CHAPTER — I

INTRODUCTION

1.1. Preliminaries.

The subject of intersection graphs has been an important area of study for the last three decades. Given a family of sets, each assigned to a vertex, the intersection graph of the family of sets has an edge between two of these vertices iff the corresponding sets intersect. Interval graphs form a class of intersection graphs and have a long and rich history. They arose from purely mathematical considerations by Hajös [1957], and independently from a problem of genetics by the Noble prize winner, Benzer [1959]. A special issue of *Discrete Mathematics* [1985] is devoted to the deduction of significant new results on interval graphs and other related classes of graphs. For an overview of this issue one may consult the paper by Golumbic [1985]. The idea of studying intersection graphs of boxes, spheres and other geometrical objects was introduced by Hadwiger, DeBrunner and Klee [1964]. This idea was subsequently developed by Roberts [1969a, 1969b, 1976] where he took boxes and cubes in the Euclidean space R^n as his objects. There have also been a number of variants of intersection graphs. The present chapter of this dissertation defines most of these variants and summarizes them in order to provide the background material for this dissertation.

In our thesis, we start with directed graphs and wish to represent a directed graph in terms of intersection of certain families of objects. While a lot of work has been done on various aspects, viz., characterization, recognition algorithm and intersection parameters such as interval number, boxicity, cubicity and sphericity and many other parameters of intersection graphs, practically no significant work has been done on directed graphs from the perspective of intersection representation. Since undirected graphs are special types of directed graphs, the problem of finding an analogous and natural translation of the theory of intersection graphs along with its abundant literature to directed graphs is very well worth investigating. Directed graphs having intersection representation will be termed *intersection digraphs*, and will be seen to be the natural analogue of intersection graphs.Beineke and Zamfirescu[1982] first introduced this idea under the name *connection digraphs* in the context of proving certain results on the line graph of a graph.

In the second chapter, we study interval digraphs, which are intersection digraphs of intervals on the real line. In this chapter, we shall give several characterizations of interval digraphs analogous to the results of interval graphs. One such characterization will require the notion of Ferrers digraphs and the Ferrers dimension of a digraph. The <u>Ferrers dimension</u> of a digraph is the minimum number of Ferrers digraphs whose intersection is the given digraph. We will see in this characterization how the complement of an interval digraph can be decomposed into two Ferrers digraphs.

Depending upon the decomposition of the complement of an interval digraph into two Ferrers digraphs, we introduce in the third chapter, the notion of interior edges of these two Ferrers digraphs and show how these interior edges are related to interval digraphs.

In the fourth chapter, we introduce the notion of circular-arc digraphs. We first rephrase our characterization of interval digraphs in a way that leads to a natural characterization of circular-arc digraphs. A circular-arc digraph is characterized in terms of the generalized circular ones property. This characterization is closely related to Tucker's [1971] characterization of circular-arc graphs.

Since the area we study is virtually a new one we leave many questions unresolved and unanswered. In conclusion, we pose some of the problems that come naturally as we proceed alongwith the important question of complexity.

1.1.1. Definitions.

Given a graph G(V,E), V = V(G) will denote its vertex set and E = E(G) will denote its edge set.

A graph H is a *subgraph* of a graph G(V,E) if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The graph G is then said to contain H. A subgraph H will be said to be a *generated subgraph* or an *induced subgraph* of G if $V(H) \subset V(G)$ and two vertices x and *y* are adjacent in H iff they are adjacent in G. The subgraph H is then said to be induced or generated by V(H).

The *complement* G(V,E) of a graph G(V,E) has the same vertex set V and two vertices are adjacent iff they are not adjacent in G. A set of vertices in a graph G is said to be an *independent set* (or a *stable set*) of vertices if no two vertices in the set are adjacent.

A *bipartite graph* is a graph G(V,E) whose vertex set V can be partitioned into two disjoint sets V_0 and V_1 such that every edge of G joins V_0 with V_1. A *complete bipartite graph* (CBG) is a bipartite graph G such that G contains every possible edge between V_0 and V_1. A *complete bipartite subgraph (CBS)* of a graph G is a subgraph B of G such that B is itself a complete bipartite graph. A *complete m-partite graph* of a graph G can be similarly defined.

A graph G(V,E) is a threshold graph if there exists a threshold assignment [a;t] consisting of a labelling a of the vertices by non-negative integers and an integer threshold t such that for a subset X of V

$$X \text{ is stable} \iff \sum_{x\in X} a(x) \leq t$$

A directed graph (or digraph) D(V,E) consists of a finite nonempty set V of points together with a collection E of ordered pairs of points. For a digraph D(V,E), an edge uv $\in$E is represented by a line segment between u and v with an arrow directed from u to v. All throughout, we assume that a digraph may have loops but no multiple edges.

The converse D′(V,E′) of a digraph D(V,E) has the same vertex set V and an edge uv is in D′ if and only if the edge vu is in D.

The successor set of a vertex v is the set of vertices u such that vu is an edge of the digraph D(V,E). The predecessor set of a vertex v is the set of vertices u such that uv is an edge of the digraph D(V,E).

A digraph D(V,E) is a Ferrers digraph if for every $x,y,z,w \in V$,

$$xy,\ zw \in E \Rightarrow xw \text{ or } zy \in E.$$

The <u>adjacency</u> <u>matrix</u> $A(D) = (a_{ij})$ of a digraph $D(V,E)$ with n vertices is a nxn matrix , where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E \\ 0, & \text{otherwise} \end{cases}$$

The <u>augmented</u> <u>adjacency</u> <u>matrix</u> $A^*(D)$ of a digraph $D(V,E)$ is the adjacency matrix $A(D) = (a_{ij})$ together with $a_{ii} = 1$.

Other definitions on graph theory will be given as and when needed throughout the chapters. For background materials on graph theory one is referred to Harary [1969], Roberts [1976] and Golumbic [1980].

1.1.2. Notations.

In general any graph theoretic notation is that of Harary [1969]. For convenience, the most frequently used notations are listed here.

$G(V,E)$	:	A graph whose vertex set is V and edge set is E
$D(V,E)$	:	A digraph with vertex set V and edge set E
Adj x	:	The set of all vertices adjacent to x
$H \subseteq D$	:	H is a subdigraph of D
K_n	:	A complete graph with n vertices
$\bar{D}$	:	Complement of D
$H(D)$	:	A graph associated with D
$H_1 \cup H_2$	:	Union of H_1 and H_2
$H_1 \cap H_2$	:	Intersection of H_1 and H_2
$\lfloor x \rfloor$	:	Greatest integer less than or equal to x
$\lceil x \rceil$	:	Least integer greater than or equal to x

1.2. Intersection graphs.

Given a finite family of sets $\mathscr{F}$, a graph $G(V,E)$ is an intersection graph of the family $\mathscr{F}$ if there exists a function S that assigns to each vertex $v \in V(G)$ a set $S_v \in \mathscr{F}$ such that for all $u,v \in V(G)$,

$$uv \in E \iff S_u \cap S_v \neq \phi.$$

Note that S_u and S_v may be equal even if $u \neq v$. The first obvious question that arises in this connection is : "Is every graph an intersection graph of some family of sets $\mathscr{F}$?". Marczewski [1945] answered this question affirmatively. On the other hand one may specify the types of families of sets and then ask under what conditions a graph may be an intersection graph of families of sets of this type. A large part of this vast topic of intersection graphs is devoted to answering this question for specific types of families of sets. An excellent introduction to the theory and problems of intersection graphs may be found in Golumbic [1980].

