CHAPTER - V

Large amplitude free vibrations of elastic plates.

Nomenclature :

The following nomenclature are used throughout this paper.

w = deflection, normal to the middle plane,

u, v = displacements corresponding to the directions of co-ordinate axes,

h = thickness of the plate,

D = flexural rigidity of the plate $= \dfrac{Eh^3}{12(1-\sigma^2)}$

E = Young's modulus,

σ = Poisson's ratio,

ρ = density of the plate material.

Introduction :

Berger's (1955) approximate plate theory for the large deflection of isotropic plates has been extended to orthotropic plate problems by Iwinski and Nowinski (1957). Nowinski (1958) has also solved some boundary value problems associated with circular and rectangular plates undergoing large deflections. Nash and Modeer (1960) found the large amplitude free vibrations of rectangular and circular plates applying the technique exhibited by Berger.

In this paper an attempt has been made to investigate the large amplitude free vibrations of triangular, elliptic and semi-circular plates.

<u>Theory :</u>

Let us consider the free vibrations of flat elastic plates with hinged, immovable edges. The deflections are considered to have the order of magnitude of the plate thickness.

The sum of the membrane and bending energies in a thin plate undergoing large deflections can be written in the form,

$$V = \frac{D}{2}\iint\left\{\left[(\nabla^2 \omega)^2 + \frac{12}{h^2}e^2\right] - 2(1-\sigma)\left[\frac{12}{h^2}e_2 + \frac{\partial^2\omega}{\partial x^2}\cdot\frac{\partial^2\omega}{\partial y^2} - \left(\frac{\partial^2\omega}{\partial x\partial y}\right)^2\right]\right\}dx\,dy \qquad \ldots (1)$$

The kinetic energy of the plate is

$$T = \frac{\rho h}{2}\iint\left(\dot{u}^2 + \dot{v}^2 + \dot{\omega}^2\right)dx\,dy \qquad \ldots (2)$$

It is now possible to form the Lagrangian function

$$L = T - V \qquad \ldots (3)$$

According to Hamilton's principle $\delta\int_{t_1}^{t_2} L\,dt = 0$ ... (4)

If we set $A = \int_{t_1}^{t_2} L\,dt$, then $\delta A = 0$ ... (5)

Neglecting e_2 and applying Euler's variational equations we get the following equations

$$\nabla^4 w - \alpha^2 f(t) \cdot \nabla^2 w + \frac{12}{h^2 e_\xi^2} \cdot \frac{\partial^2 w}{\partial t^2} = 0 \qquad \ldots (6)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{\alpha^2 h^2}{12} f(t) \qquad \ldots (7)$$

[Nash and Modeer (1960)]

where $\alpha =$ constant,

$e_\xi^{-2} = \frac{\rho h^3}{12 D}$

It is to be noted that the terms corresponding to inertia effects in the plane of the plate have been neglected for the equations (6) and (7).

Problem :

1. Large amplitude free-vibration of an isoceles right-angled triangular plate.*

Let us consider the free vibration of a flat isoceles right-angled triangular plate with hinged, immovable edges. The equal sides of the plate are considered to be of lengths a in the directions of x and y .

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For the simply-supported edges, the boundary conditions are

$$\left.\begin{aligned} u = w = \frac{\partial^2 w}{\partial x^2} &= 0 \quad \text{at} \quad x = 0 \\ v = w = \frac{\partial^2 w}{\partial y^2} &= 0 \quad \text{at} \quad y = 0 \\ u + v = w = \frac{\partial^2 w}{\partial \nu^2} &= 0 \quad \text{at} \quad x + y = a \end{aligned}\right\} \qquad \ldots(8)$$

where $\dfrac{\partial}{\partial \nu} = \dfrac{1}{\sqrt{2}}\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}\right)$

