

Chapter 1

Introduction

1.1 Compact stars: Historical Background and Recent Developments

Compact stars, i.e., white dwarfs and neutron stars, are born when normal stars die. When most of the nuclear fuel of a normal star has been consumed it cannot support it against gravitational collapse. Consequently, the star shrinks down and its density increases. At a sufficiently high density and low temperature, it produces nonthermal pressure due to pressure of degenerate fermions which supports it against further collapse [2, 3]. White dwarfs are stabilised by degenerate electron gas. There is a mass limit for white dwarfs beyond which even the degenerate electron gas cannot prevent the star from collapsing to a neutron star or a black hole. Soon after the discovery of neutron by Chadwick in 1932 and from the interpretation of Heisenberg that neutrons are also spin- $\frac{1}{2}$ particles, the idea of degenerate electron gas suggested

by Chandrasekhar was extended to the case of degenerate neutrons. The possibility of a star, whose gravitational collapse is prevented by the pressure of the cold degenerate neutron gas was first predicted by Landeau. Baade and Zwicky [4] gave the name neutron star and proposed that a transition from an ordinary star to a neutron star might occur during supernova explosion. The typical mass of a neutron star is $\sim 1 - 3 M_{\odot}$ ($M_{\odot} = \text{Solar mass}$), radius $\sim 10 \text{ km}$ and mean density $\sim 10^{14} \text{ gm cm}^{-3}$.

The structure, material composition and the relevant equation of state (EOS), i.e., the functional dependence of pressure on energy density of the internal material of a neutron star as well as its stability are subjects of fundamental importance in astrophysics. It is known that gravitational interaction is very weak compared to other fundamental interactions, and it is reasonable to neglect it while studying atomic or nuclear processes. However, very strong gravitational fields may occur in highly condensed matter present inside compact stars. A description of these stars cannot be given by the Newtonian theory and one needs the general theory of relativity to study the structure as well as the stability of these compact objects. For construction of theoretical models of relativistic stars, one may refer to the pioneering work of Schwarzschild. In 1916, he gave analytic solutions to the Einstein field equations, describing a star of uniform density. The first numerical calculations for a neutron star within the context of general relativity were made by Oppenheimer and Volkoff [5]. They studied the gravitational equilibrium of neutron stars using the specific solutions for static spherical fluid given by Tolman [6]. They obtained a maximum stable mass of $\approx 0.75 M_{\odot}$ for a neutron star, by making use of the equation of state for a cold Fermi gas. Buchdahl [7] and Bondi [8] generalised the interior Schwarzschild

solution to more general static fluid spheres and obtained useful inequalities involving the mass concentration, central density and central pressure. Intensive studies have since been made of the EOS and the structure of neutron stars.

A typical neutron star has an outer crust (nuclei and electrons), an inner crust (nuclei, neutrons and electrons and muons), an interior part consisting of superfluid protons, neutrons, normal leptons, and a more dense hadronic core. The equation of state (EOS) of the neutron star matter below the saturation density of nuclear matter is relatively well known. But physical properties of the matter at higher densities are uncertain. For more compact star, the core may contain several new exotic states of matter, such as, bose condensate of pions and kaons, occurrence of hyperons and transition from hadronic to quark matter. This will be discussed in sections 1.4 and 1.5

The study of strange quark matter(SQM) in the core of a very compact star and the possible existence of strange stars have become active area of research after the suggestion of Bodmer [9] and Witten[10] that SQM may be the true ground state of quantum chromodynamics(QCD). It is known that among nuclei, the nucleus ^{56}Fe has one of the highest binding energy per nucleon (E/A), given by $M(^{56}\text{Fe})c^2/56 = 930.4 \text{ MeV}$. But for strange matter, described by the Bag Model, $E/A = 829 \text{ MeV}$ for the Bag constant $B = 57.5 \text{ MeV}/\text{fm}^3$ and $E/A = 915 \text{ MeV}$ for $B = 85.3 \text{ MeV}/\text{fm}^3$, both lower than that in ^{56}Fe implying that SQM is more stable. The possible existence of strange (quark) stars was studied by many workers (e.g.,[11, 12]) and it was found that very compact strange stars are possible. These are smaller than normal neutron stars because they are strongly self-bound system with small gravitational

