

## CHAPTER - I

### STRESSES DUE TO NUCLEUS OF THERMOELASTIC STRAIN IN PRESENCE OF ELASTIC NUCLEUS

*Paper 1 : Thermoelastic stress concentration due to nucleus of thermoelastic strain in an infinite elastic solid with two rigid circular inserts.*

*Paper 2 : Thermoelastic stress concentration due to nucleus of thermoelastic strain in an infinite solid with two spherical elastic inclusions*

THERMOELASTIC STRESS CONCENTRATION DUE TO NUCLEUS OF THERMOELASTIC  
STRAIN IN AN INFINITE ELASTIC SOLID WITH  
TWO RIGID CIRCULAR INSERTS.

## .1.SOLUTION

Let us consider an infinite elastic solid at zero temperature. If there be situated a nucleus of thermoelastic strain at (0,0) the corresponding displacements can be derived as the gradient of a potential function [7]  $\psi$  where

$$\psi = Q_0 r^2 \log r \quad (1)$$

in which

$$Q_0 = \frac{1}{2\pi} \frac{1+\nu}{1-\nu} \alpha_0 T_0 ds \quad (2)$$

To obtain the particular solution we introduce bipolar co-ordinates by means of the transformation [14,38]

$$x = \frac{a \sinh \alpha}{\cosh \alpha + \cos \beta}, \quad y = \frac{a \sin \beta}{\cosh \alpha + \cos \beta} \quad (3)$$

$$r^2 = a^2 \frac{\cosh \alpha - \cos \beta}{\cosh \alpha + \cos \beta} \quad (4)$$

in which the ranges of the curvilinear co-ordinates  $\alpha$  and  $\beta$  are  $-\infty < \alpha < \infty$ ,  $-\pi \leq \beta \leq \pi$ .

As it is convenient to deal with  $h\psi$  instead of  $\psi$  itself in these

particular co-ordinates, we calculate displacements and stresses for the function [35]

$$h\psi = Q_0 a \left[ \sum_{n=0}^{\infty} n^{-2} \cos n\beta + \frac{1}{2} \log a \sum_{n=2}^{\infty} n^{-1} e^{n\alpha} \right] \quad (5)$$

where

$$h^{-2} = \left[ \frac{\partial x}{\partial \alpha} \right]^2 + \left[ \frac{\partial y}{\partial \alpha} \right]^2$$

Now, the displacements and stresses are

$$(u_{\alpha})_T = (\cosh \alpha + \cos \beta) \frac{Q_0}{2} \log a \sum_{n=2}^{\infty} e^{n\alpha}$$

$$(u_{\beta})_T = -(\cosh \alpha + \cos \beta) Q_0 \sum_{n=0}^{\infty} \sin(n\beta)/n \quad (6)$$

$$(a\widehat{\alpha\alpha})_T = -Q_0 a \left[ (\cosh \alpha + \cos \beta) \sum \cos(n\beta) + \frac{1}{2} \log a \sinh \alpha \sum e^{n\alpha} - \right. \\ \left. \sin \beta \sum \sin(n\beta)/n - \cos \alpha \left\{ \sum n^{-2} \cos n\beta + \frac{1}{2} \log a \sum n^{-1} e^{n\alpha} \right\} \right]$$

$$(a\widehat{\beta\beta})_T = Q_0 a \left[ (\cosh \alpha + \cos \beta) \frac{1}{2} \log a \sum n e^{n\alpha} + \frac{1}{2} \log a \sinh \alpha \sum e^{n\alpha} \right]$$

$$- \sin \beta \sum \sin(n\beta)/n + \cos \beta \left\{ \sum n^{-2} \cos n\beta + \frac{1}{2} \log a \sum n^{-1} e^{n\alpha} \right\}$$

$$(\widehat{a\alpha\beta})_T = 0 \quad (7)$$

The stresses obtained in (7) are obviously produced by the thermal expansion. This expansion gives rise to certain stresses on the surfaces of the two circular inserts  $\alpha = \pm\alpha_1$ . It is to be noted that each of the circular inserts lies entirely on either side of the line  $\alpha=0$ . We shall, therefore, make the boundaries free from stresses by the addition of the extra terms obtained on the hypothesis that there is no temperature distribution.

Let us consider a stress function which gives no stress at infinity and no stress over  $\alpha=0$  and such that on the surface  $\alpha = \pm\alpha_1$

$$(\widehat{a\alpha\alpha})_C = - (\widehat{a\alpha\alpha})_T$$

$$(\widehat{a\alpha\beta})_C = - (\widehat{a\alpha\beta})_T$$

$$(u_\alpha)_C = - (u_\alpha)_T$$

$$(u_\beta)_C = - (u_\beta)_T \quad (8)$$

Now, considering stress function in the form

$$h\chi = \sum_{n=2}^{\infty} \left\{ \phi_n(\alpha) \cos(n\beta) + \psi_n(\alpha) \sin(n\beta) \right\} \quad (9)$$

which is a solution of the biharmonic equation

$$\left( \frac{\partial^4}{\partial \alpha^4} + 2 \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} + \frac{\partial^4}{\partial \beta^4} - 2 \frac{\partial^4}{\partial \alpha^2} + 2 \frac{\partial^4}{\partial \beta^2} + 1 \right) (h\chi) = 0 \quad (10)$$

with

$$\phi_n(\alpha) = A_n \cosh(n+1)\alpha + B_n \cosh(n-1)\alpha + C_n \sinh(n+1)\alpha + D_n \sinh(n-1)\alpha \quad (11)$$

and

$$\psi_n(\alpha) = A'_n \cosh(n+1)\alpha + B'_n \cosh(n-1)\alpha + C'_n \sinh(n+1)\alpha + D'_n \sinh(n-1)\alpha \quad (12)$$

