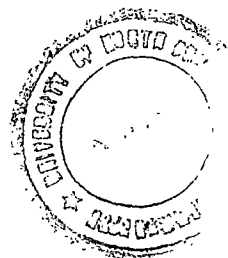


**SOME LINEAR AND NONLINEAR PROBLEMS OF THIN ELASTIC PLATES
PLACED ON ELASTIC FOUNDATION**

SRIPATI DATTA



**THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY (ENGINEERING)
OF THE
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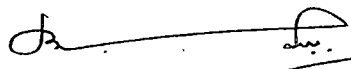
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From : Dr. B. Sen,
Principal, and
Dean of Engineering, University
of North Bengal.

This is to certify that Shri Sripati Datta has carried out research work in the Department of Mechanical Engineering, Jalpaiguri Government Engineering College and has prepared the thesis entitled "Some Linear and Non-linear Problems of Thin Elastic Plates placed on Elastic Foundation" in fulfilment of the requirement in accordance with the regulation of the University of North Bengal for the Degree of Doctor of Philosophy(Engineering). I forward the thesis for consideration of the University.


(B. Sen)
Principal, and
Dean of Engineering.

31.5.77

P R E F A C E

This thesis which contains five chapters is concerned with some linear and non-linear problems of thin elastic plates placed on elastic foundation.

The first chapter contains one linear problem on dynamical response of an orthotropic infinite plate placed on elastic foundation and subjected to time dependent concentrated force. The first work on the problem of foundation characterization was done by Winkler (1867) who assumed the foundation to act like a fluid, the reaction at any point being directly proportional to the deflection at that point. But in the case of an elastic continuum such a hypothesis approximates but crudely the actual behaviour of the continuum. Subsequently, many investigators such as Vlasov (1949), Pasternak (1954), Sokolnikoff (1956), Kerr (1964), Herrmann (1967), Fletcher (1971) and others have proposed more complicated foundation models in attempts to provide a more realistic representation of real materials. These investigators considered static loading and observed that Winkler type of foundation could be adjusted to give a reasonable estimate of the maximum deflection under the load point, but the shape of the overall deflection curve could not be well represented. In the paper of this chapter dynamic loading is considered with the foundation having the properties of a semi-infinite elastic body and a comparative study has been made with Winkler type of foundation. It is observed that Winkler type of foundation gives a reasonable estimate of maximum deflection under the load point in the case of dynamic loading too. Hence for all the

problems in the subsequent chapters, analysis has been made for foundations of Winkler type.

The second chapter contains three problems on elastic stability of thin plates of various shapes and under different types of loading. Stability of thin elastic plates having complicated shapes can be investigated with the help of various approximate numerical techniques such as finite difference method, finite element method etc. For such numerical techniques reference may be made to the works of Salvadori (1957), Weingarten (1957), Bradley (1963). But these numerical methods are time consuming. On the other hand, elastic stability problems for complicated shapes can easily be investigated with less labour following complex variable method given by Laura and Ghahady (1969) and the results obtained by different numerical techniques applied to boundary value problems involving irregular shapes can also be verified by this method.

The first paper of this chapter deals with the critical buckling conditions of a thin elastic equilateral triangular plate placed on elastic foundation with clamped edges, and under the action of uniform compression parallel to one of the edges. The same problem without the elastic foundation was investigated by Bradley (1963) by finite difference method. In the present paper, the solution is obtained by the use of conformal mapping and Galerkin's procedure. In the second paper of this chapter critical buckling conditions of a non-homogeneous simply supported rectangular plate under the action of combined bending and compression and placed on an elastic foundation have been investigated with the help of error function.

For the same problem of homogeneous rectangular plates without elastic foundation reference may be made to the works of Johnson and Noel (1953) and Timoshenko and Gere (1961). The third paper of this chapter is devoted to thermal buckling of thin plates placed on elastic foundation. Two cases are considered : a heated equilateral triangular plate and an elliptic plate. Assuming the temperature distribution to be a function of the plate thickness, the solution is obtained by complex variable theory. For thermal buckling of circular and rectangular plates reference may be made to the works of Klesner and Ferray (1953), Nowacki (1962) and Mansfield (1964).

The third chapter contains six non-linear problems of thin plates of various shapes and under different types of loading. In solving these large deflection problems, the approximate method proposed by Berger (1955) has been followed. Berger's method is based on the neglect of the second strain invariant of the middle surface strains in the expression corresponding to the total potential energy of the system. Although no complete explanation of this method is given, the results obtained so far for different problems are in good agreement with those obtained from practical analysis. Following this approximate method, many non-linear plate problems have been solved with remarkable ease by many investigators such as Iwinski and Nowinski (1957), Nowinski (1958), Nash and Modeer (1959), Basuli (1961), Sinha (1963), Banerjee (1967) and others. It is to be noted that Berger's method is still an intriguing subject. (e.g. a recent note in Jour. of Appl. Mech., June, 1974, P. 521).

The first paper of this chapter is devoted to the large deflection of a uniformly loaded circular plate placed on elastic foundation and supported at several points along the boundary. The problems of circular and rectangular plates without any edge discontinuity were investigated by Sinha (1963). In the present paper the solution is obtained in terms of an infinite series involving Bessel's functions. A particular case, where the number of supports is two, has been treated fully. The second paper deals with the large deflection of a clamped circular plate on elastic foundation under a concentrated load at the centre of the plate. The corresponding problem without any elastic foundation was treated by Basuli (1961) and the problem of an orthotropic plate was investigated by Banerjee (1967). In the second paper, the deflections are obtained involving Bessel's functions and the theoretical results have been verified experimentally. The third paper is devoted to the large deflections of a clamped circular plate on elastic foundation and under symmetrical loads. The corresponding problem without any elastic foundation was investigated by Banerjee (1967). The fourth paper deals with the large deflections of a simply supported semi-circular plate placed on elastic foundation and subjected to a uniform load. The deflections have been obtained in terms of an infinite series involving Bessel's functions and Lommel's functions. The fifth paper of this chapter is concerned with the large deflections of a simply supported equilateral triangular orthotropic plate on elastic foundation and under uniform load. Trilinear co-ordinates as shown by Sen, B. (1968) has been used in solution of this problem.

The sixth paper of this chapter deals with the large deflection of a heated clamped elliptic plate placed on elastic foundation. Berger's technique of neglecting the second invariant of the middle surface strains has been extended by Basuli (1968) to the large deflection problems of heated rectangular, circular and right-angled triangular plates without any elastic foundation and under uniform pressure and stationary temperature distribution. Assuming stationary temperature distribution the deflection in the present paper is obtained in terms of Mathieu function of the first kind and of zero order.

The fourth chapter contains only one non-linear problem on large deflections of clamped elliptic plates exhibiting rectilinear orthotropy. The plate is placed on an elastic foundation of the Winkler type and is subjected to uniform loading. The investigation is based on von Karman's (1910) differential equations extended to rectilinear orthotropy. Following von Karman's equations Weil and Newmark (1956) and Nash and Cooley (1959) investigated by numerical methods the large deflections of isotropic elliptic plates under uniform load and without elastic foundation. In the present paper the stress function is found from the compatibility equation and the final solution is obtained by Galerkin's method. Results for isotropic elliptic plates and circular plates have been deduced from those obtained for orthotropic elliptic plates. These results are in excellent agreement with the known results for isotropic plates. For circular plates without elastic foundation reference may be made to the works of Way (1934) and Schmidt (1968). For circular plates

with foundation.  reference may be made to the works of Bolton (1972).

The fifth and the last chapter of this thesis contains only one problem on large amplitude free vibrations of irregular plates placed on elastic foundation. Following Berger's equations the approximate values of the lower natural frequency for circular, square, and cornered plates with simply supported and clamped edges are obtained by the conformal mapping technique. Theoretical results thus obtained have been verified experimentally. The same problem without elastic foundation was treated by Nash and Modar (1960) and Chu and Herrmann (1956) who followed Berger's technique and von Karman's equations respectively. For non-linear vibration problems of different elastic plates reference may be made to the works of Nowinski (1963) who followed von Karman's field equations to obtain his results.

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CHAPTER - I.

DYNAMICAL RESPONSE OF AN ORTHOTROPIC INFINITE PLATE PLACED ON ELASTIC FOUNDATION AND SUBJECTED TO TIME DEPENDENT CONCENTRATED FORCE.

Introduction :

The analysis of a thin elastic plate supported by a flexible elastic foundation has numerous engineering applications. At the present time, the accepted method of approximate analysis represents the supporting material as a Winkler (1867) foundation. Winkler approximated the foundation by a series of closely packed linear springs attached to a rigid base. The springs are assumed to act independently so that the foundation acts like a fluid, the reaction at any point being directly proportional to the deflection at that point. But in the case of a coherent subgrade such a hypothesis approximates but crudely the actual behaviour of the subgrade.

The elastic modulus and Poisson's ratio of the foundation material are often known and a better approximation of the foundation reaction can be obtained on the assumption that the foundation has the properties of a semi-infinite elastic body. Numerous investigators such as Kerr (1964), Herrmann (1967), Fletcher (1971) have attempted to provide a more realistic representation of the behaviour of the elastic continuum in terms of the elastic constants of the foundation material. All these investigators considered the loading to be static.

The object of this paper is to study the dynamic response of an orthotropic infinite rectangular plate, such as reinforced concrete, under the action of a time-dependent concentrated force and placed on an elastic foundation. These types of problems are of practical importance for the design of concrete paving slabs for roads, aircraft runways, bed plates for machine tools, engines etc. Livesley (1953) has discussed the case in which the loading is static and the foundation reaction is directly proportional to the local plate deflection in details for both infinite and semi-infinite rectangular plate of various boundary conditions. In this paper the foundation is assumed to have the properties of a semi-infinite elastic body. Winkler type of foundation is also considered for comparison. Owing to the complexities of mathematics involved, the problem has been confined to axial symmetrical cases. The solutions are obtained by a semi-inverse method with the help of Laplace and Hankel transforms.

Analysis :

Consider an infinitely large orthotropic plate in a state of axial symmetry. The plate is placed on an elastic foundation and subjected to the action of a dynamic surface loading. Following Timoshenko and Woinowsky-Krieger (1959, P.365) the differential equation for equilibrium of an element of the plate can be written in rectangular co-ordinates as

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + \rho h \frac{\partial^2 w}{\partial t^2} = q(x, y, t) - p(x, y, t) \dots (1.1)$$

where $D_x = \frac{E_x' h^3}{12}$, w is the deflection in z - direction,

$$H = \frac{E'' h^3}{12} + \frac{G h^3}{6}; \quad D_y = \frac{E_y' h^3}{12}, \quad \rho = \text{density of the plate}$$

material, h = plate thickness, $q(x, y, t)$ denotes the given surface loading, $p(x, y, t)$ the reaction of the foundation and E_x' , E_y' , E'' and G are elastic constants of the material.

For an orthotropic material such as reinforced concrete

$$H = \sqrt{D_x D_y}$$

and introducing $x = x_1$, $y_1 = \left(\frac{D_x}{D_y}\right)^{\frac{1}{2}} y$ as new variables in eq.(1.1)

one gets

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\right)^2 W + \frac{\rho h}{D_x} \frac{\partial^2 W}{\partial t^2} = \frac{q(x_1, y_1, t)}{D_x} - \frac{p(x_1, y_1, t)}{D_x} \quad \dots (1.2)$$

Putting $x_1 = r_1 \cos \theta_1$, $y_1 = r_1 \sin \theta_1$ in eq.(1.2) one gets for axial symmetry

$$\left(\frac{\partial^2}{\partial r_1^2} + \frac{1}{r_1} \frac{\partial}{\partial r_1}\right)^2 W + \frac{\rho h}{D_x} \frac{\partial^2 W}{\partial t^2} = \frac{q-p}{D_x} \quad \dots (1.3)$$

Multiplying both sides of eq.(1.3) by $D_x r_1 J_0(\alpha r_1)$ and integrating between the limits 0 to ∞ one gets

$$\bar{W} [D_x \alpha^4 K(\alpha) + 1] + \rho h K(\alpha) \frac{\partial^2 \bar{W}}{\partial t^2} = K(\alpha) Q(\alpha) F(t) \quad \dots (1.4)$$

where $\bar{W} = \int_0^{\infty} w r_1 J_0(\alpha r_1) dr_1$, J_0 denoting the Bessel function of zero order; the term $K(\alpha) = \frac{V}{P}$ depends on the nature of subgrade; the term $Q(\alpha) = \bar{q}(r_1) = \int_0^{\infty} q r_1 J_0(\alpha r_1) dr_1$ depends on the intensity of loading; and $q = q(r_1) F(t)$.

Eq. (1.4) can be written as

$$\frac{\partial^2 \bar{W}}{\partial t^2} + a^2 \bar{W} = c F(t) \quad \dots (1.5)$$

where $a^2 = \frac{D_x \alpha^4 K(\alpha) + 1}{\rho h K(\alpha)}$; $C = \frac{q(\alpha)}{\rho h}$

Taking the intensity of loading to be periodic

$$F(t) = \sin \omega_0 t \quad \dots \quad \dots (1.6)$$

The initial conditions are

$$W = \frac{\partial W}{\partial t} = 0 \quad \text{at } t = 0 \quad \dots \quad \dots (1.7)$$

One gets from eq.(1.5) after applying Laplace transform method

$$\bar{W} = \frac{C \omega_0}{a^2 \omega_0^2} \left[\frac{\sin \omega_0 t}{\omega_0} - \frac{\sin at}{a} \right] \quad \dots (1.8)$$

If the load is concentrated at the origin

$$q(\alpha) = \frac{P}{2\pi} \quad \dots \quad \dots (1.9)$$

and in the case of an isotropic semi-infinite medium [Ref.5, P.230]

$$K(\alpha) = \frac{1}{K_0 \alpha} \quad \dots \quad \dots (1.10)$$

K_0 being the elastic constant of the medium defined by $K_0 = \frac{E_0}{2(1-\nu_0^2)}$,

E_0 being the modulus of elasticity and ν_0 the Poisson's ratio of the medium.

Considering eqs.(1.9) and (1.10) and applying Hankel inversion theorem one gets from eq.(1.8) for semi-infinite medium

$$W(r_1, t) = \frac{P}{2\pi} \int_0^\infty \frac{\sin \omega_0 t \alpha J_0(\alpha r_1) d\alpha}{\beta} - \frac{P \omega_0}{2\pi} \int_0^\infty \frac{\sin(\psi/\rho h)^{\frac{1}{2}} t (\rho h)^{\frac{1}{2}} \alpha J_0(\alpha r_1) d\alpha}{\psi^{\frac{1}{2}} \beta} \quad \dots (1.11)$$

where $\beta = D_x \alpha^4 + K_0 \alpha - \rho h \omega_0^2$

$$\psi = D_x \alpha^4 + K_0 \alpha$$

and for Winkler type of foundation having the reaction K per unit area per unit deflection

$$W(r_1, t) = \frac{P}{2\pi} \int_0^{\infty} \frac{\sin \omega_0 t \alpha J_0(\alpha r_1) d\alpha}{\beta_1} - \frac{P \omega_0}{2\pi} \int_0^{\infty} \frac{\sin(\psi_1 / \rho h)^{\frac{1}{2}} t (\rho h)^{\frac{1}{2}} \alpha J_0(\alpha r_1) d\alpha}{\psi_1^{\frac{1}{2}} \beta_1} \dots (1.12)$$

$$\text{where } \beta_1 = D_x \alpha^4 + K - \rho h \omega_0^2$$

$$\psi_1 = D_x \alpha^4 + K$$

The distribution of pressure $p(r_1, t)$ at any point is readily obtained from eq.(1.1)

$$p(r_1, t) = \frac{PK_0}{2\pi} \int_0^{\infty} \frac{\sin \omega_0 t \alpha^2 J_0(\alpha r_1) d\alpha}{\beta} - \frac{P \omega_0 K_0}{2\pi} \int_0^{\infty} \frac{\sin(\psi / \rho h)^{\frac{1}{2}} t (\rho h)^{\frac{1}{2}} \alpha^2 J_0(\alpha r_1) d\alpha}{\psi^{\frac{1}{2}} \beta} \dots (1.13)$$

After integrating eqs.(1.11), (1.12) and (1.13) numerically one gets finally the following results :

For semi-infinite medium

$$\frac{W}{h} = \frac{P \sin \omega_0 t}{2\pi K_0 l_0 h} \left[\frac{J_0(0.58 r_1 / l_0)}{0.71 - \mu^2} \left\{ 0.39 - 0.74 \mu \frac{\sin(0.84 \omega_0 t)}{\sin \omega_0 t} \right\} + \frac{J_0(3.4 r_1 / l_0)}{137 - \mu^2} \left\{ 15.23 - 1.3 \mu \frac{\sin(11.7 \omega_0 t)}{\sin \omega_0 t} \right\} \right] \dots (1.14)$$

$$\text{where } l_0^3 = \frac{D_x}{K_0}, \frac{K_0}{\rho h l_0} = \omega_n^2, \mu = \frac{\omega_0}{\omega_n}$$

and for Winkler foundation

$$\frac{W}{h} = \frac{P \sin w_0 t}{2 \pi K l^2 h} \left[\frac{J_0(0.58 r_1/l)}{1.13 - \mu^2} \left\{ 0.89 - 0.35 \mu \frac{\sin(1.06 w_{nt})}{\sin w_0 t} \right\} + \frac{J_0(3.4 r_1/l)}{134.6 - \mu^2} \left\{ 15.28 - 1.3 \mu \frac{\sin(11.7 w_{nt})}{\sin w_0 t} \right\} \right] \dots (1.15)$$

where $\mu^4 = \frac{D_x}{K}$, $\frac{K}{\rho h} = w_n^2$

The distribution of pressure is obtained as

$$p(r_1, t) = \frac{P \sin w_0 t}{2 \pi l_0^2} \left[\frac{J_0(0.58 r_1/l_0)}{0.71 - \mu^2} \left\{ 0.52 - 0.43 \mu \frac{\sin(0.84 w_{nt})}{\sin w_0 t} \right\} + \frac{J_0(3.4 r_1/l_0)}{137 - \mu^2} \left\{ 0.52 - 4.3 \mu \frac{\sin(11.7 w_{nt})}{\sin w_0 t} \right\} \right] \dots (1.16)$$

For an isotropic infinite plate under the concentrated load P at the origin, the maximum static deflection is obtained from eq.(1.14)

$$W = 0.214 \frac{P l_0^2}{D} \dots \dots (1.17)$$

compared to the corresponding result obtained by Timoshenko,

[Ref. 5, P. 280]

$$W = 0.192 \frac{P l_0^2}{D} \dots \dots (1.18)$$

NUMERICAL RESULTS

Steady state amplitudes at the origin for various values of frequency ratio are calculated both for semi-infinite medium and Winkler model by eqs.(1.14) and (1.15) respectively. These results are presented in Fig. 1-1.

From Fig.1.1 it is observed that Winkler foundation as well as semi-infinite medium give practically the same dynamic response under the load point when the disturbing frequency is higher than the first natural frequency. When the disturbing frequency is lower than the first natural frequency, Winkler model gives a somewhat lower value of amplitude compared to the semi-infinite medium.

