

## CHAPTER - IV

### Time - hardening and time - softening elastic plates.\*

#### Nomenclature :

The following nomenclature are used in this paper.

- $W$  = deflection, normal to the middle plane,  
 $D$  = flexural rigidity of the plate =  $\frac{Eh^3}{12(1-\sigma^2)}$ ,  
 $\rho$  = density, supposed constant,  
 $h$  = thickness of the plate,  
 $E$  = Young's modulus,  
 $\sigma$  = Poisson's ratio.

#### Introduction :

Earlier periods in the development of the theory of Elasticity were mainly concerned with materials of homogeneous elastic properties. After then, the generalization was done from isotropy to anisotropy especially in problems on crystal Elasticity.

A recent generalization has been made from homogeneity to non-homogeneity of the material in which elastic properties vary from point to point. This has helped much in the improvement of the design of structures.

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A still another generalization of the elastic properties has been made by Paria (1966). The Young's modulus has been assumed to be function of time but independent of co-ordinates. If the Young's modulus increases as time increases, the material will be said to be time-hardening. In the time-softening material, Young's modulus is assumed to decrease with increasing time. The physical justification for such assumption may be found in concrete, for example, in which the elastic properties vary during its aging periods. The elastic properties of a material may vary from season to season during the year due to variations in temperature, moisture contents and similar other varying factors. To illustrate his assumption Paria (1966) has solved the problem of free vibration of elastic rod with Young's modulus as the function of time.

In this paper the author has attempted to introduce this new idea to the case of different elastic plates.

### Theory.

Let us consider the transverse vibration of elastic plates. The differential equation for such free vibration is

[Love, A.E.H. (1927), PP.464 ]

$$D\nabla^4 W + Sh \frac{\partial^2 W}{\partial t^2} = 0$$

... (1)

If the Young's modulus depends on time, the above equation reduces to

$$\nabla^4 W + \frac{\lambda}{E(t)} \cdot \frac{\partial^2 W}{\partial t^2} = 0 \quad \dots (2)$$

where

$$\lambda = \frac{12 \beta (1 - \sigma^2)}{h^2} = \text{constant}$$

Let

$$E(t) = E_0 (1 + \alpha e^{-t/t_0}) \quad \dots (3)$$

Where  $E_0$  and  $t_0$  are constants,  $\alpha$  being a parameter.

$E(t)$  varies from  $E_0(1 + \alpha)$  at  $t = 0$  to  $E_0$  as  $t \rightarrow \infty$ . If  $\alpha$  is negative, the material is time-hardening and if  $\alpha$  is positive, it is time-softening.

Putting (3) in (2) we have,

$$E_0 (1 + \alpha e^{-t/t_0}) \nabla^4 W + \lambda \frac{\partial^2 W}{\partial t^2} = 0 \quad \dots (4)$$

Let

$$W = \omega(x, y) \cdot \phi(t) \quad \dots (5)$$

Substituting (5) in (4) we have,

$$\frac{d^2 \phi}{dt^2} + \frac{\nabla^4 \omega}{\omega} \cdot \frac{E_0 (1 + \alpha e^{-t/t_0})}{\lambda} \phi = 0 \quad \dots (6)$$

1. Free vibration of a rectangular plate.

We consider the transverse vibration of a rectangular plate having its boundary simply-supported.

Let the centre of the plate be taken as the origin.

Let us assume  $w(x,y)$  in the form

$$w(x,y) = A \cos \frac{(2m+1)\pi x}{2a} \cdot \cos \frac{(2n+1)\pi y}{2b} \quad \dots (7)$$

It is evident that this form of  $w(x,y)$  satisfies the following boundary conditions for the simply-supported edges.

$$\left. \begin{aligned} w = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at } x = \pm a \\ w = \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at } y = \pm b \end{aligned} \right\} \quad \dots (8)$$

