Chapter 1

Introduction

1.1 Basic definitions

At the outset we review some basic terminology about graphs, directed graphs and relations used throughout this dissertation.

A *graph* $G(V, E)$ consists of a set of *vertices* V (or V(G)) and a collection of *edges* E (or E(G)) which in turn consists of distinct unordered pairs of distinct elements of V. Most authours call this a *simple graph*, we call it simply a *graph*. A graph will often be denoted by an ordered pair that indicates both the vertex set and edge set: $G(V, E)$. In terms of relations, a graph G is an irreflexive symmetric relation E on V. To denote an edge we juxtapose two vertices and say the vertices are *adjacent*; that is, for $u, v \in V$, $uv \in E$ denotes u and v are adjacent or uv is an edge. A graph G' is a *subgraph* of G if $V(G') \subset V(G)$ and $E(G') \subset E(G)$. A subgraph G' is a *generated subgraph* or an *induced subgraph* of G if $V(G') \subset V(G)$ and two vertices are adjacent in G' if and only if they are adjacent in G. The *complement* $\overline{G}(V, E)$ of a graph G has the same vertex set as G and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

A *stable set* (*independent set*) is a set of vertices of G such that no two vertices in the set are adjacent. A *bipartite* graph is a graph $H(V, E)$ whose vertex set V can be partitioned into two stable sets X and Y and we write $H = H(X, Y, E)$. A *complete bipartite* graph is a bipartite graph $H(X, Y, E)$ where every pair of vertices that belong to different partite sets are adjacent.

A *directed graph* or *digraph* $D(V\ E)$ is a generalization of a graph in which E, the set of *edges or arcs* consists of ordered pairs of V, and for vertices u and v, if the ordered pair $(u, v) \in E$, then we denote this by $u \to v$. We use $u \nrightarrow v$ to mean $(u, v) \notin E$. So, a digraph $D(V, E)$ is simply a binary relation E on V having no restriction; that is, the relation could be reflexive for some elements of V and not others, and $(u, v) \in E$ does not imply $(v, u) \notin E$.

We generally represent a graph with a drawing in which vertices are depicted by small circles and if two vertices are adjacent, then a line connecting them is drawn. A digraph is typically represented in the same way, but with arrows indicating the order; that is, if $(u, v) \in E$, an arrow is drawn from u to v.

The *successor* set of a vertex v in a digraph $D(V, E)$ is the set of vertices u such that $vu \in E$. The *predecessor* set of a vertex v in $D(V, E)$ is the set of vertices u such that $uv \in E$.

The *adjacency* matrix of a (di)graph on a vertex set $V = \{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ (0, 1)-matrix with 1 in the (i, j) position if and only if (v_i, v_j) is an edge of the (di)graph.

For any undefined term, one is referred to Roberts [69], Golumbic [35] or West [96].

1.2 Background

The subject of intersection graphs has been an important area of study for the last few decades. An *intersection graph* is a graph G whose vertices correspond to a family of sets $\mathfrak{F}$ and two vertices are adjacent if and only if their corresponding sets intersect; $\mathfrak{F}$ is called a representation of G when G is the intersection graph of $\mathfrak{F}$. It was shown by Marczewski [58] that all graphs are intersection graphs; for each vertex of G, if we associate the set of all edges incident to the vertex, then G is the intersection graph of this family of sets. Interesting classes of graphs are evolved by restricting $\mathfrak{F}$ or by modifying the rule that determine adjacency. McKee and McMorris [60] evinces the diversity and importance of the intersection graph perspective. Probably the most well known intersection graph comes up when $\mathfrak{F}$ is restricted to intervals of $\mathbb{R}$ (or arcs of a circle) and the corresponding intersection graph is known as an *interval graph* (or *circulr-arc graph*). For detailed discussion on this topic see Golumbic [35] and Fishburn [30]. A survey by Trotter [89] summarizes a variety of results and open problems.

As an extension of the study of intersection graphs, the mathematical community picked up and started running with the idea of representing digraphs from the view point of intersection representation. Intersection digraphs of a family of ordered pair of intervals on the real line and of arcs of a circle, are called *interval digraphs* and *circular arc digraphs* respectively and have been introduced and studied by Sen, West and others. Harary et al. introduced the concept of interval bigraphs quite early, which is equivalent to the concept of interval digraphs. Then Hell and Huang [47] characterized and studied this class from the bigraph point of view. Their findings provide the background material of this thesis.

1.3 Interval graphs and related topics

A graph $G = (V, E)$ is an interval graph if and only if there is a family $\mathbb{I}$ of real intervals $\{I_v : v \in V\}$ such that $uv \in E \iff I_u \cap I_v \neq \phi$.

Interval graphs have a long and rich history. Discrete Mathematics (1985) published a special issue on interval graphs and related topics. They have been introduced independently by Benzer [2], a molecular biologist and Hajos [41], a mathematician. Benzer got the idea of interval graphs from an application point of view. Hajos, from purely mathematical consideration, asked basically, what graphs have a representation by collection of intervals of $\mathbb{R}$.

Interval graphs have found applications to a wide variety of modeling real world problem. It has been used in seriation problem by Kendall [51] and Hubert [50], in archaeology by Skrien [82, 83] and in developmental psychology by Coombs and Smith [16] to name a few. Stoffers [88] and Roberts [70, 71] used interval graphs to find a solution to general traffic phasing problem. More applications in this field can be found in [34] and [95].

1.3.1 Some characterization of interval graphs

All graphs are not interval graphs. For example, it is easy to verify that C_4 is not an interval graph. First characterization of interval graphs was due to Lekkerkerker and Boland [52].

A graph G is said to be *chordal* or *triangulated* if it has no induced $C_n, n \geq 4$. An *asteroidal triple* (AT) of a graph G is a set of three vertices such that there is a path between any pair that avoids the neighbourhood of the third. Lekkerkerker and Boland [52] characterized interval graphs as those chordal graphs that have no asteroidal triple.

Theorem 1.3.1 (Lekkerkerker and Boland [52]). *A graph G is an interval graph if and only if G is chordal and does not contain any asteroidal triple.*

In that paper, they also provided a complete set of forbidden subgraphs for interval graphs as given in the following Figure 1.1.