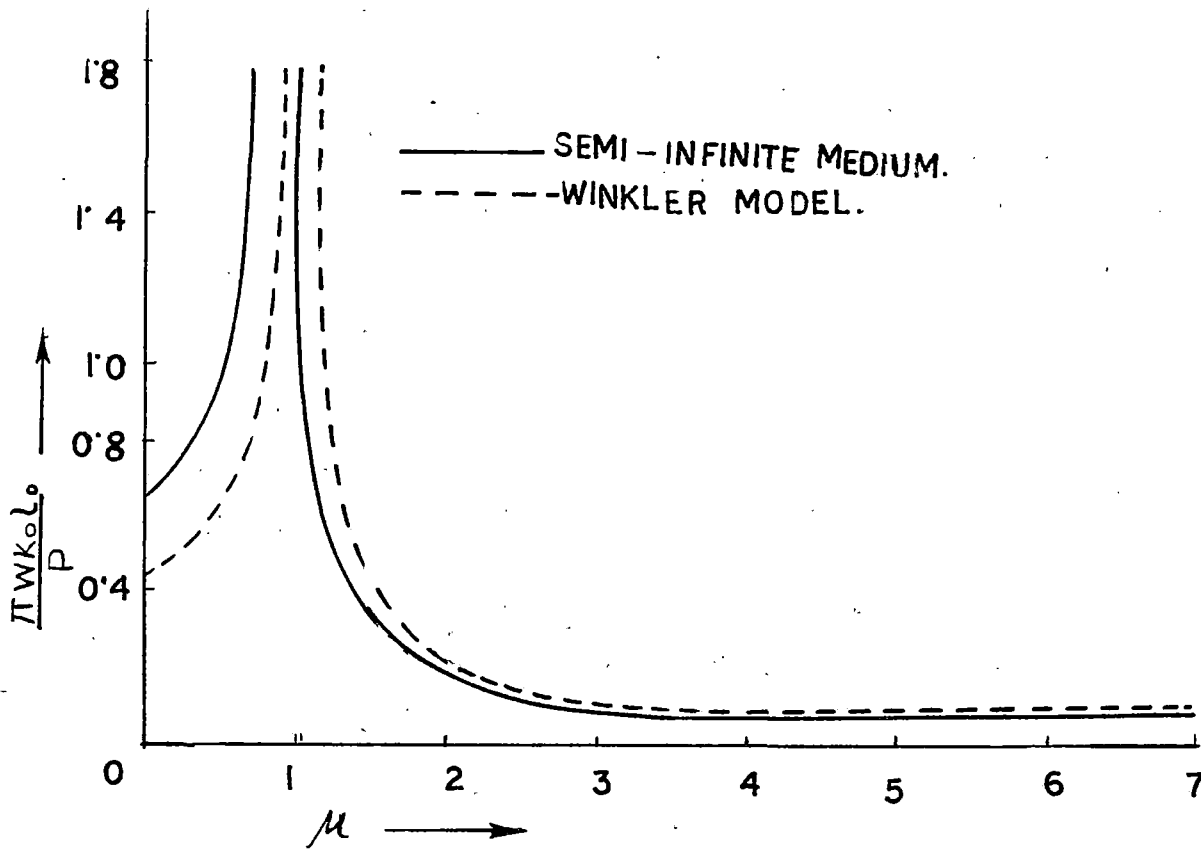


FIG. 1.1 AMPLITUDE-FREQUENCY CURVE.

CHAPTER - II

ELASTIC STABILITY OF A TRIANGULAR PLATE PLACED ON ELASTIC FOUNDATION.*

PAPER - I

Introduction :

Stability of thin elastic triangular plates can be investigated with the help of finite difference method. In this approximate method the differential equation is replaced by its finite difference approximation, rather than satisfying the differential equation and the boundary conditions at every point. The difference equation may be satisfied at the node points of a superposed triangular grid, as has been done by Bradley (1963), Salvadori (1951), Weingarten (1957), Timoshenko and Gere (1961) solved these types of problems by other numerical methods. To obtain accurate results by these methods, high order matrices are involved for which the solution is time consuming. Laura and Shahady (1969) have shown that complex variable method may be used for solution of such problems with less labour.

In this paper the complex variable theory has been applied to investigate the stability of a thin elastic equilateral triangular plate placed on an elastic foundation with clamped edges, and under the action of uniform compression parallel to one of the edges. The given domain is

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conformally transformed onto the unit circle and solution has been obtained with the help of error function. The foundation is assumed to be of the Winkler type.

Theory :

The differential equation, assuming small deflections, for the middle plane of a thin plate with no lateral loading but with force resultants in the middle plane is (cf. Timoshenko and Krieger, 1959)

$$\nabla^4 w = \frac{1}{D} \left(N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \quad \dots (2.1)$$

in which w denotes the deflection of the middle plane of the plate ; N_x , N_y are the normal forces per unit length in the middle plane of the plate, N_{xy} is the shear force per unit length in the middle plane of the plate ; D , the plate constant, is defined by

$$D = \frac{E h^3}{12 (1 - \nu^2)} \quad \dots (2.2)$$

with E denoting the modulus of elasticity, h referring to the thickness, and ν representing Poisson's ratio ; and the biharmonic operator ∇^4 is given by

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

For a plate, shown in Fig. 2.1, with a uniform compression parallel to the base, eq. (2.1) becomes,

$$\nabla^4 w + \frac{N_x}{D} \frac{\partial^2 w}{\partial x^2} = 0 \quad \dots (2.3)$$

In the case of a clamped edge boundary

$$W = 0 = \frac{\partial W}{\partial \eta}, \quad \text{on the boundary } \left[\text{cf. Timoshenko and Krieger (1959), p. 87} \right]$$

If $Z = x + iy$, $\bar{Z} = x - iy$, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \quad \frac{\partial}{\partial y} = i \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right)$$

so that

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

and therefore

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$$

Thus eq. (2.3) changes into

$$16 \frac{\partial^4 W}{\partial z^2 \partial \bar{z}^2} + \frac{N_x}{D} \left(\frac{\partial^2 W}{\partial z^2} + 2 \frac{\partial^2 W}{\partial z \partial \bar{z}} + \frac{\partial^2 W}{\partial \bar{z}^2} \right) = 0 \quad \dots (2.4)$$

$$\text{Let } Z = f(\zeta)$$

... (2.4a)

be the analytic function which maps the given shape in the z - plane onto a unit circle in the ζ - plane. Let the plate be placed on an elastic foundation having the foundation reaction k_1 per unit area per unit deflection. Substitution of eq. (2.4a) into eq. (2.4) yields

$$\begin{aligned} & \frac{\partial^4 W}{\partial \zeta^2 \partial \bar{\zeta}^2} \cdot \frac{dz}{d\zeta} \cdot \frac{d\bar{z}}{d\bar{\zeta}} - \frac{\partial^3 W}{\partial \zeta^2 \partial \bar{\zeta}} \cdot \frac{d^2 z}{d\zeta^2} \cdot \frac{d\bar{z}}{d\bar{\zeta}} - \frac{\partial^3 W}{\partial \zeta \partial \bar{\zeta}^2} \cdot \frac{d^2 \bar{z}}{d\bar{\zeta}^2} \cdot \frac{dz}{d\zeta} + \\ & + \frac{\partial^2 W}{\partial \zeta \partial \bar{\zeta}} \cdot \frac{d^2 z}{d\zeta^2} \cdot \frac{d^2 \bar{z}}{d\bar{\zeta}^2} + \frac{N_x}{D} \left[- \frac{\partial W}{\partial \zeta} \cdot \frac{d^2 z}{d\zeta^2} \cdot \left(\frac{d\bar{z}}{d\bar{\zeta}} \right)^3 + \right. \\ & \left. + \frac{\partial^2 W}{\partial \zeta^2} \cdot \frac{dz}{d\zeta} \cdot \left(\frac{d\bar{z}}{d\bar{\zeta}} \right)^3 - \frac{\partial W}{\partial \bar{\zeta}} \cdot \frac{d^2 \bar{z}}{d\bar{\zeta}^2} \cdot \left(\frac{dz}{d\zeta} \right)^3 + \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{\partial^2 w}{\partial \bar{z}^2} \left(\frac{dz}{d\bar{z}} \right)^3 \frac{d\bar{z}}{d\bar{z}} + 2 \frac{\partial^2 w}{\partial z \partial \bar{z}} \left(\frac{dz}{d\bar{z}} \right)^2 \left(\frac{d\bar{z}}{d\bar{z}} \right)^2 \Big] + \\
& + \frac{K_1 w}{16D} \left(\frac{dz}{d\bar{z}} \right)^3 \left(\frac{d\bar{z}}{d\bar{z}} \right)^3 = 0
\end{aligned}
\quad \dots (2.5)$$

in which

$$z = \eta e^{i\theta}, \quad \bar{z} = \eta e^{-i\theta}, \quad \eta \text{ being the radius of the circle.}$$

The transformed boundary conditions in the η - plane are

$$w(\eta, \bar{\eta}) \Big|_{\eta=1} = \frac{\partial w(\eta, \bar{\eta})}{\partial \eta} \Big|_{\eta=1} = 0 \quad \dots (2.6)$$

Method of solution :

Since an exact solution of eq. (2.5) is, at best, very difficult, it is convenient to use an approximate method to solve it. Galerkin's method is used in this study. The procedure is as follows : $w(\eta, \bar{\eta})$ is approximated by a linear combination of independent coordinate functions which identically satisfy the boundary conditions, i.e.

$$w(\eta, \bar{\eta}) \simeq \sum_{n=1}^k B_n \chi_n(\eta, \bar{\eta}) \quad \dots (2.7a)$$

Substituting eq. (2.7a) into eq. (2.5) results in an expression which does not vanish in general since eq.(2.7a) is not an exact solution of the partial differential equation. This nonvanishing expression is usually defined as the "error" or "residual function" and will be denoted by $\epsilon_n(\eta, \bar{\eta})$. Galerkin's method requires that the "error function" be orthogonal with respect to each co-ordinate function over the domain under consideration, i.e.,

$$\int_c \epsilon_n(\xi, \bar{\xi}) X_n(\xi, \bar{\xi}) d\xi = 0 \quad (\eta = 1, 2, 3, \dots, k) \quad \dots (2.7b)$$

Eq. (2.7b) generates a (K x K) determinantal equation. The lowest root of this equation is the critical buckling coefficient. In the case of a clamped plate it is convenient to take

$$W(\xi, \bar{\xi}) \simeq \sum_{\eta=1}^k B_{\eta} [1 - (\xi, \bar{\xi})^{\eta}]^2 \quad \dots (2.7c)$$

This form of w clearly satisfies the clamped edge boundary conditions in eq. (2.6).

Application :

In order to illustrate the procedure for the determination of the critical buckling condition, an equilateral triangular plate with clamped edges, shown in Fig. 2.1, with a uniform compression parallel to one of the sides is considered.

The mapping function is given by

$$Z = f(\xi) = 1.1352a \left[\xi + \frac{1}{6} \xi^4 + \frac{5}{63} \xi^7 + \frac{4}{81} \xi^{10} \right] \quad \dots (2.8)$$

For first approximation, let

$$K=1 \quad \text{i.e., } W = B_1 (1 - \xi, \bar{\xi})^2 \quad \dots (2.9)$$

With the mapping function in eq.(2.8), putting eq.(2.9) in eq.(2.5) one gets the following error function

$$\begin{aligned}
 \epsilon_1(\xi, \bar{\xi}) = & 4B_1(1.1352a)^2(\Phi\Psi - \xi\Phi_1\Psi - \bar{\xi}\Psi_1\Phi) + \\
 & + 2B_1(1.1352a)^2(2\xi\bar{\xi} - 1)\Phi_1\Psi_1 + \\
 & + \frac{B_1 N_x}{16D} \left[2(1.1352a)^4 \left\{ (\bar{\xi} - \xi\bar{\xi}^2)\Phi_1\Psi^3 + \right. \right. \\
 & + (\xi - \xi^2\bar{\xi})\Psi_1\Phi^3 + \bar{\xi}^2\Phi\Psi^3 + \xi^2\Phi^3\Psi + \\
 & \left. \left. + 2(2\xi\bar{\xi} - 1)\Phi^2\Psi^2 \right\} \right] + \\
 & + \frac{B_1(1.1352a)^6 k_1}{16D} (1 - \xi\bar{\xi})^2 \Phi^3 \Psi^3 \quad \dots (2.10)
 \end{aligned}$$

where

$$\begin{aligned}
 \Phi &= 1 + \frac{2}{3}\xi^3 + \frac{5}{9}\xi^6 + \frac{40}{81}\xi^9 \\
 \Psi &= 1 + \frac{2}{3}\bar{\xi}^3 + \frac{5}{9}\bar{\xi}^6 + \frac{40}{81}\bar{\xi}^9 \\
 \Phi_1 &= 2\xi^2 + \frac{10}{3}\xi^5 + \frac{40}{9}\xi^8 \\
 \Psi_1 &= 2\bar{\xi}^2 + \frac{10}{3}\bar{\xi}^5 + \frac{40}{9}\bar{\xi}^8
 \end{aligned}$$

Let us now follow Galerkin's procedure given in eq.(2.7b).

Multiplying eq.(2.10) by $(1 - \xi\bar{\xi})^2 r dr$ and integrating between the limits 0 to 2π and 0 to 1, the critical buckling condition is obtained as

$$\frac{N_{cr} a^2}{D} = 20.96 + 1.79 \frac{k_1 a^4}{D} \quad \dots (2.11)$$

In terms of the altitude of the plate, eq.(2.11) gives the critical buckling condition as

$$\frac{N_{cr} a^2}{D} = 188.64 + 16.11 \frac{k_1 a^4}{D} \quad \dots (2.12)$$

If the plate is considered without resting on the elastic foundation,

$$N_{cr} = 188.64 \frac{D}{a^2} \quad \dots (2.13)$$

Eq.(2.13) is in excellent agreement with the corresponding result obtained by Bradley [cf. Bradley (1963) P. 55] as

$$N_{cr} = 189.0 \frac{D}{a^2} \quad \dots (2.14)$$

CONCLUSIONS

The analytical procedure presented is straightforward. The same procedure may be followed for a unified treatment of elastic instability problems involving complicated boundary shape, such as regular polygonal shape, circular boundary with flat sides, epitrochoidal boundary etc., since the co-ordinate function which satisfies the given boundary conditions will be the same for any shape. The co-ordinate function used in this study satisfying the boundary conditions is simple. Only the first term approximation is found to yield fairly accurate result and thus minimises labour. No other form of w can reduce the error so greatly with such minimum labour.

It must also be noted that the accuracy of the results obtained by different numerical techniques applied to boundary value problems involving irregular shapes can be verified by the conformal mapping technique exhibited in this study.

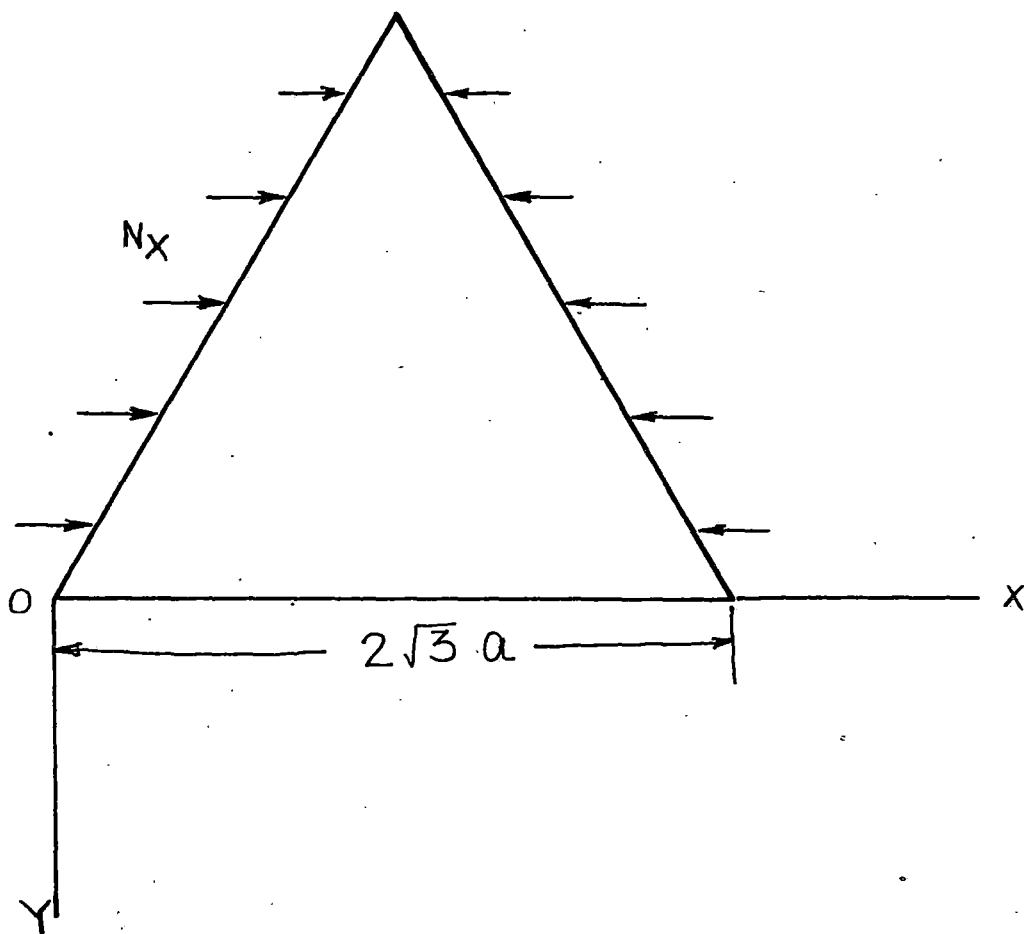


FIG.2'1 EQUILATERAL TRIANGULAR PLATE.

BUCKLING OF A NON-HOMOGENEOUS RECTANGULAR PLATE
PLACED ON ELASTIC FOUNDATION*

PAPER - II

Introduction :

Critical buckling conditions of homogeneous thin rectangular plates subjected to combined bending and compression were investigated by Timoshenko and Gere (1961), Johnson and Noel (1963) and many other investigators. The object of this paper is to use error function to obtain the approximate solutions in the case of buckling of a non-homogeneous thin rectangular plate under the action of combined bending and compression in the middle plane of the plate. The plate is placed on an elastic foundation and is simply supported. Bradley (1963) used finite difference approximations to the governing differential equations to investigate stability of equilateral triangular plates. There are other numerical methods for the solutions of these types of buckling problems. But these methods are time consuming.

Since the governing differential equation obtained in this paper cannot be exactly solved, error function and Galerkin's method have been utilised to obtain an approximate solution of the differential equation. It is observed that the results obtained from this

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method for the homogeneous plate not resting on foundation are in good agreement with the known results obtained by strain energy method. Flexural rigidity of the plate is assumed to vary exponentially and the foundation is taken of the Winkler type. Results obtained have been presented in the form of graphs.

