Chapter 4

ULTRAMETRIC CANTOR SETS: EXAMPLES

4.1 Introduction

In this chapter, we present two examples, one on the classical triadic Cantor set and another on a general class of homogeneous (p, q) Cantor sets and explain various properties of valued infinitesimals and related concepts. To justify the analytic framework of Chapter 3, we present here *independent arguments* showing the actual process how a nontrivial valuation could arise on a Cantor set. The valuation in each of these two models is shown to be related to an appropriate Cantor function $\phi(x)$. The Cantor function $\phi(x)$, in the non-archimedean framework, is also shown to be extended to a locally constant function for *any* $x \in I$. Further, we verify the multiplicative representation equation (4.3) that exists because of the nontrivial infinitesimals and the scale invariant ultrametric for every element of the Cantor set, explicitly, in either of the classical middle third set, middle α set and the (p, q) Cantor set separately. Finally, the variability of the locally constant $\phi(x)$ is reinterpreted in the usual topology as an effect of relative infinitesimals which become dominant by inversion at an appropriate log scale.

4.2 Middle third Cantor set and Cantor function

4.2.1 Valuation

Let us now investigate in detail the well known triadic Cantor set C in the light of the analytic framework developed in Chapter 3 [24]. Indeed, we are going to show in detail how the concept of relative infinitesimals and associated valuations may actually arise in the context of the classical Cantor set. The relation of the valuation with the corresponding Cantor function will also be explained.

Suppose we begin with the set $C_0 = [0, 1]$. In relation to the *scale* 1, C_0 is essentially considered to be a doublet $\{0, 1\}$, in the sense that real numbers $0 < x < 1$ are *undetectable* in the assigned scale, and hence all such numbers might be identified with 0. We denote this 0 as $0_0 = [0, 1)$, the set of *infinitesimals.* However, the possible existence of *infinitesimals* are ignored at this scale and so 0 is considered simply as a singleton $\{0\}$ only. At the next level, we choose a smaller scale $\epsilon = 1/3$ (say), so that only the elements in $[0, 1/3) \subset C_0$ are now identified with 0, so that $0_1 = [0, 1/3)$, which is actually $0_1 = 0_0$ in the unit of 1/3. Relative to this nontrivial scale 1/3, we now assign the ultrametric valuation v to 0_1. In principle, all possible ultrametric valuations are admissible here. One has to make *a priori choice* to *select* the most appropriate valuation in a given application. In the context of the triadic Cantor set, there happens to be a unique choice relating it to the Cantor's function, as explained below.

Recall that the valuation induces a nontrivial topology in 0_1. Accordingly, the set is covered by n number of disjoint clopen intervals of *valued* infinitesimals. At the level 1, $n = 1$, which is actually the clopen interval I_{11} of length $1/3$ and displaced appropriately to the middle of the 1/3rd Cantor set, viz, $I_{11} = [1/3, 2/3]$ (in the ordinary representation this is the deleted open interval, including the two end points of neighbouring closed intervals). The value assigned to these valued infinitesimals is the constant $v(I_{11}) = 1/2$, where, of course, $v(0) = 0$. In principle, again, v could assume any constant value. Our choice is guided by the triadic Cantor function. Thus the valued set of infinitesimals, at the scale $1/3$, turns out to be $0_1 = \{0, 1/2\}$.

How does this valued set of infinitesimals enlight the ordinary construction of the Cantor set? Let $C_1 = F_{11} \cup F_{12}$ where $F_{11} = [0, 1/3]$ and $F_{12} = [2/3, 1]$. The value awarded to the deleted middle open interval is now inherited by these two closed (clopen) intervals, and so $||F_{11}|| = 1/3^s$ and $||F_{12}|| = 1/3^s$, recalling that $2 = 3^s$, s being the Hausdorff dimension $s = \log 2/\log 3$.