It may be readily seen that the conditions  $(\widehat{\alpha\alpha})_c$ ,  $(\widehat{\alpha\beta})_c$  shall vanish over  $\alpha=0$  is satisfied, if  $\phi_n(0)=0$ ,  $\phi'_n(0)=0$  and  $\psi_n(0)=0$ ,  $\psi'_n(0)=0$  for  $n \geq 2$ , and hence from (11) and (12),

$$A_n + B_n = 0$$

$$(n+1)C_n + (n-1)D_n = 0$$

$$A'_n + B'_n = 0$$

$$(n+1)C'_n + (n-1)D'_n = 0$$

So,

$$h\chi = \sum_{n=2}^{\infty} \left\{ A_n [\cosh(n+1)\alpha - \cosh(n-1)\alpha] + E_n [(n-1)\sinh(n+1)\alpha - (n+1)\sinh(n-1)\alpha] \right\}$$

$$\left. \cos(\eta\beta) + \sum_{n=2}^{\infty} \left\{ A'_n [\cosh(n+1)\alpha - \cosh(n-1)\alpha] + E'_n [(n-1)\sinh(n+1)\alpha - (n+1)\sinh(n-1)\alpha] \right\} \sin(\eta\beta) \right\} \sin(\eta\beta) \quad (13)$$

where

$$E_n = C_n / (n-1) \quad \text{and} \quad E'_n = C'_n / (n-1)$$

So, the complementary displacements and stresses are given by

$$(u_{\alpha})_c = a^{-1} (\cosh \alpha + \cos \beta) \sum_{n=2}^{\infty} \left\{ [A'_n \sin \eta\beta + A_n \cos \eta\beta] [(n+1) \sinh(n+1)\alpha - (n-1) \sinh(n-1)\alpha] + (n^2 - 1) [E'_n \sin \eta\beta + E_n \cos \eta\beta] [\cosh(n+1)\alpha - \cosh(n-1)\alpha] \right\}$$

$$(u_{\beta})_c = a^{-1} (\cosh \alpha + \cos \beta) \sum_{n=2}^{\infty} n \left\{ [A'_n \cos \eta\beta - A_n \sin \eta\beta] [\cosh(n+1)\alpha - \cosh(n-1)\alpha] + [E'_n \cos \eta\beta - E_n \sin \eta\beta] [(n-1) \sinh(n+1)\alpha - (n+1) \sinh(n-1)\alpha] \right\} \quad (14)$$

$$(\widehat{a\alpha})_c =$$

$$-(\cosh \alpha + \cos \beta) \sum_{n=2}^{\infty} n^2 \left\{ [A'_n \sin n\beta + A_n \cos n\beta] [\cosh(n+1)\alpha - \cosh(n-1)\alpha] + \right.$$

$$\left. [E'_n \sin n\beta + E_n \cos n\beta] [(n-1) \sinh(n+1)\alpha - (n+1) \sinh(n-1)\alpha] \right\} + \sin \beta \times$$

$$\sum_{n=2}^{\infty} n \left\{ [A_n \sin n\beta - A'_n \cos n\beta] [\cosh(n+1)\alpha - \cosh(n-1)\alpha] + [E_n \sin n\beta - E'_n \cos n\beta] \right.$$

$$\left. [(n-1) \sinh(n+1)\alpha - (n+1) \sinh(n-1)\alpha] \right\} - \sinh \alpha \sum_{n=2}^{\infty} \left\{ [A'_n \sin n\beta + A_n \cos n\beta] \right.$$

$$\left. [(n+1) \sinh(n+1)\alpha - (n-1) \sinh(n-1)\alpha] + (n^2 - 1) [E'_n \sin n\beta + E_n \cos n\beta] \right.$$

$$\left. [\cosh(n+1)\alpha - \cosh(n-1)\alpha] \right\} + \cosh \alpha \sum_{n=2}^{\infty} \left\{ [A'_n \sin n\beta + A_n \cos n\beta] [\cosh(n+1)\alpha - \right.$$

$$\left. \cosh(n-1)\alpha] + [E_n \cos n\beta + E'_n \sin n\beta] [(n-1) \sinh(n+1)\alpha - (n+1) \sinh(n-1)\alpha] \right\}$$

$$\begin{aligned}
(\widehat{a\beta\beta})_c &= (\cosh \alpha + \cos \beta) \sum_{n=2}^{\infty} \left\{ [A'_n \sin n\beta + A_n \cos n\beta] [(n+1)^2 \cosh(n+1)\alpha - \right. \\
&(n-1)^2 \cosh(n-1)\alpha] + (n^2 - 1) [E'_n \sin n\beta + E_n \cos n\beta] [(n+1) \sinh(n+1)\alpha - \\
&(n-1) \sinh(n-1)\alpha] \left. \right\} - \sin \beta \sum_{n=2}^{\infty} n \left\{ [A_n \sin n\beta - A'_n \cos n\beta] [\cosh(n+1)\alpha - \right. \\
&\cosh(n-1)\alpha] + [E_n \sin n\beta - E'_n \cos n\beta] [(n-1) \sinh(n+1)\alpha - (n+1) \sinh(n-1)\alpha] \left. \right\} \\
&- \sinh \alpha \sum_{n=2}^{\infty} \left\{ [A'_n \sin n\beta + A_n \cos n\beta] [(n+1) \sinh(n+1)\alpha - (n-1) \sinh(n-1)\alpha] \right. \\
&+ (n^2 - 1) [E'_n \sin n\beta + E_n \cos n\beta] [\cosh(n+1)\alpha - \cosh(n-1)\alpha] \left. \right\} + \cos \beta \times \\
&\sum_{n=2}^{\infty} \left\{ [A'_n \sin n\beta + A_n \cos n\beta] [\cosh(n+1)\alpha - \cosh(n-1)\alpha] + [E_n \cos n\beta + E'_n \right.
\end{aligned}$$