Analysis :

Consider a simply supported rectangular plate of varying flexural rigidity and along whose sides $x = 0$ and $x = a$ (Fig.2.2) distributed forces, acting in the middle plane of the plate, are applied, their intensity being given by the equation

$$N_x = N_0 \left(1 - \alpha \frac{y}{b}\right) \quad \dots (2.15)$$

where N_0 is the intensity of compressive force at edge $y = 0$ and α is a numerical factor. The plate is placed on an elastic foundation having the reaction, K_1 per unit area per unit deflection and is subjected to a uniform transverse load, q .

The governing differential equation of equilibrium of an element of the plate not resting on foundation is [Timoshenko and Krieger (1959) P. 379]

$$\frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -(q + N_x \frac{\partial^2 w}{\partial x^2}) \quad \dots (2.16)$$

where

$$\left. \begin{aligned} M_x &= -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \\ M_y &= -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \\ M_{xy} &= D(1 - \nu) \frac{\partial^2 w}{\partial x \partial y} \end{aligned} \right\} \quad \dots (2.17)$$

Substituting Eqs. (2.15) and (2.17) in Eq.(2.16) and observing that the flexural rigidity is a function of the co-ordinates x and y , one gets the differential equation of equilibrium for a plate resting on elastic foundation in the following form

$$\nabla^2(D\nabla^2W) - (1-\nu)\left\{\frac{\partial^2W}{\partial y^2}\frac{\partial^2D}{\partial x^2} - 2\frac{\partial^2D}{\partial x\partial y}\frac{\partial^2W}{\partial x\partial y} + \frac{\partial^2D}{\partial y^2}\frac{\partial^2W}{\partial x^2}\right\} + K_1W = q + N_0\left(1 - \alpha\frac{y}{b}\right)\frac{\partial^2W}{\partial x^2} \quad \dots (2.18)$$

For simply supported edges the deflection can be represented by the double series

$$W = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \dots (2.19)$$

As the flexural rigidity is variable, let

$$D = D_0 e^{-2\alpha_1 \frac{x}{a}} \quad \dots (2.20)$$

where D_0 and α_1 are constants.

Eq. (2.19) is an approximate solution of Eq. (2.18) and therefore substitution of Eq.(2.19) into Eq.(2.18) results the following error function,

$$\begin{aligned} E(x,y) = & C_{mn} D_0 \left[\left\{ \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right\}^2 - \frac{4\alpha_1^2}{a^2} \left\{ \left(\frac{m\pi}{a}\right)^2 + \nu \left(\frac{n\pi}{b}\right)^2 \right\} \right] \times \\ & \times e^{-2\alpha_1 \frac{x}{a}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} + \\ & + C_{mn} K_1 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} - q - \\ & - C_{mn} N_0 \left(\frac{m\pi}{a}\right)^2 \left(1 - \alpha\frac{y}{b}\right) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \dots (2.21) \end{aligned}$$

According to Galerkin's principle, the following conditions are imposed on the error function, $E(x, y)$

$$\int_0^a \int_0^b E(x, y) W(x, y) dx dy = 0 \quad \dots (2.22)$$

Substituting Eq.(2.19) into Eq.(2.22) and observing that

$$\begin{aligned} \int_0^b y \sin \frac{i\pi y}{b} \sin \frac{j\pi y}{b} dy &= \frac{b^2}{4} \quad \text{for } i=j \\ &= 0 \quad \text{for } i \neq j \text{ and } i \pm j \text{ an even} \\ &\quad \text{number} \\ &= -\frac{4b^2}{\pi^2} \frac{ij}{(i^2 - j^2)^2} \quad \text{for } i \neq j \text{ and} \\ &\quad i \pm j \text{ an odd number} \end{aligned}$$

One gets the following

$$\begin{aligned} &\frac{\pi^6}{2\alpha_1} (1 - e^{-2\alpha_1}) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \left[\pi^2 (m^2 + n^2 \frac{a^2}{b^2})^2 - 4\alpha_1^2 (m^2 + n^2 \frac{a^2}{b^2}) \right] \times \\ &\times \frac{m^2}{(\pi^2 m^2 + \alpha_1^2)} + \pi^2 K_F \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} - \frac{16qa^4}{D_0} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2 n^2} - \\ &- \frac{N_0}{D_0} \pi^4 a^2 \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 C_{mn} - \frac{\alpha}{2} \sum_{m=1}^{\infty} m^2 \left\{ \sum_{n=1}^{\infty} C_{mn} - \frac{16}{\pi^2} \sum_{n=1}^{\infty} \sum_i \frac{C_{mini}}{(n^2 - i^2)^2} \right\} \right] \\ &= 0 \quad \dots (2.23) \end{aligned}$$

Where the nondimensional foundation modulus,

$$K_F = \frac{K_1 a^4}{D_0} \quad \text{and } n \pm i \text{ is always odd.}$$

Taking $\eta = i$, the deflection, W is obtained from Eq.(2.23)

$$\begin{aligned} W = & \frac{16qa^4}{D_0} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m\eta} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \times \\ & \times \frac{1}{\left[\frac{\pi^6(1-e^{-2\alpha_1})}{2\alpha_1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\pi^2 \left(m^2 + n^2 \frac{a^2}{b^2} \right)^2 - 4\alpha_1^2 \left(m^2 + \nu n^2 \frac{a^2}{b^2} \right) \right]} \right]} \times \\ & \times \left[\frac{m^2}{(\pi^2 m^2 + \alpha_1^2)} + \pi^2 K_F - \frac{N_0 \pi^4 a^2}{D_0} \left\{ \sum_{m=1}^{\infty} m^2 \left(1 - \frac{\alpha}{2} \right) \right\} \right] \end{aligned}$$

... (2.24)

From Eq. (2.23) the critical buckling condition is obtained when

$$\begin{aligned} \frac{N_0 a^2}{D_0} \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 c_{m\eta} - \frac{\alpha}{2} \sum_{m=1}^{\infty} m^2 \left\{ \sum_{n=1}^{\infty} c_{m\eta} - \frac{16}{\pi^2} \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{c_{mi} n_i}{(n^2 - i^2)^2} \right\} \right] \\ = \frac{\pi^2(1-e^{-2\alpha_1})}{2\alpha_1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{m\eta} \left[\pi^2 \left(m^2 + n^2 \frac{a^2}{b^2} \right)^2 - 4\alpha_1^2 \left(m^2 + \nu n^2 \frac{a^2}{b^2} \right) \right] \times \end{aligned}$$

$$\times \left[\frac{m^2}{(\pi^2 m^2 + \alpha_1^2)} + \frac{1}{\pi^2} K_F \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{m\eta} \right]$$

... (2.25)

The plate may buckle in such a way that there can be several half-waves in the direction of compression but only one half-wave in the perpendicular direction. For one half wave buckling $m = 1$, for two half-waves buckling $m = 2$ and so on.

If the plate buckles in one half-wave, one gets from Eq.(2.25) by taking $m = 1$ a system of equations of the following kind :

$$C_{1n} \left[\frac{(1-e^{-2\alpha_1})}{2\alpha_1(\pi^2+\alpha_1^2)} \left\{ \pi^2 \left(1+n^2\frac{a^2}{b^2}\right)^2 - 4\alpha_1^2 \left(1+\nu\frac{a^2n^2}{b^2}\right) \right\} + \frac{K_F}{\pi^4} - \right. \\ \left. - \sigma_{cn} \frac{a^2h}{\pi^2 D_0} \left(1-\frac{\alpha}{2}\right) \right] - 8\alpha \sigma_{cn} \frac{a^2h}{\pi^4 D_0} \sum_{l=1}^{\infty} \frac{C_{li} n_i}{(n^2-l^2)^2} = 0 \quad \dots (2.26)$$

where $\sigma_{cn} = \frac{(N_0)_{cn}}{h}$, h being the plate thickness.

The lowest root of the determinantal equation thus formed will determine the critical buckling load. From the first approximate lowest root one gets by taking $n = 1$

$$\sigma_{cn} = K \frac{\pi^2 D_0}{b^2 h} \quad \dots (2.27)$$

where

$$K = \frac{1}{1-\frac{\alpha}{2}} \left[\frac{1-e^{-2\alpha_1}}{2\alpha_1(\pi^2+\alpha_1^2)} \left\{ \pi^2 \left(\frac{b}{a} + \frac{a}{b}\right)^2 - 4\alpha_1^2 \left(\frac{b^2}{a^2} + \nu\right) \right\} + \frac{K_F}{\pi^4} \cdot \frac{b^2}{a^2} \right]$$

Thus the buckling load is a function of $\frac{a}{b}$ and the foundation modulus, K_F .

For $\alpha = 0$, the critical buckling load σ_{cn} is obtained from Eq.(2.25) by taking $n = 1$

$$\sigma_{cn} = K \frac{\pi^2 D_0}{b^2 h} \quad \dots (2.28)$$

where

$$K = \frac{1 - e^{-2\alpha_1}}{2\alpha_1(\pi^2 m^2 + \alpha_1^2)} \left\{ \pi^2 \left(m^2 \frac{b}{a} + \frac{a}{b} \right)^2 - 4\alpha_1^2 \left(m^2 \frac{b^2}{a^2} + \nu \right) \right\} + \frac{K_F}{\pi^4} \cdot \frac{1}{m^2} \cdot \frac{b^2}{a^2} \quad (m = 1, 2, 3, \dots) \quad \dots (2.29)$$

For homogeneous material, $D_0 \rightarrow D$ when $\alpha_1 \rightarrow 0$. Setting $\alpha_1 \rightarrow 0$ in Eq.(2.27) one gets the critical buckling load for a homogeneous plate on elastic foundation for one half-wave buckling

$$\sigma_{cr} = K \frac{\pi^2 D}{b^2 h} \quad \dots (2.30)$$

where

$$K = \frac{1}{1 - \alpha_1^2} \left[\left(\frac{b}{a} + \frac{a}{b} \right)^2 + \frac{K_F}{\pi^4} \cdot \frac{b^2}{a^2} \right]$$

For $K_F = 0$, Eq.(2.30) is the result obtained by Timoshenko and Gere. For n half-waves buckling, K in Eq.(2.30) can be expressed as

$$K = \frac{1}{m^2} \left(m^2 \frac{b}{a} + \frac{a}{b} \right)^2 + \frac{K_F}{\pi^4} \cdot \frac{1}{m^2} \cdot \frac{b^2}{a^2} \quad \dots (2.31)$$

The ratio $\frac{a}{b}$ for which σ_{cr} becomes a minimum for uniform compression is obtained from Eq.(2.29) and denoting this ratio by

$\left(\frac{a}{b}\right)_{cr}$, one gets for a homogeneous plate

$$\left(\frac{a}{b}\right)_{cr} = \frac{1}{\pi} \left(K_F + \pi^4 m^4 \right)^{1/4} \quad \dots (2.32)$$

and for a nonhomogeneous plate

$$\left(\frac{a}{b}\right)_{cr} = \left[m^4 + \frac{K_F}{\pi^6 m^2} \cdot \frac{2\alpha_1(\pi^2 m^2 + \alpha_1^2)}{(1 - e^{-2\alpha_1})} - \frac{4\alpha_1^2 m^2}{\pi^2} \right]^{1/4} \quad \dots (2.33)$$

The ratio $\frac{a}{b}$ at which the transition from n to $n+1$ half-waves buckling occur can also be computed from Eq. (2.29). For homogeneous plate under uniform compression, transition from one to two half-waves occurs when

$$\frac{a}{b} = \left(4 - \frac{k_F}{\pi^4}\right)^{1/4}$$

and transition from two to three half-waves occurs when

$$\frac{a}{b} = \left(36 - \frac{k_F}{\pi^4}\right)^{1/4}$$

Eq.(2.27) gives satisfactory results for small values of α . An improved result is obtained by taking two equations of the system Eq.(2.26) with coefficients C_{11} and C_{12} and setting the determinant equal to zero. Thus for one half-wave buckling

$$\sigma_{cr} = K \frac{\pi^2 D_0}{b^2 h} \quad \dots (2.34)$$

where

$$K = \left[\frac{1}{\{.0065\alpha^2 - .2(1-\alpha/2)^2\}} \times \left\{ .01(1-\alpha/2)^2 \left(B + \frac{2k_F}{\pi^4}\right)^2 + \right. \right. \\ \left. \left. + [.0013\alpha^2 - .041(1-\alpha/2)^2] \times \left[A + B \frac{k_F}{\pi^4} + \frac{k_F^2}{\pi^8}\right] \right\}^{1/2} - \right. \\ \left. - \frac{(1-\alpha/2) \left(B + \frac{2k_F}{\pi^4}\right)}{.064\alpha^2 - 2(1-\alpha/2)^2} \right] \times \frac{b^2}{a^2}$$

and

$$A = \left\{ \frac{1 - e^{-2\alpha_1}}{2\alpha_1(\pi^2 + \alpha_1^2)} \right\}^2 \left\{ \pi^2 \left(1 + \frac{a^2}{b^2}\right)^2 - 4\alpha_1^2 \left(1 + \nu \frac{a^2}{b^2}\right) \right\} \left\{ \pi^2 \left(1 + \frac{4a^2}{b^2}\right)^2 - \right. \\ \left. - 4\alpha_1^2 \left(1 + 4\nu \frac{a^2}{b^2}\right) \right\}$$

$$B = \frac{1 - e^{-2\alpha_1}}{2\alpha_1(\pi^2 + \alpha_1^2)} \left\{ \pi^2 \left(2 + 10 \frac{a^2}{b^2} + 17 \frac{a^4}{b^4}\right) - 4\alpha_1^2 \left(2 + 5\nu \frac{a^2}{b^2}\right) \right\}$$

For pure bending when $\alpha = 2$ Eq.(2.34) reduces to

$$\sigma_{cn} = \frac{\pi^2 D_0}{b^2 h} \left[2.77 \frac{b^2}{a^2} \left(A + B \frac{K_F}{\pi^4} + \frac{K_F^2}{\pi^8} \right)^{1/2} \right] \dots (2.35)$$

Setting $\alpha_1 \rightarrow 0$ in Eq.(2.35) one gets the buckling load under pure bending for a homogeneous plate for one half-wave buckling

$$\sigma_{cn} = K \frac{\pi^2 D}{b^2 h} \dots (2.36)$$

where

$$K = 2.77 \frac{b^2}{a^2} \left[\left(1 + \frac{a^2}{b^2} \right)^2 \left(1 + \frac{4a^2}{b^2} \right)^2 + \left(2 + 10 \frac{a^2}{b^2} + 17 \frac{a^4}{b^4} \right) \times \right. \\ \left. \times \frac{K_F}{\pi^4} + \frac{K_F^2}{\pi^8} \right]^{1/2}$$

For m half-wave buckling K in Eq.(2.36) can be written as

$$K = \frac{1}{m^2} \times 2.77 \frac{b^2}{a^2} \left[\left(m^2 + \frac{a^2}{b^2} \right)^2 \left(m^2 + \frac{4a^2}{b^2} \right)^2 + \right. \\ \left. + \left(2m^4 + 10m^2 \frac{a^2}{b^2} + 17 \frac{a^4}{b^4} \right) \frac{K_F}{\pi^4} + \frac{K_F^2}{\pi^8} \right]^{1/2} \dots (2.37)$$

The presence of the foundation modulus, K_F in Eq.(2.24) reduces the deflection, w and hence the bending stress. Therefore Eqs.(2.34, 2.35 and 2.36) resulting from two term approximation of Eq.(2.26) may be taken as fairly accurate.

RESULTS AND CONCLUSIONS

Numerical results are obtained for two cases : a) when the plate is under uniform compression, and b) when it is under pure bending. For uniform compression of a homogeneous plate, the values of K are

calculated for different values of $\frac{a}{b}$ with the help of Eq.(2.30) for one half-wave buckling, taking $K_F = \pi^4$, and $\alpha = 0$. For two half-waves and three half-waves buckling Eq.(2.31) is used with the same value of K_F . These results are presented in the form of graphs in Fig. 2.3. For a non-homogeneous plate under uniform compression, the values of K are calculated for one half-wave, two and three half-waves buckling by taking the same value of K_F and $\alpha_1 = 0.1$ in Eq.(2.29). These results are presented in Fig.2.4. The values of K when the plates are not on the foundation are also presented in Fig. 2.3 and 2.4 for comparison.

For pure bending of a homogeneous plate Eq. (2.37) is used for calculation of K for different values of $\frac{a}{b}$ taking $K_F = \pi^4$ and the results are presented in Fig. 2.5. In the same figure the corresponding results for K without foundation are also presented for comparison.

From the foregoing analysis and from Figs. 2.3, 2.4 and 2.5 the following conclusions may be drawn :

- i) Foundation increases the buckling load
- ii) Resistance offered by the foundation is more for one half-wave buckling compared to multiple half-waves buckling. When buckling is in more than one half-wave, the foundation resistance remains practically constant.
- iii) Foundation increases the $(\frac{a}{b})_{cr}$ ratio and reduces the $\frac{a}{b}$ ratio at which transition takes place as compared to a plate not resting on foundation.
- iv) A non-homogeneous plate will have lower buckling load as compared to a homogeneous plate.
- v) Use of error function reduces labour considerably compared to any other numerical method.

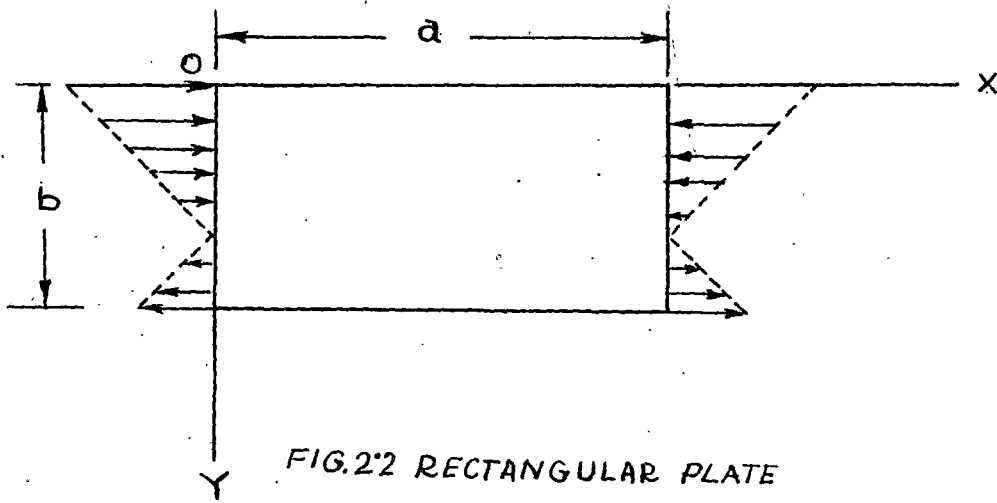


FIG.22 RECTANGULAR PLATE

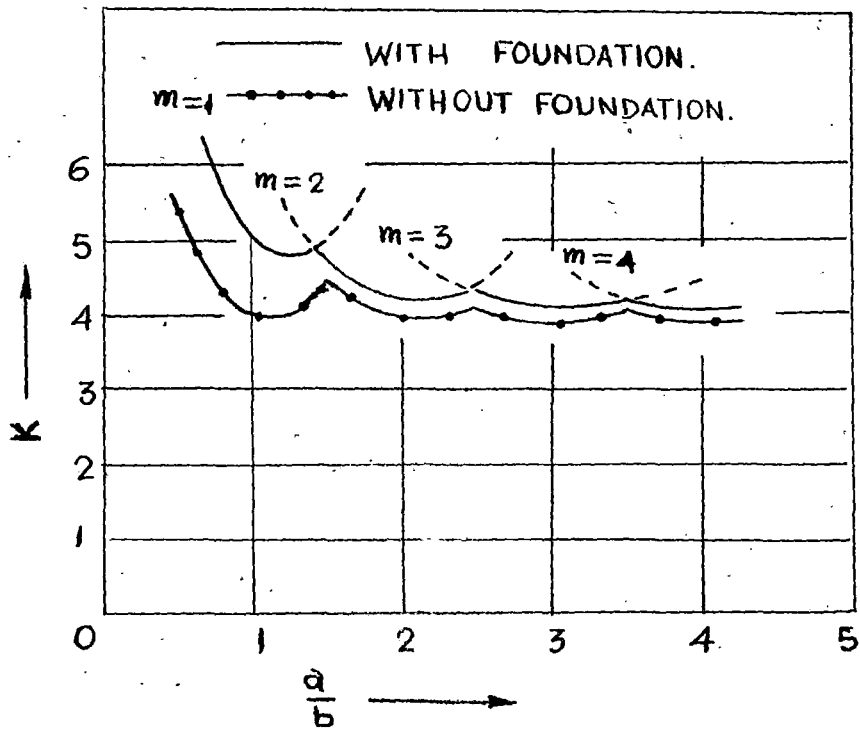


FIG.23 HOMOGENEOUS PLATE UNDER UNIFORM COMPRESSION.

