Chapter 7

GROWTH OF MEASURE: APPLICATIONS

7.1 Introduction

A point of the original Cantor set C is identified with the closure of the set of gaps of $\tilde{C}$. The increments on such an ultrametric space is accomplished by inversion rule. An interesting phenomenon, called *growth of measure*, is studied on such an ultrametric space [26]. Using the reparametrisation invariance of the valuation it is shown how the scale factors of a Lebesgue measure zero Cantor set might get *deformed* leading to a *deformed* Cantor set with a positive measure. The definition of a new *valuated exponent* is introduced which is shown to yield the fatness exponent in the case of a positive measure (fat) Cantor set.

7.2 Reparametrisation Invariance and Measure

We studied the valued ultrametric structure of a measure zero Cantor set. Here we study a few more general properties of the valued ultrametricity. We note, at first, that the valuation $v(\tilde{x})$ is LC Cantor function corresponding to a homogeneous Cantor set C. As a consequence, v satisfies the equation

$$\frac{d}{dx}v(\tilde{x}(x)) = 0. \tag{7.1}$$

and hence, v is, not only a LCF, but more importantly is a *reparametrisation* invariant object (c.f., page 33) As a result, v does not require to be an explicit function of the original variable x but may be a function instead of *any* monotonic, continuously first differentiable function of x. By the same token, v does not depend explicitly on the scale ϵ inherited from the original (mother) Cantor set, as we did in the examples of chapter 4. In the following example, we show that relative infinitesimals may instead live in a positive measure Cantor set. Notice that in the general representation of the valued ultrametric in equation (3.2), the parameter may be a constant independent of an explicit ϵ.

Example 5. *Suppose that equations (4.5) and (4.6) are replaced by*

$$\tilde{x} = \beta^n \left(\beta^{n^\delta}\right)^{n\beta_n(1+\gamma_m)} \times a \tag{7.2}$$

where $a = (1-\beta)\left(1 + \overset{\infty}{\underset{1}{\Sigma}} a_i \beta^i\right)$, $a_i \in \{0,1\}$, $\beta = \frac{1}{2}(1-\alpha)$, *and* β_n *and* γ_n *are two non-increasing sequence of positive numbers such that* $\beta_n \to 0$ *as* $n \to \infty$ *and* m *may be independent of* n *or may vary with* n *more slowly, and* $\delta > 1$ *is a constant.*

Although $\beta_n \to 0$ *as* $n \to \infty$, *the valuation* $v(\tilde{x})$ *could be non-trivial, since*

$$\begin{aligned} v(\tilde{x}) &= \lim_{n\to\infty} \log_{\beta^{-n}} \frac{\beta^n}{\tilde{x}} \\ &= \lim_{n\to\infty} \left[n^\delta \beta_n (1+\gamma_m) + \log_{\beta^{-n}} a \right] \\ &= l + \tilde{\gamma}_{m_n}(\delta) \end{aligned} \tag{7.3}$$

when we assume $n^\delta \beta_n \to l$ *as* $n \to \infty$ *and* $n^\delta \beta_n \gamma_m \to \tilde{\gamma}_{m_n}(\delta)$ *is a subdominant slowly varying non-increasing sequence, for a real* $m_n > 0$. *The representation (7.2) tells that a scale free infinitesimal* $\frac{\tilde{x}}{\beta^n}$ *may live in a Cantor set* $\tilde{C}_p$, *so that* $m(\tilde{C}_p) = l$. *Let the original Cantor set be a middle* α *set* C_α *with the uniform scale factor* $\beta = \frac{1}{2}(1-\alpha)$. *For the positive measure set* $\tilde{C}_p$ *the scale factor at the nth iteration is* $\tilde{\beta}_n = 2^{-n} \sum_{i=1}^{n} (1-\alpha_i)$ *and* $l = m\left(\tilde{C}_p\right) = \prod_{i=1}^{\infty}(1-\alpha_i) = \lim_{n\to\infty} 2^n \tilde{\beta}_n$. *Let us choose* $\delta > 1$ *such that* $\tilde{\beta}_n = \beta^{n^\delta}$. *Then* $n^\delta \beta_n \to l$ *tells that* $\beta_n \approx \frac{l \log \beta}{\log \tilde{\beta}_n}$ *as* $n \to \infty$. *Thus the dominant term* l *of the valuation* $v(\tilde{x})$ *is a constant while the subdominant asymptotic* $\tilde{\gamma}_m(\delta)$ *could be a genuine LCF (i.e. a Cantor function for a sub dominant Cantor like set* C_s *(say)), precise determination of which depends on the explicit model of the Cantor set* $\tilde{C}_p$. *It follows, therefore, from equations (4.3) and (4.4) the ultrametric valuation of* $x \in C_\alpha$ *now has the form*