The boundary conditions are satisfied by the configurations of the form

$$u(x,y,t) = \sum_{K=1,3,5,\ldots}^{\infty} B_K \sin\frac{K\pi x}{a}\left(\cos\frac{K\pi y}{a} + \sin\frac{K\pi x}{a} - \frac{K\pi}{4}\right) H(t) \qquad \ldots(9)$$

$$v(x,y,t) = \sum_{K=1,3,5,\ldots}^{\infty} B_K \sin\frac{K\pi y}{a}\left(\cos\frac{K\pi x}{a} - \sin\frac{K\pi y}{a} + \frac{K\pi}{4}\right) G(t) \qquad \ldots(10)$$

$$w(x,y,t) = \sum_{m=1,3,\ldots}^{\infty} A_m\left(\sin\frac{2m\pi x}{a}\cdot\sin\frac{m\pi y}{a} + \sin\frac{2m\pi y}{a}\cdot\sin\frac{m\pi x}{a}\right) F(t) \qquad \ldots(11)$$

The equations (9), (10) and (11) may now be substituted in equation (7) to yield

$$F^2(t) = G(t) = H(t) = f(t) \qquad \ldots(12)$$

Let us investigate the fundamental mode of vibration by putting $m = 1$ in equation (11).

Substituting equations(9), (10) and (11) in (7), considering equation (12) and integrating over the surface of the plate we have,

$$\alpha^2 = \frac{15 A_1^2 \pi^2}{a^2 h^2} \qquad \ldots(13)$$

If we now substitute equations (11), (12) and (13) in (6) with $m = 1$, we obtain,

$$\frac{12}{h^2 c_p^2} \cdot \frac{d^2F}{dt^2} + \frac{49\pi^4}{a^4} \cdot F(t) + \frac{75 A_1^2 \pi^4}{a^4 h^2} F^3(t) = 0 \qquad \ldots(14)$$

This equation is of the form

$$\ddot{F} + \lambda F + \mu F^3 = 0 \qquad \ldots(15)$$

which is to be solved subject to the initial conditions

$$F(0) = 1, \quad \dot{F}(0) = 0 \qquad \ldots(16)$$

Solution of (15) can be put in the form

$$F(t) = C_n(\omega_1 t, K) \qquad \ldots(17)$$

[Nash and Modeer (1960)]

where $$\omega_1^2 = \lambda + \mu = \frac{c_p^2 \pi^4 h^2}{12 a^4}\left(49 + 75 \frac{A_1^2}{h^2}\right) \qquad \ldots(18)$$

$$K^2 = \frac{\mu}{2(\lambda+\mu)} = \frac{75 A_1^2}{2h^2\left(49 + 75 \frac{A_1^2}{h^2}\right)} \qquad \ldots(19)$$

Here ω_1 and K are positive constants and C_n is Jacobi's elliptic function.

The period T' is given by $T' = \dfrac{4K}{\omega_1}$ ... (20)

where K is the complete elliptic integral of the first kind.

Hence $$T' = \frac{4Ka^2\sqrt{12}}{\varphi h \pi^2\left(49+75\frac{A_1^2}{h^2}\right)^{1/2}}$$ ... (21)

The usual linear period $T = \dfrac{2\pi}{\omega_2}$ ... (22)

is found from the equation

$$\nabla^4 \omega + \frac{12\ddot{\omega}}{h^2 \varphi^2} = 0$$

with $$\omega = A_1\left[\sin\frac{2\pi x}{a}\cdot\sin\frac{\pi y}{a} + \sin\frac{2\pi y}{a}\cdot\sin\frac{\pi x}{a}\right]\cos\omega_2 t$$ ... (23)

in the form $$T = \frac{2\pi a^2\sqrt{12}}{\pi^2 h \varphi\cdot 7}$$ ... (24)

Hence $$\frac{T'}{T} = \frac{\frac{2K}{\pi}}{\left[1+\frac{75}{49}\cdot\frac{A_1^2}{h^2}\right]^{1/2}} = \frac{\frac{2K}{\pi}}{\left[1+\frac{75}{49}\cdot\beta_1^2\right]^{1/2}}$$ ... (25)

where $\beta_1 = \dfrac{A_1}{h}$ ... (26)

The ratio T'/T given by (25) is plotted against various values of β_1 in (26). The graph is shown in the figure No.8