effect. The details have been considered in Chapter 4. The physics of strange quark matter in compact stars and its possible signals are reviewed in [13, 14]. Kapoor and Shukre [15] claim that none of the neutron star EOS is good enough to explain the observed mass-radius constraints of known pulsars. Recent observations suggest that there are many compact astrophysical objects (Table 1.1) like Her X-1 [16, 17, 18], 4U 1820-30 [19, 20, 21], SAX J 1808.4-3658 [22], 4U 1728-34 [23], PSR 0943+10 [24, 25] and RX J185635-3754 [22, 26], earlier thought to be neutron stars, are actually good strange star candidates whose mass-radius estimations cannot be explained by the ordinary neutron star model. Recently, in the context of modern quantum field theoretical treatment of quark matter, including color superconductivity and a vector mean field, Blaschke *et al* [27] have given a microscopic description of these hybrid stars which allows a stiff hybrid equation of state. After the suggestion of the phenomenon of color superconductivity where color charged quarks may form a color superconducting state and change the properties of quark matter [28, 29, 30, 31], neutron stars with color superconducting strange quark matter have been studied by many authors in the last few years [32, 33, 28, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43].

In an ordinary neutron star it may not always be possible to use a single EOS to describe the entire star. The possibility of a compact star having a quark-diquark core surrounded by a low density envelope of nucleons has been discussed by Kastor and Traschen [44]. Using the solution of Mukherjee *et al* [45] a core-envelope model, relevant for compact stars having a deconfined quark core surrounded by less compact hadronic matter, has been presented in ref. [46]. The possibility of the existence of more compact stars and the appearance of exotic matter have been studied in

modern relativistic field-theoretical models [47]. In a recent article it has been claimed that hyperons may appear around twice normal nuclear matter density [48]. Bose condensation of kaons in dense, cold matter has been investigated by Lee *et al* [49].

Most of the neutron star observations are based on the measurements of radio pulsars and X-ray binaries. After the discovery of pulsar by Hewish *et al* [56] and on realisation that these are rotating neutron stars, theoretical investigation of superdense stars has been done using a general relativistic treatment by both numerical and analytical methods and various parameters of neutron star have been determined. Till now nearly two thousand pulsars are known from the observations by the Hubble Space Telescope (HST), Rosi X-ray Timing Explorer (RXTE), X-ray satellites Chandra and XMM-Newton and by ground based optical telescopes as the European Southern Observatory (ESO). Studies with these orbiting observatories on the spectral lines from these astronomical sources of radio and x-ray emission, constraints on the compactness (mass-radius relation) and hence on the equation of states are determined. Thorsett and Chakrabarty [57] have derived the mass constraints of 26 neutron stars in the binary radio pulsar systems with a remarkably narrow underlying Gaussian mass distribution at $M = 1.35 \pm 0.04 M_{\odot}$. However, more massive compact stars have since been observed. Although neutron star masses can be measured in the radio pulsar binary systems with high precision, the measurement of radius is still uncertain. There is no method so far to directly measure the radius of a neutron star. Because of these uncertainties in determining the radius the accurate measurement of the neutron star EOS is a difficult task. However, with the modern X-ray observatories, a few mass-radius relationships have been appropriately measured.

Compact stars	M/M_{\odot}	Radius(km)
HerX-1 [16]	(i) 0.98 ± 0.12	(ii) 6.7 ± 1.2
HerX-1 [17, 18]	(ii) 0.85 ± 0.15	(ii) 8.1 ± 0.41
4U 1820-30 [19, 20]	$0.8 - 1.8$	< 10
4U 1820-30 [21]	1.58 ± 0.06	9.11 ± 0.40
4U 1728-34 [23]	~ 1.1	< 10
4U 1608-52 [50]	1.74 ± 0.14	9.3 ± 1.0
SAX J1808.4-3658 [22]	(i) 1.435(SS1) (ii) 1.323(SS2)	(i) 7.07 (ii) 6.55
SAX J1808.4-3658 [51]	0.9 ± 0.3	7.951 ± 1.0
SAX J1748.9-2021 [52]	(i) 1.78 ± 0.3 (ii) 1.33 ± 0.33	(i) 8.18 ± 1.62 (ii) 10.93 ± 2.09
PSR 1937+21 [23]	< 2.4	< 11.5
RX J1856-37 [22, 26]	0.9 ± 0.2	$< 6_{+2}^{-1}$
EXO 0748-675 [53]	2.10 ± 0.28	13.8 ± 0.18
EXO 1745-248 [54]	1.7	9
XTE J1739-285 [55]	1.51	< 10.9

