

CHAPTER - 3

RADIATIVE TRANSFER PROBLEMS

IN

FINITE ATMOSPHERES

3.1 Solution of a radiative transfer problem in finite atmosphere with

Isotropic scattering using a modified form of spherical harmonic method.

Kourganoff [1963] analyzed the method of single interval spherical harmonics method for solving the equation of transfer and suggested possible modification. Wilson and Sen [1963,1964a,1965b,1965a,1965b,1965c] considered different forms of intensity and applied the double interval spherical harmonics method to solve several radiative transfer problems in plane and spherical geometry. Bishnu [1968] solved the equation of transfer using a different approximate form of intensity. Wan et al. [1977,1986] used another form of intensity to get the solution of the equation of transfer. Karanjai and Talukdar [1992] solved the equation of transfer with general phase function using the form of intensity given by Bisnhu and deduced the results with phase functions like the (i) planetary (ii) Rayleigh and (iii) Henyey - Greenstein from the general solution. Karanjai and Biswas (1992,1993) applied the same method with the form of intensity given by Wan et al [1986] to solve the equation of transfer with (i) Rayleigh phase function and (ii) the phase function of the form given by

$$1 + \omega_1 P_1(\mu)P_2(\mu') + \omega_2 P_2(\mu)P_2(\mu').$$

Here we would like to introduce the following form of intensity:

$$I^+(\tau, \mu) = I(0,0) \left[\phi(\tau) + \psi(\mu) + \sum_{l=0}^{l=L} (2l+1) I_l^+(\tau) \mu P_l(2\mu-1) \right], \quad 0 \leq \mu \leq 1 \quad (3a)$$

$$I^-(\tau, \mu) = I(0,0) \left[\phi(\tau) + \psi(\mu) + \sum_{l=0}^{l=L} (2l+1) I_l^-(\tau) \mu P_l(2\mu+1) \right], \quad -1 \leq \mu \leq 0 \quad (3b)$$

where $I(0,0)$ is some constant, $\phi(\tau)$ is a function of τ only and $\psi(\mu)$ is given by

$$\psi(\mu) = \begin{cases} 1 & \text{if } 0 \leq \mu \leq 1 \\ 0 & \text{if } -1 \leq \mu \leq 0 \end{cases} \quad (3c)$$

3.1.1 The equation of transfer and the boundary conditions.

The equation of transfer for plane parallel atmosphere in this case is

$$\frac{dI(\tau, \mu)}{d\mu} = I(\tau, \mu) - \frac{1}{2} \int_{-1}^1 I(\tau, \mu') d\mu' \quad (3.1.1.1)$$

where the symbols have their usual meanings. The equation of transfer (3.1.1.1) is to be solved subject to the boundary condition,

(a) Absence of incident radiation from outside at the free surface $\tau = 0$, i.e.,

$$I(\tau, \mu) \equiv 0 \quad \text{for } -1 \leq \mu \leq 0 \quad (3.1.1.2a)$$

(b) No incoming radiation at $\tau = \tau_0$ i.e.

$$I(\tau_0, \mu) = 0, \quad -1 \leq \mu \leq 0; \quad (3.1.1.2b)$$

We will seek a solution to equation (3.1.1.1) so that $I(\tau, \mu)$ can be represented by two different expressions $I^+(\tau, \mu)$, $I^-(\tau, \mu)$ for μ in the interval $[0, 1]$ and $[-1, 0]$ respectively. We will use the two forms given by (3a) and (3b).

Therefore the form of the equation of transfer now becomes,

$$\mu \frac{dI^+(\tau, \mu)}{d\tau} = I^+(\tau, \mu) - \frac{1}{2} \left[\int_{-1}^0 I^-(\tau, \mu') d\mu' + \int_0^1 I^+(\tau, \mu') d\mu' \right] \quad (3.1.1.3a)$$

$$\mu \frac{dI^-(\tau, \mu)}{d\tau} = I^-(\tau, \mu) - \frac{1}{2} \left[\int_{-1}^0 I^-(\tau, \mu') d\mu' + \int_0^1 I^+(\tau, \mu') d\mu' \right] \quad (3.1.1.3b)$$

Using (3a) and (3b) in equations (3.1.1.3) we obtain the following equations

$$\mu \frac{dI^+(\tau, \mu)}{d\tau} = I^+(\tau, \mu) - \frac{1}{2} I(0,0) \left[2\phi(\tau) + 1 + \frac{1}{2} (I_0^+ + I_1^+ - I_0^- + I_1^-) \right] \quad (3.1.1.4a)$$

and

$$\mu \frac{dI^-(\tau, \mu)}{d\tau} = I^-(\tau, \mu) - \frac{1}{2} I(0,0) \left[2\phi(\tau) + 1 + \frac{1}{2} (I_0^+ + I_1^+ - I_0^- + I_1^-) \right] \quad (3.1.1.4b)$$

Again using (3a) in (3.1.1.4a) we obtain

$$\begin{aligned} \mu \left[\phi'(\tau) + \sum_{l=0}^{l=L} (2l+1) I_l^+(\tau) P_l(2l-1) \right] = \\ = \psi(\mu) - \frac{1}{2} + \sum_{l=0}^{l=L} (2l+1) I_l^+(\tau) \mu P_l(2\mu-1) \end{aligned} \quad (3.1.1.5a)$$

Similarly using (3b) in (3.1.1.4b) we get

$$\begin{aligned} & \mu \left[\Phi'(\tau) + \sum_{l=0}^{l=L} (2l+1) I_l^{-'}(\tau) P_l(2l+1) \right] = \\ & = \Psi(\mu) - \frac{1}{2} + \sum_{l=0}^{l=L} (2l+1) I_l^{-}(\tau) \mu P_l(2\mu+1) \end{aligned} \quad (3.1.1.5b)$$

Next, multiplying (3.1.1.5a) by $P_l(2\mu-1)$ and integrating over $[0,1]$ we obtain

$$\begin{aligned} & \Phi'(\tau) \int_0^1 \mu P_l(2\mu-1) d\mu + \frac{1}{4(2l+1)} \left[\frac{l^2-1}{2l-1} I_{l-2}^{+'} + 2l I_{l-1}^{+'} + \right. \\ & \left. \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)} I_l^{+'} + 2(l+1) I_{l+1}^{+'} + \frac{l^2+3l+2}{2l+3} I_{l+2}^{+'} \right] = \\ & = \int_0^1 \Psi(\mu) P_l(2\mu-1) d\mu + \frac{1}{2l+1} \left[l I_{l-1}^{+'} + (2l+1) I_l^{+'} + (l+1) I_{l+1}^{+'} \right] - \\ & - \frac{1}{2} \delta_{0l} - \frac{1}{4} \delta_{0l} \left[I_0^{+'} + I_1^{+'} - I_0^{-} + I_1^{-} \right], \quad l=0,1,2,\dots,L \end{aligned} \quad (3.1.1.6a)$$

Similarly multiplying (3.1.1.5b) by $P_l(2\mu+1)$ and then integrating over $[-1,0]$ we get

$$\Phi'(\tau) \int_{-1}^0 \mu P_l(2\mu+1) d\mu + \frac{1}{4(2l+1)} \left[\frac{l^2-1}{2l-1} I_{l-2}^{-'} - 2l I_{l-1}^{-'} + \right.$$

$$\begin{aligned}
& \left. \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^{-'} - 2(l+1)I_{l+1}^{-'} + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^{-'} \right] = \\
& = \int_{-1}^0 \psi(\mu) P_l(2\mu + 1) d\mu + \frac{1}{2l+1} \left[lI_{l-1}^- - (2l+1)I_l^- + (l+1)I_{l+1}^- \right] - \\
& - \frac{1}{2} \delta_{0l} - \frac{1}{4} \delta_{0l} \left[I_0^+ + I_1^+ - I_0^- + I_1^- \right], \quad l=0,1,2,\dots,L
\end{aligned} \tag{3.1.1.6b}$$

Let us set different values of l .

For $l=0,1$ from (3.1.1.6a) we get the following equations

$$\left(\frac{4}{3} I_0^{+'} + 2I_1^{+'} + \frac{2}{3} I_2^{+'} \right) - \left(I_0^+ + I_1^+ + I_0^- - I_1^- \right) = -2\phi'(\tau) + 2 \tag{3.1.1.7a}$$

$$\left(2I_0^{+'} + \frac{24}{5} I_1^{+'} + 4I_2^{+'} + \frac{6}{5} I_3^{+'} \right) - 2 \left(I_0^+ + 3I_1^+ + 2I_2^+ \right) = -2\phi'(\tau) \tag{3.1.1.7b}$$

And for $l \neq 0, 1$ from (3.1.1.6a)

$$\phi'(\tau) \int_0^1 \mu P_l(2\mu - 1) d\mu + \frac{1}{4(2l+1)} \left[\frac{l^2 - 1}{2l-1} I_{l-2}^{+'} + 2lI_{l-1}^{+'} + \right.$$

$$\left. \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^{+'} + 2(l+1)I_{l+1}^{+'} + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^{+'} \right] =$$

$$= \int_0^1 \psi(\mu) P_l(2\mu - 1) d\mu + \frac{1}{2l+1} \left[l I_{l-1}^+ + (2l+1) I_l^+ + (l+1) I_{l+1}^+ \right] \quad (3.1.1.7c)$$

Similarly for $l=0,1$ from (3.1.1.6b)

$$\left(\frac{4}{3} I_0^{-'} - 2 I_1^{-'} + \frac{2}{3} I_2^{-'} \right) + \left(I_0^+ + I_1^+ + I_0^- - I_1^- \right) = 2\Phi'(\tau) - 2 \quad (3.1.1.8a)$$

$$\left(2 I_0^{-'} - \frac{24}{5} I_1^{-'} + 4 I_2^{-'} - \frac{6}{5} I_3^{-'} \right) - 2 \left(I_0^- - 3 I_1^- + 2 I_2^- \right) = -2\Phi'(\tau) \quad (3.1.1.8b)$$

And for $l \neq 0, 1$ from (3.1.1.6b)

$$\begin{aligned} \Phi'(\tau) \int_{-1}^0 \mu P_l(2\mu + 1) d\mu + \frac{1}{4(2l+1)} \left[\frac{l^2 - 1}{2l-1} I_{l-2}^{-'} - 2l I_{l-1}^{-'} + \right. \\ \left. \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^{-'} - 2(l+1) I_{l+1}^{-'} + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^{-'} \right] = \\ = \int_{-1}^0 \psi(\mu) P_l(2\mu + 1) d\mu + \frac{1}{2l+1} \left[l I_{l-1}^- - (2l+1) I_l^- + (l+1) I_{l+1}^- \right] \end{aligned} \quad (3.1.1.8c)$$

The equations (3.1.1.8) to (3.1.1.9) are to be solved subject to the boundary condition and we will take

$$\phi(\tau) = A e^{-\tau} + B e^{\tau} \quad (3.1.1.9)$$

where τ is small and A and B are constants which are to be determined. We assume a trial solution of the form

$$I_l^+(\tau) = g_{l,\alpha} e^{-k\tau} + g_{l,\beta} e^{-\tau} + g_{l,\gamma} e^{\tau} \quad (3.1.1.10a)$$

$$I_l^-(\tau) = h_{l,\alpha} e^{-k\tau} + h_{l,\beta} e^{-\tau} + h_{l,\gamma} e^{\tau} \quad (3.1.1.10b)$$

where $g_{l,\alpha}, g_{l,\beta}, g_{l,\gamma}, h_{l,\alpha}, h_{l,\beta}, h_{l,\gamma}$ are constants.

Further from the boundary condition

$$\sum_{r=1}^{r=n-1} h_{l,\alpha}^{(r)} + h_{l,\beta} + h_{l,\gamma} = 0 \quad (3.1.1.11)$$

3.1.2 First Approximation: Here $L = 1$. We will neglect I_{L+1}^+, I_{L+1}^-

We have for $l = 0, 1$ from (3.1.1.7a) and (3.1.1.7b)

$$\left(\frac{4}{3} I_0^{+'} + 2 I_1^{+'} \right) - \left(I_0^+ + I_1^+ + I_0^- - I_1^- \right) = -2\phi'(\tau) + 2 \quad (3.1.2.1)$$

and

$$\left(2 I_0^{+'} + \frac{24}{5} I_1^{+'} \right) - 2 \left(I_0^+ + 3 I_1^+ \right) = -2\phi'(\tau) \quad (3.1.2.2)$$

Similarly for $l = 0, 1$ from (3.1.1.8a) and (3.1.1.8b) we obtain respectively

$$\left(\frac{4}{3} I_0^{-'} - 2 I_1^{-'} \right) + \left(I_0^+ + I_1^+ + I_0^- - I_1^- \right) = 2\phi'(\tau) - 2 \quad (3.1.2.3)$$

and

$$\left(2I_0^- - \frac{24}{5}I_1^- \right) - 2(I_0^- - 3I_1^-) = -2\phi'(\tau) \quad (3.1.2.4)$$

Substituting the forms of $\phi(\tau), I_1^+, I_1^-$ in equations (3.1.2.1) to (3.1.2.4) and comparing the coefficients of $e^{-k\tau}$ we get

$$\left(\frac{4k}{3} + 1 \right) g_{0,\alpha} + (2k + 1)g_{1,\alpha} + h_{0,\alpha} - h_{1,\alpha} = 0 \quad (3.1.2.5i)$$

$$(2k + 2)g_{0,\alpha} + \left(\frac{24k}{5} + 6 \right) g_{1,\alpha} = 0 \quad (3.1.2.5ii)$$

$$g_{0,\alpha} + g_{1,\alpha} + \left(-\frac{4k}{3} + 1 \right) h_{0,\alpha} + (2k - 1)h_{1,\alpha} = 0 \quad (3.1.2.5iii)$$

$$(2k - 2)h_{0,\alpha} + \left(6 - \frac{24k}{5} \right) h_{1,\alpha} = 0 \quad (3.1.2.5iv)$$

The above system will have non trivial solution if $D_1(k) = 0$, where $D_1(k)$ is given by (2.1.2.7).

