Chapter 10

DIFFERENTIAL EQUATION AND CANTOR SET

10.1 Introduction

A multiplicative representation (8.9) for the solution of Cauchy problem (8.1) is discussed in detail in the previous chapter. Because of it's multiplicative structure there is a direct relevance of the said problem in the context of a Cantor set (or, equivalently to an ultrametric space). Here we assume that the problem IVP (8.1) is defined on a Cantor set **C** which is realized as an inequivalent ultrametric space. As a consequence, each point of a Cantor set is replaced by the infinitesimal copy of an inverted Cantor set $\mathbf{C}_i$. Thus, the said scale invariant equation of chapter 9 actually is well defined on a closed set of the form $\widetilde{\mathbf{C}} = \mathbf{C} + \mathbf{C}_i$ that replaces a given Cantor set and so the above multiplicative model (8.8) is given a novel interpretation. As already established (c.f., Sec. 7.2), $\widetilde{C}$ is almost surely I or a positive measure set C_p. The results of this chapter are presented in Ref.[26].

10.2 Mathematical results

We study the relationship of the scale free DE of the form

$$x\frac{dX}{dx} = X \tag{10.1}$$

with a Cantor set C. As a preparation, let us recall how the simplest Cauchy problem

$$\frac{dX}{dx} = 1, \ X\ (1) = 1 \tag{10.2}$$

is solved on the interval $I = [0, 1]$. One considers a partition $0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$. The desired result $X\ (x) = x$ is obtained as a limit of a sum: $\lim_{\substack{\Delta x_j \to 0 \\ n \to \infty}} \sum_{j=1}^{i} \Delta\, x_j$ where $x_{i-1} < x < x_i$. The scale free Cauchy problem

$$x\frac{dX}{dx} = X, \ X\ (1) = 1 \tag{10.3}$$

is also solved exactly in an analogous fashion.

First, we note that the neighbourhood of a point x_0 is mapped to that of $x = 1$ by a rescaling $x \to \frac{x}{x_0}$. So we concentrate only in the neighbour of $x = 1$. Let $x_\pm = 1 \pm \eta$, $X_\pm = X\ (x_\pm)$. Then the DE in (10.3) splits into two branches

$$x_\pm \frac{dX_\pm}{dx_\pm} = X_\pm. \tag{10.4}$$

The equation is already solved in chapter 8 and the solution is expressed in a non-trivial form

$$X_- = \prod_{i=0}^{\infty} \frac{1}{1 + \eta^{2^i}}. \tag{10.5}$$

The right hand branch, however, has the form

$$X_+ = \frac{1}{1-\eta}\prod_{i=1}^{\infty}\frac{1}{1+\eta^{2^i}}. \tag{10.6}$$

The infinite product representation of X_-, for instance, is interpreted as follows. The first iterated value exceeds the exact value by an amount $\frac{\eta^2}{1\pm\eta}$ which is canceled progressively in a self -similar manner over smaller and smaller inverted scales log $\left(1-\eta^{2^i}\right)^{-1}$, $i = 1, 2, 3, \cdots$.

We note that the higher order correction factors $X_c = \prod_{i=1}^{\infty} \frac{1}{1+\eta^{2^i}}$ may therefore be re-interpreted as a deletion process: viz; a portion of a line segment is deleted progressively and self -similarly, analogous to the formation of a Cantor set. Alternately, a product of the form $(1-\eta)\ (1+\eta) = 1-\eta^2$ could be considered to represent a deletion: a length of size η in the neighbourhood of 1^-, is deleted progressively as $(1-\eta)(1+\eta)(1+\eta^2)\cdots\left(1+\eta^{2^{n-1}}\right) = \left(1-\eta^{2^n}\right)$, $n\to\infty$.

A second possibility is to interpret the multiplicative iteration process defined above as a dynamical process in which the dynamic (independent) variable undergoes increments not by the usual linear translations but by inversions (hoppings) over smaller and smaller sizes. This then provides one with a mechanism of deletion process stated above. Actually, these logarithmic scales inhabits concomitant smaller scales of the from η^i. To justify this in a greater detail, let us assume that the scale free problem (10.3) is now defined on a closed subset $\tilde{\mathbf{C}} \subseteq I$, called an *inverted Cantor set*, where $\tilde{\mathbf{C}} = \bigcup_i \tilde{I}_i$ is a countable union of disjoint closed intervals