$$\times \sin \eta\beta \left\{ [(n-1)\sinh(n+1)\alpha - (n+1)\sinh(n-1)\alpha] \right\}$$

$$(\widehat{a\alpha\beta})_c = (\cosh \alpha + \cos \beta) \sum_{n=2}^{\infty} n \left\{ [A_n \sin \eta\beta - A'_n \cos \eta\beta] [(n+1)\sinh(n+1)\alpha - (n-1)\sinh(n-1)\alpha] + (n^2-1) [E_n \sin \eta\beta - E'_n \cos \eta\beta] [\cosh(n+1)\alpha - \cosh(n-1)\alpha] \right\}$$

(15)

Using (8) on the surface  $\alpha = \pm\alpha_1$ , we have

$$E_n = -aQ_0 \log a \frac{\cosh(n\alpha_1) \cos(\eta\beta)}{2(n^2-1) [\cosh(n+1)\alpha_1 - \cosh(n-1)\alpha_1]}$$

$$E'_n = -aQ_0 \log a \frac{\cosh(n\alpha_1) \sin(\eta\beta)}{2(n^2-1) [\cosh(n+1)\alpha_1 - \cosh(n-1)\alpha_1]}$$

$$A_n = aQ_0 \left\{ 2[(1-n^{-2})\cosh \alpha_1 + \cos \beta] \cos^2 \eta\beta + \log a \left( \sinh \alpha_1 \sinh n\alpha_1 - \frac{1}{n} \times \cosh \alpha_1 \cosh n\alpha_1 \right) \cos \eta\beta \right\} \left\{ [\cosh(n+1)\alpha_1 - \cosh(n-1)\alpha_1] [2\cosh \alpha_1 - 2n^2 \times \right.$$

$$(\cosh \alpha_1 + \cos \beta) - 2\sinh \alpha_1 [(n+1) \sinh(n+1)\alpha_1 - (n-1) \sinh(n-1)\alpha_1] \}^{-1}$$

$$= aQ_0 \frac{\cos^2 \eta\beta}{n^2 [\cosh(n+1)\alpha_1 - \cosh(n-1)\alpha_1]}$$

$$A'_n = aQ_0 \left\{ [(1-n^{-2}) \cosh \alpha_1 + \cos \beta] \sin 2\eta\beta + \log_a [\sinh \alpha_1 \sinh n\alpha_1 - n^{-1}$$

$$\cosh \alpha_1 \cosh n\alpha_1] \sin \eta\beta \right\} \left\{ [\cosh(n+1)\alpha_1 - \cosh(n-1)\alpha_1] [2\cosh \alpha_1 - 2n^2 \times$$

$$(\cosh \alpha_1 + \cos \beta) - 2\sinh \alpha_1 [(n+1) \sinh(n+1)\alpha_1 - (n-1) \sinh(n-1)\alpha_1] \}^{-1}$$

$$= aQ_0 \frac{\cos(\eta\beta) \sin(\eta\beta)}{n^2 [\cosh(n+1)\alpha_1 - \cosh(n-1)\alpha_1]}$$

(16)

With these values of the constants complementary stresses are

known,. Thus the components of resultant displacement and stress on the surface  $\alpha = \pm\alpha_1$  are given by

$$\begin{aligned} (\widehat{a\alpha\alpha}) &= (\widehat{a\alpha\alpha})_C + (\widehat{a\alpha\alpha})_T \\ (\widehat{a\beta\beta}) &= (\widehat{a\beta\beta})_C + (\widehat{a\beta\beta})_T \\ (\widehat{a\alpha\beta}) &= (\widehat{a\alpha\beta})_C + (\widehat{a\alpha\beta})_T \end{aligned} \quad (17)$$

$$\begin{aligned} u_\alpha &= (u_\alpha)_T + (u_\alpha)_C \\ u_\beta &= (u_\beta)_T + (u_\beta)_C \end{aligned} \quad (18)$$

From (17) we have

$$\begin{aligned} a(\widehat{\alpha\alpha+\beta\beta}) &= Q_0 a (\cosh \alpha_1 + \cos \beta) (n \log a \cosh n\alpha_1 - 2 \cos n\beta) + Q_0 a (\cosh \alpha_1 + \\ &\cos \beta) \left\{ [(1-n^{-2}) \cosh \alpha_1 + \cos \beta] \cos n\beta + \log a \left[ \sinh \alpha_1 \sinh n\alpha_1 - n^{-1} \cosh \alpha_1 \right. \right. \\ &\left. \left. \cosh n\alpha_1 \right] \right\} \left\{ [(2n+2) \cosh(n+1)\alpha_1 - (2n-2) \cosh(n-1)\alpha_1] \right\} \left\{ [\cosh(n+1)\alpha_1 - \right. \\ &\left. \cosh(n-1)\alpha_1] \right\} [\cosh \alpha_1 - n^2 (\cosh \alpha_1 + \cos \beta)] - \sinh \alpha_1 [(n+1) \sinh(n+1)\alpha_1 - \end{aligned}$$

$$\left. (n-1) \sinh(n-1)\alpha_1 \right\}^{-1} \quad (19)$$

## 2. NUMERICAL RESULTS

The numerical results of equation (19) when  $\alpha_1=1, a=1, n=2$  for different values of  $\beta$  between  $0^\circ$  and  $180^\circ$  are tabulated below:

