CHAPTER - IV

"A Simplified Method for Solving Non-linear Problems using "Constant Deflection Contour" Method.

As stated earlier the method of " Lines of Equal Deflection" is one of the existing methods applied in studying the non-linear behaviour of structures subject to moderately large vibrations. With reference to the idea expressed by Banerjee and Rogerson [122] equations (3.11) and (3.12) or equations (3.12) and (3.14) may be applied to study the vibration analysis of structures. However the present investigator has some reservations in accepting the free hand use of any one and the present investigator has to add that the first choice of using equations (3.11) and (3.12), though simplifies the mathematical computations, in the sense that it involves third order ordinary differential equations, may not yield the diserable and accurate result in comparison with the second set of equations (3.12) and (3.13). In the foregoing chapters both the set of governing equations will be used for the analysis and a comparative study will be made thereafter.

In the present chapter all the problems considered here will involve the first set of governing equations viz. equations (3.11) and (3.12).

PROBLEM 4.1

Non - linear Vibrations of Rigid Elliptic Plate With Uniform Thickness

Let us consider a problem in establishing the applicability of this method to one of the useful structures, such as " an elliptic plate with uniform thickness vibrating at large amplitudes."

As usual the dynamic Von-Karman equations for a plate subjected to a normal uniform load may be put in the following form [vide equation (3.1) and (3.2)]

$$D\nabla^4 w = h\mathcal{L}(F,w) + p - \rho h w_{,tt}$$

$$\nabla^4 F = -\frac{E}{2}\mathcal{L}(w,w) \qquad \text{where } D = \frac{Eh^3}{12(1-\nu^2)}$$

With in-plane inertial effect ignored, where w is the deflection function, F is the stress function, p is the load, h is the plate thickness, ρ is the mass density. D is the flexural rigidity, E is the plate modulus of elasticity. ν is the Poison's ratio.

For an elliptic plate clamped along the edges, the family of contour lines of the deflected surface may be represented by

$$u = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \qquad [4.1.1]$$

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Where u = 0 defines the boundary and u = 1 defines the centre of plate, the boundary conditions imposed are
w = 0 at u = 0

$$\text{and } \frac{dw}{du} = 0, \text{ at } u = 0, 1 \qquad [4.1.2]$$

Then performing the integrations of equations(3·9a) and (3·9b), using equation (4.11) one may arrive at the following equations after a lengthy calculations (for brevity the trivial but lengthy calculations being ommited)

$$\frac{2D(3a^4+3b^4+2a^2b^2)}{a^4b^4}\left[(1-u)^2\frac{d^3w}{du^3} - 2(1-u)\frac{d^2w}{du^2}\right]$$

$$+\frac{8h}{a^2b^2}(1-u)\frac{dF}{du}\frac{dw}{du} + p(1-u) + \rho h\int_1^u w_{tt}\,du = 0 \qquad [4.1.3]$$

$$\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}\left[(1-u)^2\frac{d^3F}{du^3} - 2(1-u)\frac{d^2F}{du^2}\right]$$

$$= \frac{2E}{a^2b^2}(1-u)\left(\frac{dw}{du}\right)^2 \qquad [4.1.4]$$

Nowinski [121] has shown that when a plate vibrates principally in the transverse directions and in-plane movements are restricted then without only loss of generality the spacial part of the deflection as well as of the stress function may be considered as the same

$$w = \sum_{i=2}^{\alpha} A_i u^i \psi(t) \cong Au^2\psi(t)$$

$$F = Au^2\phi(t) \qquad [4.1.5]$$

A is a constant, $\Phi(t)$ and $\psi(t)$ are unknown functions of time.

Since equation (4.1.5) does not represent the exact solution, Galerkin procedure may be applied to minimize the error. Substitution of equation (4.1.5) in equation (4.1.4) and performing the required integration a relation between $\Phi(t)$ and $\psi(t)$ is first established

$$\phi(t) = -\frac{6}{5}\,\frac{a^2 b^2 AE}{(3a^4+3b^4+2a^2b^2)}\,\psi^2(t) \qquad [4.1.6]$$

while equation (4.1.3) will then reduce to

$$\frac{2}{3} D \frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4} A\psi(t) + 1{\cdot}28 \frac{Eh}{(3a^4+3b^4+2a^2b^2)} A^3\psi^3(t)$$

$$+\frac{1}{18}\rho h A \psi_{,tt}(t) = \frac{p}{12} \qquad [4.1.7]$$

Equation (4.1.7) may be put in a simplified form

$$\psi_{,tt}(t) + B_1\psi(t) + B_3\psi^3(t) = Cp \qquad [4.1.8]$$

where $B_1 = \left(\dfrac{12D}{\rho h}\right)\left(\dfrac{3a^4+3b^4+2a^2b^2}{a^4b^4}\right)$

$$B_3 = 23{\cdot}04\,\frac{EA^2}{\rho(3a^4+3b^4+2a^2b^2)}$$

$$C = \frac{3}{2\rho h A}$$

4.1. a) ***Free Linear Vibration :***

For free vibration p = o and equation (4.8) becomes.

$$\psi_{,tt}(t) + B_1\psi(t) + B_3\psi^3(t) = 0 \qquad [4.1.9]$$

For free linear vibration (with $B_3 = 0$) equation (4.1.9) reduces to

$$\psi_{,tt}(t) + B_1\psi(t) = 0$$

The linear frequency parameter is given by

$$B_1^{1/2} = \left[\left(\frac{12D}{\rho h}\right)\left(\frac{3a^4+3b^4+2a^2b^2}{a^4b^4}\right)\right]^{1/2}$$

$$\text{or,}\quad B_1^{1/2} = \left[\left(\frac{12D}{\rho h a^4}\right)\left(3m^4+2m^2+3\right)\right]^{1/2}$$

where $m = \frac{a}{b}$

Putting m = 1 we get linear frequency for a circular plate as $9.797\sqrt{\frac{D}{\rho h a^4}}$

4.1.b) ***Free Non-Linear Vibration :***

If T and T* be the corresponding time periods of linear and non-linear oscillations then, the ratio as

$$\frac{T^*}{T} = \left[1+\frac{3}{4}\frac{B_3}{B_1}\right]^{-1/2} \qquad [4.1.10]$$

$$\text{where } \frac{B_3}{B_1} = \frac{23.04\,(1-\nu^2)\,m^4}{(3m^4+2m^2+3)^2}\left(\frac{A_0}{h}\right)^2 \qquad [4.1.11]$$

$\xi = \frac{A_o}{\lambda}$ represents the relative amplitude

Numerical results have been computed and shown in table [1&2]

4.1.c) ***Static Case :***

Neglecting the inertial terms in equation (4.1.8) one gets, for analyzing the large deflection behaviour

$$B_1 \psi(t) + B_3 \psi^3(t) = C_p \qquad [4.1.12]$$

On further simplification one gets the relation between the non-dimensional central defelction $\left(\frac{W_0}{h}\right)$ and the load parameter $\left[\frac{pa^4}{Eh^4}\right]$ as

$$\frac{2}{3}\frac{(3m^4+2m^2+3)}{(1-\nu^2)}\left(\frac{W_0}{h}\right) + 15.36\frac{m^4}{(3m^4+2m^2+3)}\left(\frac{W_0}{h}\right)^3$$

$$= \frac{pa^4}{Eh^4} \qquad [4.1.13]$$

Numerical results have been shown in tables [4&5]

For circular plate (m=1) and for $\upsilon = 0.3$ equation (4.13) reduces to

$$5.8608\left(\frac{W_0}{h}\right) + 1.92\left(\frac{W_0}{h}\right)^3 = \frac{pa^4}{Eh^4} \qquad [4.1.14]$$

while $5.848\left(\frac{W_0}{h}\right) + 2.754\left(\frac{W_0}{h}\right)^3 = \frac{pa^4}{Eh^4}$ [22]

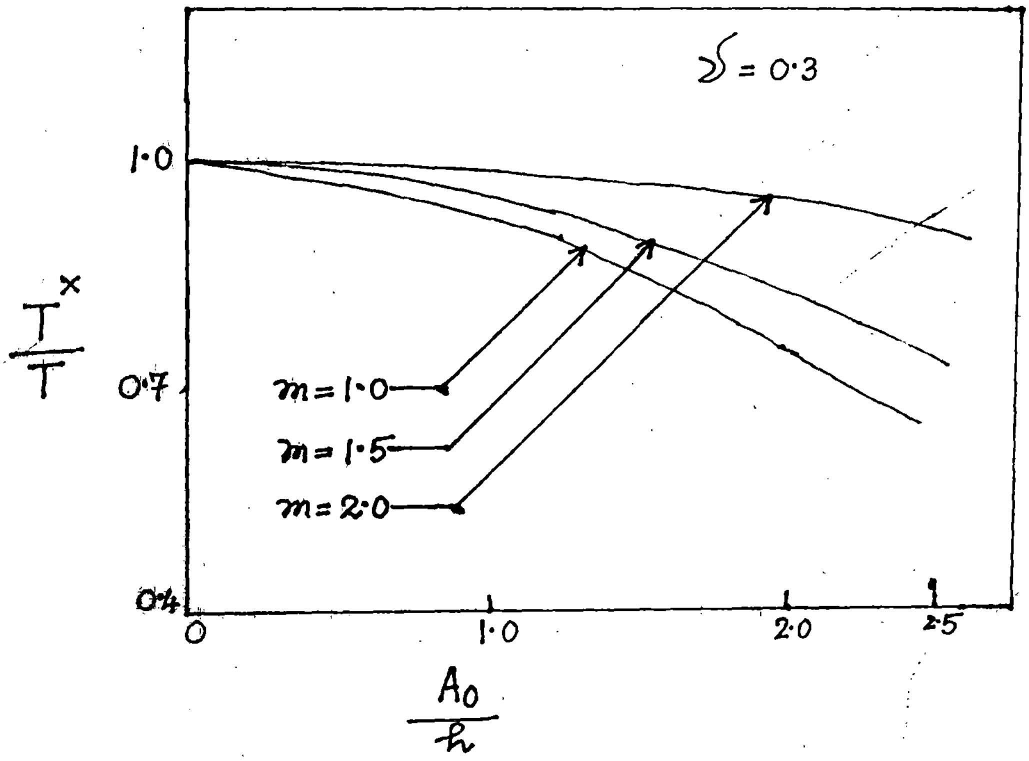

Figure I : Time Period Ratio against Relative Amplitude for Elliptic Plates for Different Values of 'm'.

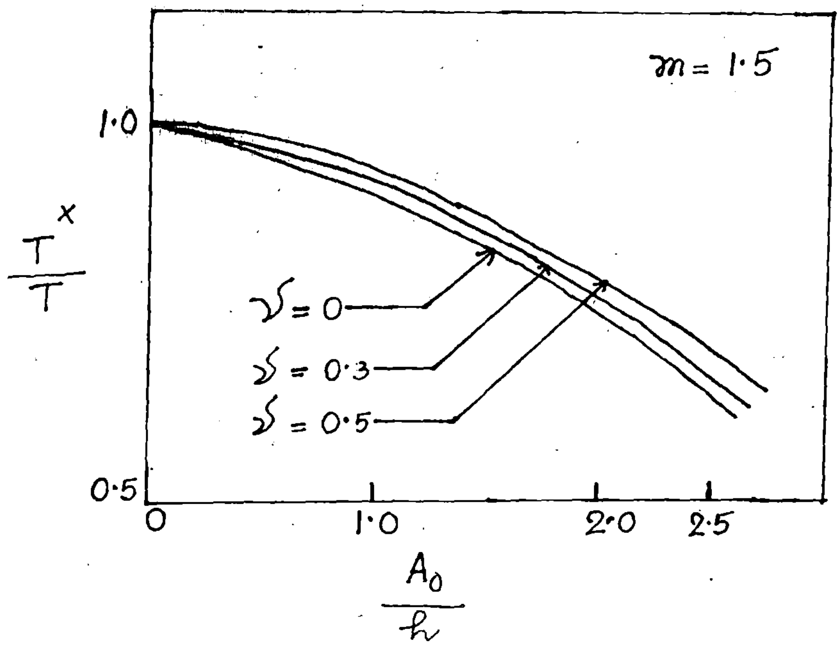

Figure II : Loa Time Period Ratio against Relative Amplitude $\frac{A_0}{h}$ for an Elliptic Plate for Different Values of Poisson's Ratio.