At the next level, when the scale is $\epsilon = 1/3^2$, the above interpretation can be easily extended. The zero set is now made of 3 clopen sets $0_2 = I_{20} \cup I_{21} \cup I_{22}$ where $I_{20} = [1/9, 2/9]$, $I_{21} = [3/9, 6/9]$ and $I_{22} = [7/9, 8/9]$. The value assigned to each of these sets are respectively, $v(I_{20}) = 1/4$, $v(I_{21}) = 2/4$ and $v(I_{22}) = 3/4$, so that the valued infinitesimals are given by $0_2 = \{0, 1/4, 2/4, 3/4\}$. Notice that the new members of the valued infinitesimals are derived as the mean value of two consecutive values from those (including 1 as well) at the previous level. These valued infinitesimals now, in turn, assign equal value to the 4 closed intervals in the ordinary level 2 Cantor set $C_2 = F_{20} \cup F_{21} \cup F_{22} \cup F_{23}$ where $F_{20} = [0, 1/9]$ and etc, viz. $||F_{2i}|| = 1/2^2 = 1/3^{2s}$, $i = 0, 1, 2, 3$. Notice that, in the sense of Sec. 2, the valued infinitesimals 0_2 induces a *fine structure* in the neighbourhood of F_{2i} : for a $x \in F_{2i}$, we now have valued neighbours $X^{\pm} = xx^{\pm k3^{-2s}}$, $k = 1, 2, 3$. Clearly, $||F_{2i}|| = ||x|| = 1/3^{2s}$, the infimum of all possible valued members, so misses the above fine structures. It also follows that the limit set of this triadic construction reproduces the Cantor function (c.f., Example 2) as the the valuation $v : [0, 1] \to [0, 1]$, defined originally on the *inverted* Cantor set $\mathbf{0} = \bigcap_n \bigcup_k I_{nk}$, and then extended on [0,1] by continuity.

Remark 5: The continuity in the present ultrametric topology is defined in the usual manner (*c.f.*, Definition 11). Further, v on $\mathbf{0}$ is an example of *locally constant* function relative to the $||.||$-topology and will be shown (in the next section) to satisfy the differential equation

$$x\frac{dv(x)}{dx} = 0. \tag{4.1}$$

We may interpret this as follows: Considered as a function on $\mathbf{0}$ (or C), v is constant in clopen sets I_{nk}(or $\mathrm{F_{nk}}$) for fixed values of both n and k, but experiences variability as either of these vary. This variability is not only continuous, but continuously first order differentiable as well. In contrast, v on $\{I_{nk}$ or $(F_{nk})\}$ is a discontinuous function in the usual topology.

4.2.2 Multiplicative structure

Let C be the standard middle $\frac{1}{3}$rd Cantor set. As will become clear our discussion will apply generically to any measure zero Cantor set (*c.f.*, *Sec.*4.2.3). The Cantor set C offers us with a privileged set of scales $\epsilon_n = 3^{-n}$. For a sufficiently large n viz : as $n \to \infty$, suppose an infinitesimal gap of the form $(0, \epsilon_{n+m})$ is decomposed into a finite number M of open subintervals $\tilde{I}_i$ of relative infinitesimals with constant valuations defined by

$$v(\tilde{x}_i) = i\ 2^{-n}\ ,\ i = 1, 2, \cdots, M,\ \tilde{x}_i \in I_i. \tag{4.2}$$

The valuation assigned by (4.2) is the triadic Cantor function $\phi : I \to I$ so that $M = 2^m - 1$ corresponding to the scale $\epsilon_m = 3^{-m}$. We call infinitesimals $\tilde{x}$ leaving in the island $\tilde{I}_i \subset (0, \epsilon_{n+m})$ of *valued infinitesimals* having the valuation (4.2) induced by the Cantor function. Any element x of the original Cantor set C is now endowed with a set of valued neighbours

$$X^i_{\pm} = x.x^{\mp v(\tilde{x}_i)}. \tag{4.3}$$

Finally, the element x is assigned the ultrametric valuation

$$\| x \| = \inf_i \log_{x^{-1}} \frac{X^i_+}{x} = \inf_i \log_{x^{-1}} \frac{x}{X^i_-} \tag{4.4}$$

so that $\| x \| = 2^{-n} = 3^{-ns}$ where $s = \frac{\log 2}{\log 3}$, the Hausdorff dimension of the triadic Cantor set C and $n \to \infty$. As it turns out, this valuation exactly reproduces the nontrivial measure of [17] derived in the context of noncommutative geometry (c.f., definition of valued measure in Sec.3.2.).