$\beta$	$0^\circ$	$30^\circ$	$60^\circ$
$\frac{(\widehat{\alpha\alpha} + \widehat{\beta\beta})}{Q_0}$	-0.5508	-0.1234	-0.4558

  

$\beta$	$90^\circ$	$120^\circ$	$150^\circ$	$180^\circ$
$\frac{(\widehat{\alpha\alpha} + \widehat{\beta\beta})}{Q_0}$	-2.905	-2.565	3.418	-2.580

THERMOELASTIC STRESS CONCENTRATION DUE TO NUCLEUS OF THERMOELASTIC  
STRAIN IN AN INFINITE ELASTIC SOLID WITH TWO SPHERICAL  
ELASTIC INCLUSIONS

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*Communicated for publications.*

## 1.SOLUTION:

Let an infinite elastic solid at zero temperature contain an element of volume  $dr$  at  $(a,0,0)$  which is heated to a temperature  $T$ . In this case, the displacement is given by the gradient of the scalar function  $\chi$  [17] where

$$\chi = \frac{A}{2G} \frac{1}{\sqrt{r^2 - 2r a \cos\theta + a^2}} \quad (1)$$

in which

$$A = - \frac{2G}{4\pi} \frac{\alpha_1 (1+\nu)}{1-\nu} T dr$$

$\alpha_1, G, \nu$  being respectively coefficient of linear thermal expansion, coefficient of elasticity in shear and Poisson's ratio.

The components of displacement and stress in space polar coordinates due to  $\chi$  are given by [86].

$$2Gu_r = - \frac{A(r - a \cos\theta)}{(r^2 - 2r a \cos\theta + a^2)^{3/2}}$$

$$2Gu_\theta = - \frac{Aa \sin\theta}{(r^2 - 2r a \cos\theta + a^2)^{3/2}}$$

$$u_{\phi} = 0 \quad (2)$$

$$\widehat{r r} = A \frac{2r^2 + 3a^2 \cos \theta - 4r a \cos \theta - a^2}{(r^2 - 2r a \cos \theta + a^2)^{5/2}}$$

$$\widehat{\theta \theta} = A \left[ \frac{3a^2 \sin^2 \theta}{(r^2 - 2r a \cos \theta + a^2)^{5/2}} - \frac{1}{(r^2 - 2r a \cos \theta + a^2)^{3/2}} \right]$$

$$\widehat{\phi \phi} = - \frac{A}{(r^2 - 2r a \cos \theta + a^2)^{3/2}}$$

$$\widehat{r \theta} = A \frac{3a \sin \theta (r - a \cos \theta)}{(r^2 - 2r a \cos \theta + a^2)^{5/2}},$$

$$\widehat{r \phi} = \widehat{\theta \phi} = 0. \quad (3)$$

We denote this displacement and stress fields derived from  $\chi$  by  $\bar{\chi}$ .

In case of torsion free, rotational symmetry and in the absence of body forces, the general solution of the displacement equation of equilibrium following Boussinesq, is represented as the sum of the two displacement fields [64]

$$2G[u,v,w] = \text{grad } \phi \quad (4)$$

$$2G[u,v,w] = \text{grad } (z,\psi) - [0,0,4(1-\nu)\psi] \quad (5)$$

provided  $\phi(r,z)$  and  $\psi(r,z)$  with  $r = \sqrt{x^2+y^2+z^2}$ , are arbitrary harmonic functions [64], i.e.

$$\nabla^2 \phi = 0, \quad \nabla^2 \psi = 0.$$

Now, introducing Spherical Dipolar Coordinates by means of the transformations [67]

$$x = \frac{\bar{p}}{q-p} \cos \gamma$$

$$y = \frac{\bar{p}}{q-p} \sin \gamma$$

$$z = -\frac{\bar{q}}{q-p}$$

where

$$q = \cosh \alpha, \quad \bar{q} = \sinh \alpha$$

$$p = \cos \beta, \quad \bar{p} = \sin \beta \quad (7)$$

The ranges of the curvilinear coordinates  $(\alpha, \beta, \gamma)$  are  $-\infty < \alpha < \infty$ ,  $0 \leq \beta \leq \pi$ ,  $0 \leq \gamma \leq 2\pi$  so that

$$1 \leq q < \infty, \quad -\infty < \bar{q} < \infty$$

$$-1 \leq p \leq 1, \quad 0 \leq \bar{p} \leq 1.$$

To obtain the particular solution, we introduce the exterior and interior spherical dipolar harmonics of integral order as generating stress functions.

We consider the dipolar harmonic stress functions [64]

$$\phi_n(\alpha, \beta) = \mu C_n(\alpha) P_n(p) \quad (\text{exterior})$$

$$\psi_n(\alpha, \beta) = \mu S_n(\alpha) P_n(p), \quad n=0, 1, 2, \dots \quad (7)$$

and

$$\phi'_n(\alpha, \beta) = \mu C_n(\alpha) P_n(p) \quad (\text{interior})$$

$$\psi'_n(\alpha, \beta) = \mu S_n(\alpha) P_n(p), \quad n=0, 1, 2, \dots \quad (8)$$

where

$$\mu = \sqrt{q-p}, \quad C_n(\alpha) = \cosh(n+1/2)\alpha, \quad S_n(\alpha) = \sinh(n+1/2)\alpha$$

and  $P_n(p)$  is the Legendre polynomial of degree  $n$ .