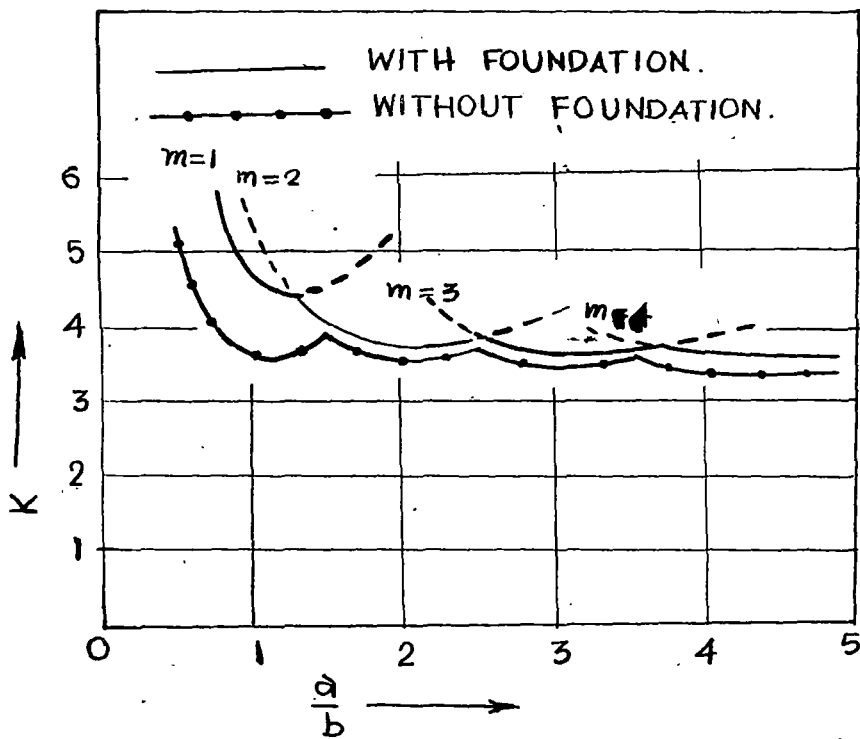


FIG. 24 NON HOMOGENEOUS PLATE UNDER UNIFORM COMPRESSION.

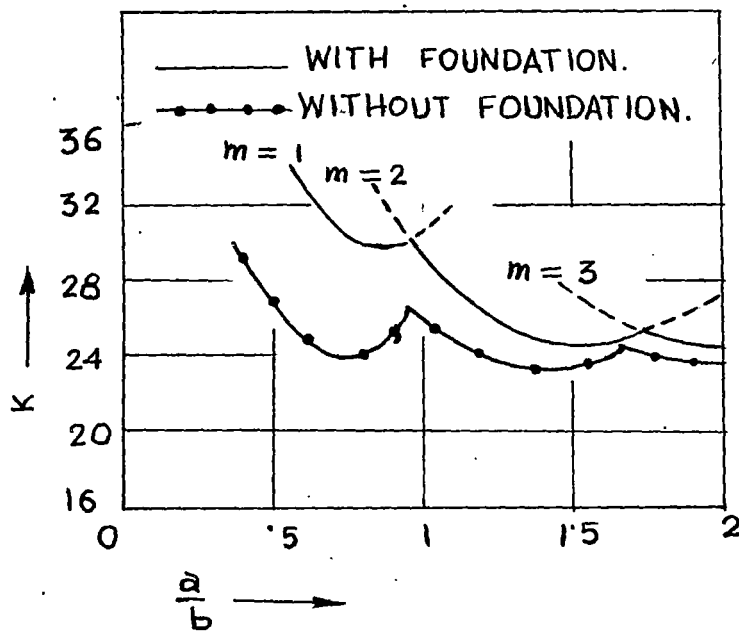


FIG.2'5 HOMOGENEOUS PLATE UNDER PURE BENDING.

THERMAL BUCKLING OF SOME HEATED PLATES PLACED
ON ELASTIC FOUNDATION*

PAPER - III

INTRODUCTION :

Thermal Buckling of thin elastic plates is of much practical importance in modern engineering. Nowacki (1962) has discussed the thermal buckling of a rectangular plate under different boundary conditions. Mansfield (1964) has investigated the buckling and curling of a heated thin circular plate of constant thickness. Klosner and Furray (1958) have studied the thermal buckling of simply supported plates under symmetrical temperature distribution.

In this paper thermal buckling of a heated equilateral triangular plate of simply-supported edges and a clamped elliptic plate placed on elastic foundation has been investigated. The foundation is assumed to be of the Winkler type. The boundary has been transformed conformally onto the unit circle and the stability criterion has been obtained with the help of Galerkin's procedure.

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THEORY :

Let us consider a plate of thickness, h , subjected to a temperature distribution T which is independent of x and y , but varies arbitrarily through the thickness, i.e.,

$$T = T(z)$$

The plate is subjected to no external load and motion of all supports in the plane of the plate is prevented. It justifies then, that under the above condition there are no displacements in the plane of the plate, i.e.,

$$u = v = 0$$

On the above propositions the differential equation for the displacement W [Boley and Weiner, 1967] is

$$D \nabla^4 W + \frac{N_T}{1-\nu} \nabla^2 W = 0 \quad \dots (2.33)$$

For a plate placed on elastic foundation having the foundation reaction, K_1 ; Eq. (2.33) becomes

$$D \nabla^4 W + \frac{N_T}{1-\nu} \nabla^2 W + K_1 W = 0 \quad \dots (2.39)$$

Eq. (2.39) may be written as

$$(\nabla^2 + P_1^2)(\nabla^2 + P_2^2)W = 0 \quad \dots (2.40)$$

in which

$$\rho_1^2 \rho_2^2 = \frac{K_1}{D} \quad \dots (2.41)$$

$$\rho_1^2 + \rho_2^2 = \frac{N_T}{D(1-\nu)} \quad \dots (2.42)$$

and

$$N_T = \alpha E \int_{-h/2}^{+h/2} T dz \quad \dots (2.43)$$

α being the coefficient of thermal expansion.

If $Z = x + iy$, $\bar{Z} = x - iy$,

Eq.(2.40) changes into

$$\left(4 \frac{\partial^2}{\partial z \partial \bar{z}} + \rho_1^2\right) \left(4 \frac{\partial^2}{\partial z \partial \bar{z}} + \rho_2^2\right) W = 0 \quad \dots (2.44)$$

Let $Z = f(\xi)$ be the analytic function which maps the given shape in the ξ plane onto a unit circle.

Thus Eq.(2.44) transforms into

$$\begin{aligned} & \frac{\partial^4 W}{\partial \xi^2 \partial \bar{\xi}^2} \frac{dz}{d\xi} \frac{d\bar{z}}{d\bar{\xi}} - \frac{\partial^3 W}{\partial \xi^2 \partial \bar{\xi}} \frac{d^2 z}{d\xi^2} \frac{d\bar{z}}{d\bar{\xi}} - \frac{\partial^3 W}{\partial \xi \partial \bar{\xi}^2} \frac{d^2 \bar{z}}{d\bar{\xi}^2} \frac{dz}{d\xi} + \\ & + \frac{\partial^2 W}{\partial \xi \partial \bar{\xi}} \frac{d^2 z}{d\xi^2} \frac{d^2 \bar{z}}{d\bar{\xi}^2} + \frac{(\rho_1^2 + \rho_2^2)}{4} \left(\frac{dz}{d\xi}\right)^2 \left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2 \frac{\partial^2 W}{\partial \xi \partial \bar{\xi}} + \\ & + \rho_1^2 \rho_2^2 \left(\frac{dz}{d\xi}\right)^3 \left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3 W = 0 \end{aligned}$$

... (2.45)

Let us assume

$$W = \sum_{n=1}^{\infty} B_n [1 - (\bar{r}\bar{r})^n] \quad \dots (2.46)$$

Clearly the above form of W satisfies the edge condition $W = 0$ at $r = 1$. Eq.(2.46) is an admissible function for the simply supported edge condition in the sense that this satisfies the kinematic boundary $W = 0$ at $r = 1$, but does not satisfy the force boundary condition $M_n = 0$. Putting Eq.(2.46) in Eq.(2.45) one gets the error function, $\epsilon_{n,0}$. Galerkin's procedure requires that the error function to be orthogonal over the domain, i.e.,

$$\int_c \epsilon_{n,0}(\bar{r}\bar{r}) w(\bar{r}\bar{r}) dc = 0 \quad (\eta = 1, 2, \dots, k) \quad \dots (2.47)$$

This generates ($K \times K$) determinantal equation. The lowest root of this gives the critical buckling temperature.

APPLICATIONS :

A. Let us consider a simply-supported equilateral triangular plate of side $2\sqrt{3}a$ placed on an elastic foundation.

To solve the differential equation for W let us put

$$W = W_1 + W_2 \quad \dots (2.48)$$

From Eq.(2.40) one gets

$$(\nabla^2 + P_1^2) W_1 = 0 \quad \dots (2.49)$$

$$(\nabla^2 + P_2^2) W_2 = 0 \quad \dots (2.50)$$

For the edge condition $W = 0$ along the boundary, let

$$W_2 = \sum_{n=1}^K B_n [1 - (\zeta\bar{\zeta})^n] = \sum_{n=1}^K B_n (1 - \lambda^{2n}) \quad \dots (2.51)$$

For 1st term approximation it is sufficient to solve either Eq.(2.49) or Eq.(2.50). Changing either of the two Eqs.(2.49) and (2.50) in the complex coordinates one gets

$$\nabla^2 W_1(\zeta\bar{\zeta}) + \rho_2^2 \left(\frac{dz}{d\zeta}\right)^2 W_1(\zeta\bar{\zeta}) = 0 \quad \dots (2.52)$$

The mapping function Z Laura and Shahady, 1969

$$Z = 1.1352a \left[\zeta + \frac{1}{6} \zeta^4 + \frac{5}{63} \zeta^7 + \frac{4}{81} \zeta^{10} \right] \quad \dots (2.52a)$$

maps an equilateral triangular plate into a unit circle in the ζ - plane. With this mapping function, putting Eq.(2.51) in Eq.(2.52) and remembering $\zeta = \pi e^{i\theta}$ one gets the required error function. After evaluating the integral given by Eq.(2.47) and taking $K=1$ one gets

$$\lambda_2^2 = 6 \quad \dots (2.53)$$

where

$$\lambda_2^2 = \rho_2^2 (1.1352a)^2$$

From Eq.(2.53) one gets the following critical buckling temperature

$$(N_T)_{cr} = D(1-\nu) \left[\frac{6}{(1.1352a)^2} + \frac{K_1(1.1352a)^2}{6D} \right] \quad \dots (2.54)$$

For $K_1 = 0$, Eq.(2.54) reduces to

$$(N_T)_{er} = \frac{4.6}{a^2} D(1-\nu) \quad \dots (2.54a)$$

Expressing this value in terms of the altitude h of the triangle one gets

$$(N_T)_{er} = \frac{41.4}{h^2} D(1-\nu) \quad \dots (2.54b)$$

which agrees well with the result obtained by Banerjee (1975) who used tri-linear co-ordinates to obtain the results for the corresponding problem without elastic foundation.

B. Let us consider an elliptic plate having centre at the origin. Let h be the thickness of the plate. For clamped edge boundary condition let us take w in the following form

$$w = \sum_{n=1}^k B_n [1 - (\bar{r}\bar{r})^n]^2 \quad \dots (2.55)$$

Clearly the above form of w satisfies the clamped edge conditions

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{at } r = 1$$

Now for the ellipse

$$\frac{x^2}{\frac{4}{3}} + \frac{y^2}{\frac{4}{5}} = 1$$

mapping function is [Laura and Shahady, 1969]

$$Z = 0.99b [\xi + 0.12\xi^3 + 0.03\xi^5 + 0.01\xi^7] \quad \dots (2.56)$$

which maps the above ellipse a unit circle in the ξ - plane. With this mapping function, putting Eq.(2.55) in Eq.(2.45) and remembering $\xi = \eta e^{i\theta}$ one gets the required error function. After evaluating the integral given by Eq.(2.47) and taking $K = 2$, the following determinant is obtained.

$$\begin{vmatrix} \frac{32}{3} - \frac{2}{3}(\lambda_1^2 + \lambda_2^2) + \frac{\lambda_1^2 \lambda_2^2}{10} & \frac{256}{15} - \frac{4}{5}(\lambda_1^2 + \lambda_2^2) + \frac{33}{70} \lambda_1^2 \lambda_2^2 \\ \frac{256}{15} - \frac{4}{5}(\lambda_1^2 + \lambda_2^2) + \frac{29}{576} \lambda_1^2 \lambda_2^2 & \frac{5632}{105} - \frac{4}{3}(\lambda_1^2 + \lambda_2^2) + \frac{127}{315} \lambda_1^2 \lambda_2^2 \end{vmatrix} = 0 \quad \dots (2.57)$$

where

$$\lambda_1^2 = P_1^2 (0.99b)^2$$

$$\lambda_2^2 = P_2^2 (0.99b)^2$$

Solving Eq.(2.57) for the lowest root, the critical buckling temperature is obtained as

$$(N_T)_{cr} = D(1-\nu) \left[\frac{44.6}{b^2} + 0.094 b^2 \frac{K_1}{D} - \sqrt{\left\{ \frac{29.2}{b^2} + 0.094 \frac{K_1}{D} \right\}^2 + 0.106 b^4 \left(\frac{K_1}{D} \right)^2} \right] \quad \dots (2.58)$$

CONCLUDING REMARKS

Solutions obtained in this study are only approximate, because only the first term of the mapping function is considered and K is taken to be 2. More accurate results are obtained by considering the remaining terms of the mapping function and taking K more than 2. Solution of the eigenvalue problem governing the stability of the thin elastic plates having various configurations, such as regular polygonal shape, circular boundary with flat sides, epitrochoidal boundary etc, is easily accomplished with the help of the complex variable theory applied in this study.

CHAPTER - IIILARGE DEFLECTION OF A CIRCULAR PLATE ON ELASTIC FOUNDATION
AND SUPPORTED AT SEVERAL POINTS ALONG THE BOUNDARY *

PAPER - I

Introduction :

Small deflections of thin plates placed on elastic foundations have been examined by S. Timoshenko and S. Woinowsky - Krieger (1959) and several other authors on the assumption that strain due to stretching of the middle surface of the plate is negligible. When the deflections are moderately large, that is, on the order of the thickness of the plate, then the forces in the middle surface of the plate must be taken into account. In the case of such large deflections of plates, three differential equations for displacements and deflections may be written, but it is usually difficult to obtain the solutions of these equations because of their nonlinear character.

On the other hand, various problems of large deflections of plates not resting on elastic foundations have been examined by S. Way (1934), S. Levy (1942) and many other authors. But the methods used by them require considerable computation. A simple and approximate, yet fairly accurate, method of analysing large deflections of plates was suggested by H. M. Berger (1955). The method uses the technique

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of neglecting the second strain invariant of the middle surface strains in the expression corresponding to the total potential energy of the system. Berger's method reduces computation and although no complete explanation of this method is offered in, the stresses and deflections obtained for both rectangular and circular plates are in good agreement with those found in practical analysis. This approximate method has been applied successfully by Nowinski (1953) to his plate problems. Nash and Modeer (1959) investigated the problems having no axial symmetry following this method.

The technique of neglecting the second strain invariant has been successfully applied by Sinha (1963) to determine large deflection of circular and rectangular plates placed on elastic foundations and under uniform lateral loads.

In this paper large deflection of a circular plate placed on elastic foundation and supported at several points along the boundary has been solved. The load is assumed to be uniformly distributed and the foundation is of the Winkler type. A complete analysis of a particular case, where the number of supports is two, is given.

Formulation of problem :

For moderately large deflections, the strain displacement relationships and the strain energy of the middle plane of the plate are

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \quad \dots (3.1)$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \quad \dots (3.2)$$

$$V_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \quad \dots (3.3)$$

$$V_1 = \frac{D}{2} \iint \left[(\nabla^2 W)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu) \left\{ \frac{12}{h^2} e_2 + \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} \right] dx dy \quad \dots (3.4)$$

in which e_1 and e_2 are the first and second middle surface strain invariants, respectively. Neglecting e_2 and by adding the potential energy of the transverse load and of the foundation reaction, K the modified energy equation becomes [Sinha (1963)].

$$V = \frac{D}{2} \iint \left[(\nabla^2 W)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu) \left\{ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} + \frac{K}{D} W^2 - \frac{2qW}{D} \right] dx dy \quad \dots (3.5)$$

Applying Euler's variational method to Eq.(3.5) the following differential equations in polar co-ordinates are obtained

$$\nabla^4 W - \alpha^2 \nabla^2 W + \frac{K}{D} W = \frac{q}{D} \quad \dots (3.6)$$

where α is a constant given by

$$\frac{\alpha^2 h^2}{12} = \frac{\partial u}{\partial \eta} + \frac{1}{2} \left(\frac{\partial w}{\partial \eta} \right)^2 + \frac{u}{\eta} + \frac{1}{\eta} \frac{\partial v}{\partial \theta} + \frac{1}{2\eta^2} \left(\frac{\partial w}{\partial \theta} \right)^2 \quad \dots (3.7)$$

$$\nabla^2 = \frac{\partial^2}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial}{\partial \eta} + \frac{1}{\eta^2} \frac{\partial^2}{\partial \theta^2}$$

Solution of problem :

Let the circular plate (Fig. 3.1) be of radius a , supported at several points along the boundary and placed on the elastic foundation. Let the centre of the plate be the origin and a diameter as the initial line, $\theta = 0$. Let the general solution of Eq.(3.6) be in the form

$$W = W_0 + W_1 \quad \dots (3.9)$$

in which W_0 is the large deflection of a plate placed on elastic foundation and simply supported along the entire boundary and W_1 satisfies the equation

$$\nabla^4 W_1 - \alpha^2 \nabla^2 W_1 + \frac{K}{D} W_1 = 0 \quad \dots (3.9)$$

Eq. (3.9) can be written in the form

$$(\nabla^2 - \rho_1^2)(\nabla^2 - \rho_2^2) W_1 = 0 \quad \dots (3.10)$$

where

$$\rho_1^2 + \rho_2^2 = \alpha^2 \quad \dots (3.11)$$

$$\rho_1^2 \rho_2^2 = \frac{K}{D} \quad \dots (3.12)$$

Considering the number of points of support is i , and denoting the concentrated reactions at these points N_1, N_2, \dots, N_i , the expression for each reaction N_i is

$$\frac{N_i}{\pi a} \left[\frac{1}{2} + \sum_{m=1}^{\infty} \cos m \theta_i \right] \quad \dots (3.13)$$

where $\theta_i = \theta - \psi_i$, ψ_i is the angle defining the position of the support i .