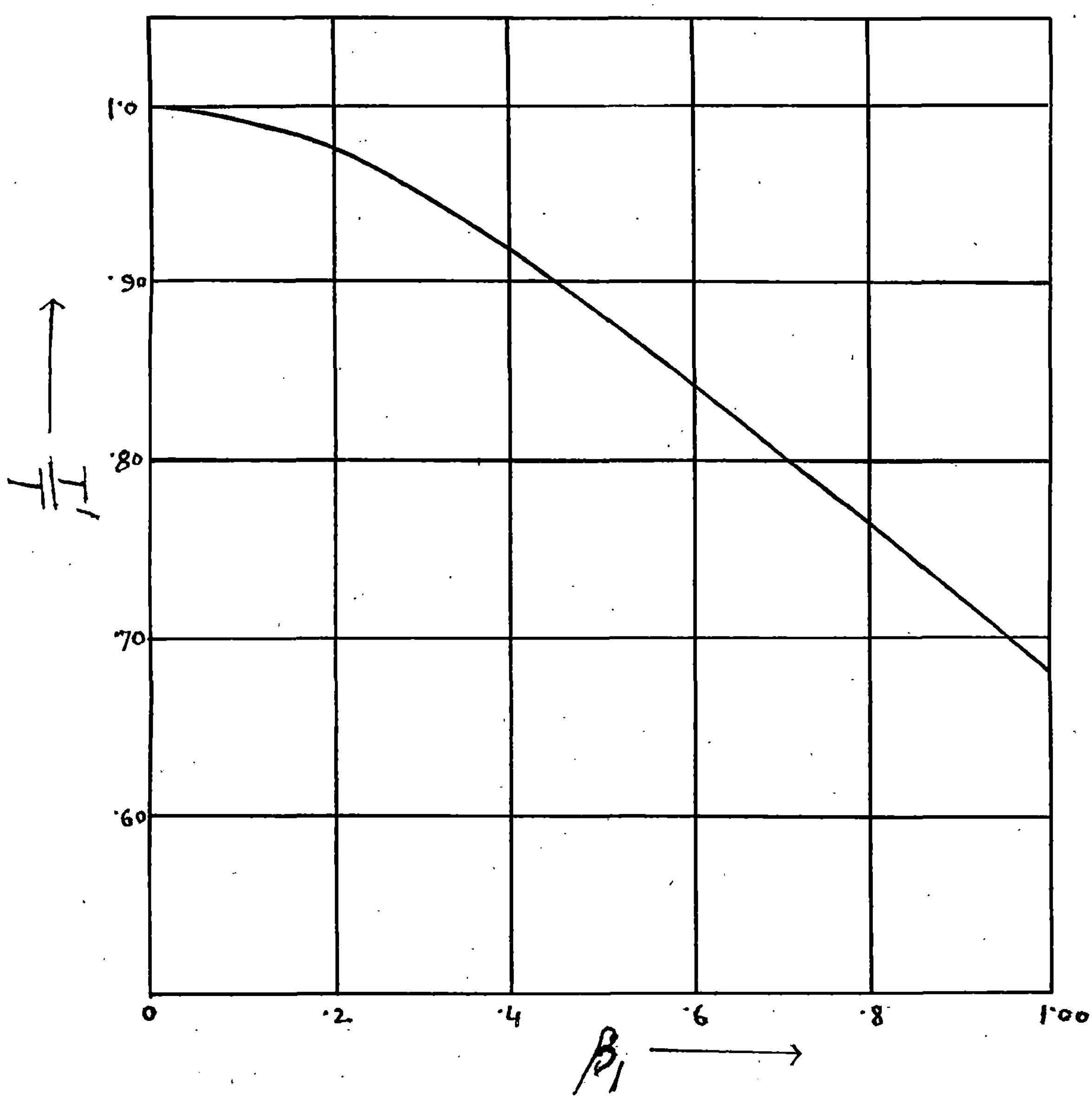

FIG. 8

Graph showing $\frac{T'}{T}$ against β_1

2. Large amplitude free-vibrations of elliptic plates.*

Let us consider the free vibration of an elliptic plate having its boundary elastically restrained against rotation.