This gives $k = 0, 0, \pm 1.8257$. To satisfy the boundary condition we will take the positive root only. Next the boundary conditions in this case are

$$I_0^+(0) = I_1^+(0) = 0 \quad (3.1.2.6a)$$

$$I_0^-(\tau_0) = I_1^-(\tau_0) = 0 \quad (3.1.2.6b)$$

and

$$h_{0,\alpha} + h_{0,\beta} + h_{0,\gamma} = 0 \quad (3.1.2.7a)$$

$$h_{1,\alpha} + h_{1,\beta} + h_{1,\gamma} = 0 \quad (3.1.2.7b)$$

Using the boundary condition (3.1.2.6b) we obtain

$$A = 5.5166547, B = 4.5166547$$

Again equating the coefficients of e^τ and $e^{-\tau}$ on each side we have

$$\frac{7}{3}g_{0,\beta} + 3g_{1,\beta} + h_{0,\beta} - h_{1,\beta} = 2A \quad (3.1.2.8a)$$

$$2g_{0,\beta} + \frac{27}{5}g_{1,\beta} = -A \quad (3.1.2.8b)$$

$$g_{0,\beta} + g_{1,\beta} - \frac{1}{3}h_{0,\beta} + h_{1,\beta} = -2A \quad (3.1.2.8c)$$

$$\frac{4}{5}h_{1,\beta} = 2A \quad (3.1.2.8d)$$

and

$$\frac{7}{3}g_{0,\gamma} + 3g_{1,\gamma} + h_{0,\gamma} - h_{1,\gamma} = 2B \quad (3.1.2.9a)$$

$$\frac{3}{5}g_{0,\gamma} = -B \quad (3.1.2.9b)$$

$$g_{0,\gamma} + g_{1,\gamma} + \frac{7}{3}h_{0,\gamma} - h_{1,\gamma} = 2B \quad (3.1.2.9c)$$

$$-2h_{0,\gamma} + \frac{27}{5}h_{1,\gamma} = B \quad (3.1.2.9d)$$

Therefore solving the systems of equations (3.1.2.8) and (3.1.2.9) and from the boundary conditions we obtain

$$g_{0,\alpha} = -39.9259 \quad h_{0,\alpha} = -13.0211$$

$$g_{1,\alpha} = 16.3993 \quad h_{1,\alpha} = -7.4875$$

$$g_{0,\beta} = 21.1952 \quad h_{0,\beta} = -46.6974$$

$$g_{1,\beta} = -8.8717 \quad h_{1,\beta} = 13.7915$$

$$g_{0,\gamma} = 18.6307 \quad h_{0,\gamma} = 48.0470$$

$$g_{1,\gamma} = -7.5276 \quad h_{1,\gamma} = -5.6467$$

Next, we compute the source function. Using the definition, we have

$$\mathfrak{S}(\tau) = \frac{1}{2} \int_{-1}^1 I(\tau, \mu) d\mu \quad (3.1.2.10)$$

This gives

$$\mathfrak{S}(\tau) = I(0,0) \left[A e^{-\tau} + B e^{\tau} + \frac{1}{2} - 4.4983 e^{-k\tau} + 18.2031 e^{-\tau} - 10.6477 e^{\tau} \right]$$

where τ is small and other constants are also determined. So we can find the source function and once this is found we can calculate the emergent intensity from any surface.

3.1.3. Second Approximation

Here $L = 2$ and putting successively $l = 0, 1, 2$ in equations (3.1.1.7) and (3.1.1.8) we obtain

the following equations

$$\left(\frac{4}{3}I_0^{+'} + 2I_1^{+'} + \frac{2}{3}I_2^{+'} \right) - (I_0^+ + I_1^+ + I_0^- - I_1^-) = -2\phi'(\tau) + 2 \quad (3.1.3.1)$$

$$\left(2I_0^{+'} + \frac{24}{5}I_1^{+'} + 4I_2^{+'} \right) - 2(I_0^+ + 3I_1^+ + 2I_2^+) = -2\phi'(\tau) \quad (3.1.3.2)$$

$$\left(\frac{1}{3}I_0^{+'} + 2I_1^{+'} + \frac{80}{21}I_2^{+'} \right) - (2I_1^+ + 5I_2^+) = 0 \quad (3.1.3.3)$$

$$\left(\frac{4}{3}I_0^{-'} - 2I_1^{-'} + \frac{2}{3}I_2^{-'} \right) + (I_0^+ + I_1^+ + I_0^- - I_1^-) = -2 + 2\phi'(\tau) \quad (3.1.3.4)$$

$$\left(-2I_0^{-'} + \frac{24}{5}I_1^{-'} - 4I_2^{-'} \right) - 2(I_0^- - 3I_1^- + 2I_2^-) = -2\phi'(\tau) \quad (3.1.3.5)$$

$$\left(\frac{1}{3}I_0^{-'} - 2I_1^{-'} + \frac{80}{21}I_2^{-'} \right) - (2I_1^- - 5I_2^-) = 0 \quad (3.1.3.6)$$

Using the forms of $\phi(\tau)$ and $I_l^+(\tau), I_l^-(\tau)$ in equations (3.1.3.1) to (3.1.3.6) and comparing the coefficients of $e^{-k\tau}$ we obtain the following set of equations

$$\left(\frac{4k}{3} + 1\right)g_{0,\alpha} + (2k+1)g_{1,\alpha} + \frac{2k}{3}g_{2,\alpha} + h_{0,\alpha} - h_{1,\alpha} = 0 \quad (3.1.3.7a)$$

$$(2k+2)g_{0,\alpha} + \left(\frac{24k}{5} + 6\right)g_{1,\alpha} + (4k+4)g_{2,\alpha} = 0 \quad (3.1.3.7b)$$

$$\frac{k}{3}g_{0,\alpha} + (2k+2)g_{1,\alpha} + \left(\frac{80k}{21} + 5\right)g_{2,\alpha} = 0 \quad (3.1.3.7c)$$

$$g_{0,\alpha} + g_{1,\alpha} + \left(-\frac{4k}{3} + 1\right)h_{0,\alpha} + (2k-1)h_{1,\alpha} - \frac{2k}{3}h_{2,\alpha} = 0 \quad (3.1.3.7d)$$

$$(2k-2)h_{0,\alpha} + \left(-\frac{24k}{5} + 6\right)h_{1,\alpha} + (4k-4)h_{2,\alpha} = 0 \quad (3.1.3.7e)$$

$$-\frac{k}{3}h_{0,\alpha} + (2k-2)h_{1,\alpha} + \left(-\frac{80k}{21} + 5\right)h_{2,\alpha} = 0 \quad (3.1.3.7f)$$

The above system of equations (3.1.3.7) will have non-trivial solution if $D_2(k) = 0$ where $D_2(k)$ is given by (2.1.3.8).

$$\text{This gives } 7.5232k^6 - 112.8488k^4 + 192k^2 = 0$$

$k = 0, 0, \pm 3.6116, \pm 1.3988$. As before we will take

$$k_1 = 3.6116 \quad k_2 = 1.3988$$

Again equating the coefficients of e^{τ} and $e^{-\tau}$ on each side of the equations (3.1.3.1) to (3.1.3.6) we have two sets of equations

$$\frac{7}{3}g_{0,\beta} + 3g_{1,\beta} + \frac{2}{3}g_{2,\beta} + h_{0,\beta} - h_{1,\beta} = -2A \quad (3.1.3.8a)$$

$$4g_{0,\beta} + \frac{54}{5}g_{1,\beta} + 8g_{2,\beta} = -2A \quad (3.1.3.8b)$$

$$\frac{1}{3}g_{0,\beta} + 4g_{1,\beta} + \frac{185}{21}g_{2,\beta} = -5A \quad (3.1.3.8c)$$

$$g_{0,\beta} + g_{1,\beta} - \frac{1}{3}h_{0,\beta} + h_{1,\beta} - \frac{2}{3}h_{2,\beta} = 2A \quad (3.1.3.8d)$$

$$\frac{6}{5}h_{1,\beta} = 2A \quad (3.1.3.8e)$$

$$\frac{1}{3}h_{0,\beta} + \frac{25}{21}h_{2,\beta} = -5A \quad (3.1.3.8f)$$

and

$$\frac{1}{3}g_{0,\gamma} + g_{1,\gamma} + \frac{2}{3}g_{2,\beta} + h_{0,\beta} - h_{1,\beta} = -2B \quad (3.1.3.9a)$$

$$\frac{6}{5}g_{1,\gamma} = -2B \quad (3.1.3.9b)$$

$$\frac{1}{3}g_{0,\gamma} - \frac{25}{21}g_{2,\gamma} = -5B \quad (3.1.3.9c)$$

$$g_{0,\gamma} + g_{1,\gamma} + \frac{7}{3}h_{0,\gamma} - 3h_{1,\gamma} + \frac{2}{3}h_{2,\gamma} = 2B \quad (3.1.3.9d)$$

$$-4h_{0,\gamma} - \frac{54}{5}h_{1,\gamma} - 8h_{2,\gamma} = -2B \quad (3.1.3.9e)$$

$$\frac{1}{3}h_{0,\gamma} - 4h_{1,\gamma} + \frac{185}{21}h_{2,\gamma} = 5B \quad (3.1.3.9f)$$

The boundary condition in this case becomes

$$h_{0,\alpha}^{(1)} + h_{0,\alpha}^{(2)} + h_{0,\beta} + h_{0,\gamma} = 0 \quad (3.1.3.10a)$$

$$h_{1,\alpha}^{(1)} + h_{1,\alpha}^{(2)} + h_{1,\beta} + h_{1,\gamma} = 0 \quad (3.1.3.10b)$$

$$h_{2,\alpha}^{(1)} + h_{2,\alpha}^{(2)} + h_{2,\beta} + h_{2,\gamma} = 0 \quad (3.1.3.10c)$$

Solving (3.1.3.8) and (3.1.3.9) we find different constants which are given below.

As in case of first approximations we obtain the required constants

$$g_{0,\alpha}^{(1)} = -180.8898, \quad h_{0,\alpha}^{(1)} = -9.3888$$

$$g_{0,\alpha}^{(2)} = -11.5438, \quad h_{0,\alpha}^{(2)} = -2.9256$$

$$g_{1,\alpha}^{(1)} = 101.95577, \quad h_{1,\alpha}^{(1)} = -12.2123$$

$$g_{1,\alpha}^{(2)} = -6.1007, \quad h_{1,\alpha}^{(2)} = 3.0180$$

$$g_{2,\alpha}^{(1)} = -38.5290, \quad h_{2,\alpha}^{(1)} = 13.6550$$

$$g_{2,\alpha}^{(2)} = -13.8553, \quad h_{2,\alpha}^{(2)} = 6.0737$$

$$g_{0,\beta} = -18.9443, \quad h_{0,\beta} = 12.2901$$

$$g_{1,\beta} = -11.7276, \quad h_{1,\beta} = 9.1943$$

$$g_{2,\beta} = -7.7392, \quad h_{2,\beta} = -19.7287$$

$$g_{0,\gamma} = -36.2250, \quad h_{0,\gamma} = -155.7106$$

$$g_{1,\gamma} = -7.5277, \quad h_{1,\gamma} = -8.8699$$

$$g_{2,\gamma} = -29.1130, \quad h_{2,\gamma} = -33.8258$$

Finally, we calculate the source function. This is obtained by using the known constants. Thus ,

$$\begin{aligned} \mathfrak{S}(\tau) = I(0,0) & \left[A e^{\tau} + B e^{-\tau} + \frac{1}{2} - 20.4398 e^{-k_1 \tau} - \right. \\ & \left. - 2.9252 e^{-k_2 \tau} - 2.5781 e^{-\tau} + 25.7722 e^{\tau} \right] \end{aligned}$$

3.2 Solution of a radiative transfer problem in finite atmosphere with

Rayleigh phase function using a modified form of spherical harmonic method.

