q_0, \qquad x \in I_1, I_3 \tag{3.15a, b}$$

$$\frac{d}{dx}\int_0^b h_1(t)\log\left|\frac{t+x}{t-x}\right| dt + 2\int_c^1 \frac{ug_1(u^2)}{u^2-x^2}\,du = -\frac{4P}{\pi}\left[\int_0^b th_0(t)\,dt + \int_c^1 ug_0(u^2)\,du\right],$$

$$x \in I_1, I_3 \tag{3.16a, b}$$

and

$$\int_c^1 g_i(u^2)\,du = 0 \qquad (i = 0, 1). \tag{3.17a, b}$$

Rewriting equation (3.15a) as

$$\int_0^b h_0(t)\log\left|\frac{t+x}{t-x}\right| dt = \pi F_1(x), \qquad x \in I_1 \tag{3.18}$$

where

$$F_1(x) = -\int_0^x \left[\frac{p_0}{\mu_{12}\theta} + \frac{2}{\pi}\int_c^1 \frac{ug_0(u^2)}{u^2-y^2}\,du\right] dy.$$

The solution of the integral equation (3.18) with the help of Cook's result [15] is found to be

$$h_0(t) = -\frac{p_0}{\mu_{12}\theta}\frac{t}{(b^2-t^2)^{1/2}} - \frac{2}{\pi}\frac{t}{(b^2-t^2)^{1/2}}\int_c^1 \frac{\sqrt{u^2-b^2}\,g_0(u^2)}{u^2-t^2}\,du. \tag{3.19}$$

Substitution of the value of $h_0(t)$ from (3.19) in (3.15b) with the aid of the result

$$\int_0^b \frac{1}{(b^2-t^2)^{1/2}}\frac{t^2\,dt}{(x^2-t^2)(u^2-t^2)} = \frac{\pi}{2}\left[\frac{x}{(x^2-b^2)^{1/2}} - \frac{u}{(u^2-b^2)^{1/2}}\right], \qquad x \in I_3$$

yields the singular integral equation

$$\int_c^1 \frac{\sqrt{u^2-b^2}\,g_0(u^2)}{u^2-x^2}\,du = -\frac{\pi}{2}\frac{p_0}{\mu_{12}\theta}, \qquad x \in I_3. \tag{3.20}$$

Next using the finite Hilbert transform technique [13] the solution of the integral equation is found to be

$$g_0(u^2) = -\frac{p_0}{\mu_{12}\theta}\sqrt{\frac{u^2(u^2-c^2)}{(u^2-b^2)(1-u^2)}} + \frac{uD_1}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}} \tag{3.21}$$

where D_1 is unknown constant to be determined from equation (3.17a).

Now substituting the value of $g_0(u^2)$ from (3.21) in (3.19) and performing the integrations, $h_0(t)$ is obtained in the following form

$$h_0(t) = -\frac{p_0}{\mu_{12}\theta}\sqrt{\frac{t^2(c^2-t^2)}{(b^2-t^2)(1-t^2)}} + \frac{tD_1}{(b^2-t^2)(c^2-t^2)(1-t^2)}. \tag{3.22}$$

By the procedure similar to one which led to the derivations of the solutions of (3.15) as given

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by (3.21) and (3.22), the solutions of equation (3.16a, b) can also be obtained and they are found to be

$$h_1(t) = -\frac{4PR}{\pi^2}\sqrt{\frac{t^2(c^2-t^2)}{(b^2-t^2)(1-t^2)}} - \frac{tD_2}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}} \tag{3.23}$$

$$g_1(u^2) = -\frac{4PR}{\pi^2}\sqrt{\frac{u^2(u^2-c^2)}{(u^2-b^2)(1-u^2)}} + \frac{uD_2}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}} \tag{3.24}$$

where

$$R = -\frac{p_0}{\mu_{12}\theta}[I_0^b + I_c^1] - D_1[J_0^b - J_c^1]$$

$$I_m^n = \int_m^n \frac{t^2\sqrt{(c^2-t^2)}}{\sqrt{(b^2-t^2)(1-t^2)}}\,dt$$

$$J_m^n = \int_m^n \frac{t^2\,dt}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}}. \tag{3.25}$$

The constant D_2 is to be determined from equation (3.17b).

In order to determine the values of the unknown constants D_1 and D_2, $g_0(u^2)$ and $g_1(u^2)$ as given by (3.21) and (3.24) respectively are substituted in (3.17a, b) and it is found that

$$D_j = A_j\left[(1-b^2)\frac{E}{F} - (c^2-b^2)\right], \qquad (j = 1, 2) \tag{3.26}$$

and

$$A_1 = \frac{p_0}{\mu_{12}\theta}, \qquad A_2 = \frac{4PR}{\pi^2} \tag{3.27}$$

where $F = F\left(\frac{\pi}{2}, q\right)$ and $E = E\left(\frac{\pi}{2}, q\right)$ are the elliptic integrals of first and second kind respectively and $q = \sqrt{\frac{1-c^2}{1-b^2}}$. Substitution of the values of D_j $(j = 1, 2)$ given by equations (3.26) in equations (3.21)–(3.24) yields

$$h_{j-1}(t) = -A_j\left[(1-b^2)\frac{E}{F} + (b^2-t^2)\right]\frac{t}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}} \qquad (j = 1, 2) \tag{3.28}$$

$$g_{j-1}(u^2) = -A_j\left[(1-b^2)\frac{E}{F} - (u^2-b^2)\right]\frac{u}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}} \qquad (j = 1, 2). \tag{3.29}$$

4. STRESS INTENSITY FACTORS AND CRACK OPENING DISPLACEMENTS

The stress intensity factors are defined as (in physical units)

$$N_b = \lim_{x\to b+}\left[\frac{\sqrt{(x-b)}\,\tau_{yy}(x,0)}{p_0}\right]_{b<x<c} \tag{4.1}$$

$$N_c = \lim_{x\to c-}\left[\frac{\sqrt{(c-x)}\,\tau_{yy}(x,0)}{p_0}\right]_{b<x<c} \tag{4.2}$$

$$N_1 = \lim_{x\to 1+}\left[\frac{\sqrt{(x-1)}\,\tau_{yy}(x,0)}{p_0}\right]_{x>1} \tag{4.3}$$

and the crack opening displacement can now be shown to be given by

$$\Delta v(x,0) = v(x,0+) - v(x,0-) = 2\int_x^b h(t)\,\mathrm{d}t, \qquad 0 \le x \le b \tag{4.4}$$

$$= 2\int_x^1 g(u^2)\,\mathrm{d}u, \qquad c \le x \le 1. \tag{4.5}$$

Substituting the values of the function $h(t)$ and $g(u^2)$, the stress component τ_{yy} can be evaluated from the expressions (2.13), (2.21) and (3.1). After evaluation of the value of τ_{xy}' and putting it in relations (4.1)–(4.3) it is found that

$$N_b = \sqrt{\frac{b(1-b^2)}{2(c^2-b^2)}}\frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)}\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2) \tag{4.6}$$

$$N_c = \sqrt{\frac{c}{2(c^2-b^2)(1-c^2)}}\left[(1-b^2)\frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)} - (c^2-b^2)\right]\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2) \tag{4.7}$$

$$N_1 = \sqrt{\frac{(1-b^2)}{2(1-c^2)}}\left[1 - \frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)}\right]\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2) \tag{4.8}$$

where

$$M_2 = \left[I_0^b + I_c^1 + \left\{(1-b^2)\frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)} - (c^2-b^2)\right\}(J_0^b - J_c^1)\right].$$

Expressions (4.4–4.5) with the aid of the equations (3.28)–(3.29) yield

$$\Delta v(x,0) = \frac{2p_0}{\mu_{12}\theta}\left[\sqrt{(1-b^2)}\,F(\beta,q)\left\{\frac{E(\beta,q)}{F(\beta,q)} - \frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)}\right\} - \sqrt{\frac{(1-x^2)(b^2-x^2)}{(c^2-x^2)}}\right]$$
$$\times\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2), \qquad 0 \le x \le b \tag{4.9}$$

and

$$\Delta v(x,0) = \frac{2p_0}{\mu_{12}\theta}\left[\sqrt{(1-b^2)}\,F(\lambda,q)\left\{\frac{E\left(\frac{\pi}{2},q\right)}{F\left(\frac{\pi}{2},q\right)} - \frac{E(\lambda,q)}{F(\lambda,q)}\right\}\right]$$
$$\times\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2), \qquad c \le x \le 1 \tag{4.10}$$

where

$$\sin\beta = \sqrt{\frac{b^2-x^2}{c^2-x^2}} \quad \text{and} \quad \sin\lambda = \sqrt{\frac{1-x^2}{1-b^2}}.$$

When $b \to 0$, we recover the stress intensity factor and the crack opening displacement for two Griffith cracks occupying the region $c \le |x| \le 1$, $y = 0$, $|z| < \infty$:

$$N_c = -\frac{\left[c^2 - \frac{E}{F}\right]}{\sqrt{2c(1-c^2)}}\left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\}k_s^2 \log k_s\right] + O(k_s^2)$$

$$N_1 = -\frac{\left[1 - \frac{E}{F}\right]}{\sqrt{2(1-c^2)}}\left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\}k_s^2 \log k_s\right] + O(k_s^2) \tag{4.11}$$

and

$$\Delta v(x, 0) = \frac{2p_0}{\mu_{12}\theta}\left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\}k_s^2 \log k_s\right] \times \left[\frac{E\left(\frac{\pi}{2}, q\right)}{F\left(\frac{\pi}{2}, q\right)}F(\lambda, q) - E(\lambda, q)\right] + O(k_s^2), \qquad c \le x \le 1 \tag{4.12}$$

where $M_2 = \frac{\pi}{4}(1 + c^2 - 2E/F)$ has been used.

It is noted that if further $c \to 0$, the cracks merge into a single crack of width two units. In this case $F \to \infty$ and $M_2 \to \pi/4$; so the results for stress intensity factor and crack opening displacements corresponding to the single crack are found to be

$$N_1 = \frac{1}{\sqrt{2}}\left[1 - \frac{P}{\pi}k_s^2 \log k_s\right] + O(k_s^2) \tag{4.13}$$

and

$$\Delta v(x, 0) = -\frac{2p_0}{\mu_{12}\theta}\sqrt{(1-x^2)}\left[1 - \frac{P}{\pi}k_s^2 \log k_s\right] + O(k_s^2), \qquad 0 \le x \le 1. \tag{4.14}$$

The results given by (4.11)–(4.14) are found to be in agreement with the results of Sarkar *et al.* [16].

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) N_b, N_c and N_1 given by (4.6), (4.7) and (4.8) at the tips of the cracks and crack opening displacements (COD) given by (4.9) and (4.10) have been plotted against dimensionless frequency k_s and distance respectively for three different types of orthotropic materials whose engineering constants have been listed in Table 1.

Table 1. Engineering elastic constants

	E_1 (Pa)	E_2 (Pa)	μ_{12} (Pa)	ν_{12}
Type I	Modulite II graphite–epoxy composite:			
	15.3×10^9	158.0×10^9	5.52×10^9	0.033
Type II	E-Type glass–epoxy composite:			
	9.79×10^9	42.3×10^9	3.66×10^9	0.063
Type III	Stainless steel–aluminium composite:			
	79.76×10^9	85.91×10^9	30.02×10^9	0.31

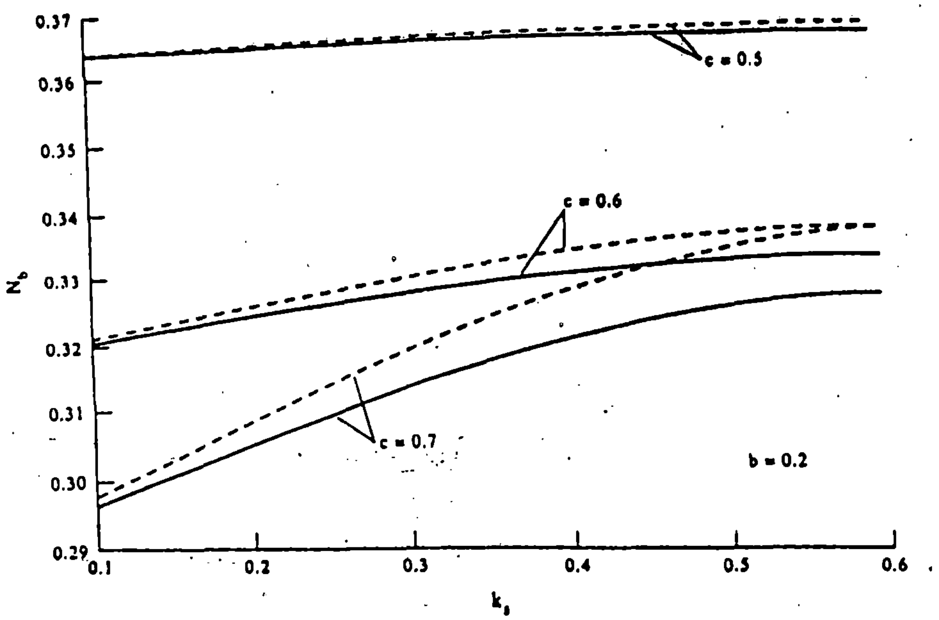

Fig. 2. Stress intensity factor N_b vs frequency k_s for generalized plane stress. (——) type I; (·····) type III.

Keeping the length of the central crack fixed ($b = 0.2$) SIFs at the tips of the central and outer cracks have been plotted against frequency k_s ($0.1 \leq k_s \leq 0.6$) for different lengths ($c = 0.5, 0.6, 0.7$) of the outer crack (Figs 2–4). It is noted from the graphs (Figs 2–4) that with the decrease in the value of outer crack length, i.e. with the increase in the value of the distance between inner and outer cracks the rate of increase in the SIF is higher with the increase in the value of the frequency k_s.

The same nature of SIFs are seen (Figs 5–7) in the case when the length of the outer cracks

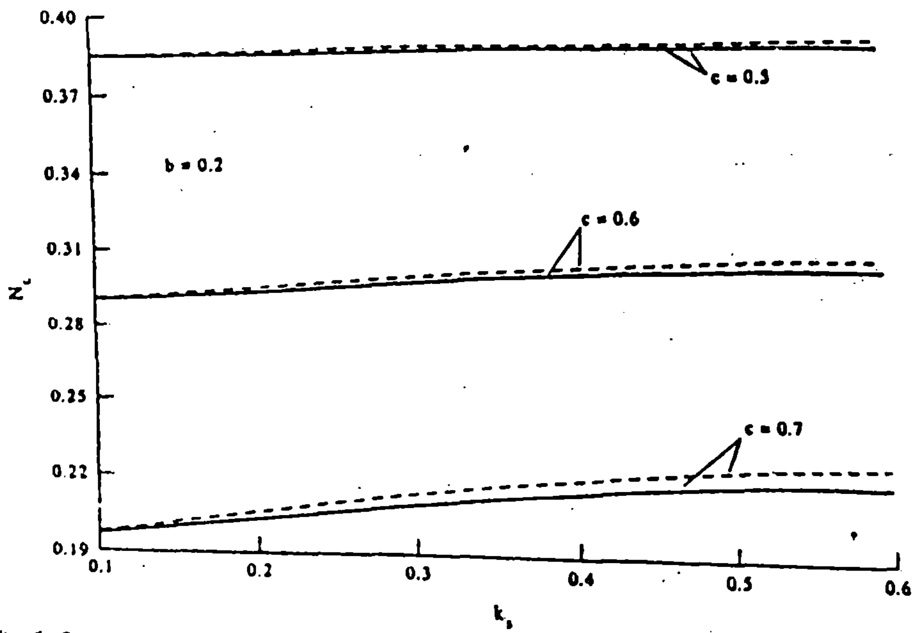

Fig. 3. Stress intensity factor N_c vs frequency k_s for generalized plane stress. (——) type I; (·····) type III.

∞ 62371 adj = + 11pts

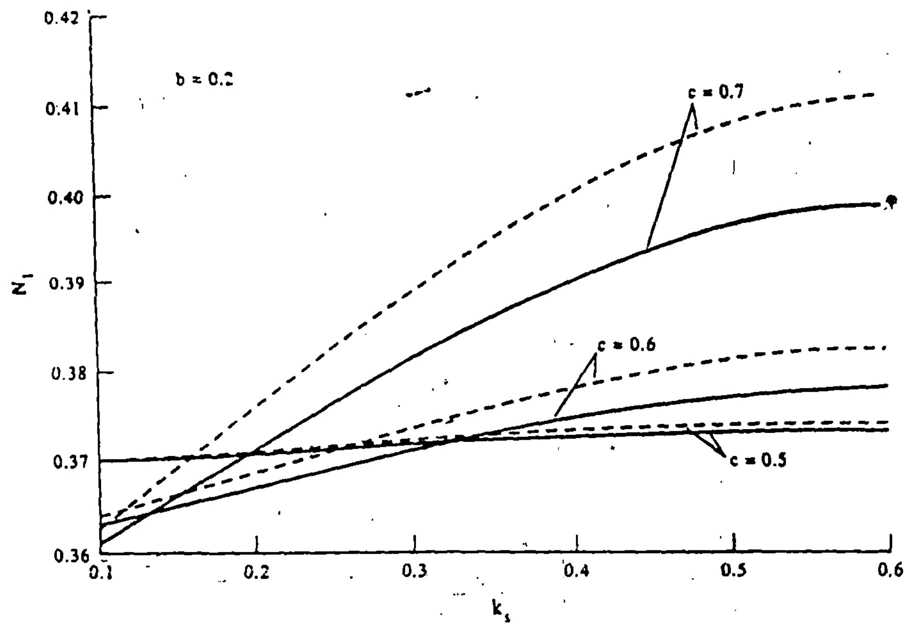

Fig. 4. Stress intensity factor N_1 vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

are fixed ($c = 0.7$) and the length of the central crack increases ($b = 0.3$, 0.4, 0.5). It is interesting to note that for fixed c ($= 0.7$) the SIFs N_b and N_c increase with the increase in the value of b, but the effect is just reverse in case of N_1.

The COD $\mu_{12}\Delta v(x, 0)/p_0$ has been plotted for different crack lengths. It is found from Figs 8 and 9 that with the increase in the value of crack length the value of COD increases. For a fixed material the variation of COD with frequency is found to be insignificant.

In all the cases where different values of crack length have been considered the variation of COD is found to be prominant for different orthotropic materials.

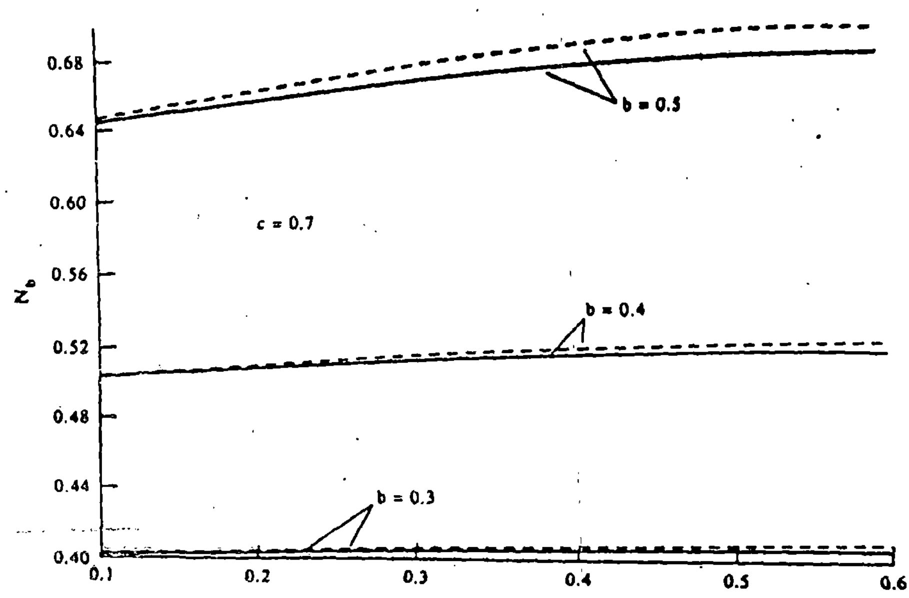

Fig. 5. Stress intensity factor N_b vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

ooo 62371 adj = + 6pts

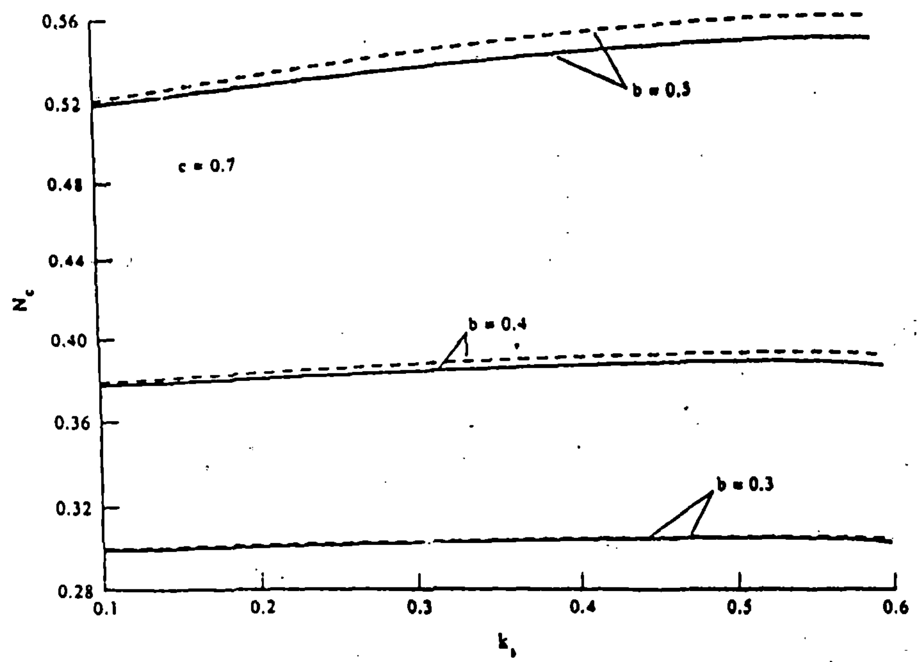

Fig. 6. Stress intensity factor N_6 vs frequency k_s for generalized plane stress. (——) type I, (-----) type III.