A *transitive orientation* F of a graph $G = (V, E)$ is an assignment of a direction, or orientation, to each edge in E such that if $xy \in F$ and $yz \in F$ then $xz \in F$. A graph is called a *comparability graph* if it has a transitive orientation. For example, the even length chordless cycles C_{2k} $(k \geq 2)$ are comparability graphs, but the odd length chordless cycles C_5, C_7, etc. are not comparability graphs. Comparability graphs are also known as *transitively orientable* (TRO) graphs.

Gilmore and Hoffman in 1964 gave another characterization of interval graph.

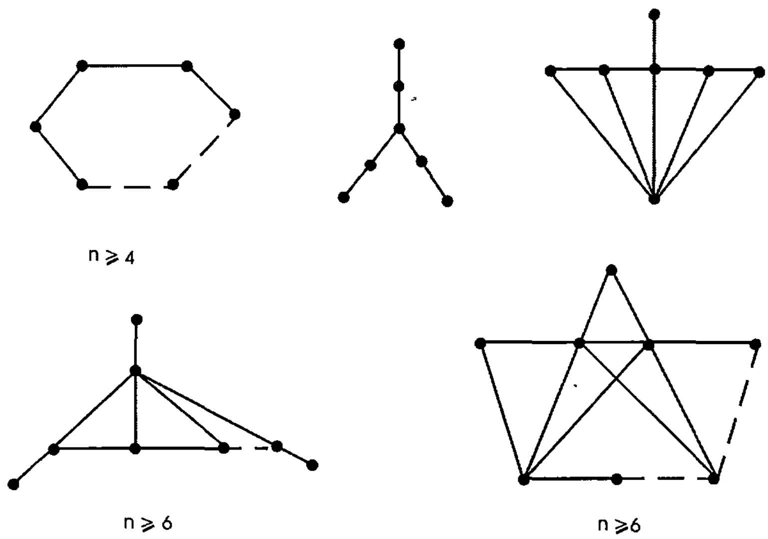


Figure 1.1: Forbidden subgraphs for interval graphs

Theorem 1.3.2 (Gilmore and Hoffman [33]**).** *A graph G is an interval graph if and only if G is chordal and its complement $\overline{G}$ is a comparability graph.*

A maximal clique of a graph is a complete subgraph which is not contained in any larger such subgraph. For a graph G, its vertex-clique incident matrix $M = (m_{ij})$ is the matrix whose rows and columns correspond to the vertices and the maximal cliques respectively of the graph such that

$$m_{ij} = \begin{cases} 1 & \text{if the } i^{th} \text{ vertex belongs to } j^{th} \text{ clique} \\ 0 & \text{otherwise} \end{cases}$$

A matrix is said to have consecutive 1's property for rows if there is a permutation of its columns so that the 1's in each row appear consecutively.

Theorem 1.3.3 (Fulkerson and Gross [31]**).** *A graph is an inteval graph* *if and only if its vertex-clique incidence matrix has a consecutive* 1*'s property for rows.*

1.3.2 Indifference graphs, unit interval graphs and proper interval graphs

R. D. Luce [55] developed a model for preference motivated by the concept of psychology. He contended that, for a set of things X and the preference relation $\mathcal{R}$, one seek a function $f : X \to \mathcal{R}$ and a just-noticeable tolerance $\delta > 0$ with $x \mathcal{R} y$ (x is preferred to y) if and only if $f(x) > f(y) + \delta$; i.e., if the value placed on x is sufficiently larger than the value placed on y. This presentation led to the development of an order called semi-order. A *semi-order* $\prec$ on X is a binary relation having the properties: For $x, y, z, w \in X$, (1) $\prec$ is irreflexive, (2) $x \prec y$ *and* $z \prec w \implies x \prec w$ *or* $z \prec y$, (3) $x \prec y$ *and* $y \prec z \implies x \prec w$ *or* $w \prec z$. Motivated by this idea, Roberts [68] in his thesis, studied graphs with adjacency determined by the rule: vertices u and v are adjacent if and only if $|f(u) - f(v)| \leq \delta$. It is observed that a preference relation represented by f gives rise to a transitive graph. A graph G where vertices u and v are adjacent if and only if $|f(u) - f(v)| \leq \delta$ is called an *indifference graph* and f is called an *indifference representation* of G. An interval graph G that has a representation in which each interval has the same (unit) length is called a *unit interval graph.* Similarly, if G has representation in which no interval properly contains another interval, G is called a *proper interval graph.* Clearly, a unit representation is also proper. It is easy to verify that the bipartite graph $K_{1,3}$ does not have a proper interval representation. The following classical result is due to Roberts [68].

Theorem 1.3.4 (Roberts [68]). *Let $G(V, E)$ be an undirected digraph. The following conditions are equivalent:*

1) G is a unit interval graph;

2) G is a proper interval graph;

3) G is an indifference graph;

4) G is an interval graph and is $K_{1,3}$-free;

5) $\overline{G}$ is a comparability graph and every transitive orientation of $\overline{G} = (V \overline{E})$ is a semiorder.

1.3.3 Probe interval graphs

The interest in many classes of intersection graphs are application driven. Advancement of molecular biology, and genetics in particular, has driven scientists to find new models. In continuation of the evoluation of interval graphs to model ideas about the fine structure of gene [2], P. Zhang [98] introduced *probe interval graphs.* He attempted to aid a problem called cosmid contig mapping, a particular component of the physical mapping of DNA [98]. A *probe interval graph* is a graph $G = (V, E)$, where V can be partitioned into (P, N), such that there is an interval corresponding to each vertex and two vertices are adjacent if and only if their corresponding intervals intersect and at least one of the vertices belongs to P. Probe interval graphs generalize interval graphs (taking $N = \phi$) and provide an instance of an intersection graph with a modified adjacency rule. Now several research works are continuing on this topic and some special classes of it may be found in [8, 11, 61]. S. Ghosh et al. [32] characterized the adjacency matrix of a probe interval graph.