3.2.1 The problem and development of equations.

We know that Rayleigh phase function is a three-term phase function and given by

$$p(\mu, \mu') = 1 + P_2(\mu)P_2(\mu')$$

The equation of transfer in case of plane parallel atmosphere with spherical symmetry is as usual given by

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2} \int_{-1}^1 p(\mu, \mu') I(\tau, \mu') d\mu' \quad (3.2.1.1)$$

where the symbols have their usual meanings. The equation (3.2.1.1) will be solved subject to the boundary condition which are stated in section 3.1 but they are again restated

(a) Absence of incident radiation from outside at the free surface $\tau = 0$, i.e.,

$$I(\tau, \mu) \equiv 0 \quad \text{for } -1 \leq \mu \leq 0 \quad (3.2.1.2a)$$

(b) No incoming radiation at $\tau = \tau_0$ i.e.

$$I(\tau_0, \mu) = 0, \quad 0 \leq \mu \leq 1; \quad (3.2.1.2b)$$

The Rayleigh phase function is given by

$$p(\mu, \mu') = 1 + P_2(\mu)P_2(\mu') = \frac{3}{8} \left[(3 - \mu^2) + (3\mu^2 - 1)\mu'^2 \right] \quad (3.2.1.3)$$

We consider the two forms of intensity given by (3a) and (3b). With these forms the equation of transfer (3.2.1.1) becomes

$$\mu \frac{dI^+(\tau, \mu)}{d\tau} = I^+(\tau, \mu) - \frac{1}{2} \left[\int_{-1}^0 p(\mu, \mu') I^-(\tau, \mu') d\mu' + \int_0^1 p(\mu, \mu') I^+(\tau, \mu') d\mu' \right] \quad (3.2.1.4)$$

and

$$\mu \frac{dI^-(\tau, \mu)}{d\tau} = I^-(\tau, \mu) - \frac{1}{2} \left[\int_{-1}^0 p(\mu, \mu') I^-(\tau, \mu') d\mu' + \int_0^1 p(\mu, \mu') I^+(\tau, \mu') d\mu' \right] \quad (3.2.1.5)$$

Using (3a) and (3b) in equations (3.2.1.4) and (3.2.1.5) we obtain the same set of ordinary differential equations as obtained in Sec (2.2), but for clarity we restate them below.

$$\begin{aligned} \mu \left[\Phi'(\tau) + \sum_{l=0}^{l=L} (2l+1) I_l^+(\tau) P_l(2\mu-1) \right] &= \Psi(\mu) - \frac{1}{2} + \sum_{l=0}^{l=L} (2l+1) I_l^+(\tau) \mu P_l(2\mu-1) \\ &- \frac{3}{64} (\mu^2 + 5) (I_0^+ - I_0^-) - \frac{3}{320} (17\mu^2 + 21) (I_1^+ + I_1^-) \\ &+ (3\mu^2 - 1) \left[-\frac{3}{64} (I_2^+ - I_2^-) - \frac{3}{320} (I_3^+ + I_3^-) \right] \end{aligned} \quad (3.2.1.6a)$$

and

$$\mu \left[\Phi'(\tau) + \sum_{l=0}^{l=L} (2l+1) I_l^-(\tau) P_l(2\mu+1) \right] = \Psi(\mu) - \frac{1}{2} + \sum_{l=0}^{l=L} (2l+1) I_l^-(\tau) \mu P_l(2\mu+1) -$$

$$\begin{aligned}
& -\frac{3}{64}(\mu^2 + 5)(I_0^+ - I_0^-) - \frac{3}{320}(17\mu^2 + 21)(I_1^+ + I_1^-) + \\
& + (3\mu^2 - 1)\left[-\frac{3}{64}(I_2^+ - I_2^-) - \frac{3}{320}(I_3^+ + I_3^-)\right]
\end{aligned} \tag{3.2.1.6b}$$

We will use the recurrence formula for Legendre polynomials and their orthogonal properties. Therefore, multiplying (3.2.1.6a) by $P_l(2\mu - 1)$ and integrating over $[0, 1]$ and using (3c) we obtain

$$\begin{aligned}
\Phi'(\tau) \int_0^1 \mu P_l(2\mu - 1) d\mu & + \frac{1}{4(2l+1)} \left[\frac{l^2 - l}{2l-1} I_{l-2}^{+'} + 2I_{l-1}^{+'} + \right. \\
& \left. + \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^{+'} + 2(l+1)I_{l+1}^{+'} + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^{+'} \right] = \\
& = \int_0^1 \Psi(\mu) P_l(2\mu - 1) d\mu + \frac{1}{2l+1} \left[lI_{l-1}^{+'} + (2l+1)I_l^{+'} + (l+1)I_{l+1}^{+'} \right] - \\
& - \frac{1}{2} \delta_{0l} - S^+, \quad l=0, 1, 2, \dots, L
\end{aligned} \tag{3.2.1.7a}$$

In a similar manner if we multiply (3.2.1.6b) by $P_l(2\mu + 1)$ and integrate over $[-1, 0]$ we have

$$\Phi'(\tau) \int_{-1}^0 \mu P_l(2\mu + 1) d\mu + \frac{1}{4(2l+1)} \left[\frac{l^2 - l}{2l-1} I_{l-2}^{-'} - 2I_{l-1}^{-'} + \right.$$

$$\begin{aligned}
& + \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^- - 2(l+1)I_{l+1}^- + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^- \Big] = \\
& = \int_{-1}^0 \Psi(\mu) P_l(2\mu+1) d\mu + \frac{1}{2l+1} \left[l I_{l-1}^- - (2l+1) I_l^- + (l+1) I_{l+1}^- \right] - \\
& - \frac{1}{2} \delta_{0l} - S^-, \quad l=0,1,2,\dots,L \tag{3.2.1.7b}
\end{aligned}$$

where S^+, S^- are defined in section (2.2.1). The equations (3.2.1.7a) and (3.2.1.7b) are to be solved subject to the boundary conditions (3.2.1.2a) and (3.2.1.2b). Also in case of L-th approximations we will assume

$$I_{L+1}^+ = I_{L+1}^- = 0 \tag{3.2.1.8}$$

Also we assume a trial solution of the form

$$I_l^+(\tau) = g_{l,\alpha} e^{-k\tau} + g_{l,\beta} e^{-\tau} + g_{l,\gamma} e^{\tau} \tag{3.2.1.9a}$$

$$I_l^-(\tau) = h_{l,\alpha} e^{-k\tau} + h_{l,\beta} e^{-\tau} + h_{l,\gamma} e^{\tau} \tag{3.2.1.9b}$$

where $g_{l,\alpha}, g_{l,\beta}, g_{l,\gamma}, h_{l,\alpha}, h_{l,\beta}, h_{l,\gamma}$ are constants. From the boundary condition we have

$$\sum_{r=1}^{n-1} h_{l,\alpha}^{(r)} + h_{l,\beta} + h_{l,\gamma} = 0 \tag{3.2.1.10}$$

where k_r are the $n-2$ roots of the determinant of order N . It will be found that there will be two zero

roots for all order of approximation L. Here will consider two approximations, viz. L = 1 and L = 2.

The source function is defined by

$$\mathfrak{S}(\tau, \mu) = \frac{3}{8} \left[(3 - \mu^2) J(\tau) + (3\mu^2 - 1) K(\tau) \right] \quad (3.2.1.11)$$

where

$$J(\tau) = \frac{1}{2} \int_{-1}^1 I(\tau, \mu) d\mu \quad (3.2.1.12)$$

and

$$K(\tau) = \frac{1}{2} \int_{-1}^1 I(\tau, \mu) \mu^2 d\mu \quad (3.2.1.13)$$

3.2.2 First Approximation.

In this case L = 1, and we neglect I_2^+ and I_2^- .

For $l = 0, 1$ we have from (3.2.1.7a)

$$\left(\frac{4}{3} I_0^{+'} + 2I_1^{+'} \right) - (I_0^+ + I_1^+ + I_0^- - I_1^-) = 2 - 2\phi'(\tau) \quad (3.2.2.1)$$

and

$$\left(2I_0^{+'} + \frac{24}{5} I_1^{+'} \right) - \frac{61}{32} I_0^+ - \frac{909}{160} I_1^+ - \frac{3}{32} I_0^- + \frac{51}{160} I_1^- = -2\phi'(\tau) \quad (3.2.2.2)$$

For $l = 0, 1$ from (3.2.1.7b) we get

$$\left(\frac{4}{3} I_0^{-'} - 2I_1^{-'} \right) + (I_0^+ + I_1^+ + I_0^- - I_1^-) = -2 + 2\phi'(\tau) \quad (3.2.2.3)$$

$$\left(-2I_0^- + \frac{24}{5}I_1^- \right) - \frac{3}{32}I_0^+ - \frac{51}{160}I_1^+ - \frac{61}{32}I_0^- + \frac{909}{160}I_1^- = -2\Phi'(\tau) \quad (3.2.2.4)$$

Inserting equation (3.2.1.9a) and (3.2.1.9b) in the above four equations and equating the coefficients of $e^{-k\tau}$ we obtain the following set of equations

$$\left(\frac{4k}{3} + 1 \right) g_{0,\alpha} + (2k + 1)g_{1,\alpha} + h_{0,\alpha} - h_{1,\alpha} = 0 \quad (3.2.2.5i)$$

$$\left(2k + \frac{61}{32} \right) g_{0,\alpha} + \left(\frac{24k}{5} + \frac{909}{160} \right) g_{1,\alpha} + \frac{3}{32}h_{0,\alpha} - \frac{51}{160}h_{1,\alpha} = 0 \quad (3.2.2.5ii)$$

$$g_{0,\alpha} + g_{1,\alpha} + \left(-\frac{4k}{3} + 1 \right) h_{0,\alpha} + (-2k - 1)h_{1,\alpha} = 0 \quad (3.2.2.5iii)$$

$$\frac{3}{32}g_{0,\alpha} + \frac{51}{160}g_{1,\alpha} + \left(-2k + \frac{61}{32} \right) h_{0,\alpha} + \left(\frac{24k}{5} - \frac{909}{160} \right) h_{1,\alpha} = 0 \quad (3.2.2.5iv)$$

The equations (3.2.2.5) will have a nontrivial solution if

$$D_1(k) = 0 \quad (3.2.2.6)$$

where $D_1(k)$ is given by the equation (2.2.1.6). Again, in a similar manner if we compare the coefficients of $e^{-\tau}$, e^{τ} then we obtain,

$$\frac{7}{3}g_{0,\beta} + 3g_{1,\beta} + h_{0,\beta} - h_{1,\beta} = -2A \quad (3.2.2.7a)$$

$$\frac{125}{32}g_{0,\beta} + \frac{1677}{160}g_{1,\beta} + \frac{3}{32}h_{0,\beta} - \frac{51}{160}h_{1,\beta} = -2A \quad (3.2.2.7b)$$

$$\frac{3}{32}g_{0,\beta} + \frac{5}{160}g_{1,\beta} + \frac{3}{32}h_{0,\beta} - \frac{141}{160}h_{1,\beta} = -2A \quad (3.2.2.6c)$$

$$g_{0,\beta} + g_{1,\beta} - \frac{1}{3}h_{0,\beta} + h_{1,\beta} = 2A \quad (3.2.2.6d)$$

and

$$\frac{1}{3}g_{0,\gamma} + g_{1,\gamma} - h_{0,\gamma} + h_{1,\gamma} = -2B \quad (3.2.2.7a)$$

$$\frac{3}{32}g_{0,\gamma} - \frac{141}{160}g_{1,\gamma} - \frac{3}{32}h_{0,\gamma} + \frac{51}{160}h_{1,\gamma} = -2B \quad (3.2.2.7b)$$

$$\frac{3}{32}g_{0,\gamma} + \frac{51}{160}g_{1,\gamma} + \frac{125}{32}h_{0,\gamma} - \frac{1677}{160}h_{1,\gamma} = 2B \quad (3.2.2.7c)$$

$$g_{0,\gamma} + g_{1,\gamma} + \frac{7}{3}h_{0,\gamma} + h_{1,\gamma} = 2B \quad (3.2.2.7d)$$

Using the boundary condition we find that

$$h_{0,\alpha} + h_{0,\beta} + h_{0,\gamma} = 0 \quad (3.2.2.8a)$$

$$h_{1,\alpha} + h_{1,\beta} + h_{1,\gamma} = 0 \quad (3.2.2.8b)$$

From (3.2.2.6) we have $k = 0, 0, \pm 2.9167$. To satisfy the boundary condition we will take only the positive root of k , i.e., $k = 2.9167$. From the boundary condition (3.3.1.2b) taking $\tau_0 = .1$ we obtain

$$A = 5.5166, B = -4.5166 \quad (3.2.2.9)$$

Solving the equations (3.2.2.5), (3.2.2.6), (3.2.2.7) and using (3.2.2.9) we obtain

$$g_{0,\alpha} = -7.0829, \quad h_{0,\alpha} = 7.6376$$

$$g_{1,\alpha} = 2.5995, \quad h_{1,\alpha} = -9.2267$$

$$g_{0,\beta} = 0.7397, \quad h_{0,\beta} = -2.1987$$

$$g_{1,\beta} = -.9827, \quad h_{1,\beta} = 12.0092$$

$$g_{0,\gamma} = 32.4842, \quad h_{0,\gamma} = -9.8363$$

$$g_{1,\gamma} = -8.8476, \quad h_{1,\gamma} = -2.7826$$

Now we are in a position to calculate the source function. From section 2.2.1 we see that this is given by

$$\mathfrak{S}(\tau, \mu) = \frac{3}{8} \left[(3 - \mu^2)J(\tau) + (3\mu^2 - 1)K(\tau) \right] \quad (3.2.2.10)$$

where

$$J(\tau) = I(0,0) \left[\phi(\tau) + \frac{1}{2} - 5.3369e^{-k\tau} + 21.3919e^{-\tau} + 7.6726e^{\tau} \right] \quad (3.3.2.11)$$

and

$$K(\tau) = I(0,0) \left[\frac{\phi(\tau)}{3} + \frac{1}{6} - 3.3312e^{-k\tau} + 2.2986e^{-\tau} + 2.6733e^{\tau} \right] \quad (3.3.2.12)$$

thereby the source function and consequently the intensity at any surface can be found out.