The intensity of the reactive forces at any point of the boundary is then given by the expression

$$\sum_{i=1}^i \frac{N_i}{\pi a} \left[\frac{1}{2} + \sum_{m=1}^{\infty} \cos m\theta_i \right] \quad \dots (3.13a)$$

in which the summation is extended over all the concentrated reactions. Assuming that the plate is solid and considering that deflections and moments at the centre must be finite, the appropriate solution of Eq.(3.9) can be taken in the form

$$\begin{aligned} W_1 = & A_0 I_0 (P_1 \pi) + B_0 I_0 (P_2 \pi) + \\ & + \sum_{m=1}^{\infty} [A_m I_m (P_1 \pi) + B_m I_m (P_2 \pi)] \cos m\theta + \\ & + \sum_{m=1}^{\infty} [A'_m I_m (P_1 \pi) + B'_m I_m (P_2 \pi)] \sin m\theta \quad \dots (3.14) \end{aligned}$$

in which I_0 is the modified Bessel function of the first kind and zero order, and I_m is of the first kind and m th order.

For determining the constants we have the following conditions at the boundary.

$$(W)_{\substack{\pi=a \\ \theta=0,\pi}} = 0 \quad \dots (3.15)$$

$$\left[\frac{\partial^2 W_1}{\partial \pi^2} + \frac{\nu}{\pi} \frac{\partial W_1}{\partial \pi} + \frac{\nu}{\pi^2} \frac{\partial^2 W_1}{\partial \theta^2} \right]_{\pi=a} = 0 \quad \dots (3.16)$$

$$\left[Q_{\pi} - \frac{1}{\pi} \frac{\partial}{\partial \theta} M_{\pi\theta} \right]_{\pi=a} = - \sum_{i=1}^i \frac{N_i}{\pi a} \left[\frac{1}{2} + \sum_{m=1}^{\infty} \cos m\theta_i \right] \quad \dots (3.17)$$

where

$$Q_n = -D \frac{\partial}{\partial n} \left[(\nabla^2 - \alpha^2) w_1 \right] \quad \dots (3.17a)$$

$$M_{n\theta} = (1-\nu)D \left[\frac{1}{r} \frac{\partial^2 w_1}{\partial n \partial \theta} - \frac{1}{r^2} \frac{\partial w_1}{\partial \theta} \right] \quad \dots (3.17b)$$

consider a particular case when the plate is supported at two points which are the two end points of the diameter taken as the initial line from which θ is measured. Then

$$\psi_1 = 0, \quad \psi_2 = \pi$$

Considering the above boundary conditions and remembering that deflection is zero at these two points of support one gets after solving for the constants

$$A_0 = \frac{P}{\pi D a} \beta \psi_0(a) \quad \dots (3.18)$$

$$B_0 = -\frac{P}{\pi D a} \beta \phi_0(a) \quad \dots (3.19)$$

$$A_m = -\frac{P}{\pi D a} \frac{\mu_m(a)}{\beta_m(a)\mu_m(a) - \lambda_m(a)\eta_m(a)} \quad \dots (3.20)$$

$$B_m = \frac{P}{\pi D a} \frac{\lambda_m(a)}{\beta_m(a)\mu_m(a) - \lambda_m(a)\eta_m(a)} \quad \dots (3.21)$$

$$A'_m = 0 = B'_m \quad \dots (3.22)$$

where $P = \lambda a^2 q =$ total load on the plate

$$\beta = \frac{[\lambda_m(a)I_m(P_2a) - \mu_m(a)I_m(P_1a)]}{[\beta_m(a)\mu_m(a) - \lambda_m(a)\eta_m(a)] \times [I_0(P_2a)\Phi_0(a) - I_0(P_1a)\Psi_0(a)]} \dots (3.23)$$

$$\Psi_0(a) = P_2^2 I_0''(P_2a) + \frac{\nu}{a} P_2 I_1(P_2a) \dots (3.24)$$

$$\Phi_0(a) = P_1^2 I_0''(P_1a) + \frac{\nu}{a} P_1 I_1(P_1a) \dots (3.25)$$

$$\mu_m(a) = P_2^2 I_m''(P_2a) + \frac{\nu}{a} P_2 I_m'(P_2a) - \frac{\nu m^2}{a^2} I_m(P_2a) \dots (3.26)$$

$$\beta_m(a) = P_2^2 P_1 I_m'(P_1a) - (1-\nu) \left\{ \frac{m^2}{a^3} I_m(P_1a) - \frac{P_1 m^2}{a^2} I_m'(P_1a) \right\} \dots (3.27)$$

$$\lambda_m(a) = P_1^2 I_m''(P_1a) + \frac{\nu}{a} P_1 I_m'(P_1a) - \frac{\nu m^2}{a^2} I_m(P_1a) \dots (3.28)$$

$$\eta_m(a) = P_1^2 P_2 I_m'(P_2a) - (1-\nu) \left\{ \frac{m^2}{a^3} I_m(P_2a) - \frac{P_2 m^2}{a^2} I_m'(P_2a) \right\} \dots (3.29)$$

Thus the complete solution of Eq.(3.6) is obtained in the following form

$$W = W_0 + A_0 I_0(P_1 r) + B_0 I_0(P_2 r) + \sum_{m=2,4,6,\dots}^{\infty} [A_m I_m(P_1 r) + B_m I_m(P_2 r)] \cos m\theta \quad \dots (3.30)$$

where

$$W_0 = \frac{q}{k} + A'_0 I_0(P_1 a) + B'_0 I_0(P_2 a) \quad \dots (3.31)$$

$$A'_0 = -\frac{q}{k} \left[\frac{P_2^2 I_0''(P_2 a) + P_2 \frac{\nu}{a} I_1(P_2 a)}{\Phi(Pa)} \right] \quad \dots (3.31a)$$

$$B'_0 = \frac{q}{k} \left[\frac{P_1^2 I_0''(P_1 a) + P_1 \frac{\nu}{a} I_1(P_1 a)}{\Phi(Pa)} \right] \quad \dots (3.31b)$$

$$\Phi(Pa) = \{ I_0(P_1 a) P_2^2 I_0''(P_2 a) - I_0(P_2 a) P_1^2 I_0''(P_1 a) \} +$$

$$+ \frac{\nu}{a} \{ P_2 I_1(P_2 a) I_0(P_1 a) - P_1 I_1(P_1 a) I_0(P_2 a) \} \quad \dots (3.31c)$$

substituting the values of the constants A'_0 , B'_0 , A_0 , B_0 , A_m , and B_m into Eq.(3.30) one gets taking $K_F = \frac{Ka^4}{D}$

$$\begin{aligned} \frac{W}{h} = & \left(\frac{qa^4}{Dh} \right) \left[\frac{1}{K_F} \left\{ 1 + \frac{[P_1^2 I_0''(P_1 a) + P_1 \frac{\nu}{a} I_1(P_1 a)] I_0(P_1 \pi)}{\Phi(P_1 a)} - \right. \right. \\ & - \left. \frac{[P_2^2 I_0''(P_2 a) + P_2 \frac{\nu}{a} I_1(P_2 a)] I_0(P_2 \pi)}{\Phi(P_2 a)} \right\} + \\ & + \frac{1}{a^3} \left\{ \beta \Psi_0(a) I_0(P_1 \pi) - \beta \Phi_0(a) I_0(P_2 \pi) - \right. \\ & - \left. \sum_{m=2,4,6,\dots}^{\infty} \left[\frac{\mu_m(a) I_m(P_1 \pi) - \lambda_m(a) I_m(P_2 \pi)}{\beta_m(a) \mu_m(a) - \lambda_m(a) \eta_m(a)} \right] \cos m\theta \right\} \end{aligned} \quad \dots (3.32)$$

As $P_1 \rightarrow 0$ and $P_2 \rightarrow 0$, Eq.(3.32) reduces to

$$\begin{aligned} W = & W_0 + \frac{Pa^2}{2\pi D} \frac{1}{(3+\nu)} \left\{ 2 \log 2 - 1 + \frac{1+\nu}{1-\nu} \left(2 \log 2 - \frac{\pi^2}{12} \right) - \right. \\ & - \left. \sum_{m=2,4,6,\dots}^{\infty} \left[\frac{1}{m(m-1)} + \frac{2(1+\nu)}{m^2(m-1)(1-\nu)} - \frac{\left(\frac{\pi}{a}\right)^2}{m(m+1)} \right] \left(\frac{\pi}{a}\right)^m \cos m\theta \right\} \end{aligned} \quad \dots (3.33)$$

as obtained by Timoshenko (1959) in the corresponding small deflection problem for a plate without foundation and supported at two points on the boundary.

The normalised constant α can be determined from Eqs.(3.7) and (3.3). Since we are interested only in the lateral displacement w , the radial and cross radial displacements u and v have been eliminated by choosing suitable expressions for u and v , compatible with their boundary conditions and integrating over the whole area of the plate. The radial and cross radial displacements have been assumed in the forms

$$u = \sum U(r) \cos m\theta \quad \dots (3.34)$$

$$v = \sum V(r) \sin m\theta \quad \dots (3.35)$$

subject to the boundary conditions $U(a) = V(a) = 0$

Multiplying both sides of the Eq.(3.7) by $r dr d\theta$ and integrating between the limits 0 to a and 0 to 2π , one gets

$$\begin{aligned} & \int_0^a \int_0^{2\pi} r U'(r) \cos m\theta dr d\theta + \int_0^a \int_0^{2\pi} U(r) \cos m\theta dr d\theta \\ & + \int_0^a \int_0^{2\pi} m V(r) \cos m\theta dr d\theta + \frac{1}{2} \int_0^a \int_0^{2\pi} r \left(\frac{\partial w}{\partial r}\right)^2 dr d\theta \\ & + \frac{1}{2} \int_0^a \int_0^{2\pi} \frac{1}{r} \left(\frac{\partial w}{\partial \theta}\right)^2 dr d\theta = \int_0^a \int_0^{2\pi} \frac{\alpha^2 h^2}{12} r dr d\theta \end{aligned}$$

After evaluating the integrals the following equation leading to α is obtained.

$$\begin{aligned}
 & -\frac{1}{2} A_0'^2 P_1^2 a^2 \left\{ \frac{1}{4} [I_0(P_1 a) + I_2(P_1 a)]^2 - \left[1 + \frac{1}{P_1^2 a^2} \right] I_1^2(P_1 a) \right\} - \\
 & -\frac{1}{2} B_0'^2 P_2^2 a^2 \left\{ \frac{1}{4} [I_0(P_2 a) + I_2(P_2 a)]^2 - \left[1 + \frac{1}{P_2^2 a^2} \right] I_1^2(P_2 a) \right\} + \\
 & + 2 A_0' B_0' P_1 P_2 \frac{a}{P_2^2 - P_1^2} \left[-\frac{1}{2} P_1 I_1(P_2 a) \{ I_0(P_1 a) + I_2(P_1 a) \} + \right. \\
 & \left. + \frac{1}{2} P_2 I_1(P_1 a) \{ I_0(P_2 a) + I_2(P_2 a) \} \right] + \\
 & + \sum_{m=2,4,6,\dots}^{\infty} \left[A_m^2 P_1^2 \left\{ -\frac{1}{8} a^2 \left[\frac{1}{4} \{ I_{m-2}(P_1 a) + I_m(P_1 a) \}^2 - \right. \right. \right. \\
 & \left. \left. \left. - \left\{ 1 + \frac{(m-1)^2}{P_1^2 a^2} \right\} I_m^2(P_1 a) \right] - \right. \right. \\
 & \left. \left. - \frac{1}{8} a^2 \left[\frac{1}{4} \{ I_m(P_1 a) + I_{m+2}(P_1 a) \}^2 - \left\{ 1 + \frac{(m+1)^2}{P_1^2 a^2} \right\} I_{m+1}^2(P_1 a) \right] + \right. \right. \\
 & \left. \left. + \frac{1}{2} \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \left(\frac{P_1}{2} \right)^{2m+2n+2t} \cdot \Phi \right\} \frac{1}{2} + \right.
 \end{aligned}$$

$$\begin{aligned}
& + B_m^2 P_2^2 \left\{ -\frac{1}{8} a^2 \left[\frac{1}{4} \left\{ I_{m-2}(P_2 a) + I_m(P_2 a) \right\}^2 + \left\{ 1 + \frac{(m-1)^2}{P_2^2 a^2} \right\} I_{m-1}^2(P_2 a) \right] - \right. \\
& - \frac{1}{8} a^2 \left[\frac{1}{4} \left\{ I_m(P_2 a) + I_{m+2}(P_2 a) \right\}^2 + \left\{ 1 + \frac{(m+1)^2}{P_2^2 a^2} \right\} I_{m+1}^2(P_2 a) \right] + \\
& + \frac{1}{2} \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \left(\frac{P_2}{2} \right)^{2m+2n+2t} \cdot \Phi \left. \right\} \frac{1}{2} + \\
& + \frac{1}{2} A_m B_m P_1 P_2 \left\{ \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \left[\frac{\left(\frac{P_1}{2} \right)^{m+2n-1} \left(\frac{P_2}{2} \right)^{m+2t-1} a^{2m+2n+2t}}{(2m+2n+2t)! n! t! \Gamma(m+n) \Gamma(m+t)} + \right. \right. \\
& + \frac{\left(\frac{P_1}{2} \right)^{m+2n-1} \left(\frac{P_2}{2} \right)^{m+2t+1} a^{2m+2n+2t+2}}{(2m+2n+2t+2)! n! t! \Gamma(m+t) \Gamma(m+n+2)} + \\
& \left. \left. + \frac{\left(\frac{P_1}{2} \right)^{m+2n+1} \left(\frac{P_2}{2} \right)^{m+2t+1} a^{2m+2n+2t+4}}{(2m+2n+2t+4)! n! t! \Gamma(m+n+2) \Gamma(m+t+2)} \right] \right\} \frac{1}{2} + \\
& + A_m^2 \left\{ \sum_{n=0}^{\infty} \left(\frac{P_1}{2} \right)^{2m+4n} \cdot \Phi_1 + \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \left(\frac{P_1}{2} \right)^{2m+2n+2t} \cdot \Psi \right\} \frac{m^2}{2} + \\
& + B_m^2 \left\{ \sum_{n=0}^{\infty} \left(\frac{P_2}{2} \right)^{2m+4n} \cdot \Phi_1 + \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \left(\frac{P_2}{2} \right)^{2m+2n+2t} \cdot \Psi \right\} \frac{m^2}{2} + \\
& + 2 A_m B_m \left\{ \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \left(\frac{P_1}{2} \right)^{m+2n} \left(\frac{P_2}{2} \right)^{m+2t} \cdot \Psi \right\} \frac{m^2}{2} = \frac{\alpha^2 h^2 a^2}{12}
\end{aligned}$$

where

$$\Phi = \frac{a^{2m+2n+2t+2}}{(2m+2n+2t+2) \Gamma(m+n) \Gamma(m+t+2)}$$

$$\Phi_1 = \frac{a^{2m+4n}}{(2m+4n) \{ \Gamma(m+n+1) \}^2}$$

$$\Psi = \frac{a^{2m+2n+2t}}{(2m+2n+2t) \Gamma(m+n+1) \Gamma(m+t+1)}$$

Thus the deflection, w is completely determined. The expressions for the bending and twisting moments are

$$M_{\eta} = -D [P_1^2 (A'_0 + A_0) I_0''(P_1 \eta) + P_2^2 (B'_0 + B_0) I_0''(P_2 \eta)]$$

$$+ \sum_{m=2,4,6}^{\infty} [P_1^2 A_m I_m''(P_1 \eta) + P_2^2 B_m I_m''(P_2 \eta)] \cos m\theta$$

$$+ \nu \left\{ \frac{P_1}{\eta} (A'_0 + A_0) I_1'(P_1 \eta) + \frac{P_2}{\eta} (B'_0 + B_0) I_1'(P_2 \eta) \right.$$

$$+ \frac{1}{\eta} \sum_{m=2,4,6}^{\infty} [P_1 A_m I_m'(P_1 \eta) + P_2 B_m I_m'(P_2 \eta)] \cos m\theta$$

$$\left. - \frac{1}{\eta^2} \sum_{m=2,4,6}^{\infty} m^2 [A_m I_m(P_1 \eta) + B_m I_m(P_2 \eta)] \cos m\theta \right\}$$

... (3.37)

$$\begin{aligned}
M_{\theta} = & -D \left[\frac{P_1}{\pi} (A'_0 + A_0) I'_1(P_1\pi) + \frac{P_2}{\pi} (B'_0 + B_0) I'_1(P_2\pi) \right. \\
& + \frac{1}{\pi} \sum_{m=2,4,6}^{\infty} \left[P_1 A_m I'_m(P_1\pi) + P_2 B_m I'_m(P_2\pi) \right] \cos m\theta \\
& - \frac{1}{\pi^2} \sum_{m=2,4,6}^{\infty} m^2 \left[A_m I_m(P_1\pi) + B_m I_m(P_2\pi) \right] \cos m\theta \\
& + \nu \left\{ P_1^2 (A'_0 + A_0) I''_0(P_1\pi) + P_2^2 (B'_0 + B_0) I''_0(P_2\pi) \right. \\
& \left. + \sum_{m=2,4,6}^{\infty} \left[P_1^2 A_m I''_m(P_1\pi) + P_2^2 B_m I''_m(P_2\pi) \right] \cos m\theta \right\} \\
& \dots (3.38)
\end{aligned}$$

$$\begin{aligned}
M_{r\theta} = & (1-\nu) D \left[-\frac{1}{\pi} \sum_{m=2,4,6}^{\infty} m \left[P_1 A_m I'_m(P_1\pi) + P_2 B_m I'_m(P_2\pi) \right] \sin m\theta \right. \\
& \left. + \frac{1}{\pi^2} \sum_{m=2,4,6}^{\infty} m \left[A_m I_m(P_1\pi) + B_m I_m(P_2\pi) \right] \sin m\theta \right] \\
& \dots (3.39)
\end{aligned}$$

The stresses due to M_r , M_{θ} and $M_{r\theta}$ can be calculated from the expressions

$$(\sigma_r) = \frac{6M_r}{h^2}, \quad (\sigma_{\theta}) = \frac{6M_{\theta}}{h^2}, \quad T_{r\theta} = \frac{6M_{r\theta}}{h^2} \quad \dots (3.40)$$

RESULTS

To obtain deflection for a given value of plate radius 'a' and foundation modulus 'K_F' one has to start from the Eq.(3.36) with an assumed value of 'α' in order to obtain the corresponding value of the load function $\frac{qa^4}{Dh}$. Once this relationship is obtained the corresponding deflection $\frac{w}{h}$ can be calculated from Eq.(3.32). For a = 50 mm, h = 0.75 mm, ν = 0.3, and K_F = 80 deflections have been presented in Fig. 3.2.