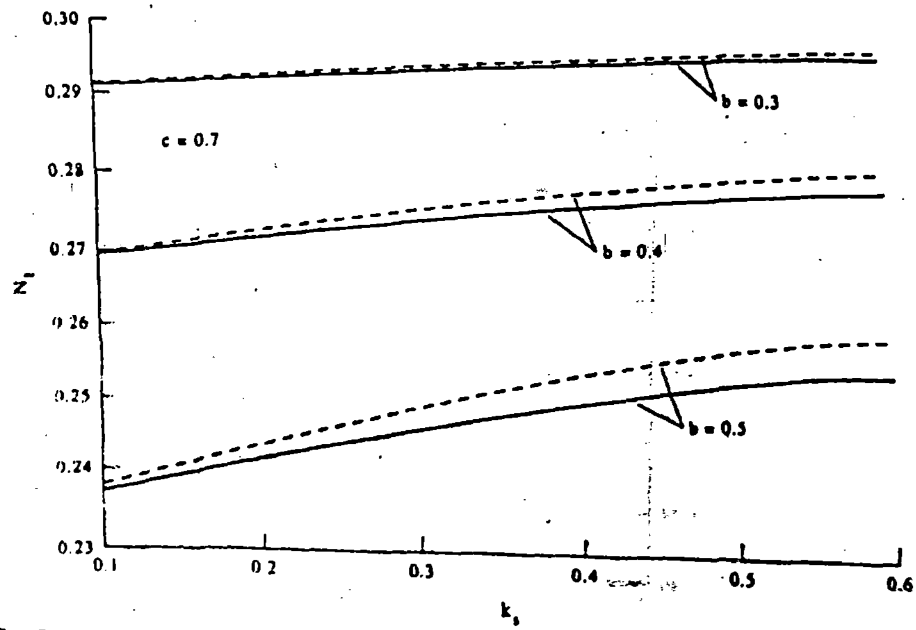

Fig. 7. Stress intensity factor N_1 vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

000 62371 adj = + 4pts

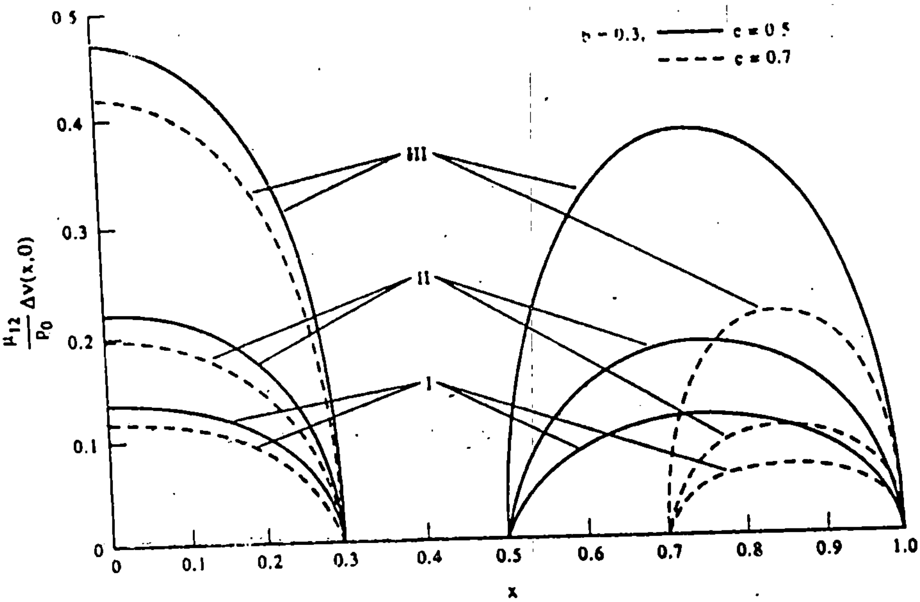

Fig. 8. Crack opening displacement vs distance for generalized plane stress ($k_s = 0.5$, b = 0.3, c = 0.5, 0.7).

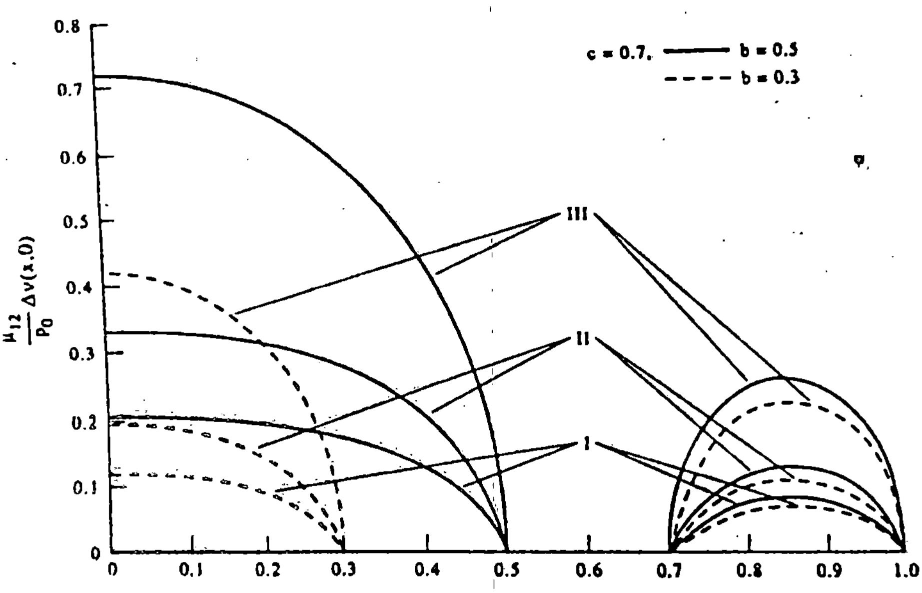

Fig. 9. Crack opening displacement vs distance for generalized plane stress ($k_s = 0.5$, b = 0.3, 0.5, c = 0.7).

000 62371 adj = + 633pts change = + 633pts

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INTERACTION OF ELASTIC WAVES WITH TWO COPLANAR GRIFFITH CRACKS IN AN ORTHOTROPIC MEDIUM

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Abstract—The problem of diffraction of normally incident elastic waves by two coplanar Griffith cracks situated in an infinite orthotropic medium has been analyzed. Fourier and Hilbert transforms have been used to solve this mixed boundary value problem. Approximate analytical results for stress intensity factors and crack opening displacement have been derived when the wave lengths are large compared to the crack length. Numerical values of stress intensity factors and the crack opening displacement for several orthotropic materials have been calculated and plotted graphically to show the effect of material orthotropy.

INTRODUCTION

DYNAMIC fracture problems involving anisotropic materials weakened by crack-like imperfections have drawn much attention by investigators because of the increased usage of macroscopically anisotropic construction materials such as fibre reinforced composites. The different possible location of cracks with respect to the planes of material symmetry introduce great modifications in the strain and stress distribution. The problems are also of considerable interest in seismology and exploration geophysics. The problems involving single or two Griffith cracks in isotropic elastic medium have been studied by many authors [1–6]. Mathematical difficulties encountered in solving the governing equations of the anisotropic elasticity theory are responsible for the availability of few results only for special classes of materials. Kassir and Bandyopadhyay [7] have studied the elastodynamic response of an infinite orthotropic solid containing a crack under the action of impact loading and the elastodynamic problem of a finite Griffith crack in an orthotropic strip under normal impact was investigated by Shindo [8]. The problem involving a moving Griffith crack in an orthotropic strip has also been studied by De and Patra [9]. Recently, Kundu and Bostrom [10] solved the problem of scattering of elastic waves by a circular crack situated in a transversely isotropic solid.

In our paper, the diffraction of normally incident time harmonic elastic waves by two coplanar Griffith cracks in an infinite orthotropic medium has been investigated. The faces of each of the cracks are assumed to be separated by a small distance so that, during small deformations of the solid, the crack faces do not come into contact. The resulting mixed boundary value problem is reduced to the solution of a triple integral equation which has further been reduced to the solution of an integro-differential equation. Iterative solution valid for low frequency has been obtained. Analytical formulae for stress intensity factor and crack opening displacement have been derived. Making the distance between two crack zero, the corresponding results for single crack have been presented. Finally, choosing the engineering elastic constants of the orthotropic material suitably the results for isotropic material have been deduced and compared with the results obtained by Jain and Kanwal [5]. To display the influence of the material orthotropy numerical values of stress intensity factors and crack opening displacement have been plotted for several orthotropic materials.

STATEMENT AND FORMULATION OF THE PROBLEM

Consider the plane problem of diffraction of normally incident longitudinal wave by two symmetrical co-planar Griffith cracks situated in an infinite orthotropic elastic medium. The cracks are assumed to occupy the region $b \leqslant |X| \leqslant a$, $Y = 0$, $|Z| < \infty$. It is convenient to normalize all lengths with respect to "a" and so setting $X/a = x$, $Y/a = y$, $Z/a = z$, $b/a = c$, the new positions of the cracks are defined by $c \leqslant |x| \leqslant 1$, $y = 0$, $|z| < \infty$ (Fig. 1).

Let a plane time harmonic elastic wave originating at $y = -\infty$ be incident normally on the two cracks, and is defined by $v_0 = \exp[i(ky - \omega t)]$ where $k = a\hat{\omega}/c_s\sqrt{c_{22}}$, $c_s = (\mu_{12}/\rho)^{1/2}$ with ρ being the density of the material. In the isotropic solid, c_s represents the velocity of the shear wave.

The non-zero stress components τ_{yy} and τ_{xy} are given by

$$\tau_{yy}/\mu_{12} = c_{12}u_{,x} + c_{22}v_{,y}$$
$$\tau_{xy}/\mu_{12} = u_{,y} + v_{,x}, \tag{1}$$

where u, v denote the component of the displacement in the x, y directions, respectively and comma denotes partial differentiation with respect to the co-ordinates or time; $c_{ij}(i, j = 1, 2)$ are non-dimensional parameters related to the elastic constants by the relations

$$c_{11} = E_1/\mu_{12}(1 - v_{12}^2 E_2/E_1)$$
$$c_{22} = E_2/\mu_{12}(1 - v_{12}^2 E_2/E_1) = c_{11}E_2/E_1 \tag{2}$$
$$c_{12} = v_{12}E_2/\mu_{12}(1 - v_{12}^2 E_2/E_1) = v_{12}c_{22} = v_{21}c_{11}$$

for generalized plane stress, and by

$$c_{11} = (E_1/\Delta\mu_{12})(1 - v_{23}v_{32})$$
$$c_{22} = (E_2/\Delta\mu_{12})(1 - v_{13}v_{31})$$
$$c_{12} = E_1(v_{21} + v_{13}v_{32}E_2/E_1)/\Delta\mu_{12} \tag{3}$$
$$= E_2(v_{12} + v_{23}v_{31}E_1/E_2)/\Delta\mu_{12}$$
$$\Delta = 1 - v_{12}v_{21} - v_{23}v_{32} - v_{31}v_{13} - v_{12}v_{23}v_{31} - v_{13}v_{21}v_{32}$$

for plane strain. In the above equations E_i, μ_{ij} and v_{ij} $(i, j = 1, 2, 3)$ denote the engineering elastic constants of the material where the subscripts 1, 2, 3 correspond to the x, y, z directions which coincide with the axes of material orthotropy and the constants E_i and v_{ij} satisfy the Maxwell's relation

$$v_{ij}/E_i = v_{ji}/E_j. \tag{4}$$

The equations of motion for orthotropic material, in terms of displacements are

$$c_{11}u_{,xx} + u_{,yy} + (1 + c_{12})v_{,xy} = \frac{a^2}{c_s^2}u_{,tt}$$
$$c_{22}v_{,yy} + v_{,xx} + (1 + c_{12})u_{,xy} = \frac{a^2}{c_s^2}v_{,tt}. \tag{5}$$

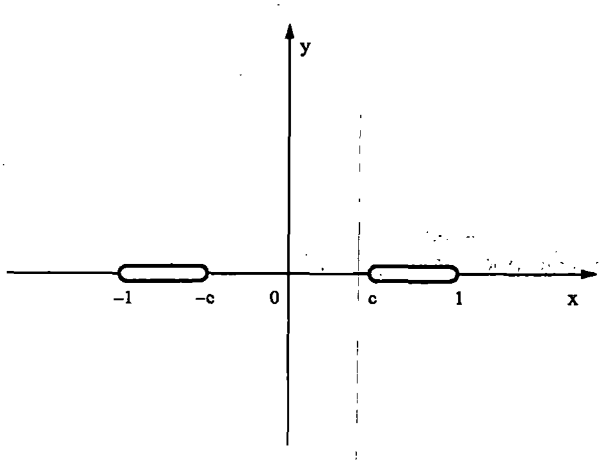

Fig. 1. Geometry of the cracks.

Therefore, substituting $u(x, y, t) = u(x, y)\exp(-i\omega t)$ and $v(x, y, t) = v(x, y)\exp(-i\omega t)$ in eq. (5) we obtain

$$c_{11}u_{,xx} + u_{,yy} + (1 + c_{12})v_{,xy} + k_s^2 u = 0 \tag{6}$$

and

$$c_{22}v_{,yy} + v_{,xx} + (1 + c_{12})u_{,xy} + k_s^2 v = 0$$

where $k_s^2 = a^2\omega^2/c_s^2$.

The boundary conditions of the problem are

$$\tau_{xy}(x, 0) = 0, \quad |x| < \infty \tag{7}$$

$$\tau_{yy}(x, 0) + \tau_{yy}^{(0)}(x, 0) = 0, \quad c \leqslant |x| \leqslant 1 \tag{8}$$

$$v(x, 0) = 0, \quad |x| < c, \quad |x| > 1. \tag{9}$$

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables would be omitted in the sequel.

The solutions of eqs (6) can be taken as

$$u(x, y) = \frac{2}{\pi}\int_0^\infty [A_1(\xi)\exp(-\gamma_1|y|) + A_2(\xi)\exp(-\gamma_2|y|)]\sin \xi x \, d\xi \tag{10}$$

$$v(x, y) = \pm\frac{2}{\pi}\int_0^\infty \frac{1}{\xi}[\alpha_1 A_1(\xi)\exp(-\gamma_1|y|) + \alpha_2 A_2(\xi)\exp(-\gamma_2|y|)]\cos \xi x \, d\xi, \quad y \gtrless 0 \tag{11}$$

where

$$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1 + c_{12})\gamma_i}, \quad i = 1, 2 \tag{12}$$

and $A_i(\xi)(i = 1, 2)$ are the unknown functions to be determined, γ_1^2, γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1 + c_{22})k_s^2\}\gamma^2 + (c_{11}\xi^2 - k_s^2)(\xi^2 - k_s^2) = 0. \tag{13}$$

From the boundary condition (7) it is found that

$$A_2(\xi) = -\beta A_1(\xi) \tag{14}$$

where

$$\beta = \frac{\gamma_1 + \alpha_1}{\gamma_2 + \alpha_2}. \tag{15}$$

Employing eq. (14) the expressions for displacements and stresses reduce to

$$u(x, y) = \frac{2}{\pi}\int_0^\infty [\exp(-\gamma_1|y|) - \beta\exp(-\gamma_2|y|)]A_1(\xi)\sin \xi x \, d\xi, \tag{16}$$

$$v(x, y) = \pm\frac{2}{\pi}\int_0^\infty \frac{1}{\xi}[\alpha_1\exp(-\gamma_1|y|) - \beta\alpha_2\exp(-\gamma_2|y|)]A_1(\xi)\cos \xi x \, d\xi, \quad y \gtrless 0 \tag{17}$$

$$\tau_{xy}/\mu_{12} = \mp\frac{2}{\pi}\int_0^\infty (\gamma_1 + \alpha_1)[\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)]A_1(\xi)\sin \xi x \, d\xi, \quad y \gtrless 0 \tag{18}$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty \Bigg[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \beta\left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\Bigg]A_1(\xi)\cos \xi x \, d\xi. \tag{19}$$

We further substitute

$$A(\xi) = \frac{\alpha_1 - \beta\alpha_2}{\xi}A_1(\xi)$$

so that the boundary conditions (8) and (9) yield the following integral equations in $A(\xi)$

$$\int_0^\infty A(\xi)\cos \xi x \, d\xi = 0, \quad |x| < c, \quad |x| > 1 \tag{20}$$

and

$$\int_0^\infty H(\xi)A(\xi)\cos\xi x\,d\xi = -\frac{\pi p_0}{2\mu_{12}}, \quad c \leqslant |x| \leqslant 1 \tag{21}$$

where $p_0 = ik\mu_{12}c_{22}$
and

$$H(\xi) = \frac{c_{12}\xi^2 - c_{22}\alpha_1\gamma_1 - \beta(c_{12}\xi^2 - c_{22}\alpha_2\gamma_2)}{(\alpha_1 - \beta\alpha_2)}. \tag{22}$$

METHOD OF SOLUTION

In order to solve the set of integral eqs (20) and (21), assume

$$A(\xi) = \frac{1}{\xi}\int_c^1 h(t^2)\sin(\xi t)\,dt \tag{23}$$

where $h(t^2)$ is an unknown function to be determined from the boundary conditions.

Inserting the value of $A(\xi)$ from eq. (23) in eq. (20) and using the following result [11]

$$\int_0^\infty \frac{\sin(\xi t)\cos(\xi x)}{\xi}\,d\xi = \begin{cases} \frac{\pi}{2}, & t > x \\ 0, & t < x \end{cases}$$

it is found that the choice of $A(\xi)$ leads to the equation

$$\int_c^1 h(t^2)\,dt = 0. \tag{24}$$

Further substitution of $A(\xi)$ from eq. (23) in eq. (21) leads to

$$\int_c^1 h(t^2)\,dt \int_0^\infty \sin(\xi t)\cos(\xi x)\,d\xi = q_0$$
$$-\frac{d}{dx}\int_c^1 h(t^2)\,dt \int_0^\infty \xi H_1(\xi)\frac{\sin(\xi t)\sin(\xi x)}{\xi^2}\,d\xi, \quad c \leqslant |x| \leqslant 1 \tag{25}$$

where

$$q_0 = -\frac{\pi p_0}{2\theta\mu_{12}} \tag{26}$$

$$H_1(\xi) = \frac{H(\xi)}{\xi\theta} - 1 \tag{27}$$

$$\theta = \frac{(c_{12}^2 + c_{12} - c_{11}c_{22})(c_{12}N_1N_2 - c_{11}) - c_{22}[c_{12}N_1^2N_2^2 + c_{11}(N_1^2 + N_1N_2 + N_2^2)]}{c_{11}(1 + c_{12})(N_1 + N_2)} \tag{28}$$

$$N_1^2 = \frac{1}{2c_{22}}[-(c_{12}^2 + 2c_{12} - c_{11}c_{22}) + \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}}]$$

$$N_2^2 = \frac{1}{2c_{22}}[-(c_{12}^2 + 2c_{12} - c_{11}c_{22}) - \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}}]. \tag{29}$$

Using the relation

$$\frac{\sin\xi x\sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_0(\xi w)J_0(\xi v)\,dv\,dw}{(x^2 - w^2)^{1/2}(t^2 - v)^{1/2}} \tag{30}$$

eq. (25) can be rewritten in the following form

$$\int_c^1 \frac{th(t^2)}{t^2 - x^2}\,dt = q_0 - \frac{d}{dx}\int_c^1 h(t^2)\,dt \int_0^x\int_0^t \frac{vwL(v,w)\,dw\,dv}{(x^2 - w^2)^{1/2}(t^2 - v^2)^{1/2}}, \quad c \leqslant |x| \leqslant 1 \tag{31}$$

where

$$L(v,w) = \int_0^\infty \xi H_1(\xi)J_0(\xi w)J_0(\xi v)\,d\xi. \tag{32}$$

Applying a contour integration technique, the infinite integral in $L(v, w)$ can be converted to the following finite integrals (details given in the Appendix)

$$L(v, w) = -ik_s^2\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - c_{22}\bar{\alpha}_1\bar{\gamma}_1 - \bar{\beta}(c_{12}\eta^2 - c_{22}\bar{\alpha}_2\bar{\gamma}_2)}{\theta(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)} \times J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\mathrm{d}\eta - \int_{1/\sqrt{c_{11}}}^1 \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\mathrm{d}\eta\right], \quad w > v \tag{33}$$

where

$$\begin{aligned}
\bar{\gamma}_1 &= [\tfrac{1}{2}\{R_1 - (R_1^2 - 4\bar{R}_2)^{1/2}\}]^{1/2} \\
\bar{\gamma}_2 &= [\tfrac{1}{2}\{R_1 + (R_1^2 - 4\bar{R}_2)^{1/2}\}]^{1/2} \\
\hat{\gamma}_1 &= [\tfrac{1}{2}\{-R_1 + (R_1^2 + 4R_2')^{1/2}\}]^{1/2} \\
\hat{\gamma}_2 &= [\tfrac{1}{2}\{R_1 + (R_1^2 + 4R_2')^{1/2}\}]^{1/2} \\
R_1 &= \frac{1}{c_{22}}\{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\eta^2 + (1 + c_{22})\} \\
\bar{R}_2 &= \frac{c_{11}}{c_{22}}(1 - \eta^2)\left(\frac{1}{c_{11}} - \eta^2\right) \\
R_2' &= \frac{c_{11}}{c_{22}}(1 - \eta^2)\left(\eta^2 - \frac{1}{c_{11}}\right) \\
\bar{\alpha}_i &= \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_i^2}{(1 + c_{12})\bar{\gamma}_i} \; (i = 1, 2) \\
\hat{\alpha}_i &= \frac{c_{11}\eta^2 - 1 + (-1)^i\hat{\gamma}_i^2}{(1 + c_{12})\hat{\gamma}_i} \; (i = 1, 2) \\
\bar{\beta} &= \frac{\bar{\alpha}_1 - \bar{\gamma}_1}{\bar{\alpha}_2 - \bar{\gamma}_2} \\
\hat{\beta} &= \frac{\hat{\alpha}_1 + \hat{\gamma}_1}{\hat{\alpha}_2 - \hat{\gamma}_2}.
\end{aligned} \tag{34}$$

The corresponding expression of $L(v, w)$ for $w < v$ follows from eq. (33) by interchanging w and v.