3.2.3. Second approximation.

Here $L=2$. Following the same procedure (Section 2.2.3) we obtain the same set of ordinary differential equations and these are described below

$$\left(\frac{4}{3}I_0^{+'} + 2I_1^{+'} + I_2^{+'} \right) - (I_0^+ + I_1^+ + I_0^- - I_1^-) = 2 - 2\phi'(\tau) \quad (3.2.3.1)$$

$$\begin{aligned} \left(2I_0^{+'} + \frac{24}{5}I_1^{+'} + 4I_2^{+'} \right) - \frac{61}{32}I_0^+ - \frac{909}{160}I_1^+ - \frac{119}{32}I_2^+ - \\ - \frac{3}{32}I_0^- + \frac{51}{160}I_1^- - \frac{9}{32}I_2^- = -2\phi'(\tau) \end{aligned} \quad (3.2.3.2)$$

$$\begin{aligned} & \left(2k + \frac{61}{32}\right)g_{0,\alpha} + \left(\frac{24k}{5} + \frac{909}{160}\right)g_{1,\alpha} + \left(4k + \frac{119}{32}\right)g_{2,\alpha} + \\ & + \frac{3}{32}h_{0,\alpha} - \frac{51}{160}h_{1,\alpha} + \frac{9}{32}h_{2,\alpha} = 0 \end{aligned} \quad (3.2.3.7b)$$

$$\begin{aligned} & \left(\frac{k}{3} - \frac{1}{64}\right)g_{0,\alpha} + \left(2k + \frac{623}{320}\right)g_{1,\alpha} + \left(\frac{80k}{21} + \frac{317}{64}\right)g_{2,\alpha} + \\ & + \frac{1}{64}h_{0,\alpha} - \frac{17}{320}h_{1,\alpha} + \frac{3}{64}h_{2,\alpha} = 0 \end{aligned} \quad (3.2.3.7c)$$

$$g_{0,\alpha} + g_{1,\alpha} + \left(-\frac{4k}{3} + 1\right)h_{0,\alpha} + (2k - 1)h_{1,\alpha} - \frac{2k}{3}h_{2,\alpha} = 0 \quad (3.2.3.7d)$$

$$\begin{aligned} & \frac{3}{32}g_{0,\alpha} + \frac{51}{160}g_{1,\alpha} + \frac{9}{32}g_{2,\alpha} + \\ & + \left(-2k + \frac{61}{32}\right)h_{0,\alpha} + \left(\frac{24k}{5} - \frac{909}{160}\right)h_{1,\alpha} + \left(-4k + \frac{119}{32}\right)h_{2,\alpha} = 0 \end{aligned} \quad (3.2.3.7e)$$

$$\frac{1}{64}g_{0,\alpha} + \frac{17}{320}g_{1,\alpha} + \frac{3}{64}g_{2,\alpha} + \left(-\frac{k}{3} - \frac{1}{64}\right)h_{0,\alpha} + \left(2k - \frac{623}{320}\right)h_{1,\alpha} + \left(-\frac{80k}{21} + \frac{317}{64}\right)h_{2,\alpha} = 0 \quad (3.2.3.7f)$$

The above set of equations will have nontrivial solution if $D_2(k) = 0$, where $D_2(k)$ is given in section 2.2.3. Therefore

$$D_2(k) = 7.5193k^6 - 104.9134k^4 + 155.4331k^2 = 0$$

This gives

$$k = 0, 0, \pm 1.298, \pm 3.503 \quad (3.2.3.8)$$

As usual we will take only the positive roots $k_1 = 1.298$, $k_2 = 3.503$

Next, equating the coefficients of $e^{-k\tau}$, e^τ we obtain respectively the following equations (3.2.3.9) and (3.2.3.10). These are given below

$$2.33g_{0,\beta} + 3g_{1,\beta} + 2.33g_{2,\beta} + h_{0,\beta} - h_{1,\beta} = -2A \quad (3.2.3.10a)$$

$$3.91g_{0,\beta} + 10.48g_{1,\beta} + 7.72g_{2,\beta} - .09h_{0,\beta} + .32h_{1,\beta} - .28h_{2,\beta} = -2A \quad (3.2.3.10b)$$

$$.31g_{0,\beta} + 3.95g_{1,\beta} + 8.76g_{2,\beta} - .02h_{0,\beta} - .05h_{1,\beta} - .05h_{2,\beta} = 0 \quad (3.2.3.10c)$$

$$g_{0,\beta} + g_{1,\beta} - .33h_{0,\beta} + h_{1,\beta} - .66h_{2,\beta} = -2A \quad (3.2.3.10d)$$

$$-0.09g_{0,\beta} - .32g_{1,\beta} - .28g_{2,\beta} + .09h_{0,\beta} + .88h_{1,\beta} + .28h_{2,\beta} = 2A \quad (3.2.3.10e)$$

$$.02g_{0,\beta} + .05g_{1,\beta} + .47g_{2,\beta} - .35h_{0,\beta} + .05h_{1,\beta} - 1.14h_{2,\beta} = 0 \quad (3.2.3.10f)$$

and comparing the coefficients of e^τ The following equations are given

$$.33g_{0,\gamma} + g_{1,\gamma} + .66g_{2,\gamma} - h_{0,\gamma} + h_{1,\gamma} = -2B \quad (3.2.3.11a)$$

$$0.09g_{0,\gamma} - .79g_{1,\gamma} + .28g_{2,\gamma} - .09h_{0,\gamma} + .32h_{1,\gamma} - .28h_{2,\gamma} = -2B \quad (3.2.3.11b)$$

$$0.35g_{0,\gamma} + .05g_{1,\gamma} - 1.14g_{2,\gamma} + .02h_{0,\gamma} + .05h_{1,\gamma} - .05h_{2,\gamma} = 0 \quad (3.2.3.11c)$$

$$g_{0,\gamma} + g_{1,\gamma} - 2.33h_{0,\gamma} - 3h_{1,\gamma} - .66h_{2,\gamma} = -2B \quad (3.2.3.11d)$$

$$.09g_{0,\gamma} - .32g_{1,\gamma} + .28g_{2,\gamma} + .09h_{0,\gamma} + .88h_{1,\gamma} + .28h_{2,\gamma} = -2B \quad (3.2.3.11e)$$

$$0.02g_{0,\gamma} + .05g_{1,\gamma} + .47g_{2,\gamma} - .31h_{0,\gamma} - .05h_{1,\gamma} - 1.14h_{2,\gamma} = 0 \quad (3.2.3.11f)$$

Next, from the boundary condition we have,

$$\left. \begin{aligned} h_{0,\alpha}^{(1)} + h_{0,\alpha}^{(2)} + h_{0,\beta} &= 0 \\ h_{1,\alpha}^{(1)} + h_{1,\alpha}^{(2)} + h_{1,\beta} &= 0 \\ h_{2,\alpha}^{(1)} + h_{2,\alpha}^{(2)} + h_{2,\beta} &= 0 \end{aligned} \right\} \quad (3.2.3.12)$$

Solving (3.2.3.9), (3.2.3.10), (3.2.3.11) and (3.2.3.12) we have

$$g_{0,\alpha}^{(1)} = -4.2634, \quad h_{0,\alpha}^{(1)} = 2.6597$$

$$g_{0,\alpha}^{(2)} = -2.3824, \quad h_{0,\alpha}^{(2)} = -24.9132$$

$$g_{1,\alpha}^{(1)} = 3.0652, \quad h_{1,\alpha}^{(1)} = -0.5578$$

$$g_{1,\alpha}^{(2)} = 1.2377, \quad h_{1,\alpha}^{(2)} = -17.9606$$

$$g_{2,\alpha}^{(1)} = -1.2908, \quad h_{2,\alpha}^{(1)} = -2.2660$$

$$g_{2,\alpha}^{(2)} = -0.4011, \quad h_{2,\alpha}^{(2)} = -4.7422$$

$$g_{0,\beta} = -19.4683, \quad h_{0,\beta} = 20.8799$$

$$g_{1,\beta} = 8.2659, \quad h_{1,\beta} = 9.3488$$

$$g_{2,\beta} = -3.0141, \quad h_{2,\beta} = 3.4685$$

$$g_{0,\gamma} = 15.1486, \quad h_{0,\gamma} = 1.3736$$

$$g_{1,\gamma} = -5.6184, \quad h_{1,\gamma} = 8.0399$$

$$g_{2,\gamma} = 4.5346, \quad h_{2,\gamma} = 3.5398$$

The source function in this case is obtained from (3.2.2.10) but the mean intensity $J(\tau)$ and the K integral $K(\tau)$ are given by

$$J(\tau) = I(0,0) \left[\phi(\tau) + \frac{1}{2} - 1.1039e^{-k_1\tau} + 1.2720e^{-k_2\tau} - 5.6834e^{-\tau} + 4.0476e^{\tau} \right]$$

and

$$K(\tau) = I(0,0) \left[\frac{\phi(\tau)}{3} + \frac{1}{6} - 0.1793e^{-k_1\tau} - 1.6792e^{-k_2\tau} - 1.0234e^{-\tau} + 2.3897e^{\tau} \right]$$

Substituting the last two relations (3.2.2.10) we can find the source function from which the intensity at any surface can be calculated.

3.3 Solution of a radiative transfer problem in finite atmosphere with

Planetary phase function using a modified form of spherical harmonic method.

3.3.1 The problem and the basic equations.

Here we consider the phase function which is of the form

$$p(\mu, \mu') = 1 + aP_1(\mu)P_1(\mu') = 1 + a\mu\mu'$$

where a is some constant. The equation of transfer in case of plane parallel atmosphere with anisotropic scattering is given by

$$\frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2} \int_{-1}^1 p(\mu, \mu') I(\tau, \mu') d\mu' \quad (3.3.1.1)$$

where the symbols have their usual meanings. The equation (3.3.1.1) will be solved subject to the boundary conditions which are stated in section 3.1 but they are again restated

(a) Absence of incident radiation from outside at the free surface $\tau = 0$, i.e.,

$$I(\tau, \mu) = 0 \quad \text{for } -1 \leq \mu \leq 0 \quad (3.3.1.2a)$$

(b) No incoming radiation at $\tau = \tau_0$, i.e.,

$$I(\tau_0, \mu) = 0, \quad 0 \leq \mu \leq 1; \quad (3.3.1.2b)$$

We consider the forms of intensity given by (3a) and (3b). Using these two forms, the equation of transfer now takes the forms

$$\mu \frac{dI^+(\tau, \mu)}{d\tau} = I^+(\tau, \mu) - \frac{1}{2} \left[\int_{-1}^0 p(\mu, \mu') I^-(\tau, \mu') d\mu' + \int_0^1 p(\mu, \mu') I^-(\tau, \mu') d\mu' \right] \quad (3.3.1.3)$$

and $\psi(\mu)$ is given by (3c); the form of $\phi(\tau)$ will be specified later.

Next, multiplying (3.3.1.5) by $P_l(2\mu - 1)$ and integrating over $[0, 1]$ and using (3c) and the orthogonal property of Legendre polynomials in $[0, 1]$ we obtain

$$\begin{aligned} & \phi'(\tau) \int_0^1 \mu P_l(2\mu - 1) d\mu + \frac{1}{4(2l+1)} \left[\frac{l^2 - l}{2l-1} I_{l-2}^{+'} + 2l I_{l-1}^{+'} + \right. \\ & \left. \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^{+'} + 2(l+1) I_{l+1}^{+'} + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^{+'} \right] = \\ & = \int_0^1 \psi(\mu) P_l(2\mu - 1) d\mu + \frac{1}{2(2l+1)} \left[l I_{l-1}^{+'} + (2l+1) I_l^{+'} + (l+1) I_{l+1}^{+'} \right] - \frac{1}{2} \delta_{0l} - \\ & - \frac{1}{4} \delta_{0l} \left[I_0^{+'} + I_1^{+'} - I_0^{-'} + I_1^{-'} \right] - aT \int_0^1 \mu P_l(2\mu - 1) d\mu, \quad l=0, 1, 2, \dots, L \end{aligned} \quad (3.3.1.7)$$

Similarly multiplying (3.3.1.6) by $P_l(2\mu + 1)$ and integrating over $[-1, 0]$ we have

$$\begin{aligned} & \phi'(\tau) \int_{-1}^0 \mu P_l(2\mu + 1) d\mu + \frac{1}{4(2l+1)} \left[\frac{l^2 - l}{2l-1} I_{l-2}^{-'} - 2l I_{l-1}^{-'} + \right. \\ & \left. + \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^{-'} - 2(l+1) I_{l+1}^{-'} + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^{-'} \right] = \end{aligned}$$

$$\begin{aligned}
&= \int_{-1}^0 \Psi(\mu) P_l(2\mu + 1) d\mu + \frac{1}{2(2l+1)} \left[II_{l-1}^- - (2l+1)I_l^- + (l+1)I_{l+1}^- \right] - \frac{1}{2} \delta_{0l} - \\
&- \frac{1}{4} \delta_{0l} \left[I_0^+ + I_1^+ - I_0^- + I_1^- \right] - aT \int_{-1}^0 \mu P_l(2\mu + 1) d\mu, \quad l=0,1,2,\dots,L \quad (3.3.1.8)
\end{aligned}$$

where I_l' denotes the derivative with respect to the optical thickness and δ is the Kronecker delta.