Substituting (7) in (6) we have,

$$\frac{d^2 \phi}{dt^2} + \frac{K_{mn}^2}{\lambda} \cdot E_0 (1 + \alpha e^{-t/t_0}) \phi = 0 \quad \dots (9)$$

where

$$K_{mn}^2 = \frac{\pi^4}{16} \left[ \frac{(2m+1)^2}{a^2} + \frac{(2n+1)^2}{b^2} \right]^2$$

To solve (9), let us put

$$\tau = e^{-t/2t_0} \quad \dots (10)$$

Substituting (10) in (9), we have

$$\frac{d^2\phi}{d\tau^2} + \frac{1}{\tau} \frac{d\phi}{d\tau} + \frac{4K_{mn}^2 E_0 t_0^2}{\lambda} \left( \alpha + \frac{1}{\tau^2} \right) \phi = 0 \quad \dots (11)$$

If  $\alpha$  is positive, we write  $\alpha = \alpha_1^2$ . The equation (11) reduces to

$$\frac{d^2\phi}{d\tau^2} + \frac{1}{\tau} \frac{d\phi}{d\tau} + q_{mn}^2 \left( \alpha_1^2 + \frac{1}{\tau^2} \right) \phi = 0 \quad \dots (12)$$

where

$$q_{mn}^2 = \frac{4K_{mn}^2 E_0 t_0^2}{\lambda}$$

If  $\alpha$  is negative, we write  $\alpha = -\alpha_2^2$ . Then the equation (11) reduces to

$$\frac{d^2\phi}{d\tau^2} + \frac{1}{\tau} \frac{d\phi}{d\tau} - q_{mn}^2 \left( \alpha_2^2 - \frac{1}{\tau^2} \right) \phi = 0 \quad \dots (13)$$

We shall consider the case only when  $\alpha$  is positive. The case when  $\alpha$  is negative can be treated in the similar manner by replacing Bessel functions by Modified Bessel functions.

The solution of (12) can be put in the form

$$\phi = A_1 J_{iq_{mn}}(z_1) + A_2 J_{-iq_{mn}}(z_1) \quad \dots (14)$$

where

$$z_1 = \alpha_1 q_{mn} z = \alpha_1 q_{mn} e^{-t/2t_0} \quad \dots (15)$$

Hence

$$W = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn} \cos \frac{(2m+1)\pi x}{2a} \cdot \cos \frac{(2n+1)\pi y}{2b} \left[ A_{mn} J_{iq_{mn}}(z_1) + B_{mn} J_{-iq_{mn}}(z_1) \right] \quad \dots (16)$$

Let the initial conditions be

$$\left. \begin{aligned} W(x, y, 0) &= \sum \sum C_{mn} \cos \frac{(2m+1)\pi x}{2a} \cdot \cos \frac{(2n+1)\pi y}{2b} \\ \frac{\partial}{\partial t} W(x, y, 0) &= \sum \sum D_{mn} \cos \frac{(2m+1)\pi x}{2a} \cdot \cos \frac{(2n+1)\pi y}{2b} \end{aligned} \right\} \quad \dots (17)$$

where  $C_{mn}$  and  $D_{mn}$  are known constants.

Combining equations (16) and (17) and solving for the constants we have,

$$A_{mn} = \frac{1}{\alpha_1 \Delta_1} \left[ \frac{2D_{mn}t_0}{q_{mn}} J_{-iq_{mn}}(\alpha_1 q_{mn}) + \alpha_1 C_{mn} J'_{-iq_{mn}}(\alpha_1 q_{mn}) \right]$$

$$B_{mn} = -\frac{1}{\alpha_1 \Delta_1} \left[ \frac{2D_{mn}t_0}{q_{mn}} J_{iq_{mn}}(\alpha_1 q_{mn}) + \alpha_1 C_{mn} J'_{iq_{mn}}(\alpha_1 q_{mn}) \right]$$