Employing the series expansions for the Bessel function J_0 and the Hankel function $H_0^{(1)}$ in eq. (33), it is found that

$$L(v, w) = \frac{2}{\pi}Pk_s^2\log k_s + O(k_s^2) \tag{35}$$

where

$$P = \frac{1}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - c_{22}\bar{\alpha}_1\bar{\gamma}_1 - \bar{\beta}(c_{12}\eta^2 - c_{22}\bar{\alpha}_2\bar{\gamma}_2)}{(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)}\mathrm{d}\eta - \int_{1/\sqrt{c_{11}}}^1 \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)}\mathrm{d}\eta\right].$$

Now, let us expand $h(t^2)$ in the form

$$h(t^2) = h_0(t^2) + k_s^2\log k_s h_1(t^2) + O(k_s^2). \tag{36}$$

Inserting the above expansion of $h(t^2)$ and the value of $L(v, w)$ given by eq. (35) into eq. (31) and equating the coefficients of like powers of k_s, we obtain the equations

$$\int_c^1 \frac{th_0(t^2)}{t^2 - x^2}\mathrm{d}t = q_0, \quad c \leqslant |x| \leqslant 1 \tag{37}$$

and

$$\int_c^1 \frac{th_1(t^2)}{t^2 - x^2}\mathrm{d}t = -\frac{2P}{\pi}\int_c^1 th_0(t^2)\mathrm{d}t, \quad c \leqslant |x| \leqslant 1. \tag{38}$$

Using the finite Hilbert transform technique [12], the solutions of the above integral equations can be obtained as

$$h_0(t^2) = \frac{2}{\pi} q_0 \left(\frac{t^2 - c^2}{1 - t^2} \right)^{1/2} + \frac{D_1}{\sqrt{(1 - t^2)(t^2 - c^2)}} \tag{39}$$

$$h_1(t^2) = -\frac{2}{\pi} P \left[\frac{q_0(1 - c^2)}{\pi} + D_1 \right] \left(\frac{t^2 - c^2}{1 - t^2} \right)^{1/2} + \frac{D_2}{\sqrt{(1 - t^2)(t^2 - c^2)}}, \tag{40}$$

where D_1 and D_2 are constants to be determined using the condition given by eq. (24) so that

$$\int_c^1 h_0(t^2)\mathrm{d}t = 0 \tag{41}$$

and

$$\int_c^1 h_1(t^2)\mathrm{d}t = 0.$$

Substitution of the values of $h_0(t^2)$ and $h_1(t^2)$ given by eqs (39) and (40) in (41), yields

$$D_1 = \frac{2}{\pi} q_0 \left[c^2 - \frac{E}{F} \right] \tag{42}$$

$$D_2 = \frac{2}{\pi^2} q_0 \left[1 + c^2 - \frac{2E}{F} \right] \left[\frac{E}{F} - c^2 \right], \tag{43}$$

where $F = F[\pi/2, \sqrt{1 - c^2}]$ and $E = E[\pi/2, \sqrt{1 - c^2}]$ are the elliptic integrals of first and second kind, respectively. Substituting the value of D_1 and D_2 given by eqs (42) and (43) into eqs (39) and (40), we obtain

$$h_0(t^2) = -\frac{p_0}{\mu_{12}\theta} \frac{\left[t^2 - \frac{E}{F} \right]}{\sqrt{(1 - t^2)(t^2 - c^2)}} \tag{44}$$

$$h_1(t^2) = -\frac{P p_0}{\pi \mu_{12}\theta} \frac{\left[1 + c^2 - \frac{2E}{F} \right] \left[t^2 - \frac{E}{F} \right]}{\sqrt{(1 - t^2)(t^2 - c^2)}}. \tag{45}$$

CRACK OPENING DISPLACEMENT AND STRESS INTENSITY FACTORS

The crack opening displacement and the normal stress component in the plane of the crack can be written as

$$\Delta v(x, 0) = v(x, 0+) - v(x, 0-) = 2 \int_x^1 h(t^2)\mathrm{d}t, \quad c \leqslant x \leqslant 1 \tag{46}$$

and

$$\tau_{yy}(x, 0) = \frac{2\mu_{12}\theta}{\pi} \int_c^1 \frac{t h(t^2)}{t^2 - x^2} \mathrm{d}t, \quad 0 < x < c \tag{47}$$

$$= -\frac{2\mu_{12}\theta}{\pi} \int_c^1 \frac{t h(t^2)}{x^2 - t^2} \mathrm{d}t, \quad x > 1. \tag{48}$$

Expressions (47) and (48) with the aid of the eqs (36), (44) and (45) yield

$$\tau_{yy}(x, 0) = -p_0 \left[1 + \frac{\left[x^2 - \frac{E}{F} \right]}{\sqrt{(1 - x^2)(c^2 - x^2)}} \right] \left[1 - \frac{P}{\pi} \left\{ 1 + c^2 - \frac{2E}{F} \right\} k_s^2 \log k_s \right] + O(k_s^2), \quad 0 < x < c \tag{49}$$

$$\tau_{yy}(x, 0) = -p_0 \left[1 - \frac{\left[x^2 - \frac{E}{F} \right]}{\sqrt{(x^2 - 1)(x^2 - c^2)}} \right] \left[1 - \frac{P}{\pi} \left\{ 1 + c^2 - \frac{2E}{F} \right\} k_s^2 \log k_s \right] + O(k_s^2), \quad x > 1. \tag{50}$$

The stress intensity factors are defined as (in physical units)

$$K_c = \lim_{x \to c-} \left[\frac{\sqrt{(c-x)}\tau_{yy}(x,0)}{p_0} \right]_{0<x<c} \tag{51}$$

$$K_1 = \lim_{x \to 1+} \left[\frac{\sqrt{(x-1)}\tau_{yy}(x,0)}{p_0} \right]_{x>1}. \tag{52}$$

Substituting eqs (49) and (50) into (51) and (52) it can be shown that

$$K_c = -\frac{\left[c^2 - \dfrac{E}{F}\right]}{\sqrt{2c(1-c^2)}} \left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\} k_s^2 \log k_s \right] + O(k_s^2) \tag{53}$$

$$K_1 = \frac{\left[1 - \dfrac{E}{F}\right]}{\sqrt{2(1-c^2)}} \left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\} k_s^2 \log k_s \right] + O(k_s^2). \tag{54}$$

Further substituting eqs (36), (44) and (45) in the expression given by eq. (46), the crack opening displacement is obtained as

$$\Delta v(x,0) = \frac{2p_0}{\mu_{12}\theta} \left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\} k_s^2 \log k_s \right] \left[\frac{E\left(\dfrac{\pi}{2}, q\right)}{F\left(\dfrac{\pi}{2}, q\right)} F(\lambda, q) - E(\lambda, q) \right] + O(k_s^2), \quad c \leqslant x \leqslant 1 \tag{55}$$

where

$$\sin \lambda = \sqrt{\frac{1-x^2}{1-c^2}} \text{ and } q = \sqrt{(1-c^2)}.$$

Letting $c \to 0$ in the expression for stress intensity factor and crack opening displacement, the results for a single crack occupying the region $|x| \leqslant 1, y = 0, |z| < \infty$ are found to be

$$K_1 = \frac{1}{\sqrt{2}} \left[1 - \frac{P}{\pi} k_s^2 \log k_s \right] + O(k_s^2) \tag{56}$$

$$\Delta v(x,0) = -\frac{2p_0}{\mu_{12}\theta} \sqrt{(1-x^2)} \left[1 - \frac{P}{\pi} k_s^2 \log k_s \right] + O(k_s^2), \quad 0 \leqslant x \leqslant 1. \tag{57}$$

For isotropic medium, putting

$$c_{11} = c_{22} = \frac{\lambda + 2\mu}{\mu}, \quad \mu_{12} = \mu, \quad c_{12} = c_{11} - 2 = \frac{\lambda}{\mu}$$

so that

$$\alpha_1 = \gamma_1, \quad \alpha_1 = \xi^2/\gamma_2, \quad k_s = m_2, \quad k_s/\sqrt{c_{11}} = m_1, \quad \tau^2 = \frac{1}{c_{11}}$$

$$N_1 = 1 = N_2, \quad \theta = -2(1-\tau^2) \text{ and } P = \frac{\pi}{2} c_1,$$

where

$$c_1 = \frac{3\tau^4 - 4\tau^2 - 3}{4(1-\tau^2)}, \quad \gamma_i = (\xi^2 - m_i^2)^{1/2} \text{ and } m_i = \frac{a\omega}{c_i} (i = 1, 2)$$

the expressions for displacement and stress are found to be

$$v(x, \pm 0) = \mp \frac{p_0}{2\mu(1-\tau^2)} \left[1 - \frac{c_1}{2}\left\{1 + c^2 - \frac{2E}{F}\right\} m_2^2 \log m_2 \right]$$

$$\times \left[\frac{E\left(\dfrac{\pi}{2}, q\right)}{F\left(\dfrac{\pi}{2}, q\right)} F(\lambda, q) - E(\lambda, q) \right] + O(m_2^2), \quad c \leqslant x \leqslant 1$$

$$= 0, \quad |x| < c, \quad |x| > 1$$

and

$$\tau_{yy}(x,0) = -p_0\left[1+\frac{\left[x^2-\dfrac{E}{F}\right]}{\sqrt{(1-x^2)(c^2-x^2)}}\right]\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]+O(m_2^2)\ 0<x<c$$

$$= -p_0,\quad c\leqslant |x|\leqslant 1$$

$$= -p_0\left[1-\frac{\left[x^2-\dfrac{E}{F}\right]}{\sqrt{(x^2-1)(x^2-c^2)}}\right]\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]+O(m_2^2),\quad |x|>1.$$

Now, the crack opening displacement and stress intensity factors are found to be

$$\Delta v(x,0) = -\frac{p_0}{\mu(1-\tau^2)}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]$$
$$\times\left[\frac{E\left(\dfrac{\pi}{2},q\right)}{F\left(\dfrac{\pi}{2},q\right)}F(\lambda,q)-E(\lambda,q)\right]+O(m_2^2),\quad c\leqslant x\leqslant 1$$

and

$$K_c = -\frac{\left[c^2-\dfrac{E}{F}\right]}{\sqrt{2c(1-c^2)}}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]+O(m_2^2)$$

$$K_1 = \frac{\left[1-\dfrac{E}{F}\right]}{\sqrt{2(1-c^2)}}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]+O(m_2^2)$$

which coincide with the results obtained by Jain and Kanwal [5] up to the order of $m_2^2\log m_2$ in the isotropic case.

When $c\to 0$, we recover the stress intensity factor and crack opening displacement for a single crack

$$K_1 = \frac{1}{\sqrt{2}}\left[1-\frac{c_1}{2}m_2^2\log m_2\right]+O(m_2^2)$$

$$\Delta v(x,0) = \frac{p_0}{\mu(1-\tau^2)}\sqrt{(1-x^2)}\left[1-\frac{c_1}{2}m_2^2\log m_2\right]+O(m_2^2),\quad 0\leqslant x\leqslant 1$$

which agrees with the result of Mal [2].

NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) K_c and K_1 given by eqs (53) and (54) at the inner and outer tips of the cracks, and crack opening displacement (COD) given by eq. (55) have been plotted against dimensionless frequency k_s and distance, respectively for three different types of orthotropic materials whose engineering constants have been listed in Table 1.

From Fig. 2 it is found that SIF K_c at the inner tip of the crack increases at a slow rate with the increase in the value of frequency $k_s(0.1\leqslant k_s\leqslant 0.6)$. On the other hand the rate of increase of

Table 1. Engineering elastic constants

	E_1 (Pa)	E_2 (Pa)	μ_{12} (Pa)	ν_{12}
Type I	Modulite II graphite–epoxy composite:			
	15.3×10^9	158.0×10^9	5.52×10^9	0.033
Type II	E-Type glass–epoxy composite:			
	9.79×10^9	42.3×10^9	3.66×10^9	0.063
Type III	Stainless steel–aluminium composite:			
	79.76×10^9	85.91×10^9	30.02×10^9	0.31

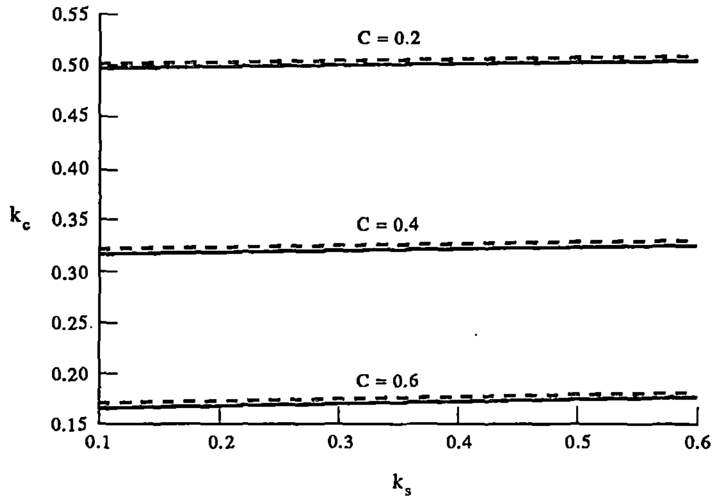

Fig. 2. Stress intensity factor K_c vs frequency k_s for generalized plane stress. (—, Type I; - - - -, Type II).

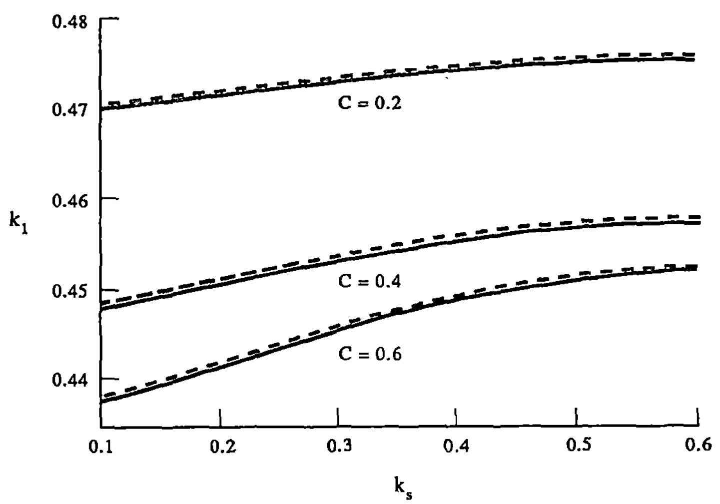

Fig. 3. Stress intensity factor K_1 vs frequency k_s for generalized plane stress. (—, Type I; - - - -, Type II).

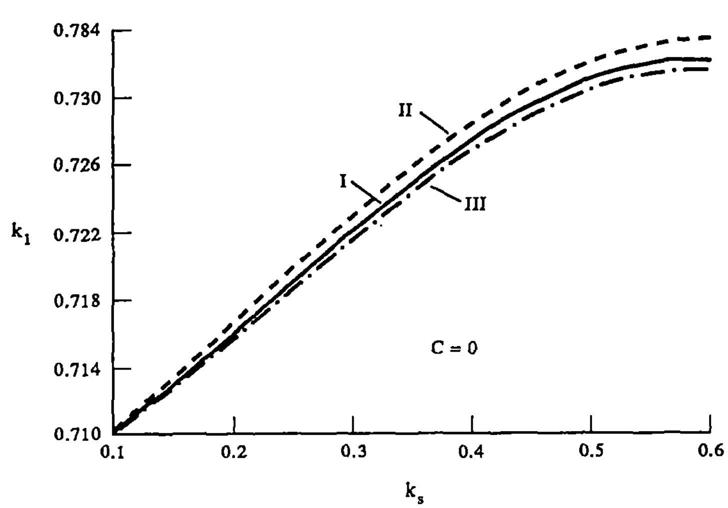

Fig. 4. Stress intensity factor K_1 vs frequency k_s for generalized plane stress. (Single crack, $c = 0$).

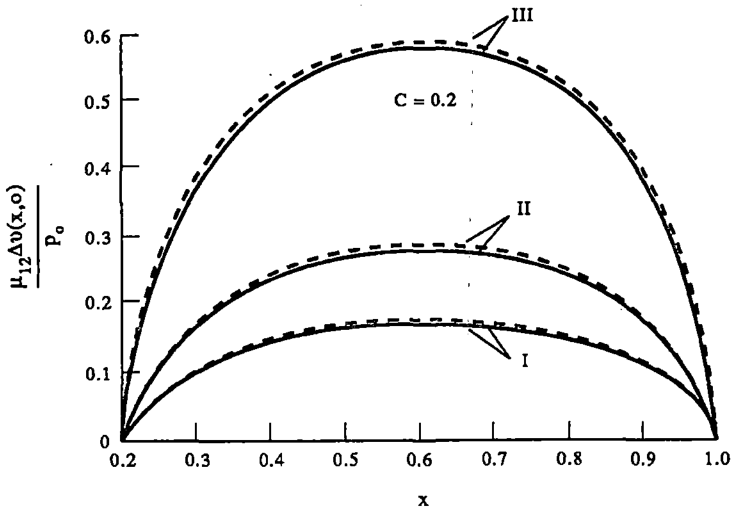

Fig. 5. Crack opening displacement (COD) vs distance ($c = 0.2$) for generalized plane stress. (—, $k_s = 0.2$; - - - -, $k_s = 0.6$).

the SIF K_I (Fig. 3) with frequency k_s at the outer tip of the crack is found to be higher than that of K_c.

In both the cases the value of SIF is higher for small values of c, i.e. for greater crack length SIF is higher. But it is interesting to note that for different materials the variation of SIFs in both the cases are not significant. In the case of single crack ($c = 0$) the variation of SIF with material properties has been shown in Fig. 4.

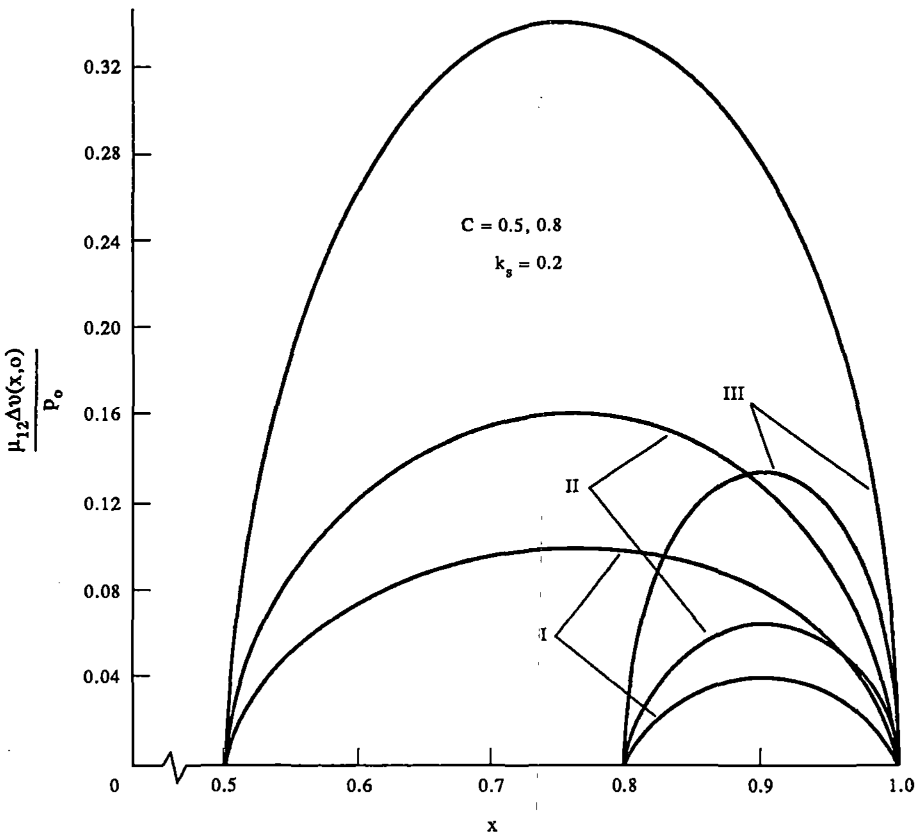

Fig. 6. Crack opening displacement (COD) vs distance ($c = 0.5$ and $c = 0.8$) for generalized plane stress.

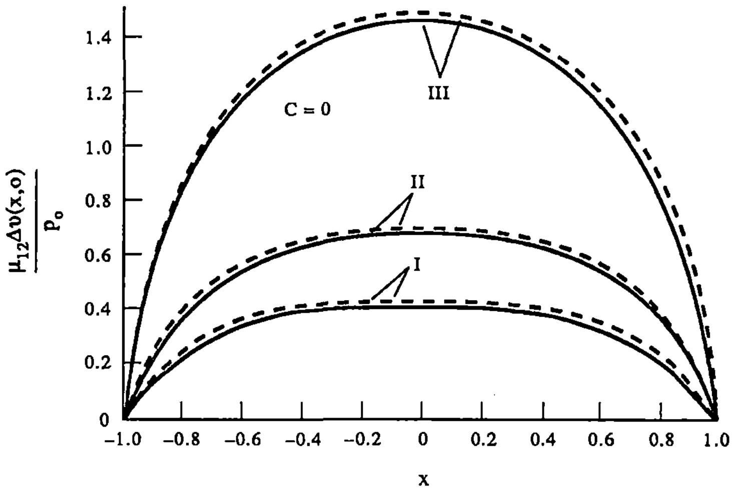

Fig. 7. Crack opening displacement (COD) vs distance (single crack, $c = 0$) for generalized plane stress. (—, $k_s = 0.2$; - - - -, $k_s = 0.6$).

The COD has been plotted for different crack lengths. In each case COD increases gradually from zero, attains maximum value and then decreases to zero. It is found that with the increase in the values of c (i.e. for small crack length) the values of COD decrease (Figs 5 and 6). For a fixed material the variation of COD with frequency is found to be insignificant, but it is noticed that for smaller values of c (Figs 5 and 7) the variation of COD with frequency is palpable; $c = 0$ (Fig. 7) corresponds to the case of single crack.

In all the cases where different values of c have been considered the variation of COD is found to be prominent for different orthotropic materials.