For example setting successively $l=0$, $l=1$, and $l=2$ in (3.3.1.7) we obtain

$$\left(\frac{4}{3} I_0^{+'} + 2I_1^{+'} + \frac{2}{3} I_2^{+'} \right) - (I_0^+ + I_1^+ + I_0^- - I_1^-) + 2aT = -2\Phi'(\tau) + 2 \quad (3.3.1.9)$$

$$\left(2I_0^{+'} + \frac{24}{5} I_1^{+'} + 4I_2^{+'} + \frac{6}{5} I_3^{+'} \right) - 2(I_0^+ + 3I_1^+ + 2I_2^+) + 2aT = -2\Phi'(\tau) \quad (3.3.1.10)$$

$$\left(\frac{1}{3} I_0^{+'} + 2I_1^{+'} + \frac{80}{21} I_2^{+'} + 6I_2^{+'} + \frac{12}{5} I_4^{+'} \right) - (2I_1^+ + 5I_2^+ + 3I_3^+) = 0 \quad (3.3.1.11)$$

and for $l \neq 0, 1, 2$ we have from (3.3.1.7)

$$\Phi'(\tau) \int_0^1 \mu P_l(2\mu - 1) d\mu + \frac{1}{4(2l+1)} \left[\frac{l^2 - l}{2l-1} I_{l-2}^{+'} + 2II_{l-1}^{+'} + \right.$$

$$\begin{aligned}
& + \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^{+'} + 2(l+1)I_{l+1}^{+'} + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^{+'} \Big] = \\
& = \int_0^1 \Psi(\mu) P_l(2\mu - 1) d\mu + \frac{1}{2(2l+1)} \left[II_{l-1}^{+'} + (2l+1)I_l^{+'} + (l+1)I_{l+1}^{+'} \right] + \\
& \qquad \qquad \qquad + aT \int_0^1 \mu P_l(2\mu - 1) d\mu \qquad \qquad \qquad (3.3.1.12)
\end{aligned}$$

Again setting successively $l=0$, $l=1$, and $l=2$ in (3.3.1.8) we obtain the following ordinary differential equations

$$\left(\frac{4}{3} I_0^{-'} - 2I_1^{-'} + \frac{2}{3} I_2^{-'} \right) + (I_0^{+'} + I_1^{+'} + I_0^{-} - I_1^{-}) - 2aT = -2 + 2\Phi'(\tau) \qquad (3.3.1.13)$$

$$\left(-2I_0^{-'} + \frac{24}{5} I_1^{-'} - 4I_2^{-'} + \frac{6}{5} I_3^{-'} \right) - 2(I_0^{-} - 3I_1^{-} + 2I_2^{-}) + 2aT = -2\Phi'(\tau) \qquad (3.3.1.14)$$

$$\left(\frac{1}{3} I_0^{-'} - 2I_1^{-'} + \frac{80}{21} I_2^{-'} - 6I_3^{-'} + \frac{12}{5} I_4^{-'} \right) - (2I_1^{-} - 5I_2^{-} + 3I_3^{-}) = 0 \qquad (3.3.1.15)$$

and for $l \neq 0, 1, 2$ we have from (3.3.1.8)

$$\Phi'(\tau) \int_{-1}^0 \mu P_l(2\mu + 1) d\mu + \frac{1}{4(2l+1)} \left[\frac{l^2 - l}{2l-1} I_{l-2}^{-'} - 2II_{l-1}^{-'} + \right.$$

$$\begin{aligned}
& + \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^- - 2(l+1)I_{l+1}^- + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^- \Big] = \\
& = \int_{-1}^0 \psi(\mu) P_l(2\mu+1) d\mu + \frac{1}{2l+1} \left[l I_{l-1}^- - (2l+1) I_l^- + (l+1) I_{l+1}^- \right] + \\
& \quad + aT \int_{-1}^0 \mu P_l(2\mu+1) d\mu \tag{3.3.1.16}
\end{aligned}$$

We solve equations (3.3.1.7) to (3.3.1.8) subject to the boundary conditions (3.3.1.2a) and (3.3.1.2b) and the form of $\phi(\tau)$ is taken as

$$\phi(\tau) = A e^{-\tau} + B e^{\tau} \tag{3.3.1.17}$$

Let us assume the trial solution as

$$I_l^+(\tau) = g_{l,\alpha} e^{-k\tau} + g_{l,\beta} e^{-\tau} + g_{l,\gamma} e^{\tau} \tag{3.3.1.18a}$$

$$I_l^-(\tau) = h_{l,\alpha} e^{-k\tau} + h_{l,\beta} e^{-\tau} + h_{l,\gamma} e^{\tau} \tag{3.3.1.18b}$$

where $g_{l,\alpha}, g_{l,\beta}, g_{l,\gamma}, h_{l,\alpha}, h_{l,\beta}, h_{l,\gamma}$ are constants. As before, when we are working in L th approximations we will neglect I_{L+1}^+, I_{L+1}^- .

3.3.2. First approximate solution.

Here $L = 1$. Therefore putting successively $l = 0, 1$ in the equations (3.3.1.9) and (3.3.1.10) we obtain following the ordinary differential equations.

$$\left(\frac{4}{3}I_0^{+'} + 2I_1^{+'} \right) - \left(I_0^+ + I_1^+ + I_0^- - I_1^- \right) + aR = 2 - 2\phi'(\tau) \quad (3.3.2.1)$$

$$\left(2I_0^{+'} + \frac{24}{5}I_1^{+'} \right) - 2\left(I_0^+ + 3I_1^+ \right) + aR = -2\phi'(\tau) \quad (3.3.2.2)$$

$$\left(\frac{4}{3}I_0^{-'} - 2I_1^{-'} \right) + \left(I_0^+ + I_1^+ + I_0^- - I_1^- \right) - aR = -2 + 2\phi'(\tau) \quad (3.3.2.3)$$

$$\left(-2I_0^{-'} + \frac{24}{5}I_1^{-'} \right) - 2\left(I_0^- - 3I_1^- \right) + aR = -2\phi'(\tau) \quad (3.3.2.4)$$

where

$$R = \left[\frac{1}{2} + \frac{I_0^+ + I_0^-}{3} + \frac{I_1^+ - I_1^-}{2} \right] \quad (3.3.2.5)$$

Substituting the forms (3.3.1.18a) and (3.3.1.18b) in (3.3.2.1) to (3.3.2.4) and comparing the coefficients of $e^{-k\tau}$ we obtain the following linear equations

$$\left(\frac{4k}{3} + 1 - \frac{a}{3}\right)g_{0,\alpha} + \left(2k + 1 - \frac{a}{2}\right)g_{1,\alpha} + \left(1 - \frac{a}{3}\right)h_{0,\alpha} + \left(-1 + \frac{a}{2}\right)h_{1,\alpha} = 0 \quad (3.3.2.5a)$$

$$\left(2k + 2 - \frac{a}{3}\right)g_{0,\alpha} + \left(\frac{24k}{5} + 6 - \frac{a}{2}\right)g_{1,\alpha} - \frac{a}{3}h_{0,\alpha} + \frac{a}{2}h_{1,\alpha} = 0 \quad (3.3.2.5b)$$

$$\left(1 - \frac{a}{3}\right)g_{0,\alpha} + \left(1 - \frac{a}{2}\right)g_{1,\alpha} + \left(-\frac{4k}{3} + 1 - \frac{a}{3}\right)h_{0,\alpha} + \left(2k - 1 + \frac{a}{2}\right)h_{1,\alpha} = 0 \quad (3.3.2.5c)$$

$$\frac{a}{3}g_{0,\alpha} + \frac{a}{2}g_{1,\alpha} + \left(2k - 1 + \frac{a}{3}\right)h_{0,\alpha} + \left(-\frac{24k}{5} + 6 + \frac{a}{2}\right)h_{1,\alpha} = 0 \quad (3.3.2.5d)$$

The above system (3.3.2.5) will have non-trivial solution if the determinant of the coefficients vanishes, i.e, if $D_1(k) = 0$ where

$$D_1(k) = \begin{vmatrix} 1 + \frac{4k}{3} - \frac{a}{3} & 2k - \frac{a}{2} & 1 - \frac{a}{3} & -1 + \frac{a}{2} \\ 2k + 2 - \frac{a}{3} & \frac{24k}{5} + 6 - \frac{a}{2} & -\frac{a}{3} & \frac{a}{2} \\ 1 - \frac{a}{3} & 1 - \frac{a}{2} & -\frac{4k}{3} + 1 - \frac{a}{3} & 2k - 1 + \frac{a}{2} \\ \frac{a}{3} & \frac{a}{2} & 2k - 2 + \frac{a}{3} & -\frac{24k}{5} + 6 - \frac{a}{2} \end{vmatrix}$$

Let us take $a = 2.319461$. Then $D_1(k) = 0$ implies that

$$5.76k^4 - 6.030229k^2 = 0$$

$$\text{or, } k = 0, 0, \pm 1.0238165 \quad (3.3.2.6)$$

If we compare the coefficients of $e^{-\tau}$ from (3.3.2.1) (3.3.2.4) we obtain the following equations

$$\left(\frac{7}{3} - \frac{a}{3}\right)g_{0,\beta} + \left(3 - \frac{a}{2}\right)g_{1,\beta} + \left(1 - \frac{a}{2}\right)h_{0,\beta} + \left(-1 + \frac{a}{2}\right)h_{1,\beta} = -2A \quad (3.3.2.7a)$$

$$\left(-4 + \frac{a}{3}\right)g_{0,\beta} + \left(-\frac{54}{5} + \frac{a}{2}\right)g_{1,\beta} + \frac{a}{3}h_{0,\beta} - \frac{a}{2}h_{1,\beta} = 2A \quad (3.3.2.7b)$$

$$\left(1 - \frac{a}{3}\right)g_{0,\beta} + \left(1 - \frac{a}{2}\right)g_{1,\beta} + \left(-\frac{1}{3} - \frac{a}{3}\right)h_{0,\beta} + \left(1 + \frac{a}{2}\right)h_{1,\beta} = -2A \quad (3.3.2.7c)$$

$$\frac{a}{3}g_{0,\beta} + \frac{a}{2}g_{1,\beta} + \frac{a}{2}h_{0,\beta} + \left(\frac{6}{5} - \frac{a}{2}\right)h_{1,\beta} = 2A \quad (3.3.2.7d)$$

and comparing the coefficients of e^{τ} we have the following equations

$$\left(\frac{1}{3} + \frac{a}{3}\right)g_{0,\gamma} + \left(1 + \frac{a}{2}\right)g_{1,\gamma} + \left(-1 + \frac{a}{3}\right)h_{0,\gamma} + \left(1 - \frac{a}{2}\right)h_{1,\gamma} = -2B \quad (3.3.2.8a)$$

$$\frac{a}{3}g_{0,\gamma} + \left(-\frac{6}{5} + \frac{a}{2}\right)g_{1,\gamma} + \frac{a}{3}h_{0,\gamma} - \frac{a}{2}h_{1,\gamma} = -2B \quad (3.3.2.8b)$$

$$\left(1 - \frac{a}{3}\right)g_{0,\gamma} + \left(1 - \frac{a}{2}\right)g_{1,\gamma} + \left(\frac{7}{3} - \frac{a}{3}\right)h_{0,\gamma} + \left(-3 + \frac{a}{2}\right)h_{1,\gamma} = 2B \quad (3.3.2.8c)$$

$$\frac{a}{3}g_{0,\gamma} + \frac{a}{2}g_{1,\gamma} + \left(-\frac{4}{3} + \frac{a}{3}\right)h_{0,\gamma} + \left(\frac{54}{5} - \frac{a}{2}\right)h_{1,\gamma} = -2B \quad (3.3.2.8d)$$

Now, from the boundary condition (3.3.1.2a) we have

$$h_{0,\alpha} + h_{0,\beta} + h_{0,\gamma} = 0 \quad (3.3.2.9a)$$

$$h_{1,\alpha} + h_{1,\beta} + h_{1,\gamma} = 0 \quad (3.3.2.9b)$$

From the boundary condition (3.3.1.2b) taking $\tau_0 = .1$ we obtain

$$A = 5.5166547, B = -4.5166547$$

Finally, solving (3.2.2.5), (3.2.2.7) and (3.2.2.8) we have

$$g_{0,\alpha} = -194.6346, \quad h_{0,\alpha} = -23.3784$$

$$g_{1,\alpha} = 64.5706, \quad h_{1,\alpha} = 9.1931$$

$$g_{0,\beta} = -18.1239, \quad h_{0,\beta} = 23.3738$$

$$g_{1,\beta} = 5.6911, \quad h_{1,\beta} = 9.1931$$

$$g_{0,\gamma} = 19.1405, \quad h_{0,\gamma} = -14.8358$$

$$g_{1,\gamma} = -7.5267, \quad h_{1,\gamma} = -4.6595$$

Finally, we can calculate the source function and this is given by

$$\begin{aligned} \mathfrak{S}(\tau, \mu) = & \left[A e^{-\tau} + B e^{\tau} + \frac{1}{2} + \left(-6.4747 e^{-k\tau} - 6.6534 e^{-\tau} + 5.4475 e^{\tau} \right) + \right. \\ & \left. + \alpha \mu \left(-5.7858 e^{-k\tau} - 0.0010 e^{-\tau} + 0.0031 e^{\tau} \right) \right] \quad (3.3.2.10) \end{aligned}$$

3.4. Solution of a radiative transfer problem in finite atmosphere with

Henyeey-Greenstein phase function using a modified form of spherical harmonic method.