where  $\Delta_1 = -\frac{2i \sinh \pi q_{mn}}{\pi q_{mn} \alpha_1}$

Hence

$$\begin{aligned}
 W(x, y, t) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos \frac{(2m+1)\pi x}{2a} \cdot \cos \frac{(2n+1)\pi y}{2b} \\
 &\times \frac{\pi q_{mn}}{2i \sinh \pi q_{mn}} \left[ \frac{2D_{mn}t_0}{q_{mn}} \left\{ J_{iq_{mn}}(z_0) \cdot J_{-iq_{mn}}(z_1) \right. \right. \\
 &\quad \left. \left. - J_{iq_{mn}}(z_1) J_{-iq_{mn}}(z_0) \right\} + \alpha_1 C_{mn} \left\{ J_{-iq_{mn}}(z_1) J'_{iq_{mn}}(z_0) \right. \right. \\
 &\quad \left. \left. - J_{iq_{mn}}(z_1) J'_{-iq_{mn}}(z_0) \right\} \right] \dots (18)
 \end{aligned}$$

where  $z_1 = \alpha_1 q_{mn} e^{-t/2t_0}$ ,  $z_0 = \alpha_1 q_{mn}$

Now for numerical computation it will be convenient to replace the products of Bessel functions of imaginary orders by the equivalent series involving Hypergeometric functions as follows.

$$\begin{aligned}
 &J_{-iq_{mn}}(\alpha_1 q_{mn}) J_{iq_{mn}}(\alpha_1 q_{mn} e^{-t/2t_0}) \\
 &= \frac{e^{-itq_{mn}/2t_0}}{\Gamma(1+iq_{mn})} \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda \left(\frac{\alpha_1 q_{mn}}{2}\right)^{2\lambda}}{\Gamma(\lambda) \Gamma(1-iq_{mn}+\lambda)} {}_2F_1\left(-\lambda, iq_{mn}-\lambda; iq_{mn}+1; e^{-t/t_0}\right) \dots (19)
 \end{aligned}$$

Similarly

$$\begin{aligned}
 & J_{iq_{mn}}(\alpha_1 q_{mn}) \cdot J_{-iq_{mn}}(\alpha_1 q_{mn} e^{-t/2t_0}) \\
 &= \frac{e^{\frac{itq_{mn}}{2t_0}}}{\Gamma(1-iq_{mn})} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\alpha_1 q_{mn}}{2}\right)^{2n}}{\Gamma(n) \Gamma(1+iq_{mn}+n)} \\
 & \quad \times {}_2F_1(-n, -iq_{mn}-n; -iq_{mn}+1; e^{-t/t_0}) \dots (20)
 \end{aligned}$$

Also

$$\begin{aligned}
 & J'_{-iq_{mn}}(z_0) J_{iq_{mn}}(z_1) - J'_{iq_{mn}}(z_0) J_{-iq_{mn}}(z_1) \\
 &= \frac{1}{2} \left[ \left\{ J_{-1-iq_{mn}}(z_0) \cdot J_{iq_{mn}}(z_1) - J_{-1+iq_{mn}}(z_0) \cdot J_{-iq_{mn}}(z_1) \right\} \right. \\
 & \quad \left. - \left\{ J_{1-iq_{mn}}(z_0) \cdot J_{iq_{mn}}(z_1) - J_{1+iq_{mn}}(z_0) \cdot J_{-iq_{mn}}(z_1) \right\} \right] \dots (21)
 \end{aligned}$$

But

$$\begin{aligned}
 & J_{-1-iq_{mn}}(\alpha_1 q_{mn}) \cdot J_{iq_{mn}}(\alpha_1 q_{mn} e^{-t/2t_0}) \\
 &= \frac{2e^{\frac{-itq_{mn}}{2t_0}}}{\alpha_1 q_{mn} \Gamma(1+iq_{mn})} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\alpha_1 q_{mn}}{2}\right)^{2n}}{\Gamma(n) \Gamma(n-iq_{mn})} \times {}_2F_1(-n, 1-n+iq_{mn}; 1+iq_{mn}; e^{-t/t_0}) \dots (22)
 \end{aligned}$$