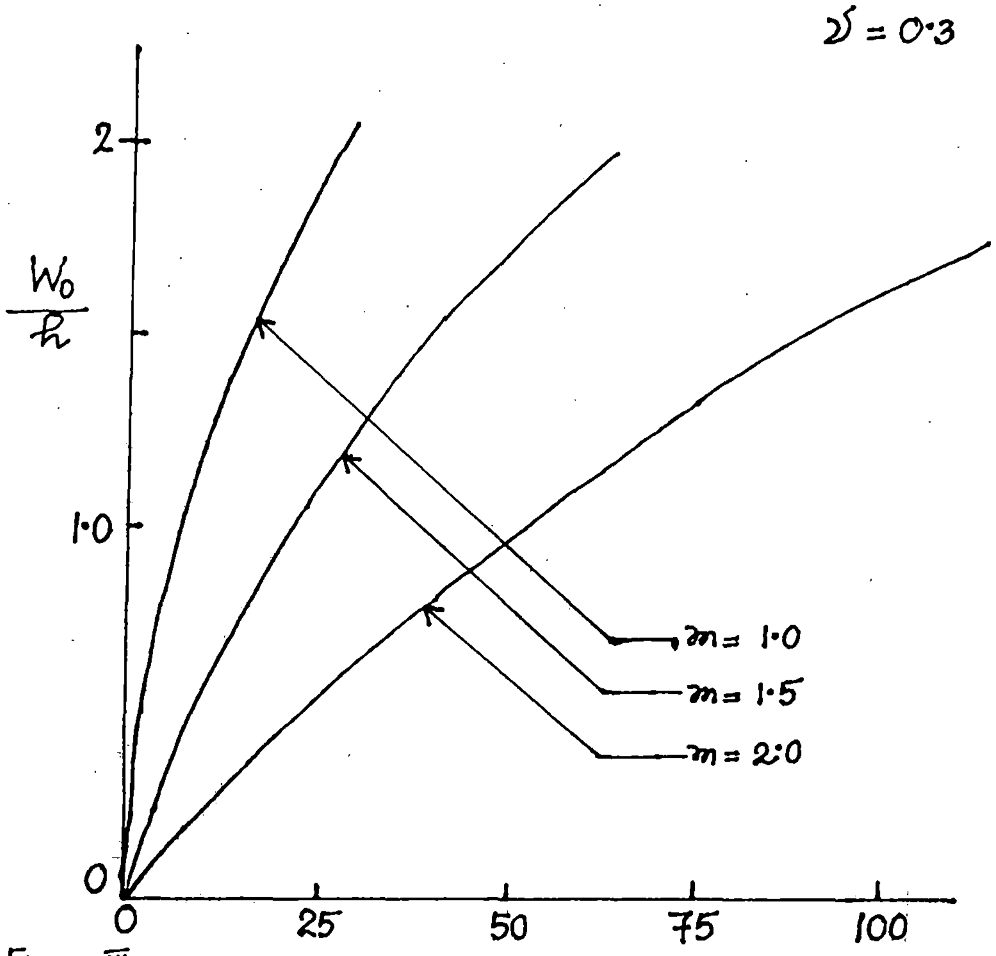

Figure III

$\frac{pa^4}{Eh^4}$

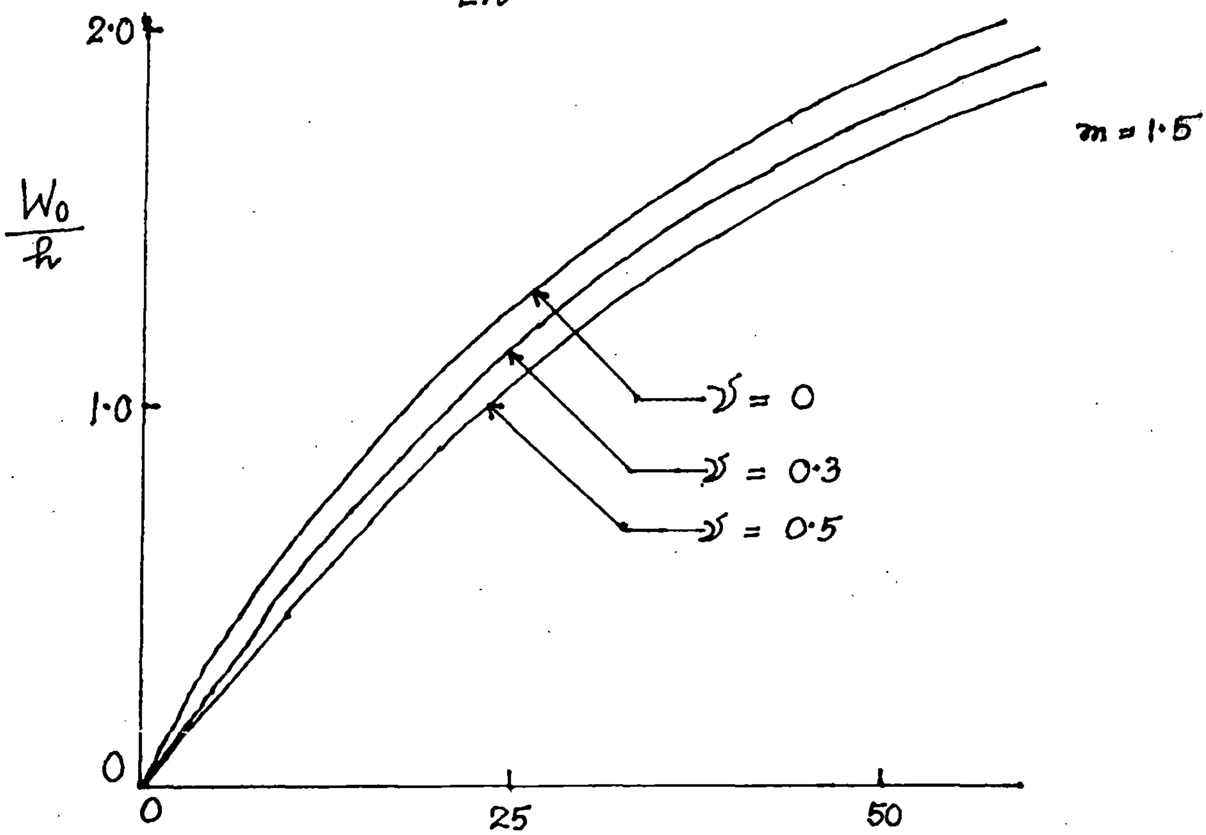

Figure IV

$\frac{pa^4}{Eh^4}$

Figure III & IV : Non-linear Static Behaviour of Elliptic Plates.

Numerical Results :

Table [1]: Dependence of the relative time period of nonlinear and linear vibrations ($T^*/_T$) on relative amplitude ($\frac{A_0}{h}$) for different values of m = ($a/_h$), υ = 0.3

$\frac{A_0}{h}$	$T^*/_T$			
	m = 1	m = 1.5	m = 2	m = 2.5
0	1.000	1.000	1.000	1.000
0.5	0.9705	0.9811	0.9911	0.9956
1.0	0.8946	0.9300	0.9656	0.9829
1.5	0.7989	0.8593	0.9296	0.9627
2.0	0.7045	0.7822	0.8795	0.9362
2.5	0.6209	0.7076	0.8276	0.9050

Table [2]: Dependence of relative time period (T*/T) on $\left(\frac{A_0}{h}\right)$ for different values of ν, m = 2.0

$\frac{A_0}{h}$	$T^*/_T$			
	υ= 0.2	υ = 0.3	υ = 0.4	υ = 0.5
0	1.000	1.000	1.000	1.000
0.5	0.9905	0.9911	0.9917	0.9926
1.0	0.9636	0.9656	0.9680	0.9713
1.5	0.9230	0.9269	0.9318	0.9385
2.0	0.8734	0.8795	0.8870	0.8973
2.5	0.8197	0.8276	0.8375	0.8513

Table [3]: Comparative study of relative time periods of non-linear and linear vibration [$T^*/_T$] versus [$\frac{A_0}{\sqrt{\mu_1}}$] where μ_1 = 12 ($3m^4 + 2m^2 + 3$) for a circular plate as obtained in the present study and the results give by Sinharay and Banerjee [124], υ = 0.3, m = 1

$\frac{A_0}{\sqrt{\mu_1}}$	$T^*/_T$	
	Present Study	[124]
0	1.00	1.000
0.02	0.9937	0.992
0.04	0.9757	0.970
0.06	0.9477	0.936
0.08	0.912	0.894
0.10	0.8722	0.848
0.12	0.8296	0.7997
0.14	0.7865	0.752
0.16	0.7443	0.707

Table [4] : Dependence of Central deflection $\frac{W_0}{h}$ on load parameter $\frac{pa^4}{Eh^4}$ for different values of m and $\upsilon = 0.3$

$\frac{W_0}{h}$	pa^4/Eh^4			
	m = 1	m = 1.5	m = 2	m = 2.5
0	0	0	0	0
0.2	1.187	3.3513	8.6772	19.476
0.4	2.446	6.866	17.5542	39.168
0.6	3.93	10.7107	26.8305	58.608
0.8	5.67	15.047	36.7059	80.064
1.0	7.78	20.040	47.385	101.722
1.4	13.47	32.652	71.923	148.416
1.6	17.24	40.6003	86.1913	173.952
2.0	27.08	60.600	119.7200	230.400
2.4	40.606	87.166	161.2358	295.488

Table [5] : Dependence of central deflection W_0/h on load parameter pa^4/Eh^4 for different values of υ, m = 1.5

$\frac{W_0}{h}$	pa^4/Eh^4		
	$\upsilon = 0.2$	$\upsilon = 0.3$	$\upsilon = 0.5$
0	0	0	0
0.2	3.1784	3.3513	4.0598
0.4	6.3240	6.866	8.2867
0.6	10.1934	10.7107	12.8476
0.8	14.3589	15.047	17.9097
1.0	19.1826	20.04	23.6400
1.2	24.8287	25.8537	30.2054
1.4	31.4619	32.652	37.7731
1.6	39.2469	40.6003	46.5100
2.0	58.2926	60.600	67.8400
2.4	85.1927	87.1660	96.4915

Discussions :

Tables (1,2,3) and figures (I,II) show the dependence of the ratio of non-linear to linear time periods T^*/T on the relative amplitude A_0/h for elliptic plates. It may be observed in table (1) and figure (I) that there is hardly any effect of non linearity so far as the free oscillations of elliptic plates having higher eccentricity is concerned. However, in case of circular plate the non-linear effect is notable. In table (3) it may also be noted that the results of the present study for circular plates are in excellent agreement with those obtained by Sinharay and Banerjee [124]

Tables [4& 5] and figures (III & IV) show the variation of central deflection (W_0/h) for different values of m and υ. Table (4) and figure (III) shows that for a particular value of central deflection, the value of load parameter (pa^4/Eh^4) increases with the increase of the vlaues of m. If implies that to obtain a particular central deflection, more load is needed for an elliptic plate than for a circular plate. Again table (5) shows that for a particular value of load parameter, the central deflection for the plate having higher Possion ratio are smaller than that for the plate meterial having lower Poisson ratio.

From equation [4.1.4], the results for a rigid circular plate do not tally with those of Yamaki[22]. The reason for this may be due to the procedural difference. Also the assumption of retaining the same spacial part for the deflection function and stress function may not be valid for the present case . Also as indicated in the beginning of this chpater the use of equation (3.11) and (3.12) for simplification appears to be unjustified. The use of equation (3.12) and (3.13) will be made later on to justify the above agreement and the results become more accurate.

Problem - 4.2

Non-linear Vibrations of Plates on Elastic Foundation

With the increasing demands for improved efficiency in material usage in structures and the emphasis on high strength/weight ratio, the geometric non-linear behaviour of plates has become more significant. The majority of analysis into the non-linear deflection of plates on elastic foundation has been restricted to the determination of static deflection for an elastic foundation with a Winkler deflection characteristic at the plate/foundation interface. Non-linear static or dynamic analysis of plates under viscous damping and placed on an elastic foundation of Pasternak model is presented.

The 'Constant Deflection Contour' method will be used here in support of its application to a little more complicated problem for which the governing differential equation are of Von Karman type extended to a dynamic case including the effect of elastic foundation and vicous damping. As it has already been stated that this method can easily be applied to investigate large amplitude behaviour of vibrating plates having uncommon or complicated boundary.

The dynamic Von Karman equations for plate placed in an elastic foundation of the Pasternak model and subjected to a normal uniform load may be put in following form [125].

$$D\nabla^4 w - h\mathcal{L}(F,w) - p + \rho h w_{,tt} + \rho h K_v w_{,t} + Kw - G\nabla^2 w = 0 \qquad [4.2.1]$$

$$\nabla^4 F = -\frac{E}{2}\mathcal{L}(w,w) \qquad [4.2.2]$$

with in-plane inertia effect ignored; where, D is the flexural rigidity $= \frac{Eh^3}{12(1-\nu^2)}$

p is the applied load, E is the plate modulus of elasticity, ($Kw - G\nabla^2 w$) is the linear foundation reaction for Pasternak model, K and G are foundation constants, w is the vertical deflection, ρ is the density of the plate material, υ is the Poisson's ratio, K_v is the viscous damping constant.

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It is assumed that the set of Equations (4.2.1) and (4.2.2) satisfied in every region bounded by the contour line C_u. Using the transformations in Chapter III and integrating over the region equations (4.2.1) and (4.2.2) will reduce to

$$\iint_{\Omega}\left[A_1\left(\frac{d^4w}{du^4}\right)+A_2\left(\frac{d^3w}{du^3}\right)+A_3\left(\frac{d^2w}{du^2}\right)+A_4\left(\frac{dw}{du}\right)\right]d\Omega$$

$$=\iint_{\Omega}\left[A_5\left(\frac{dw}{du}\right)\left(\frac{dF}{du}\right)+A_6\frac{d}{du}\left(\frac{dw}{du}\frac{dF}{du}\right)+\left(p-\rho h w_{,tt}-\rho h K_v w_{,t}\right)\right.$$

$$\left.-Kw+A_7\left(\frac{dw}{du}\right)+A_8\left(\frac{d^2w}{du^2}\right)\right]d\Omega \qquad [4.2.3]$$

$$\iint_{\Omega}\left[A_1\left(\frac{d^4F}{du^4}\right)+A_2\left(\frac{d^3F}{du^3}\right)+A_3\left(\frac{d^2F}{du^2}\right)+A_4\left(\frac{dF}{du}\right)\right]d\Omega$$

$$=-\frac{E}{2}\iint_{\Omega}\left[A_9\left(\frac{dw}{du}\right)^2+A_{10}\frac{d}{du}\left(\frac{dw}{du}\right)^2\right]d\Omega \qquad [4.2.4]$$

where $A_1 = \left(u_{,x}^2+u_{,y}^2\right)^2$

$$A_2 = 6\left(u_{,x}^2 u_{,xx}+u_{,y}^2 u_{,yy}\right)+2\left(u_{,x}^2 u_{,yy}+u_{,y}^2 u_{,xx}\right)+8u_{,x}u_{,y}u_{,xy}$$

$$A_3 = 4\left(u_{,x}u_{,xxx}+u_{,y}u_{,yyy}\right)+3\left(u_{,xx}^2+u_{,yy}^2\right)+2u_{,xx}u_{,yy}+u_{,xy}^2+4\left(u_{,x}u_{,xyy}+u_{,y}u_{,xxy}\right)$$

$$A_4 = u_{,xxxx}+u_{,yyyy}+2u_{,xxyy}$$

$$A_5 = 2\left(u_{,xx}u_{,yy}-u_{,xy}^2\right)$$

$$A_6 = u_{,xx} u_{,y}^2 + u_{,yy} u_{,x}^2 - 2u_{,x} u_{,y} u_{,xy}$$

$$A_7 = G\left(u_{,xx} + u_{,yy}\right)$$

$$A_8 = G\left(u_{,x}^2 + u_{,y}^2\right)$$

$$A_5 = A_9$$

$$A_6 = A_{10} \qquad [4.2.5]$$

Equations (4.2.3), (4.2.4) can be further reduced to simpler form on application of Green's theorem wherever possible.