We introduce in Chapter II, the notion of intersection digraphs analogous to undirected intersection graphs. Given a family $\mathscr{F}$ of ordered pairs of subsets (S_v, T_v) of a set X, a digraph $D(V,E)$ is an *intersection digraph* of the family $\mathscr{F} = \{(S_v, T_v)\}$ when there is a one-one correspondance between V and $\mathscr{F}$ and $uv \in E$ iff the corresponding sets have a non-empty intersection.

Naturally, as in the case of undirected graphs here also the question arises :"Is every digraph an intersection digraph?" and we will answer to this question again in the affirmative. Here also one may start with specific types of sets and search for finding conditions under which a digraph can be represented as an intersection digraph of a family taken from these sets. A large part of our thesis mainly centers round answering these questions where the sets are intervals on real line and arcs on a circle.

1.3. Interval graphs.

1.3.1. *Interval graphs and their applications.*

An interval graph is the intersection graph of a family of intervals on the real line. The intervals may be open, closed or half-open. This idea was first introduced by Hajös [1957]. Again, at about the same time, during the investigation of the fine structure of the gene, the well-known molecular biologist Benzer [1959] introduced independently the same idea of interval graphs. It can easily be shown that not every graph is an interval graph. So the question arises, under what conditions can one associate to each vertex v of a graph an interval I_v of the real line such that

$$uv \in E \iff I_u \cap I_v \neq \phi \ ?$$

Benzer [1959] posed the question : "Are the subelements within the gene arranged in a linear order ?" By making recombination tests of the mutation of the gene, he determined overlap data on the mutations, that is, whether or not the substructures of the corresponding mutation overlap. If we define a graph G whose vertices are the substructures studied in the tests and whose edges are pairs of overlapping substructures, then G is an interval graph iff the substructures can be linearly arranged. He found he did have interval graphs.

Since that time interval graphs have been used in many diverse areas such as psychology, archeology, management sciences, computer science, ecology etc. For instance, interval graphs have been used in seriation using overlap data, by Kendall [1963,1969a,b,1971a,b,c] Hubert [1974] and Skrien [1980] in archaeology, and by Coombs and Smith [1973] in developmental psychology. Roberts [1976,1978b,1979c], Stoffers [1968] and Opsut and Roberts [1980] used interval graphs in arriving at solutions to general traffic phasing problems. Another general type of problem whose solutions have incorporated the interval graph is that of fleet maintenance (Golumbic [1980], Opsut and Roberts [1980]).

Still another application of interval graphs is to the frequency assignment problem. This problem is exactly analogous to the fleet maintenance problem and is studied, among others, by Gilbert [1972], Pennotti [1976] and Opsut and Roberts [1980].

Since every graph is the intersection graph of some family of sets $\mathcal{F}$, one might ask just how likely is it that a graph is an interval graph. Cohen et.al. [1979],using exact analysis, asymptotic theory and Monte Carlo simulation, estimated the probability of a random graph to be an interval graph.

1.3.2. Some characterizations of interval graphs.

In 1957, Hajös [1957] showed that interval graphs are necessarily triangulated graphs. A graph is a <u>triangulated graph</u> (or a chordal graph) if every cycle of length strictly greater than 3 possesses a chord. With the help of this result, Lekkerkerker and Boland [1962] first characterized interval graphs by defining an asteroidal triple. A collection of three pairwise non- adjacent vertices v_1, v_2, v_3 of a graph G is called an <u>asteroidal triple</u> of the given graph G, if for each pair v_i, v_j $(i \neq j)$ there exists a path joining them, such that none of its vertices is adjacent to the third vertex v_k $(k = i, j)$. Fig.1.1. shows two graphs each with an asteroidal triple v_1, v_2, v_3.

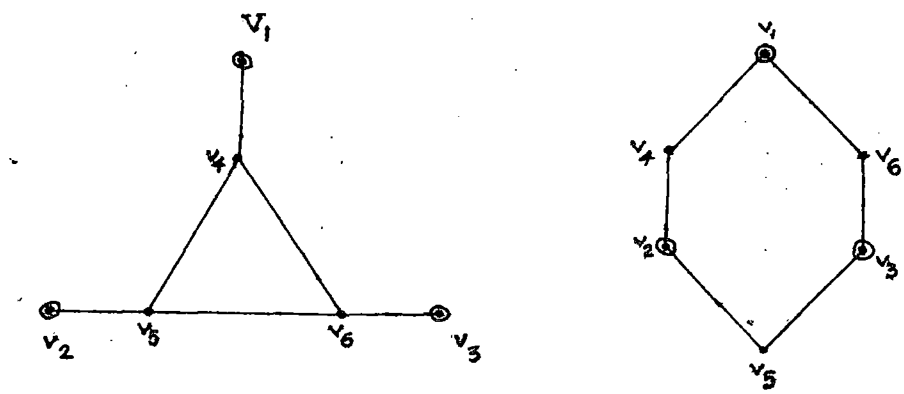


Fig.1.1.

Theorem 1.1. [Lekkerkerker Boland, 1962]. *A graph G is an interval graph if and only if G is a triangulated graph and contains no asteroidal triple.*

In that paper, they again provided a complete set of forbidden subgraphs for interval graphs. Actually, they proved the following :

Theorem 1.2. *A graph G is an interval graph if and only if G contains none of the forbidden subgraphs of Fig.* 1.2.*as an induced subgraph.*