For this case, let us assume w in the form

$$w = W(x, y) \cdot F(t) \qquad \ldots(27)$$

Substituting equation (27) in (6) we have

$$\nabla^4 W \cdot F(t) - \alpha^2 F^3(t)\, \nabla^2 W + \frac{12}{h^2 c_p^2} \cdot \frac{d^2 F}{dt^2} \cdot W = 0 \qquad \ldots(28)$$

where $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y}\right)$ and $F^2(t) = f(t)$

A solution of equation (28) is possible if

$$\frac{\nabla^4 W}{W} = K^4, \quad \frac{\nabla^2 W}{W} = -K^2 \qquad \ldots(29)$$

From (29)

$$\nabla^2 W + K^2 W = 0 \qquad \ldots(30)$$

Changing into elliptic co-ordinates (ξ, η), we have

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) W + 2p^2 \left(\cosh 2\xi - \cos 2\eta\right) W = 0 \qquad \ldots(31)$$

where $p = \frac{Kd}{2}$, $2d$ being the interfocal distance of the ellipse.

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Solution of (31) can be written as

$$W = \sum_{m=0}^{\infty} \bar{C}_{2m}\, Ce_{2m}(\xi, q)\, ce_{2m}(\eta, q) \qquad \dots (32)$$

where $Ce_{2m}(\xi, q)$ and $ce_{2m}(\eta, q)$ are the Modified Mathieu function and Mathieu function of the first kind of order $2m$ and $q = p^2 = \frac{k^2 d^2}{4}$.

While solving a problem of bending of a plate with elliptic hole, instead of taking Mathieu functions of all orders, taking a single Mathieu function of 2nd order, Naghdi (1955) has shown that the results obtained are satisfactory for larger elliptic hole. In our present problem also we can make similar approximation by taking a single Mathieu function of Zero order.

Hence (32) reduces to

$$W = c_1 Ce_0(\xi, q)\, ce_0(\eta, q) \qquad \dots (33)$$

Combining equations (28) and (29) we have the following differential equation for determining $F(t)$.

$$\frac{d^2F}{dt^2} + \frac{h^2 c_p^2 k^4}{12} F(t) + \frac{\alpha^2 h^2 c_p^2 \cdot k^2}{12} F^3(t) = 0 \qquad \dots (34)$$

The equation is of the form

$$\ddot{F} + \lambda_1 F + \mu F^3 = 0 \qquad \dots (35)$$

which is to be solved subject to initial conditions

$$F(0) = 1\ , \quad \dot{F}(0) = 0 \qquad \dots (36)$$

Solution of (35) can be put in the form

$$F(t) = C_\eta(\omega_3 t, \lambda_2) \qquad \ldots (37)$$

where
$$\left.\begin{aligned}(\omega_3)^2 &= \frac{h^2 c_\rho^2 \cdot K^4}{12}\left(1+\frac{\alpha^2}{K^2}\right)\\ \lambda_2^2 &= \frac{1}{2\left(1+\frac{K^2}{\alpha^2}\right)}\end{aligned}\right\} \qquad \ldots (38)$$

Here ω_3 and λ_2 are positive contants and C_η is Jacobi's elliptic function.

The period T_1 is given by $T_1 = \dfrac{4K}{\omega_3}$ ... (39)

K being the complete elliptic integral of the first kind.

Hence $\omega = c_1 Ce_0(\xi, q) ce_0(\eta, q) C_\eta(\omega_3 t, \lambda_2)$ ... (40) is known.

For ω to vanish on the boundary $\xi = \xi_0$

q must be root of $Ce_0(\xi_0, q) = 0$ ... (41)

To determine α, we know that

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial \omega}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \omega}{\partial y}\right)^2 = \frac{\alpha^2 h^2}{12} f(t)$$

