INTRODUCTION

The theory of thermo-elasticity is concerned with the influence of the thermal state upon the distribution of stress and strain and with the inverse effect ,that of deformation upon the thermal state of an elastic medium. Duhamel [46] in 1838, initiated the subject deriving equation for the distribution of strain in an elastic medium containing temperature gradients. Subsequently, these results were rediscovered by several authors but Neumann gave the present form known as Duhamel-Neumann relations. The basic theory was applied by Duhamel to a number of problems and later used by Neumann and other authors as the basis of the detailed study. These investigations were instrumental in developing techniques for the solution of thermo-elastic problems but not until the present century did the subject received the practical stimulus. There have been a rapid development of thermo-elasticity stimulated by various engineering sciences in the post war years. A considerable progress in the field of air-craft and machine structures,mainly with gas and steam turbines,highway engineering especially in the preparation of air base,and the emergence of new topics in chemical and nuclear engineering have given rise to numerous problems in which thermal stress play an important role and frequently even a primary role.

For most practical problems,the effect of the stresses and deformations upon the temperature distribution is quite small and can be neglected. The procedure allows the determination of the temperature distribution in the solid resulting from prescribed thermal condition to become first,an independent step of a thermal stress analysis;the second step is then the determination of the stresses and deformation of the body due to this temperature distribution.Before proceeding further,it will be worthwhile mentioning briefly equation of heat conduction and steady state, dynamic state of thermo-elasticity.

EQUATION OF HEAT CONDUCTION

Let in the space (X_r) a solid body B be bounded by the surface S and $T(X_r,t)$ denote the temperature at the point (X_r) and at the time t. Then temperature differences between the points of the region B results in a flow of heat. Across a surface element $d\sigma$ at the point(X_r) the quantity of heat flowing in the time interval Δt is

$$\nabla Q = -\lambda T,_{n} d\sigma \ \Delta t$$

where λ is the coefficient of internal heat conduction, $T,_{n} = \frac{\partial T}{\partial n}$ is the normal derivative of the temperature at the point $(X_r)$of the surface element , in the direction of heat flow.

Now we investigate the equilibrium due to heat in a region B_1

bounded by S_1, constituting a part of B. The quantity of heat flowing into the region B_1 across the boundary S_1 in the time Δt is given by

$$\Delta Q' = \lambda \int_{S_1} T,_n d\sigma \, \Delta t$$

If W denotes the quantity of heat generated in unit volume in unit time, then the quantity of heat generated inside the region under consideration is

$$\Delta Q'' = \int_{B_1} W \, dv \, \Delta t.$$

On the other hand , $\Delta Q = \Delta Q' + \Delta Q''$ can be determined from

$$\Delta Q = \int_{B_1} c\rho\dot{T} \, dv \, \Delta t$$

where ρ is the density and c is the specific heat of the body. The condition $\Delta Q = \Delta Q' + \Delta Q''$ implies the equation

$$\int_{B_1} (c\rho\dot{T} - W) dv - \lambda \int_{S_1} T,_n \, d\sigma = 0$$

which by divergence theorem becomes

$$\int_{B_1} (c\rho\dot{T} - W - \lambda T,_{kk})\, dv = 0$$

Since this is true for all arbitrary region B_1, hence

$$T,_{kk} - \dot{T}/s = -Q/s \qquad (1)$$

where $s = \lambda/\rho c$, $W = Qc\rho$.

We have used tensor notation ,i.e

$$T,_i = \frac{\partial T}{\partial X_i}, T,_{kk} = \nabla^2 T$$

in a cartesian coordinate system. Dots represent derivatives with respect to time.

Solution of equations (1) determine temperature as a function of position and time. If the temperature is independent of time and if there are no heat sources inside the region B, then (1) can be by Laplace equation

$$T,_{kk} = 0 \qquad (2)$$

and hence in this case, temperature function is a potential function.

EQUATIONS OF THERMO-ELASTICITY.

Generation of stress and strain in a body takes place due to non-uniform distribution of temperature. The temperature T represents the increment of the temperature from the initial stress less state. We assume that the change in temperature is small and therefore it has no influence on the mechanical and thermal properties of the body.