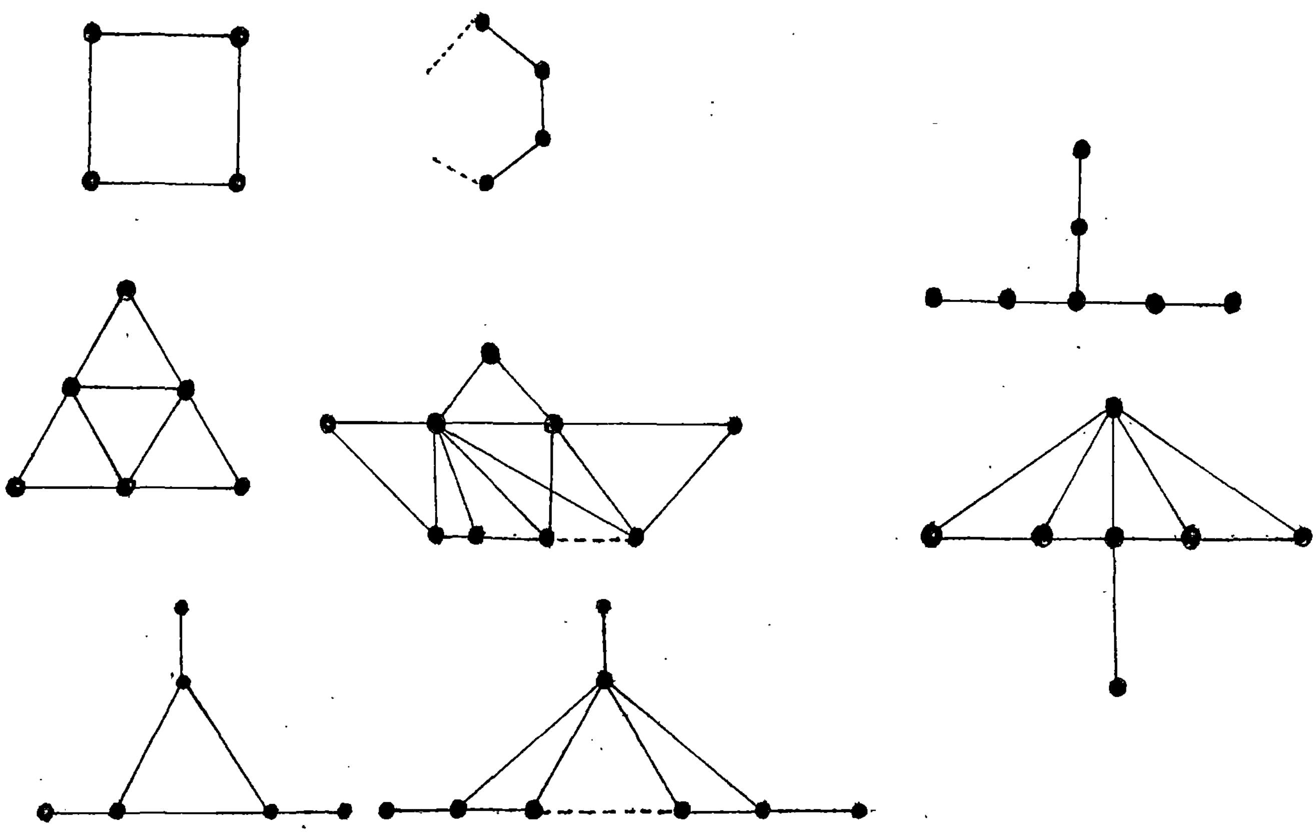


Fig.1.2.

Another characterization of interval graphs was given by Gilmore and Hoffman [1964] in which they relate interval graphs to another graph, known as comparability graph.

A digraph is <u>transitive</u> if whenever there is an arc from u to v and an arc from v to w, $u \neq w$, there is an arc from u to w. An <u>orientation</u> of a graph G is an assignment of a direction to each of the edges of the graph to get a digraph. A graph has a <u>transitive orientation</u> if there is an orientation of its edges so that the resulting digraph is transitive. Graphs which have transitive orientations are sometimes called <u>comparability graphs</u>. They have been characterized by Ghouila-Houri [1962] and Gilmore-Hoffman [1964].

Theorem.1.3 [Gilmore-Hoffman, 1964] *An undirected graph G is an interval graph if and only if G is triangulated and its complement $\bar{G}$ is a comparability graph.*

A <u>clique</u> of a graph is a set of vertices which form a complete induced subgraph. A clique is called <u>maximal</u> if it is not contained in any larger clique of the graph. An ordering $K_1, K_2, \ldots, K_n$ of maximal cliques is *consecutive* if for $p < q < r$ and for any vertex v that belongs to K_p and K_r, v belongs to K_q. If G is a graph, its <u>vertex maximal clique incidence matrix</u> $M = (m_{ij})$ is the matrix whose rows and columns correspond to the vertices and the maximal cliques respectively of the graph and

$$m_{ij} = \begin{cases} 1, & \text{if the i-th vertex} \in \text{ the j-th column} \\ 0, & \text{otherwise} \end{cases}$$

A matrix is said to have consecutive ones property for rows if it is possible to permute the columns of the matrix so that ones in each row appear consecutively. Fulkerson and Gross [1965] gave an efficient algorithm to test whether a (0,1) matrix has this property or not. Ryser [1969] also studied consecutive ones property and certain generalizations. Tucker [1972] has characterized the consecutive ones problem in terms of forbidden configurations. Note that the vertex-maximal clique incidence matrix has the consecutive ones property for rows if and only if G has an ordering of the maximal cliques that is consecutive.

Theorem 1.4. [Fulkerson and Gross, 1965]. *A graph G is an interval graph if and only if its vertex-maximal clique incidence matrix has the consecutive ones property for rows.*

Mirkin [1972] also obtained a characterization of the interval graphs in terms of the augmented adjacency matrix of the graph. A matrix is said to be quasi- diagonalizable if for simultaneous row and column permutations, consecutive ones appear in each row, starting at the main diagonal.

Theorem 1.5. [Mirkin,1972]. *A graph is an interval graph if and only if its augmented adjacency matrix is quasi-diagonalizable.*

Smadici [1987] defined a set N(G) corresponding to a graph G as the set of all edges of G with the property that for each end vertex of such an edge there exists a vertex adjacent to it and non -adjacent to the other end vertex. He then used this notion to obtain yet another characterization of an interval graph.

Scheinerman [1988] introduced two equivalent models of random interval graphs. Several results about the number of edges, degrees, chromatic number and other indices of almost all interval graphs were also established.

Booth and Lueker [1976] used the Fulkerson and Gross [1965] characterization to get a linear time algorithm [Sec. 1.5] to recognize an interval graph.

Theorem 1.6. [Booth-Lueker, 1976] *Interval graphs can be recognized in linear time.*

This recognition method was based on the general consecutive arrangement problem, which is as follows : Given a finite set X and a collection $\mathcal{M}$ of subsets of X, does there exist a permutation π on X in which the members of each subset $M \in \mathcal{M}$ appear as a consecutive subsequence of π ? The consecutive arrangement and consecutive ones problems are equivalent.

In the interval graph problem, X is the set of maximal cliques and $\mathcal{M} = \{M(v)\}_{v \in V}$, where M(v) is the set of all maximal cliques containing v. An extensive survey of algorithmic aspects of comparability graphs and interval graphs will be found in Golumbic [1980] and Möhring [1985].

1.3.3 Interval graphs and interval orders.

Fishburn [1970] and Mirkin [1970] introduced independently the concept of interval orders, with the help of which they obtained several characterizations of interval graphs.

A binary relation (A,P) is called an <u>interval order</u> if for all a,b,c,d $\in$ A the following condition hold :

(S_1) P is irreflexive

(S_2) aPb and cPd $\Longrightarrow$ aPd or cPb.

Fishburn [1970] gave the following characterization of interval orders.

Theorem 1.7. [Fishburn,1970]. *A partial order $\langle V,P\rangle$ is an interval order if and only if the elements $v \in V$ can be represented by intervals I_v on the real line such that*

$$uv \in P \iff I_u \text{ lies entirely to the left of } I_v \quad (I_u < I_v).$$

He also proved the following :

Theorem 1.8. [Fishburn, 1970]. *A partial order is an interval order if and only if it does not contain a suborder isomorphic to Fig.* 1.3.