$$\begin{aligned}
 & J_{-1+iq_{mn}}(\alpha_1 q_{mn}) \cdot J_{-iq_{mn}}(\alpha_1 q_{mn} e^{-t/2t_0}) \\
 &= \frac{2e^{\frac{itq_{mn}}{2t_0}}}{\alpha_1 q_{mn} \Gamma(1-iq_{mn})} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\alpha_1 q_{mn}}{2}\right)^{2n}}{\Gamma(n) \Gamma(n+iq_{mn})} \times {}_2F_1(-n, 1-n-iq_{mn}; 1-iq_{mn}; e^{-t/t_0}) \dots (23)
 \end{aligned}$$

$$\begin{aligned}
& J_{1-iq_{mn}}(\alpha_1 q_{mn}) \cdot J_{iq_{mn}}(\alpha_1 q_{mn} e^{-t/2t_0}) \\
&= \frac{\alpha_1 q_{mn} e^{-\frac{itq_{mn}}{2t_0}}}{2\Gamma(1+iq_{mn})} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\alpha_1 q_{mn}}{2}\right)^{2n}}{\Gamma(n)\Gamma(n+2-iq_{mn})} {}_2F_2\left(-n, -1-n+iq_{mn}; 1+iq_{mn}; e^{-t/t_0}\right) \dots (24)
\end{aligned}$$

$$\begin{aligned}
& J_{1+iq_{mn}}(\alpha_1 q_{mn}) \cdot J_{-iq_{mn}}(\alpha_1 q_{mn} e^{-t/2t_0}) \\
&= \frac{\alpha_1 q_{mn} e^{\frac{itq_{mn}}{2t_0}}}{2\Gamma(1-iq_{mn})} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\alpha_1 q_{mn}}{2}\right)^{2n}}{\Gamma(n)\Gamma(n+2+iq_{mn})} {}_2F_2\left(-n, -1-n-iq_{mn}; 1-iq_{mn}; e^{-t/t_0}\right) \dots (25)
\end{aligned}$$

If  $\alpha \rightarrow 0$ , equation (18) reduces to ( setting  $n = 0$  in the equivalent Hypergeometric series )

$$\begin{aligned}
W &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos(2m+1)\frac{\pi x}{2a} \cdot \cos(2n+1)\frac{\pi y}{2b} \times \\
& \times \left[ \frac{2D_{mn}t_0}{q_{mn}} \sin \frac{q_{mn}t}{2t_0} + C_{mn} \cos \frac{q_{mn}t}{2t_0} \right] \dots (26)
\end{aligned}$$

which is the classical result.

If the parameter  $\alpha$  in the assumed law is assumed to be small, the perturbed deflection of various orders may be obtained by collecting the different powers of  $\alpha$ , in the given Hypergeometric series as follows.

The first perturbation is obtained by putting  $n = 1$  in (19), (20), (22) and (23) and  $n = 0$  in (24) and (25).

Thus the first perturbed deflection is found to be

$$\begin{aligned}
 W(x, y, t) = & -\frac{\alpha_1^2}{4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_{mn}^2 \cos(2m+1)\frac{\pi x}{2a} \cdot \cos(2n+1)\frac{\pi y}{2b} \\
 & \times \left[ \frac{D_{mn} 2t_0}{q_{mn}(1+q_{mn}^2)} \left\{ (1+e^{-t/t_0}) \sin \frac{q_{mn} t}{2t_0} - (1-e^{-t/t_0}) \cos \frac{q_{mn} t}{2t_0} \right\} \right. \\
 & + \frac{q_{mn} C_{mn}}{1+q_{mn}^2} \left\{ q_{mn} \left( \sin \frac{q_{mn} t}{2t_0} + \cos \frac{q_{mn} t}{2t_0} \right) \right. \\
 & \left. \left. + e^{-t/t_0} \left( \cos \frac{q_{mn} t}{2t_0} - q_{mn} \sin \frac{q_{mn} t}{2t_0} \right) \right\} \right] \dots (27)
 \end{aligned}$$

Collecting terms of higher order of  $\alpha$  in like manner, the higher perturbations may be obtained.