Here we shall be concerned with another type of phase function called Henyeey - Greenstein phase function. This phase function contains a parameter (g). In our discussion we shall be concerned with only one value of g which is associated with the fundamental equations which we are going to describe.

3.4.1. The problem and the basic equations.

The equation of transfer appropriate to the problem is

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2} \int_{-1}^1 p(\mu, \mu') I(\tau, \mu) d\mu' \quad (3.4.1.1)$$

where the symbols have their usual meanings and this equation is to solved subject to the boundary conditions given by

(a) Absence of incident radiation from outside at the free surface $\tau = 0$, i.e.

$$I(\tau, \mu) \equiv 0 \quad \text{for } -1 \leq \mu \leq 0 \quad (3.4.1.2a)$$

(b) No incoming radiation at $\tau = \tau_0$ i.e.

$$I(\tau_0, \mu) = 0, \quad 0 \leq \mu \leq 1; \quad (3.4.1.2b)$$

Here we shall take the same two forms of intensity given by (3a) and (3b). Using these two forms of intensity the equation transfer becomes

$$\mu \frac{dI^+(\tau, \mu)}{d\tau} = I^+(\tau, \mu) - \frac{1}{2} \left[\int_{-1}^0 p(\mu, \mu') I^-(\tau, \mu') d\mu' + \int_0^1 p(\mu, \mu') I^+(\tau, \mu') d\mu' \right] \quad (3.4.1.2)$$

$$\mu \frac{dI^-(\tau, \mu)}{d\tau} = I^-(\tau, \mu) - \frac{1}{2} \left[\int_{-1}^0 p(\mu, \mu') I^-(\tau, \mu') d\mu' + \int_0^1 p(\mu, \mu') I^+(\tau, \mu') d\mu' \right] \quad (3.4.1.3)$$

The form of the Henyey-Greenstein phase function is

$$p(\mu, \mu') = \sum_{k=0}^3 w_k P_k(\mu) P_k(\mu') \quad (3.4.1.4)$$

where w_k are constants .

We take $w_0 = 0$, $w_1 = 3g$, $w_2 = 5g^2$, $w_3 = 7g^3$, and $w_k = 0$ for $k \geq 4$

Here g is called the anisotropy and is defined by

$$g = \langle \cos(\theta) \rangle$$

Inserting the equations (3a) and (3b) in equations (3.4.1.2) and (3.4.1.3) we obtain

$$\mu \left[\Phi'(\tau) + \sum_{i=0}^{l=L} (2i+1) \mu I_i^+(\tau) P_i(2\mu-1) \right] = \Psi(\mu) + \sum_{i=0}^{l=L} (2i+1) I_i^+(\tau) \mu P_i(2\mu-1) - \alpha_0 - \alpha_1 \mu - \alpha_2 \mu^2 - \alpha_3 \mu^3 \quad (3.4.1.5)$$

and

$$\mu \left[\phi'(\tau) + \sum_{l=0}^{l=L} (2l+1) \mu I_l^{-}(\tau) P_l(2\mu+1) \right] = \Psi(\mu) +$$

$$+ \sum_{l=0}^{l=L} (2l+1) I_l^{-}(\tau) \mu P_l(2\mu+1) - \alpha_0 - \alpha_1 \mu - \alpha_2 \mu^2 - \alpha_3 \mu^3 \quad (3.4.1.6)$$

where

$$\alpha_0 = \frac{1}{2} + \frac{1}{4} (I_0^+ + I_1^+ - I_0^- + I_1^-) - \frac{g^2}{32} \left[5(I_0^+ - I_0^-) + \right.$$

$$\left. + 17(I_1^+ + I_1^-) + 15(I_2^+ - I_2^-) + 3(I_3^+ + I_3^-) \right] \quad (3.4.1.7a)$$

$$\alpha_1 = \frac{3g}{4} + \frac{21g^3}{32} + \frac{g}{2} (I_0^+ + I_0^-) + \left(\frac{3g}{4} - \frac{21g^3}{16} \right) (I_1^+ - I_1^-) +$$

$$+ \left(\frac{g}{4} - \frac{39g^3}{16} \right) (I_2^+ + I_2^-) - \frac{21g^3}{16} (I_3^+ - I_3^-) - \frac{3g^3}{16} (I_4^+ + I_4^-) \quad (3.4.1.7b)$$

$$\alpha_2 = \frac{3g^2}{32} \left[5(I_0^+ - I_0^-) + 17(I_1^+ + I_1^-) + 15(I_2^+ - I_2^-) + 3(I_3^+ + I_3^-) \right] \quad (3.4.1.7c)$$

$$\alpha_3 = \frac{5g^3}{16} \left[-\frac{7}{2} + 7(I_1^+ - I_1^-) + 13(I_2^+ + I_2^-) + 7(I_3^+ - I_3^-) + (I_4^+ + I_4^-) \right] \quad (3.4.1.7d)$$

In the next few steps we will make use of the recurrence formula for Legendre polynomials

$$\mu P_l(2\mu \pm 1) = \frac{1}{2l+1} \left[\frac{l+1}{2} P_{l-1}(2\mu \pm 1) \mp \frac{2l+1}{2} P_l(2\mu \pm 1) + \frac{l}{2} P_{l-1}(2\mu \pm 1) \right]$$

Multiplying (3.4.1.5) by $P_l(2\mu - 1)$ and integrating over $[0,1]$ and using the orthogonal properties of Legendre polynomials in $[0,1]$ we get

$$\begin{aligned} \Phi'(\tau) \int_0^1 \mu P_l(2\mu - 1) d\mu + \frac{1}{4(2l+1)} \left[\frac{l^2-1}{2l-1} I_{l-2}^{*'} + 2l I_{l-1}^{*'} + \right. \\ \left. \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^{*'} + 2(l+1) I_{l+1}^{*'} + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^{*'} \right] = \\ = \int_0^1 \Psi(\mu) P_l(2\mu - 1) d\mu + \frac{1}{2l+1} \left[l I_{l-1}^{*'} + (2l+1) I_l^{*'} + (l+1) I_{l+1}^{*'} \right] - \\ - \int_0^1 (\alpha_0 + \alpha_1 \mu + \alpha_2 \mu^2 + \alpha_3 \mu^3) P_l(2\mu - 1) d\mu, \quad l=0,1,2,\dots,L \quad (3.4.1.8) \end{aligned}$$

Similarly multiplying (3.4.1.6) by $P_l(2\mu + 1)$, using the orthogonal property of Legendre polynomial in $[-1,0]$ and integrating over $[-1,0]$ we obtain

$$\begin{aligned}
\Phi'(\tau) & \int_{-1}^0 \mu P_l(2\mu+1) d\mu + \frac{1}{4(2l+1)} \left[\frac{l^2-1}{2l-1} I_{l-2}^- - 2l I_{l-1}^- + \right. \\
& \left. + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)} I_l^- - 2(l+1) I_{l+1}^- + \frac{l^2+3l+2}{2l+3} I_{l+2}^- \right] = \\
& = \int_{-1}^0 \Psi(\mu) P_l(2\mu+1) d\mu + \frac{1}{2l+1} \left[l I_{l-1}^- - (2l+1) I_l^- + (l+1) I_{l+1}^- \right] - \\
& - \int_{-1}^0 (\alpha_0 + \alpha_1 \mu + \alpha_2 \mu^2 + \alpha_3 \mu^3) P_l(2\mu+1) d\mu, \quad l=0,1,2,\dots,L \quad (3.4.1.9)
\end{aligned}$$

The equations (3.4.1.8) and (3.4.1.9) are to be solved subject to the boundary conditions (3.4.1.2a) and (3.4.1.2b). Further we assume the form of $\phi(\tau)$ as

$$\phi(\tau) = A e^{-\tau} + B e^{\tau},$$

where τ is small and A and B are constants which are to be determined. We assume a trial solution of the form

$$I_l^+(\tau) = g_{l,\alpha} e^{-k\tau} + g_{l,\beta} e^{-\tau} + g_{l,\gamma} e^{\tau} \quad (3.4.1.10a)$$

$$I_l^-(\tau) = h_{l,\alpha} e^{-k\tau} + h_{l,\beta} e^{-\tau} + h_{l,\gamma} e^{\tau} \quad (3.4.1.10b)$$

where $g_{l,\alpha}, g_{l,\beta}, g_{l,\gamma}, h_{l,\alpha}, h_{l,\beta}, h_{l,\gamma}$ are constants.

Further from the boundary condition (3.4.1.2a)

$$\sum_{r=1}^{r=n-1} h_{l,\alpha}^{(r)} + h_{1,\beta} + h_{l,\gamma} = 0 \quad (3.4.2.11)$$

3.4.2. First Approximate solution

In this case $L = 1$ and from (3a) and (3b) we have respectively,

$$I^+(\tau, \mu) = I(0,0) \left[I(\tau) + \psi(\mu) + \sum_{l=0}^1 (2l+1) I_l^+(\tau, \mu) \mu P_l(2\mu-1) \right] \quad (3.4.2.1)$$

and

$$I^-(\tau, \mu) = I(0,0) \left[\phi(\tau) + \psi(\mu) + \sum_{l=0}^1 (2l+1) I_l^-(\tau, \mu) \mu P_l(2\mu+1) \right] \quad (3.4.2.2)$$

Putting successively $l=0, 1$ in equations (3.4.1.8) and (3.4.1.9) we get the following ordinary differential equations

$$\begin{aligned} \left(\frac{4}{3} I_0^{+'} + 2 I_1^{+'} \right) + (g-1) (I_0^+ + I_0^-) + \left(-1 + \frac{3g}{2} - \frac{7g^3}{16} \right) (I_1^+ - I_1^-) = \\ = -2\phi'(\tau) + 2 - \frac{3g}{2} - \frac{7g^3}{16} \end{aligned} \quad (3.4.2.1)$$

$$\left(2I_0^{+'} + \frac{24}{5} I_1^{+'} \right) + \left(-2 + g + \frac{15g^2}{16} \right) I_0^+ + \left(g - \frac{15g^2}{16} \right) I_0^- +$$

$$\begin{aligned}
& + \left(-6 + \frac{3g}{2} + \frac{51g^2}{16} + \frac{21g^3}{21} \right) I_1^+ + \left(-\frac{3g}{2} + \frac{51g^2}{16} - \frac{21g^3}{16} \right) I_1^- \\
& = -2\phi'(\tau) - \frac{3g}{2} + \frac{21g^3}{32}
\end{aligned} \tag{3.4.2.2}$$

$$\begin{aligned}
& \left(\frac{4}{3} I_0^- - 2I_1^- \right) + (1-g)(I_0^+ + I_0^-) + \left(1 - \frac{3g}{2} + \frac{7g^3}{16} \right) (I_1^+ - I_1^-) = \\
& = 2\phi'(\tau) - 2 + \frac{3g}{2} + \frac{7g^3}{16}
\end{aligned} \tag{3.4.2.3}$$

$$\begin{aligned}
& \left(-2I_0^- + \frac{24}{5} I_1^- \right) + \left(g - \frac{15g^2}{16} \right) I_0^+ + \left(\frac{3g}{2} - \frac{51g^2}{16} + \frac{21g^3}{16} \right) I_1^+ + \\
& + \left(-2 + g + \frac{15g^2}{16} \right) I_0^- + \left(6 - \frac{3g}{2} - \frac{51g^2}{16} - \frac{21g^3}{16} \right) I_1^- \\
& = -2\phi'(\tau) - \frac{3g}{2} + \frac{21g^3}{32}
\end{aligned} \tag{3.4.2.4}$$

If we vary the parameter g then we get a system of differential equations for various values of g . We shall be concerned with only one value of g . Now comparing the various parts of we get sets

of linear equations from which we have to determine the constants. The steps are exactly same which we all along dealt with. Therefore we have, by using (3.4.1.10a) and (3.4.1.10b) in equations (3.4.2.1) to (3.4.2.4) and comparing the coefficients of $e^{-k\tau}$ we obtain

$$\left(\frac{4k}{3} + .5\right)g_{0,\alpha} + (2k + .3047)g_{1,\alpha} + .5h_{0,\alpha} - .3047h_{1,\alpha} = 0 \quad (3.4.2.7a)$$

$$(2k + 1.2656)g_{0,\alpha} + \left(\frac{24k}{5} + 4.2891\right)g_{1,\alpha} - .2656h_{0,\alpha} + .1172h_{1,\alpha} = 0 \quad (3.4.2.7b)$$

$$.5g_{0,\alpha} + .3047g_{1,\alpha} + \left(-\frac{4k}{3} + .5\right)h_{0,\alpha} + (2k - .3047)h_{1,\alpha} = 0 \quad (3.4.2.7c)$$

$$.2656g_{0,\alpha} + .1172g_{1,\alpha} + (2k - 1.2656)h_{0,\alpha} + \left(-\frac{24k}{5} + 4.2891\right)h_{1,\alpha} = 0 \quad (3.4.2.5d)$$

The determinant of the coefficients $g_{0,\alpha}, g_{1,\alpha}, h_{0,\alpha}, h_{1,\alpha}$ In equation (3.4.1.5) is given by

$$D_1(k) = \begin{vmatrix} \frac{4k}{3} + .5 & 2k + .3047 & .5 & -.3047 \\ 2k + 1.2656 & \frac{24k}{5} + 4.2891 & -.2656 & .1172 \\ .5 & .3047 & -\frac{4k}{3} + .5 & 2k - .3047 \\ .2656 & .1172 & 2k - 1.2656 & -\frac{24k}{5} + 4.2891 \end{vmatrix}$$

If the system of equations (3.4.2.5) have non-trivial solution then we must have

$$D_1(k) = 0 \quad (3.4.2.6)$$

Therefore we have $k = 0, 0, \pm 1.4329$.