Table 1.1: Some candidates for Compact Stars

Using three such mass-radius relationships that include the apparent radius, the gravitational redshift and the averaged mass density, Zhang *et al* [58] have constrained the mass and radius of the X-ray sources, 1E 1207.4-5209, Aq1 X-1 and EXO 0748-678. The fact that neutron stars in low-mass X-ray binaries are the progenitors of millisecond period pulsars was confirmed after the discovery of the millisecond pulsar in the X-ray binary SAXJ1808.4-3658 (hereafter SAX) during its outburst in 1998 [59]. SAX was observed by RXTE during the 1998, 2002, 2005 and 2008 outbursts. Morsink and Leahy [60] analysed the pulse shapes detected during the multiple outbursts of SAX in order to constrain the neutron star mass and radius and derived constraint on mass $0.8M_{\odot} < M < 1.7M_{\odot}$ and radius $5 \text{ km} < b < 13 \text{ km}$. Using the time resolved spectroscopy of thermonuclear X-ray burst observed from SAXJ1748.9-2021 Güver and Özel [52] obtained two solutions for the neutron star mass (M) and radius (b) as $M = 1.78 \pm 0.3 M_{\odot}$, $b = 8.18 \pm 1.62 \text{ km}$ and $M = 1.33 \pm 0.33 M_{\odot}$, $b = 10.93 \pm 2.09 \text{ km}$. The evidence of neutron stars with high mass $M = 2.1 \pm 0.2 M_{\odot}$ for PSR J0751+1807 in a neutron star - white dwarf binary system [61] and large radius $b > 12 \text{ km}$ for the isolated neutron star RX J1856.5-3754 [62] point to a stiff EOS. This has raised a question whether the existence of quark matter in compact stars can be excluded as well. Recent developments in the field of compact stars: astrophysical data, the EOS of dense matter and its impact on the global properties of compact stars has been discussed in ref. [63]. Using modified Recharadson potential along with density dependent quark masses allowing chiral symmetry restoration Gangopadhyay *et al* [18] have considered some pulsars whose masses are recently determined accurately to predict their radii by fitting with their strange star EOS. They have obtained the

radii of these stars smaller compared to neutron stars. Some candidates for compact stars of masses (M) in solar mass (M_{\odot}) unit and radii within the observed ranges are listed in Table 1.1. It is expected that in the near future we will get more and more accurate estimations of masses and radii of neutron stars from observation so that the matter compositions and the EOS of the stars can be fixed exactly.

Motivated by the recent theoretical as well as observational developments, since their discovery, compact stars have been recognised as promising laboratories for studying matter under extreme conditions. Considerable attention is now directed to the study of compact astrophysical objects which include strange stars [64, 20], quark-diquark stars [65], hybrid stars [66], etc.

In following sections we highlight briefly the problem of maximum mass of compact stars, properties of anisotropic stars and strange stars, which will form the major topics of this thesis.

1.2 Maximum Mass of Cold Compact Objects

The existence of a mass limit for a compact star is an important issue in relativistic astrophysics. This can be understood from a simple argument proposed by Landau in 1932, that applies to the fermion stars (white dwarfs and neutron stars). The argument is based on the assumption that in the high density region, the EOS is given by that of a relativistic fermi gas. This gives the maximum number of degenerate fermions that can be bound gravitationally as

$$N_{max} \sim \left(\frac{\hbar c}{G m_B^2} \right)^{3/2} \sim 2 \times 10^{57}, \quad (1.1)$$

and hence the maximum mass $M_{max} \sim N_{max} m_B \sim 1.5 M_\odot$, where m_B is the mass of the baryons.