Fig. 1.3

A relation (V,R) is known as Ferrers relation if the sets of successors $S(v) = \{u \in V \,/\, vu \in R\}$, $v \in V$ (or equivalently, the sets of predecessors $P(v) = \{u \in V \,/\, uv \in R\}$, $v \in V$) are linearly ordered by inclusion. Mirkin [1970] called it quasi-linear relation. Interval orders can be completely characterized by Ferrers relations as shown by Mirkin [1970].

Theorem 1.9. [Mirkin, 1970]. *An irreflexive relation is an interval order if and only if it is a Ferrers relation.*

The digraphs corresponding to Ferrers relations are known as Ferrers digraphs. As our work is very much related to the theory of Ferrers digraphs, we shall discuss these digraphs in detail in Sec.1.9. Fishburn [1970] explored the close relationship between interval graphs and interval orders. For a given interval graph G, an interval order can be obtained from a transitive orientation of $\tilde{G}$, the complement of G; on the other hand the symmetric complement of a given interval order can be seen to produce an interval graph.

These two results together show us how the two notions of interval graphs and interval orders are closely related.

Theorem 1.10. [Fishburn,1970]. *A graph $G(V,E)$ is an interval graph if and only if there exists a binary relation R on $V(G)$ such that (V,R) is an interval order and $\bar{E} = R \cup R^{-1}$ where $\bar{G}(V,\bar{E})$ is the complement of the graph G.*

For further reference on the relationship between interval graphs and interval order the recent paper by Trotter [1988] may be seen.

1.3.4. Notions related to interval graphs.

The solution of many real world problems are seen to depend on graphs which are either interval graphs or are related to interval graphs and so the scope of research in this area is widening regularly. In this section, a brief survey of several graphs related to interval graphs in a natural way are given.

A well-known gereralization of interval graphs, introduced independently by Trotter and Harary [1979] and Griggs and West [1980], is to represent each point not by a single interval but by a finite set of intervals. A graph is a <u>t-interval graph</u> if it is the intersection graph of, at most, t-intervals on the real line. In particular. a <u>double interval graph</u> is a graph which possesses an intersection representation of its vertices by two intervals. It can be easily seen that any graph is a t-interval graph for some t. So the natural parameter in studying multiple interval graphs is the interval number i(G) of a given graph G, defined as the least t for which the given graph G is a t-interval graph. For an interval graph. the interval number i(G) is 1. Multiple interval graphs and interval numbers have also been studied by Griggs [1979], Harary and Kabell [1980], Scheinerman [1984,1985a,b] Erdös and West [1985]. For some specific graphs, interval numbers have been computed by Mathews and Trotter [1978],Hopkins and Trotter [1981], and Scheinerman and West [1983], Hopkins, Trotter and West [1984] and Thomas [1987].

Cozzens [1981] defined a <u>cointerval graph</u> as the complement of an interval graph and gave a list of forbidden subgraphs characterizing cointerval graphs. Benzaken, Hammer and deWerra [1985] characterized the class of interval graphs whose complements are also interval graphs. They gave a short proof that this class is equivalent to the class of split graphs with Dilworth number at most 2. The <u>Dilworth number</u> of a graph G is the minimum k for which there exists a partition of the vertex set into k chains. A forbidden subgraph characterization and a linear time recognition algorithm [Sec.1.5] were also presented for these graphs. Harary and McMorria [1987] again studied the same problem of graphs and bigraphs having interval and cointerval properties. They showed that both a graph G and its complement $\bar{G}$ are interval graphs iff G has no induced subgraph isomorphic to any of a list of seven graphs. They also extended the question to bipartite graphs.

Skrien [1984] characterized those graphs whose line graphs are interval graphs. The <u>line graph</u> L(G) of a graph G is the intersection graph of the family of edges of G. Skrien and Gimble [1985] called an interval graph G <u>homogeneously representable</u> if for every vertex v there exists an interval representation of G in which the interval representing v is the left-most (or right-most) interval. They proved that an interval graph is homogeneously representable if and only if it has no induced subgraph isomorphic to the graphs given by the following graphs :

Fig. 1.4

An <u>unbounded interval graph</u> is the intersection graph of a family of halflines in R^1. Klavzar and Petkovsêk [1987] showed that unbounded interval graphs are precisely the triangulated graphs whose complements are bipartite. Further characterization in terms of forbidden subgraphs were given. The intersection graph of a finite family of halflines in R^2 which is simply a join of families of unbounded interval graphs were also characterized in terms of forbidden subgraphs.

Fishburn [1984] studied the number of interval lengths used in a representation of an interval order. <u>Interval count</u> of an interval graph is the minimum number of interval lengths in an interval representation of the graph. Graphs with interval count 2 were studied by Lebowitz [1978] and Skrien [1984]. Fishburn and Graham [1985] studied the classes of interval graphs that arise when the lengths of the representing intervals are closed real intervals with lengths in $[1,\alpha]$.

Recall that for an interval representation $\{(I_v)\}_{v\in V}$ of a graph $G(V,E)$, an interval order (V,P) can be obtained by orienting all the edges of the complementary graph $\bar{G}$ according to the following rule :

$$uv \in P \iff I_u < I_v.$$

Given an interval graph G(V,E), different interval representations of G will produce different interval orders. Fishburn [1985b] considered the number of interval orders that have the same interval graph. Those interval graphs which have an essentially unique interval order (V,P) were characterized in two ways. The first follows from Hanlon's [1982] analysis on counting interval graphs. In a very large paper, Hanlon extensively enumerated various sorts of interval graphs alongwith unit interval graphs [sec.1.4]. The enumeration makes use of a structural decomposition of interval graphs which leads to a characterization of those interval graphs having a unique interval order representation. The second was given by Fishburn [1985b] which is based on an equivalence relation on ordered pairs of points in V. He also considered the problems of the bounds on the lengths of the representing intervals, the number of distinct left end points that have to be used in a representation, and extremization problems for interval graphs and interval orders.

In chapter-II, we study intersection representation of digraphs by a family of ordered pairs of intervals on the real line, i.e., the *interval digraphs*.

1.4. Unit interval graphs.

Unit interval graphs form a restricted class of interval graphs. A graph G is a <u>unit interval graph</u> if it is the intersection graph of a set of closed intervals of unit length (like interval graphs, closedness and openness is not arbitrary for unit interval graphs and we specify closed intervals). These graphs were characterized by Roberts [1969a] where he also showed that this class is equivalent to other two classes of graphs known as proper interval graphs and indifference graphs.

A graph is a <u>proper interval graph</u> if it can be represented by a family of intervals on the real line such that no interval properly contains another. A graph $G(V,E)$ is an <u>indifference graph</u> if given $\delta > 0$, there is an assignment f of real numbers to the vertices of G such that for all $x, y \in V$,

$$xy \in E \iff |f(x) - f(y)| < \delta.$$

Apart from showing that these three classes of graphs are exactly the same, he also gave the following characterizations of a unit interval graph :

Theorem 1.11. [Roberts, 1969a]. *A graph G is a unit interval graph (indifference graph, proper interval graph) if and only if G is an interval graph and contains no* $K_{1,3}$ *as an induced subgraph.*

Maehara [1980] also did some work on this class of graphs under the name of time graphs, much of which was nevertheless anticipated by Roberts.