We shall confine ourselves to an isotropic homogeneous body with respect to both its mechanical and thermal properties. Let u_i $(i=1,2,3)$ be the components of displacement vector $\vec{u}$, e_{ij} $(i,j=1,2,3)$ be the components of displacement of strain tensor and σ_{ij} $(i,j=1,2,3)$, the components of stress tensor.

In the linear theory of elasticity, the strain tensor e_{ij} is considered with the displacement vector by the relation

$$e_{ij} = (u_{i,j} + u_{j,i})/2, \qquad i,j = 1,2,3 \tag{3}$$

The strain tensor is symmetric, i.e, $e_{ij} = e_{ji}$, The components of strain tensor can not be arbitrary, since they should have the following six relations— the so called comparability conditions:

$$e_{ij,kl} + e_{kl,ij} - e_{jl,ik} - e_{ik,jl} = 0, \qquad i,j,k,l = 1,2,3 \tag{4}$$

which are satisfied identically if e_{ij} is expressed by u_i in accordance with (3) when the displacement field is continuous.

In thermo-elasticity strain tensors are made up of two parts. The first part e^o_{ij} is a uniform expansion proportional to the temperature rise T. Since this expansion is the same in all directions for an isotropic body, only normal strains and no shearing strains arise in this manner. If α_t is the coefficient of linear expansion and δ_{ij} is the Kronecker's symbol, then

$$e^o_{ij} = \alpha_t T \ \delta_{ij} \ , \quad i,j=1,2,3 \tag{5}$$

The second part e'_{ij} comprises the strains required to maintain the continuity of the body as well as those arising because of external loads. These strains are related to the stresses by means of the Hooke's law of linear isothermal elasticity. Hence

$$e'_{ij} = \left[\sigma_{ij} - \frac{\nu}{1+\nu}\,\theta\delta_{ij}\right]/2\mu_1, \quad i,j=1,2,3 \tag{6}$$

where μ_1 is the shear modulus, ν is the Poission's ratio and $\theta = \sigma_{kk}$ is the sum of the normal stresses. Hence finally we have

$$e_{ij} = e^o_{ij} + e'_{ij} = \alpha_t T \ \delta_{ij} + \left[\ \sigma_{ij} - \frac{\nu}{1+\nu}\,\theta\delta_{ij}\ \right]/2\mu_1 \tag{7}$$

the so called Duhamel-Neumann relation.

Denoting $\Theta = e_{kk}$, we have from (7)

$$\Theta - 3\alpha_t T = \frac{1-2\nu}{E}\,\theta, \; E = 2\mu_1(1+\nu) \tag{8}$$

where E is the Young's modulus

Solving (7) for stresses, we have

$$\sigma_{ij} = 2\mu_1 e_{ij} + (\lambda\Theta - \gamma T)\delta_{ij}, \qquad i,j=1,2,3 \tag{9}$$

where λ, γ are Lame's elastic constants given by the relations

$$\nu = \frac{\lambda}{2(\lambda+\mu_1)}, \; \gamma = (3\lambda+2\mu_1)\alpha_t$$

Now, in order to find the equations of elastic equilibrium, let us consider a body B with boundary S loaded in an arbitrary way and placed in a stationary temperature field. Let us consider the equilibrium of a sub-domain B_1 with boundary S_1. If F_i denotes the components of the body force per unit volume and p_i the components of surface tractions acting on the surface S_1, then from the condition of equilibrium we obtain the following three equations for the region B_1:

$$\int_{B_1} F_i \, dv + \int_{S_1} p_i \, d\sigma = 0, \quad i = 1,2,3.$$

Taking into account that $p_i = \sigma_{ij} n_j$, where n_j denotes the components of unit normal vector of surface S_1, we get, on making use of divergence theorem

$$\int_{B_1} (F_i + \sigma_{ij,j}) \, dv = 0$$

Since this is true for an arbitrary region B_1 , the equilibrium equations take the form

$$\sigma_{ij,j} + F_i = 0 \quad , \qquad i = 1,2,3 \tag{10}$$

If in these equilibrium equations, we express stresses by strains and then by displacements, we obtain a system of three equations in which the unknown functions are the components of displacement vector:

$$\mu_1 u_{i,kk} + (\lambda + \mu) u_{k,ki} + F_i - \gamma T_{,i} = 0 \tag{11}$$

$$i,k = 1,2,3.$$

In cylindrical coordinates (r,θ,z), let u,v,w represent the components of displacement vector $\vec{u}$, σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} represent normal stresses and τ_{rz}, $\tau_{\theta z}$, $\tau_{r\theta}$ represent shear stresses. In the case of axial symmetry about the z-axis, equations (11) reduce to two equations.