Now, to make contact with the absolute value (3.2) and the inversion rule (3.1), let for a sufficiently large but fixed n, $\tilde{x}_i \in \tilde{I}_i$ has the form (we *set for definiteness $m = 0$)*

$$\tilde{x}_i = 3^{-n} \cdot 3^{-n \cdot i 2^{-r}} \times a_i \tag{4.5}$$

where $ni = 2^r \cdot k_i$, k_i being, in general, a sufficiently large real number and $a_i = \Sigma a_{ij} 3^{-j} \in O_i$, a gap of size $3^{-r}$of the Cantor set C and $a_{ij} \in \{0, 1, 2\}$. Then $0 < \tilde{x}_i < 3^{-n}$ and $v(\tilde{x}_i) = i.2^{-r}$. One also verifies that for an $a_i = 1 + a_{0i}$, $v(a_i) = \lim_{n \to \infty} \log_{3^n}(a_i/3^{-n}) = 1$. By the inversion rule (3.1) the elements $\tilde{x}_i$ of the ball $\tilde{I}_i$ now connect to an $x_i \in C$ given by

$$x_i = 3^{-n} \cdot 3^{-n(- \cdot i 2^{-r})} \times b_i, \; b_i = \Sigma \; b_{ij} \; 3^{-j}, \; b_{ij} \in \{0, 2\} \tag{4.6}$$

where $\lambda = a_i \times b_i \in (0, 1)$. (Infinitesimal) Scales $\epsilon_n = 3^{-n}$, are the *primary* scales when the scales 3^{-k_i}(or equivalently 2^{-r}) are the *secondary* scales.

Finally, to verify the new multiplicative representation one notes that there exists $y_i \in C$ in a neighbourhood of x_i so that

$$y_i = 3^{-n} \; c_i, \quad c_i = \Sigma \; c_{ij} \; 3^{-j}, \; c_{ij} \in \{0, 2\}$$

and

$$x_i = y_i . y_i^{-i.2^{-r}} \tag{4.7}$$

so as to satisfy the identity

$$c_i^{n-k_i} = b_i^n. \tag{4.8}$$

To verify that (4.8) is not empty we note that for the end points $\frac{1}{3}$ and $\frac{2}{3}$, both belonging to C, (4.8) means $\left(\frac{2}{3}\right)^n = \left(\frac{1}{3}\right)^{n-k_1}$ yielding $k_1 = ns$, $s = \frac{\log 2}{\log 3}$. For this value of k_1, (4.8) now tells that $c_i^{1-s} = b_i$ so that $c_i = \left(\frac{1}{3}\right)^r$ and $b_i = \left(\frac{2}{3}\right)^r$ for a suitable r. Similar estimates for k_i are available for other (consecutive) end points of (higher order) gaps. It thus follows that the representation (4.3) is realized at the level of the finite Hausdorff measure of the set, when the value of the constant k is real (rather than a natural number).

4.2.3 Multiplicative Structure: Middle α set

Next, we consider C_α Cantor set. Here in each iteration we removes an open interval of length proportional to α from a closed interval $I = [0, 1]$, leaving out two open intervals of size β each. Therefore this Cantor set offers us with a privileged set of scales $\epsilon_n = r^{-n}$. For a sufficiently large n viz : as $n \to \infty$, suppose an infinitesimal gap of the form $(0, \epsilon_{n+m})$ is decomposed into a finite number M of open subintervals $\tilde{I}_i$ of relative infinitesimals with constant valuations defined by

$$v(\tilde{x}_i) = i\ 2^{-n}\ ,\ i = 1, 2, \cdots, M,\ \ \tilde{x}_i \in I_i. \tag{4.9}$$

The valuation assigned by equation (4.9) is the Cantor function $\phi : I \to I$ so that $M = 2^m - 1$ corresponding to the scale $\epsilon_m = r^{-m}$. We call infinitesimals $\tilde{x}$ leaving in the island $\tilde{I}_i \subset (0, \epsilon_{n+m})$ of *valued infinitesimals* having the valuation equation (4.9) induced by the Cantor function. Any element x of the original Cantor set is now endowed with a set of valued neighbours

$$X^i_\pm = x \cdot x^{\mp v(\tilde{x}_i)}. \tag{4.10}$$

Finally, the element x is assigned the ultrametric valuation

$$\| x \| = \inf_i \log_{x^{-1}} \frac{X^i_+}{x} = \inf_i \log_{x^{-1}} \frac{x}{X^i_-} \tag{4.11}$$

so that $\| x \| = 2^{-n} = \beta^{ns}$ where $s = \frac{\log 2}{\log \frac{1}{\beta}}$, the Hausdorff dimension of the C_α Cantor set and $n \to \infty$.