Roychaudhuri [1987] defined the k-th power of a graph G as the graph G^k with the vertex set $V(G^k) = \{v_1, \ldots, v_n\}$ and the edge set $E(G^k)$ such that $v_i v_j \in E(G^k)$ iff the distance $d_G(v_i, v_j) \leq k$. She then proved that if G^{k-1} is an interval (respectively unit interval) graph then G^k is an interval (respectively unit interval) graph.

1.4.1. Semiorder

In developing theoretical models for the measurement of preference, Scott and Suppes [1958] studied the following representation problem : Given a binary relation (A,P) and a positive number δ, under what condition does there exist a real valued function f on the set A such that for all $x,y \in A$,

$$xPy \iff f(x) > f(y) + \delta \text{ ?} \qquad \ldots\ldots\ldots\ldots (1)$$

They showed that the representable pairs (A,P) correspond precisely to the notion of semiorder. A binary relation (A,P) is called a *semiorder* if for all $a,b,c,d \in A$, the following conditions hold :

(S_1) P is irreflexive

(S_2) apb and cPd $\Rightarrow$ aPd or cPb

(S_3) aPb and bPc $\Rightarrow$ aPc or cPd

Note that a binary relation is a semiorder if and only if it is an interval order [sec.1.3.] and satisfies the condition

(S_3). The concept of semiorder was originally introduced by Luce [1956] in a rather different way. The definition given above is due to Scott and Suppes [1958].They proved the following :

Theorem 1.12. [Scott and Suppes, 1958]. *If A is finite, then $\langle A,P\rangle$ is representable in the form (1) if and only if $\langle A,P\rangle$ is a semiorder.*

As an immediate consequence of the celebrated Scott Suppes Theorem [1958] on semiorder, Roberts obtained the following characterization of unit interval graphs.

Theorem 1.13. [Roberts, 1969a] *A graph $G\langle V,E\rangle$ is an indifference graph if and only if there exists an orientation P of the edges of the complementary graph $\bar{G}$ such that P defines a semiorder on V.*

Indifference graphs have applications in the frequency assignment problem. Another major application of indifference graphs is in solving problems in seriation [Hubert,1974]. The approach to seriation which uses overlap data is used in archaeology by Kendall [1969,1971], in developmental psychology by Coombs and Smith [1973] and in organizing opinion data in political science. It is hoped that indifference graphs will find lot of applications in the area of decision making i.e. to measure preference, utility, compatibility etc.

Ducamp and Falmagne [1969] introduced a generalization of semiorder, called bisemiorder. A <u>bisemiorder</u> is a quadruple (S,E,R,T) where S and E are two sets and R and T are two relations on $S \cup E$, $R \subseteq S \times E$ satisfying certain axioms ; it was shown that it is equivalent to finding functions $s : S \longrightarrow R$ and $e : E \longrightarrow R$ and two positive numbers δ_1, δ_2 $(\delta_1 > \delta_2)$ such that for all $a \in S$, $b \in E$

$$aRb \iff s(a) > e(b) + \delta_1$$

$$aTb \iff s(a) > e(b) + \delta \; .$$

Cozzens and Roberts [1981] discussed <u>double semiorder</u>, considering a pair of binary relations P_1 and P_2 on the same set A and seeking necessary and sufficient conditions for the existence of a real valued function f on A and two positive numbers δ_1, δ_2 $(\delta_1 > \delta_2)$ so that for all $x, y \in A$

$$xP_1Y \iff f(x) > f(y) + \delta_1$$

$$xP_2y \iff f(x) > f(y) + \delta_2$$

For a graph theoretical version of the above problem, they again introduced the notion or double indifference graph, a multigraph of certain type, as an obvious analogue of an indifference graph.

1.5. Efficiency of algorithms.

An algorithm is a step by step procedure to solve a problem. The time complexity is a measure of time required to execute an algorithm. The algorithm whose time complexity is bounded above by some polynomial P(n), where n is the input variable, is known as a polynomial time algorithm or *efficient algorithm*. The algorithms of other problems for which the execution time increases exponentially with respect to n are known as exponential time algorithms or *inefficient algorithms*. A problem which can be solved by a polynomial time algorithm is called a tractable (or *computationally easy*) problem. The class of these problems is denoted by P. A problem for which no efficient algorithm exists is an intractable problem and the class of these problems is denoted by NP, signifying a non-deterministic polynomial. The class NP includes all the problems of the class P i.e. $P \leq NP$.

The theory of NP-completeness was initiated by Cook [1971] and after that Karp [1972] presented a large collection of NP-complete problems. These problems have the property that all known algorithms for solving them require exponential time. This property imposes severel limitations on the size of the graph that can be handled by such a programme. What makes the NP-complete problems so important is that it has been shown that these problems are either all tractable or all intractable [Cook, 1971]. That is, these problems have the property that if any one of them can be solved by a polynomial algorithm, then a polynomial algorithm exists for each of them, whereas if it can be proved that no polynomial algorithm is possible for any of them, then the same will be true for all of them. It is now conjectured that all NP-complete problems are intractable.

The class NP-complete contains many classical problems, such as travelling salesman problem and the integer programming problem in operations research, vertex colouring problem in graph theory and the multiple copy file allocation problem in computer science. In fact, proving a problem is NP-complete says that the problem is just as hard as a large number of other problems that are widely recognized as being difficult. Indeed, the collection of NP-complete problems is growing regularly with time. A partial catalogue of these problems may be found in Garey and Johnson [1979] as well as a detailed description of the theory of NP-completeness.

1.6. Boxicity, Cubicity, and other dimensions.

Interval graphs have been generalized in several ways. One such generalization is an <u>n-dimensional interval graph</u> (or *Box representation*) of a graph, introduced by Roberts

[1969b]. This type of graph is the intersection graph of a family of boxes with sides parallel to the coordinate axes in n-dimensional Euclidean space R^n. Precisely, a <u>box in R^n</u> is the n-cartesian product $\overset{n}{\underset{i=1}{X}}(J_i)$, where J_i's are intervals (open, closed or half-open) on the real line. In the plane, a box is simply a rectangle with sides parallel to the coordinate axes. It can easily be established that any graph G with n vertices is an n-dimensional interval graph and consequently it becomes meaningful to find the least n, for which G is an intersection graph of boxes in R^n. The <u>boxicity</u> of a graph G, denoted by b(G), is the smallest integer k such that G is representable as an intersection graph of boxes in R^k. Note that $b(G) \leq 1$ iff G is an interval graph. If every vertex of a graph can be represented by one and the same real number, we say $b(G)= 0$ iff G is a complete graph. Quest and Wegner [1990] gave a characterization of the graphs with $b(G) \leq 2$, by means of the arrangements of zeros and ones in some special matrices attached to the graph. Roberts [1969b] showed that boxicity is well defined for each graph and a graph G with n vertices has boxicity at most $\lfloor n/2 \rfloor$. This result was rediscovered by Wittenshausen [1980]. Roberts [1969b] also observed that if $n \geq 1$, a graph G is representable as the intersection graph of boxes in R^n if and only if G is the intersection or n interval graphs. The notion of boxicity

has applications to problems of niche overlap in ecology [Cohen,1978], Roberts [1978b] and to fleet maintenance problems in operation research [Opsut and Roberts,1980].