1.3.4 Tolerance graphs and permutation graphs

Golumbic and Monma [37] introduced tbe concept of *tolerence graphs* to generalize some of the well-known applications associated with interval graphs. Their original motivation was the need to solve scheduling problems in which resources such as rooms, vehicles, support personnel, etc may be needed on an exclusive basis, but where a measure of flexibility or tolerence would allow for sharing or relinquishing the resource if a solution is not otherwise possible. Tolerence graphs are constructed from intersecting intervals in a manner similar to interval graphs, but putting an edge between two vertices depends on measuring the size of the intersection of the two intervals. Informally, if both intervals are willing to "tolerate"or ignore the intersection, then no edge is added between their vertices in the graph. The formal definition of tolerance graphs is as follows:

A graph $G = (V, E)$ is a *tolerance graph*, if each vertex $v \in V$ can be assigned a closed interval I_v and a tolerance $t_v \in R^+$ so that $xy \in E$ if and only if $|I_x \cap I_y| \geq min\{t_x, t_y\}$. Such a collection $\langle I, t\rangle$ of intervals and tolerances is called *tolerance representation* where $I = \{I_x : x \in V\}$ and $t = \{t_x : x \in V\}$. If graph G has a tolerance representation with $t_v \leq |I_v|$ $\forall$ $v \in V$, then G is called a *bounded tolerance graph.* The topic has assumed much interest to the researchers and Golumbic and Trenk [39] has written a book on it.

A graph $G = (V, E)$ is a *permutation graph* if there is a permutation

π of $V = \{1, 2, 3, \ldots, n\}$ so that for vertices i, j we have $ij \in E$ if and only if the order of i and j are reversed in π. If a graph G is a permutation graph using π, then its complement $\overline{G}$ is also a permutation graph using the reversal of π. We will take special interest on this topic in our thesis and we will discuss it again in chapter 2.

The following theorems due to Golumbic and Monma [37] show that the class of bounded tolerance graphs is a simultaneous generalization of interval graphs and permutation graphs.

Theorem 1.3.5. *The following conditions are equivalent about a graph G:*

1) G is an interval graph;

2) G is a tolerance graph with constant tolerances;

3) G is a bounded tolerance graph with constant tolerances.

Theorem 1.3.6. *The following are equivalent statements about a graph G:*

1) G has a tolerance representation with $t_i = |I_i|$ *for all* $i \in V(G)$*;*

2) G is an interval containment graph;

3) G is a permutation graph.

1.3.5 Homogeneous graphs

In a family of intervals, a *left-end interval* is an interval whose left end-point is leftmost among all end-points of intervals in the family. Similarly, a *right-end interval* is an interval whose right end-point is the rightmost among the intervals in the family. An *end interval* is a left-end or a right-end interval.

It is well known that an interval representation of an interval graph is not unique and an interval graph may have many interval representations that differ in the order of the end points of the intervals on the real line. An excellent account of this area of study is given by P. C. Fishburn [30]. D. Skrien and J. Gimbel [84] characterized those graphs for which every vertex v, there is an interval representation such that v is an end interval. This family of graphs is defined to be *vertex-homogeneous.* S. Olariu [64] also obtained the results of Skrien and Gimbel by approaching from a different point of view.

1.3.6 Circular-arc graphs

A circular-arc graph is the intersection graph of a family of arcs on a circle. Extensive work on circular-arc graphs was done by Tucker [91, 92, 93, 94]. After a long period of thorough research by different mathematicians to find the recognition algorithm of circular-arc graph, it was R.M.McConnel [59] who could finally solve the problem in linear time. Additional references for recognition algorithms of special classes of these graphs can be found in [22, 25]. For an excellent survey of this topic see [53].

1.4 Interval bigraphs/digraphs

A bipartite graph (in short, *bigraph*) $B = (X, Y, E)$ is an *intersection bigraph* if there exists a family $\mathcal{F} = \{I_v : v \in X \cup Y\}$ of sets such that $uv \in E$ if and only if $I_u \cap I_v \neq \phi$. An intersection bigraph is an *interval bigraph* (recpectively, a *circular-arc bigraph*) if $\mathcal{F}$ is a family of intervals on the real line (respectively, arcs on a circle). Let $B = (X, Y, E)$ be a bigraph with bipartition X and Y. Then the submatrix of the adjacency matrix of B containing rows corresponding to the vertices of X and columns corresponding to the vertices of Y is the *biadjacency matrix* or the *reduced adjacency matrix* or simply the *matrix* of B. This concept was introduced in 1982 by Harary et al. [44] and later studied by Hell and Huang [47].

An *interval digraph*, analogous to the concept of interval bigraph, was introduced in 1989 by Sen et al. [75]. An interval digraph is a directed graph with an ordered pair of intervals (S_u, T_u) corresponding to each vertex u such that $u \to v$ if and only if $S_u \cap T_v \neq \phi$. We note that the models for both interval bigraphs and interval digraphs are essentially the same.

Let D be a digraph and $B(D)$ be the *associated bipartite graph* with bipartition (X, Y), obtained from D by replacing each vertex $v_i \in V(D)$ by two vertices $x_i \in X$ and $y_i \in Y$ and each arc v_iv_j of D by an edge x_iy_j. Then it is clear from the definition that D is an interval digrph if and only if $B(D)$ is an interval bigraph. Similarly, if H is a bipartite graph with bipartition (X, Y), we can orient all edges from X to Y, and observe that the resulting digraph is an interval digraph if and only if H is an interval bigraph, since for a vertex $x \in X$ only S_x matters and for a vertex $y \in Y$, it is only T_y.

1.4.1 Ferrers digraphs/ bigraphs and Ferrers dimension

Ferrers digraphs and Ferrers dimensions [14, 15, 56] play an important role in our study. This special class of digraphs was introduced independtly by Guttman [40] and Riguet [67]. Riguet defined a Ferrers digraph to be a digraph $D(V, E)$ in which $\forall x, y, z$ and $t \in V$; $xy, zt \in E \Longrightarrow xt \in E$ or $zy \in E$ (inclusive). Riguet characterized Ferrers digraphs as those whose successor sets (or predecessor sets) are linearly ordered by inclusion. This suggests that by independent row and column permutation of the adjacency matrix $A(D)$, the 1's (or 0's) are clustered in the lower left sector forming a Ferrers diagram as shown in Figure 1.2.