Now, to make contact with the absolute value (3.2) and the inversion rule (3.1), let for a sufficiently large but fixed n, $\tilde{x}_i \in \tilde{I}_i$ has the form (we set for definiteness $m = 0$)

$$\tilde{x}_i = \beta^n \cdot \beta^{n \cdot i 2^{-r}} \times a_i \tag{4.12}$$

where $ni = 2^r \cdot k_i$, k_i being, in general, a sufficiently large real number and $a_i = \Sigma a_{ij} (1 - \beta) \beta^j \in O_i$, a gap of size $\beta^r$of the Cantor set and $a_{ij} \in \{0, 1, 2\}$. Then $0 < \tilde{x}_i < \beta^n$ and $v(\tilde{x}_i) = i.2^{-r}$. One also verifies that for an $a_i = 1 + a_{0i}$, $v(a_i) = \lim_{n \to \infty} \log_{\beta^{-n}} (a_i / \beta^n) = 1$. By the inversion rule (3.1) the elements $\tilde{x}_i$ of the ball $\tilde{I}_i$ now connect to an $x_i \in C$ given by

$$x_i = \beta^n \cdot \beta^{n(-\cdot i p^{-r})} \times b_i, \; b_i = \Sigma \, b_{ij} \, (1 - \beta) \beta^j, \; b_{ij} \in \{0, 2, \} \tag{4.13}$$

where $\lambda = a_i \times b_i \in (0, 1)$. (Infinitesimal) Scales $\epsilon_n = r^{-n}$, are the *primary* scales when the scales β^{k_i}(or equivalently 2^{-r}) are the *secondary* scales.

Finally, to verify the new multiplicative representation one notes that there exists $y_i \in C$ in a neighbourhood of x_i so that

$$y_i = \beta^n \, c_i, \quad c_i = \Sigma \, c_{ij} \, (1-\beta) \, \beta^j, \;\; c_{ij} \in \{0, 2\}$$

and

$$x_i = y_i . y_i^{-i.2^{-r}} \tag{4.14}$$

so as to satisfy the identity

$$c_i^{n-k_i} = b_i^n . \tag{4.15}$$

Accordingly, it follows that a gap O in I/C (which is a connected interval in the usual topology) containing a point x of the Cantor set C is indeed realized as an "infinitesimal" Cantor set in the valuation defined by the Cantor function associated with the original Cantor set C itself. One thus concludes that

Proposition 3. *Any element x of an ultrametric Cantor set C is endowed with a class of valued neighbours having the multiplicative representation of the form (4.3) (or (4.10)) and the non-archimedean absolute value $||x|| = \inf_i v(\tilde{x}_i)$.*

4.3 (p,q) Cantor set and Cantor function

We first show that the value $v(x)$ awarded to the valued infinitesimals $X \in B_i$, $i = 1, 2, \ldots, n$ is given by the Cantor function $\phi : I \to I$ with points of discontinuity in $\phi'(x)$, in the usual sense, are in C [26]. In the new formalism this discontinuity is removed in a scale invariant way using logarithmic differentiability over (valued) infinitesimal open line segments replacing each $x \in C$. Our definition of $v(x)$ is guided by the given Cantor set C so as to retrieve the finite Hausdorff measure uniquely via the construction of the valued measure.

Let us denote the valued scale free infinitesimals by $[0,1)$, denoted here by $\tilde{C}$. The interval $[0,1)$ here is a copy of the scale free infinitesimals $\mathbf{I}^+$ for an arbitrary small ϵ_0 (say). The valued infinitesimals in $[0,1)$ then introduce a new set of scales of the form r^{-n} (in the unit of ϵ_0) so that the scales introduced in definition 1 are now parameterized as $\epsilon = \epsilon_0\, r^{-n}$. The choice of the 'secondary' scales r^{-n} are motivated by the finite level Cantor set C. At the ordinary level i.e. at the scale 1 (corresponding to $n = 0$), there is no valued infinitesimal (at the level of ordinary real calculus) except the trivial 0. So relative to the finite scale (given by $\delta = \frac{\epsilon}{\epsilon_0} = 1$) $[0,1)$ reduces to the singleton $\{0\}$. At the next level, we choose the smaller scale $\delta = \frac{1}{r}$. Consequently, elements in $[0,\frac{1}{r})$ are undetectable and identified with 0, again in the usual sense. Presently we have, however, the following.