On transformation to line integrals equations (5.3) and (5.4) will then become.

$$f_1(u)\frac{d^3w}{du^3} + f_2(u)\frac{d^2w}{du^2} + f_3(u)\frac{dw}{du}\frac{dF}{du} + f_4(u)p + f_5(u)\frac{dw}{du} + \rho h\int_1^u w_{,tt}\,du + \rho h K_v \int_1^u w_{,t}\,du + K\int_1^u w\,du = 0 \qquad [4.2.6]$$

$$g_1(u)\frac{d^3F}{du^3} + g_2(u)\frac{d^2F}{du^2} + g_3(u)\frac{dF}{du} + g_4(u)\left(\frac{dw}{du}\right)^2 = 0 \qquad [4.2.7]$$

where $f_i(u)$ and $g_i(u)$ are functions of u only

Equations (4.2.6) and (4.2.7) are the two basic equations to study the dynamic response of structures of arbitrary shape. Hereforth, unless the contour lines are defined one cannot proceed further. The following illustration may be cited for studying the dynamic response of a given shape.

Damped Oscillation of elliptic Plates on An Elastic Foundation :

Considered here an elliptic plate clamped along the edges. The family contour lines of deflected surface may be represented as usual, by

$$u(x,y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

where u = 0 defines the boundary. The boundary conditions imposed are

$$w = 0 \quad at \quad u = 0$$

$$\frac{dw}{du} = 0 \quad at \quad u = 0, 1 \qquad [4.2.8]$$

Then performing the integratins of equations (4.2.3 , 4.2.4) one may arrive at the following equations after a lengthy calculations

$$2D\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}\left[(1-u)^2\frac{d^3w}{du^3} - 2(1-u)\frac{d^2w}{du^2}\right]$$

$$+\frac{8h}{a^2b^2}(1-u)\frac{dw}{du}\frac{dF}{du} + p(1-u) + \rho h\int_1^u\left[w_{,tt} + K_v\, w_{,t}\right]du$$

$$+ K\int_1^u w\,du - 2G\left(\frac{1}{a^2}+\frac{1}{b^2}\right)(1-u)\frac{dw}{du} = 0 \qquad [4.2.9]$$

$$\frac{3a^4+3b^4+2a^2b^2}{a^4b^4}\left[(1-u)^2\frac{d^3F}{du^3} - 2(1-u)\frac{d^2F}{du^2}\right]$$

$$= \frac{2E}{a^2b^2}(1-u)\left(\frac{dw}{du}\right)^2 \qquad [4.2.10]$$

Considering that the plate vibrates primarily in the transverse direction and the plate is restrained from in-plane movements, one can assume without any loss of generality [121]

$$w = Au^2\Psi(t)$$
$$F = Au^2\Phi(t) \qquad [4.2.11]$$

Since equation (4.2.11) does not represent the exact solution, Galerkin procedure may be applied to minimize the error. Substituting equation (4.2.11) in equation (4.2.10) and (4.2.9) and performing the required integrations a relation between $\Phi(t)$ and $\psi(t)$ is first established

$$\Phi(t) = -\frac{6}{5}\,\frac{a^2 b^2 AE}{(3a^4+3b^4+2a^2b^2)}\,\psi^2(t) \qquad [4.2.12]$$

and equation (4.2.9) will then reduce to

$$\left[\frac{2D}{3}\,\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}+\frac{K}{18}+\frac{G}{5}\,\frac{(a^2+b^2)}{a^2b^2}\right]\psi(t)$$

$$+\,1{\cdot}28\,\frac{hE}{(3a^4+3b^4+2a^2b^2)}\,A^2\psi^3(t)-\frac{p}{12}+\frac{\rho h}{18}\left[\psi_{,tt}(t)+K_v\psi_{,t}(t)\right]$$

$$=0 \qquad [4.2.13]$$

Equation (4.2.13) can be put in a simplified form

$$\psi_{,tt}(t)+\mu_0\psi_{,t}(t)+\mu_1^2\psi(t)+\mu_3A^2\psi^3(t)=0 \qquad [4.2.14]$$

where p has been set to zero for free vibration and

$$\mu_0 = K_v$$

$$\mu_1^2 = \frac{12D}{\rho h}\,\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}+\frac{K}{\rho h}+\frac{18G}{5\rho h}\,\frac{(a^2+b^2)}{a^2b^2}$$

$$\mu_3 = 23{\cdot}04\,\frac{EA^2}{\rho\,(3a^4+3b^4+2a^2b^2)}$$

The solution of equation (4.2.14) may be taken as [68]

$$\psi(t) = a_0 e^{-\frac{\mu_0 t}{2}} \sin\left[\mu_1 t\left(1+\frac{3}{8} a_0 A^2 \frac{\mu_3}{\mu_1^2} e^{-\mu_0 t}\right)+\psi_0\right] \qquad [4.2.15]$$

If T and T* be the corresponding time periods of linear and non linear oscillations then

$$\frac{T^*}{T} = \frac{1}{1+\frac{3}{8} a_0 A^2 \frac{\mu_3}{\mu_1^2} e^{-\mu_0 t}}$$

The Dependence of T*/T on the relative amplitude has been presented in Tables [11 and 12]

Static Case

Neglecting the inertial term in equation (4.2.13) the static deflection is given by

$$\left[\frac{2}{3}\frac{(3m^4+2m^2+3)}{(1-\nu^2)} + \frac{0.0555}{(1-\nu^2)}K^* + \frac{1}{5}\frac{(1+m^2)}{(1-\nu^2)}G^*\right]\frac{W_0}{h} + \frac{15.38\, m^4}{(3m^4+2m^2+3)}\left(\frac{W_0}{h}\right)^3 = \frac{pa^4}{Eh^4} \qquad [4.2.16]$$

where $m = \frac{a}{b}$, $K^* = \frac{Ka^4}{D}$, $G^* = \frac{Ga^2}{D}$

$\frac{W_0}{h}$ presents the central deflection and K* and G* are dimensionless parameters

Numerical Results :

Table 6 : Dependence of Central Deflection ($\frac{W_0}{h}$) on Load Parameter ($\frac{pa^4}{Eh^4}$) for different values of K*. υ 0.3, G* =0 , m = 1

$\frac{W_0}{h}$	pa^4/Eh^4					
	K*=0	K*=40	K*=80	K*=120	K*=160	K*=200
0.2	1.74	2.16	2.65	3.13	3.62	4.25
0.4	2.47	3.44	4.42	5.39	6.37	7.34
0.6	3.23	5.39	6.85	8.32	9.78	10.95
0.8	5.67	7.62	9.57	11.52	13.47	15.43
1.0	7.78	10.22	12.66	15.10	17.54	19.98
	9.0[125]	11.8[125]	14.7 [125]	17.0 [125]	20.0 [125]	22.7[125]
1.2	10.34	13.27	16.20	19.13	22.06	24.98
1.4	13.47	16.88	20.30	23.72	27.13	30.55
1.6	17.24	21.14	25.04	28.95	32.86	36.76
1.8	21.75	26.13	30.53	34.92	39.31	43.71

Table7 : Dependence of Central Deflection on Load Parameter for diferent values of K*, G* = 20, υ = 0.3, m = 1.5

$\frac{W_0}{h}$	pa^4/Eh^4		
	K* = 50	K* = 100	K* = 150
0.2	6.82	7.43	8.04
0.4	13.80	15.02	16.24
0.6	21.11	22.94	24.77
0.8	28.91	31.36	33.79
1.0	37.38	40.43	43.47
1.2	46.65	50.32	53.97
1.4	56.65	61.20	65.46
1.6	68.33	73.22	79.01
2.0	95.27	101.37	167.46

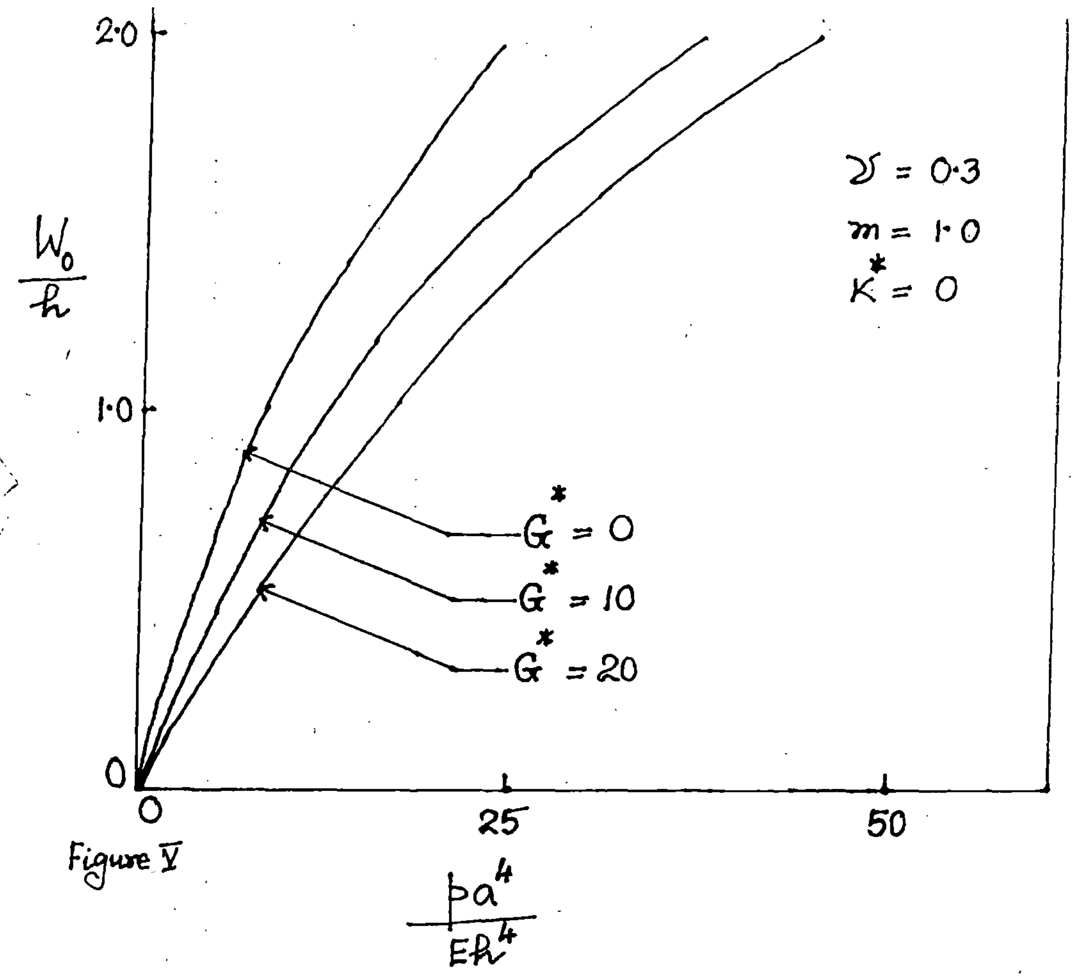

Figure V

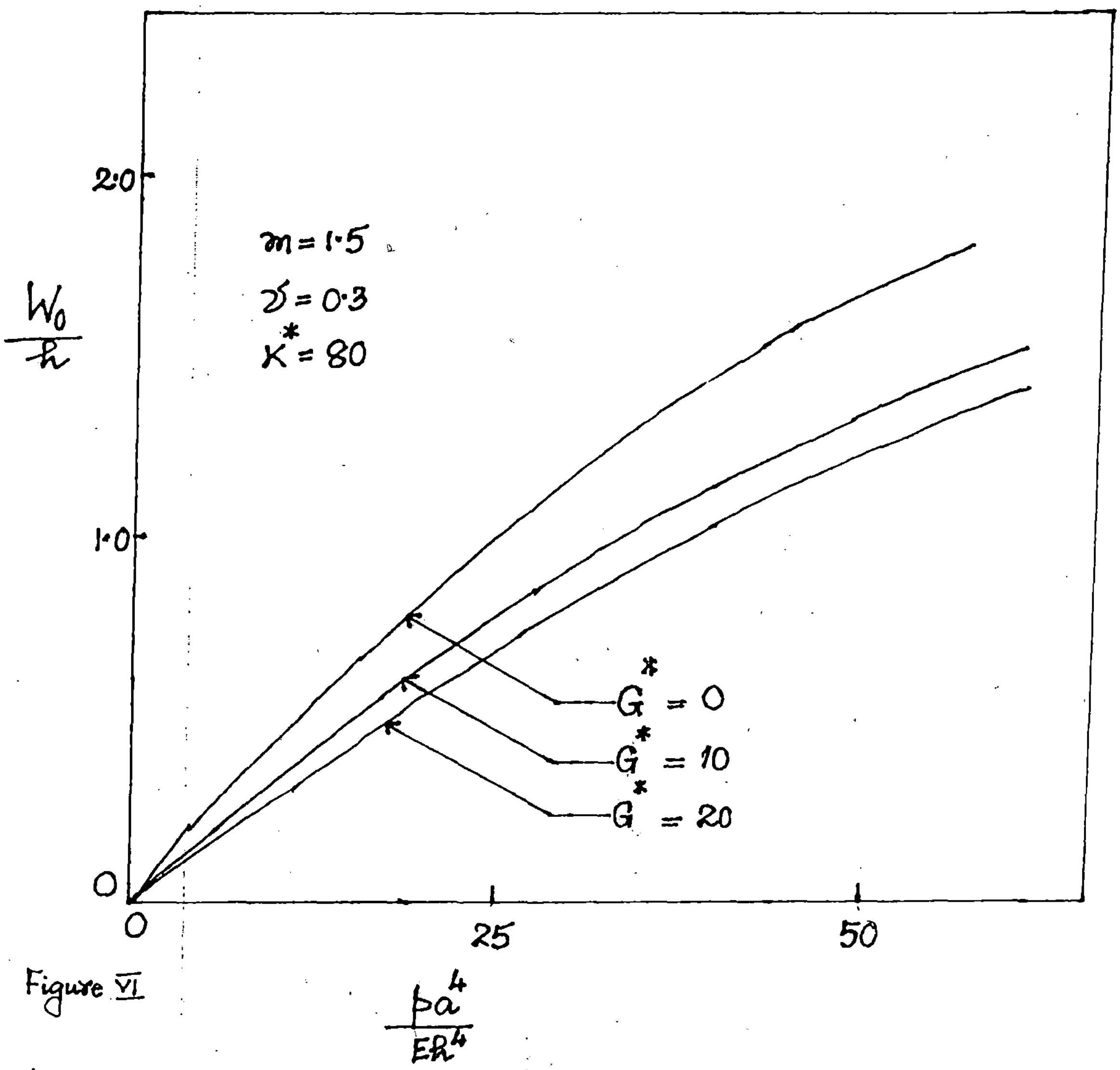

Figure VI

Figure V & VI : Non-linear Central Deflection vs Uniform Lateral Load (Clamped) for Different values of Pasternak Foundation Parameter G^*