Changing into elliptic co-ordinates, the above equation reduces to

$$h_1 h_2\left[\frac{\partial}{\partial \xi}\left(\frac{u_\xi}{h_2}\right) + \frac{\partial}{\partial \eta}\left(\frac{u_\eta}{h_1}\right)\right] + \frac{1}{2} h_1 h_2\left[\left(\frac{\partial \omega}{\partial \xi}\right)^2 + \left(\frac{\partial \omega}{\partial \eta}\right)^2\right]$$

$$= \frac{\alpha^2 h^2}{12} f(t) \qquad \ldots (42)$$

where

$$h_1 = h_2 = \frac{1}{d\sqrt{\sinh^2\xi + \sin^2\eta}}$$

Boundary conditions for u_ξ and u_η are

$$u_\xi = u_\eta = 0 \quad \text{at } \xi = \xi_0 \qquad \dots (43)$$

Let us assume that

$$\left.\begin{aligned} u_\xi &= \sum_{n=0}^{\infty} P(\xi) \cos 2n\eta \, F^2(t) \\ u_\eta &= \sum_{n=1}^{\infty} G(\xi) \sin 2n\eta \, F^2(t) \end{aligned}\right\} \qquad \dots (44)$$

Subject to the conditions $P(\xi_0) = G(\xi_0) = 0$

Combining equations (37), (40), (42) and (44)

and integrating equation (42) over the surface of the plate we have

$$\int_0^{2\pi}\int_0^{\xi_0} \left[\left(\frac{\partial \omega}{\partial \xi}\right)^2 + \left(\frac{\partial \omega}{\partial \eta}\right)^2\right] d\xi \, d\eta$$

$$= \frac{\alpha^2 h^2 d^2}{6} \int_0^{2\pi}\int_0^{\xi_0} \left(\sinh^2\xi + \sin^2\eta\right) d\xi \, d\eta \qquad \dots (45)$$

After evaluating the integrals we get the following equation to determine α;

$$C_1^2\Bigg[\sum_{r=1}^{\infty}\left\{A_{2r}^{(0)}\right\}^2 r\,\mathrm{Sinh}\,4r\xi_0+\sum_{r=1}^{\infty}\sum_{\substack{s=1\\ r\neq s}}^{\infty}\frac{A_{2r}^{(0)}.\,A_{2s}^{(0)}.\,4rs}{r^2-s^2},$$

$$\times\left\{r\,\mathrm{Sinh}\,2r\xi_0\,\mathrm{Cosh}\,2s\xi_0-s\,\mathrm{Sinh}\,2s\xi_0\,\mathrm{Cosh}\,2r\xi_0\right\}$$

$$+\sum_{r=1}^{\infty}4r^2\left(A_{2r}^{(0)}\right)^2\Bigg\{\sum_{r=1}^{\infty}\left(A_{2r}^{(0)}\right)^2\left(\xi_0+\frac{\mathrm{Sinh}\,4r\xi_0}{4r}\right)$$

$$+\;2A_0^{(0)}\sum_{r=1}^{\infty}\frac{A_{2r}^{(0)}\,\mathrm{Sinh}\,2r\xi_0}{r}+\sum_{r=1}^{\infty}\sum_{\substack{s=1\\ r\neq s}}^{\infty}\frac{A_{2r}^{(0)}.\,A_{2s}^{(0)}}{r^2-s^2}$$

$$\times\left(r\,\mathrm{Sinh}\,2s\xi_0\,\mathrm{Cosh}\,2r\xi_0-s\,\mathrm{Sinh}\,2r\xi_0\,\mathrm{Cosh}\,2s\xi_0\right)\Bigg\}\Bigg]$$

$$=\frac{\alpha^2h^2d^2\,\mathrm{Sinh}\,2\xi_0}{12}$$

OR,

$$C_1^2F_1(\xi_0)=\frac{\alpha^2h^2d^2\,\mathrm{Sinh}\,2\xi_0}{12} \qquad \ldots(46)$$

where $A_{2r}^{(0)}$ **are the Fourier coefficients in the expansion of** $ce_0(\xi,q)$

Considering equations (38), (39) and (46) we get

$$T_1 = \frac{4K\sqrt{12}}{h c_p K^2\left(1+\frac{12C_1^2 F_1(\xi_0)}{K^2 h^2 d^2 \sinh 2\xi_0}\right)^{1/2}} \qquad \text{...(47)}$$