$\tilde{I}_i$ of varying sizes. $\tilde{I}_i$ in fact, is the closure of a corresponding gap O_i (inclusive of the end points) of the original Cantor set C. Suppose, a dynamic variable (say, a particle) in motion on this set $\tilde{\mathbf{C}}$, hops between the end points of each of such disjoint closed intervals $\tilde{I}_i$, following the scale free DE (10.3). Let $|\ \tilde{I}_0 \ |= \eta$ be the maximum hopping size. Then the smaller hopping sizes are η proportion of the remaining sizes of the set $\tilde{\mathbf{C}}$ viz., $|\ \tilde{I}_i \ |= \eta\ (1-\eta)^{i-1} \equiv \eta_i$ (say), when we assume $|\ \tilde{\mathbf{C}}\ | = 1$. Because of the rescaling symmetry (scale invariance) of the DE (10.3) each of the component intervals $\tilde{I}_i$ could be imagined to have been symmetrically placed at 1 with end points, say, at $x_{0\pm} = \frac{1}{2}\ (1 \pm \eta)$. Now, the particle at left end points x_{0-} of $\tilde{I}\ _0$ hops to the right end point x_{0+} following the rule

$$x_{0-} \ \rightarrow x_{0-}^{-1} \ = x_{0+}\ X_1 \tag{10.7}$$

so that we have, using equation (10.6)

$$(1-\eta)^{-1} = (1+\eta)\ X_c,\ X_c = \prod_{i=1}^{\infty} \left(1+\eta^{2^i}\right) \tag{10.8}$$

because of the scale invariance. Equation (10.8) tells that hopping motion of the type considered above, of any given size η is accomplished by an infinite cascade of self similar smaller scale inverted motions of sizes η^{2^i}, $i = 1, 2, \cdots$. The total length covered by all these self-similar jumps , viz., 1, is reached multiplicatively i.e. as $1 = \lim_{n\to\infty} \left(1-\eta^{2^n}\right)$, reminiscent of an ultrametric limiting process. Notice that in the ordinary sense, the total jump size is determined additively as an infinite series viz., $\sum_1^{\infty} \eta\ (1-\eta)^{i-1} = \Sigma\eta_i \ = \ 1.$

Accordingly, each $x \in C$ is replaced by infinitesimal copy of an inverted Cantor set $\tilde{\mathbf{C}}$. Because of scale invariance, the DE (10.3) at an $x \in C$, which is actually not defined in the usual (even in the natural ultrametric) sense, is now raised to an equation which is well defined on a closed set of the form $\tilde{\mathbf{C}}$.

We note that the solution (10.5) and (10.6) is the standard solution derived in an unconventional way and interpreted non-trivially. On a Cantor set, however, the equation (10.3) can accommodate a host of new solutions in consonance with the multiplicative model interpretation. The origin of these new solutions could be explained in the context of locally constant functions (LCF). To justify, in a most natural way, the existence of locally constant functions, let us write a solution of equation (10.3) in the form

$$X = x.x^{\phi(x^{-1})}. \tag{10.9}$$

The function $\phi(x)$ here represents a LCF and is defined by the scale free equation on logarithmic variables, viz:

$$\log x^{-1} \frac{d\phi}{d \log x^{-1}} = \phi. \tag{10.10}$$

Clearly $\phi(x)$ corresponds to our non-trivial valuation (3.2) denoted $v(\tilde{x}) = | \tilde{x} |_u$. To verify $v(\tilde{x})$, indeed is a LCF, we note that

$$\frac{d}{dx} v(\tilde{x}) = \lim_{\epsilon \to 0} \frac{d}{dx} \left(\frac{\log x}{\log \epsilon} + 1 \right) = 0. \tag{10.11}$$

Equation (10.10), on the other hand, reveals the variability of a LCF over smaller logarithmic scales. Of course, the valuation also passes this test

$$\log v(\tilde{x}) = \log\log \tfrac{x}{\epsilon} + \log\log_\epsilon \lambda - \log\log \epsilon^{-1}$$

leading to equation (10.10) in the inverted rescaled real variable $\frac{x}{\epsilon}$ (in the log log scale). We have already seen that $v(\tilde{x})$ relates to an appropriate Cantor function. Consequently, a Cantor function $\phi(x)$ is shown to be a LCF with variability over log log scales. Equation (10.9) constitutes an ultrametric extension not only of a Cantor set, but of any connected interval of R.

We summerise the above findings as a theorem thus extending theorem 1 in a significant manner.

Theorem 3. *An element of an ultrametric Cantor set C is replaced by the set of gaps of the Cantor set C_i where relative infinitesimals are supposed to live in. Increments on such an extended Cantor set C is accomplished by following an inversion rule of the form (3.1). A scale free differential equation of the form equation (10.3) is well defined on such an ultrametric space and accommodates Cantor functions as locally constant functions. The associated infinitesimal valuation $v(\tilde{x})$ is a locally constant function with variability over double logarithmic scales.*