Recall that a pair of edges of a graph G is said to be *independent* if they have no vertices in common. An *independent set of edges* is a set of edges, each pair of which is independent. Gabai [1974] presented a number of bounds on boxicity. In particular, he showed that if $\bar{G}$ contains an independent set of k edges as a generated subgraph, then $b(G) \geq k$. After Gabai's work, very little was added to what was known about boxicity until Trotter [1979] characterized those graphs G on n vertices such that $b(G) = \lfloor n/2 \rfloor$.

Cozzens and Roberts [1983] showed that the problem of computing boxicity is NP-complete. They developed some techniques for computing boxicity which give useful bounds. They are based on the observation that $b(G) \leq k$ if and only if there is an edge covering of $\bar{G}$ by subgraphs of $\bar{G}$, each of which is a cointerval graph, the complement of an interval graph.

Scheinerman [1984] proved that every planer graph has an intersection representation by sets, each of which is the union of two boxes in the plane. Thomassen [1986] showed that every planar graph is the intersection graph of a collection of three dimensional boxes, with intersections occuring only in the boundaries of the boxes.

The analogue to unit intervals in higher diminsional space is unit boxes with sides parallel to coordinate axes. The <u>cubicity</u> of a graph G, denoted by c(G) is the least integer k such that G is the intersection graph of unit boxes in R^k. Note that G is the intersection graph of unit boxes in R^k iff G is the intersection of k unit interval graphs. Kabell [1980] called a graph with cubicity k, a k-dimensional unit interval graph. Clearly, $b(G) \leq c(G)$, for if G is the intersection graph of unit boxes in R^k, then it is the intersection graph of boxes in R^k. Roberts [1969b] first introduced the notion of cubicity and proved that cubicity is well- defined for any graph G. In fact, he showed that if c(G) denotes the cubicity of a graph G, then $c(G) \leq \lfloor 2n/3 \rfloor$, where n is the number of vertices in G. This upper bound is attained for p-partite graph k(3,3,...3,i) where n = 3k + i, $0 \leq i \leq 2$. Cozzens [1981] gave methods of calculating the cubicity of some classes of graphs. A detailed and excellent survey on all aspects of cubicity and boxicity can be found in the dissertation of Cozzens [1981].

Wegner [1967] took convex regions in three dimensional Euclidean space and showed that every graph is the intersection graph of convex regions in three dimensional space but not in two dimensional space. However, no theorem similar to Wegner's is possible for boxes in three dimensional space, as there are graphs which are intersection graphs of boxes in higher dimensional space. Instead of extending unit intervals to unit boxes, one could look at closed unit spheres in Euclidean space E^n.Havel and Kuntz [1980] defined the (*unit*) sphericity of a graph G to be the least integer k such that G is the intersection graph of closed unit spheres in k-dimensional space. Havel [1982] showed that there are graphs of sphericity 2 with arbitrary large cubicity whereas Fishburn [1983] showed that there are graphs of cubicity 2 and 3 with sphericity larger than cubicity. Maehara [1984] used the term space graph to describe intersection graphs of spheres and derived certain bounds on the sphericity for several classes of graphs.

Sen [1984] studied intersection graphs of subcubes of a unit cube Q_n defined recursively as

$$\begin{aligned} Q_1 &= \{0,1\} \\ Q_2 &= Q_1 \times Q_1 \\ &\cdots\cdots\cdots \\ Q_k &= Q_{k-1} \times Q_1 \end{aligned}$$

He proved that any graph G is the intersection graph of a family of subcubes of a unit cube Q_k. He introduced the concept of *n-index of G* analogous to boxicity, which is the minimum k such that the graph G is the intersection graph of subcubes of Q_k. He also proved that n-ind of a graph G is equal to the minimum number of complete bipartite subgraphs of $\bar{G}$ whose union is $\bar{G}$. This is analogous to the works on boxicity by Cozzens and Roberts [1983] who proved that the boxicity of a graph G is equal to the minimum number of interval graphs whose intersection is G. In terms of the complement, this means that boxicity of a graph G is equal to the minimum number of cointerval graphs whose union is $\bar{G}$. Sen [1984] has replaced these cointerval graphs by complete bipartite subgraphs in his work and this provides a setting in which n-index of a graph is perhaps more natural than boxicity.

In conclusion,we must mention an excellent and a very fundamental work done by Cozzens and Roberts [1989] where they introduced the notion of a dimensional property of graphs. A <u>dimensional property</u> of graphs is a property P such that every graph G is the intersection graph having property P. If P is a dimensional property, they described a general method for computing the least integer k so that G is the intersection of k graphs having property P. They then gave simple applications of the method to computing the boxicity, the cubicity and a number of such other dimensions of a graph.

1.7 Circular-arc graphs.

A natural topological extension of interval graphs is the circular-arc graph. A graph is a circular-arc graph if it is the intersection graph of a family of arcs on a circle. The problem of characterizing circular-arc graphs was posed by Klee [1969] in which he asked : "what graphs are circular-arc graphs?" These graphs have important applications in testing for circular arrangements of genetic molecules [Stahl,1967], in circular indifference situations such as colour wheels and musical tones [Hubert,1974 and Luce 1971].

Circular-arc graphs are also used in designing traffic signals by Stoffers [1968] and Roberts [1976,1979c] and in cyclic scheduling problems by Tucker [1978]. To determine the dimension of a partial ordered set, Trotter and Moore [1976] was led to the use of circular arc models. At present, these graphs have found important applications in computer science for designing a compiler or any other basic software system [Tucker,1978]. Before 1970, the problem of characterizing circular-arc graphs was a difficult one, but during the last two decades Tucker did extensive work on circular- arc graphs to solve the problems of characterization and recognition algorithm [Tucker,1971,1974,1978, 1980].

A (0,1) matrix A is said to have the quasi-circular ones property if the sets $U_1,\ldots,U_n$, $V_1,\ldots,V_n$ contain all the ones in A, where U_i is the set of ones in column i starting at the main diagonal and going down (possibly around) until the first 0 is reached, and V_i is the set of ones in row i starting at the main diagonal and going to the right (possibly around) as far as possible.

Theorem 1.14. [Tucker,1971]. *A graph G is a circular-arc graph if and only if the vertex can be indexed so that the augmented adjacency matrix has the quasi-circular ones property.*

Notwithstanding this, no structure theorem, that is, characterization in terms of forbidden configuration of a circular-arc graph was known until Kabell [1980] could obtain a satisfactory solution to the problem. However, previous to that, Trotter and Moore [1976] determined a collection of forbidden subgraphs of a circular-arc graph, which consist of graphs with clique covering number 2; but the list is not complete. And again Tucker [1974] could obtain a structure theorem for some special types of circular-arc graphs, viz. unit circular-arc graphs in which the arcs are of unit length and proper circular-arc graphs in which no arc properly contains another. It is to be noted that these two concepts do not coincide here as they would for interval graphs.