The usual linear period is given by

$$T_2 = \frac{2\pi}{\omega_4} \qquad \text{...(48)}$$

ω_4 being found out from the following equation

$$\nabla^4 w + \frac{12}{h^2 c_p^2}\ddot{w} = 0 \qquad \text{...(49)}$$

in the form

$$\omega_4^2 = \frac{K^4 h^2 c_p^2}{12} \qquad \text{...(50)}$$

Hence

$$\frac{T_1}{T_2} = \frac{\frac{2K}{\pi}}{\left(1+\frac{12C_1^2 F_1(\xi_0)}{K^2 h^2 d^2 \sinh 2\xi_0}\right)^{1/2}} \qquad \text{...(51)}$$

If $d \to 0$, $\xi \to \infty$ the ellipse degenerates to a circle of radius R (Say). In that case

$$Ce_0(\xi, q) \longrightarrow P_0' J_0(kr)$$

where $P_0' = \dfrac{ce_0(0,q)\, ce_0(\pi/2,q)}{A_0(0)}$

and $ce_0(\eta, q) \longrightarrow A_0(0) \longrightarrow \frac{1}{\sqrt{2}}$

Hence in the limiting case, equation (40) reduces to

$$W = C_1 P_0' J_0(kr) \frac{1}{\sqrt{2}} C_n(\omega^{*}t, \lambda)$$

$$= A J_0(kr) C_n(\omega^{*}t, \lambda) \qquad \text{...(52)}$$

Also equation (45) can be written as

$$C_1^2 \left[\left\{ ce_0'(\xi, q)\, ce_0(\xi, q) \right\}_0^{\xi_0} + 2q \int_0^{\xi_0} c^2e_0(\xi, q) \left\{ \cosh 2\xi - \theta_0 \right\} d\xi \right]$$

$$= \frac{\alpha^2 h^2 d^2 \sinh 2\xi_0}{12} \qquad \text{...(53)}$$

where $\theta_0 = A_0^{(0)} \cdot A_2^{(0)} + \sum_{r=0}^{\infty} A_{2r}^{(0)} \cdot A_{2r+2}^{(0)}$

Since $ce_0(\xi_0, q) = 0;\ d^2 \sinh 2\xi_0 \to 2R^2,\ \theta_0 \to 0,\ A_{2r}^{(0)} \to 0$

and $\cosh 2\xi\, d\xi \to 2r\,dr/d^2$ as $\xi \to \infty,\ d \to 0,$

equation (53) reduces to, in the limiting case,

$$C_1^2 \left[K^2 P_0'^2 \int_0^R r J_0^2(kr)\, dr \right] = \frac{\alpha^2 h^2 R^2}{6}$$

OR, $$\frac{\alpha^2 h^2}{6} = A^2 K^2 J_1^2(KR) \qquad \text{...(54)}$$

Since $J_0(KR) = 0$

Hence

$$T_1/T_2 = \frac{2K/\pi}{\left(1+6A^2J_1^2(KR)/h^2\right)^{1/2}} \qquad \ldots(55)$$

Equations(52), (54) and (55) are the corresponding results for a circular plate as obtained by Willam A. Nash and James R. Modeer (1960).

Numerical results :

Since $ce_0(\xi_0, q) = 0$, $q = 6{\cdot}4$

when $\xi_0 = 3{\cdot}12$ and $d = 2{\cdot}11$

Let $\alpha = 2\sqrt{2}$

Putting all these values in (46)

$$\frac{c_1}{h} = {\cdot}74$$

Corresponding T_1/T_2 is found from equation (55) in the form

$$\frac{T_1}{T_2} = {\cdot}97$$

3. Large amplitude free-vibrations of semi-circular plates.

Let us consider a plate in the form of a semi - circle having its boundary elastically restrained against rotation. Let us take the centre as pole and the bounding diameter as initial line.

For the above case, let us assume w in the form

$$w = W(r,\theta) \cdot F(t) \qquad \ldots (56)$$

Substituting (56) in (6) we have

$$\nabla^4 W \cdot F(t) - \alpha^2 F^3(t) \cdot \nabla^2 W + \frac{12}{h^2 c_p^2} \frac{d^2 F}{dt^2} \cdot W = 0 \qquad \ldots (57)$$

where $f(t) = F^2(t)$ and $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

A solution of (57) is possible if

$$\frac{\nabla^4 W}{W} = K^4 \quad \text{and} \quad \frac{\nabla^2 W}{W} = -K^2 \qquad \ldots (58)$$

From (58) $\qquad \nabla^2 W + K^2 W = 0 \qquad \ldots (59)$

To solve (59) let us put

$$W = \sum_{m=1,3,\cdots}^{\infty} R_m \sin m\theta \qquad \ldots (60)$$

R_m being the function of r only.