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APPENDIX

Evaluation of L(v, w)

The integral $L(v, w)$ given by eq. (32) is

$$L(v, w) = \int_0^\infty M(\xi, \gamma_1, \gamma_2) J_0(\xi w) J_0(\xi v) \mathrm{d}\xi \tag{A1}$$

where

$$M(\xi,\gamma_1,\gamma_2)=\xi H_1(\xi)=\frac{c_{12}\xi^2-c_{22}\alpha_1\gamma_1-\beta(c_{12}\xi^2-c_{22}\alpha_2\gamma_2)}{\theta(\alpha_1+\beta\alpha_2)}-\xi \tag{A2}$$

$$\gamma_1=[\tfrac{1}{2}\{-B_1+(B_1^2-4B_2)^{1/2}\}]^{1/2}$$

$$\gamma_2=[\tfrac{1}{2}\{-B_1-(B_1^2-4B_2)^{1/2}\}]^{1/2}$$

$$B_1=\frac{1}{c_{22}}\{(c_{12}^2+2c_{12}-c_{11}c_{22})\xi^2+(1+c_{22})k_s^2\} \tag{A3}$$

$$B_2=\frac{1}{c_{22}}(\xi^2-k_s^2)(c_{11}\xi^2-k_s^2).$$

To evaluate the integral (A1) we consider two contour integrals

$$I_1=\int_{\Gamma_1}M(\xi,\gamma_1,\gamma_2)J_0(\xi v)H_0^{(1)}(\xi w)\mathrm{d}\xi,\quad w>v$$

$$I_2=\int_{\Gamma_2}M(\xi,\gamma_1,\gamma_2)J_0(\xi v)H_0^{(2)}(\xi w)\mathrm{d}\xi,\quad w>v, \tag{A4}$$

where Γ_1 and Γ_2 are the closed contours defined in Fig. 8, and $H_0^{(1)}$, $H_0^{(2)}$ are the zero order Hankel functions of the first and second kind, respectively.

Assuming the relation

$$\left\{\frac{(c_{12}^2+2c_{12}-c_{11}c_{22})(1+c_{22})}{c_{22}^2}+\frac{2(1+c_{11})}{c_{22}}\right\}^2-\left\{\frac{(c_{12}^2+2c_{12}-c_{11}c_{22})^2}{c_{22}^2}-\frac{4c_{11}}{c_{22}}\right\}\times\left\{\frac{(1+c_{22})^2}{c_{22}^2}-\frac{4}{c_{22}}\right\}<0 \tag{A5}$$

it is noted that the branch points $\xi=\lambda_i(i=1-4)$ corresponding to the roots of the equation $B_1^2-4B_2=0$ are always complex. Now, the branch points corresponding to the roots of the equations

$$-B_1+(B_1^2-4B_2)^{1/2}=0\text{ and }-B_1-(B_1^2-4B_2)^{1/2}=0$$

are $\xi=\pm k_s$ and $\xi=\pm k_s/\sqrt{c_{11}}$, respectively where it has been assumed that

$$(c_{11}c_{22}-c_{12}^2-2c_{12})>(1+c_{22}) \tag{A6}$$

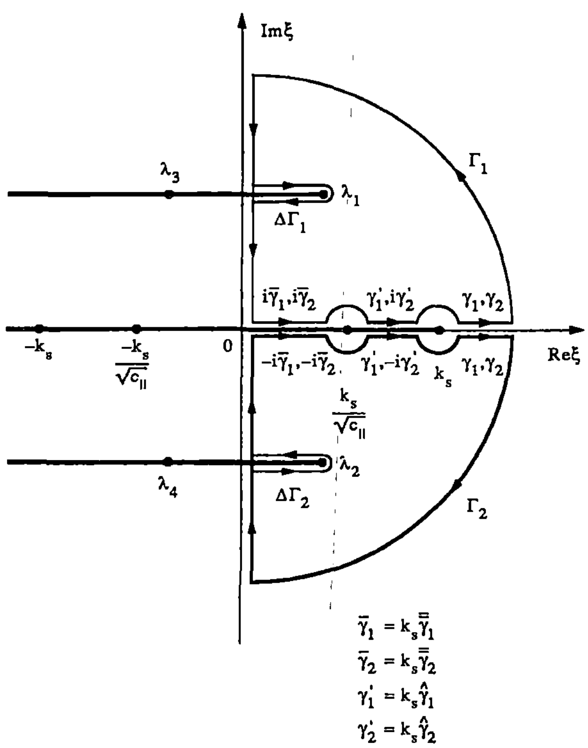

Fig. 8. Contours of integration for integral in eq. (A1).

and

$$c_{12}^2 + 2c_{12} + c_{11} > 0.$$

Therefore under the above conditions, $\xi = \pm k_s/\sqrt{c_{11}}$ and $\xi = \pm k_s$ are the branch points of γ_1 and γ_2, respectively. Equations (A5) and (A6) are true for most of the orthotropic materials. The integrals in eq. (A4) can be shown to be zero on the contours $\Delta\Gamma_1$ and $\Delta\Gamma_2$ (Fig. 8) around the branch cuts from λ_1 and λ_2. Thus integrating along the contours Γ_1 and Γ_2 the integral $L(v, w)$ for $w > v$ can be finally written as

$$L(v, w) = -ik_s^2 \left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - c_{22}\bar{\alpha}_1\bar{\gamma}_1 - \bar{\beta}(c_{12}\eta^2 - c_{22}\bar{\alpha}_2\bar{\gamma}_2)}{\theta(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)} \times J_0(k_s\eta v) H_0^{(1)}(k_s\eta w)\,\mathrm{d}\eta \right.$$
$$\left. - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)} J_0(k_s\eta v) H_0^{(1)}(k_s\eta w)\,\mathrm{d}\eta \right], \quad w > v$$

where $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\beta}, \hat{\beta}, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\gamma}_2$ are given by eq. (34).

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Steady State Propagation of a Series of Parallel Cracks in Anti-Plane State of Strain in an Inhomogeneous Elastic Medium

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Abstract

The problem of a series of semi-infinite, parallel and equally spaced cracks subjected to identical loads satisfying the conditions of anti-plane state of strain and steadily propagating in an infinite inhomogeneous medium has been solved by the application of Wiener-Hopf technique. Elastic moduli and density are assumed to vary exponentially in the direction of propagation of the cracks. The problem of crack propagation in the case of constant strain on the crack edges has been treated. Expressions of the stress and crack opening displacement have been derived in closed form and the effect of the inhomogeneity of the medium has been shown by means of graphs.

1. Introduction

Many authors have studied the dynamic crack propagation in a homogeneous elastic medium. The problem presents an interest for better understanding of the brittle behaviour of materials. Scattering of elastic wave by a single crack has been studied in great detail. But the literature involving the scattering of elastic waves by a series of cracks is very few. It is only recently that Angel and Achenbach [1] studied the reflection and transmission of elastic waves by a periodic array of cracks. Matczynski [2] also considered the quasi static problem of an infinite homogeneous elastic medium weakened by an infinite number of semi-infinite equally spaced parallel cracks. However, natural or artificial materials are

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generally inhomogeneous and propagation of cracks in an inhomogeneous medium has not been studied. Recently, steady state solutions have been derived by Atkinson [3] for crack propagation in media with spatially varying elastic moduli when the crack propagates in a plane where the elastic moduli are constant. Atkinson and List [4] also considered the steady state crack propagation in variable moduli media when the crack moves in the direction of the modulus variation. Steady state crack propagation due to shear waves in a medium of monoclinic type has recently been studied by Chattopadhyay and Bandyopadhyay [5].

In our paper, we have considered the steady state propagation of a series of semi-infinite, rectilinear parallel and uniformly spaced cracks in an infinite inhomogeneous medium. Cracks are assumed to move steadily in the direction of modulus variation, it being assumed that the moduli vary exponentially. We further assume that the medium possesses constant elastic wave speeds. These assumptions are necessary for the steady state solution to exist. We assume that the loading is such that Mode III conditions prevail. Mode III is the simplest mode to analyze mathematically. Nevertheless, it can be expected that the results for the stress intensity factor obtained here will be qualitatively similar to other modes, even though the specific structure of the stress variation near the crack tip will differ in each case. Following Atkinson and List [4], we have also assumed in our paper that the edges of the cracks are loaded on their entire length by constant strain.

2. Formulation of the Problem

Consider an infinite elastic medium with spatially varying density and elastic moduli divided partially by an infinite number of semi-infinite, rectilinear, parallel and uniformly spaced cracks.

The semi-infinite cracks are situated parallel to the negative x_1-axis at $2h$ distance apart and move along positive x_1-direction at a constant velocity $c < c_2$.

The cracks are assumed to propagate steadily in the direction of modulous variation. We assume that the elastic moduli and density both vary exponentially in the same manner; so that the medium may have constant elastic wave speeds.

Owing to symmetry of the problem, it is reduced to the problem of an infinite elastic strip of thickness $2h$ weakened in the middle plane $x_2 = 0$ by a semi-infinite crack $x_1 < 0$, the surfaces $x_2 = \pm h$ of the strip being rigidly clamped.

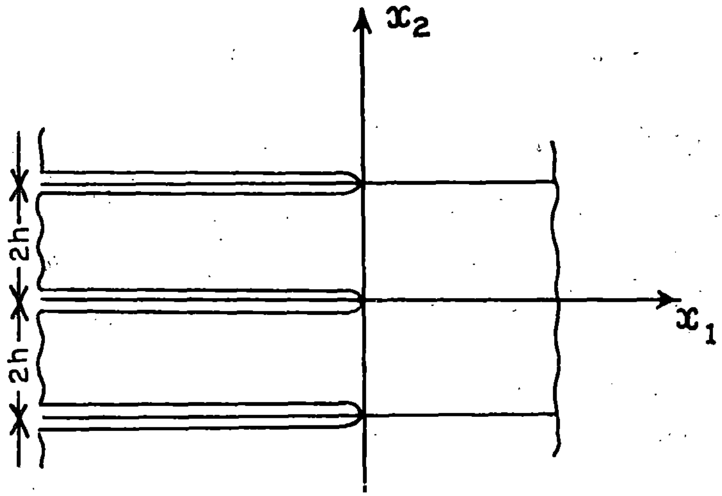

Fig.1

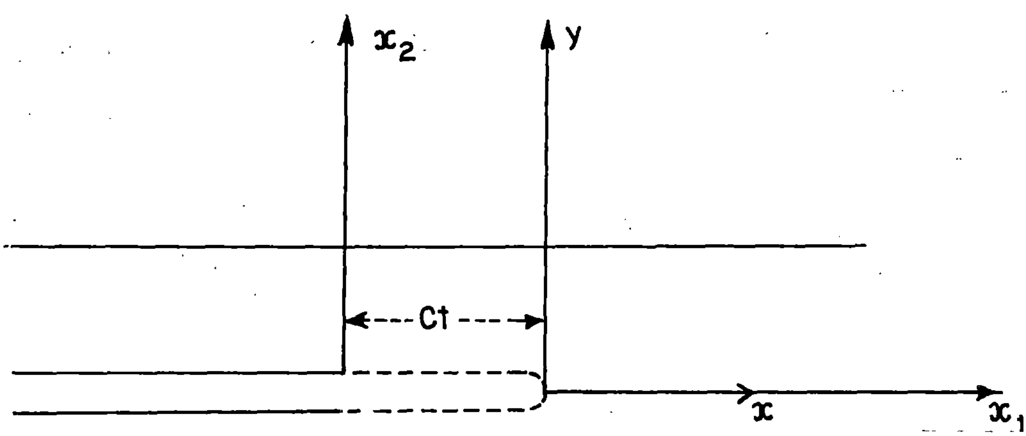

Fig.2

The displacement $\vec{U}$ in the anti-plane state of strain in a rectangular co-ordinate system (x_1, x_2, x_3) is in the form

$$\vec{U} = [0, 0, w(x_1, x_2, t)] \tag{1}$$

The non-vanishing components of this state of strain are given by the following relations:-

$$e_{13} = \frac{\partial w}{\partial x_1}, \qquad e_{23} = \frac{\partial w}{\partial x_2}$$

$$\sigma_{13} = \mu \frac{\partial w}{\partial x_1} = \mu_0 e^{2\alpha x_1} \frac{\partial w}{\partial x_1}, \quad \sigma_{23} = \mu \frac{\partial w}{\partial x_2} = \mu_0 e^{2\alpha x_1} \frac{\partial w}{\partial x_2} \tag{2}$$

where the shear modulous $\mu(x_1) == \mu_0 e^{2\alpha x_1}$, μ_0 and α are constants.

Using relation (2), the equation of motion of SH-waves is

$$\frac{\partial}{\partial x_1}\left[\mu(x_1)\frac{\partial w}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[\mu(x_1)\frac{\partial w}{\partial x_2}\right] = \rho(x_1)\frac{\partial^2 w}{\partial t^2}$$

or, $$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + 2\alpha\frac{\partial w}{\partial x_1} = c_2^{-2}\frac{\partial^2 w}{\partial t^2} \tag{3}$$

where $\rho(x_1) = \rho_0 e^{2\alpha x_1}$; so $c_2 = \sqrt{\mu(x_1)/\rho(x_1)} = \sqrt{\mu_0/\rho_0}$ is the shear wave velocity.

The fixed coordinate system may be replaced by the conventional system (x, y, z) moving with the crack tip.

$$x_1 = x + ct, \quad x_2 = y, \quad x_3 = z \tag{4}$$

Using relation (4), equation (3) becomes

$$\left(1 - \frac{c^2}{c_2^2}\right)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 2\alpha\frac{\partial w}{\partial x} = 0 \tag{5}$$

Applying complex Fourier transform in x, equation (5) becomes

$$\frac{d^2\overline{W}}{dy^2} - \beta^2\overline{W} = 0 \tag{6}$$

where $\beta^2 = (1 - \frac{c^2}{c_2^2})\,\zeta^2 + 2\,i\,\alpha\,\zeta$ (7.1)

and $\overline{W}(\zeta\,,y) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} w(x,y)\, e^{i\zeta x}\, dx$ (7.2)

The solution of equation (6) becomes

$$\overline{W}(\zeta,y) = A \sinh(\beta y) + B \cosh(\beta y) \tag{8}$$

where the constants A and B are to be determined.

3. Solution of the problem for constant strain $\frac{\partial w}{\partial y} = P$ of the crack edges $x < 0$

We now consider the problem when the constant strain given by

$$\frac{\partial w}{\partial y} = P \tag{9}$$

is applied to the crack face $y = 0$, $x < 0$.

We shall therefore consider the steady state crack propagation under the boundary conditions.

$\frac{\partial w}{\partial y} = P,$ for $x < 0\,, y = 0$ (10.1)

$w(x,y) = 0,$ for $x > 0, y = 0$ (10.2)

$w(x,y) = 0,$ for $|x| < \infty, y = h$ (10.3)

Now we can write

$\frac{\partial w}{\partial y} = P,$ for $x < 0, y = 0$

$= e(x)\,,$ for $x > 0, y = 0$

where $e(x)$ is the unknown function which is to be detemined.

In our case

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{\partial w}{\partial y} e^{i\zeta x}\, dx = (2\pi)^{-1/2} \int_{-\infty}^{0} \frac{\partial w}{\partial y} e^{i\zeta x}\, dx + (2\pi)^{-1/2} \int_{0}^{\infty} \frac{\partial w}{\partial y} e^{i\zeta x}\, dx$$

$$\frac{\partial \overline{W}}{\partial y}(\zeta, 0) = (2\pi)^{-1/2} \int_{-\infty}^{0} P\, e^{i\zeta x}\, dx + (2\pi)^{-1/2} \int_{0}^{\infty} e(x)\, e^{i\zeta x}\, dx \tag{11}$$

Therefore using (8) and writing $(2\pi)^{-1/2} \int_{0}^{\infty} e(x)\, e^{i\zeta x}\, dx = E_{+}(\zeta)$,

$$\beta A = (2\pi)^{-1/2} \frac{P}{i\zeta} + E_{+}(\zeta) \qquad \text{for } -k < \operatorname{Im}\zeta < 0 \tag{12}$$

$$\text{if } e(x) \sim O(e^{-kx}) \quad \text{as } x \to \infty$$

Using the conditions (10.2) and (10.3), it can be easily shown that

$$A = -\frac{\overline{W}_{-}(\zeta, 0)}{\tanh(\beta h)} \tag{13}$$

where $\overline{W}_{-}(\zeta, 0) = (2\pi)^{-1/2} \int_{-\infty}^{0} w(x, 0)\, e^{i\zeta x} dx$ is analytic in the lower half-plane $\operatorname{Im}\zeta < k_1$, if we assume $w(x, 0) \sim O(e^{k_1 x})$ as $x \to -\infty$.

Eliminating A by equations (12) and (13)

$$-\beta\, \frac{\overline{W}_{-}(\zeta, 0)}{\tanh(\beta h)} = \frac{-iP}{\sqrt{2\pi}} \frac{1}{\zeta} + E_{+}(\zeta) \tag{14}$$

Let $$K(\zeta) = \beta \coth(\beta h) = \frac{1}{h} \beta h \frac{\cosh(\beta h)}{\sinh(\beta h)} = \frac{1}{h} \prod_{n=1}^{\infty} \left\{ \frac{1 - \left(\frac{i\beta h}{\pi(n - 1/2)} \right)^2}{1 - \left(\frac{i\beta h}{n\pi} \right)^2} \right\} \tag{15}$$

[cf. Noble [8], eqns. (3.96a) and (3.96b), p.123]

Now consider

$$1-\left(\frac{i\beta h}{n\pi}\right)^2=1+\left(\frac{\beta h}{n\pi}\right)^2=\left(\frac{h}{n\pi}\right)^2\left[\nu^2\zeta^2+2i\alpha\zeta+\left(\frac{n\pi}{h}\right)^2\right]$$

$$=\left(\frac{\nu h}{n\pi}\right)^2\left[\zeta^2+\frac{2i\alpha\zeta}{\nu^2}+\left(\frac{n\pi}{\nu h}\right)^2\right] \quad (16)$$

where $\nu^2=1-c^2/c_2^2$.

So equation (16) can be written as

$$1-\left(\frac{i\beta h}{n\pi}\right)^2=\left(\frac{\nu h}{n\pi}\right)^2(\zeta+i\eta_n^+)(\zeta+i\eta_n^-)$$

where $\eta_n^{\pm}=\frac{\alpha}{\nu^2}\pm\left[\frac{\alpha^2}{\nu^4}+\left(\frac{n\pi}{\nu h}\right)^2\right]^{1/2}$

Similarly, $1-\left(\frac{i\beta h}{(n-1/2)\pi}\right)^2=\left(\frac{\nu h}{n\pi}\right)^2(\zeta+i\eta_{n-1/2}^+)(\zeta+i\eta_{n-1/2}^-)$

It may be noted that η_n^- and $\eta_{n-1/2}^-$ are negative real quantities.

So equation (15) becomes

$$K(\zeta)=\frac{1}{h}\prod_{n=1}^{\infty}\frac{(\zeta+i\eta_{n-1/2}^-)(\zeta+i\eta_{n-1/2}^+)\,n^2}{(\zeta+i\eta_n^-)(\zeta+i\eta_n^+)(n-1/2)^2}$$

$$=\frac{1}{h}\prod_{n=1}^{\infty}\frac{(\zeta+i\eta_{n-1/2}^-)}{(\zeta+i\eta_n^-)}\frac{n}{(n-1/2)}\cdot\prod_{n=1}^{\infty}\frac{(\zeta+i\eta_{n-1/2}^+)}{(\zeta+i\eta_n^+)}\frac{n}{(n-1/2)}$$

$$=K^-(\zeta)\,.K^+(\zeta) \qquad \text{(say)} \quad (17)$$

where $K^-(\zeta)$ is analytic in the lower half-plane given by $\operatorname{Im}\zeta<-\eta_{1/2}^-$ where as $K^+(\zeta)$ is analytic in the upper half plane given by $\operatorname{Im}\zeta>-\eta_{1/2}^+$.

Now $$K^{+}(\zeta)=\prod_{n=1}^{\infty}\frac{(\zeta+i\eta^{+}_{n-1/2})}{(\zeta+i\eta^{+}_{n})}\frac{(n-0)}{(n-1/2)}$$

$$=\prod_{n=1}^{\infty}\frac{\left[\zeta+i\left(\frac{\alpha}{\nu^2}+\left(\frac{\alpha^2}{\nu^4}+\frac{(n-1/2)^2\pi^2}{\nu^2h^2}\right)^{1/2}\right)\right](n-0)}{\left[\zeta+i\left(\frac{\alpha}{\nu^2}+\left(\frac{\alpha^2}{\nu^4}+\frac{n^2\pi^2}{\nu^2h^2}\right)^{1/2}\right)\right](n-1/2)}$$

$$=\prod_{n=1}^{\infty}\frac{\left[\frac{\zeta\nu h}{\pi}+i\left(\frac{\alpha h}{\nu\pi}+\left\{\frac{\alpha^2h^2}{\nu^2\pi^2}+(n-1/2)^2\right\}^{1/2}\right)\right](n-0)}{\left[\frac{\zeta\nu h}{\pi}+i\left(\frac{\alpha h}{\nu\pi}+\left\{\frac{\alpha^2h^2}{\nu^2\pi^2}+n^2\right\}^{1/2}\right)\right](n-1/2)}$$

Now elastic moduli and density are assumed to be varying slowly with x_1 so that αh may be assumed to be small.