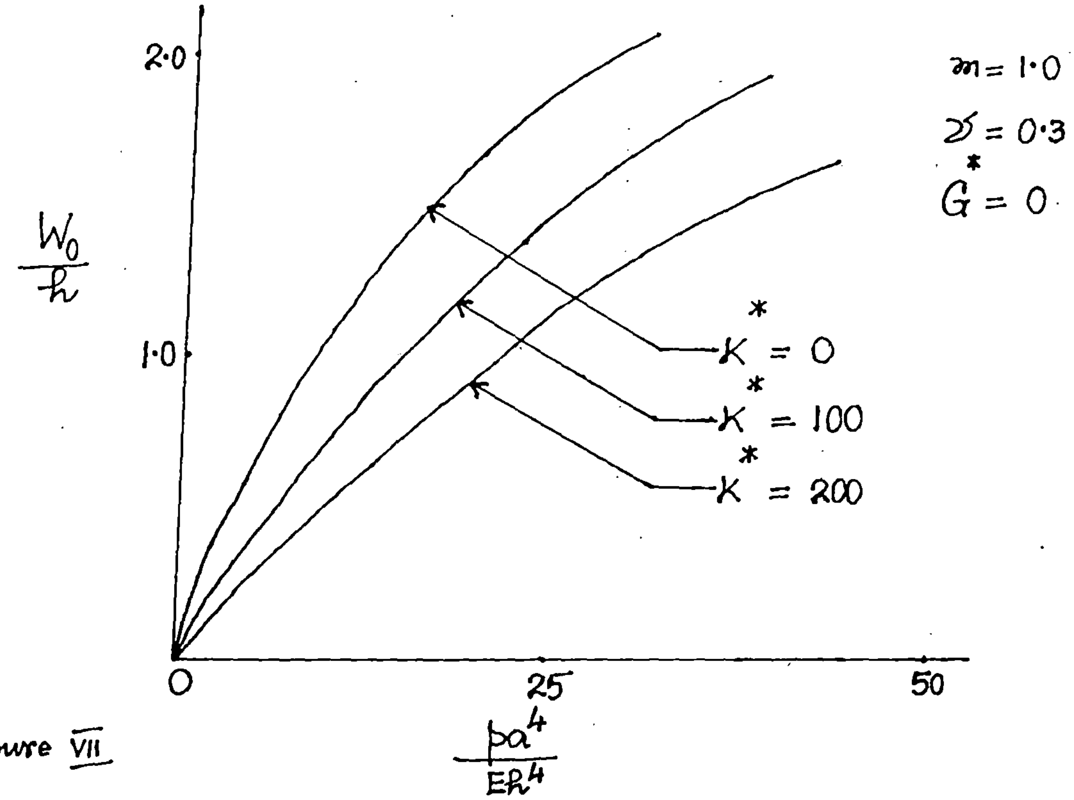

Figure VII

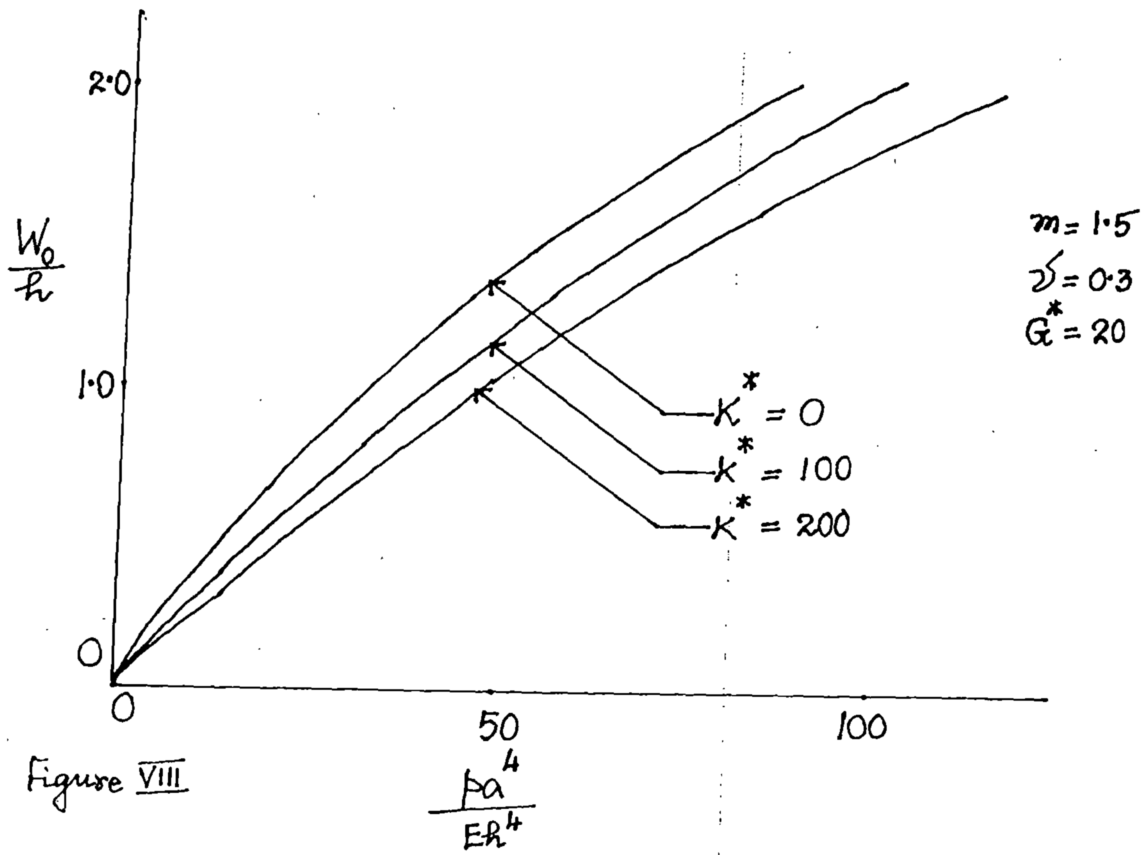

Figure VIII

Figure VII & VIII : Non-linear Central Deflection vs Uniform Lateral Load for a Clamped Elastic Plate for Different Values of Winkler Foundation Parameter K^*

Table (8) : StaticDeflection of an elliptic plate on Elastic Foundation. υ = 0.3, m= 1.5, G* = 10

pa^4/Eh^4

W_0/h	K* 0	K*=40	K*=80	K*=120	K*=160	K*=200
0.2	4.78	5.27	5.75	6.24	6.73	7.22
0.4	9.72	10.69	11.67	12.65	13.62	14.60
0.6	15.00	16.45	17.92	19.38	20.84	22.31
0.8	20.75	22.76	24.66	26.61	28.56	30.51
1.0	27.17	29.61	32.05	34.49	36.93	39.37
1.2	34.41	37.34	40.27	43.20	46.12	49.05
1.4	42.63	46.05	49.47	52.88	56.30	59.71
1.6	52.01	55.91	59.82	36.72	67.62	71.53
2.0	74.86	79.74	84.62	89.50	94.38	99.26

Table 9 : values of load parameter for various values of m and υ for K* = 80 and G*= 10

pa^4/Eh^4

W_0/h	J = 0.3			m = 1.5		
	m=1	m=1.5	m=2	ν =0.2	ν =0.3	ν =0.5
0.2	3.04	5.75	11.85	5.0189	5.75	6.98
0.4	6.18	11.67	23.91	10.0703	11.67	14.12
0.6	9.50	17.92	36.36	15.1869	17.92	21.60
0.8	13.10	24.66	49.41	20.4014	24.66	29.55
1.0	17.06	32.05	63.27	25.7454	32.05	38.17
1.2	21.49	40.27	78.11	31.2517	40.27	47.61
1.4	26.46	49.47	94.16	36.9545	49.47	58.04
1.6	32.09	59.82	111.60	42.884	59.82	69.61
2.0	45.64	84.62	151.48	55.5582	84.62	96.86

Table 10 : The nonlinear central deflection for a rigidly clamped plate on a Pasternak foundation subject to a static load p = 20, υ = 0.3, m = 1

W_0/h	G*		
	K* = 50	K* = 100	K* = 150
0.2	207.08	200.32	193.23
0.4	92.15	86.00	79.00
0.6	54.63	43.20	40.14
0.8	33.81	27.00	19.96
1.0	20.86	14.06	7.00
	20.0 [125]	14.0 [125]	5.2 [125]

Table 11 : Dependence on foundation parameter of the relative time period of nonlinear and linear vibration T*/T for elliptic plate for various values of relative amplitude $\frac{A_0}{h}$ for p=0 and m = 2 , υ = 0.3

[T*/T]

$\frac{A_0}{h}$	G*=0			K*=0		
	K*=40	K*=120	K*=200	G*=50	G*=100	G*=200
0.5	0.9994	0.9995	0.99957	0.9997	0.9998	0.9999
1	0.9979	0.9981	0.9983	0.9990	0.9994	0.9996
1.5	0.9952	0.9957	0.9961	0.9978	0.9986	0.9992
2.0	0.9916	0.9924	0.9932	0.9960	0.9975	0.9985
2.5	0.9870	0.9882	0.9894	0.9940	0.9961	0.9977

Table 12 : Comparison of variation of T*/T with relative amplitude $\frac{A_0}{h}$ between cases for circular (m=1) and an elliptic (m=2.0) plate ; υ =0.3 = 0 K* = 40 G* = 100, p= 0

A_0/h	T*/T	
	m=1	m=2
0.0	1.000	1.000
0.5	0.9965	0.9998
101.0	0.9864	0.9994
1.5	0.9700	0.9986
2.0	0.9211	0.9962

Discussions:

Tables (6-9) and figures (V,VI.VII,VIII) show the static behaviour of an elliptic plate for various values of the parameters υ , m and the foundation parameters G* and K*. The results show a very good agreements with those of Smaill [125] in the limiting case when a=b

Table (9) shows that the static deflections are as expcted dependent on the ratio $\frac{a}{b}$ appreciably.

Table (9) also depicts the dependence of the central deflection on poisson's ratio. The choice of material having higher poisson ratio increases the load bearing capacity.

Tables (6,7,8) show the static behaviour of elliptic plate for different values of Winkler foundation parameter K* when Pasternak foundation parameter G* is kept fixed.

Table (10) identifies the characteristics of Pasternak foundation G^* for different values of Winkler foundation parameter K^*. Though a single term approximation has been made the results are inquite good agreement with those of Smaill [125] for circular plate.

Table (6) when compared with Smaill's [125]. results show that the values of non-dimensional central deflections are little higher than those given by Smaill [125] for all values of K^*.

All the tables, presented here, are for undamped cases only with $K_v = 0$. This is the reson for which the results given in tables [6] differ Smaill's results (Fig-5 of Ref [125]). It appears that the central deflections are higher for undamped cases than those for a damped oscillatory motion for a fixed load.

In table(11) and (12) the dependence of $^{T^*}/_{T}$ on relative amplitude has been presented. It may be observed that there hardly any effect of nonlinearity so far as undamped free oscillation of elliptic plates are concerned, irrespective of variation in the values of foundation parameter. However, in case of circular plate the nonlinear effect may observed. [Table (12)]

Thus the concept of "Constant Deflection Contour" method may safely be applied for the study of static and dynamic behaviour of plates on elastic foundations.

Problem - 4.3

The non-linear damped vibration of moderately thick plates has been studied by using the method of "Constant Deflection Contour lines" and well-known Berger [55] method.. Berger offered a simplified approach to study the non-linear behaviour of thin plates.

<u>Some points on Berger's Method</u>

Combined the potential energy due to the bending and stretching of the middle surface of a plate/shell may be represented by

$$V = \iint \frac{D}{2}\left\{ \left(\nabla^2 w\right)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu)\left[\frac{12}{h^2} e_2 + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] \right\} dx\,dy \qquad [4.3.1a]$$

in terms of the displacement w, $e_1 = e_{xx} + e_{yy}$, $e_2 = e_{xx}\, e_{yy} - \frac{1}{4} e_{xy}^2$, e_1 and e_2 being the first and second strain invariants. And E, υ, h are Young modulus, Poisson ratio and thickness respectively and with usual notations the in-plane strain components are given by

$$e_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - wK_1$$

$$e_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - wK_2$$

$$e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \qquad [4.3.2a]$$

For the shallow shells K_1 and K_2 denote the principal curvatures at a point of the middle surface. For plate problem they are put to zero.

In 1955 Berger [55] proposed that the so called strain invariant of the membrane strain to the strain energy of the plate may be neglected without appreciably impairing the accuracy of the results. On neglection of e_2, equation (4.3 1a) will reduce to

$$V = \frac{1}{2}\iint D\left[\left(\nabla^2 w\right)^2 + \frac{12 e_1^2}{h^2} - 2(1-\nu)\left\{ \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right\} \right] dx\,dy - \iint q w \, dx\,dy \qquad [4.3.3a]$$

with $$e_1 = \left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right) + \frac{1}{2}\left\{ \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right\}$$

The governing equation will now reduce to

$$\nabla^2(D\nabla^2 w) - C f(t)\nabla^2 w - q + \rho h w_{,tt} = 0 \qquad [4.3.4a]$$

$$\text{with} \quad e_1 = C(1-\nu^2) f(t) / Eh$$

where c is a normalized constant of integration and function of time, f(t) to be determined. Here the present author aims to verify the applicability of the "Constant Deflection Contour method" to Berger equations with regard to specific problem.

Non-Linear Damped Oscillations of Moderately Thick Plate of Arbitrary Shape

Many workers utilize Berger's equation in their respective field of investigations and obtained satisfactory results. In most cases the effects of transverse shear deformation and rotatory inertia has not been taken into account. Sathya moorthy and Chia [133] show that the effect of transverse shear and rotatory inertia play an important role in the large amplitude vibrations of moderately thick plates of different shape. Banerjee and Bhattacharya [132] investigated the effect of transverse shear and rotatory inertia on large amplitude vibration of thick plates.

The works so far carried out on the theory of non-linear vibrations of thick plates are restricted to the plates of regular shapes only. The present investigation concerns with the study of the non-linear static and dynamic behaviour of moderately thick plates of arbitrary shape by using the idea of "Lines of Equal Deflection". To study the dynamic behaviour a damping factor has been introduced. Numerical results for elliptic and circular plates have been computed and compared with the other available known results.