$$\nabla^2 u - r^{-2} u + \frac{1}{(1-2\nu)} \Theta_{,r} - \frac{2(\nu+1)}{(1-2\nu)} \alpha_t T_{,r} = 0$$

$$\nabla^2 w + \frac{1}{1-2\nu} \Theta_{,z} - \frac{2(\nu+1)}{(1-2\nu)} \alpha_t T_{,z} = 0 \tag{12a,b}$$

where

$$\theta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}, \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

To solve the equations (11) in the absence of body forces i.e. $F_i=0$, Goodier [35] introduced a thermoelastic potential ϕ in terms of which the displacement vector is defined by the relation

$$u_i = \partial\phi/\partial x_i \tag{13}$$

and ϕ is a particular solution of the Poission's equation

$$\nabla^2\phi = T(x_r) \tag{14}$$

A well known particular integral of (14) is

$$\phi(x_r) = -\frac{1}{4\pi}\int_V \frac{T(\xi_r)\, dV(\xi_r)}{R(x_r,\xi_r)} \tag{15}$$

where $R(x_r,\xi_r)$ is the distance between the points (x_r) and (ξ_r).

Integrals of the type (15) were employed by Borchardt [78] in a general discussion of the theory of thermo elasticity and also to solve certain special problems involving asymmetric distribution of temperature in solids with spherical on circular boundaries. Problems concerning spheres and cylinders are dealt in [15, pp.

362-67]. the problems of thin elastic plates, under fairly general distributions of temperatures have been considered by Galarkin, Nadai, Marguerre [55], Sokolnikoff [73] and Pell [74]. Several approximate solutions of the engineering problems concerned with thermal stresses in plates and rods are discussed in chapter 14 of Timoshenko and Goodier's " Theory of Elasticity" [17].

The calculation of the steady-state thermal stresses in an isotropic elastic half-space or slab with traction free faces has been the subject of several investigations. The distribution of thermal stress due to special temperature distribution in infinite and semi-infinite solids have been discussed by a variety of authors, i.e. Mindlin and Cheng [48], Myklestad [52] , Sternberg and Mc'Dowell [70], using an extension of Boussinesq-Papkovich method of isothermal elasticity solved the problem of half-space, The basis of the method is that the solution of the equation of equilibrium (11) may be expressed in terms of the four Boussinesq-Papkovich functions, one of which is the solution of Poisson's equation and remaining three are of Laplace equation. These equations have been studied extensively, particularly in potential theory, and general procedures of their solutions are known. Sneddon and Locket [72] approached this class of problems by direct solution of the equations of thermo-elasticity using a double Fourier integral transform method, the results being transformed to Hankel type integral in the case of axial symmetry. A further approach due to

Nowinski [75] exploits the fact that in steady-state thermo-elasticity each component of the displacement vector is a bi-harmonic function which can be expressed as a combination of harmonics. Possibly the most economical method of solutions of the type of problems is that of Williams [77] who expressed the displacement vector in terms of two scalar potential functions, one of which is directly related to the temperature field. Further, Muki [20] has introduced the displacement and stress components in the form of Hankel transform for the particular solution of the thermo -elastic equations.

It is to note that Nowacki [13] has made thorough survey of the problems of both elasto-static and elasto-dynamic in presence of the temperature excellently.

ABOUT THE THESIS

In the present age of science and technology it is inevitable to have a study on the problems of thermo-elasticity because of the increasing rarge of applications of the theory and analysis of the thermal stresses in industry, and especially in advanced technologies such as Aerospace Engineering, Laser Engineering, Design of Turbines, Micro electronics Industry. The subject has tremendous importance in compliance with its application in

Geophysical and Seismological problems. The interest in this field of science has been increasing among mechanical engineers, Semi-conductor engineers and chemical engineers.