We assume that the void (emptiness) of 0 reflects in an inverted manner the structure of the Cantor set C that is available at the finite scale. That is to say, at the first iteration of C from I, q open intervals are removed leaving out p closed intervals F_{1n}, $n = 1, 2, \ldots, p$. At the scale $\frac{1}{r}$ in the void of $\tilde{C}$, on the other hand, there now emerges (by "inversion") q open islands (intervals) $\mathbf{I}_{1i}$, $i = 1, 2, \ldots, q$. By definition, $\mathbf{I}_{1i}$ contains, for each i, the so called valued infinitesimals X_i which are assigned the values $v(X_i) = \phi(X_i) = \frac{i}{p}$, $i = 1, 2, \ldots, q, X_i \in I_{1i}$.

We note that at the scale $\delta = \frac{1}{r}$, there are p voids in $\tilde{C}$. At the next level of the scale $\frac{1}{r^2}$, there emerges again in each void q islands of open intervals, so that there are now pq number of total islands $\mathbf{I}_{2i}$, $i = 1, 2, \ldots, pq$. The value assigned to each of these valued islands of infinitesimals are $v(X_j) = \phi(X_j) = \frac{j}{p^2}$, $j = 1, 2, \ldots, pq$, where $X_j \in \mathbf{I}_{2j}$. Continuing this iteration, at the n th level, the (secondary) scale is $\delta = \frac{1}{r^n}$ and the number of open intervals $\mathbf{I}_{nj}$ of infinitesimals are now $q(1 + p + p^2 + \cdots + p^n) = N$ (say) with corresponding values

$$v(X_j) = \phi(X_j) = \frac{j}{p^n}, \; j = 1, 2, \ldots, N \tag{4.16}$$

where $X_j \in \mathbf{I}_{nj}$. Thus v and hence the Cantor function ϕ is defined on the "inverted Cantor set" $\tilde{C} = \bigcap_n \bigcup_j \mathbf{I}_{nj}$ and is extended to $\phi : I \to I$ by continuity following equations like

$$\begin{aligned} \phi(\beta_n) - \phi(\alpha_n) &= \frac{1}{p^n}, \\ \beta_n - \alpha_n &= \frac{1}{r^n} \text{ where } x \in [\alpha_n, \beta_n] \subset I. \end{aligned}$$

We note that the absolute value $\|.\|$ awarded to each block of the Cantor intervals F_{nk} are

$$\| F_{nk} \| = r^{-ns} \tag{4.17}$$

for each $k = 1, 2, \ldots, p^n$ where $C = \bigcap_n \bigcup_k F_{nk}$ and so $s = \frac{\log p}{\log r}$, since the valued set of infinitesimals induces fine structures to an element in F_{nk} viz. for an $y \in F_{nk}$, we now have the infinitesimal neighbours $Y^j_{\pm} = y.\, y^{\mp jp^{-n}}, j = 1, 2, \ldots, N$.

Clearly, the absolute value in equation (4.17) corresponds to the minimum of $v(x)$ at the nth iteration. Thus the valuations defined as the associated Cantor function leads to a valued measure on C that equals the corresponding Hausdorff measure with $s = \frac{\log p}{\log r}$.

Let us now recall that the solutions of $\phi'(x) = 0$ in a non-archimedean space are locally constant functions [20]. To show that Cantor function $\phi : I \to I$ is a locally constant function, let us recall that the Cantor set C is constructed recursively as $C = \bigcap_n \bigcup_k F_{nk}$. The set I, on the other hand, is written as $I = \bigcap_n [\ (\bigcup_{k=1}^{p^n} \tilde{F}_{nk}) \cup (\bigcup_{j=1}^{N} \mathbf{I}_{nj})\ ]$, the open interval $\tilde{F}_{nk}$ being F_{nk} with end points removed (recall that $\mathbf{I}_{nj}$ are closed in the ultrametric topology). By definition $v(\mathbf{I}_{nj}) = a_{nj}$ a constant for each n and j. We set $v(\tilde{F}_{nk}) = 0$ as $n \to \infty$. This equality is to be understood in the following sense. At an infinitesimal scale $\epsilon_0 \to 0^+$ the zero value of $\tilde{F}_{nk}$ becomes finitely valued recursively for each n since a Cantor point $x \in C$ is replaced by a copy of the (inverted) Cantor set $\tilde{C}$ with finite number of closed intervals like $\mathbf{I}_{nj}$. The derivatives of ϕ vanishes not only for each n and j but *even as* $n \to \infty$ (and $\epsilon \to 0$, for each arbitrarily small but fixed ϵ_0). Thus, the equality $\phi'(x) = 0$ on I/C, in the ordinary sense, gets extended to every $x \in C$ when the Cantor set is reinterpreted as a nonarchimedean space. The removal of the usual derivative discontinuities is also explained dynamically as due to the fact that the approach to an actual Cantor set point x is accomplished in the nonarchimedean setting by inversion. That is to say, as a variable $X \in I$ approaches $x \in C$, the usual linear shift in I is replaced by infinitesimal hoppings between two nieghbouring elements of the form $X_+/x \propto x/X_-$.