The set of decoupled differential equations governing the vibrations of plates are given by R. Bhattacharya and B. Banerjee [132].

$$\nabla^4 w + \frac{6K}{5(1-\nu^2)}\frac{E}{G_c}\frac{\bar{\alpha}^2 h^2}{12}\tau(t)\nabla^4 w - \frac{6\rho}{5G_c}\frac{\partial^2}{\partial t^2}(\nabla^2 w)$$

$$-\bar{\alpha}^2\tau(t)\nabla^2 w + \frac{12}{h^2 C_p^2}\frac{\partial^2 w}{\partial t^2} = 0 \qquad [4.3.1]$$

$$\text{where} \quad \frac{\bar{\alpha}^2 h^2}{12}\tau(t) = \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \qquad [4.3.2]$$

w is the vertical deflection, k is tracing constant characterising the effects of transverse shear deformation, υ is Poisson's ratio, E is Young's modulus, G_c is shear modulus. $\bar{\alpha}$ is coupling parameter, h is the thickness of the plate, τ (t) is non-linear time dependent function, ρ is the density of the material.

$C_p = \left[\frac{E}{\rho(1-\nu^2)}\right]^{1/2}$, speed of the wave propagation along the surface of the plate. The deflections are of the same order of magnitude as the plate thickness.

∇^2 is the two dimensional Laplacian operator

Putting

$$\left.\begin{aligned} A' &= 1 + \frac{6K}{5(1-\nu^2)} \frac{E}{G_c} \frac{\bar{\alpha}^2 h^2}{12} \tau(t) \\ B' &= \frac{6\rho}{5G_c} \\ C' &= \bar{\alpha}^2 \tau(t) \\ D' &= \frac{12}{h^2 C_p^2} \end{aligned}\right\} \qquad [4.3.3]$$

Equation (3.1) becomes

$$A'\nabla^4 w - B'\frac{\partial^2}{\partial t^2}(\nabla^2 w) - C'\nabla^2 w + D'\frac{\partial^2 w}{\partial t^2} = 0$$

For damping we introduce another term in the above equation

$$A'\nabla^4 w - B'\frac{\partial^2}{\partial t^2}(\nabla^2 w) - C'\nabla^2 w + D'\frac{\partial^2 w}{\partial t^2} + K_v \frac{\partial w}{\partial t} = 0 \qquad [4.3.4]$$

where K_v is the damping constant

It is assumed that the equation (4.3.2) and (4.3.4) are satisfied in every region bounded by the contour line C_u. Using the transformations done in Chapter I and integrating over the region, eqations (4.3.4) and (4.3.2) will respectively reduce to

$$\iint_\Omega A'\left[A_1\frac{d^4w}{du^4}+A_2\frac{d^3w}{du^3}+A_3\frac{d^2w}{du^2}+A_4\frac{dw}{du}\right]d\Omega$$

$$-\iint_\Omega B'\frac{\partial^2}{\partial t^2}\left[A_5\frac{d^2w}{du^2}+A_6\frac{dw}{du}\right]d\Omega$$

$$-\iint_\Omega C'\left[A_5\frac{d^2w}{du^2}+A_6\frac{dw}{du}\right]d\Omega+\iint_\Omega D'\frac{\partial^2 w}{\partial t^2}\,d\Omega$$

$$+\iint_\Omega K_v\frac{\partial w}{\partial t}\,d\Omega=0 \qquad [4.3.5]$$

$$\iint_\Omega \frac{\bar{\alpha}^2 h^2}{12}\,\tau(t)\,d\Omega=\iint \frac{1}{2}A_5\left(\frac{dw}{du}\right)^2 d\Omega \qquad [4.3.6]$$

where

$$A_1=\left(u_{,x}^2+u_{,y}^2\right)^2$$

$$A_2=6\left(u_{,x}^2u_{,xx}+u_{,y}^2u_{,yy}\right)+2\left(u_{,x}^2u_{,yy}+u_{,y}^2u_{,xx}\right)+8u_{,x}u_{,y}u_{,xy}$$

$$A_3=4\left(u_{,x}u_{,xxx}+u_{,y}u_{,yyy}\right)+3\left(u_{,xx}^2+u_{,yy}^2\right)+4\left(u_{,x}u_{,xyy}+u_{,y}u_{,xxy}\right)$$
$$-2u_{,xx}u_{,yy}+4u_{,xy}^2$$

$$A_4=u_{,xxxx}+u_{,yyyy}+2u_{,xxyy}$$

$$A_5=u_{,x}^2+u_{,y}^2$$

$$A_6=u_{,xx}+u_{,yy}$$

On transformation to line intergals, utilising Green's theorem, equation (4.3.5) and (4.3.6) becomes

$$A'\left[f_1(u)\frac{d^3w}{du^3}+f_2(u)\frac{d^2w}{du^2}+f_3(u)\frac{dw}{du}\right]-B'\frac{\partial^2}{\partial t^2}\left[g_5\frac{dw}{du}\right]$$

$$-\,C'g_5\frac{dw}{du}+D'\int_1^u w_{,tt}\,du+K_v\int_1^u w_{,t}\,du=0 \qquad [4.3.8]$$

and

$$\frac{\bar{\alpha}^2 h^2}{12}\,\tau(t)=E'\int_1^u j(u)\left(\frac{dw}{du}\right)^2 du \qquad [4.3.9]$$

where E' is a constant and $f_i(u)$, $g_i(u)$ and $j(u)$ are functions of u only.

Illustration :
Damped Oscillation of Elliptic Plate

Considering an elliptic plate clamped along the boundary, the family of contour lines of the deflected surface may be presented by

$$u=1-\frac{x^2}{a^2}-\frac{y^2}{b^2} \qquad [4.3.10]$$

where u = 0 defines the boundary .
The boundary conditions for a clamped plate are

$$\left.\begin{aligned} &w=0 \quad \text{at} \quad u=0\\ &\text{and } \frac{dw}{du}=0 \quad \text{at} \quad u=0,1\end{aligned}\right\} \qquad [4.3.11]$$

Performing the integrations of equations (4.3.5) and (4.3.6) with the boundary conditions represented by (4.3.11). One may arrive at the following equations for elliptic plate after a lengthy calculations

$$A'\left[(1-u)^2\frac{d^3w}{du^3}-2(1-u)\frac{d^2w}{du^2}\right]-M^2(1-u)\frac{dw}{du}-N^2(1-u)\frac{dw}{du}$$

$$+P^2\int_1^u w_{,tt}\,du+Q^2\int_1^u w_{,t}\,du=0 \qquad [4.3.12]$$

$$\frac{\bar{\alpha}^2 h^2}{12}\,\tau(t)=\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\int_1^u (1-u)\left(\frac{dw}{du}\right)^2 du \qquad [4.3.13]$$

where

$$
\left.
\begin{aligned}
M^2 &= \frac{C' a^2 b^2 (a^2+b^2)}{(3a^4+3b^4+2a^2b^2)} \\
N^2 &= \frac{B' a^2 b^2 (a^2+b^2)}{(3a^4+3b^4+2a^2b^2)} \\
p'^2 &= \frac{D' a^4 b^4}{2(3a^4+3b^4+2a^2b^2)} \\
Q^2 &= \frac{K_d a^4 b^4}{2(3a^4+3b^4+2a^2b^2)}
\end{aligned}
\right\} \qquad [4.3.14]
$$

Considering that the plate vibrates primarily in the transverse direction and the plate is restrained from in-plane movements one can assume without any loss of generality

$$
w \sum_{i=2}^{\alpha} W_i u^i \psi(t) \simeq W_0 u^2 \psi(t) \qquad [4.3.15]
$$

Method of Solution

Since equation (4.3.15) does not represent the exact solution, Galerkin procedure may be applied to minimize the error. Substituting equation (4.3.15) in equation (4.3.12) and (4.3.13) and after performing the integration satisfying necessary ' " condition as required in Galerkin procedure one gets

$$
\frac{1}{3} A' \psi(t) + \frac{1}{10} M^2 \psi(t) + \left[\frac{1}{10} N^2 + \frac{1}{18} p'^2 \right] \psi_{,tt}(t)
$$

$$
+ \frac{1}{18} Q^2 \psi_{,t}(t) = 0 \qquad [4.3.16]
$$

$$
\Upsilon(t) = \frac{4 W_0}{\alpha^2 h^2} \frac{a^2+b^2}{a^2 b^2} \psi^2(t) \qquad [4.3.17]
$$

Substituting $\tau(t)$ given by equation (4.3.17) into the first two terms on the left hand side of equation (4.3.16) it reduces to

$$\psi_{,tt}(t) + \mu\psi_{,t}(t) + \mu_1^2\psi(t) + \mu_2 W_0 \psi^3(t) = 0 \qquad [4.3.18]$$

where

$$\left.\begin{aligned}
\mu &= \frac{K_d}{\dfrac{12}{h^2 C_p^2} + \dfrac{108}{25}\dfrac{\rho}{G_c}\left(\dfrac{a^2+b^2}{a^2b^2}\right)} \\[2ex]
\mu_1^2 &= \frac{3a^4+3b^4+2a^2b^2}{\dfrac{a^4b^4}{h^2C_p^2} + \dfrac{9}{25}\dfrac{\rho}{G_c}a^2b^2(a^2+b^2)} \\[2ex]
\mu_2 &= \frac{2}{5}\,\frac{3\left(\dfrac{a^2+b^2}{h}\right)^2 + \dfrac{KE}{G_c(1-\nu^2)}\left(\dfrac{a^2+b^2}{a^2b^2}\right)(3a^4+3b^4+2a^2b^2)}{\dfrac{a^4b^4}{h^2C_p^2} + \dfrac{9}{25}\dfrac{\rho}{G_c}a^2b^2(a^2+b^2)}
\end{aligned}\right\} \quad [4.3.19]$$

The solution of equation (3.18) may be taken as

$$\psi(t) = a_0 e^{-\mu t/2} \sin\left[\mu_1 t\left(1+\frac{3}{8}a_0^2 W_0^2 \frac{\mu_2}{\mu_1^2} e^{-\mu t}\right) + \psi_0\right] \qquad [4.3.20]$$

The time period of non-linear oscillation

$$T^* = \frac{2\pi}{\mu_1\left(1+\frac{3}{8}a_0^2 W_0^2 \frac{\mu_2}{\mu_1^2} e^{-\mu t}\right)}$$

The corresponding time period of linear oscillation is

$$T = \frac{2\pi}{\mu_1} \quad (\text{for linear oscillation } \mu_2 = 0)$$

Thus $$\frac{T^*}{T} = \frac{1}{1+\frac{3}{8}a_0^2 W_0^2 \frac{\mu_2}{\mu_1^2} e^{-\mu t}} \qquad [4.3.21]$$

where

$$\frac{\mu_2}{\mu_1^2} = \frac{\frac{6}{5}\left(\frac{a^2+b^2}{h}\right)^2}{(3a^4+3b^4+2a^2b^2)} + \frac{2}{5}\frac{KE}{G_c(1-\nu^2)}\frac{(a^2+b^2)}{a^2b^2}$$

$$[4.3.22]$$

Static Case

For mechanical loading the inertial terms in equation (4.3.1) are neglected to consider the static case and the required differential equation for the static deflection of a thick elastic plate is

$$\nabla^4 w + \frac{6K}{5(1-\nu^2)}\frac{E}{G_c}\frac{\bar{\alpha}^2 h^2}{12}\tau(t)\nabla^4 w - \bar{\alpha}^2\tau(t)\nabla^2 w - \frac{p}{D} = 0 \quad [4.3.23]$$

where p is the uniform load and the coupling parameter $\frac{\bar{\alpha}^2 h^2}{12}\tau(t)$ is given by equation (4.3.2) and

$D = \frac{Eh^3}{12(1-\nu^2)}$ is flexural rigidity.

Equation (4.3.23) can be written as

$$A'\nabla^4 w - C'\nabla^2 w - \frac{p}{D} = 0 \qquad [4.3.24]$$

where A' and C' are given by equation (4.3.3)

Now introducing the idea of constant deflection contour lines as it has been done in the case of dynamic loading and proceeding as before for elliptic plate where the contour lines are represented by $u = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$

One gets the equation

$$A'\left[(1-u)^2\frac{d^3w}{du^3} - 2(1-u)\frac{d^2w}{du^2}\right] - M^2(1-u)\frac{dw}{du} + q_1(1-u) = 0 \qquad [4.3.25]$$

where M^2 is given by equation (4.3.14)

and $$q_1 = \frac{a^4b^4p}{2D(3a^4+3b^4+2a^2b^2)} \qquad [4.3.26]$$

For the solution of equation (3.25) is assumed

$$w = W_0 u^2 \qquad [4.3.27]$$

Substituting equation (3.27) in equation (4.3.25) and applying Galerkin procedure to minimize the error one gets

$$\left(\frac{W_0}{h}\right) + \beta\left(\frac{W_0}{h}\right)^3 = \gamma\left(\frac{pa^4}{Eh^4}\right) \qquad [4.3.28]$$

where

$$\beta = \frac{6}{5}\frac{(a^2+b^2)^2}{(3a^4+3b^4+2a^2b^2)} + \frac{2}{5}\frac{KE}{G_c}\frac{h^2}{(1-\nu^2)}\frac{(a^2+b^2)}{a^2b^2}$$

$$\gamma = \frac{3}{2}\frac{(1-\nu^2)\,b^4}{(3a^4+3b^4+2a^2b^2)}$$

Table 13 : Free vibrations of clamped elliptical plate

$$\frac{KE}{G_c} = 0, \quad \nu = 0.3, \quad \mu = 0, \quad m = 1.5$$

$\frac{W_0 a_0}{h}$	$T^*/_T$		
	Present Study	Das and Banerjee (68)	Sathyamoorthy (135)
0	1.000	1.000	1.000
0.5	0.9502	0.9502	0.9615
1	0.8268	0.8268	0.8700
1.5	0.6797	0.6797	0.7654
2	0.5441	0.5441	0.6500

Table 14 : Free vibrations of clamped Elliptic plate

$$\frac{KE}{G_e} = 0, \quad \nu = 0.3, \quad \mu = 0, \quad m = 2$$

$\frac{W_0 a_0}{h}$		$T^*/_T$	
	Present Study	Das and Banerjee	Sathyamoonthy
0	1.000	1.000	1.000
0.5	0.9545	0.9545	0.9615
1.0	0.8399	0.8400	0.8750
1.5	0.6998	0.6999	0.7538
2.0	0.5674	0.5674	0.6500