The work of this thesis is occupied with some important and interesting problems of thermo-elasticity. In this elastic problems displacements and stresses have been derived in the presence of temperature with endeavor to obtain results which shall be practically important in applications to applied mathematics, engineering and technology in which the material of construction is solid. Here, both statical and dynamical types of problems of thermo-elasticity have been dealt with. The complete work is divided into four chapters and the problems in each chapter are relevant to each other.

In the process chapter I contains two highly interesting inclusion problems of thermo elasticity each of which is treated satisfying composite boundary conditions. The first one is a plane problem and the second one is considered in the case of solid body.

A general series form of stress function in bi-polar coordinate system was given by G.B.Geffrey [38]. It has been applied to the problems of a semi-infinite plate with a concentrated force at any point [51], a semi-infinite region with a circular hole under tension parallel to the straight edge or plane boundary [38] and under its own weight, and to the infinite plate with two holes [60], or a hole formed by two intersecting circles [45]. Solutions have been given for the circular disk subject to concentrated forces at any point to its own weight when suspended at a point [49], or in rotation about an eccentric axis [50], with and without [62] the use of bipolar coordinates, and for the effect of a circular hole in a semi-infinite plate with a concentrated force on the straight edge [79]. The equilibrium problem of thermo-elasticity for region bounded by two non-concentric circular inserts does not appear to have received any previous attention. In this particular instance, the region is infinite and contains two circular inserts of the same radius. The nucleus of heat is placed in the middle of the line joining two centers of the inserts, moreover both are assumed to be symmetric with respect to the common axis of symmetry. Bipolar coordinates [14] are used to obtain thermal stresses in the form of series following method of G.B.Geffrey [38] and then the approach of Das [30]. The series solution is based upon the boundary stress function approach. Numerical evaluations of the distribution of the plane thermal stresses have been performed by the method due to L.N.G.Fillon [84]. To begin with the second problem it is found that in a paper, E.Sternber and M.A.Sadowsky [67] determine a solution in series for the stress distribution in an infinite elastic medium which possess two spherical cavities of the same size and both are assumed to be symmetric with respect to the common axis of symmetry of the cavities and with respect to the plane of geometric symmetry perpendicular to this axis, the solution is based upon the boundary stress function approach and apparently constitutes the first application of spherical dipolar coordinates in the theory of elasticity. The thermal stress problem in an infinite elastic solid at zero temperature except for a heated region has been solved by Goodier [35]. In a paper, B.D.Sharma [69] has derived stresses and displacements on the surface of a spherical cavity when the heated element is at some finite distance from it and a solid sphere at zero temperature having a heated nucleus inside it in an infinite solid. Chatterjee and Dutta [37] have determined stresses due to a nucleus in the form of a center of dilatation in an infinite elastic solid with rigid infinite inclusion. An axially-symmetric thermo-elastic problem of the infinite cylinder has been solved due to nuclei of thermo elastic strain of unit intensity. Regarding the above solutions as the Green function, a number of particular cases of discontinuous temperature fields were investigated in detail in the paper of M. Sokolowski [63]. E.Sternberg, R.A.Eubanks, M.A.Sadowsky [64] have considered a problem where they have used spherical harmonics corresponding to either to the exterior or to the interior problem for the sphere. W.Piechocki, J.Ignaczak [66] have derived thermal stresses due to a thermal inclusion in a circular ring and a spherical shell using Heviside function for the temperature distribution. The problem of action of a nucleus of thermoelastic strain in a solid circular disk was solved in another way by B.Sen [80]. J.Ignaczak [41] has solved a problem with a hemispherical pit at the free surface in an elastic half-space in the form of series of spherical functions and analogous subsequences of solution for the problem of a hemispherical pit was given by R.A.Eubanks [42]. In a paper, stress have been determined due to a nucleus of thermoelastic strain in an infinite elastic solid with a spherical elastic inclusion of a different material by S.C.Bose [25] using spherical harmonics. The present problem is not only interesting but also important from physical point of view. Here an infinite elastic solid is considered to have two spherical inclusions of different materials while the nucleus of heat is placed in the axis of symmetry. This rotationally symmetric torsion free and mixed boundary value problem is solved by the application of spherical dipolar harmonics [64] employing Boussinesq [76] approach. Numerical calculations are made to show the distribution of displacements and stresses of this problem.