So neglecting α^2h^2 we get

$$K^{+}(\zeta)=\prod_{n=1}^{\infty}\frac{\left[\frac{\zeta\nu h}{\pi}+i\left(\frac{\alpha h}{\nu\pi}+(n-1/2)\right\}\right](n-0)}{\left[\frac{\zeta\nu h}{\pi}+i\left\{\frac{\alpha h}{\nu\pi}+n\right\}\right](n-1/2)}$$

$$=\prod_{n=1}^{\infty}\frac{\left[n-\left(\frac{1}{2}+\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right)\right](n-0)}{\left[n-\left(\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right)\right](n-1/2)} \tag{18}$$

Next using the formula

$$\prod_{n=1}^{\infty}\left\{\frac{(n-a_1)\,\ldots\ldots\ldots\ldots\ldots\,(n-a_k)}{(n-b_1)\,\ldots\ldots\ldots\ldots\ldots\,(n-b_k)}\right\}=\prod_{m=1}^{k}\frac{\Gamma(1-b_m)}{\Gamma(1-a_m)}$$

which expresses the general infinite product in terms of the Gamma functions (cf. Whittaker and Watson [6], p.239)) we obtain from (18)

$$K^{+}(\zeta) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left[1-\left(\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right)\right]}{\Gamma(1)\,\Gamma\left[\frac{1}{2}-\left(\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right)\right]} \tag{19}$$

Similarly, for small values of αh, neglecting $\alpha^2 h^2$, it can be easily shown that

$$K^{-}(\zeta) = \frac{\sqrt{\pi}}{h}\frac{\Gamma\left[1+\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[\frac{1}{2}+\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right]} \tag{20}$$

Now writing $\beta \coth(\beta h) = K(\zeta) = K^{+}(\zeta)K^{-}(\zeta)$, equation (14) becomes

$$-K^{+}(\zeta)\; K^{-}(\zeta)\; \overline{W}_{-}(\zeta,0) = -\frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta} + E_{+}(\zeta)$$

so,
$$-K^{-}(\zeta)\,\overline{W}_{-}(\zeta,0) = -\frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\frac{1}{K^{+}(\zeta)} + \frac{E_{+}(\zeta)}{K^{+}(\zeta)}$$

$$= -\frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\left[\frac{1}{K^{+}(\zeta)}-\frac{1}{K^{+}(0)}\right] - \frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\frac{1}{K^{+}(0)} + \frac{E_{+}(\zeta)}{K^{+}(\zeta)}$$

Therefore,

$$-K^{-}(\zeta)\,\overline{W}_{-}(\zeta,0) + \frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\frac{1}{K^{+}(0)} = \frac{E_{+}(\zeta)}{K^{+}(\zeta)} - \frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\left[\frac{1}{K^{+}(\zeta)}-\frac{1}{K^{+}(0)}\right] \tag{21}$$

The expression on the left hand side of equation (21) is regular in the half-plane $\operatorname{Im}\zeta < 0$ whereas R.H.S. is regular in $\operatorname{Im}\zeta > -K_1$ where $K_1 = \min(k, \eta^{+}_{1/2})$. The equation (21) holds in the strip $-K_1 < \operatorname{Im}\zeta < 0$ and therefore using analytic continuation and Liouville's theorem we can write

$$\overline{W}_{-}(\zeta,0) = \frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\frac{1}{K^{+}(0)K^{-}(\zeta)} \tag{22}$$

and $$E_+(\zeta) = \frac{iP}{\sqrt{2\pi}} \frac{1}{\zeta} \left[1 - \frac{K^+(\zeta)}{K^+(0)}\right] \tag{23}$$

Therefore, by help of (11) and (23), we obtain

$$\frac{\partial \overline{W}}{\partial y}(\zeta, 0) = -\frac{iP}{\sqrt{2\pi}} \frac{1}{\zeta} \frac{K^+(\zeta)}{K^+(0)}$$

So, $$\frac{\partial w}{\partial y} = -\frac{iP}{\sqrt{2\pi}} \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} \frac{1}{\zeta} \frac{K^+(\zeta)}{K^+(0)} e^{-i\zeta x} d\zeta \quad \text{where } -K_1 < \varepsilon < 0 \tag{24}$$

For $x < 0$, considering a semi-circular contour in the upper half ζ-plane it can easily be verified that

$$\frac{\partial w}{\partial y} = P$$

Now for $x > 0$, substituting the values of $K^+(\zeta)$ and $K^+(0)$ from (19) and (24) we obtain

$$\frac{\partial w}{\partial y} = -\frac{iP}{2\pi} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} \frac{1}{\zeta} \frac{\Gamma\left[1 - \frac{i\zeta\nu h}{\pi} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[\frac{1}{2} - \frac{i\zeta\nu h}{\pi} + \frac{\alpha h}{\nu\pi}\right]} e^{-i\zeta x} d\zeta \quad (x > 0)$$

$$= -\frac{iP}{2\pi} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} e^{-\frac{\pi x}{\nu h}\left(\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right)} \int_{-i\infty + s}^{i\infty + s} \frac{1}{\left(\frac{1}{2} - p + \frac{\alpha h}{\nu\pi}\right)} \times \frac{\Gamma\left(p + \frac{1}{2}\right)}{\Gamma(p)} e^{\frac{\pi x}{\nu h} p} dp$$

where $s = \frac{1}{2} + \frac{\alpha h}{\nu\pi} - \frac{\nu h \varepsilon}{\pi}$

$$= \frac{iP}{2\pi} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} e^{-\frac{\pi x}{\nu h}\left(\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right)} \int_{-i\infty + s}^{i\infty + s} \frac{\Gamma\left(p - \frac{1}{2} - \frac{\alpha h}{\nu\pi}\right)}{\Gamma\left(p + \frac{1}{2} - \frac{\alpha h}{\nu\pi}\right)} \times \frac{\Gamma(p + \frac{1}{2})}{\Gamma(p)} e^{\frac{\pi x}{\nu h} p} dp$$

$$= -P\frac{\Gamma\left[\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{\alpha h}{\nu\pi}\right]} e^{-\frac{\pi x}{\nu h}\left(\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right)} \frac{e^{\frac{\pi x}{\nu h}\left(\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right)}\left(1-e^{-\frac{\pi x}{\nu h}}\right)^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}$$

$$\times {}_2F_1\left(-\frac{1}{2}; -\frac{\alpha h}{\nu\pi}; \frac{1}{2}; 1-e^{-\frac{\pi x}{\nu h}}\right)$$

[cf. Erdelyi et. al. [7], formula no. 7. p.262]

$$= -\frac{P}{\sqrt{\pi}}\frac{\Gamma\left[\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{\alpha h}{\nu\pi}\right]} \frac{1}{\sqrt{1-\exp(-\pi x/\nu h)}} {}_2F_1\left(-\frac{1}{2}; -\frac{\alpha h}{\nu\pi}; \frac{1}{2}; 1-e^{-\frac{\pi x}{\nu h}}\right)$$

where ${}_2F_1\left(-\frac{1}{2}; -\frac{\alpha h}{\nu\pi}; \frac{1}{2}; 1-e^{-\frac{\pi x}{\nu h}}\right)$ is the hypergemotric function.

It is known that the series

$${}_2F_1(a,b,c,z) = 1 + \frac{a.b}{1.c}z + \frac{a(a+1)\,b(b+1)}{1.2.c(c+1)}z^2 +$$

$$+ \frac{a(a+1)\,(a+2)\,b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)}z^3 + \ldots\ldots\ldots\ldots$$

therefore neglecting $\left(\frac{\alpha h}{\nu\pi}\right)^2$ and higher power of $\frac{\alpha h}{\nu\pi}$,

$${}_2F_1\left(-\frac{1}{2}; -\frac{\alpha h}{\nu\pi}; \frac{1}{2}; 1-e^{-\frac{\pi x}{\nu h}}\right) = 1 + \frac{\alpha h}{\nu\pi}\left(\frac{z}{1} + \frac{z^2}{2.3} + \frac{z^3}{3.4} + \frac{z^4}{4.7} + \ldots\ldots\right)$$

where $z = 1 - e^{-\frac{\pi x}{\nu h}}$;

After a little algebraic simplification it can be shown that for small $\frac{\alpha h}{\nu\pi}$

$${}_2F_1\left(-\frac{1}{2}; -\frac{\alpha h}{\nu\pi}; \frac{1}{2}; z\right) = 1 + \frac{\alpha h}{\nu\pi}\left[(1+\sqrt{z})\log(1+\sqrt{z}) + (1-\sqrt{z})\log(1-\sqrt{z})\right]$$

Therefore

$$\frac{\partial w}{\partial y} = -\frac{P}{\sqrt{\pi}} \frac{\Gamma\left[\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{\alpha h}{\nu\pi}\right]} \frac{1}{\sqrt{1-\exp(-\pi x/\nu h)}}$$

$$\times\left\{1+\frac{\alpha h}{\nu h}\left[(1+\sqrt{z})\log(1+\sqrt{z})+(1-\sqrt{z})\log(1-\sqrt{z})\right]\right\} \quad (x>0) \qquad (25)$$

Next in order to determine the crack opening displacement consider equation (22) viz.

$$\overline{W}_-(\zeta,0) = \frac{iP}{\sqrt{2\pi}} \frac{1}{\zeta} \frac{1}{K^+(0)K^-(\zeta)}$$

which by help of equations (19) and (20) becomes

$$\overline{W}_-(\zeta,0) = \frac{iP}{\sqrt{2\pi}} \frac{h}{\pi} \frac{\Gamma\left[\frac{1}{2}+\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right]} \frac{\Gamma\left[\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{\alpha h}{\nu\pi}\right]} \frac{1}{\zeta}$$

Therefore

$$w(x,0) = \frac{ihP}{\pi} \frac{1}{2\pi} \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{\Gamma\left[\frac{1}{2}+\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right]} \frac{\Gamma\left[\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{\alpha h}{\nu\pi}\right]} \frac{1}{\zeta} e^{i\zeta x}\, d\zeta$$

Obviously for $x>0, w(x,0)\equiv 0$. In order to find $w(x,0)$ for $x<0$, we firstly evaluate $\frac{dw(x,0)}{dx}$ which is given by

$$\frac{dw}{dx} = \frac{hP}{\pi} \frac{1}{2\pi} \frac{\Gamma\left[\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{\alpha h}{\nu\pi}\right]} \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{\Gamma\left[\frac{1}{2}+\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{i\zeta\nu h}{\pi}-\frac{\alpha h}{\nu\pi}\right]} e^{-i\zeta x}\, d\zeta$$

so,
$$\frac{dw}{dx} = \frac{P\, e^{\frac{\pi x}{\nu h}\left(\frac{1}{2}-\frac{\alpha h}{\nu\pi}\right)}}{\nu\sqrt{\pi}} \frac{1}{2\pi i} \frac{\Gamma\left[\frac{1}{2}+\frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1+\frac{\alpha h}{\nu\pi}\right]} \int_{s-i\infty}^{s+i\infty} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p)}{\Gamma(p+\frac{1}{2})} e^{-\frac{\pi x}{\nu h}p}\, dp$$

where $p = \frac{1}{2} + \frac{i\nu\zeta h}{\pi} - \frac{\alpha h}{\nu\pi}$ and $s = \frac{1}{2} - \frac{\alpha h}{\nu\pi} + \frac{\nu h\varepsilon}{\pi}$

Using the table of inverse Laplace transform [7], we find

$$\frac{dw}{dx} = \frac{P\, e^{\frac{\pi x}{\nu h}\left(\frac{1}{2} - \frac{\alpha h}{\nu\pi}\right)}}{\nu\sqrt{\pi}} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \frac{1}{\sqrt{1 - \exp(\pi x/\nu h)}}$$

Integrating w.r.t. x we obtain

$$w(x, 0) = \frac{P}{\nu\sqrt{\pi}} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \int_0^x e^{-\frac{\alpha x}{\nu^2}} \frac{e^{\frac{\pi x}{2\nu h}}}{\sqrt{1 - \exp(\pi x/\nu h)}}\, dx \qquad (\text{for } x<0) \qquad (26)$$

Making $x \to -\infty$, it can easily be shown that

$$w(x, 0) \to -\frac{Ph}{\pi} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \frac{\Gamma\left[\frac{1}{2} - \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 - \frac{\alpha h}{\nu\pi}\right]} \qquad (27)$$

Putting $\alpha = 0$ in (25) and (26) expressions for $\frac{\partial w(x, 0)}{\partial y}$ and $w(x, 0)$ for homogeneous medium can be derived and they are found to be identical with the results given by Matczynski [2].

Crack opening displacement is obviously $\Delta w = 2w(x, 0)$ where $w(x, 0)$ is given by (26). In figs. 3-5 dimensionless values of the crack opening displacement given by $Y = \frac{\pi \Delta w}{2ph}$ have been plotted against the dimensionless distance $x' = -\frac{x}{h}$ along the length of the crack for different values of $\alpha_1 = \frac{\alpha h}{\nu\pi}$ and $c_1 = c/c_2$.

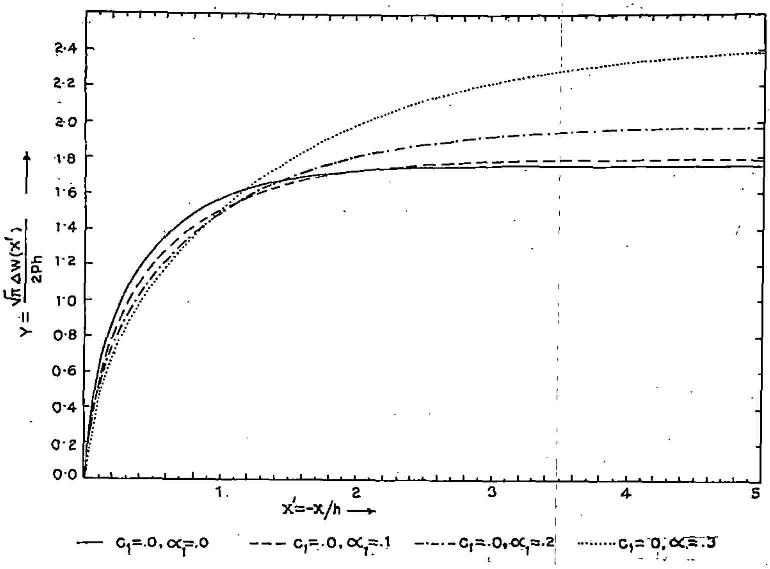
Y = √π ΔW(x′)/2Ph
x′=-x/h
C1=.0, α1=.0
C1=.0, α1=.1
C1=.0, α1=.2
C1=.0, α1=.3

Fig. 3

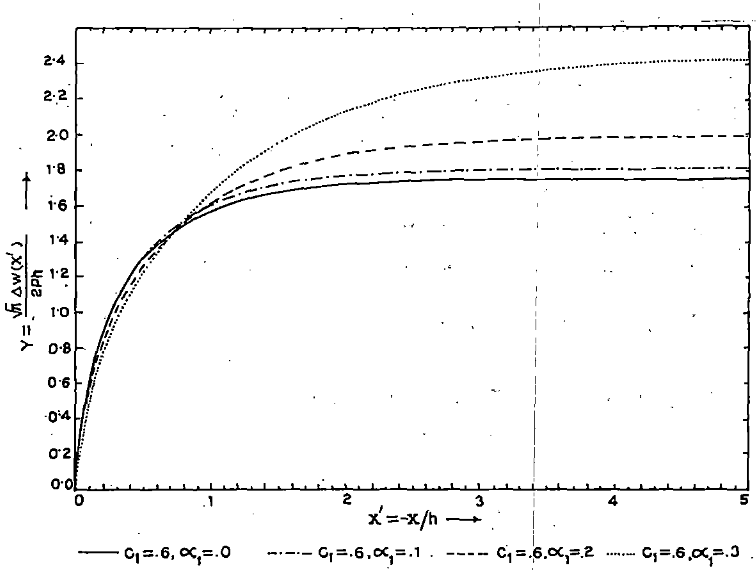
Y = √π ΔW(x′)/2Ph
x′=-x/h
C1=.6, α1=.0
C1=.6, α1=.1
C1=.6, α1=.2
C1=.6, α1=.3

Fig. 4

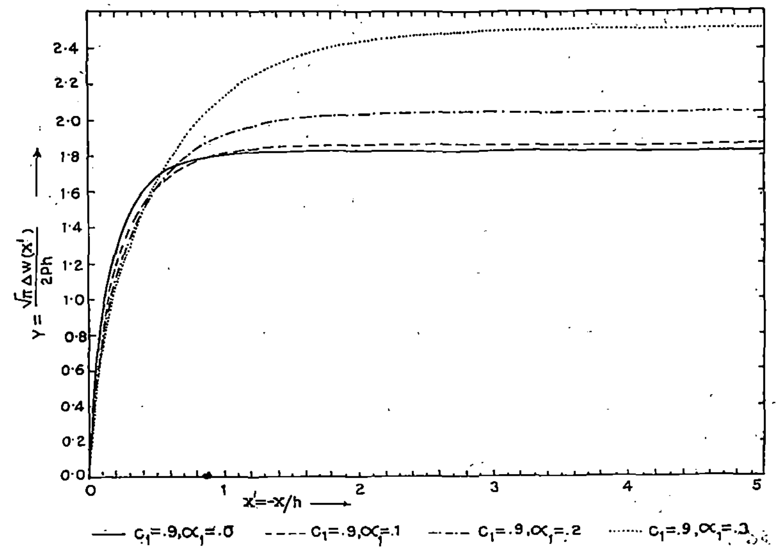

Fig. 5

It is interesting to note that for a fixed value of c_1, crack opening displacement increases with the increase in the values of the inhomogeneity parameter α_1 for large values of x' whereas for small values of x' ($x' \neq 0$), the result is just the opposite. Further it may be noted that for any given value of the inhomogeneity parameter α_1, crack opening displacement Y at any point x' increases with the increase in the crack propagation velocity.

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INPLANE PROBLEM OF DIFFRACTION OF ELASTIC WAVES BY A PERIODIC ARRAY OF COPLANAR GRIFFITH CRACKS

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Abstract—This paper represents the analysis of the problem of diffraction of longitudinal waves by a series of periodically spaced coplanar Griffith cracks in an infinite, isotropic elastic medium. Due to the periodicity of the geometry, the problem with a single crack in a strip with boundaries such that shear stress and normal displacement are zero on them. On use of Fourier transform the mixed boundary value problem for a typical strip has been reduced firstly to the solution of dual integral equations and finally to that of a Fredholm integral equations of the second kind. Numerical values of stress intensity factor and the crack opening displacement have been plotted graphically.

1. INTRODUCTION

THE PROBLEMS involving cracks or inclusions in elastodynamics are of much importance in view of their application in geophysics and earthquake engineering. Uptil now many problems have been solved involving one or two cracks in an infinite homogeneous elastic medium. Loeber and Sih [1] and Mal [2] have studied the problem of diffraction of elastic waves by a Griffith crack in an infinite medium. The problem of a finite crack at the interface of two elastic half spaces has been discussed by Srivastava *et al.* [3] and Bostrom [4]. Finite crack perpendicular to the surface of the infinitely long elastic strip has been studied by Chen [5] for impact load and by Srivastava *et al.* [6] for normally incident waves. But elastodynamic problems involving two or more Griffith cracks have not yet received much attention. Jain and Kanwal [7] have studied the problem of scattering of elastic waves by two Griffith cracks for normally incident waves and the same problem has been considered by Itou [8] for impact load. Angel and Achenbach [9] have studied the problem of reflection and transmission of elastic waves by a periodic array of cracks in an infinite isotropic medium. The problem of diffraction of SH-waves by a series of cuts in nonhomogeneous solid was investigated by De Sarkar [10]. The steady state vibration of an infinite isotropic medium with a periodic system of coplanar cracks has been discussed by Parton and Morozov [11] using the method of the finite Fourier transforms to reduce the relevant mixed relations.

In our paper, the diffraction of normally incident time harmonic elastic waves by a periodic array of coplanar Griffith cracks in infinite elastic medium has been analyzed. Due to geometrical symmetry the problem has been reduced to the solution of the problem of a single crack in a strip whose boundaries are shear free and constrained in a way not to permit normal displacement. Applying Fourier transform the problem has been converted to the solution of dual integral equations. The dual integral equations finally have been reduced to a Fredholm integral equation of second kind by applying Abel's transform. Expressions for stress intensity factor and crack opening displacement have been derived in closed form. The numerical values of stress intensity factor and crack opening displacement have been presented graphically to bring out the salient features of the problem.

2. FORMULATION OF THE PROBLEM

We consider a homogeneous, isotropic, linearly elastic, unbounded solid weakened by a infinite number of collinear cracks of equal length which are equally spaced on a line taken as the x_1-axis. The length of each crack is $2a$ and the period of the crack-array is $2h_1$ as shown in Fig. 1. The

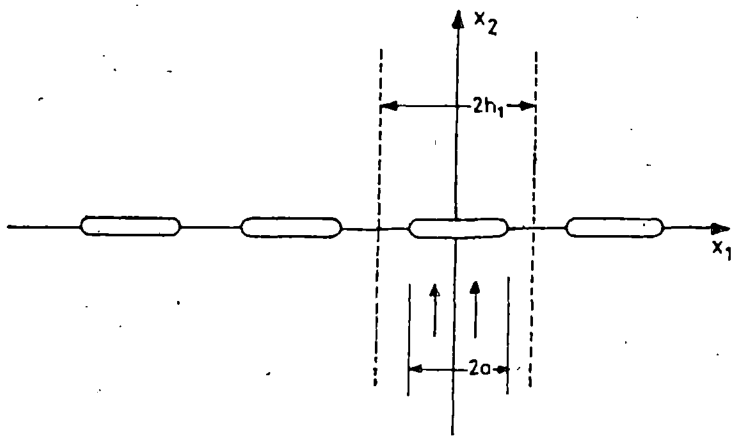

Fig. 1. Incidence of plane time-harmonic wave on a periodic array of cracks.

cracks lie in the plane $x_2 = 0$ and extend to infinity in the x_3-direction which is perpendicular to the plane of the figure. For convenience we make all the lengths dimensionless by writing

$$x_1/a = x, \quad x_2/a = y, \quad x_3/a = z, \quad h_1/a = h.$$

Let an incident time-harmonic body wave travel in the direction of the positive y-axis. The steady state term $e^{-i\omega t}$, which is common to all field variables, has been omitted in the sequel.

By simple symmetry considerations, the displacement and stress distribution due to the scattered field in the entire xy-plane can be derived by considering only the isotropic elastic strip $|x| \leqslant h$ with a central crack $|x| \leqslant 1$, $y = 0$; the boundaries of the strip $x = \pm h$ being shear free and constrained in a way not to permit normal displacement.

The displacement components are

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y} \tag{1}$$

and

$$v = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}$$

where ϕ and ψ are scalar and vector potentials satisfying the following equations.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{a^2}{c_1^2}\frac{\partial^2 \phi}{\partial t^2}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{a^2}{c_2^2}\frac{\partial^2 \psi}{\partial t^2} \tag{2}$$

where $c_1 = (\lambda + 2\mu/\rho)^{1/2}$ and $c_2 = (\mu/\rho)^{1/2}$ are the dilatational and shear wave velocities, λ, μ are the Lame's constant, ρ is the density of the material.