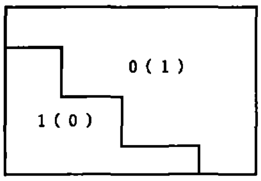


Figure 1.2: Ferrers digraph

He also showed that the above conditions are equivalent to the condition that $A(D)$ has no 2×2 permutation matrices as under:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Any of these forbidden matrices is called an *obstruction* and the zeros in an obstruction are said to form a *couple.*

In 1971, Bouchet [4] showed that any digraph D is the intersection of a family of Ferrers digraphs. This observation leads to the important notion of Ferrers dimension. The *Ferrers dimension* of a digraph D is defined to be the minimum number of Ferrers digraphs whose intersection is D. Bouchet [4, 5] obtained several interesting results on Ferrers dimension. Cogis [13] and Doignon, Ducamp and Falmagne [23] have studied extensively the digraphs of Ferrers dimension 2 and have characterized independently this class of digraphs in different contexts. Cogis associated an undirected graph $H(D)$ with D whose vertices correspond to the 0'of the adjacency matrix of D and two such vertices are joined by an edge if and only if the corresponding 0's form a couple. He proved that D is of Ferrers dimension at most 2 if and only if $H(D)$ is bipartite. The bigraph corresponding to a Ferrers digraph is a *Ferrers bigraph/chain graph.* The *Ferrers dimension/chain dimension* of a bigraph is the Ferrers dimension of the corresponding digraph. The following theorem characterizes a bigraph of Ferrers dimension at most 2.

The following conditions are equivalent.

Theorem 1.4.1 (Sen et al. [75, 78],**Cogis** [13]**).** *The following conditions are equivalent for a bipartite graph B:*

1) B has Ferrers dimension at most 2;

2) The rows and columns of its biadjacency matrix can be permuted independently so that no 0 has a 1 both to its right and below;

3) The graph $H(B)$ of couples of B is bipartite.

In chapters 2 and 5 we will further characterize bigraphs of Ferrers dimension at most 2 in terms of permutation bigraphs and in terms of complements of two-clique circular-arc graphs.

1.4.2 Characterization of interval digraphs / bigraphs

In [78], it was proved that an interval digraph is a generalization of an interval graph. In fact, they proved the following result.

Theorem 1.4.2 (Sen, Sanyal and West [78]**).** *An undirected graph G is an interval graph if and only if the corresponding digraph $D(G)$ with loops at every vertex is an interval digraph.*

Let D be a digraph and its complement be $\overline{D}$. A more interesting and useful characterization of interval digraph uses its adjacency matrix.

Characterization of interval graph and digraph so far developed, involve an order of their vertices. Sanyal and Sen [73] tried to characterize them in a different way and posed the question, "Is there any ordering among the edges of a (di)graph that characterizes an interval (di)graph?" They answered this question in the affirmative. For a digraph $D(V, E)$, they introduced the notion of a *consistent ordering* of the edges of D.

The set of all edges of a digraph D is said to have a *consistent ordering* if E has a linear ordering ($<$) such that for pq, pu, rs, $tq \in E$,

1) $pq < rs < pu \Longrightarrow ps \in E\,(q \neq u)$.

2) $pq < rs < tq \Longrightarrow rq \in E\,(p \neq t)$.

Theorem 1.4.3 (Sanyal and Sen [73]**).** *A digraph $D(V, E)$ is an interval digraph if and only if its edge set has a consistent ordering.*

There are quite a number of characterizations of interval bigraphs/digraphs and their subclasses. In order to get into them, we first explain the following notions.

A (0, 1) matrix is said to have a *zero-partition* if the rows and columns of the matrix can be permuted independently such that each 0 can be replaced by one of $\{R, C\}$ in such a way that every R has only R's to its right and every C has only C's below it. Figure 1.3 gives an example of a matrix with a zero-partition.

1	1	R	R
1	1	1	R
C	1	R	R
C	1	1	1
C	C	1	R

Figure 1.3: A matrix with a zero-partition

An alternative way of describing a zero-partition is in terms of *generalized linear ones property.* With the help of the concept of stair-partition of a matrix, we can describe this property. A *stair-partition* of a matrix is a partition of its elements into two subsets L and U by a polygonal path from the upper left to the lower right such that the set L is closed under rightward and downword movement and the set U is the complement part of L. Equivalently, U corresponds to the positions in some upper triangular matrix and L to the positions in the lower triangular matrix. This is shown in Figure 1.4.

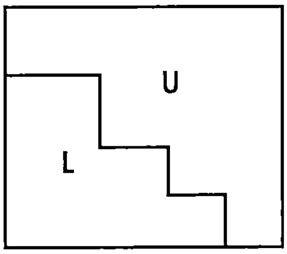


Figure 1.4: Stair partition

A (0, 1)-matrix has the *generalized linear ones property* if it has a stair partition (L, U) such that the 1's in U are consecutive and appear left-most in each row and the 1's in L are consecutive and appear top-most in each column.

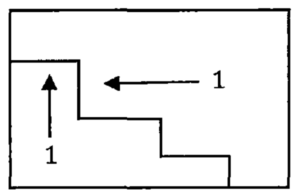


Figure 1.5: Generalized linear ones property

A complete bipartite (sub)graph is called a *biclique*. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_t\}$ be distinct induced subgraphs of a graph $G = (V, E)$. The set $\mathcal{G}$ is said to be consecutively ordered when for any $v \in V$, if $i < j < l$ and $v \in G_i \cap G_l$, then $v \in G_j$. We say $\mathcal{G}$ covers G if it forms an edge cover of G.

For a bipartite graph we have the following list of equivalent conditions characterizing an interval bigraph. The equivalences of (1), (2), (6) and (7) were obtained by Hell and Huang [47]. Das et al. [75, 76] used the digraph language to prove the equivalences of (1),(3), (4), (5) and (8).