Chapter II contains two very useful problems of thermoelasticity (i) a double layered problem (ii) a three layered problem.

The design of highways and airport runways as well as the foundation problems in soil mechanics, especially when the earth mass supporting a heavy structure has different soil strata over it, it is highly needful to look to the endurance of the solid or land on which there is generated a thermal stress either due to impulse shock or owing to some local heating nucleus. Investigation of the stress distribution in a layered system was made by Burnister [27] in a series of papers. In a later paper, Acum and Fox [22] attacked a problem in a three layered system only by the method of Burnister. R.D.Mindlin and D.H.Cheng [48] who employed Galerkin vector for the center of dilatation to obtain stresses for semi-space. Paria [59] in his paper, determined elastic stress distribution in a three layered system due to a concentrated force. But when a plane bombs from above on the surface of the two layered system or a three layered system such as highway or airport land, an emmense heat is generated on the surface and as the heat is assumed to be distributed through layers, therefore the stresses are generated and in each layer and the underlying mass also. The upper layer of the double layered system, as in high ways is considered to be concrete pavement and the underlying mass is natural soil and in case of three layered system, the upper layer is of concrete pavement, the middle layer is of gravel base course and the lower mass is the natural soil as in the case of airport runways. The method of solution consists in taking the Hankel transform [18] of the stress function instead of stress function itself. Stresses in each layer due to the distribution of temperature and ultimately total stress in the underlying mass have been determined for both the cases. The type of heat flux function, if possible from physical point of view, is taken to be linear and graphical representation of the stress -distributions in the underlying mass are shown in the figures. Experimental results of the elastic constants are taken from international critical tables [8].

Chapter III is concerned with two dynamical problems of thermo-elasticity. In the first problem, components of displacement and stress are determined due to disturbance produced by a periodic heat nucleus and the second problem is on the generation of waves produced by an impulsive heat nucleus.

The problem of calculating the components of stress at a point in an elastic solid when it is deformed by the application of surface tractions which vary with time is of considerable interest in soil mechanics, in the theory of foundations and in the branch of applied mathematics. There has been extensive discussion of the corresponding statical problems. But Sneddon [71] in his palerno lecture discusses dynamical problems of this type in a systematic way. Special problems have been solved by Lambs [42] .

In a problem, Eason, Fulton and Sneddon [33] have dealt with the determination of distribution of stresses in an infinite elastic solid when the time dependent body force act upon certain region of the solid. Assuming strains to be small, the general solution of the equation of motion for any distribution of body forces is derived by the four-dimensional Fourier transforms [14] and from that general solution is derived for the isotropic solid. The solution of the equation of motion in the case in which the distribution of the body force is symmetrical about an axis is also derived. the solutions of some typical two dimensional and three dimensional problems are considered and exact analytical expressions are found for the components of displacement and stress. In the present discussion, at first detailed solution of the three dimensional thermo-elastic problem is obtained and the distribution of displacement and stress have been derived when the time dependent body force and temperature act on certain region of the solid. Then the problem consists of deducing the displacements and stresses due to the disturbance produced by the insertion of a periodic heat nucleus in the solid. Weber's Bessels function [19] of the second kind is ultimately realized for those cases. Capitatising the above procedure another interesting three dimensional problem is taken into account to determine components of displacement and stress when an impulsive heat nucleus act in the solid. Dirac Delta functions [4] are utilized in the time dependent body force and temperature acting at the origin in order to obtain the solution of the equation of motion. In each case, strains are assumed to be infinitesimal so that the equations of the classical theory of elasticity [7] are applicable. Fourier transform technique [14] is applied in both the cases separately.

In the last chapter, thermal stresses have been derived for the problems in the elastic semi-space due to heat exposure on the bounding plane of isotropic media and assume that there are no heat sources inside the semi-space. If $T(x_1, x_2, 0) = f(x_1, x_2)$ is prescribed then the determination of the state of stress due to a heating of the plane $x_3 = 0$ has been the subject of many investigations. E.Melan and H.Parkus [81] investigated the action of a concentrated heat source situated in the plane $x_3 = 0$ of a thermally insulated semi-space and proved that in this case a plane state of stress exists. The same conclusion was obtained by A.I.Luyre [82] who applied a different method, with respect to both the semi-space and a layer. E.Sternberg and E.L.MacDowell [68] in their investigation presented a solution to the problem by means of a method which was an extension of the Boussinseq-Papkovioh method [76] to thermal problems. A different procedure employing the Fourier integral transform was chosen by I.N.Sneddon and F.J.Lockett [72]. The solution can also be derived by introducing the thermoelastic displacement potential and satisfying the boundary condition by means of the Love or Galarkin function [54].