Table 15 : Free vibration of clamped elliptic plate

$$\frac{KE}{G_c} = 1, \quad \nu = 0.3, \quad \mu = 0, \quad \frac{a}{b} = 1.5$$

$\frac{W_0 a_0}{h}$	$T^*/_T$								
	h/a=0.2			h/a=0.1			h/a=0.066		
	Present Study	Das & Banerj	Sathya moorthy	Present Study	Das & Banerjee	Sathya moorthy	Present Study	Das & Banerjee	Satya moorthy
0	1.000	1.000	1.111	1.000	1.000	1.025	1.000	1.000	1.0077
0.5	0.9454	0. 9454	1.0538	0.9490	0.9490	0.9846	0.9497	0.9497	0.9730
1	0.8124	0.8124	0.9154	0.8232	0.8238	0.8808	0.8252	0.8252	0.8700
1.5	0.6581	0.658	0.7800	0.6742	0.6742	0.7712	0.6773	0.6773	0.7700
2.0	0.5199	0.5198	0.6600	0.5379	0.5379	0.6550	0.5411	0.5411	0.6500

Table 16 : Free vibration of clamped elliptic plate

$$\frac{KE}{G_c} = 1, \quad \nu = 0.3, \quad \mu = 0, \quad \frac{a}{b} = 2$$

	$T^*/_T$								
	h/a=0.2			h/a=0.1			h/a=0.066		
	Present Study	Das & Banerj	Sathya moorthy	Present Study	Das & Banerjee	Sathya moorthy	Present Study	Das & Banerjee	Satya moorthy
0	1.000	1.000	1.1807	1.000	1.000	1.0423	1.000	1.000	1.0135
0.5	0.9470	0. 9471	1.0942	0.9526	0.9526	0.9846	0.9537	0.9537	0.9731
1	0.8172	0.8172	0.9270	0.8341	0.8341	0.8865	0.8373	0.8373	0.8570
1.5	0.6653	0.6653	0.800	0.6909	0.6908	0.7827	0.6960	0.6959	0.7769
2.0	0.5279	0.5279	0.6731	0.5570	0.5569	0.6600	0.5629	0.5628	0.6500

Table 17 : Free vibration of clamped circular plate

$m = 1, \quad \nu = 0.3, \quad \mu = 0$

$\frac{W_0 a_0}{h}$	$T^*/_T$								
	h/a=0.2 $\frac{KE}{G_c} = 8.197$			h/a=0.1, $\frac{KE}{G_c} = 8.8133$			h/a=0.066, $\frac{KE}{G_c} = 19.3165$		
	Present Study	Das & Banerjee	K.K.Raju [136]	Present Study	Das & Banerjee	K.K.Raju (136)	Present Study	Das & Banerjee	K.K.Raju (136)
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.2	0.9868	0. 9869	1.9921	0.9885	0.9885	0.9924	0.9904	0.9910	0.9927
0.4	0.9494	0.9494	0.9699	0.9556	09556	0.9710	0.9628	0.9629	0.9718
0.6	0.8929	0.89303	0.9366	0.9053	0.9054	0.9388	0.9210	0.9202	0.9402
0.8	0.8242	0.8244	0.8965	0.8433	0.8433	0.8995	0.8664	0.8664	0.9015
1.0	0.7501	0.7503	0.8533	0.7749	0.7751	0.8568	0.8058	0.8058	0.8591

Table18 : Damped oscillations of clamped elliptical plate

$K_v = 0.5, \quad \frac{1}{h^2 C_p^2} = 1, \quad \frac{\rho}{b^2 G_c} = 0.5, \quad m = 1.5, \quad \frac{h}{a} = 0.2$

$\frac{W_0 a_0}{h}$	$T^*/_T$					
	KE/Gc=2.5		KE/Gc=10		KE/Gc=20	
	Present Study	Das & Banerjee (68)	Present Study	Das & Banerjee (68)	Present Study	Das & Banerjee (68)
0	1.000	1.000	1.000	1.000	1.000	1.000
0.25	0.9862	0.9892	0.9780	0.9873	0.9673	0.9874
0.5	0.9472	0.9582	0.9177	0.9510	0.8809	0.9416
0.75	0.8885	0.9105	0.8321	0.8661	0.7790	0.8775
1	0.8177	0.8513	0.7359	0.8291	0.6492	0.8012

Table 19 : Damped oscillations of clamped elliptical plate

$$K_v = 0.5, \frac{1}{h^2 C_p^2} = 1, \frac{\rho}{b^2 G_c} = 0.5, t = 5 \text{ secs}, m = 2, \frac{h}{a} = 0.2$$

$\frac{W_0 a_0}{h}$	$T^*/_T$					
	KE/Gc=2.5		KE/Gc=10		KE/Gc=20	
	Present Study	Das & Banerjee (68)	Present Study	Das & Banerjee (68.)	Present Study	Das & Banerjee (68)
0	1.000	1.000	1.000	1.000	1.000	1.000
0.25	0.9858	0.9892	0.9733	0.9867	0.9571	0.9828
0.5	0.9455	0.9582	0.9011	0.9489	0.8479	0.9345
0.75	0.8853	0.9105	0.8020	0.8920	0.7125	0.8638
1	0.8128	0.8513	0.6949	0.8228	0.5829	0.7811

Table 20 : Damped Oscillations of clamped circular plate

$$K_v = 0.5, \quad m = 1, \quad \frac{1}{h^2 C_p^2} = 1, \quad \frac{\rho}{b^2 G_c} = 0.5, \quad t = 5 \text{ secs} \quad \frac{h}{a} = 0.2$$

$\frac{W_0 a_0}{h}$	$T^*/_T$					
	KE/Gc=2.5		KE/Gc=10		KE/Gc=20	
	Present Study	Das & Banerjee (68)	Present Study	Das & Banerjee (68.)	Present Study	Das & Banerjee (68)
0	1.000	1.000	1.000	1.000	1.000	1.000
0.25	0.9863	0.9889	0.9877	0.98122	0.9744	0.9862
0.5	0.9475	0.9570	0.9527	0.9289	0.9051	0.9470
0.75	0.8892	0.9983	0.8995	0.8531	0.8091	0.8881
1	0.8187	0.8478	0.8343	0.7655	0.7045	0.8171

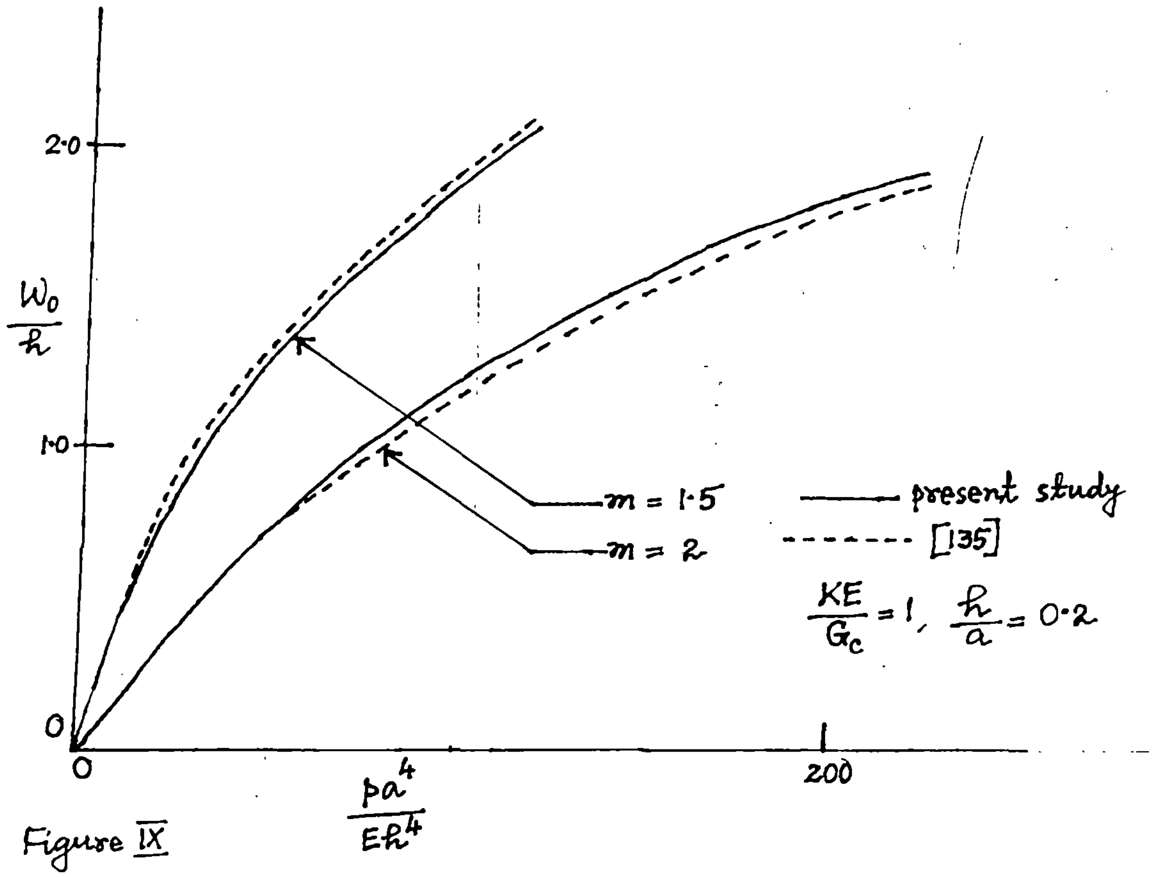

Figure IX :- Non-linear Static Behaviour of Damped Elastic Plates

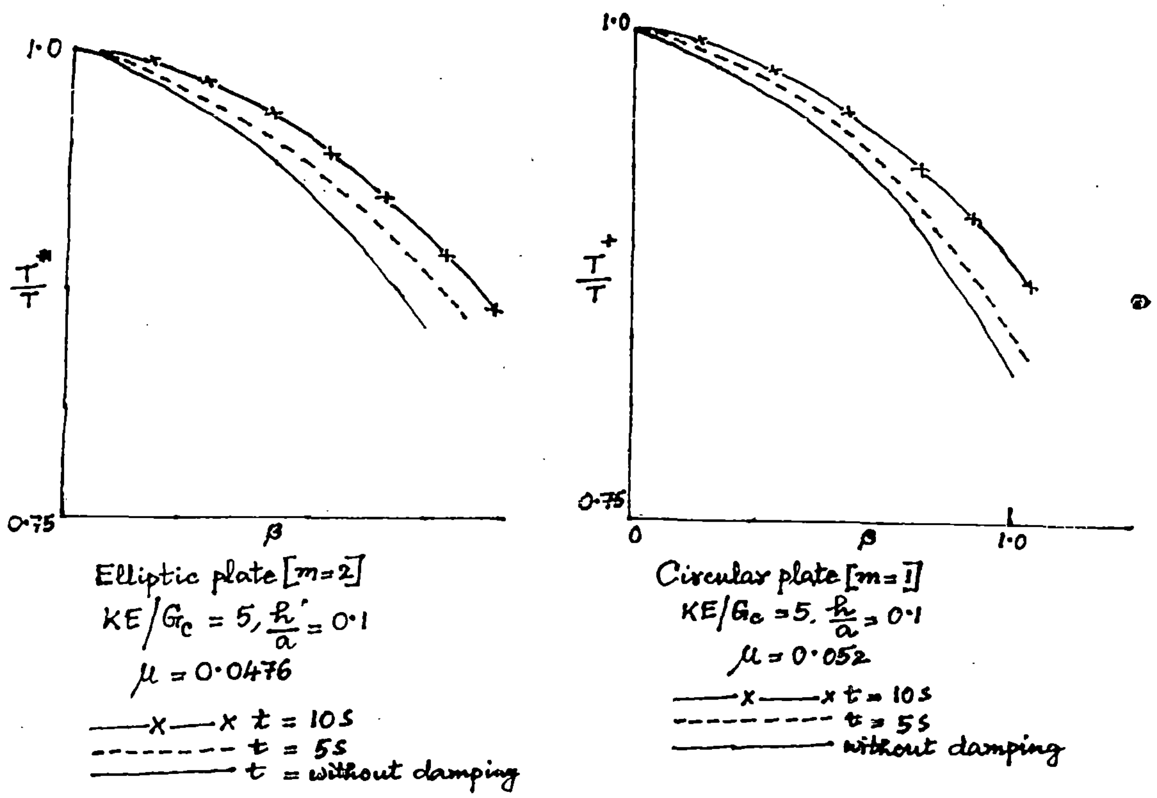

Figure X :- Time-Period Ratio vs Relative Amplitude for Elliptic & Circular Plates [Damped]

Table 21 : Static Deflection for Thick Clamped Circular Plate.

$\frac{KE}{G_c} = 1, \frac{h}{a} = 0.2, \nu = 0.3, m = 1$

W_0/h	Present Study	Sathyamoorthy [135]
0.5	3.3961	3.2756
1.0	9.5844	8.6227
1.5	21.356	18.1126
2.0	41.505	33.8169

Discussion

Tables [13-17] represent the dependence of $\frac{T^*}{T}$ on central deflection $\frac{W_0 a_0}{h}$ for elliptical and circular plates for free oscillations and the results are compared with those of Das and Banerjee [68] and Sathya-moorthy [135]. The results show a very good agreement with those of Das and Banerjee [68].

Tables [18-20] show the dependence of $\frac{T^*}{T}$ on $\frac{W_0 a_0}{h}$ for damped oscillations. ($K_v = 0.5$) for elliptic and circular plates and the results are compared with those of Das and Banerjee [68].

Table 21 shows the static behaviour of plates. The non-dimensional deflection parameters $\frac{W_0}{h}$ are obtained for different values of the load parameter pa^4/Eh^4. The results show a very good agreement with those of Sathyamoorthy [135] for small values of p.

It is observed that the numerical results of the present study showing the role of rotatory inertia are in good agreement with those obtained by other methods. The discrepancies in some cases between the present results and those of K.Kanakaraju and G. Venkateswara Rao [136], and M. Sathyamoothy [135] are due to the fact that K. KankaRaju and G. Venkateswara Rao use Fine Element Method whereas classical VonKarman equation has been solved by Sathyamoorthy, but present study uses- Berger's approximation.

The present investigation while checking the work of Das and Banerjee has observed some salient points which are unfortunatehy not in favour of the authors of Ref [68].

For example Equation (5) of referance [68] though appears to be true in the concept of "constant Deflection contour Method" but it becomes totally erroneous when equation (7) is simultaneously considered. The reason is obvious as the expressions R,G.F. in reference [68] can never be identically equal to those A_1, A_2, A_3, A_4, A_5, A_6 of equation(3.7))obtained in the present study.Probably the authors of referenee [68] have not checked the deductions.