Therefore, substituting $\phi(x, y, t) = \phi(x, y)e^{-i\omega t}$ and $\psi(x, y, t) = \psi(x, y)e^{-i\omega t}$, our problem reduces to the solution of the equations

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + k_1^2 \phi = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + k_2^2 \psi = 0 \tag{3}$$

subject to the boundary conditions

$$\tau_{yy}(x, 0) = -p(x), \quad |x| < 1 \tag{4}$$

$$\tau_{xy}(x,0)=0,\quad |x|\leqslant h \tag{5}$$

$$v(x,0)=0,\quad 1\leqslant |x|\leqslant h \tag{6}$$

$$\tau_{xy}(\pm h, y)=0,\quad |y|<\infty \tag{7}$$

$$u(\pm h, y)=0,\quad |y|<\infty \tag{8}$$

where $k_i = a\omega/c_i$ $(i=1,2)$.

Solutions of eq (3) are

$$\phi(x,y)=\sqrt{\frac{2}{\pi}}\left[\int_0^\infty A_1(\zeta)e^{-\alpha y}\cos\zeta\,x\,d\zeta+\int_0^\infty A_2(\xi)\cosh(\alpha_1 x)\cos\xi y\,d\xi\right]$$

and

$$\psi(x,y)=\sqrt{\frac{2}{\pi}}\left[\int_0^\infty B_1(\zeta)e^{-\beta y}\sin\zeta\,x\,d\zeta+\int_0^\infty B_2(\xi)\sinh(\beta_1 x)\sin\xi y\,d\xi\right] \tag{9}$$

where $A_1(\zeta)$, $A_2(\xi)$, $B_1(\zeta)$, $B_2(\xi)$ are constants and

$$\begin{aligned}
\alpha &= (\zeta^2-k_1^2)^{1/2}, & \zeta>k_1 \quad \beta &= (\zeta^2-k_2 2)^{1/2}, & \zeta>k_2\\
&= -i(k_1^2-\zeta^2)^{1/2}, & \zeta<k_2 \quad &= -i(k_2^2-\zeta^2)^{1/2}, & \zeta<k_2\\
\alpha_1 &= (\xi^2-k_1^2)^{1/2}, & \xi>k_1 \quad \beta_1 &= (\xi^2-k_2^2)^{1/2}, & \xi>k_2\\
&= -i(k_1^2-\xi^2)^{1/2}, & \xi<k_1 \quad &= -i(k_2^2-\xi^2)^{1/2}, & \xi<k_2.
\end{aligned}$$

Now the stress τ_{xy} can be expressed as

$$a\tau_{xy}(x,y)=\sqrt{\frac{2}{\pi}}\left[-\mu\int_0^\infty(-2\zeta\alpha A_1(\zeta)e^{-\alpha y}+(\zeta^2+\beta^2)B_1(\zeta)e^{-\beta y})\sin\zeta\,x\,d\zeta\right.$$
$$\left.+\mu\int_0^\infty(-2\xi\alpha_1 A_2(\xi)\sinh(\alpha_1 x)+(\xi^2+\beta_1^2)B_2(\xi)\sinh(\beta_1 x))\sin\xi y\,d\xi\right]. \tag{10}$$

The boundary condition (5) yields

$$B_1(\zeta)=\frac{2\zeta\alpha}{\zeta^2+\beta^2}A_1(\zeta). \tag{11}$$

Assuming $-\zeta A_1(\zeta)=A(\zeta)$, $\alpha_1 A_2(\xi)=C(\xi)$, $-\xi B_2(\xi)=D(\xi)$ and using the relation (11), expressions for displacements and stresses finally can be written as

$$u=\sqrt{\frac{2}{\pi}}\int_0^\infty\left[e^{-\alpha y}-\frac{2\alpha\beta}{2\zeta^2-k_2^2}e^{-\beta y}\right]A(\zeta)\sin\zeta\,x\,d\zeta$$
$$+\sqrt{\frac{2}{\pi}}\int_0^\infty[C(\xi)\sinh(\alpha_1 x)+D(\xi)\sinh(\beta_1 x)]\cos\xi y\,d\xi \tag{12}$$

$$v=\sqrt{\frac{2}{\pi}}\int_0^\infty\left[e^{-\alpha y}-\frac{2\zeta^2}{2\zeta^2-k_2^2}e^{-\beta y}\right]\alpha\zeta^{-1}A(\zeta)\cos\zeta\,x\,d\zeta$$
$$-\sqrt{\frac{2}{\pi}}\int_0^\infty[\xi\alpha_1^{-1}C(\xi)\cosh(\alpha_1 x)+\beta_1\xi^{-1}D(\xi)\cosh(\beta_1 x)]\sin\xi y\,d\xi \tag{13}$$

$$a\tau_{yy}=-\mu\sqrt{\frac{2}{\pi}}\int_0^\infty\left[(2\zeta^2-k_2^2)e^{-\alpha y}-\frac{4\alpha\beta\zeta^2}{2\zeta^2-k_2^2}e^{-\beta y}\right]\zeta^{-4}A(\zeta)\cos\zeta\,x\,d\zeta$$
$$-\mu\sqrt{\frac{2}{\pi}}\int_0^\infty\left[\frac{(2\alpha_1^2+k_2^2)}{\alpha_1}C(\xi)\cosh(\alpha_1 x)+2\beta_1 D(\xi)\cosh(\beta_1 x)\right]\cos\xi y\,d\xi \tag{14}$$

$$\alpha\tau_{xy} = -\mu\sqrt{\frac{2}{\pi}}\int_0^\infty [e^{-\alpha y} - e^{-\beta y}]2\alpha A(\zeta)\sin\zeta\times d\zeta$$

$$-\mu\sqrt{\frac{2}{\pi}}\int_0^\infty [2\xi C(\xi)\sinh(\alpha_1 x) + \xi^{-1}(2\xi^2 - k_2^2)D(\xi)\sinh(\beta_1 x)]\sin\xi y\, d\xi. \quad (15)$$

3. SOLUTION OF THE PROBLEM

The boundary conditions (4) and (6) yield the following two integral equations:

$$\int_0^\infty \frac{1}{\zeta}[1 + H(\zeta)]B(\zeta)\sin\zeta\times d\zeta = R(X), \quad 0 \leqslant |x| \leqslant 1 \quad (16)$$

$$\int_0^\infty \frac{1}{\zeta}B(\zeta)\cos\zeta\times d\zeta = 0, \quad 1 \leqslant |x| \leqslant h \quad (17)$$

where

$$B(\zeta) = \frac{2\alpha(k_1^2 - k_2^2)A(\zeta)}{2\zeta^2 - k_2^2} \quad (18)$$

$$H(\zeta) = \frac{(2\zeta^2 - k_2^2) - 4\alpha\beta\zeta^2}{2\alpha\zeta(k_1^2 - k_2^2)} - 1 \quad (19)$$

$$H(\zeta)\to 0 \quad \text{as } \zeta\to\infty.$$

$$R(x) = \sqrt{\frac{2}{\pi}}\mu^{-1}a\int_0^x p(x)dx - \int_0^\infty\left[\frac{(2\alpha_1^2 + k_2^2)}{\alpha_1^2}C(\xi)\sinh(\alpha_1 x) + 2D(\xi)\sinh(\beta_1 x)\right]d\xi. \quad (20)$$

Let us consider the solution of integral eqs (16) and (17) in the form

$$B(\zeta) = \sqrt{\frac{\pi}{2}}\zeta\int_0^1 tf(t)J_0(\zeta t)\,dt \quad (21)$$

so that the integral eq. (17) is automatically satisfied.

Now, substituting the value of $B(\zeta)$ from (21) in (16) and using Abel's transform we obtain the following Fredholm integral equation of second kind:

$$f(t) + \int_0^1 uf(u)L_1(t,u)\,du = Q(t) \quad (22)$$

where,

$$Q(t) = \frac{2a}{\mu\pi t}\frac{d}{dt}\int_0^t (t^2 - z^2)^{1/2}p(z)\,dz - \sqrt{\frac{2}{\pi}}\int_0^\infty)\alpha_1^{-1}(2\alpha_1^2 + k_2^2)I_0(\alpha_1 t)C(\xi) + 2\beta_1 I_0(\beta_1 t)D(\xi)]\,d\xi \quad (23)$$

and

$$L_1(t,u) = \int_0^\infty \zeta H(\zeta)J_0(\zeta u)J_0(\zeta t)\,d\zeta. \quad (24)$$

From the boundary conditions (7) and (8), the unknown functions $C(\xi)$ and $D(\xi)$ can be found to be related to $B(\zeta)$ as:

$$C(\xi) = \frac{2}{\pi k_2^2(k_1^2 - k_2^2)\sinh(\alpha_1 h)}\left[-\xi^2\int_0^\infty g_1(\xi,\zeta)B(\zeta)\,d\zeta + \frac{(2\xi^2 - k_2^2)}{2}\int_0^\infty g_2(\xi,\zeta)B(\zeta)\,d\zeta\right] \quad (25)$$

$$D(\xi) = \frac{2}{\pi k_2^2(k_1^2 - k_2^2)\sinh(\beta_1 h)}\left[\xi^2\int_0^\infty g_1(\xi,\zeta)B(\zeta)\,d\zeta - \xi^2\int_0^\infty g_2(\xi,\zeta)B(\zeta)\,d\zeta\right] \quad (26)$$

where,

$$g_1(\xi,\zeta) = \left\{\frac{2\beta_1^2 + k_2^2}{\zeta^2 + \beta_1^2} - \frac{2\alpha_1^2 + k_2^2}{\zeta^2 + \alpha_1^2}\right\}\sin(\zeta h)$$

$$g_2(\xi,\zeta) = \left\{\frac{2(\beta_1^2 + k_2^2)}{\zeta^2 + \beta_1^2} - \frac{2\alpha_1^2 + k_2^2}{\zeta^2 + \alpha_1^2}\right\}\sin(\zeta h). \quad (27)$$

Next, substituting the value of $B(\zeta)$ from (21) in the expressions of $C(\xi)$ and $D(\xi)$ given by (25) and (26) and using the result (Gradsteyn [12])

$$\int_0^\infty \frac{\zeta \sin(\zeta h) J_0(\zeta u)}{\zeta^2+\alpha_1^2}\,\mathrm{d}\zeta = \frac{\pi}{2} I_0(\alpha_1 u)\mathrm{e}^{-\alpha_1 h}$$

$C(\xi)$ and $D(\xi)$ can be written in terms of $f(t)$ as

$$C(\xi) = \sqrt{\frac{\pi}{2}}\frac{1}{2(k_1^2-k_2^2)}\int_0^1 [(2\alpha_1^2+k_2^2)I_0(\alpha_1 u)\mathrm{e}^{-\alpha_1 h}]\frac{uf(u)\mathrm{d}u}{\sinh(\alpha_1 h)}$$

$$D(\xi) = -\sqrt{\frac{\pi}{2}}\frac{\xi^2}{(k_1^2-k_2^2)}\int_0^1 [I_0(\beta_1 u)\mathrm{e}^{-\beta_1 h}]\frac{uf(u)\mathrm{d}u}{\sinh(\beta_1 h)}. \tag{28}$$

Using the above relations (28) in (23) we obtain

$$Q(t) = \frac{2a}{\mu\pi t}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \sqrt{t^2-z^2}p(z)\,\mathrm{d}z + \int_0^1 u[L_2(t,u)+L_9(t,u)]f(u)\,\mathrm{d}u \tag{29}$$

where,

$$L_2(t,u) = -\frac{1}{2(k_1^2-k_2^2)}\int_0^\infty [\alpha_1^{-1}(2\alpha_1^2+k_2^2)^2 I_0(\alpha_1 t)L_0(\alpha_1 u)\mathrm{e}^{-\alpha_1 h}]\frac{\mathrm{d}\xi}{\sinh(\alpha_1 h)} \tag{30}$$

$$L_9(t,u) = \frac{2}{(k_1^2-k_2^2)}\int_0^\infty [\beta_1(\beta_1^2+k_2^2)I_0(\beta_1 t)I_0(\beta_1 u)\mathrm{e}^{-\beta_1 h}]\frac{\mathrm{d}\xi}{\sinh(\beta_1 h)}. \tag{31}$$

Next substituting $Q(t)$ from (29) in (22) and assuming $p(x)=p_0$ and $f(t)=ap_0 g(t)/\mu$ we finally obtain the following Fredholm integral equation of second kind for the determination of $g(t)$:

$$g(t) + \int_0^1 ug(u)L(t,u)\,\mathrm{d}u = 1 \tag{32}$$

where

$$L(t,u) = L_1(t,u) - L_2(t,u) - L_9(t,u) \tag{33}$$

and $L_1(t,u)$, $L_2(t,u)$ and $L_9(t,u)$ are given by (24), (30) and (31) respectively.

It is to be noted that the kernal $L_1(t,u)$ represented by the semi-infinite integral given by eq. (24) has a slow rate of convergence. In order to make the numerical analysis easier, the semi-infinite integral has therefore been converted to finite integrals by using simple contour integration technique (Srivastava *et al.* [3]) and is given by

$$L_1(t,u) = -\frac{ik_2^4}{2(k_2-k_1)}\left[\int_0^\gamma \frac{(2\eta^2-1)^2}{(\gamma^2-\eta^2)^{1/2}}J_0(k_2\eta u)H_0^{(1)}(k_2\eta t)\,\mathrm{d}\eta + \int_0^1 4\eta^2(1-\eta^2)^{1/2}J_0(k_2\eta u)H_0^{(1)}(k_2\eta t)\,\mathrm{d}\eta\right],\quad t>u \tag{34}$$

where $\gamma = k_1/k_2$. The corresponding expression of $L_1(t,u)$ for $t<u$ can be obtained by interchanging t and u in (34).

4. STRESS INTENSITY FACTOR AND DISPLACEMENT

The normal stress $\tau_{yy}(x,y)$ in the plane $y=0$ in the vicinity of the crack tip can be found from eq. (14) and is given by

$$\tau_{yy}(x,0) = -\mu\sqrt{\frac{2}{\pi}}\int_0^\infty B(\zeta)\cos\zeta x\,\mathrm{d}\zeta + 0(1),\quad x>1$$

$$= -\frac{p_0 x}{\sqrt{x^2-1}}g(1) + 0(1),\quad x>1.$$

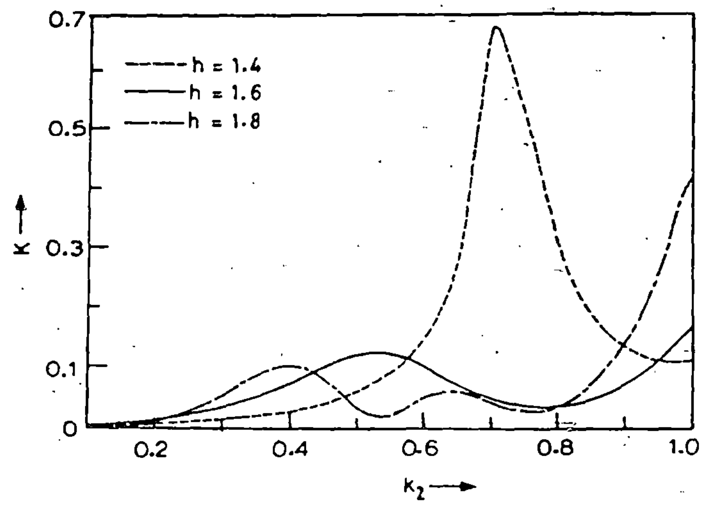

Fig. 2. Stress intensity factor K vs dimensionless frequency k_2.

Defining the stress intensity factor by

$$K = \underset{x \to 1+}{Lt} \left| \frac{\tau_{yy}(x, 0)\sqrt{x-1}}{p_0} \right|$$

it is found that

$$K = \frac{|g(1)|}{\sqrt{2}}. \tag{35}$$

Now the crack opening displacement $\Delta v(x, 0) = v(x, 0+) - v(x, 0-)$ can be obtained from (13) as

$$\Delta v(x, 0) = -\frac{k^2}{\sqrt{2\pi}(k_1^2 - k_2^2)} \int_0^\infty \frac{1}{\zeta} B(\zeta)\cos(\zeta x)\, d\zeta, \quad |x| \leqslant 1$$

which on substitution of the value of $B(\zeta)$ from (21) takes the form

$$\Delta v(x, 0) = \frac{a p_0}{\mu(1-\gamma^2)} \int_x^1 \frac{t g(t)\, dt}{(t^2 - x^2)^{1/2}}, \quad |x| \leqslant 1. \tag{36}$$

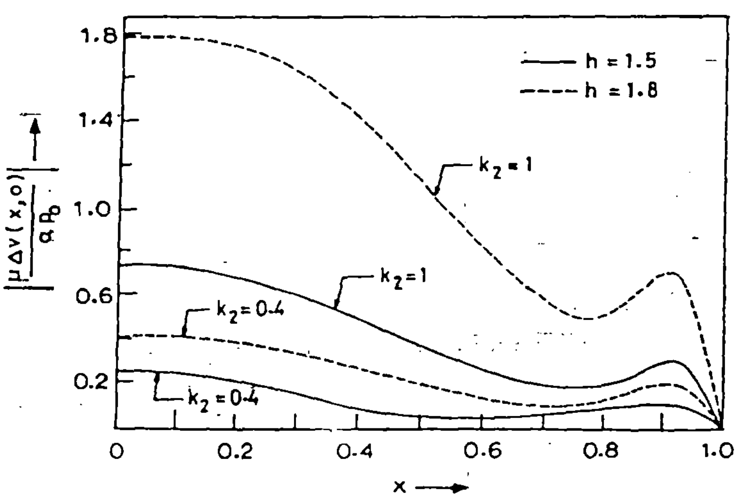

Fig. 3. Crack opening displacement vs distance.

5. NUMERICAL RESULTS AND DISCUSSION

Using the method of Fox and Goodwin [13] the Fredholm integral equation given by eq. (32) has been solved numerically for different values of dimensionless frequency k_2 and h, the separating distance of the cracks. At first the integral in (32) has been presented by a quadrature formula involving values of the desired function $g(t)$ at pivotal points inside the specified range of integration and then converted to a set of linear algebraic simultaneous equations, solving which the first approximation to the required pivotal values of $g(t)$ has been obtained. Applying difference-correction technique the first approximations has been improved. The standard numerical integration technique has been used to evaluate the kernals $L_1(t,u)$, $L_2(t,u)$ and $L_9(t,u)$ given by (34), (30) and (31). After solving the integral eq. (32) numerically, the stress intensity factor K and the crack opening displacement $\mu\Delta v(x,0)/ap_0$ have been calculated numerically and plotted separately against dimensionless frequency $k_2(0 < k_2 \leqslant 1)$ and dimensionless distance $x(0 \leqslant x \leqslant 1)$ respectively for different values of h. The value of γ is taken to be $1/\sqrt{3}$. From Fig. 2 it is interesting to note that the number of oscillations in stress intensity factor K increases with the increase in the values of h. The crack opening displacement (Fig. 3) is greater for higher values of h and also for higher values of dimensionless frequency k_2.

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DIFFRACTION OF ELASTIC WAVES BY THREE COPLANAR GRIFFITH CRACKS IN AN ORTHOTROPIC MEDIUM

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Abstract—The dynamic response of three co-planar Griffith cracks situated in an infinite orthotropic medium due to elastic waves incident normally on the cracks has been treated. The Fourier transform technique has been used to reduce the elastodynamic problem to the solution of a set of four integral equations. These integral equations have been solved by using the finite Hilbert transform technique and Cook's result. The analytical forms of crack opening displacement and stress intensity factors have been derived for low frequency vibration. Numerical results of crack opening displacement and stress intensity factors for several orthotropic materials have been calculated and plotted graphically to display the influence of the material orthotropy.

1. INTRODUCTION

Recently, with the increased usage of macroscopically anisotropic construction materials such as fibre-reinforced materials, the study of diffraction of elastic waves with cracks or inclusions has attracted the attention of scientists. The different possible location of cracks with respect to the planes of material symmetry is of great interest in Seismology and Exploration Geophysics. The problem of scattering of elastic waves by cracks of finite dimension in isotropic medium has been investigated by several investigators. Many investigators [1–6] have solved the diffraction problem involving single or two cracks in an isotropic medium. Dhawan and Dhaliwal [7] solved the statical problem involving three coplanar cracks in an infinite transversely isotropic medium. The dynamic problem of singular stresses around cracks in orthotropic medium are few in number. Kassir and Bandyopadhyay [8] solved the problem of elastodynamic response of an infinite orthotropic solid containing a crack under the action of impact loading. The problem of normal impact response of a finite Griffith crack in an orthotropic strip has been solved by Shindo [9]. De and Patra [10] have also solved the problem involving a moving Griffith crack in an orthotropic strip. Recently Kundu and Bostrom [11] treated the diffraction problem of a circular crack in orthotropic medium.

To the best knowledge of the authors, the problem of diffraction of elastic waves by three coplanar Griffith cracks in an orthotropic material has not been considered. In our paper, the interaction of normally incident time harmonic elastic waves with three coplanar Griffith cracks in an orthotropic medium has been investigated. It is assumed that the faces of each of the cracks do not come into contact during small deformation of the solid. The resulting mixed boundary value problem is reduced to the solution of a set of four integral equations which has been reduced to the solution of an integro-differential equation. Iteration method has been used to obtain the low frequency solution of the problem. This enables us to obtain approximate value of the crack opening displacements and stress intensity factors. Making the length of the central crack tend to zero, the corresponding results for two Griffith cracks have been obtained. Numerical results for stress intensity factors and crack opening displacements have been plotted against dimensionless frequency and distance respectively for different orthotropic materials which have been shown graphically.

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2. STATEMENT AND FORMULATION OF THE PROBLEM

Consider the interaction of normally incident longitudinal wave with three coplanar Griffith cracks situated in an infinite orthotropic elastic medium. The cracks are assumed to occupy the position $|X| \leq d_1$, $d_2 \leq |X| \leq d$, $Y = 0$, $|Z| < \infty$. Let E_i, μ_{ij} and ν_{ij} $(i, j = 1, 2, 3)$ denote the engineering elastic constants of the material where the subscripts 1, 2, 3 correspond to the X, Y, Z directions chosen to coincide with the axes of material orthotropy. Normalizing all the lengths with respect to 'd' and setting $X/d = x$, $Y/d = y$, $Z/d = z$, $d_1/d = b$, $d_2/d = c$, the cracks are defined by $|x| \leq b$, $c \leq |x| \leq 1$, $y = 0$, $|z| < \infty$ (Fig. 1).