Until the work of Tucker [1979,1980], determining whether a graph is a circular-arc graph was a difficult problem, but now a polynomial time algorithm exists to test for circular-arc graphs. Unfortunately, the algorithm does not yield a specific characterization of circular-arc graphs as we have for interval graphs.

Gavril [1974a] defined some subclasses of circular-arc graphs and succeeded in characterizing these classes viz., Δ-circular-arc graphs and Θ-circular-arc graphs. A graph is a *Δ-circular-arc graph* if it is the intersection graph of a family of arcs on a circle, so that for three arcs, if every pair intersects then the intersection of the three arcs is non-empty. A graph is a *Θ-circular-arc graph* if it is a circular-arc graph and if for every clique, the intersection of the arcs corresponding to the vertices of the clique is non-empty. Note that a Θ-circular-arc graph is also a Δ-circular-arc graph, but the converse is not true. Moreover, he provided efficient algorithms for recognizing these two classes and for finding a maximum clique, a maximum independent set and a minimum covering by cliques of circular-arc graphs. In a recent paper, Golumbic and Hammer [1988] considered the same problem of finding the maximum stable set in a circular -arc graph and they gave two algorithms for the problem. The first algorithm assumes that a representation of the circular-arc is given and under this assumption they propose a solution of the problem in O(nlogn) time; while the second algorithm solves the problem in O(n+e) time with the assumption that only the circular-arc graph is given.

As in the case of interval graphs, circular-arc graphs were also generalized in n-dimensional space R^n. Feinberg [1979] showed that any graph G is the intersection graph of the products of arcs on a sphere, called 'patches' on a sphere. He then used the notion of Circular dimension of a graph G as the least integer k such that G is the intersection graph of patches on k-sphere. Trotter [1979], Shearer [1980] and Cozzens and Roberts [1989] obtained bounds of circular dimension for several graphs.

In Chapter - IV, we introduce *circular-arc digraph*, the intersection digraph of a family of arcs on a circle. We obtain in this chapter, a characterization of circular-arc digraphs analogous to a characterization of circular-arc graphs in terms of quasi- circular ones property given by Tucker [1971].

1.8. Related Topics.

Many families of sets other than intervals have been studied for intersection graphs. Instead of discussing all of them exhaustively, we mention below only several lines of research.

Several classes of intersection graphs arising from families of paths in a tree have been studied; distinct classes of graphs arise by considering paths to be the sets of vertices or the edges and by allowing the underlying tree to be undirected or directed. Rooted trees will produce other classes. The undirected vertex path graphs or the intersection graph of a family of vertex paths in a tree, were studied by Renz [1970], Gavril [1978] and Lobb[1982]. An efficient recognition algorithm was given by Gavril [1978]. For directed trees, the class of intersection graphs were characterized by Gavril [1975] where an efficient algorithm was also given for recognizing them.

Recall that a graph is a *chordal graph* if every cycle of length greater than 3 possess a chord. Dirac [1961] and Rose [1970] characterized chordal graphs and an efficient algorithm [Rose,1970] for these graphs also exists. The minimum colouring, maximum clique, maximum clique cover and maximum independent set problems for chordal graphs were solved efficiently by Gavril [1972]. Venimadhavan and Srineevasha Kumar [1988] analyzed clique structure of chordal graphs, and its subclasses.

The intersection graph of a family of subtrees of a tree is called a <u>subtree graph</u>. Gavril [1974] showed that these graphs are exactly the chordal graphs. The intersection graph of a family of edge paths in a tree were studied by Golumbic and Jamison [1985a,b], Tarjan [1985] and Syslo [1970,1985]. Monma and Wei [1986] gave a unified approach for studying different types of such graphs.

In a fundamental paper, already mentioned in sec.1.7, Trotter and Moore [1976] discussed a number of characterization problem involving interval graphs, circular-arc graphs, 0,1-matrices, the representation of families of sets by points or intervals of the real line and other notions. They showed that despite their apparent diversity, these characterization problems are intimately related.

Intersection graphs find their usefulness in applications mainly because such graphs result out of intersections between several objects of similar type (from the real world). However, intersection is only one type of interaction between objects and several graph theorists started thinking of other types of interactions which might produce different classes of graphs and their corresponding theories. In this section, the graphs obtained from several interactions other than intersection will be discussed. It is to be noted that some of these graphs eventually become intersection graphs of certain families.

Evan and Itai [1971] introduced the concept of overlap graphs, different from intersection graphs. A graph G(V,E) is an <u>overlap graph</u> if its vertices can be put into a one-one correspondance with a collection of intervals on the real line in such a way that two vertices are adjacent if and only if the corresponding intervals intersect but neither contains the other. Overlap graphs were studied by Gavril [1973], Fournier [1978] and Buckingham [1980]. It turns out, nevertheless, that this class of graphs is exactly the same as the class of <u>circle graphs</u>, the intersection graphs of a finite collection of chords on a circle.

Golumbic and Monma [1982] introduced the concept of measured interaction and defined tolerance graphs to generalize both interval graphs and permutation graphs. A graph G(V,E) is a <u>tolerance graph</u> if there exists assignment of intervals I_v

and tolerances t_v to each vertex v such that

$$uv \in E \iff |I_u \cap I_v| \geq \min\{t_u, t_v\},$$

where $|I|$ is the length of the interval I. They noted that the requirement of equal tolerances for all vertices yields, precisely, the interval graphs, while requiring $t_x = |I_x|$ yields the permutation graphs.

Suppose π is permutation of numbers $1,2,\ldots,n$. Let $G[\pi]$ be an undirected graph constructed from π in the following manner : $G[\pi]$ has vertices numbered from 1 to n; two vertices are joined by an edge if the larger of their corresponding number is to the left of the smaller in π. An undirected graph G is called a <u>permutation graph</u> if there exist a permutation π such that $G = G[\pi]$.

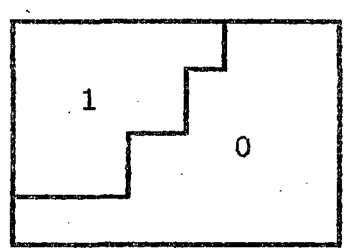
Golumbic [1984,1985] studied another type of graph called <u>containment graph</u> ; it is a graph where each vertex is assigned to a set S_v of a family of sets F in such a way that two vertices u and v are adjacent if and only if either $S_u \subset S_v$ or $S_v \subset S_u$.

1.9. Ferrers digraphs and Ferrers dimension.

Guttman [1944] and Riguet [1951] introduced independently a special claas of digraphs, known as Ferrers digraphs. Riguet defined <u>Ferrers digraphs</u> to be a digraph D(V,E) in which for all $x, y, z, w \in V$,

$$xy,\ zw \in E \implies xw \text{ or } zy \in E.$$

It is to be noted that the vertices are not necessarily distinct. Doignon, Ducamp and Falmagne [1984] called it a *biorder*. Ferrers digraphs have many characterizations. Riguet [1951] characterized Ferrers digraphs as those whose successor sets (or equivalently, predecessor sets) are linearly ordered by inclusion. In other words, if the rows of the adjacency matrix are indexed by vertices in decreasing order of outdegree, and the columns by vertices in decreasing order of indegree, then the rearranged adjacency matrix takes the form



The representation of 1's (or 0's) in the above form will be referred to as a Ferrers diagram.