Theorem 1.4.4. *The following statements are equivalent for a bipartite graph $H = (X, Y, E)$ with n vertices:*

1) H is an interval bigraph;

2) $\overline{H}$ is a two-clique circular arc graph in which no two arcs cover the whole circle;

3) $\overline{H}$ is the union of two disjoint Ferrers bigraphs;

4) The biadjacency matrix of H has a zero-partition;

5) There is a cover of H consisting of bicliques that can be consecutively ordered;

6) The vertices of H can be ordered $v_1 < v_2 < \ldots < v_n$ *so that there do not exist* $a < b < c$*with* v_a, v_b *in the same partite set and* $v_a v_c \in E$*, but* $v_b v_c \notin E$ *(see Figure 1.6);*

7) The vertices of G can be ordered $v_1 < v_2 < \ldots < v_n$ *so that there do not exist* $a < b < c$ *with any of the four structures in Figure 1.7;*

8) The biadjacency (reduced adjacency) matrix has generalized linear one's property;

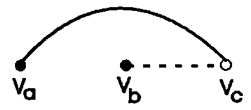


Figure 1.6: Forbidden graph of Theorem 1.4.4(6)

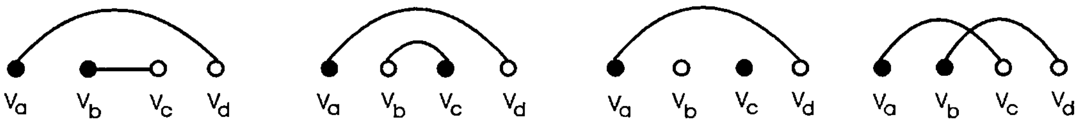


Figure 1.7: Forbidden graphs of Theorem 1.4.4(7)

Amongs the above, conditions (2) and (4) are the important ones and we will use them frequently in this thesis.

Unit interval bigraphs

Now we consider unit interval bigraphs, a special subclass of interval bigraphs. A unit interval bigraph is an interval bigraph in which all the intervals are of same length. This idea is so simple and pervasive that many researchers have converged to this notion studying from different view points. So much so that Brown in his thesis [6] has cited about 15 different but equivalent classes of structured graphs of this family. We will list only a few of them in Theorem 1.4.5. In fact, a fine account of the results of this topic of interval bigraphs and its subclasses can be found in [6].

We need to spell out some notions here.

Proper interval bigraphs are interval bigraphs having representations such that no interval properly contains another interval of the same partite set.

Indifference bigraphs are those for which there exists a real valued function on $X \cup Y$ such that for $u \in X$ and $v \in Y$, uv is an edge if and only if $|f(u) - f(v)| \leq 1$.

Analogous to Robert's work [68], these graphs have been proved to be equivalent and have been characterized in terms of monotonic consecutive arrangement [77]. A (0, 1)-matrix has a *monotone consecutive arrangement* if there exist independent row and column permutation such that the resulting matrix has the following structure: The 0's can be labeled by R and C such that every position above or to the right of an R is an R, and every position below or to the left of a C is a C.

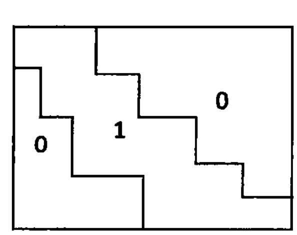


Figure 1.8: Monotone consecutive arrangement

A proper *circular-arc graph* is a circular-arc graph, which has a representation by arcs such that no arc properly contains another.

In the following, the equivalence of (1), (2), (3) and (5) is to be found in [77] while Hell and Huang [47] obtained the equivalences of (1), (4), (6) and (8). Lastly, the condition (7) was obtained by Steiner [87] who incidentally provided a linear time algorithm for recognition of these classes of bigraphs. Again, the condition (8) was obtained by Lin and West [54].

Theorem 1.4.5 ([77, 47, 54, 87]). *For a bipartite graph H, the following are equivalent:*

1) H is a unit interval bigraph;

2) H is a proper interval bigraph;

3) H is an indifference bigraph;

4) $\overline{H}$ is a proper circular-arc graph;

5) The biadjacency matrix of H has a monotone consecutive arrangement;

6) H is asteroidal triple free;

7) H is a permutation graph;

8) H does not contain an induced cycle of length at least 6 or any of the graphs of Figure 1.9 as an induced subgraph.

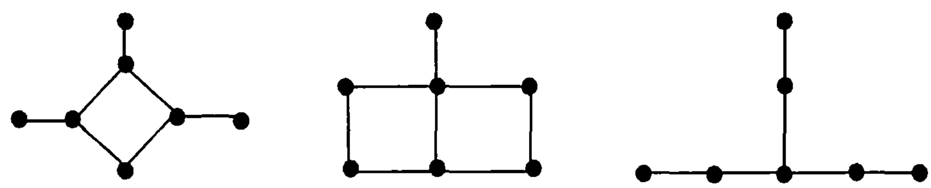


Figure 1.9: Forbidden graphs of Theorem 1.4.5(8)

1.4.3 Interior edges

Das and Sen [18] introduced the concept of *interior edges* and used this notion to find out the digraphs that are of Ferrers dimjension 2 but are not interval digraphs. Cogis [14] proved that a digraph D is of Ferrers dimension at most 2 if and only if its associated graph $H(D)$ is bipartite. Also it is known that for a digraph D having Ferrers dimension 2, the complement $\overline{D}$ is the union of two Ferrers digraphs (not necessarily disjoint). These two Ferrers digraphs are called *realization* of $\overline{D}$. Obviously realization of $\overline{D}$ is not unique. The graph $H(D)$ may have more than one connected component as well as isolated vertex(vertices) (the 0's of $A(D)$ that donot belong to any obstruction). The set of all isolated vertices is denoted by $\mathcal{I}(H)$ or $\mathcal{I}$. The graph obtained by deleting the isolated vertices from $H(D)$ is called the *bare graph* associated with D and is denoted by H^b