The particular solution generated by  $\phi_n, \psi_n$  and  $\phi'_n, \psi'_n$  will be distinguished by  $\bar{\phi}_n, \bar{\psi}_n$  and  $\bar{\phi}'_n, \bar{\psi}'_n$ . Solutions for  $\bar{\phi}_n$  are

$$u_{\alpha} = \frac{\mu}{2G} \left[ \bar{q} C_n(\alpha) P_n(p) + k\mu^2 S_n(\alpha) P_n(p) \right],$$

$$u_{\beta} = \frac{\bar{p}\mu}{4G} \left[ C_n(\alpha) P_n(p) - 2\mu^2 C_n(\alpha) P'_n(p) \right],$$

$$u_{\gamma} = 0.$$

(9)

$$\sigma_{\alpha} = \mu \left\{ \left[ \frac{k^2+1}{4} \mu^4 + \frac{\bar{q}^2}{2} - \frac{\bar{p}^2}{4} \right] C_n(\alpha) P_n(p) + \bar{p}^2 \mu^2 C_n(\alpha) P'_n(p) + k\bar{q}\mu^2 S_n(\alpha) P_n(p) \right\},$$

$$\sigma_{\beta} = \mu \left\{ \left[ -\frac{k^2}{4} \mu^4 - \frac{\bar{q}^2}{4} + \frac{\bar{p}^2}{2} \right] C_n(\alpha) P_n(p) + (p\mu^2 - 2\bar{p}^2) \mu^2 C_n(\alpha) P'_n(p) - \right.$$

$$\left. \frac{1}{2} k\bar{q}\mu^2 S_n(\alpha) P_n(p) \right\},$$

$$\sigma_{\gamma} = -\mu^3 \left\{ \frac{q}{2} C_n(\alpha) P_n(p) + (pq-1) C_n(\alpha) P'_n(p) + \frac{1}{2} k\bar{q} S_n(\alpha) P_n(p) \right\},$$

$$\tau_{\alpha\beta} = \bar{p}\mu \left[ \frac{3}{4} \bar{q} C_n(\alpha) P_n(p) - \frac{3}{2} \bar{q}\mu^2 C_n(\alpha) P_n'(p) + \frac{3}{4} k\mu^2 S_n(\alpha) P_n(p) - \frac{1}{2} k\mu^4 S_n(\alpha) P_n'(p) \right] \quad (10)$$

where  $k = 2n+1$ ,  $n=0,1,2,\dots$  and

$$\tau_{\beta\gamma} = \tau_{\gamma\alpha} = 0.$$

Solutions for  $\bar{\psi}_n$  are

$$u_\alpha = -\frac{1}{4G\mu} \left[ \left\{ (7-8\nu)(qp-1) + q\mu^2 \right\} S_n(\alpha) P_n(p) + k\bar{q}\mu^2 C_n(\alpha) P_n(p) \right],$$

$$u_\beta = -\frac{\bar{p}\bar{q}}{4G\mu} \left[ (7-8\nu) S_n(\alpha) P_n(p) - 2\mu^2 S_n(\alpha) P_n'(p) \right],$$

$$u_\gamma = 0. \quad (11)$$

$$\sigma_\alpha = -\mu^{-1} \left\{ \left[ \frac{k^2-4+8\nu}{4} q\bar{q}\mu^2 + \frac{k^2+2-4\nu}{4} \bar{q}p\mu^2 + \frac{7-8\nu}{4} \bar{q}^3 \right] S_n(\alpha) P_n(p) \right.$$

$$+ (1-2\nu)\bar{q}\bar{p}^2\mu^2 S_n(\alpha)P'_n(p) + k\left[\bar{q}^2+(1-\nu)(qp-1)\right]\mu^2 C_n(\alpha)P_n(p)\left\}$$

$$\sigma_\beta = \mu^{-1} \left\{ \left[ \frac{k^2-6+8\nu}{4} q\bar{q} \mu^2 - \frac{k^2+6-4\nu}{4} \bar{q}p \mu^2 + \frac{7-8\nu}{4} \bar{q}^3 \right] S_n(\alpha)P_n(p) \right.$$

$$\left. + \left[ (3-2\nu)\bar{p}^2-qp+1 \right] \bar{q}\mu^2 S_n(\alpha)P'_n(p) + \frac{1}{2}k\left[\bar{q}^2-2\nu(qp-1)\right]\mu^2 C_n(\alpha)P_n(p) \right\}$$

$$\sigma_\gamma = \mu \left\{ (q-2\nu p)\frac{\bar{q}}{2} S_n(\alpha)P_n(p) + (pq-1+2\nu\bar{p}^2)\bar{q}S_n(\alpha)P'_n(p) + \right.$$

$$\left. \frac{1}{2}k\left[\bar{q}^2-2\nu(qp-1)\right]C_n(\alpha)P_n(p) \right\}$$

$$\tau_{\alpha\beta} = \bar{p}\mu^{-1} \left\{ \left[ \frac{1-2\nu}{2} q\mu^2 - \frac{7-8\nu}{4} \bar{q}^2 \right] S_n(\alpha)P_n(p) + \frac{1}{2}\left[ 3\bar{q}^2+(2-4\nu)(qp-1) \right] \right.$$

$$\left. \mu^2 S_n(\alpha)P'_n(p) - \frac{k(5-4\nu)}{4} \bar{q}\mu^2 C_n(\alpha)P_n(p) + \frac{1}{2}k\bar{q}\mu^2 C_n(\alpha)P'_n(p) \right\} \quad (12)$$

where  $k = 2n+1$ ,  $n=0,1,2,\dots$ , and

$$\tau_{\beta\gamma} = \tau_{\gamma\alpha} = 0.$$

Similar solutions will be found for  $\bar{\phi}'_n, \bar{\psi}'_n$ .

## 2. SOLUTION OF THE PROBLEM

Infinite elastic solid with two axisymmetric spherical inclusions, the heated element being situated outside the spherical surface is considered in this section.