Again, using (3.4.1.10a) and (3.4.1.10b) in equations (3.4.2.1) and (3.4.2.4) and comparing the coefficients of $e^{-\tau}, e^{\tau}$ we get

$$\begin{aligned} \left(\frac{7}{3} - g\right)g_{0,\beta} + \left(3 - \frac{3g}{2} + \frac{7g^3}{16}\right)g_{1,\beta} + (1-g)h_{0,\beta} + \\ + \left(-1 + \frac{3g}{2} - \frac{7g^3}{16}\right)h_{1,\beta} = -2A \end{aligned} \quad (3.4.2.7a)$$

$$\begin{aligned} \left(4 - g - \frac{15g^2}{16}\right)g_{0,\beta} + \left(\frac{54}{5} - \frac{3g}{2} - \frac{51g^2}{16} - \frac{21g^3}{16}\right)g_{1,\beta} + \left(\frac{15g^2}{16} - g\right)h_{0,\beta} + \\ + \left(\frac{3g}{2} - \frac{51g^2}{16} + \frac{21g^3}{16}\right)h_{1,\beta} = -2A \end{aligned} \quad (3.4.2.7b)$$

$$\begin{aligned} (1-g)g_{0,\beta} + \left(1 - \frac{3g}{2} + \frac{7g^3}{16}\right)g_{1,\beta} + \left(-\frac{1}{3} - g\right)h_{0,\beta} + \\ + \left(1 + \frac{3g}{2} - \frac{7g^3}{16}\right)h_{1,\beta} = -2A \end{aligned} \quad (3.4.2.7c)$$

$$\begin{aligned} & \left(g - \frac{15g^2}{16} \right) g_{0,\beta} + \left(\frac{3g}{2} - \frac{51g^2}{16} + \frac{21g^3}{16} \right) g_{1,\beta} + \left(g + \frac{15g^2}{16} \right) h_{0,\beta} + \\ & + \left(\frac{6}{5} - \frac{3g}{2} - \frac{51g^2}{16} - \frac{21g^3}{16} \right) h_{1,\beta} = 2A \end{aligned} \quad (3.4.2.7d)$$

and equating the coefficients of e^τ we get

$$\begin{aligned} & \left(\frac{1}{3} + g \right) g_{0,\gamma} + \left(1 + \frac{3g}{2} - \frac{7g^3}{16} \right) g_{1,\gamma} + (g-1)h_{0,\gamma} + \\ & + \left(1 - \frac{3g}{2} + \frac{7g^3}{16} \right) h_{1,\gamma} = -2B \end{aligned} \quad (3.4.2.8a)$$

$$\begin{aligned} & \left(g + \frac{15g^2}{16} \right) g_{0,\gamma} + \left(-\frac{6}{5} + \frac{3g}{2} + \frac{51g^2}{16} + \frac{21g^3}{16} \right) g_{1,\gamma} + \left(g - \frac{15g^2}{16} \right) h_{0,\gamma} + \\ & + \left(-\frac{3g}{2} + \frac{51g^2}{16} - \frac{21g^3}{16} \right) h_{1,\gamma} = -2B \end{aligned} \quad (3.4.2.8b)$$

$$(1-g)g_{0,\gamma} + \left(1 - \frac{3g}{2} + \frac{7g^3}{16} \right) g_{1,\gamma} + \left(\frac{7}{3} - g \right) h_{0,\gamma} +$$

$$+ \left(-3 + \frac{3g}{2} - \frac{7g^3}{16} \right) h_{1,\gamma} = 2B \quad (3.4.2.8c)$$

$$\left(g - \frac{15g^2}{16} \right) g_{0,\gamma} + \left(\frac{3g}{2} - \frac{51g^2}{16} + \frac{21g^3}{16} \right) g_{1,\gamma} + \left(-4 + g + \frac{15g^2}{16} \right) h_{0,\gamma} +$$

$$+ \left(\frac{54}{5} - \frac{3g}{2} - \frac{51g^2}{16} - \frac{21g^3}{16} \right) h_{1,\gamma} = -2B \quad (3.4.2.8d)$$

Now, from the boundary condition (3.4.1.2a) we have

$$h_{0,\alpha} + h_{0,\beta} + h_{0,\gamma} = 0 \quad (3.3.2.9a)$$

$$h_{1,\alpha} + h_{1,\beta} + h_{1,\gamma} = 0 \quad (3.3.2.9b)$$

From the boundary condition (3.4.1.2b) taking $\tau_0 = .1$ we obtain

$$A = 5.5166547, B = -4.5166547$$

Finally, solving (3.4.2.5), (3.4.2.7) and (3.4.2.8) we have

$$g_{0,\alpha} = 4.2656, \quad h_{0,\alpha} = -13.7708$$

$$g_{1,\alpha} = -1.8193, \quad h_{1,\alpha} = -7.9130$$

$$g_{0,\beta} = 21.1762, \quad h_{0,\beta} = 31.1084$$

$$g_{1,\beta} = 7.1262, \quad h_{1,\beta} = 13.7475$$

$$g_{0,\gamma} = 25.4694, \quad h_{0,\gamma} = -17.3376$$

$$g_{1,\gamma} = -11.2555, \quad h_{1,\gamma} = -5.8345$$

Therefore the source function is given by

$$\begin{aligned} \mathfrak{S}(\tau, \mu) = I(0,0) & \left[A e^{-\tau} + B e^{\tau} + \left(.25 + 0.4570\mu - 0.1367\mu^2 \right) + \right. \\ & \left. + \mathfrak{R}_1 e^{-k\tau} + \mathfrak{R}_2 e^{-\tau} + \mathfrak{R}_3 e^{\tau} \right] \end{aligned} \quad (3.3.2.10)$$

where

$$\mathfrak{R}_1 = 2.6614 - 1.0934\mu - 1.7645\mu^2 + 1.6660\mu^3 \quad (3.3.2.11a)$$

$$\mathfrak{R}_2 = 0.3517 + 11.6747\mu + 7.1520\mu^2 - 1.8103\mu^3 \quad (3.3.2.11b)$$

$$\mathfrak{R}_3 = 7.0251 + 0.8896\mu - 1.7917\mu^2 - 1.4821\mu^3 \quad (3.3.2.11c)$$

The intensity at any surface can now be calculated.

3.5 Solution of a radiative transfer problem in finite atmosphere with

General phase function using a modified form of spherical harmonic method.

We will consider this in case of finite atmosphere. Here the treatment of the problem is the same as that we have considered in section 2.3 except that the boundary conditions are different. Even the general equations from which we are going to deduce the spherical harmonic equations for different cases, will be the same. Therefore, at the beginning we state the basic equation of transfer with the appropriate boundary conditions and thereafter we jump few steps (these are identical and have already been deduced in section 2.3.1) and state the necessary equations.

3.5.1. The problem and the basic equations.

The equation of transfer in case of plane parallel atmosphere with spherical symmetry is as usual given by

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2} \int_{-1}^1 p(\mu, \mu') I(\tau, \mu') d\mu' \quad (3.5.1.1)$$

where the symbols have their usual meanings. The equation (3.5.1.1) will be solved subject to the boundary condition which are stated in section 3.1 but they are again restated

(a) Absence of incident radiation from outside at the free surface $\tau = 0$, i.e.,

$$I(\tau, \mu) \equiv 0 \quad \text{for } -1 \leq \mu \leq 0 \quad (3.5.1.2a)$$

(b) No incoming radiation $\tau = \tau_0$ at, i.e.

$$I(\tau_0, \mu) = 0, \quad 0 \leq \mu \leq 1; \quad (3.5.1.2b)$$

We will take the same two forms given by equations (3a) and (3b). We consider the phase function in the following form

$$P(\mu, \mu') = \sum_{k=0}^{\infty} w_k P_k(\mu) P_k(\mu') \quad (3.5.1.3)$$

From section (2.3.1) we have the following equations

$$\begin{aligned} \Phi'(\tau) \int_0^1 \mu P_l(2\mu - 1) d\mu + \frac{1}{4(2l+1)} \left[\frac{l^2 - l}{2l-1} I_{l-2}^{+'} + 2l I_{l-1}^{+'} + \right. \\ \left. + \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^{+'} + 2(l+1) I_{l+1}^{+'} + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^{+'} \right] = [\Phi(\tau) + 1] \delta_{0,l} + \\ + \frac{1}{2(2l+1)} \left[l I_{l-1}^{+'} + (2l+1) I_l^{+'} + (l+1) I_{l+1}^{+'} \right] - \frac{1}{2} \sum_{k=0}^{\infty} w_k S_{lk}^{+'} \Pi, \quad l=0,1,2,\dots,L \quad (3.5.1.4) \end{aligned}$$

and

$$\begin{aligned} \Phi'(\tau) \int_{-1}^0 \mu P_l(2\mu + 1) d\mu + \frac{1}{4(2l+1)} \left[\frac{l^2 - l}{2l-1} I_{l-2}^{-'} - 2l I_{l-1}^{-'} + \right. \\ \left. + \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^{-'} - 2(l+1) I_{l+1}^{-'} + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^{-'} \right] = \Phi(\tau) \delta_{0,l} + \\ + \frac{1}{2(2l+1)} \left[l I_{l-1}^{-'} - (2l+1) I_l^{-'} + (l+1) I_{l+1}^{-'} \right] - \frac{1}{2} \sum_{k=0}^{\infty} w_k S_{lk}^{-'} \Pi, \quad l=0,1,2,\dots,L \quad (3.5.1.5) \end{aligned}$$

These are two basic equations which will be solved subject to the prescribed boundary conditions. Further, in this section we will take two other phase functions apart from isotropic and Rayleigh; these are i) Planetary phase function and ii) Henyey-Greenstein phase functions. In other words we will deduce spherical harmonic equations for each of these phase functions from this

general phase function and show that they are identical with those already obtained in chapter 3. Also the notations $S_{l,k}^+, S_{l,k}^-, \Pi$ are described in section (2.3.1)

3.5.2. Different cases.

3.5.2.1. Isotropic scattering.

The phase function in this case is

$$p(\mu, \mu') = P_0(\mu)P_0(\mu') = 1 \quad (3.5.2.1.1)$$

We have as before $w_0 = 1$, and $w_k = 0$, for $k \geq 1$

A. First approximations.

Here $L = 1$, The spherical harmonics equations were already obtained in section 2.3.2. We are not giving details of their development but state only the equations in both approximations.

$$\left(\frac{4}{3}I_0^{+'} + 2I_1^{+'} \right) - \left(I_0^+ + I_1^+ + I_0^- - I_1^- \right) = 2 - 2\Phi'(\tau) \quad (3.5.2.1.2)$$

$$\left(2I_0^{+'} + \frac{24k}{5}I_1^{+'} \right) - 2\left(I_0^+ + 3I_1^+ \right) = -2\Phi'(\tau) \quad (3.5.2.1.3)$$

$$\left(\frac{4}{3}I_0^{-'} - 2I_1^{-'} \right) + \left(I_0^- + I_1^- + I_0^+ - I_1^+ \right) = -2 + 2\Phi'(\tau) \quad (3.5.2.1.4)$$

$$\left(-2I_0^{-'} + \frac{24k}{5}I_1^{-'} \right) - 2(I_0^{-} - 3I_1^{-}) = -2\phi'(\tau) \quad (3.5.2.1.5)$$

The equations (3.5.2.2) to (3.5.2.5) are same as those obtained in section 3.1.1

B. Second approximations.

Here $L = 2$. For $l = 0, 1, 2$ from (3.5.1.4) and (3.5.1.5) we get the following spherical harmonics equations

$$\left(\frac{4}{3}I_0^{+'} + 2I_1^{+'} + \frac{2}{3}I_2^{+'} \right) - (I_0^{+} + I_1^{+} + I_0^{-} - I_1^{-}) = 2 - 2\phi'(\tau) \quad (3.5.2.1.6)$$

$$\left(2I_0^{+'} + \frac{24}{5}I_1^{+'} + 4I_2^{+'} \right) - 2(I_0^{+} + 3I_1^{+} + 2I_2^{+}) = -2\phi'(\tau) \quad (3.5.2.1.7)$$

$$\left(\frac{1}{3}I_0^{+'} + 2I_1^{+'} + \frac{80}{21}I_2^{+'} \right) - (2I_1^{+} + 5I_2^{+}) = 0 \quad (3.5.2.1.8)$$

$$\left(\frac{4}{3}I_0^{-'} - 2I_1^{-'} + \frac{2}{3}I_2^{-'} \right) + (I_0^{+} + I_1^{+} + I_0^{-} - I_1^{-}) = -2 + 2\phi'(\tau) \quad (3.5.2.1.9)$$

$$\left(-2I_0^{-'} + \frac{24}{5}I_1^{-'} - 4I_2^{-'} \right) - 2(I_0^{-} - 3I_1^{-} + 2I_2^{-}) = -2\phi'(\tau) \quad (3.5.2.1.10)$$

and

$$\left(\frac{1}{3}I_0^- - 2I_1^- + \frac{80}{21}I_2^- \right) - (2I_1^- - 5I_2^-) = 0 \quad (3.5.2.1.11)$$

Again these equations (i.e., (3.5.2.1.6) to (3.5.2.1.11)) are identical with the equations obtained in the second approximation in the section (3.1.3). There we have found their solutions.