An examination of the Eq.3.32 will reveal that the deflection $\frac{w}{h}$ depends on K_F, the plate radius 'a', and on the value of the angle, θ. For a given value of the load function Eq.(3.32) can be written as

$$\left(\frac{w}{h}\right)_{\substack{\eta=0 \\ \theta=0}} = K_1 \left(\frac{qa^4}{Dh}\right), \quad \left(\frac{w}{h}\right)_{\substack{\eta=a \\ \theta=\pi/2}} = K_2 \left(\frac{qa^4}{Dh}\right) \quad \dots (3.41)$$

where K₁ and K₂ are two numerical constants, K₂ being greater than K₁. Because of the reactive forces at the two points of support, deflection on the diameter at θ = 0 will be less than those on the diameter at θ = π/2. Maximum deflection will occur at the boundary at θ = ±π/2. Deflections according to the linear theory have also been plotted in Fig.3.2 and it is clear that the errors of the linear theory increases as the load increases. In order to study the variation of moments, Eqs.(3.37),(3.38) and (3.39) are plotted in Fig.3.3 for various values of (r/a) and for the angles at which they become maximum. It is observed that the maximum bending moments,

their magnitudes being unequal, are developed at $\eta = 3a/4$, $\theta = \pm \pi/2$ and the twisting moment is maximum at $r = a$, $\theta = \pm \pi/4$, $\pm 3\pi/4$.

As the plate must be in equilibrium on the supports, the foregoing analysis for two simple supports represents the worst condition when the deflections and stresses are maximum for a given load function. With the increase in the number of supports, w_1 in Eq.3.8 decreases. For an infinitely large number of supports, w in Eq.3.8 will approach to w_0 in the limit and the point of maximum bending moments will shift to the centre of the plate, $(M_r)_{\max}$ being equal to $(M_0)_{\max}$ in that case.

The present study can be extended to any number of supports, provided the supports are so chosen as not to disturb the equilibrium of the plate. For example, if three equidistant supports are chosen $\psi_1 = 0$, $\psi_2 = 120^\circ$, $\psi_3 = 240^\circ$, the differential equations together with the boundary conditions remaining unchanged. If the plate is clamped on the supports, the boundary conditions and the concentrated reactions at the supports will change totally demanding a separate investigation.

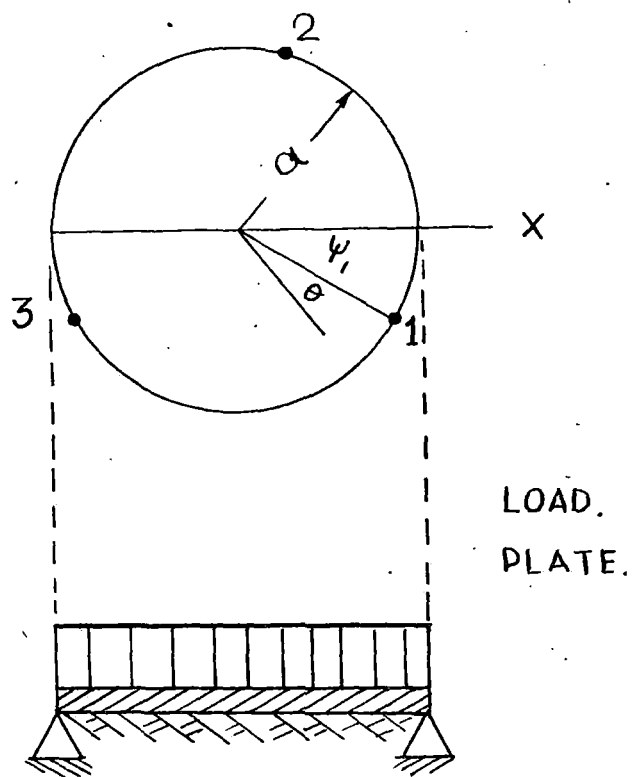


FIG.31 CIRCULAR PLATE ON
FOUNDATION.

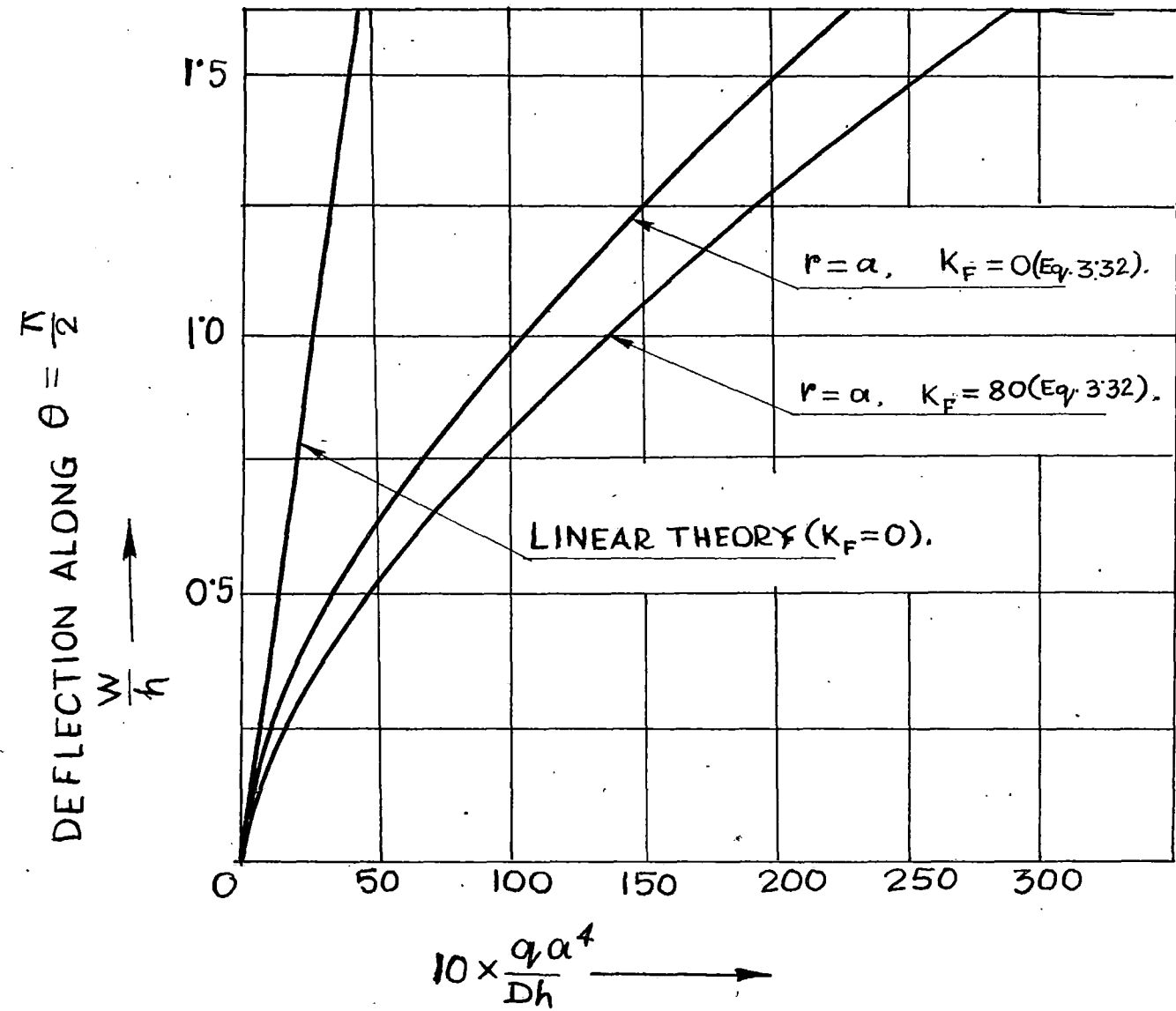


FIG.32 LOAD DEFLECTION CURVE.

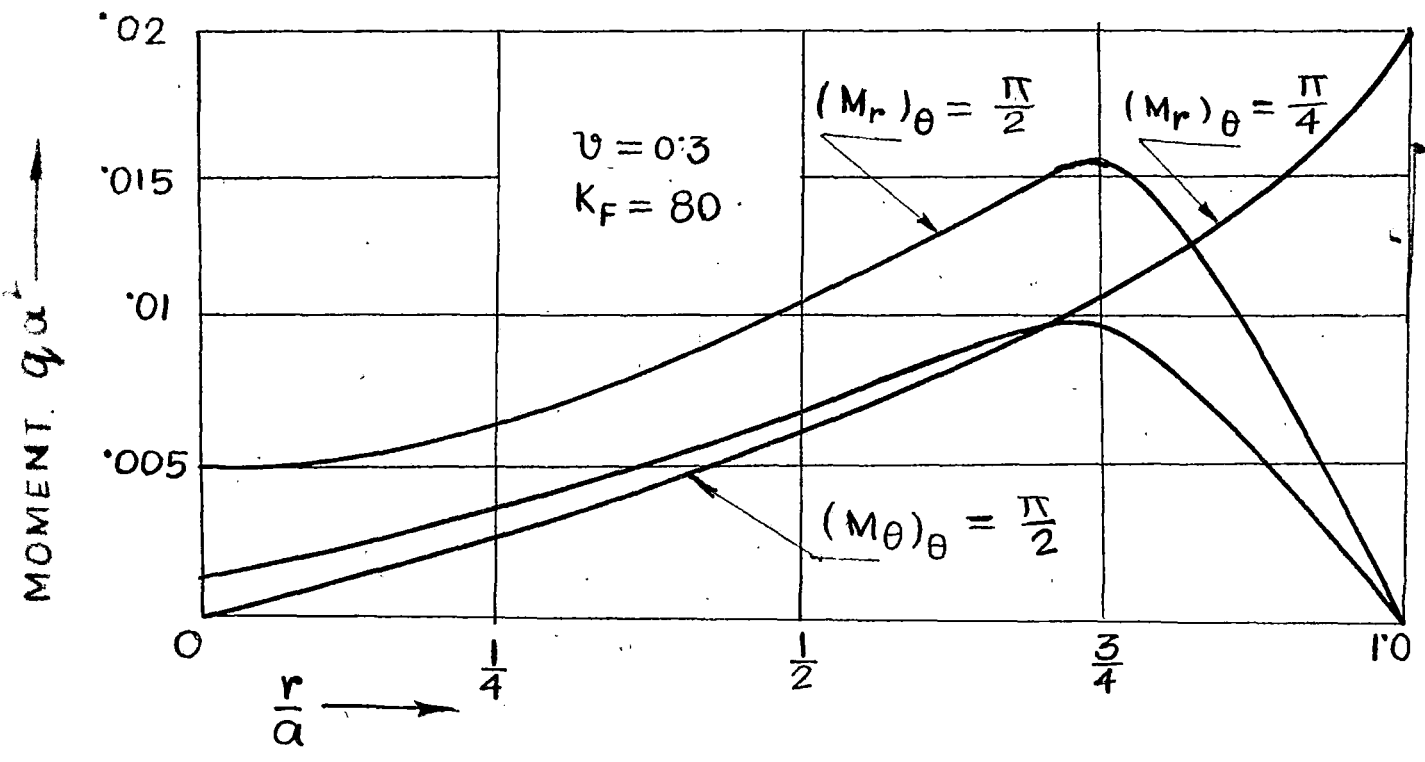


FIG. 3.3 MOMENT CURVE

LARGE DEFLECTION OF A CIRCULAR PLATE ON ELASTIC
FOUNDATION UNDER A CONCENTRATED LOAD AT THE
CENTRE*

PAPER - II

INTRODUCTION :

In this paper the large deflection of a clamped circular plate on elastic foundation of Winkler type under a concentrated load at the centre of the plate has been investigated following Berger's approximate method. The deflections are obtained involving Bessel's functions and the theoretical results have been verified experimentally. Graphs are plotted both for theoretical and experimental values for a given value of foundation modulus. The theoretical results have been compared with other known results.

FORMULATION OF PROBLEM

For moderately large deflections of plates under a concentrated load Eq.(3.4) is modified to

$$V = \frac{D}{2} \iint \left[(\nabla^2 w)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu) \left\{ \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} + \frac{K}{D} w^2 \right] dx dy \quad \dots (3.42)$$

*Published in the Journal of Applied Mechanics, Trans.
A.S.M.E., Vol. 42, No. 2, 1975.

Applying Euler's variational method to Eq.(3.42) and using polar co-ordinates the following differential equations have been obtained

$$\nabla^2 (\nabla^2 - \alpha^2) w + \frac{K}{D} w = 0 \quad \text{except at the load point.} \quad \dots (3.43)$$

$$\frac{u}{r} + \frac{du}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 = \frac{\alpha^2 h^2}{12} \quad \dots (3.44)$$

where

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \quad \text{and}$$

$\alpha =$ a constant given by Eq.(3.44)

Considering the radial stress and shearing stress on a concentric circular area of radius r , the concentrated load P at the centre, and since u and $\frac{dw}{dr}$ are both zero at the centre, one gets

$$\lim_{r \rightarrow 0} Dr \frac{d}{dr} \left[(\nabla^2 - \alpha^2) \right] = \frac{P}{2\pi} \quad \dots (3.45)$$

SOLUTION OF PROBLEM

To solve the problem for a circular plate of radius 'a' and thickness 'h' solution of Eq.(3.43) can be taken in the following convenient form

$$W = A I_0 (P_1 r) + B I_0 (P_2 r) + C \left[K_0 (P_1 r) - K_0 (P_2 r) \right] \quad \dots (3.46)$$

$$\text{where } P_1^2 + P_2^2 = \alpha^2 \quad \dots (3.47)$$

$$P_1^2 P_2^2 = \frac{K}{D} \quad \dots (3.48)$$

A, B, and C are constants to be determined from the boundary conditions

Boundary conditions for clamped edge are

$$(W)_{r=a} = 0 = \left(\frac{dw}{dr} \right)_{r=a} \quad \dots (3.49)$$

Imposing Eq.(3.45) on Eq.(3.43) one gets

$$C = \frac{P}{2 \pi D (P_2^2 - P_1^2)} \quad \dots (3.50)$$

Considering Eq.(3.46) and Eq.(3.49) one gets

$$A = C \left[\frac{\frac{1}{a} - K_1(P_1 a) P_1 I_0(P_2 a) - K_0(P_1 a) P_2 I_1(P_2 a)}{P_2 I_1(P_2 a) I_0(P_1 a) - P_1 I_1(P_1 a) I_0(P_2 a)} \right] \quad \dots (3.51)$$

$$B = C \left[\frac{\frac{1}{a} - K_0(P_2 a) P_1 I_1(P_1 a) - K_1(P_2 a) P_2 I_0(P_1 a)}{P_2 I_1(P_2 a) I_0(P_1 a) - P_1 I_1(P_1 a) I_0(P_2 a)} \right] \quad \dots (3.52)$$

Thus

$$W = C \left[\left\{ \frac{\frac{1}{a} - K_1(P_1 a) P_1 I_0(P_2 a) - K_0(P_1 a) P_2 I_1(P_2 a)}{P_2 I_1(P_2 a) I_0(P_1 a) - P_1 I_1(P_1 a) I_0(P_2 a)} \right\} I_0(P_1 r) \right. \\ + \left\{ \frac{\frac{1}{a} - K_0(P_2 a) P_1 I_1(P_1 a) - K_1(P_2 a) P_2 I_0(P_1 a)}{P_2 I_1(P_2 a) I_0(P_1 a) - P_1 I_1(P_1 a) I_0(P_2 a)} \right\} I_0(P_2 r) \\ \left. + \left\{ K_0(P_1 r) - K_0(P_2 r) \right\} \right] \quad \dots (3.53)$$

is determined completely.

Setting $r \rightarrow 0$ in Eq.(3.45) one gets the maximum deflection at the centre of the plate.

$$\text{Thus } W_{\text{max.}} = A + B + C \log_e \left[\frac{P_2}{P_1} \right] \quad \dots (3.54)$$

If $P_1 \rightarrow 0$, $P_2 \rightarrow \infty$ or $P_1 \rightarrow \infty$, $P_2 \rightarrow 0$, one gets the corresponding large deflection for isotropic circular plate not resting on elastic foundation as obtained by Basuli (1961) in the form

$$W = - \frac{P}{2\pi D \alpha^3 a I_1(\alpha a)} \left[I_0(\alpha a) - 1 - I_0(\alpha r) + \right. \\ \left. + I_0(\alpha r) \alpha a K_1(\alpha a) + \alpha a I_1(\alpha a) \log_e \left(\frac{r}{a} \right) + K_0(\alpha r) \right. \\ \left. + \alpha a I_1(\alpha a) \right] \quad \dots (3.55)$$

If $r \rightarrow 0$, Eq.(3.55) leads to

$$W_{\text{max.}} = - \frac{P}{2\pi D \alpha^3 a I_1(\alpha a)} \left[I_0(\alpha a) - 2 + \alpha a K_1(\alpha a) - \right. \\ \left. + \alpha a I_1(\alpha a) \log_e \left(\frac{\alpha a}{2} \right) \right] \quad \dots (3.56)$$

To determine the displacement u , one gets from Eq.(3.44)

$$u r = \frac{\alpha^2 a^2 r^2}{24} - \frac{1}{4} \int \left(\frac{dw}{dr} \right)^2 r dr + K^1 \quad \dots (3.57)$$

where K^1 is the constant of integration.