We consider two spherical inclusions of the same radius and of different materials whose surfaces are taken as

$$\alpha = \alpha_0, \quad \alpha = -\alpha_0, \quad \alpha_0 > 0.$$

The heated element is at  $(0,0,0)$ .

In order to fit the conditions on the spherical surfaces, we must have on  $\alpha = \pm\alpha_0$

$$(\sigma_\alpha)_i = (\sigma_\alpha)_e,$$

$$(\tau_{\alpha\beta})_i = (\tau_{\alpha\beta})_e,$$

$$(u_\alpha)_i = (u_\alpha)_e,$$

$$(u_\beta)_i = (u_\beta)_e.$$

$\alpha = \pm\alpha_0$ ,  $0 \leq \beta \leq \pi$ ,  $0 \leq \gamma \leq 2\pi$ , i and e stand for interior and exterior respectively

(13)

Also, all the components of displacement and stress must be finite when  $\alpha \leq \pm\alpha_0$  and zero at infinity. In the region  $\alpha \geq \pm\alpha_0$  the stress systems  $\bar{\phi}_n$  and  $\bar{\psi}_n$  are uniform and hence we can superpose them on the solutions S and S' to the problem subject to the boundary conditions (13) in the forms

$$\bar{S}' = \sum [ a'_n \bar{\phi}'_n + b'_n \bar{\psi}'_n ] \quad (\text{inside}) \quad (14)$$

$$\bar{S} = \bar{\chi} + \sum [ a_n \bar{\phi}_n + b_n \bar{\psi}_n ] \quad (\text{outside}) \quad (15)$$

Now, for the solution  $\bar{\chi}$  on  $\alpha = \pm\alpha_0$

$$u_\alpha = \frac{A \mu_0}{2\sqrt{2}G^2} \sum \left[ -\bar{q}_0 S_n(\alpha_0) P_n(p) - \mu_0^2 k C_n(\alpha_0) P_n(p) \right] \cos(n\pi),$$

$$u_\beta = \frac{A \mu_0}{2\sqrt{2}G^2} \bar{p} \sum \left[ C_n(\alpha_0) P_n(p) - 2 \mu_0^2 C_n(\alpha_0) P'_n(p) \right] \cos(n\pi),$$

$$\sigma_{\alpha} = \frac{\sqrt{2A} \mu_0}{G} \Sigma \left\{ \left[ \frac{k^2+1}{4} \mu_0^4 + \frac{\bar{q}_0^2}{2} - \frac{\bar{p}^2}{4} \right] C_n(\alpha_0) P_n(p) + \bar{p}^2 \mu_0 C_n(\alpha_0) P'_n(p) + k \bar{q}_0 \mu_0^2 S_n(\alpha_0) P_n(p) \right\} \cos(n\pi)$$

$$\tau_{\alpha\beta} = \frac{\sqrt{2A} \mu_0}{G} \bar{p} \Sigma \left[ -\frac{3}{4} \bar{q}_0 S_n(\alpha_0) P_n(p) + \frac{3}{2} \bar{q}_0 \mu_0^2 S_n(\alpha_0) P'_n(p) \right.$$

$$\left. - \frac{3}{4} k \mu_0^2 C_n(\alpha_0) P_n(p) + \frac{1}{2} k \mu_0^4 C_n(\alpha_0) P'_n(p) \right] \cos(n\pi) \quad (16)$$

So, if we satisfy the four conditions of (13) the values of the four constants  $a_n, a'_n, b_n, b'_n$  of (14) and (15) are determined. Thus

$$a'_n \mu \left\{ \left[ \frac{k^2+1}{4} \mu^4 + \frac{\bar{q}^2}{2} - \frac{\bar{p}^2}{4} \right] C_n(\alpha) P_n(p) + \bar{p}^2 \mu^2 C_n(\alpha) P'_n(p) + k \bar{q} \mu^2 S_n(\alpha) P_n(p) \right\} -$$

$$- b'_n \mu^{-1} \left\{ \left[ \frac{k^2-4+8\nu'}{4} \bar{q} \bar{q} \mu^2 + \frac{k^2+2-4\nu'}{4} \bar{q} \bar{p} \mu^2 + \frac{7-8\nu'}{4} \bar{q}^3 \right] S_n(\alpha) P_n(p) + \right.$$

$$+ (1-2\nu') \bar{q} \bar{p}^2 \mu^4 S_n(\alpha) P'_n(p) + k \left[ \bar{q}^2 + (1-\nu')(qp-1) \right] \mu^2 C_n(\alpha) P_n(p) \left. \right\}$$

$$= \frac{\sqrt{2A} \mu_0}{G} \left\{ \left[ \frac{k^2+1}{4} \mu_0^4 + \frac{\bar{q}_0^2}{2} - \frac{\bar{p}^2}{4} \right] C_n(\alpha_0) P_n(p) + \bar{p}^2 \mu_0 C_n(\alpha_0) P'_n(p) + \right.$$

$$\left. + k \bar{q}_0 \mu_0^2 S_n(\alpha_0) P_n(p) \right\} \cos(n\pi) + a_n \mu \left\{ \left[ \frac{k^2+1}{4} \mu^4 + \frac{\bar{q}^2}{2} - \frac{\bar{p}^2}{4} \right] C_n(\alpha) P_n(p) + \right.$$

$$\left. + \bar{p}^2 \mu^2 C_n(\alpha) P'_n(p) + k \bar{q} \mu^2 S_n(\alpha) P_n(p) \right\} - b_n \mu^{-1} \left\{ \left[ \frac{k^2-4+8\nu}{4} \bar{q} \bar{q} \mu^2 + \frac{k^2+2-4\nu}{4} \right. \right.$$