Since a digraph of Ferrers dimension at most 2 is equivalent to the existence of independent row and column permutation of the adjacency matrix so that the resulting matrix has no 0 with a 1 both to its right and below, it can not have an obstruction of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It was shown that an interval digraph is necessarily a digraph of Ferrers dimension at most 2 but the converse is not true. As a matter of fact, it was proved that a digraph D is an interval digraph if and only if it is the intersection of two Ferrers digraphs whose union is complete or equivalently the complement $\overline{D}$ is the union of two disjoint Ferrers digraphs. Let $H(B)$ be bipartite and $(R,\, C)$ be a bicolouration of H^b. Cogis showed that there is a bicolouration $(R,\, C)$ of H^b such that each of $R \cup \mathcal{I}$ and $C \cup \mathcal{I}$ is a Ferrers bigraph and their union is the complement $\overline{B}$ of B. Note that not all bicolouration of H^b have this property and a bicolouration $(R,\, C)$ of H^b for which $R \cup \mathcal{I}$ and $C \cup \mathcal{I}$ are Ferrers bigraphs is called *satisfactory bicolouration.*

We know that a bigraph B is an interval bigraph if and only if its complement is the union of two disjoint Ferrers bigraphs. This means that there is a bicolouration $(R,\, C)$ of H^b and a partition $\mathcal{I}$ into two disjoint subsets $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $R_1 = R \cup \mathcal{I}_1$ and $C_1 = C \cup \mathcal{I}_2$ are two disjoint Ferrers bigraphs whose union is the complement $\overline{B}$ of B. An edge I of $\mathcal{I}_1$ is said to be an *interior edge of* R_1 and is denoted by $\mathcal{I}_r$ if it has a configuration

$$\begin{pmatrix} 1 & R \\ R & \mathcal{I}_r \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} R & \mathcal{I}_r \\ 1 & R \end{pmatrix}$$

in $\overline{B}$. Similarly, an edge I of $\mathcal{I}_2$ is said to be an interior edge of C_1 and is denoted by $\mathcal{I}_c$ if it has a configuration

$$\begin{pmatrix} 1 & C \\ C & \mathcal{I}_c \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} C & \mathcal{I}_c \\ 1 & C \end{pmatrix}$$

With this notion of interior edges, Das and Sen [18] proved that if a digraph D of Ferrers dimension 2 is an interval digraph then for any satisfactory bicolouration of H^b, $\mathcal{I}_r \cap \mathcal{I}_c = \phi$. But the converse is not true. They gave an example of eight vertex digraph with Ferrers dimension 2 which is not an interval digraph and for which $\mathcal{I}_r \cap \mathcal{I}_c = \phi$. In [20] this idea of interior edges were used to find additional configurations of a digraph having Ferrers dimension 2 which are not interval digraphs. For further discussion see subsection 1.4.4

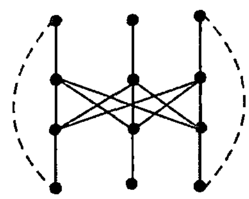
1.4.4 Forbidden subgraphs and forbidden substructures.

Analogue for an AT (see page 5) is an *asteroidal triple of edges*(ATE): a set of three edges such that there is a path between any two that avoids the neighbourhood of the third edge, where if $e = uv \in E$, then the neighbourhood of e is $N(e) = N(u) \cup N(v)$.

In 1997, Haiko Müller [63] gave a polynomial time $O(n^5 m^6 \log n)$ recognition algorithm for interval bigraphs. He introduced a family of graphs, called insects (shown in Figure 1.10) and conjectured the following characterization of interval bigraphs:

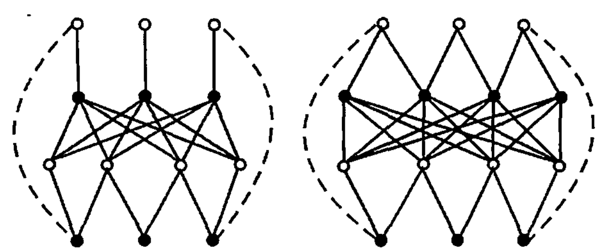
Conjecture 1.4.6. *A bipartite graph is an interval bigraph if and only if it contains no asteroidal triple of edges and no induced insects.*

This conjecture was refuted in [17, 47] who found a counter examples to this conjecture.Hell and Huang [47] obtained a class of graphs called bugs (see Figure 1.11), which are not interval bigraphs. Also they do not contain induced insects and asteroidal triple of edges and so provide counter examples



Dashed edges may or may not be present

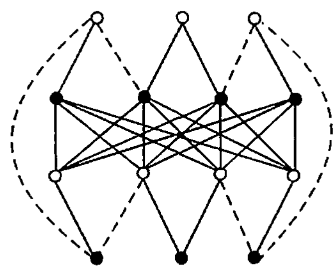
Figure 1.10: Insects



Dashed edges may or may not be present

Figure 1.11: Bugs

to Müller's conjecture. They have used the term extended insect to mean either an insect or bug. Thus the list of forbidden structure for an interval bigraph has been extended. S.Das, A.Das et al. [18, 20] started from the class of bigraphs of Ferrers dimension 2 and tried to obtain a forbidden configuration characterization of interval bigraphs using biadjacency matrix. In doing so, they improved upon the results of Hell and Huang [47] by obtaining a large number of forbidden subgraphs which include the extended insects of [47]. In fact, they obtained five forbidden configurations $F^i\,(i = 0, 1, 2, 3, 4)$ of the biadjacency matrix of an interval bigraph. Specially, the class of bigraphs of configuration F^0 is given in Figure 1.12. It can be easily seen that all the graphs of the extended insects form a subclass of the class of Figure 1.12 and can be obtained by different combinations of inclusions/exclusions of the dashed edges of Figure 1.12. However, the complete list is still elusive and it remains an interesting open problem till date.