Displacement components are also made dimensionless with respect to 'd' so that dimensionless components of displacement in x, y directions are assumed to be u, v respectively, where

$$u = u(x, y, t) \quad \text{and} \quad v = v(x, y, t).$$

Let a time harmonic plane elastic wave originating at $y = -\infty$ and incident normally on the three cracks be given by $v = v_0 \exp[i(ky - \omega t)]/d$ where $k = d\omega/c_s\sqrt{c_{22}}$, $c_s = (\mu_{12}/\rho)^{1/2}$, v_0 is a constant, ω and v_0/d are the frequency and dimensionless amplitude of the incident wave respectively, ρ being the density of the material. In the isotropic solid, c_s represents the velocity of the shear wave.

The non-zero stress components τ_{yy} and τ_{xy} are given by

$$\tau_{yy}/\mu_{12} = c_{12}u_{,x} + c_{22}v_{,y}$$

$$\tau_{xy}/\mu_{12} = u_{,y} + v_{,x} \tag{2.1}$$

where u, v denote the component of the displacement in the x, y directions respectively and comma denotes partial differentiation with respect to the coordinates or time ; $c_{ij}(i, j = 1, 2)$ are

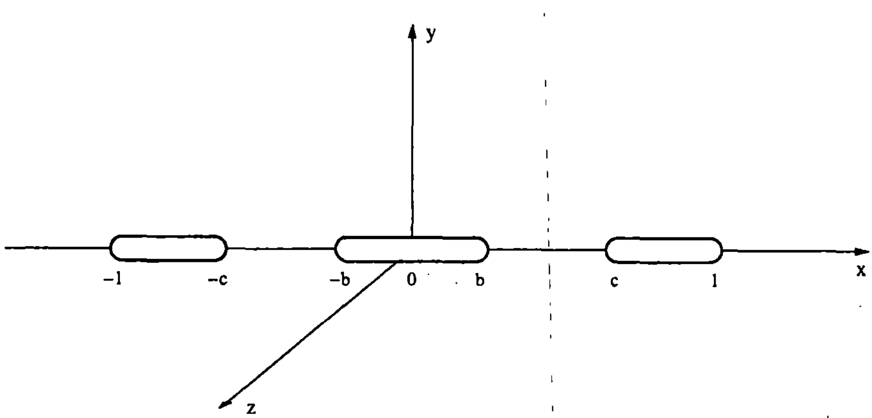

Fig. 1. Geometry of the cracks.

nondimensional parameters related to the elastic constant by the relations:

$$c_{11} = E_1/\mu_{12}(1 - \nu_{12}^2 E_2/E_1)$$

$$c_{22} = E_2/\mu_{12}(1 - \nu_{12}^2 E_2/E_1) = c_{11}E_2/E_1$$

$$c_{12} = \nu_{12}E_2/\mu_{12}(1 - \nu_{12}^2 E_2/E_1) = \nu_{12}c_{22} = \nu_{21}c_{11} \tag{2.2}$$

for generalized plane stress, and by

$$c_{11} = (E_1/\Delta\mu_{12})(1 - \nu_{23}\nu_{32})$$

$$c_{22} = (E_2/\Delta\mu_{12})(1 - \nu_{13}\nu_{31})$$

$$c_{12} = E_1(\nu_{21} + \nu_{13}\nu_{32}E_2/E_1)/\Delta\mu_{12} = E_2(\nu_{12} + \nu_{23}\nu_{31}E_1/E_2)/\Delta\mu_{12}$$

$$\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{21}\nu_{32} \tag{2.3}$$

for plane strain. The constants E_i and ν_{ij} satisfy Maxwell's relation:

$$\nu_{ij}/E_i = \nu_{ji}/E_j. \tag{2.4}$$

The displacement equations of motion for orthotropic material are

$$c_{11}u_{,xx} + u_{,yy} + (1 + c_{12})v_{,xy} = \frac{d^2}{c_s^2}u_{,tt}$$

$$c_{22}v_{,yy} + v_{,xx} + (1 + c_{12})u_{,xy} = \frac{d^2}{c_s^2}v_{,tt}. \tag{2.5}$$

Substitution of $u(x, y, t) = u(x, y)\exp(-i\omega t)$ and $v(x, y, t) = v(x, y)\exp(-i\omega t)$ in equations (2.5) reduces them to

$$c_{11}u_{,xx} + u_{,yy} + (1 + c_{12})v_{,xy} + k_s^2 u = 0$$

$$c_{22}v_{,yy} + v_{,xx} + (1 + c_{12})u_{,xy} + k_s^2 v = 0 \tag{2.6}$$

with $k_s^2 = d^2\omega^2/c_s^2$, which are to be solved subject to the boundary conditions

$$v(x, 0) = 0, \quad b \leq |x| \leq c, \qquad |x| \geqslant 1 \tag{2.7}$$

$$\tau_{xy}(x, 0) = 0, \qquad |x| < \infty \tag{2.8}$$

$$\tau_{yy}(x, 0) + \tau_{yy}^{(0)}(x, 0) = 0, \qquad |x| < b, \qquad c < |x| < 1. \tag{2.9}$$

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables would be omitted in the sequel.

Using the condition (2.8), the solutions of equations (2.6) may be written as

$$u(x, y) = \frac{2}{\pi}\int_0^\infty [\exp(-\gamma_1|y|) - \beta\exp(-\gamma_2|y|)]A_1(\xi)\sin(\xi x)\,d\xi \tag{2.10}$$

$$v(x, y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}[\alpha_1\exp(-\gamma_1|y|) - \beta\alpha_2\exp(-\gamma_2|y|)]A_1(\xi)\cos(\xi x)\,d\xi, \qquad y > 0 \tag{2.11}$$

and the stress components are given by

$$\tau_{xy}/\mu_{12} = -\frac{2}{\pi}\int_0^\infty (\gamma_1+\alpha_1)[\exp(-\gamma_1|y|)-\exp(-\gamma_2|y|)]A_1(\xi)\sin(\xi x)\,d\xi, \qquad y>0 \quad (2.12)$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty \left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \beta\left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\right]A_1(\xi)\cos(\xi x) \quad (2.13)$$

where

$$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1+c_{12})\gamma_i}, \qquad i=1,2 \quad (2.14)$$

$$\beta = \frac{\gamma_1+\alpha_1}{\gamma_2+\alpha_2} \quad (2.15)$$

$A_1(\xi)$ is the unknown function to be determined, and γ_1^2, γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1+c_{22})k_s^2\}\gamma^2 + (c_{11}\xi^2 - k_s^2)(\xi^2 - k_s^2) = 0. \quad (2.16)$$

With the aid of the boundary conditions, (2.7) and (2.9) $A(\xi)$ is found to satisfy the integral equations

$$\int_0^\infty A(\xi)\cos(\xi x)\,d\xi = 0, \qquad x \in I_2, I_4 \quad (2.17)$$

and

$$\int_0^\infty H(\xi)A(\xi)\cos(\xi x)\,d\xi = -\frac{\pi p_0}{2\mu_{12}}, \qquad x \in I_1, I_3 \quad (2.18a, b)$$

where $I_1 = (0, b)$, $I_2 = (b, c)$, $I_3 = (c, 1)$, $I_4 = (1, \infty)$ and

$$p_0 = ik\mu_{12}c_{22}v_0/d \quad (2.19)$$

$$A(\xi) = \frac{\alpha_1 - \beta\alpha_2}{\xi}A_1(\xi) \quad (2.20)$$

$$H(\xi) = \frac{c_{12}\xi^2 - c_{22}\alpha_1\gamma_1 - \beta(c_{12}\xi^2 - c_{22}\alpha_2\gamma_2)}{(\alpha_1 - \beta\alpha_2)}. \quad (2.21)$$

3. METHOD OF SOLUTION

The solution of the integral equations (2.17) and (2.18) is taken in the form

$$A(\xi) = \frac{1}{\xi}\int_0^b h(t)\sin(\xi t)\,dt + \frac{1}{\xi}\int_c^1 g(u^2)\sin(\xi u)\,du \quad (3.1)$$

where $h(t)$ and $g(u^2)$ are the unknown functions to be determined. Substituting the value of $A(\xi)$ from (3.1) in (2.17) and using the following result [12]

$$\int_0^\infty \frac{\sin(\xi t)\cos(\xi x)}{\xi}\,d\xi = \begin{cases} \frac{\pi}{2}, & t>x \\ 0, & t<x \end{cases}$$

it is found that the choice of $A(\xi)$ leads to the equation

$$\int_c^1 g(u^2)\,\mathrm{d}u = 0. \tag{3.2}$$

Further substituting $A(\xi)$ from (3.1) in (2.18a) and using the result [13]

$$\int_0^\infty \xi^{-1}\sin(\xi u)\sin(\xi x)\,\mathrm{d}\xi = \frac{1}{2}\log\left|\frac{u+x}{u-x}\right|$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h(t)\log\left|\frac{t+x}{t-x}\right|\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(u^2)\log\left|\frac{u+x}{u-x}\right|\mathrm{d}u \\ = 2\left[q_0 - \frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h(t)\,\mathrm{d}t\int_0^\infty H_1(\xi)\xi^{-1}\sin(\xi t)\sin(\xi x)\,\mathrm{d}\xi\right. \\ \left. - \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(u^2)\,\mathrm{d}u\int_0^\infty H_1(\xi)\xi^{-1}\sin(\xi u)\sin(\xi x)\,\mathrm{d}\xi\right], \qquad x\in I_1\end{aligned} \tag{3.3}$$

where

$$q_0 = -\frac{\pi p_0}{2\theta\mu_{12}} \tag{3.4}$$

$$H_1(\xi) = \frac{H(\xi)}{\xi\theta} - 1 \to 0 \quad \text{as} \quad \xi\to\infty \tag{3.5}$$

$$\theta = \frac{(c_{12}^2 + c_{12} - c_{11}c_{22})(c_{12}N_1N_2 - c_{11}) - c_{22}[c_{12}N_1^2N_2^2 + c_{11}(N_1^2 + N_1N_2 + N_2^2)]}{c_{11}(1+c_{12})(N_1+N_2)} \tag{3.6}$$

$$\begin{aligned}N_1^2 &= \frac{1}{2c_{22}}\{c_{11}c_{22} - c_{12}^2 - 2c_{12} + [(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}]^{1/2}\} \\ N_2^2 &= \frac{1}{2c_{22}}\{c_{11}c_{22} - c_{12}^2 - 2c_{12} - [(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}]^{1/2}\}.\end{aligned} \tag{3.7}$$

Using the relation

$$\frac{\sin\xi x\sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_0(\xi w)J_0(\xi v)\,\mathrm{d}v\,\mathrm{d}w}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}$$

equation (3.3) can now be rewritten in the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h(t)\log\left|\frac{t+x}{t-x}\right|\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(u^2)\log\left|\frac{u+x}{u-x}\right|\mathrm{d}u \\ = 2\left[q_0 - \frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h(t)\,\mathrm{d}t\int_0^x\int_0^t \frac{vwL(v,w)\,\mathrm{d}w\,\mathrm{d}v}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}\right. \\ \left. - \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(u^2)\,\mathrm{d}u\int_0^x\int_0^u \frac{vwL(v,w)\,\mathrm{d}w\,\mathrm{d}v}{(x^2-w^2)^{1/2}(u^2-v^2)^{1/2}}\right], \qquad x\in I_1\end{aligned} \tag{3.8}$$

where

$$L(v,w) = \int_0^\infty \xi H_1(\xi)J_0(\xi w)J_0(\xi v)\,\mathrm{d}\xi \tag{3.9}$$

and $J_0(\;)$ is the Bessel function of order zero.

Applying a contour integration technique [14], the infinite integral in $L(v,w)$ can be converted to the following finite integrals

$$L(v,w) = -ik_s^2\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - \bar{\alpha}_1\bar{\gamma}_1 c_{22} - \bar{\beta}(c_{12}\eta^2 - \bar{\alpha}_2\bar{\gamma}_2 c_{22})}{\theta(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,\mathrm{d}\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,\mathrm{d}\eta\right], \qquad w > v \quad (3.10)$$

where

$$\bar{\gamma}_1 = \left[\frac{1}{2}\{R_1 - (R_1^2 - 4\bar{R}_2)^{1/2}\}\right]^{1/2}$$

$$\bar{\gamma}_2 = \left[\frac{1}{2}\{R_1 + (R_1^2 - 4\bar{R}_2)^{1/2}\}\right]^{1/2}$$

$$\hat{\gamma}_1 = \left[\frac{1}{2}\{-R_1 + (R_1^2 + 4R_2')^{1/2}\}\right]^{1/2}$$

$$\hat{\gamma}_2 = \left[\frac{1}{2}\{R_1 + (R_1^2 + 4R_2')^{1/2}\}\right]^{1/2}$$

$$R_1 = \frac{1}{c_{22}}\{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\eta^2 + (1 + c_{22})\}$$

$$\bar{R}_2 = \frac{c_{11}}{c_{22}}(1-\eta^2)\left(\frac{1}{c_{11}} - \eta^2\right)$$

$$R_2' = \frac{c_{11}}{c_{22}}(1-\eta^2)\left(\eta^2 - \frac{1}{c_{11}}\right)$$

$$\bar{\alpha}_i = \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_i^2}{(1 + c_{12})\bar{\gamma}_i}, \qquad i = 1, 2$$

$$\hat{\alpha}_i = \frac{c_{11}\eta^2 - 1 + (-1)^i\hat{\gamma}_i^2}{(1 + c_{12})\hat{\gamma}_i}, \qquad i = 1, 2$$

$$\bar{\beta} = \frac{\bar{\alpha}_1 - \bar{\gamma}_1}{\bar{\alpha}_2 - \bar{\gamma}_2}$$

$$\hat{\beta} = \frac{\hat{\alpha}_1 + \hat{\gamma}_1}{\hat{\alpha}_2 - \hat{\gamma}_2}. \quad (3.11)$$

The corresponding expression of $L(v,w)$ for $w < v$ is obtained by interchanging v and w in (3.10).

Employing the series expansions for the Bessel function J_0 and the Hankel function $H_0^{(1)}$ in equation (3.10), it is found that

$$L(v,w) = \frac{2}{\pi}Pk_s^2 \log k_s + O(k_s^2) \quad (3.12)$$

where

$$P = \frac{1}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - \bar{\alpha}_1\bar{\gamma}_1 c_{22} - \bar{\beta}(c_{12}\eta^2 - \bar{\alpha}_2\bar{\gamma}_2 c_{22})}{(\bar{\alpha}_1 - \bar{\beta}\bar{\alpha}_2)}\,\mathrm{d}\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{(\hat{\alpha}_1 - \hat{\beta}\hat{\alpha}_2)}\,\mathrm{d}\eta\right]. \quad (3.13)$$

Let us now expand $h(t)$ and $g(u^2)$ in the form

$$h(t) = h_0(t) + k_s^2 \log k_s\, h_1(t) + O(k_s^2)$$

and

$$g(u^2) = g_0(u^2) + k_s^2 \log k_s\, g_1(u^2) + O(k_s^2). \tag{3.14}$$

Substituting the above equations (3.14) and the value of $L(v, w)$ given by (3.10) in equations (3.8) and (3.2) and equating the coefficients of like powers of k_s, the following equations are derived.

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h_0(t)\log\left|\frac{t+x}{t-x}\right|\mathrm{d}t + 2\int_c^1 \frac{ug_0(u^2)}{u^2-x^2}\mathrm{d}u = 2q_0, \qquad x \in I_1, I_3 \tag{3.15a, b}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^b h_1(t)\log\left|\frac{t+x}{t-x}\right|\mathrm{d}t + 2\int_c^1 \frac{ug_1(u^2)}{u^2-x^2}\mathrm{d}u = -\frac{4P}{\pi}\left[\int_0^b th_0(t)\,\mathrm{d}t + \int_c^1 ug_0(u^2)\,\mathrm{d}u\right],$$

$$x \in I_1, I_3 \tag{3.16a, b}$$

and

$$\int_c^1 g_i(u^2)\,\mathrm{d}u = 0 \qquad (i = 0, 1). \tag{3.17a, b}$$

Rewriting equation (3.15a) as

$$\int_0^b h_0(t)\log\left|\frac{t+x}{t-x}\right|\mathrm{d}t = \pi F_1(x), \qquad x \in I_1 \tag{3.18}$$

where

$$F_1(x) = -\int_0^x\left[\frac{p_0}{\mu_{12}\theta} + \frac{2}{\pi}\int_c^1 \frac{ug_0(u^2)}{u^2-y^2}\mathrm{d}u\right]\mathrm{d}y.$$

The solution of the integral equation (3.18) with the help of Cook's result [15] is found to be

$$h_0(t) = -\frac{p_0}{\mu_{12}\theta}\frac{t}{(b^2-t^2)^{1/2}} - \frac{2}{\pi}\frac{t}{(b^2-t^2)^{1/2}}\int_c^1 \frac{\sqrt{u^2-b^2}\,g_0(u^2)}{u^2-t^2}\mathrm{d}u. \tag{3.19}$$

Substitution of the value of $h_0(t)$ from (3.19) in (3.15b) with the aid of the result

$$\int_0^b \frac{1}{(b^2-t^2)^{1/2}}\frac{t^2\,\mathrm{d}t}{(x^2-t^2)(u^2-t^2)} = \frac{\pi}{2}\left[\frac{x}{(x^2-b^2)^{1/2}} - \frac{u}{(u^2-b^2)^{1/2}}\right], \qquad x \in I_3$$

yields the singular integral equation

$$\int_c^1 \frac{\sqrt{u^2-b^2}\,g_0(u^2)}{u^2-x^2}\mathrm{d}u = -\frac{\pi}{2}\frac{p_0}{\mu_{12}\theta}, \qquad x \in I_3. \tag{3.20}$$

Next using the finite Hilbert transform technique [13] the solution of the integral equation is found to be

$$g_0(u^2) = -\frac{p_0}{\mu_{12}\theta}\sqrt{\frac{u^2(u^2-c^2)}{(u^2-b^2)(1-u^2)}} + \frac{uD_1}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}} \tag{3.21}$$

where D_1 is unknown constant to be determined from equation (3.17a).

Now substituting the value of $g_0(u^2)$ from (3.21) in (3.19) and performing the integrations, $h_0(t)$ is obtained in the following form

$$h_0(t) = -\frac{p_0}{\mu_{12}\theta}\sqrt{\frac{t^2(c^2-t^2)}{(b^2-t^2)(1-t^2)}} + \frac{tD_1}{(b^2-t^2)(c^2-t^2)(1-t^2)}. \tag{3.22}$$

By the procedure similar to one which led to the derivations of the solutions of (3.15) as given by (3.21) and (3.22), the solutions of equation (3.16a, b) can also be obtained and they are found to be

$$h_1(t) = -\frac{4PR}{\pi^2}\sqrt{\frac{t^2(c^2-t^2)}{(b^2-t^2)(1-t^2)}} - \frac{tD_2}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}} \tag{3.23}$$

$$g_1(u^2) = -\frac{4PR}{\pi^2}\sqrt{\frac{u^2(u^2-c^2)}{(u^2-b^2)(1-u^2)}} + \frac{uD_2}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}} \tag{3.24}$$

where

$$R = -\frac{p_0}{\mu_{12}\theta}[I_0^b + I_c^1] - D_1[J_0^b - J_c^1]$$

$$I_m^n = \int_m^n \frac{t^2\sqrt{(c^2-t^2)}}{\sqrt{(b^2-t^2)(1-t^2)}}\,dt$$

$$J_m^n = \int_m^n \frac{t^2\,dt}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}}. \tag{3.25}$$

The constant D_2 is to be determined from equation (3.17b).