$$\| x \| = l + \tilde{\gamma}_{m_n}(\delta). \tag{7.4}$$

For larger and larger values of n $(\to \infty)$, *we can disregard the subdominant term (since* $\tilde{\gamma}_{m_n} \to 0$ *as* $m_n \to \infty$*) so that*

$$\| x \| = l \quad \forall\, x \in C_\alpha,\ x \neq 0. \tag{7.5}$$

Clearly the trivial ultrametric (7.5) reveals that the mother set C_α *must get deformed to a positive measure set* C_p *so that* $\mu_v(C_p) = m(C_p) = l$, *when the reparametrisation invariance of LC correction factors is invoked. Indeed, we have* $\| x - y \| = l$ *for any two* $x, y \in C_p$. *Thus, any single clopen* *ball* $B(x_0)$, $x_0 \in C_p$ *(say) covers the compact* C_p *and hence* $\mu_v(C_p) = d_u(B(x_0)) = l$.

To summarize, we have shown that any element $x \in C_\alpha$ when deformed by the non-trivial, reparametrisation invariant valuation of relative infinitesimals, is identified with an element of a 1-set C_p. Because of this invariance, the relative infinitesimals may be assumed to live in a positive measure set $\tilde{C}_p$, which, in turn, determines the measure (size) of the deformed set C_p. Since each element $x \in C \subset [0,1]$ is written as the arithmetic sum of two elements $x_0 \in C_\alpha$ and $x_1 \in C_{\alpha'}$ ($C_{\alpha'}$ being the Cantor set of infinitesimal neighbours of x_0), it follows from a theorem of Solomyak [46] that for $\beta = \frac{1}{2}(1-\alpha) \in \left(0, \frac{1}{2}\right)$, there exists $C_{\alpha'}$ for a.e. $\beta' = \frac{1}{2}(1-\alpha') \in \left(0, \frac{1}{2}\right)$ so that $C_\alpha + C_{\alpha'}$, has positive measure and $\frac{1}{\log\frac{1}{\beta}} + \frac{1}{\log\frac{1}{\beta'}} > \frac{1}{\log 2}$. This, therefore, constitutes an alternative proof for the said assertion. Indeed, in the above construction, the set of infinitesimals $C_{\alpha'}$ itself is a 1-set $\tilde{C}_p$.

It follows, accordingly, that a slower rate of removal of middle open sets compared to a measure zero Cantor set hides a positive measure in an infinitesimal scaling factor which is exposed under the present scale invariant valuation. The uniform rate of deletion in the case of a measure zero set is violated because of the underlying reparametrisation invariance. Further, in a dynamical process leading to a Cantor set, a positive measure Cantor set C_p is favoured a.s (almost surely) compared to a measure zero set C_α since relative infinitesimal neighbours a.s. lie in a Cantor set $C_{\alpha'}$ satisfying the above constraints.