## 2. Free-vibration of isocoles right-angled triangular Plate.

Let us consider the transverse vibration of an isocoles right-angled triangular plate having its boundary simply-supported. The equal sides of the plate are considered to be of lengths  $a$  in the directions of  $X$  and  $Y$ . Let us assume  $w(x, y)$  in the form

$$W = C \left[ \sin \frac{2m\pi x}{a} \cdot \sin \frac{m\pi y}{a} + \sin \frac{2m\pi y}{a} \cdot \sin \frac{m\pi x}{a} \right] \dots (28)$$

where  $m$  is an odd integer.

It is evident that this form of  $\omega$  satisfies the following boundary conditions for the simply-supported edges.

$$\left. \begin{aligned} \omega &= \frac{\partial^2 \omega}{\partial x^2} = 0 && \text{at } x = 0 \\ \omega &= \frac{\partial^2 \omega}{\partial y^2} = 0 && \text{at } y = 0 \\ \omega &= \frac{\partial^2 \omega}{\partial v^2} = 0 && \text{at } x+y = a \end{aligned} \right\} \dots (29)$$

where  $\frac{\partial}{\partial v} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)$

Substituting (28) in equation (6) and proceeding as before  $\phi$  is determined in the form,

$$\phi = C_1 J_{iq_m}(z_2) + C_2 J_{-iq_m}(z_2) \dots (30)$$

where

$$q_m^2 = \frac{4k_m^2 E_0 t_0^2}{\lambda}, \quad k_m^2 = \frac{49\pi^4 m^4}{a^4}$$

$$z_2 = \alpha_1 q_m e^{-t/2t_0}$$

Hence

$$W = \sum_{m=1,3,5,\dots}^{\infty} \left( \sin \frac{2m\pi x}{a} \cdot \sin \frac{m\pi y}{a} + \sin \frac{2m\pi y}{a} \cdot \sin \frac{m\pi x}{a} \right) \times$$

$$\times \left[ A_m J_{iq_m}(z_2) + B_m J_{-iq_m}(z_2) \right] \dots (31)$$

Let the initial conditions be

$$\left. \begin{aligned} W(x, y, 0) &= \sum C_m \left( \sin \frac{2m\pi x}{a} \cdot \sin \frac{m\pi y}{a} + \sin \frac{2m\pi y}{a} \cdot \sin \frac{m\pi x}{a} \right) \\ \frac{\partial}{\partial t} W(x, y, 0) &= \sum D_m \left( \sin \frac{2m\pi x}{a} \cdot \sin \frac{m\pi y}{a} + \sin \frac{2m\pi y}{a} \cdot \sin \frac{m\pi x}{a} \right) \end{aligned} \right\} \dots (32)$$

where  $C_m$  and  $D_m$  are known constants.

Putting (31) in (32) and solving for the constants, we have

$$A_m = \frac{l}{\alpha_1 \Delta_2} \left[ \frac{D_m t_0}{q_m} J_{-iq_m}(\alpha_1 q_m) + \alpha_1 C_m J'_{-iq_m}(\alpha_1 q_m) \right]$$

$$B_m = -\frac{l}{\alpha_1 \Delta_2} \left[ \frac{D_m t_0}{q_m} J_{iq_m}(\alpha_1 q_m) + \alpha_1 C_m J'_{iq_m}(\alpha_1 q_m) \right]$$

where

$$\Delta_2 = -\frac{2i \sinh \pi q_m}{\pi q_m \alpha_1}$$

Hence  $W(x, y, t)$  is known.

To obtain the classical result as well as the numerical computation together with different perturbed deflections, it will be convenient to replace the products of Bessel functions of imaginary orders by the equivalent series involving Hypergeometric functions as shown in the case of rectangular plate.