4.3.1 Multiplicative structure in (p, q) Cantor set

We now show that any element of the (p, q) Cantor set also has the multiplicative representation. We divide the interval $I = [0, 1]$ into r

number of closed subintervals each of length $\frac{1}{r}$ and delete q number of open subintervals from them so that $p+q=r$. Therefore this Cantor set offers us with a privileged set of scales $\epsilon_n = r^{-n}$. For a sufficiently large n viz : as $n \to \infty$, suppose an infinitesimal gap of the form $(0, \epsilon_{n+m})$ is decomposed into a finite number M of open subintervals $\tilde{I}_i$ of relative infinitesimals with constant valuations defined by

$$v(\tilde{x}_i) = i\ p^{-n}\ ,\ i = 1, 2, \cdots , M,\ \ \tilde{x}_i \in I_i. \tag{4.18}$$

The valuation assigned by (4.18) is the triadic Cantor function $\phi : I \to I$ so that $M = q(1 + p + p^2 + \cdots + p^m)$ corresponding to the scale $\epsilon_m = r^{-m}$. We call infinitesimals $\tilde{x}$ leaving in the island $\tilde{I}_i \subset (0, \epsilon_{n+m})$ *valued infinitesimals* having the valuation (4.18) induced by the Cantor function. Any element x of the original Cantor set is now endowed with a set of valued neighbours

$$X^i_{\pm} = x \cdot x^{\mp v(\tilde{x}_i)}. \tag{4.19}$$

Finally, the element x is assigned the ultrametric valuation

$$\| x \| \ = \ \inf_i \log_{x^{-1}} \frac{X^i_+}{x} \ = \ \inf_i \log_{x^{-1}} \frac{x}{X^i_-} \tag{4.20}$$

so that $\| x \| = p^{-n} = r^{-ns}$ where $s = \frac{\log p}{\log r}$, the Hausdorff dimension of the (p,q) Cantor set and $n \to \infty$. As it turns out, this valuation exactly reproduces the nontrivial measure of [17] derived in the context of noncommutative geometry (c.f., definition of valued measure in Sec.3.2.)

Now, to make contact with the absolute value (3.2) and the inversion rule (3.1), let for a sufficiently large but fixed n, $\tilde{x}_i \in \tilde{I}_i$ has the form (we set for definiteness $m = 0$)

$$\tilde{x}_i = r^{-n} \cdot r^{-n \cdot ip^{-t}} \times a_i \tag{4.21}$$

where $ni = 2^t \cdot k_i$, k_i being, in general, a sufficiently large real number and $a_i = \Sigma a_{ij} r^{-j} \in O_i$, a gap of size $r^{-t}$of the Cantor set and $a_{ij} \in \{0, 1, 2, \ldots, N\}$ where $N = r - 1$. Then $0 < \tilde{x}_i < r^{-n}$ and $v(\tilde{x}_i) = i.p^{-t}$. One also verifies that for an $a_i = 1 + a_{0i}$, $v(a_i) = \lim_{n \to \infty} \log_{r^n}(a_i/r^{-n}) = 1$. By the inversion rule (3.1) the elements $\tilde{x}_i$ of the ball $\tilde{I}_i$ now connect to an $x_i \in C$ given by

$$x_i = r^{-n} \cdot r^{-n(-\cdot ip^{-t})} \times b_i, \; b_i = \Sigma \, b_{ij} \, r^{-j}, \; b_{ij} \in \{0, 2, 4 \ldots N\} \tag{4.22}$$

where we consider the special case that the closed interval $I = [0, 1]$ is divided into r number of closed intervals with r odd and $\frac{r-1}{2}$ number of open intervals in the even places are deleted, and $\lambda = a_i \times b_i \in (0, 1)$. Infinitesimal scales $\epsilon_n = r^{-n}$, are the *primary* scales when the scales r^{-k_i}(or equivalently p^{-t}) are the *secondary* scales.