After evaluating the integrals and using the boundary condition $u \rightarrow 0$ as $r \rightarrow a$, the following equation determining u is obtained.

$$\begin{aligned}
 ur = & \frac{\alpha h r^2}{24} - \frac{1}{4} \left[A^2 P_1^2 \left\{ \frac{r I_0(P_1 r) I_1(P_1 r)}{P_1} + \frac{1}{2} I_1^2(P_1 r) r^2 - \frac{1}{2} I_0^2(P_1 r) r^2 \right\} + \right. \\
 & + B^2 P_2^2 \left\{ \frac{r I_0(P_2 r) I_1(P_2 r)}{P_2} + \frac{1}{2} I_1^2(P_2 r) r^2 - \frac{1}{2} I_0^2(P_2 r) r^2 \right\} + \\
 & + C^2 P_1^2 \left\{ \frac{1}{2} r^2 K_1^2(P_1 r) + \frac{1}{2} K_0^2(P_1 r) r^2 - \frac{r}{P_1} K_0(P_1 r) K_1(P_1 r) \right\} + \\
 & + C^2 P_2^2 \left\{ \frac{1}{2} r^2 K_1^2(P_2 r) + \frac{1}{2} K_0^2(P_2 r) r^2 - \frac{r}{P_2} K_0(P_2 r) K_1(P_2 r) \right\} + \\
 & + \frac{2ABP_1 P_2 r}{P_2^2 - P_1^2} \left\{ P_2 I_1(P_1 r) I_0(P_2 r) - P_1 I_1(P_2 r) I_0(P_1 r) \right\} - \\
 & - \frac{2C^2 P_1 P_2 r}{P_2^2 - P_1^2} \left\{ P_1 K_1(P_2 r) K_0(P_1 r) - P_2 K_1(P_1 r) K_0(P_2 r) \right\} - \\
 & - \frac{2ACP_1 P_2 r}{P_2^2 - P_1^2} \left\{ P_2 I_1(P_1 r) K_0(P_2 r) + P_1 K_1(P_2 r) I_0(P_1 r) \right\} - \\
 & - \frac{2BC P_1 P_2 r}{P_2^2 - P_1^2} \left\{ P_1 I_1(P_2 r) K_0(P_1 r) + P_2 K_1(P_1 r) I_0(P_2 r) \right\} + \\
 & + BC \left\{ P_2^2 r^2 \left[I_1(P_2 r) K_1(P_2 r) + I_0(P_2 r) K_0(P_2 r) \right] - \right.
 \end{aligned}$$

$$\begin{aligned}
& - P_2 r \left[I_1(P_2 r) K_0(P_2 r) - I_0(P_2 r) K_1(P_2 r) \right] \} \\
& - AC \left\{ P_1^2 r^2 \left[I_1(P_1 r) K_1(P_1 r) + I_0(P_1 r) K_0(P_1 r) \right] - \right. \\
& \left. - P_1 r \left[I_1(P_1 r) K_0(P_1 r) - I_0(P_1 r) K_1(P_1 r) \right] \right\} + \\
& + A^2 P_1^2 \left[\frac{-a I_0(P_1 a) I_1(P_1 a)}{P_1} + \frac{1}{2} I_1^2(P_1 a) a^2 - \frac{1}{2} I_0^2(P_1 a) a^2 \right] + \\
& + B^2 P_2^2 \left[\frac{-a I_0(P_2 a) I_1(P_2 a)}{P_2} + \frac{1}{2} I_1^2(P_2 a) a^2 - \frac{1}{2} I_0^2(P_2 a) a^2 \right] + \\
& + C^2 P_1^2 \left[\frac{1}{2} a^2 K_1^2(P_1 a) - \frac{1}{2} K_0^2(P_1 a) a^2 - \frac{a}{P_1} K_0(P_1 a) K_1(P_1 a) \right] + \\
& + C^2 P_2^2 \left[\frac{1}{2} a^2 K_1^2(P_2 a) - \frac{1}{2} K_0^2(P_2 a) a^2 - \frac{a}{P_2} K_0(P_2 a) K_1(P_2 a) \right] + \\
& + \frac{2ABP_1 P_2 a}{P_2^2 - P_1^2} \left[P_2 I_1(P_1 a) I_0(P_2 a) - P_1 I_1(P_2 a) I_0(P_1 a) \right] - \\
& - \frac{2C^2 P_1 P_2 a}{P_2^2 - P_1^2} \left[P_1 K_1(P_2 a) K_0(P_1 a) - P_2 K_1(P_1 a) K_0(P_2 a) \right] - \\
& - \frac{2AC P_1 P_2 a}{P_2^2 - P_1^2} \left[P_2 I_1(P_1 a) K_0(P_2 a) + P_1 K_1(P_2 a) I_0(P_1 a) \right] - \\
& - \frac{2BC P_1 P_2 a}{P_2^2 - P_1^2} \left[P_1 I_1(P_2 a) K_0(P_1 a) + P_2 K_1(P_1 a) I_0(P_2 a) \right] +
\end{aligned}$$

$$\begin{aligned}
& + BC \sqrt{-P_2^2} a^2 \left\{ I_1(P_2 a) K_1(P_2 a) + I_0(P_2 a) K_0(P_2 a) \right\} - \\
& - P_2 a \left\{ I_1(P_2 a) K_0(P_2 a) - I_0(P_2 a) K_1(P_2 a) \right\} \sqrt{-} - \\
& - AC \sqrt{-P_1^2} a^2 \left\{ I_1(P_1 a) K_1(P_1 a) + I_0(P_1 a) K_0(P_1 a) \right\} - \\
& - P_1 a \left\{ I_1(P_1 a) K_0(P_1 a) - I_0(P_1 a) K_1(P_1 a) \right\} \sqrt{-} - \frac{\alpha^2 h^2 a^2}{12} \dots (3.53)
\end{aligned}$$

Also as $r \rightarrow 0$, $u \rightarrow 0$ from symmetry. Thus the equation for α is given by

$$\begin{aligned}
\frac{\alpha^2 h^2 a^2}{12} &= A^2 P_1^2 \sqrt{-} \frac{a I_0(P_1 a) I_1(P_1 a)}{P_1} + \frac{1}{2} I_1^2(P_1 a) a^2 - \frac{1}{2} I_0^2(P_1 a) a^2 \sqrt{-} + \\
& + B^2 P_2^2 \sqrt{-} \frac{a I_0(P_2 a) I_1(P_2 a)}{P_2} + \frac{1}{2} I_1^2(P_2 a) a^2 - \frac{1}{2} I_0^2(P_2 a) a^2 \sqrt{-} + \\
& + C^2 P_1^2 \sqrt{-} \frac{1}{2} a^2 K_1^2(P_1 a) - \frac{1}{2} K_0^2(P_1 a) a^2 - \frac{a}{P_1} K_0(P_1 a) K_1(P_1 a) \sqrt{-} + \\
& + C^2 P_2^2 \sqrt{-} \frac{1}{2} a^2 K_1^2(P_2 a) - \frac{1}{2} K_0^2(P_2 a) a^2 - \frac{a}{P_2} K_0(P_2 a) K_1(P_2 a) \sqrt{-} + \\
& + \frac{2AB P_1 P_2 a}{P_2^2 - P_1^2} \sqrt{-} P_2 I_1(P_1 a) I_0(P_2 a) - P_1 I_1(P_2 a) I_0(P_1 a) \sqrt{-} - \\
& - \frac{2C^2 P_1 P_2 a}{P_2^2 - P_1^2} \sqrt{-} P_1 K_1(P_2 a) K_0(P_1 a) - P_2 K_1(P_1 a) K_0(P_2 a) \sqrt{-} -
\end{aligned}$$

$$\begin{aligned}
& - \frac{2AC P_1 P_2 a}{P_2^2 - P_1^2} \left[P_2 I_1(P_1 a) K_0(P_2 a) + P_1 K_1(P_2 a) I_0(P_1 a) \right] - \\
& - \frac{2BC P_1 P_2 a}{P_2^2 - P_1^2} \left[P_1 I_1(P_2 a) K_0(P_1 a) + P_2 K_1(P_1 a) I_0(P_2 a) \right] + \\
& + BC \left[P_2^2 a^2 \left\{ I_1(P_2 a) K_1(P_2 a) + I_0(P_2 a) K_0(P_2 a) \right\} - \right. \\
& - P_2 a \left\{ I_1(P_2 a) K_0(P_2 a) - I_0(P_2 a) K_1(P_2 a) \right\} \left. \right] - \\
& - AC \left[P_1^2 a^2 \left\{ I_1(P_1 a) K_1(P_1 a) + I_0(P_1 a) K_0(P_1 a) \right\} - \right. \\
& - P_1 a \left\{ I_1(P_1 a) K_0(P_1 a) - I_0(P_1 a) K_1(P_1 a) \right\} \left. \right] + \\
& + \frac{2ACP_1^2}{P_2^2 - P_1^2} + \frac{2BC P_2^2}{P_2^2 - P_1^2} + AC \\
& - BC - C^2 \left[1 + \frac{P_2^2 + P_1^2}{P_2^2 - P_1^2} \log_e \frac{P_1}{P_2} \right] \dots (3.59)
\end{aligned}$$

If $P_1 \rightarrow 0$, $P_2 \rightarrow \alpha$ or $P_1 \rightarrow \alpha$, $P_2 \rightarrow 0$, Eq.(3.59) leads to

$$\begin{aligned}
\left[\frac{P a^2}{\lambda Dh} \right]^2 &= \frac{\frac{1}{3} (\alpha a)^6}{\left[\gamma + \log_e \frac{\alpha a}{2} - \frac{I_0(\alpha a) + \alpha a K_1(\alpha a) - 2}{\alpha a I_1(\alpha a)} \right]} \\
&= \frac{1}{3} \left\{ \frac{I_0(\alpha a) - 1}{I_1(\alpha a)} \right\}^2 \dots (3.60)
\end{aligned}$$

The above result was obtained by Basuli (1961) in determining the large deflection of a clamped circular plate under a concentrated load at the centre without any elastic foundation.

EXPERIMENTAL VERIFICATION

Deflection was measured experimentally by an apparatus shown in Fig.3.4. The thin circular mild steel plate (A) of 0.75 mm thickness was placed between two washers and rigidly clamped over the base (B) by means of eight bolts and a thick mild steel ring (M) of 6.0 mm thickness, the diameter of the cavity (H) being 160 mm. Load was applied at the centre of the plate through the load spindle (C). The load spindle was accurately finished and made to move vertically through the bush (D) which was closely fitted at the centre of the frame (E). The lower face of the load spindle collar (F) was accurately ground and polished. Loads were applied over the load pan (G) and the corresponding deflection was measured by means of the dial indicator (J) placed against the lower face of the collar. The dial indicator used can read upto 12 mm with an accuracy of 0.0125 mm. The readings were taken first with the cavity empty and these readings correspond to $K_p = 0$. The cavity was then completely filled in with sand and the experiment was repeated. The value of the nondimensional modulus for sand used was determined experimentally to be $K_p = 430$.

RESULTS

To calculate deflections for plates resting on elastic foundation one has to start from Eq.(3.59) with assumed values of $\frac{K}{D}$ and α leading to the corresponding values of the load function.

Once the load function is obtained, the deflection is determined from Eq.(3.54). The theoretical results for plates with elastic foundation and without elastic foundation have been verified by the experimental results. The corresponding graphs are shown in the Fig. 3.5. Results according to Timoshenko and Kireger ^(1959, P.415) and Schmidt ⁽¹⁹⁶⁸⁾ corresponding to $K_f=0$ have also been presented for comparison in the same graph.

It is observed from Fig. 3.5 that the deflections calculated both for plates with and without any elastic foundation are in good agreement with the values obtained experimentally. This justifies the assumptions Berger made that the second strain invariant of the middle plane can be neglected for practical purpose. It is also observed that the deviation of the experimental curve from the theoretical one is more with higher values of the load function. This is due to the fact that the assumption of K to be proportional to the deflection is not strictly correct as K varies nonlinearly with the deflection at the higher values of loads.

The equation for bending moment is given by

$$\begin{aligned}
 M_r = & - D \left[A P_1^2 I_0(P_1 r) + B P_2^2 I_0(P_2 r) - (1 - \nu) \frac{1}{r} A P_1 I_1(P_1 r) - \right. \\
 & - (1 - \nu) \frac{1}{r} B P_2 I_1(P_2 r) + C \left\{ P_1^2 K_0(P_1 r) - P_2^2 K_0(P_2 r) \right. \\
 & \left. \left. + (1 - \nu) \frac{1}{r} P_1 K_1(P_1 r) - (1 - \nu) \frac{1}{r} P_2 K_1(P_2 r) \right\} \right] \quad \dots (3.61)
 \end{aligned}$$

Bending moment at a point not very close to the centre of the plate may be calculated with the help of Eq.(3.61) and the corresponding bending stress is given by

$$\sigma_n = - \frac{6}{h^2} (M_r) \quad \dots (3.62)$$

On examination of the Eq.(3.61) it is observed that the bending moment becomes infinite at the centre of the plate. Hence Eq.(3.61) cannot be used to determine the maximum bending stress which occurs at the centre of the plate. The concentrated load P at the centre of the plate may be assumed to be uniformly distributed over a concentric circular area of a very small radius, c . The shear stress is given by

$$(\tau)_{rz} = \frac{6Q}{h^3} \left(\frac{h^2}{4} - z^2 \right) \quad \dots (3.63)$$

$$Q = \text{Shear force} = -D \frac{d}{dr} \left[\nabla^2 (\nabla^2 - \alpha^2) \right] w.$$

From the theory of bending of rectangular beams $(\tau)_{rz}$ is zero on the outer surfaces of the plate. The maximum compressive stress at the centre of the upper face of the plate is then given by

$$\sigma = -\sigma_1 - \frac{P}{\pi c^2} \left[\frac{1+2\nu}{2} - (1+\nu)\beta \right] \quad \dots (3.64)$$

where σ_1 is the bending stress calculated by Eq.(3.62) for points very close to the load point and β is a numerical factor depending on $\frac{2c}{h}$, the ratio of the diameter of the loaded area to the thickness of the plate [Timoshenko (1959), P.70]. The maximum tensile stress will occur at the centre of the inner surface of the plate and its value will be less than the compressive stress at the corresponding point on the upper surface. There will be a high concentration of stress surrounding the load point.

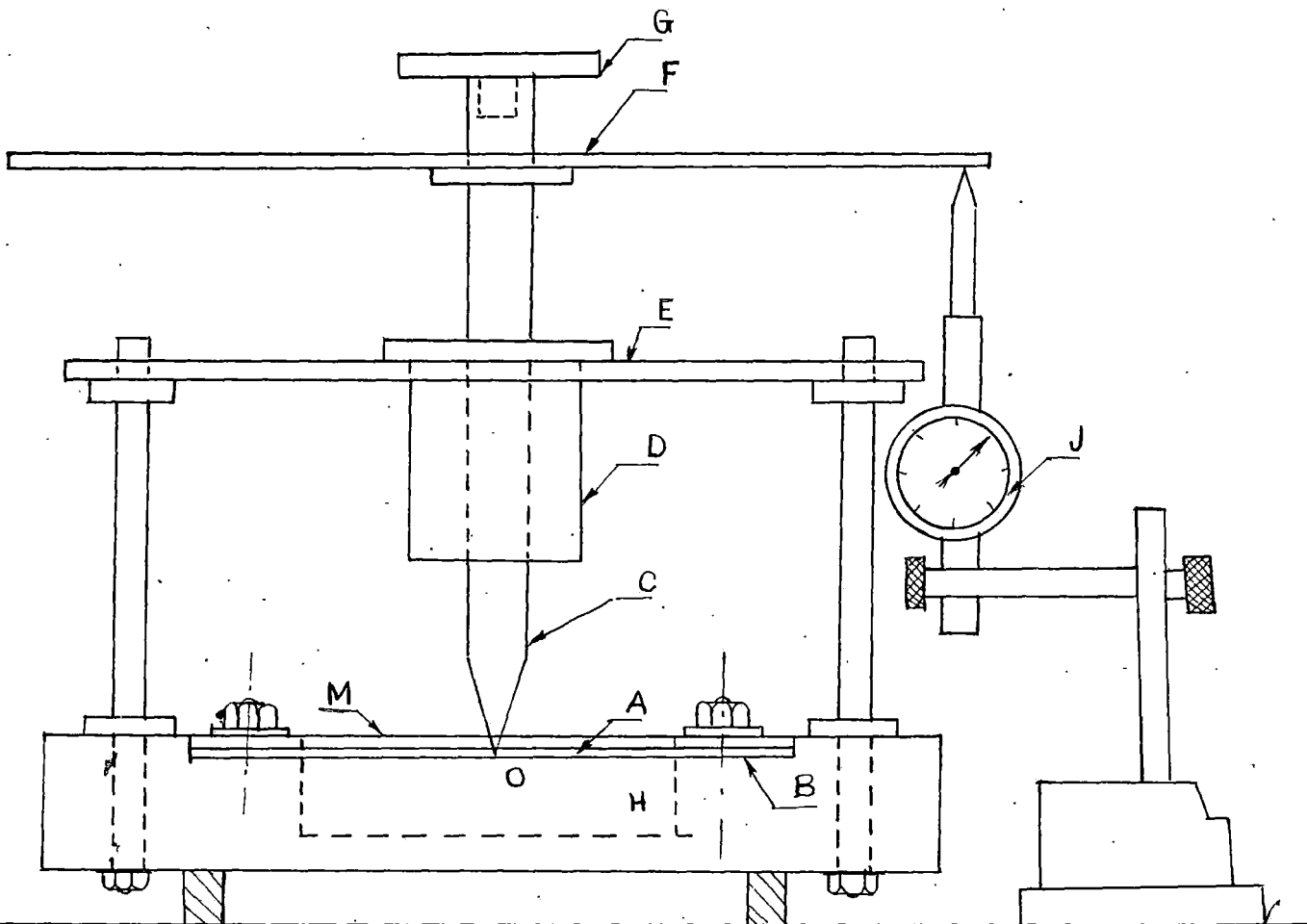


FIG. 34

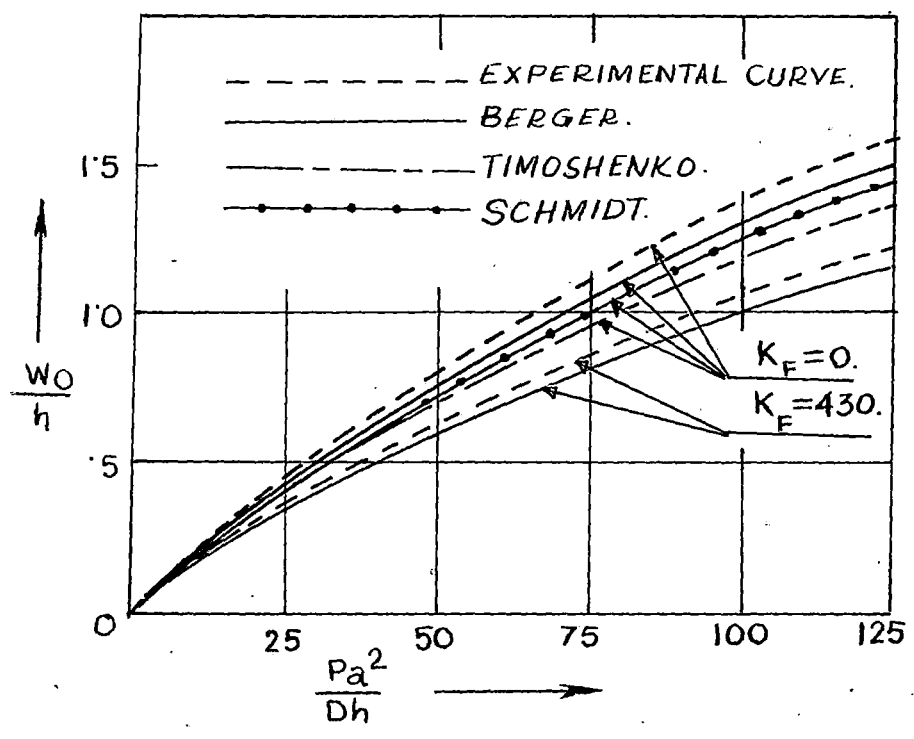


FIG.35 DEFLECTION CURVE.