The generic result that follows from this example is stated thus

Theorem 2. *Because of the reparametrisation invariance of the infinitesimal valuation, a measure zero Cantor set C_α is a.s. deformed to a positive measure Cantor set C_p, the measure of which is determined by the Cantor set $\tilde{C}_p$ in which the relative infinitesimals are supposed to live.*

Next to expose the significance of the sub-dominant term, let us first define a renormalized valuation $v_R(\tilde{x})$:

$$v_R\left(\tilde{x}\right) = \log_{\beta^n} \log_{\beta^n} \left[\frac{\tilde{x}}{\left(\beta^n\right)^{1+v_0(\tilde{x})}}\right], \quad n \to \infty \tag{7.6}$$

where $v_0\left(\tilde{x}\right) = l < 1$ is the dominant valuation of the infinitesimal $\tilde{x}$. The LCF $\tilde{\gamma}_{m_n}(\delta)$ is now given by (c.f.,(chapter 6))

$$\tilde{\gamma}_{m_n}(\delta) \ = \alpha_i \beta^{m_n \rho(\delta)} \tag{7.7}$$

where the $\delta-$dependent constant ρ is called a *renormalised valuated exponent* and the non-zero constant α_i assumes values from a finite set for a secondary scale β^{m_n}. As will become clear the valuated exponent ρ is useful to distinguish two sets with identical Hausdorff dimensions.

7.2.1 Applications

1. Middle third Cantor sets:

As an application of the renormalised valuated exponent, let us first consider a class of s- sets where $s = \log_3 2$, constructed as a slight variation of the process of Example 2, Sec.2.4. Let $I = [0, 1]$. Also let

$0 << \delta_n = 3^{-(n+1)\alpha_n} \lessapprox 1$, $n, = 1, 2, \cdots$ (so that $\delta_n^{-1} \gtrapprox 1$), be a non-increasing sequence. For definiteness, one may choose $\alpha_n = q^{-n}$, for a sufficiently large positive integer n and $q > 1$. In that case α_n may be considered to belong to the range set of an appropriate Cantor function. Delete the middle open interval of length 1/3. Next, delete a length $3^{(-2(1+\alpha_1))}$ from each of the two closed subintervals. Then, delete the length $3^{(-3(1-\alpha_2))}$ from each of 2^2 closed subintervals. Call these two operations together O_1. O_n consists of two steps: deletion of 2^{n+1} open intervals of length $3^{-(n+1)(1+\alpha_n)}$, which is succeeded by the next deletion of lengths $3^{-(n+2)(1-\alpha_{n+1})}$ from 2^{n+2} remaining closed subintervals. Notice that we are considering a set of fluctuating scale factors, i.e., in the $(n+1)$th step open intervals of slightly smaller sizes compared to the middle third set are removed. In the next step, however, open intervals of slightly bigger sizes are removed. As a consequence, we get a family of limit sets which are indistinguishable and equivalent to the middle third Cantor set at the level of the Hausdorff dimension, but nevertheless, distinguishable at the level of renormalised valuated exponents. Indeed, the total length of deleted open intervals viz., $\frac{1}{3}+\frac{2}{3^2}\delta_1+\frac{2^2}{3^3}\delta_2^{-1}+\cdots = 1+\sum u_n$ equals 1, when the series of real numbers $\sum u_n$ vanishes. The sequence u_n is determined by the sequence α_n, i.e., $\alpha_n = \log_{3^{(n+1)}}(1 - \frac{3^{(n+1)}}{2^n}\tilde{u}_n)^{-1}$ and $\alpha_{n+1} = \log_{3^{(n+2)}}(1 + \frac{3^{(n+2)}}{2^{n+1}}\tilde{u}_{n+1})$ so that $u_n = -\tilde{u}_n$, $u_{n+1} = \tilde{u}_{n+1}$. Clearly, such a series exists. Hence, all such sets are of measure zero.