3.5.2.2. Case II. Rayleigh phase function.

The Rayleigh phase function is given by

$$P(\mu, \mu') = 1 + \frac{1}{8}(3\mu^2 - 1)(3\mu'^2 - 1) = 1 + P_2(\mu)P_2(\mu') \quad (3.5.2.2.1)$$

we have

$$w_0 = 1, w_1 = 0, w_2 = \frac{1}{2} \text{ and } w_k = 0, \text{ for } k \geq 3$$

The spherical harmonics equations are obtained in the usual manner and we avoid details.

A. First approximations.

Here $L=1$. From (3.5.1.4) we have

$$l=0: \left(\frac{4}{3}I_0^+ + 2I_1^+ \right) - (I_0^+ + I_1^+ + I_0^- - I_1^-) = 2 - 2\phi'(\tau) \quad (3.5.2.2.2)$$

$$l=1: \left(2I_0^+ + \frac{24}{5}I_1^+ \right) - \frac{61}{32}I_0^+ - \frac{909}{160}I_1^+ - \frac{3}{32}I_0^- + \frac{51}{160}I_1^- = -2\phi'(\tau) \quad (3.5.2.2.3)$$

$$l=0: \left(\frac{4}{3}I_0^- - 2I_1^- \right) + (I_0^+ + I_1^+ + I_0^- - I_1^-) = -2 + 2\phi'(\tau) \quad (3.5.2.2.4)$$

$$l=1: \left(-2I_0^{-'} + \frac{24}{5}I_1^{-'} \right) - \frac{3}{32}I_0^{+'} - \frac{51}{160}I_1^{+'} - \frac{61}{32}I_0^{-} + \frac{909}{160}I_1^{-} = -2\phi'(\tau) \quad (3.5.2.2.5)$$

So we see that the equations (3.5.2.2.2) to (3.5.2.2.5) are identical with the equations in the section (3.2.2)

B. Second approximations.

Here $L = 2$

For $l = 0, 1, 2$ from (3.5.1.5) we have

$$\left(\frac{4}{3}I_0^{+'} + 2I_1^{+'} + \frac{2}{3}I_2^{+'} \right) - \left(I_0^{+'} + I_1^{+'} + I_0^{-} - I_1^{-} \right) = 2 - 2\phi'(\tau) \quad (3.5.2.2.6)$$

$$\begin{aligned} \left(2I_0^{+'} + \frac{24}{5}I_1^{+'} + 4I_2^{+'} \right) - \frac{61}{32}I_0^{+'} - \frac{909}{160}I_1^{+'} - \frac{119}{32}I_2^{+'} - \\ - \frac{3}{32}I_0^{-} + \frac{51}{160}I_1^{-} - \frac{9}{32}I_2^{-} = -2\phi'(\tau) \end{aligned} \quad (3.5.2.2.7)$$

$$\begin{aligned} \left(\frac{1}{3}I_0^{+'} + 2I_1^{+'} + \frac{80}{21}I_2^{+'} \right) + \frac{1}{64}I_0^{+'} - \frac{623}{320}I_1^{+'} - \frac{317}{64}I_2^{+'} - \\ - \frac{1}{64}I_0^{-} + \frac{17}{320}I_1^{-} - \frac{3}{64}I_2^{-} = 0 \end{aligned} \quad (3.5.2.2.8)$$

Again for $l = 0, 1, 2$ from (3.5.1.4) we obtain in a similar way the following equations

Here we consider only the first approximation $L = 1$.

For $l = 0$, we have from (3.5.1.4)

$$\left(\frac{4}{3}I_0^{+'} + 2I_1^{+'} \right) - (I_0^+ + I_1^+ + I_0^- - I_1^-) = 2 - 2\phi'(\tau) - 2 \sum_{k=1}^{\infty} w_k S_{0,k}^+ \Pi$$

Here we have obtained
$$\Pi = \frac{1}{2} + \frac{I_0^+ + I_0^-}{3} + \frac{I_1^+ - I_1^-}{2}$$

and using $S_{0,k}^+, w_k$ we get

$$\left(\frac{4}{3}I_0^{+'} + 2I_1^{+'} \right) - (I_0^+ + I_1^+ + I_0^- - I_1^-) + a\Pi = 2 - 2\phi'(\tau) \quad (3.5.2.3.2)$$

For $l = 1$ from (3.5.1.4) we find

$$\left(2I_0^{+'} + \frac{24k}{5}I_1^{+'} \right) - 2(I_0^+ + 3I_1^+) = -2\phi'(\tau) - 6 \sum_{k=1}^{\infty} w_k S_{1,k}^+ \Pi$$

And using $\Pi, S_{1,k}^+$ we obtain

$$\left(2I_0^{+'} + \frac{24k}{5}I_1^{+'} \right) - 2(I_0^+ + 3I_1^+) + a\Pi = -2\phi'(\tau) \quad (3.5.2.3.3)$$

For $l = 0$ from (3.5.1.5) we have

$$\left(\frac{4}{3}I_0^{-'} - 2I_1^{-'} \right) + (I_0^- + I_1^- + I_0^+ - I_1^+) = -2 + 2\phi'(\tau) - 2 \sum_{k=1}^{\infty} w_k S_{0,k}^- \Pi$$

Once again using $\Pi, S_{0,k}^-$ we obtain after simplification

$$\left(\frac{4}{3}I_0^- - 2I_1^- \right) + \left(I_0^- + I_1^+ + I_0^- - I_1^- \right) - a\Pi = -2 + 2\phi'(\tau) \quad (3.5.2.3.4)$$

Finally for $l=1$ from (3.5.1.5) we get

$$\left(-2I_0^- + \frac{24k}{5}I_1^- \right) - 2(I_0^- - 3I_1^-) = -2\phi'(\tau) - 6 \sum_{k=1}^{\infty} w_k S_{1,k}^- \Pi$$

using $\Pi, S_{1,k}^-$ we obtain after simplification

$$\left(-2I_0^- + \frac{24k}{5}I_1^- \right) - 2(I_0^- - 3I_1^-) + a\Pi = -2\phi'(\tau) \quad (3.5.2.3.5)$$

Thus we see that once again we get identical equations which we have obtained in section 3.3.2 if we set $R = \Pi$.

3.5.2.4. Case IV . Henyey - Greenstein Phase function.

Here the phase function is defined by

$$p(\mu, \mu') = 1 + 3gP_1(\mu)P_1(\mu') + 5g^2P_2(\mu)P_2(\mu') + 7g^3P_3(\mu)P_3(\mu') \quad (3.5.2.4.1)$$

Comparing (3.5.2.4.1) with (3.5.1.3) we get

$$w_0 = 1, w_1 = 3g, w_2 = 5g^2, w_3 = 7g^3 \text{ and } w_k = 0, \text{ for } k \geq 4$$

Here we consider only the first approximation $L = 1$.

For $l=0$, we have from (3.5.1.4)

$$\left(\frac{4}{3}I_0^{+'} + 2I_1^{+'} \right) - \left(I_0^+ + I_1^+ + I_0^- - I_1^- \right) = 2 - 2\phi'(\tau) - 2 \sum_{k=1}^{\infty} w_k S_{0,k}^+ \Pi$$

Now we have to find different Π s for k and l . Varying k and l we get

$$\Pi = \frac{1}{2} + \frac{I_0^+ + I_0^-}{3} + \frac{I_1^+ - I_1^-}{2}, \quad k=1$$

$$\Pi = \frac{I_0^+ - I_0^-}{8} + \frac{17}{40}(I_1^+ + I_1^-), \quad k=2$$

$$\Pi = -\frac{1}{8} + \frac{I_1^+ - I_1^-}{4}, \quad k=3$$

Therefore using $S_{0,k}^+, w_k$ we get

$$\begin{aligned} \left(\frac{4}{3}I_0^{+'} + 2I_1^{+'} \right) + (g-1)(I_0^+ + I_0^-) + \left(-1 + \frac{3g}{2} - \frac{7g^3}{16} \right) (I_1^+ - I_1^-) = \\ = -2\phi'(\tau) + 2 - \frac{3g}{2} - \frac{7g^3}{16} \end{aligned} \quad (3.5.2.4.2)$$

Next for $l=1$ from (3.5.1.4) as before

$$\left(2I_0^{+'} + \frac{24k}{5}I_1^{+'} \right) - 2(I_0^+ + 3I_1^+) = -2\phi'(\tau) - 6 \sum_{k=1}^{\infty} w_k S_{1,k}^+ \Pi$$

If we simplify the R.H.S of previous equation with the known values of $\Pi, S_{1,k}^+, w_k$ and obtain the following equation

$$\begin{aligned}
& \left(2I_0^+ + \frac{24}{5}I_1^+\right) + \left(-2 + g + \frac{15g^2}{16}\right)I_0^+ + \left(g - \frac{15g^2}{16}\right)I_0^- + \\
& + \left(-6 + \frac{3g}{2} + \frac{51g^2}{16} + \frac{21g^3}{21}\right)I_1^+ + \left(-\frac{3g}{2} + \frac{51g^2}{16} - \frac{21g^3}{16}\right)I_1^- \\
& = -2\phi'(\tau) - \frac{3g}{2} + \frac{21g^3}{32} \tag{3.5.2.4.3}
\end{aligned}$$

Next for $l=0$ we get from (3.5.1.5)

$$\left(\frac{4}{3}I_0^- - 2I_1^-\right) + \left(I_0^- + I_1^+ + I_0^- - I_1^-\right) = -2 + 2\phi'(\tau) - 2 \sum_{k=0}^{\infty} w_k S_{0,k}^- \Pi$$

Using $\Pi, S_{1,k}^-, w_k$ we get after little calculation

$$\begin{aligned}
& \left(\frac{4}{3}I_0^- - 2I_1^-\right) + \left(1-g\right)\left(I_0^+ + I_0^-\right) + \left(1 - \frac{3g}{2} + \frac{7g^3}{16}\right)\left(I_1^+ - I_1^-\right) = \\
& = 2\phi'(\tau) - 2 + \frac{3g}{2} + \frac{7g^3}{16} \tag{3.5.2.4.4}
\end{aligned}$$

Finally, from (3.5.1.5) for $l=1$

$$\left(-2I_0^- + \frac{24k}{5}I_1^-\right) - 2(I_0^- - 3I_1^-) = -2\phi'(\tau) - 6 \sum_{k=1}^{\infty} w_k S_{1,k}^- \Pi$$

Putting the calculated values of Π , $S_{1,k}^-$, w_k we obtain the following ordinary differential equation

$$\begin{aligned} &\left(-2I_0^- + \frac{24}{5}I_1^-\right) + \left(g - \frac{15g^2}{16}\right)I_0^+ + \left(\frac{3g}{2} - \frac{51g^2}{16} + \frac{21g^3}{16}\right)I_1^+ + \\ &+ \left(-2 + g + \frac{15g^2}{16}\right)I_0^- + \left(6 - \frac{3g}{2} - \frac{51g^2}{16} - \frac{21g^3}{16}\right)I_1^- \\ &= -2\phi'(\tau) - \frac{3g}{2} + \frac{21g^3}{32} \end{aligned} \quad (3.5.2.4.4)$$

Once again we see that the above four spherical harmonics equations are exactly the same with the ones which have been obtained in section 3.4.2. and in that section we have found their solutions.