Finally, to verify the new multiplicative representation one notes that there exists $y_i \in C$ in a neighbourhood of x_i so that

$$y_i = r^{-n} \, c_i, \quad c_i = \Sigma \, c_{ij} \, r^{-j}, \quad c_{ij} \in \{0, 2, \ldots N\}$$

and

$$x_i = y_i . y_i^{-i.p^{-t}} \tag{4.23}$$

so as to satisfy the identity

$$c_i^{n-k_i} = b_i^n. \qquad (4.24)$$

To verify that (4.24) is not empty we note that for the end points $\frac{1}{r}$ and $\frac{p}{r}$, both belonging to C, (4.24) means $\left(\frac{p}{r}\right)^n = \left(\frac{1}{r}\right)^{n-k_1}$ yielding $k_1 = ns$, $s = \frac{\log p}{\log r}$. For this value of k_1, (4.24) now tells that $c_i^{1-s} = b_i$ so that $c_i = (\frac{1}{r})^t$ and $b_i = \left(\frac{p}{r}\right)^t$ for a suitable t. Similar estimates for k_i are available for other (consecutive) end points of (higher order) gaps. It thus follows that the representation (4.19) is realized at the level of the finite Hausdorff measure of the set, when the value of the constant k is real (rather than a natural number).

4.4 Locally Constant Cantor Function and Usual topology

The variability of the locally constant function $\phi : I \to I$ may, even, be captured in the usual topology as follows [25].
Indeed, we show that

$$\frac{d\phi}{dx} = 0 \qquad (4.25)$$

for finite values of $x \in I$ is transformed into

$$\frac{d\phi}{dv(\tilde{x})} = -O(1)\phi \qquad (4.26)$$

for an infinitesimal $\tilde{x}$ satisfying $\frac{x}{\epsilon} = \lambda\ \frac{\epsilon}{\tilde{x}} = \epsilon^{-v(\tilde{x})}$, $0 < \tilde{x} < \epsilon \leq x$, $x \to 0^+$, $x \in I$, $\lambda > 0$, when one interprets 0 in relation to the scale ϵ as $O(\ \delta = \frac{\epsilon^2}{x} \log \epsilon^{-1}\ )$. However, this follows once one notes that equation (4.25) means, in the ordinary sense, $d\phi = 0 = O(\delta)$, $dx \neq 0$, for a finite $x \in I$. But, as $x \to \epsilon$, that is, as $dx \to 0 = O(\delta)$, the ordinary variable x is replaced by the ultrametric extension $x = \epsilon.\epsilon^{-v(\tilde{x})}$ so that $d\log x = dv(\tilde{x}) \log \epsilon^{-1} = O(\delta)$. On the other hand, the constant function ϕ (equation (4.25)), now, in the presence of smaller scale infinitesimals, has the form $\phi = \phi_0 \epsilon^{k_0 v(\tilde{x})}$ for a real constant k_0. Equation (4.26) thus follows. The variability of $\phi(x)$ in the usual topology is thus explained as an effect of the relative infinitesimals which are *insignificant* relative to the finite scale of $x \in C$, but attain a dominant status in the appropriate logarithmic variable $v(\tilde{x}) = \log_{\epsilon^{-1}} \frac{\epsilon}{\tilde{x}}$. It is also of interest to compare the present case with computation. In the ordinary framework, the scale ϵ stands for the level of accuracy in a computational problem. The infinitesimals in $(0, \epsilon)$ are "valueless" in the sense that these have no effect on the actual computation. The open interval $(0, \epsilon)$ is thus effectively identified with $\{0\}$. In the present framework, the zero element 0 is, however, identified with a smaller interval of the form $(0, \delta)$ where $\delta = \eta\epsilon \log \epsilon^{-1}$ and $0 < \eta \lesssim 1$. The valued infinitesimals in the interval (δ, ϵ) are already shown to have significant influence on the structure of the Cantor set. The variability of $\phi(x)$ as given by equation (4.26) is revealed, on the other hand, in relation to an infinitesimal variable lying in $(0, \delta)$.