The main purpose of the present problem is to establish the applicability of the concept of "Constant Contour Deflection Method" for the study of static and dynamic behaviour of moderately thick plate of arbitrary shape . The advantage of this proposal is that the basic equations (4.3.8) and (4.3.9) established here are ordinary differential equations of third order while equation (4.3.1) and (4.3.2) are partial differential equations of fourth order. Moreover, modified equations will describe the nature ofnonlinear oscillations of plates of arbitrary shape provided the equation of its deflection contour u,(x,y) = constant, is known. As for example if

$x(x, y) = y\,[a/2\,(2/a - y) - x^2]$, we get the results of the uniformly loaded parabolic plate with a clamped edge.

Problem - 4.4

Non liner vibrations of elastic Plates with varying Flexural Rigidity

Non-homogeneous materials, with varying flexural rigidity have received a considerable attention.The governing differential equations are of Karman type, extended to a dynamic case, including the effect of varying flexural rigidity. Assuming the Young's Modulus to be inhomogeneous, the govering differential equations are solved with the boundary conditions for clamped edge and by Galerk-in method. The "constant deflection contour" method is employed here.

The dynamic Von-Karman equations for a non-homogenous plate having varying flexural rigidity and subjected to a normal uniform load may be put in the form

$$\nabla^2\left(D\nabla^2 w\right) - (1-\nu)\mathcal{L}(D, w) = p - \rho h w_{,tt} + \mathcal{L}(F, w) \quad [4.4.1]$$

$$\nabla^2\left(\mu\nabla^2 F\right) - (1+\nu)\mathcal{L}(\mu, F) = -\frac{E}{2}\mathcal{L}(w, w) \quad [4.4.2]$$

with in plane inertia effect ignored

and $\mu = \frac{1}{h}$

considering E as function of x and y, and h and υ as constants equations (4.4.1) and (4.4.2) become

$$\frac{h^3}{12(1-\nu^2)}\Big[E\nabla^4 w + (E_{,xx}+\nu E_{,yy})w_{,xx} + (E_{,yy}+\nu E_{,xx})w_{,yy}$$

$$+2(1-\nu)E_{,xy}w_{,xy} + 2\left\{E_{,x}\frac{\partial}{\partial x}(\nabla^2 w) + E_{,y}\frac{\partial}{\partial y}(\nabla^2 w)\right\}\Big]$$

$$= p - \rho h w_{,tt} + \alpha(F,w) \qquad [4.4.3]$$

$$\nabla^4 F = hE\left(w_{,xy} - w_{,xx}w_{,yy}\right) \qquad [4.4.4]$$

It is assumed that the set of equations (4.4.3) and (4.4.4) is satisfied in every region bounded by the contour line C_u As a necess ;ity for the application of "Constant Deflection Countour" method we integrate equqtions (4.4.3) and (4.4.4)over the region.

$$\frac{h^3}{12(1-\nu^2)}\Big[\iint E\left\{A_1\frac{d^4w}{du^4} + A_2\frac{d^3w}{du^3} + A_3\frac{d^2w}{du^2} + A_4\frac{dw}{du}\right\}d\Omega$$

$$+\iint\left\{B_1\frac{dE}{du}\frac{dw}{du} + B_2\frac{d^2E}{du^2}\frac{dw}{du} + B_3\frac{dE}{du}\frac{d^2w}{du^2}\right.$$

$$\left. + B_4\frac{d^2w}{du^2}\frac{d^2E}{du^2} + B_5\frac{dE}{du}\frac{d^3w}{du^3}\right\}d\Omega\Big]$$

$$= \iint(p - \rho h w_{,tt})\,d\Omega + \iint A_5\frac{dw}{du}\frac{dF}{du}\,d\Omega$$

$$+\iint A_6\frac{d}{du}\left\{\left(\frac{dw}{du}\right)\left(\frac{dF}{du}\right)\right\}d\Omega \qquad [4.4.5]$$

$$\iint \left[A_1 \frac{d^4 F}{du^4} + A_2 \frac{d^3 F}{du^3} + A_3 \frac{d^2 F}{du^2} + A_4 \frac{dF}{du} \right] d\Omega$$

$$= - \iint \frac{Eh}{2} \left\{ A_5 \left(\frac{dw}{du} \right)^2 + A_6 \frac{d}{du} \left(\frac{dw}{du} \right)^2 \right\} d\Omega \qquad [4.4.6]$$

where

$$A_1 = \left(u_{,x}^2 + u_{,y}^2 \right)^2$$

$$A_2 = 6\left(u_{,x}^2 u_{,xx} + u_{,y}^2 u_{,yy} \right) + 2\left(u_{,x}^2 u_{,yy} + u_{,y}^2 u_{,xx} \right) + 8 u_{,x} u_{,y} u_{,xy}$$

$$A_3 = 4\left(u_{,x} u_{,xxx} + u_{,y} u_{,yyy} \right) + 3\left(u_{,xx}^2 + u_{,yy}^2 \right)$$

$$+ 4\left(u_{,x} u_{,xyy} + u_{,y} u_{,xxy} \right) + 2 u_{,xx} u_{,yy} + 4 u_{,xy}^2$$

$$A_4 = u_{,xxxx} + u_{,yyyy} + 2 u_{,xxyy}, \qquad A_5 = 2\left(u_{,xx} u_{,yy} - u_{,xy}^2 \right)$$

$$A_6 = u_{,xx} u_{,y}^2 + u_{,yy} u_{,x}^2 - 2 u_{,x} u_{,y} u_{,xy}$$

$$B_1 = u_{,xx}^2 + u_{,yy}^2 + 2\nu u_{,xx} u_{,yy} + 2(1-\nu) u_{,xy}^2 + 2 u_{,x} u_{,xxx}$$

$$+ 2 u_{,y} u_{,yyy} + 2 u_{,x} u_{,xyy} + 2 u_{,y} u_{,xxy}$$

$$B_2 = u_{,x}^2 u_{,xx} + u_{,y}^2 u_{,yy} + 2\nu u_{,y}^2 u_{,xx} + 2\nu u_{,x}^2 u_{,yy}$$

$$+ 2(1-\nu) u_{,x} u_{,y} u_{,xy}$$

$$B_3 = 7\left(u_{,x}^2 u_{,xx} + u_{,y}^2 u_{,yy} \right) + (\nu+2)\left(u_{,x}^2 u_{,yy} + u_{,y}^2 u_{,xx} \right)$$

$$+ 2(5-\nu) u_{,x} u_{,y} u_{,xy}$$

$$B_4 = \left(u_{,x}^2 + u_{,y}^2 \right)^2$$

$$B_5 = 2B_4 = 2\left(u_{,x}^2 + u_{,y}^2 \right)^2 \qquad [4.4.7]$$

Equations (4.4.5) and (4.4.6) can be further reduced to a simpler form on application of Green's theorem wherever possible. On transformation to line integrals equations (4.4.5) and (4.4.6) will become

$$f_1(u)\frac{d^3w}{du^3} + f_2(u)\frac{d^2w}{du^2} + f_3(u)\frac{dw}{du} + f_4(u)p$$

$$+ \rho h\int_1^u w_{,tt}\,du + g_5(u)\frac{dF}{du}\frac{dw}{du} = 0 \qquad [4.4.8]$$

$$g_1(u)\frac{d^3F}{du^3} + g_2(u)\frac{d^2F}{du^2} = g_3(u)\left(\frac{dw}{du}\right)^2 + \int_1^u g_4(u)\left(\frac{dw}{du}\right)^2 du$$

$$[4.4.9]$$

Equations (4.4.8) and (4.4.9) are the two basic equations to study the dynamic response of structures of arbitrary shapes.

An elliptic plate clamped along the edges is considered. The family of contour lines of the deflected surface are as usual represented by

$$u = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \qquad [4.4.10]$$

where u = 0 defines the boundary. The boundary conditions for clamped edges are equations (3.1.1) and (3.1.2)

Suppose the non-linearity is governed by the equation

$$E = E_0\left[1+\beta\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\right]$$

$$\text{i.e. } E = E_0\left[1+\beta(1-u)\right] \qquad [4.4.11]$$

Where β and E_0 are constants.

Inserting the expressions for E, u, u_x etc into equations (4.4.5) and (4.4.6) integrating over the region bounded by C_u, subject to the boundary conditions for clamped edges, one gets respectively

$$\frac{h^3}{12(1-\nu^2)}\left[2E_0 P\left\{(1-u)^2+\beta(1-u)^3\right\}\frac{d^3w}{du^3} - 2E_0 P\left\{2(1-u) + 3\beta(1-u)^2\right\}\frac{d^2w}{du^2} + 4\beta E_0 P'(1-u)\frac{dw}{du}\right] + p(1-u) + \rho h\int_1^u w_{,tt}\,du + \frac{8}{a^2b^2}(1-u)\frac{dF}{du}\frac{dw}{du} = 0 \qquad [4.4.12]$$

$$(1-u)^2\frac{d^3F}{du^3} - 2(1-u)\frac{d^2F}{du^2} = \frac{2E_0 h}{a^2b^2P}\left[\left\{(1-u)+\beta(1-u)^2\right\}\left(\frac{dw}{du}\right)^2 + \beta\int_1^u (1-u)\left(\frac{dw}{du}\right)^2 du\right] \qquad [4.4.13]$$

$$\left.\begin{aligned} P &= \frac{3a^4+3b^4+2a^2b^2}{a^4b^4} \\ P' &= \frac{a^4+b^4+2\nu a^2b^2}{a^4+b^4} \end{aligned}\right\} \qquad [4.4.14]$$

Considering that the plate vibrates primarily in the transverse direction and the plate is restained from in-plane movements, one can assume without any loss of generality [12]

$$\begin{aligned} w &= Au^2\psi(t) \\ F &= Au^2\Phi(t) \end{aligned} \qquad [4.4.15]$$

Since equation (4.4.15) does not represent the exact solution, Galerkin procedure is applied to minimise the error. Substituting equation (4.4.15) into equation (4.4.13) a relation between $\Phi(t)$ and $\psi(t)$ is first established

$$\Phi(t) = -\frac{6}{5}\frac{E_0 h}{P a^2 b^2}\left(1+\frac{7\beta}{144}\right)\psi^2(t) \qquad [4.4.16]$$

while equation (4.4.12) will reduce to

$$\frac{E_0 h^3}{6(1-\nu^2)}\left[P\left(\frac{1}{3}+\frac{\beta}{5}\right)-\beta\frac{P'}{5}\right]A\psi(t)$$

$$+\ 1{\cdot}28\,\frac{E_0 h}{P a^4 b^4}\left(1+\frac{7\beta}{144}\right)A^3\psi^3(t)+\frac{1}{18}\rho h A\psi_{,tt}(t)$$

$$= \frac{p}{12} \qquad [4.4.17]$$

Equation (4.4.17) may be put in a simple form

$$\psi_{,tt} + C_1\psi(t) + C_3\psi^3(t) = C\,p \qquad [4.4.18]$$

$$C_1 = \frac{3E_0 h^2}{\rho(1-\nu^2)}\left[P\left(\frac{1}{3}+\frac{\beta}{5}\right)-\frac{\beta}{5}P'\right]$$

$$C_3 = \frac{23{\cdot}04\,EA^2}{\rho P a^4 b^4}\left(1+\frac{7\beta}{144}\right)$$

$$C = \frac{3p}{2\rho h A} \qquad [4.4.19]$$

a) Free Linear Vibration

For free vibration p = 0 equation (4.4.18) will become

$$\psi_{,tt}(t) + C_1 \psi(t) + C_3 \psi^3(t) = 0 \qquad [4.4.20]$$

The linear frequency parameter is given by

$$\omega = B_1^{1/2} = \left[\frac{3 E_0 h^2}{\rho (1-\nu^2)} \left\{ P\left(\frac{1}{3} + \frac{\beta}{5}\right) - \frac{\beta P'}{5} \right\} \right]^{1/2} \qquad [4.4.21]$$

b) Non-Linear Vibration

If T and T* be the corresponding time periods of linear and non-linear free oscillations then the ratio

$$\frac{T^*}{T} = \left[1 + \frac{3}{4} \frac{C_3}{C_1} \right]^{-1/2} \qquad [4.4.22]$$

where

$$\frac{C_3}{C_1} = \frac{7\cdot 68 \left(1 + \frac{7\beta}{144}\right)(1-\nu^2) m^4 \left(\frac{A_0}{h}\right)^2}{(3m^4 + 2m^2 + 3)\left[(3m^4 + 2m^2 + 3)\left(\frac{1}{3} + \beta/5\right) - \beta/5 \left(m^4 + 2\nu m^2 + 1\right)\right]} \qquad [4.4.23]$$

Where m = a/b, A_0/h = represents the relative amplitude. Numerical results have been computed and shown in tables (22—31)

c) Static case

Negleting the inertial term in equation (4.4.18) one gets for analyzing the large defection behaviour

$$C_1 \Psi(t) + C_3 \Psi^3(t) = C p \qquad [4.4.24]$$

On further simplification one gets the relation between the non dimesional central deflection (W_0/h) and the load parameter (pa^4/Eh^4)

$$\frac{2}{(1-\nu^2)}\left[(3m^4+2m^2+3)\left(\frac{1}{3}+\beta/5\right) - \frac{\beta}{5}\left(m^4+2\nu m^2+1\right)\right]\frac{W_0}{h}$$

$$+ \frac{15\cdot 36\left(1+\frac{7\beta}{144}\right)m^4}{\left(3m^4+2m^2+3\right)}\left(\frac{W_0}{h}\right)^3 = \frac{pa^4}{Eh^4} \qquad [4.4.25]$$

Numerical results are shown in tables(32 — 35)

Table - [22]: Dependence of relative time period of non - linear and linear vibrations[$T^*/_T$] on relative amplitudes [A_0/h] for circular plate for different values of β, υ = 0.3, m=1

A_0/h	$T^*/_T$				
	β = -2	β = -1	β = 0	β = 1	β = 2
0	1.00	1.000	1.000	1.000	1.000
0.5	0.8782	0.9539	0.9705	0.9777	0.9818
1.0	0.6730	0.8449	0.8946	0.9184	0.9323
1.5	0.5171	0.7233	0.7989	0.8388	0.8635
2.0	0.4121	0.6165	0.7045	0.7549	0.7878
2.5	0.3399	0.5300	0.6209	0.6764	0.7142

Table - [23] Dependence of relative time period of non - linear and linear vibrations $[T^*/_T]$ on relative amplitudes $[A_0/h]$ for circular plate for different values of β, υ=0.3, m=1.5