Now, to determine the Hausdorff dimension, we first note that the scaling of closed intervals (bridges) follows the recurrence $2l_n = l_{n-1} - \delta_n^{\pm 1}3^{-(n+1)}$, where + sign goes with an odd n and the - sign with n even and l_n denotes the length of each closed interval at level n. Accordingly, $l_n = \frac{1}{3.2^n}[1 - \frac{\delta_1}{3} - \frac{2\delta^{-1}}{3^2} - \cdots \frac{2^{n-1}\delta_n^{\pm 1}}{3^n}] \approx \frac{\delta_{n+1}^{\pm 1}}{3^{n+2}}$, for a sufficiently large n. As a consequence, the scale factors behave as either $\beta_{n+1} = 3^{-(n+1)(1+\alpha_n)}$ or $\beta_{n+2} = 3^{-(n+2)(1-\alpha_{n+1})}$ respectively, and hence, the lower and upper box dimensions and the Hausdorff dimension are all equal and equal to $\lim_{n\to\infty} \frac{\log 2}{\log 3^{(1\pm\alpha_n)}} = \log_3 2$.

One may also estimate the thickness of these sets easily. Because of the above scaling, the limiting length of the closed intervals (bridges) coincides with that of the corresponding gap (viz., $\delta_{n+1}^{\pm 1} 3^{-(n+2)}$) at the nth level. It follows therefore that the ratio of sizes of bridges and gaps (c.f., Sec.2.1) has the limiting value 1. Hence, thickness of all these sets coincides with that of the classical middle third Cantor set as well.

However, a higher order (renormalised) valuated exponent can indeed reveal the local dissimilarities of such an $s-$set. Extending the representations (4.6) and (7.2) a little further to suit the present problem, we would now have for an element x of the $s-$set,

$$x_{i\pm} = 3^{-n} \cdot 3^{-n(-i2^{-m_n(1\pm\alpha_{m_n})})} \times b, \; ||b|| = 1 \tag{7.8}$$

where i assumes values from a finite set and $m_n \to \infty$ at a slower rate as $n \to \infty$, so that a renormalised valuation is defined as

$$v_R(x) = \inf_i \log_{2^{-m_n}} \log_{3^{-n}}(x_{i+}/x_0) = \alpha_{m_n}, \; x_0 = 3^{-n} \cdot 3^{-n(-i2^{-m_n})}. \tag{7.9}$$

It now follows from the definition of α_{m_n}, that one can find a sufficiently large natural number $q >> 1$ such that $\alpha_{m_n} = q^{-m_n}$. Consequently, we obtain $v_R(x) = \alpha_{m_n} = 3^{-\rho\tilde{m}_n}$, where $\rho = \log_{3^r} q > 0$ is the *valuated exponent*, for suitable positive integers r and $\tilde{m}_n$.

Now, to justify the existence of such a q, let us first assume $\tilde{u}_{2m} = u^1_m$ and $\tilde{u}_{2m+1} = u^2_m$ such that $\sum u^i_m = l$. Then $\sum_2^\infty u_n = \sum \tilde{u}_{2m+1} - \sum \tilde{u}_{2m} = l - l = 0$. Consequently, $\alpha_{2m} = \log_{3^{(2m+1)}}(1 - \frac{3^{(2m+1)}}{2^{2m}} u^1_m)^{-1}$ and $\alpha_{2m+1} = \log_{3^{(2m+2)}}(1 + \frac{3^{(2m+2)}}{2^{2m+1}} u^2_m)$.

Let $\eta_{2m} = \frac{3^{(2m+1)}}{2^{2m}} u^1_m$ and $\tilde{\eta}_{2m+1} = \frac{3^{(2m+2)}}{2^{2m+1}} u^2_m$. Then the functions $(1 - \eta_{2m})^{-1}$ and $1 + \tilde{\eta}_{2m+1}$ are identified as LCF of the form (7.1), in the neighbourhood of 1. Using scale invariance, we can then choose for x in equation (23) as $x = 3^{-n}(1 - \eta_n)$ (or $x = 3^{-n}(1 + \tilde{\eta}_n)$), and the scale factor $\epsilon = 3^{-n}$. Thus, there exists a Cantor function $\tau(\tilde{x})$, $\tilde{x} = x/\epsilon$ such that $\log_{\epsilon^{-1}} \tilde{x}^{-1} = \tau(\tilde{x})$ (or $\log_{\epsilon^{-1}} \tilde{x} = \tau(\tilde{x})$). As a result, there exists positive integers q and m_n so that the sequence $\{q^{-m_n}\} \subset \text{Range}(\tau(\tilde{x}))$. More generally, because of the local constancy, the limiting form α_n could be $\alpha_n = \tilde{l} + q^{-m_n}$, where $\tilde{l}$ is a non-negative constant, $0 \leq \tilde{l} < 1$.