In the first problem, thermal stresses in a semi-infinite solid have been obtained for an interesting problem in which there is a constant supply of heat over an elliptic area on the bounding plane surface, the rest being kept at a constant temperature. In a paper, Nowacki [54] has solved the problem of thermal stresses in an ealstic half-space, the bounding plane surface of which is kept at a constant temperature $T = T_0$ inside a circle of radius 'a', the exterior of the circle being thermally insulated. B.R.Das [32] found thermal stresses in a semi infinite elastic solid with a constant heat flow over a circular area on the plane boundary. The analysis involves dual integral equation [3] and Bessel function [11] for its solution. Goodier [35] has given a complete solution of the thermo-elastic problem of an infinite solid at temperature zero except for a heated (or cooled) region. The semi infinite solid is considered by R.D.Mindlin and D.H.Cheng [48]. The cases of such a region in the form of an ellipsoid of revolution and semi-infinite circular cylinder, uniformly hot, have been worked out by N.O.Myklested [52]. In this problem, elliptic coordinates are used and the bounding surface of the semi-infinite elastic solid is given by $z=0$, the axis of z being drawn into the body. Temperature and the potential of the thermo elastic displacement ψ are obtained in terms of Matheieu function [10]. Love function is considered for the solution of biharmonic equation [9]. Components of displacement and stresses are obtained in curvilinear coordinates as given by C.B.Ling [44]. Numerical evaluations have been made for a suitable case collecting experimental results from Bickley and Molachlan [2].

The second problem is a thermoelastic boundary value problem of three dimensions when these thermal stresses are produced in a body by unequal distribution of temperature which may be regarded as a specified function of coordinates and time. In this paper, stresses due to periodic supply of heat produced by the blow of a jet flame on the straight edge of a semi-infinite isotropic elastic thick plate distributed over a finite portion of it, have been obtained.

The third problem deals with the determination of thermal stresses due to prescribed flux of heat on the surface of a thick plate. The thermal stress problem of a thick circular plate at zero temperature except for a heated regions on the plane faces was considered by Nowacki [53]. The solution obtained by him satisfied the boundary conditions concerned with the stress on the edge surface in an approximate manner only. The object of this paper is to find the exact solution of the thermo elastic problem of a thick plate of infinite radius of an isotropic material, with stress free edges subjected to two different temperature distributions. In the first case, we assume a constant flux of heat within a circular region of exposure, the exterior of the circular region being free of any flux of heat. Secondly we assume a paraboloidal distribution of temperature within the circular region, the exterior being insulated. Numerical calculations for the variation of (rr + $\theta\theta$) on the free surface have also been obtained in the second case.

The last problem is concerned with the distribution of stresses and displacements in a semi-infinite isotropic elastic solid when a prescribed flux of heat is applied on a circular region of the upper surface. The problem of determining the steady-state thermal stresses and displacements in a semi-infinite elastic medium was treated by Sternberg and McDowell [70] by the use of Green's functions. They proved that the stress field induced by an arbitrary distribution of surface temperature is plane and parallel to the boundary and obtained the solutions in closed forms for a circular region of exposure with uniform or hemispherical distribution of temperature. This problem was discussed by B.D.Sharma [83] by using integral transform methods. He discussed the same problem in case of isotropic material. Nowacki [54] solved the problem of thermal stresses in an elastic half-space , the bounding plane surface of which is kept at a constant temperature inside a circle of radius 'a', the exterior of the circle being thermally insulated. Sneddon and Locket [72] discussed the same problem by using double Fourier transform methods and arrived at the same result. In this problem two types of flux function, one being constant and other parabolic have been prescribed in a circular region of the bounding plane, the rest of the surface being kept free of flux of heat to determine thermal stresses and numerical results have been obtained.