Thus the maximum mass obtained above depends only on the values of physical constants, which is a remarkable result. M_{max} will be almost the same for white dwarfs and neutron stars, although degenerate fermions are different in the two cases. Consideration of realistic EOS may affect this result by a factor of order one. Majority of the maximum mass calculations depend on the EOS of the dense matter inside the compact stars. In ref. [67] it has been shown that maximum allowed masses for different neutron star EOS lie within the range $M \sim (1.46 - 2.48) M_\odot$. For a non-rotating neutron star, Rhoades and Ruffini [68] have shown that this maximum mass can not be larger than $\sim 3.2 M_\odot$. Using modern EOS for neutron star matter fitted to experimental nucleon-nucleon scattering data and the properties of light nuclei, Kalogera and Baym [69] calculated the minimum upper limit on a neutron star mass equal to $2.9 M_\odot$ regarding the EOS as valid upto density $\rho = 2\rho_{nuc}$ and $2.2 M_\odot$ upto $\rho = 4\rho_{nuc}$ where, $\rho_{nuc} = 2.7 \times 10^{14} \text{ gm cm}^{-3}$ is the nuclear matter saturation density. Banerjee *et al* [70] following the energy balance arguments, calculated the maximum mass of strange stars. For a typical value of $B^{1/4} \sim 145 \text{ MeV}$, where B is the bag constant, they obtained the values maximum radius $b_{max} = 12.11 \text{ km}$, $M_{max} = 1.54 M_\odot$ and $N_{max} = 1.55 \times 10^{57}$, almost the same as for a neutron star. It was also shown that the maximum mass decreases as B increases. Some of the results for maximum mass obtained in different models are depicted in tabular form in Table

2.5 in the next chapter.

The existence of a maximum mass actually follows from the relativistic equation of hydrostatic equilibrium and other physical constraints, in particular, the requirement that within the fluid sphere, the velocity of sound be subluminal. In a very recent paper, assuming a barotropic EOS, Fujisawa *et al* [71] have decreased the Buchdahl [7] bound of compactness (total mass to surface radius ratio) $\leq 4/9$ to a new value $\simeq 0.3636403$. We have obtained a lower value of compactness $\simeq 0.3615$ in our work [72] where the problem of maximum mass is studied in the context of Vaidya-Tikekar model. This will be discussed in the next chapter. In particular, the effect of the relevant EOS of the matter inside has been explored in detail. One point to be noted here is that we have not considered the effect of rotation. If rotation is taken into account, maximum mass limit is expected to increase marginally because the centrifugal forces oppose the gravity. Baumgarte *et al* [73] have constructed relativistic equilibrium models of differentially rotating neutron stars and have shown that they can support significantly more mass than their nonrotating or uniformly rotating counterparts. It is pointed out [67] that the increase of maximum mass of a static neutron star due to rotation is about 20% and that for strange quark star is about 40%.

1.3 Anisotropic Stars

A common assumption in the study of stellar structure and evolution is that the interior of a star can be modeled as a perfect fluid, which necessarily requires that the pressure in the interior of a star to be isotropic. This approach has been used extensively in the study of polytropes, including white dwarfs, and of compact

objects such as neutron stars. Theoretical advances in the last decades indicate that, in many systems, deviations from local isotropy in the pressure, in particular, at very high densities, may play an important role in determining their properties. By anisotropy we mean that the radial component of pressure, p_r , differs from the tangential component, p_\perp . There are several reasons of this pressure difference. It may occur because of the existence of a solid core or the presence of type 3A superfluid [74], phase transition [75], pion condensation [76], slow rotation [77], mixer of two gases [78] or strong magnetic fields [79]. Another reason of anisotropy within these highly compact stars could be the presence of strong electric field [80].