A_0/h	$T/_{T^*}$				
	β= -2	β= -1	β = 0	β = 1	β = 2
0	1.000	1.000	1.000	1.000	1.000
0.5	0.9190	0.9703	0.9811	0.9858	0.9904
1	0.75643	0.8942	0.9300	0.9464	0.9633
1.5	0.60873	0.7982	0.8593	0.8894	0.9222
2.0	0.4978	0.7036	0.7822	0.8240	0.8722
2.5	0.4167	0.6200	0.7076	0.7574	0.8181

Table - [24] Dependence of $[T^*/_T]$ on relative amplitudes $[A_0/h]$ for circular plate for different values of β, υ = 0.3, m = 2

A_0/h	$T^*/_T$				
	β = -2	β = -1	β = 0	β = 1	β = 2
0	1.000	1.000	1.000	1.000	1.000
0.5	0.9569	0.9858	0.9911	0.9933	0.9945
1.0	0.8553	0.9465	0.9656	0.9739	0.9786
1.5	0.7354	0.8896	0.9269	0.9438	0.9536
2.0	0.6300	0.8244	0.8795	0.9057	0.9215
2.5	0.5435	0.7579	0.8276	0.8626	0.8843

Table - [25] Dependence of relative time period of non - linear and linear vibrations $[T^*/_T]$ on relative amplitudes $[A_0/h]$ for circular plate for different values of m, υ = 0.3, β = 1

A_0/h	$T^*/_T$		
	m = 1	m = 1.5	m = 2
0	1.000	1.000	1.000
0.5	0.9777	0.9858	0.9933
1.0	0.9184	0.9464	0.9739
1.5	0.8388	0.8894	0.9438
2.0	0.7549	0.8240	0.9057
2.5	0.6764	0.7574	0.8626

Table - [26] : Dependence of [T/$_T$] on relative amplitudes [A_0/h] for circular plate for different values of m, υ =0.3, β = 2

A_0/h	$T^*/_T$		
	m = 1	m = 1.5	m = 2
0	1.000	1.000	1.000
0.5	0.9818	0.9904	0.9945
1.0	0.9323	0.9633	0.9786
1.5	0.8635	0.9222	0.9536
2.0	0.7878	0.8722	0.9215
2.5	0.7142	0.8181	0.8843

Table - [27] : Dependence of relative time period of non - linear and linear vibrations[$T^*/_T$] on relative amplitudes [A_0/h] for circular plate for different values of m, υ =0.3, β = -1

A_0/h	$T^*/_T$		
	m = 1	m = 1.5	m = 2
0	1.000	1.000	1.000
0.5	0.9539	0.9703	0.9858
1.0	0.8449	0.8942	0.9465
1.5	0.7233	0.7982	0.8896
2.0	0.6165	0.7036	0.8244
2.5	0.5300	0.6200	0.7579

Table - [28] : Dependence of relative time period on relative amplitudes [A_0/h] for elliptic plate for different values of m, υ = 0.3, β = -2

A_0/h	$T^*/_T$		
	m = 1	m = 1.5	m = 2
0	1.000	1.000	1.000
0.5	0.8782	0.9190	0.9569
1.0	0.6730	0.7564	0.8553
1.5	0.5171	0.6087	0.7354
2.0	0.4121	0.4978	0.6300
2.5	0.3399	0.4167	0.5435

Table - [29] : Dependence of $T^*/_T$ on relative amplitudes [A_0/h] for circular plate for different values of m, $\upsilon = 0.3$, $\beta = -0.7$

A_0/h	$T^*/_T$		
	m = 1	m = 1.5	m = 2
0	1.000	1.000	1.000
0.5	0.9608	0.9784	0.9880
1.0	0.8647	0.9087	0.9543
1.5	0.7523	0.8221	0.9046
2.0	0.6492	0.7335	0.8460
2.5	0.5630	0.6525	0.7847

Table - [30] : Dependence of relative time period of non - linear and linear vibrations $T^*/_T$ on relative amplitudes [A_0/h] for circular plate for different values of υ, m = 1, $\beta = 2$

A_0/h	$T^*/_T$		
	$\upsilon = 0.2$	$\upsilon = 0.3$	$\upsilon = 0.5$
0	1.000	1.000	1.000
0.5	0.9808	0.9812	0.9844
1.0	0.9289	0.9323	0.9415
1.5	0.8574	0.8635	0.88032
2.0	0.7796	0.7878	0.81116
2.5	0.7047	0.7142	0.7418

Table - [31] : Dependence of relative time period of non - linear and linear vibrations $T^*/_T$ on relative amplitudes [A_0/h] for circular plate for different values of υ, $\beta = -1$, m =2

A_0/h	$T^*/_T$		
	$\upsilon = 0.2$	$\upsilon = 0.3$	$\upsilon = 0.5$
0	1.000	1.000	1.000.
.5	0.9849	0.9858	0.9886
1.0	0.9432	0.9465	0.9565
1.5	0.8835	0.8896	0.9089
2.0	0.8156	0.8244	0.85239
2.5	0.7472	0.7579	0.79267

Table [32] : Dependence of central deflection (W_0/h) on load parameter (pa^4/Eh^4) for different values of β, $\upsilon = 0.3$, m = 1

W_0/h	pa^4/Eh^4				
	$\beta = -2$	$\beta = -1$	$\beta = 0$	$\beta = 1$	$\beta = 2$
0	0	0	0	0	0
0.2	0.2365	0.3319	1.187	1.6629	2.1384
0.4	0.5563	0.7515	2.466	3.4106	4.3780
0.6	1.0425	1.3465	3.93	5.3754	6.8198
0.8	1.7782	2.2045	5.67	7.6183	9.5649
1.0	2.8468	3.4133	7.78	10.2477	12.7146
1.2	4.3313	5.0602	10.34	13.3601	16.3698
1.4	6.31507	7.2336	13.47	17.0525	20.6317
1.6	8.8814	10.0206	17.24	21.4214	25.6014
2.0	16.0935	17.7861	27.08	32.5752	38.0688

Table [33] : Dependence of central deflection (W_0/h) on load parameter (pa^4/Eh^4) for different values of β, m = 2, $\upsilon = 0.3$

W_0/h	pa^4/Eh^4				
	$\beta = -2$	$\beta = -1$	$\beta = 0$	$\beta = 1$	$\beta = 2$
0	0	0	0	0	0
0.2	1.712	5.1949	8.6772	12.1609	15.6437
0.4	3.6046	10.58	17.5542	24.1609	31.5068
0.6	5.8581	16.3456	26.5542	37.3214	47.8087
0.8	8.6531	22.6818	36.7059	50.74047	64.7686
1.0	12.17	29.779	47.385	64.998	82.606
1.2	16.5892	37.8272	5.9.0524	80.3039	101.5401
1.4	22.0884	47.016	71.923	96.8677	121.7904
1.6	28.8568	57.538	86.1913	114.8993	143.5763
2.0	46.9	83.336	119.72	156.204	192.632

Table [34] : Dependence of central deflection (W_0/h) on load parameter (Pa^4/Eh^4) for different values of m, $\upsilon = 0.3$, $\beta = 2$

W_0/h	pa^4/Eh^4		
	m = 1	m = 1.5	m = 2
0	0	0	0
0.2	2.1384	6.04	15.6437
0.4	4.378	12.2606	31.5068
0.6	6.8198	18.8421	47.8087
0.8	9.5649	25.9651	64.7686
1.0	12.7146	33.81	82.606
1.2	16.3698	42.5572	101.5401
1.4	20.6317	52.3874	121.7904
1.6	25.6014	63.464	143.5763
2.0	38.0688	90.18	192.632

Table [35] : Dependence of central deflection (W_0/h) on load parameter (Pa^4/Eh^4) for different values of υ, m = 2, $\beta = 1$

W_0/h	pa^4/Eh^4		
	$\upsilon = 0.2$	$\upsilon = 0.3$	$\upsilon = 0.5$
0	0	0	0
0.2	11.5949	12.1509	14.5771
0.4	23.3995	24.5315	29.3639
0.6	35.6234	37.3214	44.5694
0.8	48.4764	50.7404	60.4052
1.0	62.168	64.998	77.079
1.2	76.9079	80.3039	94.8011
1.4	92.9	96.8677	113.7811
1.6	110.3713	114.8993	134.2289
2.0	150.544	156.204	180.366

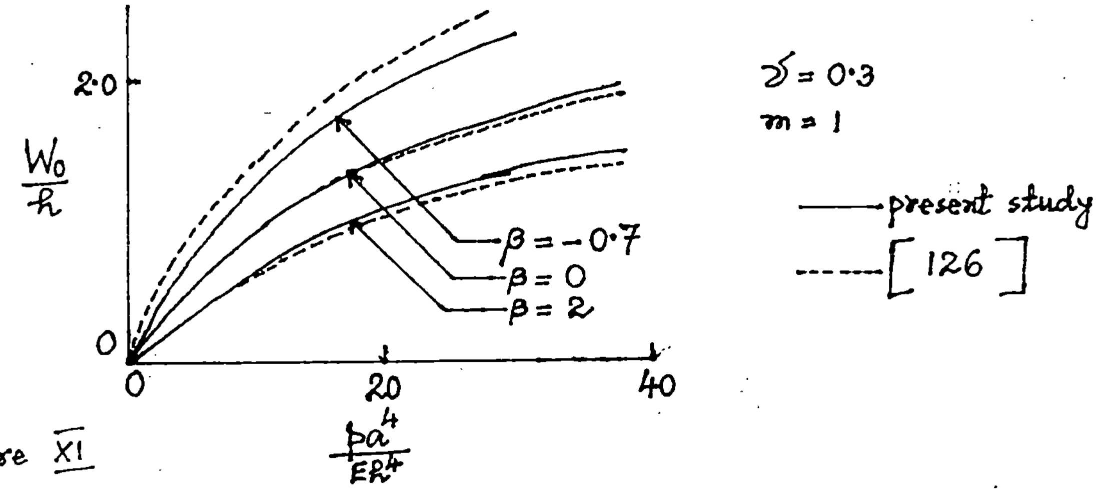

Figure XI :- Non-linear Static Behaviour of a Clamped Circular Plate with varying Flexural Rigidity

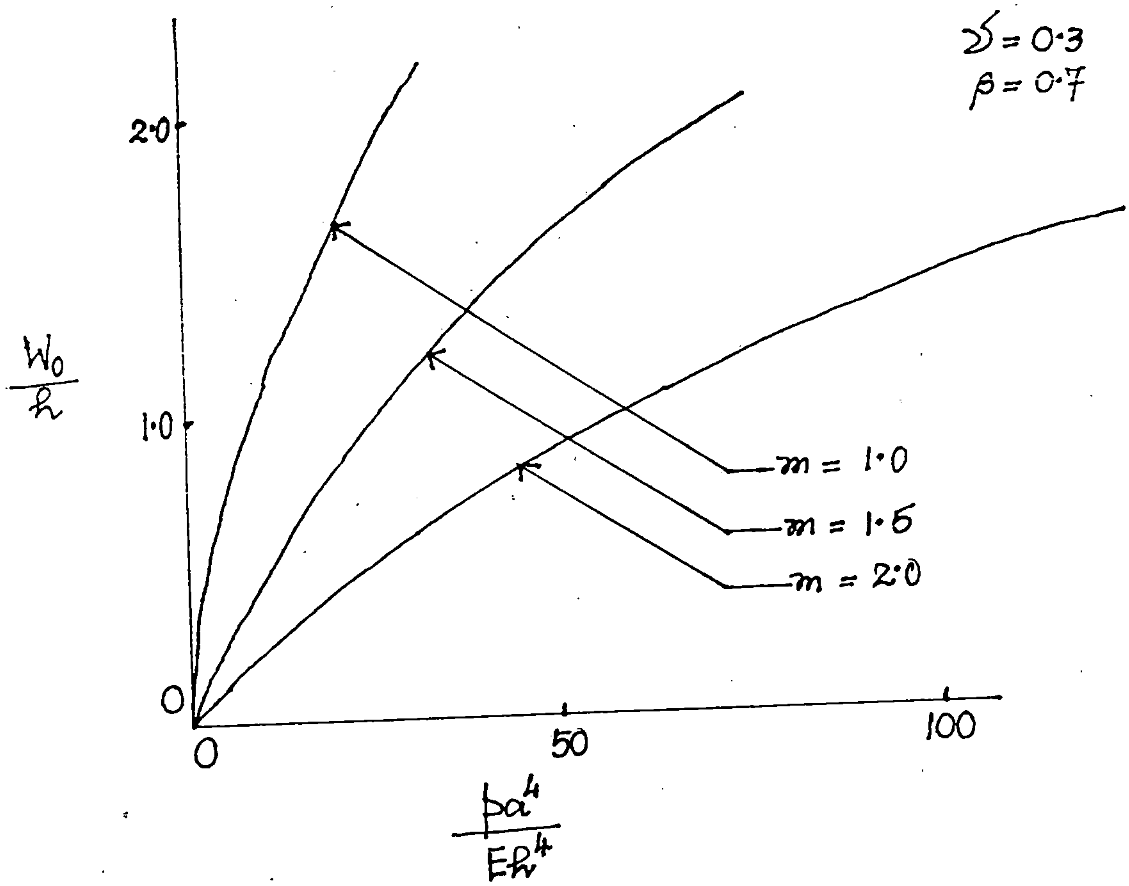

Figure XII :- Non-linear Static Behaviour of Clamped Elastic Plates with varying Flexural Rigidity for Different Values of Aspect Ratios

Discussion :

From tables(23 & 24), it is observed that for a particular value of m the effect of non-linearity increases with the decreasing value of β whereas for a particular value of β the effect of nonlinearity decreases with increasing value of m from tables(25 to 29).

Normally the variation of Poisson's ratio is overlooked since the effect in large vibration is marginal. However tables (30 & 31) show that for moderately large vibration the non-linear effect may be taken into account in the sense that the effect of non linearity appears to be appreciable when value of Poisson's ratio for the corresponding material deereases.

The numerical results for static deflection for the present study have been compared with those of [126] and what has been observed is that the present results are not in exact agreement with those of [126], particularly when $\beta < 0$, but for $\beta > 0$ the results agree well with those of [126]. The first reason for slight difference may be caused due to what has been explained in the very beginning of this chapter (page - 39)

The second reason is the procedural difference, Ohanabe el.al [126] has used Berger equations whereas in the present analysis Karman equations have been employed. Since Karman equations are reliable than those of Berger, hence the present results may be more acceptable than those presented in Ref. [126].