Substituting (60) in (59) and solving we get

$$R_m = A_m J_m(Kr) \qquad \ldots (61)$$

where A_m is a constant and J_m is the Bessel function of order m.

From (57) equation to determine $F(t)$ reduces to

$$\frac{d^2F}{dt^2} + \frac{h^2 c_p^2 k^4}{12} F + \frac{\alpha^2 h^2 c_p^2 k^2}{12} F^3 = 0 \qquad \ldots(62)$$

Solution of (62) is given in the form as in the previous case,

$$F(t) = C_n(\omega_s t, \lambda) \qquad \ldots(63)$$

where

$$\left.\begin{aligned} \omega_s^2 &= \frac{\alpha^2 c_p^2 k^4}{12}\left(1+\frac{\alpha^2}{k^2}\right) \\ \lambda^2 &= \frac{1}{2\left(1+\frac{k^2}{\alpha^2}\right)} \end{aligned}\right\} \qquad \ldots(64)$$

Thus

$$w = \sum_{m=1,3,\ldots}^{\infty} A_m J_m(kr) \sin m\theta \,.\, C_n(\omega_s t, \lambda) \qquad \ldots(65)$$

is known.

For the fundamental mode,

$$w = A_1 J_1(kr) \sin\theta \, C_n(\omega_s t, \lambda) \qquad \ldots(66)$$

w to vanish on the boundary $r = a$

K must be root of $J_1(Ka) = 0$ $\qquad \ldots(67)$

To determine α let us consider the equation (7) which in polar co-ordinates takes the form

$$\frac{\alpha^2 h^2}{12} f(t)$$

$$= \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \quad \ldots (68)$$

Let us assume

$$\left.\begin{aligned} u &= \sum U(r) \cos m\theta . F^2(t) \\ v &= \sum V(r) \sin m\theta . F^2(t) \end{aligned}\right\} \quad \ldots (69)$$

Combining equations (68), (69), (65) and (63), multiplying (68) by $r\,dr\,d\theta$ and integrating w.r.t. r between the limits 0 to a and w.r.t. θ between the limits 0 to π we get the following equation to determine α

$$A_m^2 \left[-\frac{k^2 a^2}{2} J_m^2(ka) + m J_m^2(ka) + \frac{1}{2} k^2 a^2 J_{m+1}^2(ka) + k(m+1) J_m(ka) . J_{m+1}(ka) \right]$$

$$= \frac{\alpha^2 h^2 a^2}{6} \quad \ldots (70)$$

For fundamental mode of vibration,

$$\alpha^2 = \frac{3k^2 J_2^2(ka) . A_1^2}{h^2} \quad \ldots (71)$$

using $J_1(ka) = 0$

Considering equations (63), (64) and (71) we have

$$T_3 = \frac{4K\sqrt{12}}{h c_0 K^2 \left(1 + \frac{3 J_2^2(ka) A_1^2}{h^2}\right)^{1/2}} \qquad \ldots(72)$$

As in the previous case,

$$T_4 = \frac{2\pi}{\omega_6}\,, \quad \text{where} \quad \omega_6^2 = \frac{K^4 h^2 c_0^2}{12} \qquad \ldots(73)$$

Hence

$$T_3/T_4 = \frac{\frac{2K}{\pi}}{\left(1 + \frac{3 A_1^2 J_2^2(ka)}{h^2}\right)^{1/2}} \qquad \ldots(74)$$

<u>**Numerical calculation.**</u>

Since $J_1(ka) = 0$, $Ka = 3{\cdot}81$

From table $J_2(ka) = J_2(3{\cdot}81) = {\cdot}409$

Let $a = 10$, $\alpha = {\cdot}1$

Putting all these values in (71)

$$\frac{A_1}{h} = {\cdot}58$$

Corresponding T_3/T_4 is found from (74) in the form

$$\frac{T_3}{T_4} = {\cdot}71$$