Bowers and Liang [81] have suggested that anisotropy may have non-negligible effect on such parameters as maximum equilibrium mass and surface redshift. Chan, Herrera and Santos [82] have studied the role played by the local pressure anisotropy on the onset of instabilities and showed that small anisotropy might in principle drastically change the stability of the system. Mak and Harko [83] have presented a class of exact solutions of Einstein field equation describing spherically symmetric and static anisotropic stars. They showed that the basic physical quantities (mass and radius) of their model can describe realistic astrophysical objects like neutron stars. Dev and Gleiser [84] have obtained exact solutions for a general relativistic anisotropic star and have shown that if $p_\perp > p_r$, the anisotropic star becomes more stable as compared to an isotropic configuration [85]. It was also found that the transverse pressure permits a higher compactness than that is given by Buchdahl-Bondi bound [7, 8] for perfect fluid stars. An exact solution of Einstein-Maxwell equations has been reported in ref. [86] where anisotropic pressure originates due to a charged fluid

distribution. A new class of exact solutions for a compact self-gravitating object with locally anisotropic pressure has been obtained by Petri [87]. A particular bound on the surface redshift parameter in case of realistic anisotropic star has been obtained by Ivanov [88]. Many new models have been developed in recent past to describe the anisotropic stellar structure and the role of pressure anisotropy on such configurations [89, 90, 91, 92]. Sharma and Maharaj [93] have shown that anisotropy decreases the value of the surface tension of strange stars. Recently in an article [94] we showed that in Vaidya-Tikekar model [95], maximum compactness, redshift and mass increase in the presence of anisotropic pressure which has been discussed in Chapter 3 in details where we also generalise the solution to the charged anisotropic spheres.

1.4 Compact stars: Composition

The primary concept about neutron star is that it is constituted from neutrons with small number of protons and leptons. Actually neutron star has an outer crust made of nuclei and electrons, an inner crust made of nuclei, neutrons and electrons and muons, an interior part consisting of superfluid protons, neutrons, normal leptons, and a more dense hadronic core. As the star becomes more compact, the core may contain more exotic matter, depending on the density. Various possibilities of the composition and structure are considered below [13].

Nucleon star: As the core density increases, the reaction $e^- \rightarrow k^- + \nu$, becomes energetically favourable. Thus, the fermions e^- get replaced by bosons as ν escapes. The possibility of this reaction taking place depends on the mass of K^- inside the dense matter of the star. Terrestrial laboratory experiments fortunately provide some

useful information here. The kinetic energy spectrum of produced K in Ni–Ni collision was studied by KaoS collaboration at GSI. It was shown that in the attractive nuclear matter at thrice the nuclear density, the mass of K goes down to about 200 MeV (the mass in vacuum is 495 MeV). Note that the chemical potential of electron inside the neutron star matter can very well be close to 200 MeV so that the above reaction can take place. The neutrons can now be converted back to protons [13] and the core of a newly formed neutron star could become iso-spin symmetric, with almost equal number of protons and neutrons. This lowers the energy per baryon of the nuclear matter. The neutron star of this type is called Nucleon star, being very similar to an ordinary nucleus, which contains almost equal number of protons and neutrons. The maximum mass of these stars has been estimated to be around $1.5 M_{\odot}$.

Hyperon star: At high densities inside the core, the chemical potential of neutrons may exceed the masses (modified by interactions) of the heavier members of baryon octet. One would, therefore, expect to find hyperons ($\Lambda, \Sigma, \Xi, \Delta$) in the core.

Hybrid star: Highly dense matter in the core of a neutron star may transform into quark matter. The transition of baryonic matter to quark matter begins at density about $2 \rho_n$, where ρ_n is the nuclear matter density. Core may be surrounded by a few kilometer thick shell of mixed quark-hadron matter which is surrounded by a thin shell of hadronic liquid and finally a thin outer crust of heavy ions[47, 96].

H-dibaryon star: Dibaryon is an exotic composite of six quarks, doubly strange, with baryon number two, zero spin and zero iso-spin. In the core of moderately dense neutron stars, there is a sizable number of Λ hyperons, which could combine to produce H-dibaryons. However, the H-matter is unstable against a compression and

a perturbation can easily convert a dibaryon star into a possible strange star, which will be considered in the next section.