In order to determine the values of the unknown constants D_1 and D_2, $g_0(u^2)$ and $g_1(u^2)$ as given by (3.21) and (3.24) respectively are substituted in (3.17a, b) and it is found that

$$D_j = A_j\left[(1-b^2)\frac{E}{F} - (c^2-b^2)\right], \qquad (j=1,2) \tag{3.26}$$

and

$$A_1 = \frac{p_0}{\mu_{12}\theta}, \qquad A_2 = \frac{4PR}{\pi^2} \tag{3.27}$$

where $F = F\left(\frac{\pi}{2}, q\right)$ and $E = E\left(\frac{\pi}{2}, q\right)$ are the elliptic integrals of first and second kind respectively and $q = \sqrt{\frac{1-c^2}{1-b^2}}$. Substitution of the values of D_j $(j=1,2)$ given by equations (3.26) in equations (3.21)–(3.24) yields

$$h_{j-1}(t) = -A_j\left[(1-b^2)\frac{E}{F} + (b^2-t^2)\right]\frac{t}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}} \qquad (j=1,2) \tag{3.28}$$

$$g_{j-1}(u^2) = -A_j\left[(1-b^2)\frac{E}{F} - (u^2-b^2)\right]\frac{u}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}} \qquad (j=1,2). \tag{3.29}$$

4. STRESS INTENSITY FACTORS AND CRACK OPENING DISPLACEMENTS

The stress intensity factors are defined as (in physical units)

$$N_b = \lim_{x\to b+}\left[\frac{\sqrt{(x-b)}\,\tau_{yy}(x,0)}{p_0}\right]_{b<x<c} \tag{4.1}$$

$$N_c = \lim_{x\to c-}\left[\frac{\sqrt{(c-x)}\,\tau_{yy}(x,0)}{p_0}\right]_{b<x<c} \tag{4.2}$$

$$N_1 = \lim_{x\to 1+}\left[\frac{\sqrt{(x-1)}\,\tau_{yy}(x,0)}{p_0}\right]_{x>1} \tag{4.3}$$

and the crack opening displacement can now be shown to be given by

$$\Delta v(x, 0) = v(x, 0+) - v(x, 0-) = 2\int_x^b h(t)\,dt, \qquad 0 \le x \le b \tag{4.4}$$

$$= 2\int_x^1 g(u^2)\,du, \qquad c \le x \le 1. \tag{4.5}$$

Substituting the values of the function $h(t)$ and $g(u^2)$, the stress component τ_{yy} can be evaluated from the expressions (2.13), (2.21) and (3.1). After evaluation of the value of τ_{yy} and putting it in relations (4.1)–(4.3) it is found that

$$N_b = \sqrt{\frac{b(1-b^2)}{2(c^2-b^2)}}\,\frac{E\left(\frac{\pi}{2}, q\right)}{F\left(\frac{\pi}{2}, q\right)}\left[1 - \frac{4P}{\pi^2}M_2 k_s^2 \log k_s\right] + O(k_s^2) \tag{4.6}$$

$$N_c = \sqrt{\frac{c}{2(c^2-b^2)(1-c^2)}}\left[(1-b^2)\frac{E\left(\frac{\pi}{2}, q\right)}{F\left(\frac{\pi}{2}, q\right)} - (c^2-b^2)\right]\left[1 - \frac{4P}{\pi^2}M_2 k_s^2 \log k_s\right] + O(k_s^2) \tag{4.7}$$

$$N_1 = \sqrt{\frac{(1-b^2)}{2(1-c^2)}}\left[1 - \frac{E\left(\frac{\pi}{2}, q\right)}{F\left(\frac{\pi}{2}, q\right)}\right]\left[1 - \frac{4P}{\pi^2}M_2 k_s^2 \log k_s\right] + O(k_s^2) \tag{4.8}$$

where

$$M_2 = \left[I_0^b + I_c^1 + \left\{(1-b^2)\frac{E\left(\frac{\pi}{2}, q\right)}{F\left(\frac{\pi}{2}, q\right)} - (c^2-b^2)\right\}(J_0^b - J_c^1)\right].$$

Expressions (4.4)–(4.5) with the aid of the equations (3.28)–(3.29) yield

$$\Delta v(x, 0) = \frac{2p_0}{\mu_{12}\theta}\left[\sqrt{(1-b^2)}\,F(\beta, q)\left\{\frac{E(\beta, q)}{F(\beta, q)} - \frac{E\left(\frac{\pi}{2}, q\right)}{F\left(\frac{\pi}{2}, q\right)}\right\} - \sqrt{\frac{(1-x^2)(b^2-x^2)}{(c^2-x^2)}}\right]$$
$$\times\left[1 - \frac{4P}{\pi^2}M_2 k_s^2 \log k_s\right] + O(k_s^2), \qquad 0 \le x \le b \tag{4.9}$$

and

$$\Delta v(x, 0) = \frac{2p_0}{\mu_{12}\theta}\left[\sqrt{(1-b^2)}\,F(\lambda, q)\left\{\frac{E\left(\frac{\pi}{2}, q\right)}{F\left(\frac{\pi}{2}, q\right)} - \frac{E(\lambda, q)}{F(\lambda, q)}\right\}\right]$$
$$\times\left[1 - \frac{4P}{\pi^2}M_2 k_s^2 \log k_s\right] + O(k_s^2), \qquad c \le x \le 1 \tag{4.10}$$

where

$$\sin\beta = \sqrt{\frac{b^2-x^2}{c^2-x^2}} \quad \text{and} \quad \sin\lambda = \sqrt{\frac{1-x^2}{1-b^2}}.$$

When $b \to 0$, we recover the stress intensity factor and the crack opening displacement for two Griffith cracks occupying the region $c \le |x| \le 1$, $y = 0$, $|z| < \infty$:

$$N_c = -\frac{\left[c^2 - \frac{E}{F}\right]}{\sqrt{2c(1-c^2)}}\left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\}k_s^2 \log k_s\right] + O(k_s^2)$$

$$N_1 = -\frac{\left[1 - \frac{E}{F}\right]}{\sqrt{2(1-c^2)}}\left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\}k_s^2 \log k_s\right] + O(k_s^2) \tag{4.11}$$

and

$$\Delta v(x, 0) = \frac{2p_0}{\mu_{12}\theta}\left[1 - \frac{P}{\pi}\left\{1 + c^2 - \frac{2E}{F}\right\}k_s^2 \log k_s\right] \times \left[\frac{E\left(\frac{\pi}{2}, q\right)}{F\left(\frac{\pi}{2}, q\right)}F(\lambda, q) - E(\lambda, q)\right] + O(k_s^2), \qquad c \le x \le 1 \tag{4.12}$$

where $M_2 = \frac{\pi}{4}(1 + c^2 - 2E/F)$ has been used.

It is noted that if further $c \to 0$, the crack merge into a single crack of width two units. In this case $F \to \infty$ and $M_2 \to \pi/4$; so the results for stress intensity factor and crack opening displacements corresponding to the single crack are found to be

$$N_1 = \frac{1}{\sqrt{2}}\left[1 - \frac{P}{\pi}k_s^2 \log k_s\right] + O(k_s^2) \tag{4.13}$$

and

$$\Delta v(x, 0) = -\frac{2p_0}{\mu_{12}\theta}\sqrt{(1-x^2)}\left[1 - \frac{P}{\pi}k_s^2 \log k_s\right] + O(k_s^2), \qquad 0 \le x \le 1. \tag{4.14}$$

The results given by (4.11)–(4.14) are found to be in agreement with the results of Sarkar *et al.* [16].

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) N_b, N_c and N_1 given by (4.6), (4.7) and (4.8) at the tips of the cracks and crack opening displacements (COD) given by (4.9) and (4.10) have been plotted against dimensionless frequency k_s and distance respectively for three different types of orthotropic materials whose engineering constants have been listed in Table 1.

Table 1. Engineering elastic constants

	E_1 (Pa)	E_2 (Pa)	μ_{12} (Pa)	ν_{12}
Type I	Modulite II graphite–epoxy composite:			
	15.3×10^9	158.0×10^9	5.52×10^9	0.033
Type II	E-Type glass–epoxy composite:			
	9.79×10^9	42.3×10^9	3.66×10^9	0.063
Type III	Stainless steel–aluminium composite:			
	79.76×10^9	85.91×10^9	30.02×10^9	0.31

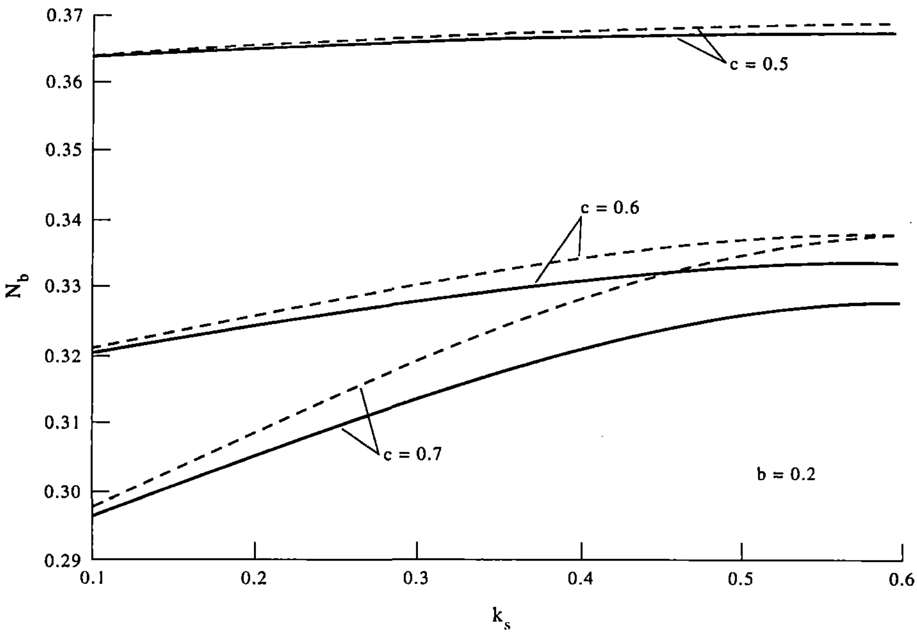

Fig. 2. Stress intensity factor N_b vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

Keeping the length of the central crack fixed ($b = 0.2$) SIFs at the tips of the central and outer cracks have been plotted against frequency k_s ($0.1 \le k_s \le 0.6$) for different lengths ($c = 0.5$, 0.6, 0.7) of the outer crack (Figs 2–4). It is noted from the graphs (Figs 2–4) that with the decrease in the value of outer crack length, i.e. with the increase in the value of the distance between inner and outer cracks the rate of increase in the SIF is higher with the increase in the value of the frequency k_s.

The same nature of SIFs are seen (Figs 5–7) in the case when the length of the outer cracks

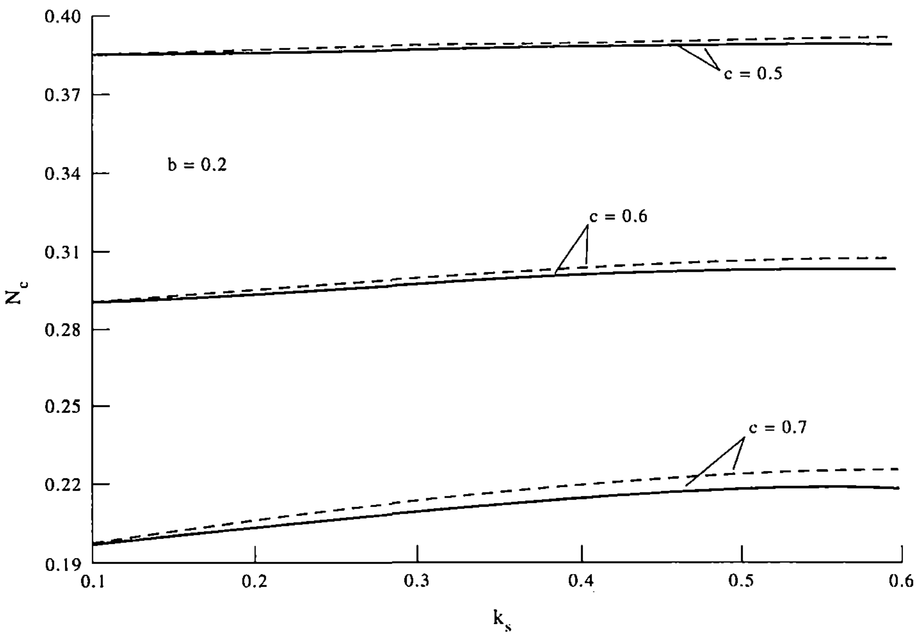

Fig. 3. Stress intensity factor N_c vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

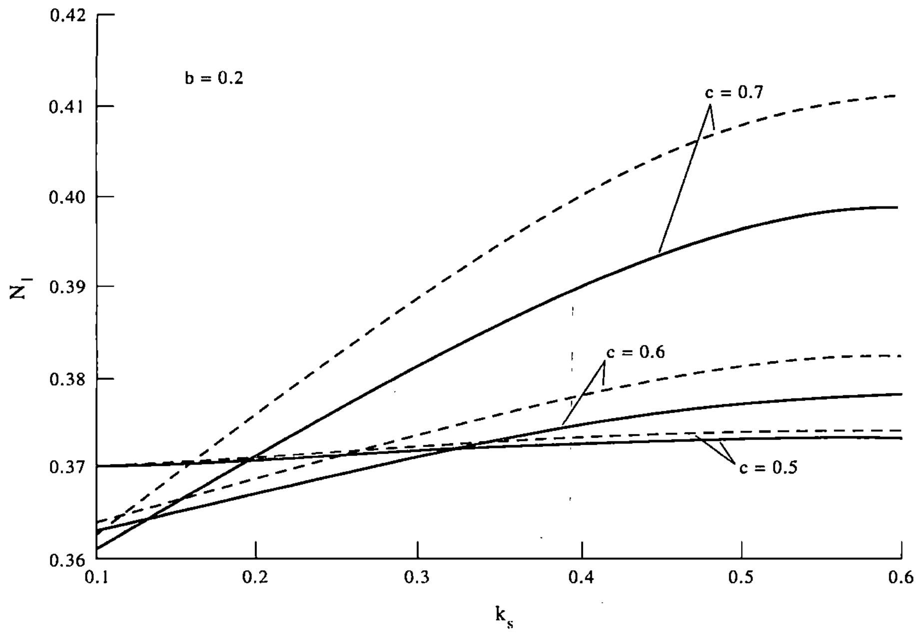

Fig. 4. Stress intensity factor N_I vs frequency k_s for generalized plane stress. (——) type I; (- - - - -) type III.

are fixed ($c = 0.7$) and the length of the central crack increases ($b = 0.3$, 0.4, 0.5). It is interesting to note that for fixed c ($= 0.7$) the SIFs N_b and N_c increase with the increase in the value of b, but the effect is just reverse in case of N_1.

The COD $\mu_{12}\Delta v(x, 0)/p_0$ has been plotted for different crack lengths. It is found from Figs 8 and 9 that with the increase in the value of crack length the value of COD increases. For a fixed material the variation of COD with frequency is found to be insignificant.

In all the cases where different values of crack length have been considered the variation of COD is found to be prominant for different orthotropic materials.

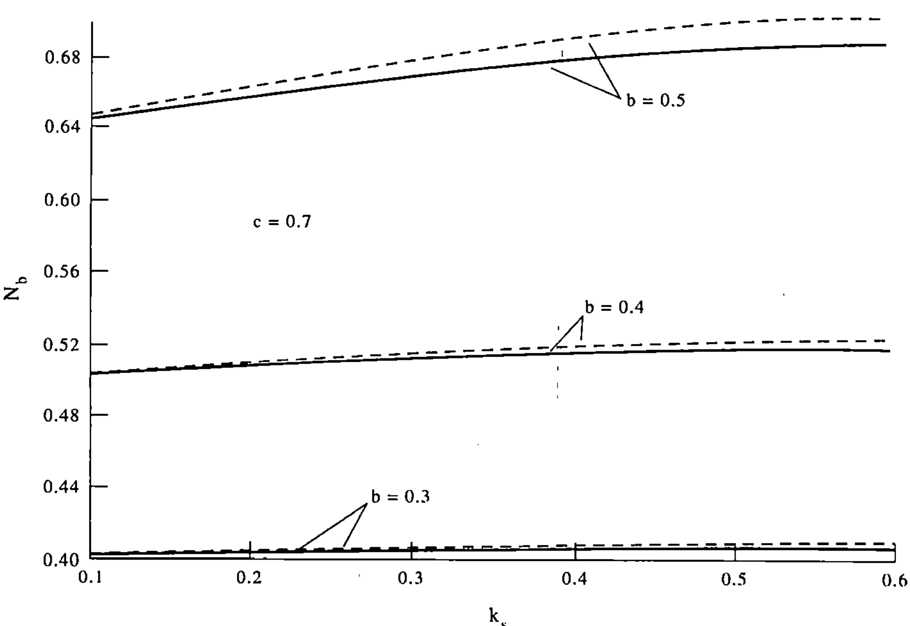

Fig. 5. Stress intensity factor N_b vs frequency k_s for generalized plane stress. (——) type I; (- - - - -) type III.

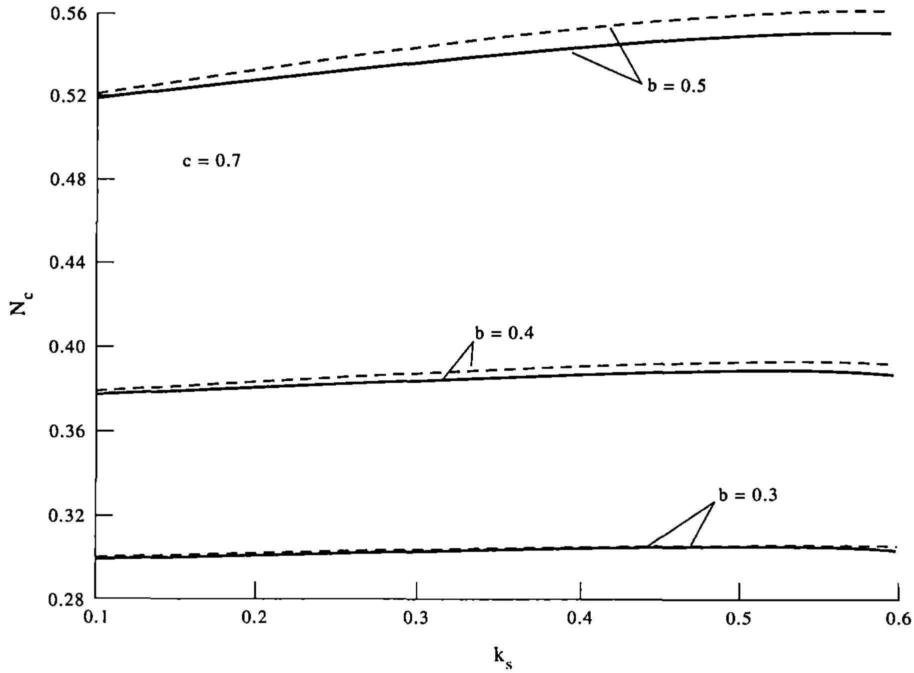

Fig. 6. Stress intensity factor N_c vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

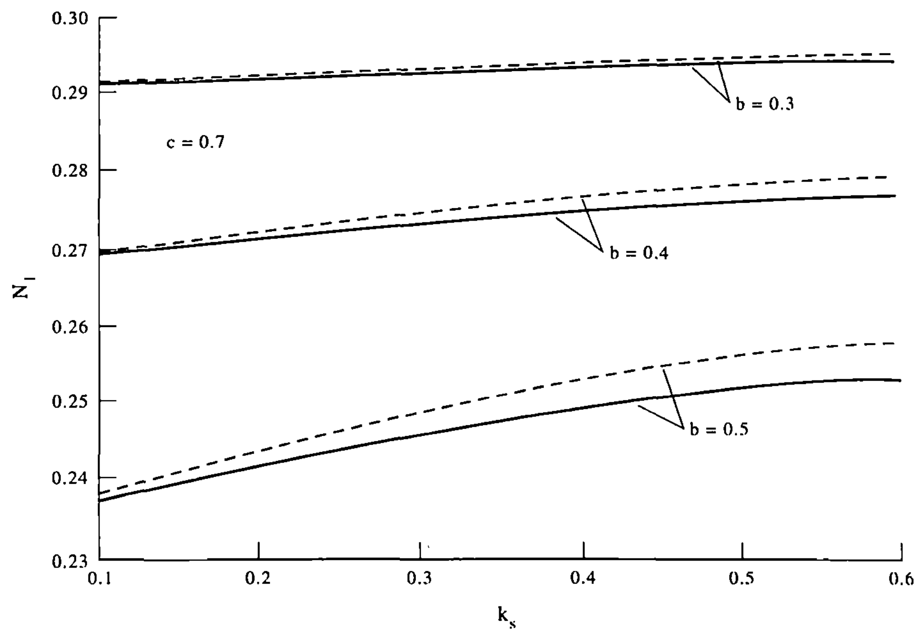

Fig. 7. Stress intensity factor N_I vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

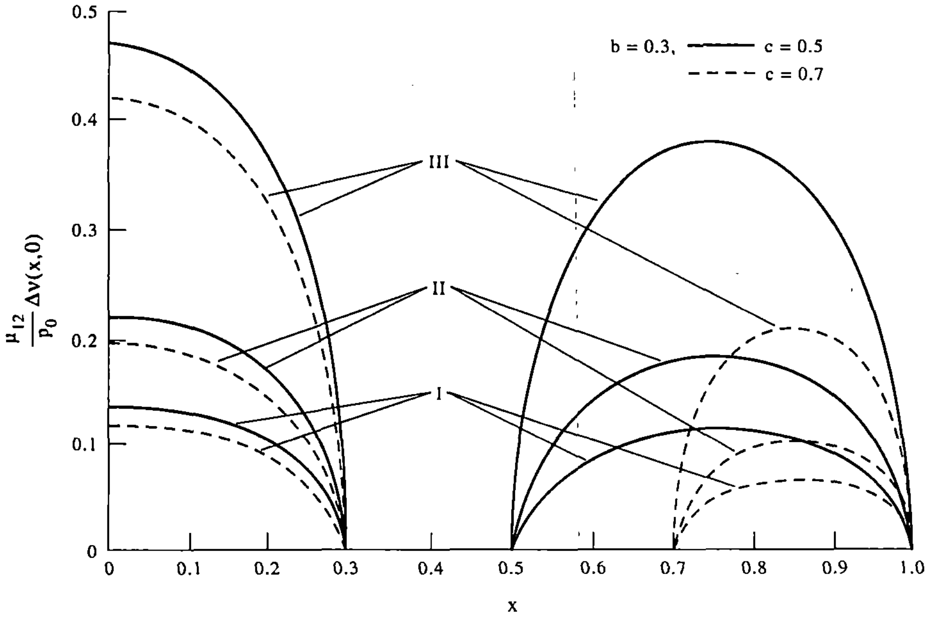

Fig. 8. Crack opening displacement vs distance for generalized plane stress ($k_s = 0.5$, $b = 0.3$, $c = 0.5$, 0.7).

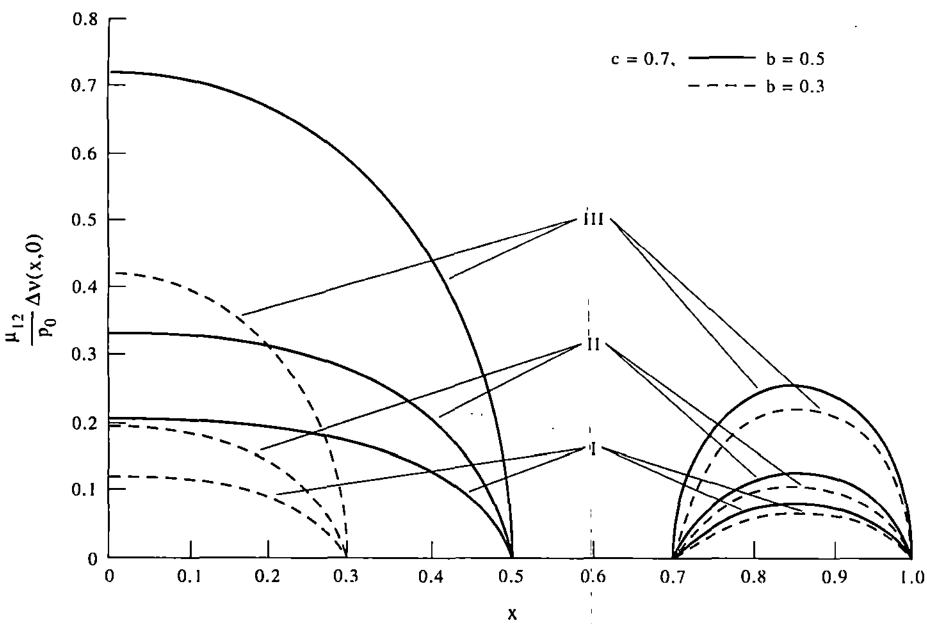

Fig. 9. Crack opening displacement vs distance for generalized plane stress ($k_s = 0.5$, $b = 0.3, 0.5$, $c = 0.7$).

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