We remark that the exponent ρ may be considered to be the inverse of the Hausdorff dimension of a *residual* Cantor set that would remain attached with infinitesimal scales in a neighbourhood of a point (of the original Cantor set). For the classical middle third Cantor set $\alpha_n = 0\ \forall n$ and so $\rho = \infty$, which is consistent with the fact that the residual set is null. Since, sets with infinite Hausdorff dimension $s = \infty$ are excluded, by definition, ρ indeed is positive $\rho > 0$.

2. 1-sets: Irregular 1-sets [3] are positive measure Cantor sets and are generally classified on the basis of fatness and/or uncertainty exponents.

The LC renormalised valuation (7.6) and (7.7) now tells that $v_R(\tilde{x})$ is a Cantor function corresponding to a subdominant residual Cantor set C_s, and so has the form $v_R(\tilde{x}) = \alpha_i \beta^{m_n \rho}$. As for the s-sets, the valuated exponent $\rho > 0$ equals the inverse of the Hausdorff dimension of the residual set C_s. For $\rho = \infty$, the double exponential factor in (7.2) drops out (i.e., reduces to the trivial factor β^n), and hence the 1-set is a regular set [3] having connected components (actually corresponds to a nonfractal set). Consequently, $0 < \rho \leq \infty$.

Now, to compare with the fatness exponent [37, 43], we first recall the relationship between the uncertainty exponent α, $0 < \alpha \leq 1$ [44] and the fatness exponent $\tilde{\beta}$, $0 < \tilde{\beta} \leq \infty$. It is shown [37] that $\tilde{\beta} = \alpha$ in [0,1], so that there is essentially the fatness exponent that has to be considered. We claim that $\rho = \tilde{\beta}$. The parameter $\tilde{\beta}$ is defined as

$$\tilde{\beta} = \lim_{\epsilon \to 0} \frac{\log[\mu(\epsilon) - \mu(0)]}{\log \epsilon} \tag{7.10}$$

where $\mu(\epsilon)$ is a LC measure which tells the scaling of smaller gap sizes when the smaller gaps are coarse grained by fattening by the amount ϵ and $\mu(0)$ equals the positive (Lebesgue) measure of the set. In our multiplicative representation (c.f.,(7.2) and (7.7)), the fattening size is $\epsilon = \beta^n$ and

$$x = \beta^n \left(\beta^n\right)^{-(l + k\beta^{n\rho})} \times b \tag{7.11}$$

where k is a constant independent of β, so that the exponent ρ is defined by (7.10) when we identify $\mu(\beta^n) = \log_{\beta^n}(x/\beta^n)$. Notice that $\mu(0) =$

$\lim_{n\to\infty} \log_{\beta^n}(x/\beta^n) = l$. Notice also that the measure μ here is nothing but the valuation of relative infinitesimals at the fattened scale ϵ, which equals the full measure of the Cantor set $\tilde{C}_p$ at the scale ϵ (c.f., Example 4) where the infinitesimals live. Because of the reparametrisation invariance, we may suppose that $\tilde{C}_p$ is determined by the original 1-set and vice versa. At the scale ϵ, the gaps of $\tilde{C}_p$ are fattened by the amount ϵ, and in the presence of a positive measure, the said valuation is determined by the sum of the fattened gap sizes. For a zero measure set, this valuation, on the other hand, is determined instead by the finite Hausdorff measure, upto a finer (double logarithmic) scale correction that arises from the possible presence of local fine structures (c.f., above application). This observation proves the claim.