1.5 Strange Star

The study of strange quark matter has become a subject of great interest after the suggestion of Bodmer [9] and Witten [10] that it is the true ground state of QCD. The essential idea is that the energy per baryon of SQM could be less than that of the most stable nucleus, ^{56}Fe , making SQM more stable. Because of the difficulty of QCD in the non-perturbative domain, phenomenological models are used to study hadrons and SQM. The most effective model is the MIT bag model given by Chodos *et al* [97]. In this model the nucleon is assumed to be a bubble of perturbative vacuum immersed in the true vacuum. Quarks are confined to the bubble by means of a net inward pressure B , the bag constant, exerted on it by surrounding vacuum. Farhi and Jaffe [98] studied SQM with the MIT bag model with various values of strange quark mass, m_s and the bag constant B . They found a region of stability in the $m_s - B$ plane in which the SQM is stable. This implies that one could expect strange matter to provide star-like self-bound systems called the strange stars that are much more compact than normal neutron stars. These self bound strange stars consist of a plasma of almost equal populations of massless u, d quarks and lightly massive s quarks and electrons.

Strange matter may come in lumps with baryon number (A), starting from $A \sim 10^2$ to enormous objects $A \sim 10^{57}$, beyond which the configurations become unstable under gravitational collapse. If strange matter hypothesis is true then all stellar con-

figuration would be metastable and all neutron stars may eventually become strange stars. Such stars will consist of 3 flavored strange matter covered by a thin crust of hadrons with density lower than the neutron drip density ($4 \times 10^{11} \text{ gm/cm}^3$). However, if a large quantity of quark matter is left in the universe, as a relic of the quark-hadron phase transition in the early universe, it could condense due to gravitational attraction and can even form invisible quark galaxies. The stars of such galaxies would be strange stars. No indication of such a scenario has been observed.

Depending on the rotational frequency and mass, a density greater than 3 - 4 times the nuclear density may occur in pulsars and it is likely that deconfinement may set in such pulsars. Using the popular model for SQM EOS, the MIT bag model, Li *et al* [16] suggested that X-ray pulsar Her X-1 is a strange star. Considering a particular interquark potential, Dey *et al* [20] developed an EOS for strange matter and proposed that X-ray pulsar Her X-1 and X-ray burster 4U1820-30 both are good strange star candidates. Using the same EOS Li *et al* [22] suggested that the low mass X-ray binary pulsar SAXJ1808.4-3658 is also a strange star. Quite a few recently observed compact stars are also good candidates for strange stars [18]. The details will be studied in Chapter 4.

In search of SQM, scientists have been studying the possibility that in the extreme pressure prevailing in the core of neutron stars, the constituents, proton, neutron and the hyperons, could melt and undergo a phase transition to produce the quark-gluon plasma. It is still not known at what density this phase transition become possible. It is expected that the existence of quark-gluon plasma will be confirmed in near future from the laboratory experiments with heavy ion colliders at CERN and, LHC

in particular. There is no definite indication yet from the Lattice QCD simulation results. However, a simple argument can provide a rough estimate of the threshold. For a characteristic nucleon radius $b \sim 1 \text{ fm}$, nuclei begin to touch each other at a density, $\rho \sim (\frac{4\pi}{3}b^3)^{-1} \sim 0.24 \text{ fm}^{-3}$. For higher densities, the nuclear boundaries should disappear and quarks may become deconfined.

1.6 Objectives of the present work

In view of the developments, discussed above, it may be useful to study compact objects in a different perspective. Instead of numerical calculations, an analytic calculation based on the generalisation of Vaidya-Tikekar approach may be useful to study various issues related to compact stars. This has been the main objective of this work. An outline of the work may be given as:

- The solution of Mukherjee *et al* [45] will be generalised to the case of anisotropic stars and different properties of anisotropic stars will be studied in this model. Although the results obtained are valid for a class of stars, the case of very compact objects can be covered.
- The problem of finding the maximum mass both for isotropic and anisotropic static spherically symmetric cold compact stellar objects will be studied. In this context our objective is to put constraints on the maximum mass and radius of ultra compact stars without going into the details of the complex EOS used to study such stars.
- Vaidya-Tikekar model provides a class of EOS, once the mass and radius of a

very compact star are given. It can be useful in the study of strange quark star, when the EOS is not known apriori. Here, strange stars will be studied in the context of Vaidya-Tikakar model with a density dependent bag constant in the EOS of color-flavour-locked(CFL) strange quark matter.

- The technique of VT model will be generalised to higher dimensions and the effects of number of dimensions on mass, radius, maximum mass, etc. of a compact star will be studied.
- The theoretical calculations will be compared numerically with those of other workers, considering, in particular, some compact neutron stars and strange stars observed and studied intensively.