Dignon, Ducamp and Falmagne [1984] proved that a digraph is a Ferrers digraph if and only if it has no alternating cycle. An <u>n-alternating</u> <u>cycle</u> of a digraph D(V,E) is a finite sequence $a_1, d_1, a_2, d_2, \ldots, a_n, d_n$ such that

$$\begin{array}{ll} a_1 d_1 \in E, & d_1 a_2 \in \bar{E}' \\ a_2 d_2 \in E, & d_2 a_3 \in \bar{E}' \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ a_n d_n \in E, & d_n a_1 \in \bar{E}' \end{array} ,$$

where a_i, $d_i \in V$ for all i.

Monjardet [1978] showed that a digraph D(V,E) is a Ferrers digraph if and only if $D \bar{D}' D$ is a subdigraph of D; that is, if U is the set of arcs of $D\bar{D}' D$, then $U \subseteq E$. The characterization of a Ferrers digraph D(V,E) in terms of forbidden subgraphs is based on the following convention : A solid arc will be drawn between two vertices a and b if $ab \in E$ and a dotted arc if $ab \notin E$. From the definition it follows that a Ferrers digraph does not contain any subdigraph isomorphic to the following graph (*Fig.* 1.5).

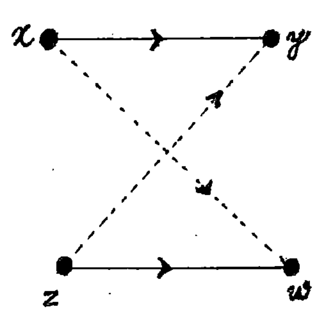


Fig. 1.5

This forbidden subgraph may degenarate in several ways, by identifying one vertex with another, thereby leading to the four diagrams of *Fig.*1.6

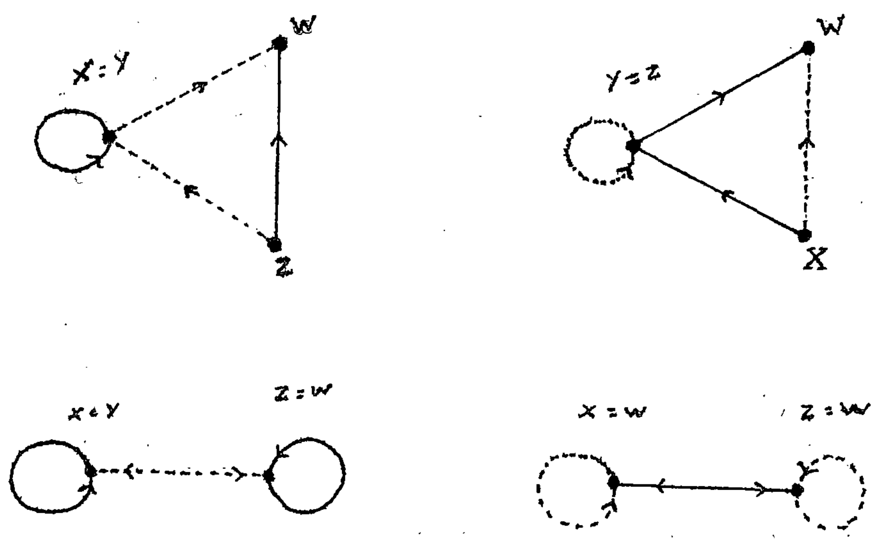


Fig. 1.6

Recall that a threshold graph is a graph G(V,E) for which there exists a real mapping $f : X \longrightarrow R$ and a threshold $t \in R$ such that for every $S \subseteq X$,

$$S \text{ is stable} \iff \sum_{x \in S} v(x) \leq t.$$

Threshold graphs were introduced by Chvàtal and Hammer [1977]. The definitions of threshold graphs and Ferrers digraphs look completely different, but their characterizations suggest remarkable similarity. Various characterizations of Ferrers digraphs led Cogis [1982] to discover that the underlying graphs of symmetric Ferrers digraphs are precisely the threshold graphs.

For each characterization of Ferrers digraphs, there is a corresponding characterization of threshold graphs and vice versa [Cogis,1982]. Intersection of Ferrers digraphs was studied by Bouchet [1971] and he showed that any digraph D is the intersection of a family of Ferrers digraphs containing D. The *Ferrers dimension* of a digraph D is the minimum number of Ferrers digraphs whose intersection is D. Bouchet [1971,1982,1984] also obtained several interesting results on Ferrers dimension. He was able to extend to any digraph Dushnik and Miller's [1941] order dimension of a partial order. The *order dimension* of a partial order G is the minimum number of linear orders whose intersection is G. Bouchet [1971] showed that the notion of Ferrers dimension of a Ferrers digraph is an extension of the order dimension. Cogis [1982] gave another proof of this result for finite digraphs (extended to infinite digraphs by Doignon, Ducamp and Falmagne [1984]). Moreover, the order dimension was proved to be polynomially equivalent to the Ferrers dimension. Yannakakis [1982] showed that the problem of designing efficient algorithms for order dimension exceeding 2 is NP-complete; but 2-dimensional posets are polynomially recognizable.

An excellent survey of the notion of order dimension and analogous parameters for graphs and directed graphs such as boxicity, threshold dimension and Ferrers dimension to name only a few, is to be found in a review paper by West [1985].

1.9.1. Characterizations of digraphs with Ferrers dimension 2.

The digraphs with Ferrers dimension (F.D) 2 were characterized independently by Cogis [1979] and also by Doignon, Ducamp and Falmagne [1984] in different contexts. Many of the results of Doignon et.al.overlap with Cogis' work. In order to characterize a digraph D of F.D.2, Cogis [1979] defined an undirected graph H(D) associated with D.

The <u>graph associated with a digraph D</u> is the graph H(D) whose vertices correspond to the 1's of the adjacency matrix of the complementary digraph $\bar{D}$, with two such vertices joined by an edge if the corresponding 1's belong to an *obstruction*; that is, the corresponding 1's belong to a configuration of the form $\begin{pmatrix}10\\01\end{pmatrix}$ or $\begin{pmatrix}01\\10\end{pmatrix}$ in $A(\bar{D})$. Riguet [1951] had shown that a digraph D is a Ferrers digraph if and only if there does not exist any obstruction in A(D). The main characterization of digraphs with F.D.2, given by Cogis, is as follows :

Theorem 1.15. [Cogis,1979]. *A digraph D is of Ferrers dimension at most 2 if and only if H(D) is bipartite.*

This results yields a polynomial algorithm [Cogis,1979] for testing whether a digraph has F.D.2 or not. The same characterization was obtained in a more general form by Doignon et.al. where the graph is not restricted to be finite.