$$\left. \bar{q} p \mu^2 + \frac{7-8\nu}{4} \bar{q}^3 \right] S_n(\alpha) P_n(p) + (1-2\nu) \bar{q} \bar{p}^2 \mu^2 S_n(\alpha) P'_n(p) + k \left[ \bar{q}^2 + (1-\nu)(qp-1) \right]$$

$$\left. \mu^2 C_n(\alpha) P_n(p) \right\}$$

(17)

$$a'_n \bar{p} \mu \left[ \frac{3}{4} \bar{q} C_n(\alpha) P_n(p) - \frac{3}{2} \bar{q} \mu^2 C_n(\alpha) P'_n(p) + \frac{3}{4} k \mu^2 S_n(\alpha) P_n(p) - \frac{1}{2} k \mu^4 S_n(\alpha) P'_n(p) \right]$$

$$+ b_n \bar{p} \mu^{-1} \left\{ \left[ \frac{1-2\nu'}{2} q \mu^2 - \frac{7-8\nu'}{4} \bar{q}^2 \right] S_n(\alpha) P_n(p) + \frac{1}{2} \left[ 3\bar{q}^2 + (2-4\nu')(qp-1) \right] \mu^2 S_n(\alpha) \right.$$

$$\left. P_n'(p) - \frac{k(5-4\nu')}{4} \bar{q} \mu^2 C_n(\alpha) P_n(p) + \frac{1}{2} k \bar{q} \mu^4 C_n(\alpha) P_n'(p) \right\}$$

$$= \frac{\sqrt{2A} \mu_0}{G} \bar{p} \left[ -\frac{3}{4} \bar{q}_0 S_n(\alpha_0) P_n(p) + \frac{3}{2} \bar{q}_0 \mu_0^2 S_n(\alpha_0) P_n'(p) - \frac{3}{4} k \mu_0^2 C_n(\alpha_0) P_n(p) + \right.$$

$$\left. \frac{1}{2} k \mu_0^4 C_n(\alpha) P_n'(p) \right] \cos(n\pi) + a_n \bar{p} \mu \left[ \frac{3}{4} \bar{q} C_n(\alpha) P_n(p) - \frac{3}{2} \bar{q} \mu^2 C_n(\alpha) P_n'(p) + \right.$$

$$\left. \frac{3}{4} k \mu^2 S_n(\alpha) P_n(p) - \frac{1}{2} k \mu^4 S_n(\alpha) P_n'(p) \right] + b_n \bar{p} \mu^{-1} \left\{ \left[ \frac{1-2\nu}{2} q \mu^2 - \frac{7-8\nu}{4} \bar{q}^2 \right] S_n(\alpha) \right.$$

$$P_n(p) + \frac{1}{2} \left[ 3\bar{q}^2 + (2-4\nu)(qp-1) \right] \mu^2 S_n(\alpha) P_n'(p) - \frac{k(5-4\nu)}{4} \bar{q} \mu^2 C_n(\alpha) P_n(p) +$$

$$\left. \frac{1}{2} k \bar{q} \mu^4 C_n(\alpha) P_n'(p) \right\}$$

(18)

$$a'_n \frac{\mu}{2G_1} \left[ \bar{q} C_n(\alpha) P_n(p) + k \mu^2 S_n(\alpha) P_n(p) \right] - \frac{b'_n}{4G_1 \mu} \left[ \left\{ (7-8\nu') (qp-1) + q\mu^2 \right\} S_n(\alpha) \right.$$

$$\left. P_n(p) + k \bar{q} \mu^2 C_n(\alpha) P_n(p) \right] = \frac{A \mu_0}{2\sqrt{2}G^2} \left[ \bar{q} C_n(\alpha_0) P_n(p) - k \mu_0^2 C_n(\alpha_0) P_n(p) \right]$$

$$\cos(n\pi) + a_n \frac{\mu}{2G} \left[ \bar{q} C_n(\alpha) P_n(p) + k \mu^2 S_n(\alpha) P_n(p) \right] - \frac{b_n}{4G\mu} \left[ \left\{ (7-8\nu) (qp-1) + q\mu^2 \right\} \right.$$

$$\left. S_n(\alpha) P_n(p) + k \bar{q} \mu^2 C_n(\alpha) P_n(p) \right] \quad (19)$$

and

$$a'_n \frac{\bar{p}\mu}{4G_1} \left[ C_n(\alpha) P_n(p) - 2\mu^2 C_n(\alpha) P'_n(p) \right] - b'_n \frac{\bar{p}\bar{q}}{4G_1 \mu} \left[ (7-8\nu') S_n(\alpha) P_n(p) - \right.$$

$$\left. 2\mu^2 S_n(\alpha) P'_n(p) \right] = \frac{A \mu_0}{2\sqrt{2}G^2} \bar{p} \left[ C_n(\alpha_0) P_n(p) - 2\mu_0^2 C_n(\alpha_0) P'_n(p) \right] \cos(n\pi) +$$

$$\frac{\bar{p}\mu}{4G} a_n \left[ C_n(\alpha) P_n(p) - 2\mu^2 C_n(\alpha) P'_n(p) \right] - \frac{\bar{p}\bar{q}}{4G\mu} b_n \left[ (7-8\nu) S_n(\alpha) P_n(p) - 2\mu^2 \right.$$

$$S_n(\alpha)P'_n(p) \quad (20)$$

where

$\nu'$  = Poisson's ratio for the material of the inclusion,

$G_1$  = Coefficients of elasticity in shear for the material of inclusion.