1.7 Mathematical Techniques

For the studies of relativistic compact stars one starts with the interior metric for a static spherically symmetric perfect fluid configuration given by

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{2\mu(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (1.2)$$

to solve the Einstein's field equations

$$G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R = T_{ij}.$$

The perfect fluid approximation demands that (i) pressure and energy density should be positive at the origin, (ii) pressure should be isotropic, (iii) the pressure and energy density profiles should be monotonically decreasing with increasing radius,

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13 APR 2017



(iv) pressure should be zero at the boundary and (v) the velocity of sound should not be greater than the velocity of light inside the configuration.

Consequently we choose the energy-momentum tensor in the form

$$T_{ij} = (\rho + p)u_i u_j + p g_{ij}, \quad (1.3)$$

where u^i is the 4-velocity of the fluid, ρ is the energy-density and p the pressure.

The Einstein's field equations for the metric (1.2) are now obtained as,

$$\rho = \frac{(1 - e^{-2\mu})}{r^2} + \frac{2\mu' e^{-2\mu}}{r} \quad (1.4)$$

$$p = \frac{2\gamma' e^{-2\mu}}{r} - \frac{(1 - e^{-2\mu})}{r^2} \quad (1.5)$$

$$p = e^{-2\mu} \left(\gamma'' + \gamma'^2 - \gamma'\mu' + \frac{\gamma'}{r} - \frac{\mu'}{r} \right). \quad (1.6)$$

Equations (1.5) and (1.6) may be combined to obtain the pressure isotropy condition as

$$\gamma'' + \gamma'^2 - \gamma'\mu' - \frac{\gamma'}{r} - \frac{\mu'}{r} - \frac{(1 - e^{2\mu})}{r^2} = 0. \quad (1.7)$$

1.7.1 TOV Equations

The conventional approach for the studies of relativistic compact stellar objects is to prescribe first an equation of state (EOS), i.e., $p = p(\rho)$, and then for a given central density ρ_c solve the general relativistic TOV (Tolman [6], Openheimer and Volkov [5]) equations given by

$$-\frac{dp}{dr} = \frac{1}{r^2}(\rho + p)(2M(r) + pr^3) \left(1 - \frac{2M(r)}{r}\right)^{-1}, \quad (1.8)$$

$$\frac{dM(r)}{dr} = \frac{1}{2}\rho(r)r^2, \quad (1.9)$$

where we have made the conventional choice, $8\pi G = c = 1$.

Given the EOS, $p = p(\rho)$, one chooses a value of $\rho_c = \rho(r = 0)$, and starts integrating the equations (1.8) and (1.9) from the centre of the star onwards, using the initial condition $M(r = 0) = 0$, and continues till $p(\rho(r))$ drops down to zero at some value of $r = b$, which we interpret as the radius of the star with a central density ρ_c . The mass of the star is given by $M = M(r = b)$ and thus, one can study the mass-radius ($M - b$) curves for various central densities.

The TOV approach has been very useful in the study of neutron stars, particularly where one can model the EOS of the constituents. Considerable results are already available for such stars. However, as the density exceeds twice the nuclear density, the composition as well as the EOS become very uncertain. A number of possible, but exotic compositions have already been considered. It is hoped that observational results will soon be available to settle the complex problem of the composition of very compact stars.

However, since the application of TOV equations requires prior knowledge of the EOS, compact objects make the exercise difficult and at most tentative. In the case of superdense stars, apart from the conventional approach based on the TOV equation it may be worthwhile to explore an alternative approach which does not require prior knowledge of EOS. Vaidya-Tikekar model can provide useful results here. This will be discussed briefly in the next section.