Dashed edges may or may not be present

Figure 1.12: Bigraphs of F^0

1.4.5 Related topics

Tolerance (di)graphs, bitolerance (di)graphs, homomorphic graphs

Bogart and Trenk [3] provided a different directed graph analogue of interval graphs based on work in tolerance graphs. A directed graph $D(V, E)$ is a *bounded bitolerance digraph* if each vertex $v \in V$ can be assigned a real interval $I_v = [L(v), R(v)]$, a left tolerant point $p(v) \in I_v$, and a right tolerant point $q(v) \in I_v$ so that $E = \{(x, y) : L(x) \leq q(y) \text{ and } R(x) \geq p(y)\}$. The representation $\{[L(v), R(v)], p(v), q(v) : v \in V\}$ is called a *bounded bitolerance representation* and an example is given in Figure 1.13. According to this definition, bounded bitolerance digraphs, have a loop at each vertex. They characterized bounded bitolerance digraphs in terms of two interval orders which is analogous to that of interval digraphs in terms of two Ferrers digraphs [75].

An ordered set $P = (V, \prec)$ consists of a set V together with a binary relation

$\prec$ on V that is irreflexive, transitive and therefore antisymmetric. Elements $x, y \in V$ are said to be *comparable* if $x \prec y$ or $y \prec x$; otherwise they are *incomparable*, which is denoted by $x \parallel y$. A *linear order* is one with no incomparabilities. The *dimension* of an order P is the smallest number of linear orders whose intersection is P. We construct a digraph $\widetilde{P_1} \cap \widetilde{P_2}$ from two ordered sets P_1 and P_2 that have the same ground set.

Let $P_1 = (V, \prec_1)$ and $P_2 = (V, \prec_2)$ be ordered sets. The digraph $\widetilde{P_1} \cap \widetilde{P_2} = (V, E)$ has $(x, y) \in E$ if and only if

(i) $x \prec_1 y$ or $x \|_1 y$ (ii) $x \prec_2 y$ or $x \|_2 y$.

Bogart and Trenk [3] proved the following result

Theorem 1.4.7. *A directed graph $D(V, E)$ is a bounded bitolerance digraph if and only if there exists interval orders $P_1 = (V, \prec_1)$ and $P_2 = (V, \prec_2)$ for which $\widetilde{P_1} \cap \widetilde{P_2} = D(V, E)$.*

In a bounded bitolerance representation, it is not necessary to have $p(v) \leq q(v)$. If in fact $p(v) \leq q(v)\ \forall\ v \in V$, then $D(V, E)$ is called a *totally bounded bitolerance digraph* and these digraphs were studied in [80, 81]. If a digraph $D(V, E)$ has an interval digraph representation $\{(S_v, T_v) : v \in V\}$ where $T_v \subseteq S_v\ \forall\ v \in V$, then D is called an *interval nest digraph* and this class was studied in [65]. Shull and Trenk [80] proved the following proposition.

Proposition 1.4.8. *A digraph $D(V, E)$ is a totally bounded bitolerance digraph if and only if $D(V, E)$ is an interval nest digraph.*

They have finally explored the relationship between interval digraphs and bounded bitolerance digraphs, both of which are directed graph analogues of

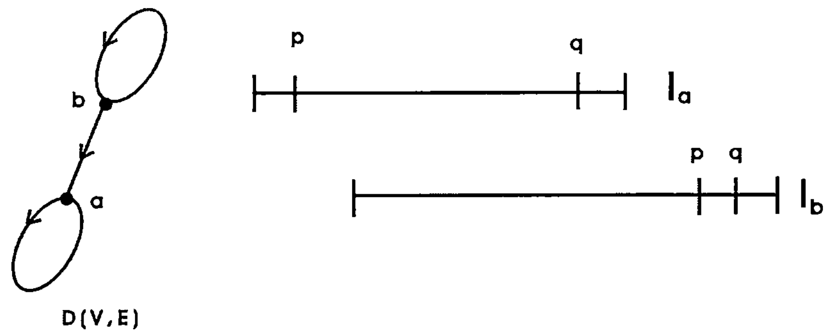


Figure 1.13: Bounded bitolerance representation

the well known interval graphs.

Theorem 1.4.9 (Shull and Trenk [81], **Sen and Sanyal** [77]**).** *The following are equivalent statements about a symmetric digraph D that has a loop at each vertex.*

1) D is a bounded bitolerance digraph;

2) D is an interval digraph;

3) The underlying simple graph is an interval graph;

4) D is a totally bounded bitolerance digraph;

5) D is an interval nest digraph.

A *homomorphism* f of a (di)graph G to a (di)graph H is a mapping $f : V(G) \to V(H)$ which preserves edges, i.e., $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. If there is a homomorphism of G to H, we write $G \to H$. For a fixed (di)graph H, the *homomorphism problem* HOMP(H) asks whether or not an input (di)graph G satisfies $G \to H$. The complexity of all homomorphism problems has been studied by P. Hell and J. Nestril [48]. HOMP(H) is polynomial time solvable if H is bipartite and is NP-complete otherwise. The *list homomorphism problem* for H is a generalization of the homomorphism problem for H and is defined as: Given (di)graph G, H and lists $L(v) \subseteq V(H)$, $v \in V(G)$, a *list homomorphism* of G to H with respect to the lists L is a homomorphism f of G to H, such that $f(v) \in L(V) \, \forall \, v \in V(G)$. For a fixed (di)graph H, the *list homomorphism problem* L–HOMP(H) asks whether or not an input (di)graph G, with lists L, admits a list homomorphism to H with respect to L. Note that, HOMP(H) is a restriction of L–HOMP(H), where $L(v) = V(H)$. Thus L–HOMP(H) is NP-complete when H is not bipartite.

Feder, Hell, Huang and Rafiey [29] introduced the concept of *adjusted interval digraphs* obtained by a slight change in the definition. Let $\{(S_v, T_v) : v \in V\}$ be an interval digraph representation of $D(V, E)$. If the intervals S_v, T_v, $v \in V$, can be chosen so that for each v, the intervals S_v and T_v have the same left end point, we say that D is an *adjusted interval digraph.* It is clear that an adjusted interval digraph must be reflexive. By contrast to interval digraphs, adjusted interval digraphs have a forbidden structure characterization, parallel to a characterization for undirected graphs. Feder et al. [29] related adjusted interval digraphs to a list homomorphism problem. They also observed that if H is an adjusted interval digraph, then the problem L-HOMP(H) is polynomial time solvable, and conjectured that for all other reflexive digraphs H, the problem L-HOMP(H) is NP-complete.