Solving (17)-(20), we get

$$a'_n = \frac{A}{G} \frac{\Delta_1}{\Delta} ; \quad b'_n = \frac{A}{G} \frac{\Delta_2}{\Delta}$$

$$a_n = \frac{A}{G} \frac{\Delta_3}{\Delta} ; \quad b_n = \frac{A}{G} \frac{\Delta_4}{\Delta}$$

where  $\Delta, \Delta_1, \Delta_2, \Delta_3$  and  $\Delta_4$  denote the determinants, the elements of which are the coefficients of  $a'_n, b'_n, a_n, b_n$  and the constant term of the equations (17)-(20). The values of these coefficients together with the equations (14), (15), (2), (3) and (9)-(12) constitute the complete solution of the problem.

The components of displacement and stress on the surface  $\alpha = \pm\alpha_0$  are

$$\frac{[u_\alpha]_\alpha}{A/G} = \pm\alpha_0 = \sum \frac{a'_n \mu_0}{A/G} \frac{1}{4G_1} \left[ \bar{q}_0 C_n(\alpha_0) + k\mu_0^2 S_n(\alpha_0) \right] P_n(p) - \sum \frac{1}{A/G} \frac{b'_n}{4G_1 \mu_0}$$

$$\left[ \left\{ (7-8\nu') (q_0 p - 1) + q_0 \mu_0^2 \right\} S_n(\alpha_0) P_n(p) + k \bar{q}_0 \mu_0^2 C_n(\alpha_0) P_n(p) \right],$$

$$\frac{[u_\beta]_\alpha}{A/G} = \pm \alpha_0 = \sum \frac{a'_n}{A/G} \mu_0 \bar{p} \left[ C_n(\alpha_0) P_n(p) - 2\mu_0^2 C_n(\alpha_0) P'_n(p) \right] - \sum \frac{b'_n}{A/G} \frac{\bar{q}_0 \bar{p}}{\mu_0}$$

$$\left[ (7-8\nu') S_n(\alpha_0) P_n(p) - 2\mu_0^2 S_n(\alpha_0) P'_n(p) \right],$$

$$\frac{[\sigma_\alpha]_\alpha}{A/G} = \pm \alpha_0 = \sum \frac{a'_n}{A/G} \mu_0 \left\{ \left[ \frac{k^2+1}{4} \mu_0^4 + \frac{\bar{q}_0^2}{2} - \frac{\bar{p}^2}{4} \right] C_n(\alpha_0) P_n(p) + \bar{p}^2 \mu_0^2 C_n(\alpha) P'_n(p) \right.$$

$$\left. k \bar{q}_0 \mu_0^2 S_n(\alpha_0) P_n(p) \right\} - \sum \frac{b'_n}{A/G} \mu_0^{-1} \left\{ \left[ \frac{k^2-4+8\nu'}{4} q_0 \bar{q}_0 \mu_0^2 + \frac{k^2+2-4\nu'}{4} \bar{q}_0 p \mu_0^2 + \frac{7-8\nu'}{4} \times \right. \right.$$

$$\left. q_0^3 \right] S_n(\alpha_0) P_n(p) + (1-2\nu') \bar{q}_0 p^2 \mu_0^2 S_n(\alpha_0) P'_n(p) + k \left[ \bar{q}_0^2 + (1-\nu') (q_0 p - 1) \right] \times$$

$$\left. \mu_0^2 C_n(\alpha_0) P_n(p) \right\}$$

$$\frac{[\tau_{\alpha\beta}]_{\alpha}}{A/G} = \pm\alpha_0 = \sum \frac{a'_n}{A/G} \mu_0 \bar{p} \left[ \frac{3}{4} \bar{q}_0 C_n(\alpha_0) P_n(p) - \frac{3}{2} \bar{q}_0 \mu_0^2 C_n(\alpha_0) P'_n(p) + \frac{3}{4} k \mu_0^2 \right.$$

$$\left. S_n(\alpha_0) P_n(p) - \frac{1}{2} k \mu_0^4 S_n(\alpha_0) P'_n(p) \right] + \sum \frac{b'_n}{A/G} \mu_0^{-1} \bar{p} \left\{ \left[ \frac{1-2\nu}{2} \bar{q}_0 \mu_0^2 - \frac{7-8\nu'}{4} \bar{q}_0^{-2} \right] \right.$$

$$S_n(\alpha_0) P_n(p) + \frac{1}{2} \left[ 3\bar{q}_0^{-2} + (2-4\nu') (\bar{q}_0 p - 1) \right] \mu_0^2 S_n(\alpha_0) P'_n(p) - \frac{k(5-4\nu')}{4} \bar{q}_0 \mu_0^2$$

$$\left. C_n(\alpha_0) P_n(p) + \frac{1}{2} k \bar{q}_0 \mu_0^4 C_n(\alpha_0) P'_n(p) \right\}$$

where  $\nu'$  and  $G_1$  are Poisson's ratio and coefficients of elasticity in shear respectively for the material of the inclusion.

### 3. NUMERICAL RESULTS

Here the numerical calculations are made for the case  $n=0$ , on the surface  $\alpha = \pm\alpha_0 = 1$  of the spherical inclusions taking numerical values [6]  $\nu' = 1/3$ ,  $\nu = 1/4$  and  $G/G_1 = 2$ .

Numerical values of  $u_\alpha, u_\beta$  and  $\sigma_\alpha$  for different values of  $\beta$  between  $0^\circ$  and  $180^\circ$  are given in the table below :

$\beta$	$0^\circ$	$30^\circ$	$60^\circ$	$120^\circ$	$150^\circ$	$180^\circ$
$\frac{u_\alpha}{A/8G_1^2}$	-10.453	-1.037	3.6904	-8.429	-17.371	-9.117
$\frac{u_\beta}{A/8G_1^2}$	-9.967	-4.241	1.381	3.629	-3.709	9.252
$\frac{\sigma_\alpha}{A/2G_1}$	-8.241	-5.975	3.333	-12.146	-35.153	13.541