1.7.2 Vaidya-Tikekar Model: An alternative approach

In this method, we choose a suitable geometry for the 3-space and then determine an appropriate equation of state for the matter in the cold compact star. This approach is convenient in cases where the equation of state is not known or uncertain. The method becomes particularly simple when the choice of the 3-geometry permits an exact solution of Einstein's equations. The ansatz of Vaidya and Tikekar [95], for the metric function,

$$e^{2\mu} = \frac{1 + kr^2/R^2}{1 - r^2/R^2} \quad (1.10)$$

has been found to be very useful. Here, k and R are two constant parameters, which specify the geometry of the $t = \text{constant}$ hypersurface, which is spheroidal when embedded in a four dimensional Euclidean space. These parameters will eventually be related to the properties of the constituent matter. Since one of the metric functions is known, using equation (1.7), one can derive a second-order differential equation for the other metric, $e^{2\nu} = \psi^2$, given by

$$(1 + k - kx^2)\Psi_{xx} + kx\Psi_x + k(k + 1)\Psi = 0, \quad (1.11)$$

where $x^2 = 1 - \frac{r^2}{R^2}$. The solution of the equation (1.11) was obtained by Vaidya and Tikekar[95] for $k = 2$, and by Tikekar [99] for $k = 7$. Maharaj and Leach [100] gave the solution for sets of values of k .

Now, equation (1.11) may be written as

$$(1 - z^2)\psi_{zz} + z\psi_z + (k + 1)\psi = 0, \quad (1.12)$$

where, $z^2 = \frac{k}{k+1} \left(1 - \frac{r^2}{R^2}\right)$ and ψ_z is the derivative of ψ w.r.t. z .

The general solution to the above equation, obtained by Mukherjee, Paul and Dadhich [45], is given by

$$e^\nu = \psi = A \left\{ \frac{\cos[(n+1)\zeta + \delta]}{n+1} - \frac{\cos[(n-1)\zeta + \delta]}{n-1} \right\}, \quad (1.13)$$

where $\zeta = \cos^{-1} z$, $n^2 = k + 2$ and A and δ are constants to be determined by matching the solution with the exterior vacuum solution of Schwarzschild at the boundary, $r = b$, i.e.,

$$e^{2\nu(b)} = 1 - \frac{2M}{b} \quad (1.14)$$

$$e^{2\mu(b)} = \left(1 - \frac{2M}{b}\right)^{-1}. \quad (1.15)$$

The energy-density and pressure are given in this model as

$$\rho = \frac{1}{R^2(1-z^2)} \left[1 + \frac{2}{(k+1)(1-z^2)} \right] \quad (1.16)$$

$$p = -\frac{1}{R^2(1-z^2)} \left[1 + \frac{2z\psi_z}{(k+1)\psi} \right]. \quad (1.17)$$

The radius of the star is determined from the condition that the pressure vanishes at the boundary, $r = b$, which gives

$$\frac{\psi'(z_b)}{\psi(z_b)} = -\frac{k+1}{2z_b}. \quad (1.18)$$

The mass of the star is given by,

$$M = \frac{(1+k)b^3/R^2}{2(1+kb^2/R^2)}. \quad (1.19)$$

There are four parameters in the solution, k , R , A and δ . Two of these are utilised to match the exterior Schwarzschild metric and one is utilised to fix the given input, radius or central density or surface density. We are thus left with one parameter, say k , which may be used to characterize the relevant equation of state, which is given implicitly by equations (1.16) and (1.17). Thus, the model describes a one-parameter family of EOS for a given M and b . It is possible that not all the EOS will be realised in nature. The method has been generalised to cover the pseudo spheroidal case (when embeded in four dimensional Euclidian space) by choosing

$$e^{2\mu} = \frac{1+kr^2/R^2}{1+r^2/R^2}. \quad (1.20)$$

It has been found useful in many cases [101, 102].

The possible existence of different kinds of highly compact stars has been a subject of study for years. The Vaidya-Tikekar model can play a very useful role here. The model, though very simple, satisfies the physical constraints of a realistic star, if $k \geq \frac{3}{17}$. Following the method of Chandrasekhar, Knutsen [103] has also shown that the Vaidya-Tikekar star with $k = 2$ is stable with respect to small radial perturbations. Calculations with the general solution (1.13) have also confirmed this stability for other values of k [104].

The simplicity of the model have inspired a number of workers [101, 105, 106, 107, 104, 108, 46, 109, 72, 110] to use it to study different aspects of compact stars. The model has also been generalised to the case where the star has a charge [111, 108]

or anisotropy of some special types [89, 94]. The model has also been generalised to higher dimensions [112, 113] which will be discussed in Chapter 5.