Interval k-graphs

Interval k-graphs, a generalization of interval bigraphs, was introduced by Brown et al. [9]. An *interval k-graph* is a k-partite graph with an interval assigned to each vertex so that two vertices are adjacent if and only if they belong to distinct partite sets and their corresponding intervals have non empty intersection. Obviously, an interval bigraph is an interval k-graph with $k = 2$.

Analogous to the idea of consecutive maximal cliques in the characterization of interval graphs, they characterized interval k-graphs in terms of consecutively ordered subgraphs. They have also proved the following result: Interval graphs $\subsetneq$ Probe interval graphs $\subsetneq$ Interval k-graphs.

Brown in his thesis [6] proved the following result.

Theorem 1.4.10. *A graph is an interval k-graph if and only if there exists a cover of complete r-partite subgraphs that can be consecutively ordered, where $1 \leq r \leq k$, for each subgraph.*

A graph G is said to be perfect if for every induced subgraph H of G, $\chi(H) = \omega(H)$, where $\chi(H)$ is the chromatic number of H (i.e., the fewest number of colours needed to properly colour H) and $\omega(H)$ is the number of vertices in the largest induced clique in G. They have also proved the following results.

Theorem 1.4.11. *Interval k-graphs are perfect.*

Theorem 1.4.12. *If a graph G has an asteroidal triple of edges, then G is not an interval k-graph.*

1.5 A brief outline of the thesis

1.5.1 Permutation bigraphs

The concept of permutation graphs have been studied by B. Dushnik, E. M. Miller [26], S. Even, A. Pnueli, A. Lempel [27]. Permutation graphs can be defined in many equivalent ways. We consider the following definition.

As defined earlier, an undirected graph $G(V,\ E)$ with n-vertices is a permutation graph, if there is a bijection $L\ : V \to \{1, 2, \ldots, n\}$ and a permutation π of $\{1, 2, \ldots, n\}$ such that two vertices u and v are adjacent in G if and only if the orders of u and v are reversed in L and π.

Let $(P\,, <)$ be a poset. A containment representation of a poset assigns each element $x \in P$, a set S_x such that $x < y$ if and only if $S_x \subset S_y$. It is well known that interval containment posets are precisely the posets of dimension 2 [57]. Furthermore, the Ferrers dimension of the comparability digraph (a digraph that is reflexive, antisymmetric and transitive) of a poset equals the order dimension of the poset [4, 23, 75]. Sen et al. proved that interval containment digraphs are precisely the digraphs of Ferrers dimension 2. From these results, it implies that a poset P is an interval containment poset if and only if its comparability digraph is an interval containment digraph.

Permutation graphs have several known characteristics and we state a few of them.

Theorem 1.5.1. *Let $G(V,\ E)$ be an undirected graph. The following conditions are equivalent:*

1) G is a permutation graph;

2) Both G and its complement $\overline{G}$ are transitively orientable;

3) G is a containment graph of intervals on the real line;

4) G is a comparability graph of poset dimension at most 2.

In this thesis, we generalize the concept of permutation graph and introduce the concept of permutation bigraph. In chapter 2 we show that this class is equivalent to the bigraph of Ferrers dimension at most 2. Then we characterize interval bigraphs and indifference bigraphs in terms of their permutation labellings.

1.5.2 Interval tournaments

A touranment is a complete oriented digraph. A tournament that is an interval digraph is an *interval tournament.* Interval tournaments were introduced and characterized by Brown, Busch and Lundgren [7]. They have characterized them by a complete list of seven forbidden sub-tournaments. They have also proved that a tournament with n-vertices is an interval tournament if and only if it has a transitive sub-tournament having $(n-1)$-vertices. In chapter 3 we provide an alternative proof of the above theorem. This approach helps us to obtain further characterizations of interval tournaments. One of these characterizations is that a tournament is an interval tournament if and only if all of its 3-cycles have a common vertex. In continuation we have obtained three forbidden subdigraphs of an interval tournament. Lastly, we characterize the complement of an interval tournament in terms of two-clique circular-arc graphs.

1.5.3 Homogeneous interval bigraphs

Motivated by the idea of homogeneous graphs (subsection 1.3.5) we have developed in chapter 4 the concept of *homogeneous interval bigraphs* and characterize them in terms of forbidden bigraphs. An interval bigraph B is said to be *vertex-homogeneous* if for every vertex v of B, there exists an interval representation of B for which the interval representing v is an end interval. Analogously, an interval bigraph B is said to be *edge-homogeneous* if for every edge $e = uv$ of the bigraph B, there exists an interval representation of B for which the interval representing both u and v have the same end.

In chapter 4, we characterize both vertex-homogeneous and edge-homogeneous interval bigraphs in terms of forbidden subbigraphs.

1.5.4 Circular-arc bigraphs and its subclasses

Circular-arc bigraphs are the intersection bigraphs of a family of arcs on a circle. The digraph version of this class was introduced by Sen et al. [76].

Tucker [91] introduced the concept of quasi circular one's property of a $(0,1)$-matrix and used it to characterize a circular-arc graph. Sen et al. [76] extended this concept of Tucker and introduced *generalized circular ones property*: Given a $(0,1)$-matrix A and a stair partion (L, U) (see Figure 1.4), let $V_i[W_j]$ be the 1's in row i [column j] that begin at the stair and continue rightward [downward](around if possible) until a first 0 is reached. A has a generalized circular one's property if V_i's and W_j's together cover all the 1's of A. They characterized circular-arc digraphs/bigraphs in terms of this property.

In the final chapter of this thesis (Chapter 5), we obtain two new characterizations of circular-arc bigraphs. One of them is the representation of a circular arc bigraph as an intersection of two-clique circular graphs while another one represents the same as a union of interval bigraph and a Ferrers bigraph. Finally, we introduce the notions of proper and unit circular arc bigraphs, characterize them and show that, as in the case of circular-arc graphs, unit circular arc bigraphs from a proper subclass of the class of proper circular arc bigraphs.