LARGE DEFLECTION OF A CIRCULAR PLATE ON ELASTIC
FOUNDATION UNDER SYMMETRICAL LOADS *

PAPER - III

INTRODUCTION

Following Berger's approximate method, large deflections of clamped circular plates on elastic foundation and subjected to some special types of symmetrical transverse loads, which are functions of the distance from the centre of the plates, distributed over a concentric circular portion of the plates have been investigated in this paper. Deflections, bending moments and bending stresses are calculated for different values of foundation modulus and these are presented in the form of graphs. The results have been compared with other known results.

SOLUTION OF PROBLEM

Let us consider a clamped circular plate of radius 'a'. The centre of the plate is taken as the origin. For moderately large deflection of plates the governing differential equations in polar co-ordinates are

$$\nabla^4 W - \alpha^2 \nabla^2 W + \frac{k}{D} W = \frac{q}{D} \quad \dots (3.65)$$

$$\frac{u}{r} + \frac{du}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 = \frac{\alpha^2 h^2}{12} \quad \dots (3.66)$$

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where

$$\nabla^2 \equiv \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}$$

If there be a symmetrical distribution of transverse load varying as $(b^2 - r^2)^\lambda$, $(\lambda > -1)$ over a concentric circular area of radius $b < a$,

$$\begin{aligned} \frac{q}{D} = \frac{\Phi(r)}{D} = f(r) &= c(b^2 - r^2)^\lambda \quad \text{when } r < b < a \quad \dots (3.67) \\ &= 0 \quad \text{when } b < r < a \end{aligned}$$

Eq.(3.65) now is written as

$$(\nabla^2 - \alpha^2) \nabla^2 W + \frac{K}{D} W = f(r) \quad \dots (3.68)$$

The boundary conditions for clamped edges are

$$(W)_{r=a} = 0 = \left(\frac{dW}{dr} \right)_{r=a} \quad \dots (3.69)$$

Let us assume the deflection W in the following form

$$W = \sum_{s=1}^{\infty} A_s \left[J_0(P_s r) - J_0(P_s a) \right] \quad \dots (3.70)$$

where J_0

is the Bessel function of the first kind and zero order and P_s is the s -th root of $J_1(Pa) = 0$, J_1 being the Bessel function of the first kind and first order.

It is evident that the boundary conditions for clamped edges are satisfied by the above configurations of W .

Since

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right] J_0(P_S r) = -P_S^2 J_0(P_S r),$$

putting Eq.(3.70) into Eq.(3.68) one gets,

$$\sum_{S=1}^{\infty} A_S \left[P_S^4 J_0(P_S r) + \alpha^2 P_S^2 J_0(P_S r) + \frac{K}{D} \left\{ J_0(P_S r) - J_0(P_S a) \right\} \right] = f(r) \quad \dots (3.71)$$

If it is possible to expand $f(r)$ in a series of Bessel function, one gets,

$$A_S \int_0^a \left[P_S^2 (P_S^2 + \alpha^2) J_0^2(P_S r) + A_S \frac{K}{D} \left\{ J_0^2(P_S r) - J_0(P_S a) J_0(P_S r) \right\} \right] r dr = \int_0^a f(r) J_0(P_S r) r dr$$

$$\text{Or } A_S \frac{a^2}{2} \left[P_S^2 (P_S^2 + \alpha^2) + \frac{K}{D} \right] J_0^2(P_S a) = \int_0^a f(r) \times J_0(P_S r) r dr \quad \dots (3.72)$$

Putting $r = b \sin \theta$ and $f(r) = C (b^2 - r^2)^\lambda$ in the integral of Eq.(3.72), one gets

$$\begin{aligned} \int_0^a f(r) J_0(P_S r) r dr &= \int_0^b C r (b^2 - r^2)^\lambda J_0(P_S r) dr \\ &= C b^{2(\lambda + 1)} \int_0^{\frac{\pi}{2}} \sin \theta \cos^{2\lambda + 1} \theta \times J_0(P_S b \sin \theta) d\theta \end{aligned}$$

$$= \frac{C(2b)^{2(\lambda+1)} J_{\lambda+1}(P_s b) 2^{\lambda} (\lambda+1)}{(P_s b)^{\lambda+1}} \dots (3.73)$$

This is a special form of Sonaine's first definite integral containing Bessel function [Watson (1952)], where $\lambda > -1$.

Substituting the value obtained from Eq.(3.73) in Eq.(3.72) one gets after simplification

$$A_3 = \frac{C(2b)^{\lambda+1} J_{\lambda+1}(P_s b) \Gamma(\lambda+1)}{a^2 \int_{P_s}^{\lambda+3} (P_s^2 + \alpha^2) + P_s^{\lambda+1} \frac{K}{D} \int J_0^2(P_s a)} \dots (3.74)$$

Thus

$$W = \frac{C(2b)^{\lambda+1} \Gamma(\lambda+1)}{a^2} \sum_{s=1}^{\infty} \frac{J_{\lambda+1}(P_s b) \int J_0(P_s r) - J_0(P_s a)}{\int_{P_s}^{\lambda+3} (P_s^2 + \alpha^2) + P_s^{\lambda+1} \frac{K}{D} \int J_0^2(P_s a)}$$

is completely determined. ... (3.75)

The deflection will be maximum at the centre. From Eq.(3.75) maximum deflection is obtained putting $r = 0$.

Thus

$$W_{\max.} = \frac{C(2b)^{\lambda+1} \Gamma(\lambda+1)}{a^2} \sum_{s=1}^{\infty} \frac{J_{\lambda+1}(P_s b) \int 1 - J_0(P_s a)}{\int_{P_s}^{\lambda+3} (P_s^2 + \alpha^2) + P_s^{\lambda+1} \frac{K}{D} \int J_0^2(P_s a)}$$

... (3.76)

To determine the displacement u , one gets from Eq.(3.66) and Eq.(3.70)

$$\frac{du}{dr} + \frac{u}{r} = \frac{\alpha^2 h^2}{12} - \frac{1}{2} \left(\frac{dw}{dr} \right)^2$$

$$= \frac{\alpha^2 h^2}{12} - \frac{1}{4} \sum_{S=1}^{\infty} A_s^2 P_s^2 J_1^2(P_s r) - \frac{1}{4} \sum_{S=1}^{\infty} \sum_{\substack{m=1, \\ S \neq m}}^{\infty} A_s A_m P_s P_m \times$$

$$\times J_1(P_s r) J_1(P_m r) \dots (3.77)$$

Multiplying Eq.(3.77) by $r dr$ and integrating with respect to r , one gets,

$$u r = \frac{\alpha^2 h^2 r^2}{24} - \frac{1}{4} \sum_{S=1}^{\infty} A_s^2 P_s^2 \int_0^r \frac{-r^2}{2} \left\{ \left(1 - \frac{1}{P_s^2 r^2}\right) \times \right.$$

$$\times \left. J_1'^2(P_s r) + J_1^2(P_s r) \right\} \int_0^r - \frac{1}{4} \sum_{S=1}^{\infty} \sum_{\substack{m=1 \\ S \neq m}}^{\infty} A_s A_m P_s P_m \times$$

$$\times \int_0^r \left\{ \frac{P_s J_2(P_s r) J_1(P_m r) - P_m J_1(P_s r) J_2(P_m r)}{P_s^2 - P_m^2} \right\} \int_0^r +$$

+ K.....Eq.(3.78) where K is the constant of integration. Boundary condition imposed on u is

$$(u)_{r=a} = 0$$

Thus

$$K = \frac{1}{4} \sum_{S=1}^{\infty} A_s^2 P_s^2 a^2 J_0^2(P_s a) - \frac{\alpha^2 h^2 a^2}{24} \dots (3.79)$$

Also as $r \rightarrow 0$, $u \rightarrow 0$, from symmetry.

Therefore the equation to determine α leads to

$$\frac{\alpha^2 h^2 a^2}{6} = \sum_{S=1}^{\infty} A_s^2 P_s^2 a^2 J_0^2(P_s a) \dots (3.80)$$

Putting in Eq.(3.75)

$$\lambda = \frac{1}{2}, \quad K = 0$$

the deflection W is given by,

$$\begin{aligned} W &= \sum_{s=1}^{\infty} A_s \left[J_0(P_s r) - J_0(P_s a) \right] \\ &= \frac{2b^3 c}{a^2} \sum_{s=1}^{\infty} \frac{P(P_s b)}{P_s^2 (P_s^2 + \alpha^2) J_0^2(P_s a)} \left[J_0(P_s r) - J_0(P_s a) \right] \end{aligned} \quad \dots (3.81)$$

$$\text{where } P(P_s b) = \frac{1}{2} \left[1 - \frac{P_s^2 b^2}{2.5} + \frac{P_s^4 b^4}{2.4.5.7} - \dots \right]$$

As α tends to zero Eq.(3.81) leads to

$$W = \frac{2b^3 c}{a^2} \sum_{s=1}^{\infty} \frac{P(P_s b) \left[J_0(P_s r) - J_0(P_s a) \right]}{P_s^4 J_0^2(P_s a)} \quad \dots (3.82)$$

Eq.(3.82) is the result obtained by Sen (1935) in his corresponding small deflection problem.

Let us examine another type of transverse load function varying as $(r^4 - b^4)$ over a Concentric Circular area of $b < a$.

$$\text{In this case } f(r) = \begin{cases} C(r^4 - b^4) & (0 \leq r \leq b < a) \\ 0 & (b \leq r \leq a) \end{cases} \quad \dots (3.83)$$

Expanding $f(r)$ in the series of Bessel function and proceeding in the same manner one gets,

$$A_s = \frac{32bc (4 - P_s^2 b^2) J_1(P_s b)}{a^2 \left[P_s^7 (P_s^2 + \alpha^2) + P_s^5 \frac{K}{D} \right] J_0^2(P_s a)} \quad \dots (3.84)$$

Thus

$$W = \frac{32bc}{a^2} \sum_{s=1}^{\infty} \frac{(4-P_s^2 b^2) J_1(P_s b) [J_0(P_s r) - J_0(P_s a)]}{\left[P_s^7 (P_s^2 + \alpha^2) + P_s^5 \frac{K}{D} \right] J_0^2(P_s a)} \quad \dots (3.85)$$

is completely determined. Deflection is maximum at the centre.

The central deflection is obtained putting $r = 0$.

Thus

$$W_{\max.} = \frac{32bc}{a^2} \sum_{s=1}^{\infty} \frac{(4-P_s^2 b^2) J_1(P_s b) [1 - J_0(P_s a)]}{\left[P_s^7 (P_s^2 + \alpha^2) + P_s^5 \frac{K}{D} \right] J_0^2(P_s a)} \quad \dots (3.86)$$

By substituting the Eq.(3.84) in Eq.(3.78) and Eq.(3.80) one gets the equation for u and α respectively for the type of load function in Eq.(3.83).

The equation for bending moment is given by

$$M_r = -D \left[\frac{\partial^2 W}{\partial r^2} + \nu \left(\frac{1}{r} \frac{\partial W}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 W}{\partial \theta^2} \right) \right] \quad \dots (3.87)$$

Since W is a function of r only, the equation for bending moment becomes

$$M_r = -D \left[\frac{d^2 W}{dr^2} + \nu \left(\frac{1}{r} \frac{dW}{dr} \right) \right] \quad \dots (3.88)$$

Considering Eq.(3.75) and Eq.(3.88) the value for bending moment for the type of loading in Eq.(3.67) is obtained as

$$M_r = \frac{DC(2b)^{\lambda+1} \Gamma(\lambda+1)}{a^2} \sum_{s=1}^{\infty} \frac{P_s J_{\lambda+1}(P_s b)}{\left[P_s^{\lambda+3} (P_s^2 + \alpha^2) + P_s^{\lambda+1} \frac{K}{D} \right] J_0^2(P_s a)}$$

$$X \left[P_s J_0(P_s r) + (\nu - 1) \frac{1}{r} J_1(P_s r) \right] \quad \dots (3.89)$$

For clamped edge the bending moment will be maximum at the centre.

Thus

$$(M_r)_{\max.} = \frac{DC(2b)^{\lambda+1} \Gamma(\lambda+1)}{a^2} \times \sum_{s=1}^{\infty} \frac{P_s^2 J_{\lambda+1}(P_s b)}{\left[P_s^{\lambda+3}(P_s^2 + \alpha^2) + P_s^{\lambda+1} \frac{K}{D} \right] J_0^2(P_s a)} \times \left[1 + \frac{\nu-1}{2} \right] \dots (3.90)$$

The maximum bending stress is given by

$$(\sigma_r)_{\max.} = - \frac{6}{h^2} (M_r)_{\max.} \dots (3.91)$$

For the load function given by Eq.(3.67), the maximum deflections, maximum bending moments and maximum bending stress for $\lambda = 1$ are given as follows :

$$W_{\max.} = \frac{4b^2 c}{a^2} \sum_{s=1}^{\infty} \frac{J_2(P_s b) \left[1 - J_0(P_s a) \right]}{\left[P_s^4 (P_s^2 + \alpha^2) + P_s^2 \frac{K}{D} \right] J_0^2(P_s a)} \dots (3.92)$$

$$(M_r)_{\max.} = \frac{4DCb^2}{a^2} \sum_{s=1}^{\infty} \frac{J_2(P_s b) \left[1 + \frac{\nu-1}{2} \right]}{\left[P_s^2 (P_s^2 + \alpha^2) + \frac{K}{D} \right] J_0^2(P_s a)} \dots (3.93)$$

$$(\sigma_r)_{\max.} = - \frac{24DCb^2}{a^2 h^2} \sum_{s=1}^{\infty} \frac{J_2(P_s b) \left[1 + \frac{\nu-1}{2} \right]}{\left[P_s^2 (P_s^2 + \alpha^2) + \frac{K}{D} \right] J_0^2(P_s a)} \dots (3.94)$$

If $\alpha \rightarrow 0$ and $K = 0$, the corresponding results for small deflections problems are obtained.

Thus for small deflections

$$W_{\max.} = \frac{4b^2c}{a^2} \sum_{s=1}^{\infty} \frac{J_2(P_s b) \left[1 - J_0(P_s a) \right]}{P_s^6 J_0^2(P_s a)} \quad \dots (3.95)$$

$$(M_r)_{\max.} = \frac{4DC b^2}{a^2} \sum_{s=1}^{\infty} \frac{J_2(P_s b)}{P_s^4 J_0^2(P_s a)} \left[1 + \frac{\nu-1}{2} \right] \quad \dots (3.96)$$

$$(\sigma_r)_{\max.} = - \frac{24DCb^2}{a^2 h^2} \sum_{s=1}^{\infty} \frac{J_2(P_s b)}{P_s^4 J_0^2(P_s a)} \times \left[1 + \frac{\nu-1}{2} \right] \quad \dots (3.97)$$

The above results will be utilised for numerical computations and for side by side comparison.

RESULTS

Numerical results are presented here for the case of the circular plates with clamped edge. The type of load function considered is as in Eq.(3.67) and the value of λ is assumed to be 1, radius of the plate being $a = 2b$. The maximum deflections, maximum bending stresses are calculated for various values of the load functions and for various values of the foundation modulus. These are presented in the form of graphs. Central deflection and maximum bending stresses are also calculated for small deflections and these are also presented

in the form of graphs for comparison. Variation of the bending moment along the radius is also calculated both for small deflection and large deflection.

In calculating the central deflection, one has to start from Eq.(3.80) with an assumed value of (αa) leading to a particular value of the load function. Once this relationship is obtained the maximum value of the deflection can be obtained from Eq.(3.92) for various values of the foundation modulus. These results are presented in Fig.3.6 to 3.9. On examination of the Eq.(3.92), it is revealed that as the radius of the plate increases, the central deflection also increases for a given value of the load function. For small deflection Eq.(3.95) is to be used for calculation of the central deflection.

In calculating the bending moment for various values of $(\frac{r}{a})$, Eq.(3.89) is to be used putting the value of $\lambda = 1$. The variation of the bending moment along the radius of the plate is presented in Fig. 3.10. Variation of the bending moment along the radius according to the linear theory can be calculated with the help of the Eq.(3.89) putting the value of $\alpha = 0$ and $\lambda = 1$. The maximum bending stresses both for large and small deflection and for various values of foundation modulus are presented in the Fig. 3.11.

For the type of loading in Eq.(3.89) the central deflection for various values of load function and foundation modulus can be calculated with the help of Eq.(3.86) in conjunction with the corresponding equation for α . Values of the bending moment and bending stresses can also be easily calculated.

From Eq.(3.92) it is observed that the central deflection of the circular plate depends on the radius 'a' of the plate and on the value of the foundation modulus. But for a given value of the foundation modulus, deflection increases mainly due to increase of radius, because the effect of the value of $p_s^2 \cdot \frac{K}{D}$ is little in comparison with other terms when the plate radius is increased. For a given value of the plate radius, as the foundation modulus increases the Eq.(3.65) behaves as the linear equation of the type

$$\nabla^4 W + \frac{K}{D} W = q/D \quad \dots (3.98)$$

This is also seen from the Fig.3.6 through 3.9 where the deflection curves for higher values of the foundation modulus K_F tend towards linearity. The nature of the curves of the Fig.3.6 through 3.9 for $K_F = 0$ are in good agreement with those as found by other authors. For value of $\lambda = \frac{1}{2}$ in Eq.(3.67) and for $K_F = 0$, the deflection curve is presented in Fig. 3.12. This is in good agreement with the result obtained by Banerjee B [(1967), 5] .

From the Eq. (3.92) and also from the Fig.3.6/it is seen that with the increase of foundation modulus, the deflections of the plates decrease. This is expected. Because better foundations will give larger upward pressures to reduce the effect of applied loads.

For the type of loading in Eq.(3.67), the load on the plate at $r = \underline{+} b$ is zero and the deflected shape of the plate is shown in the Fig.3.13. From the Fig.3.10 it is seen that the bending moment varies from + ive to - ive value along the radius. The deflection of the plate along the radius will be as shown in Fig.3.13. This shows that

the direct stresses in the upper and lower fibres of the plates will change from tensile to compressive with the value of zero stress at the point of no deflection. Maximum bending stress will occur at the centre of the plate. With the increase of load maximum bending stresses will vary linearly with the load and the stresses will vary nonlinearly with the deflection. The effect of the foundation modulus is to reduce the bending stresses. Errors in the results of the deflection obtained from the two approximate equations (3.65) and (3.66) will be less for the values of K_p other than zero. But the results for the bending moments and bending stresses will not be as accurate as the deflection because their values depend on the derivative of the displacement and deflection.

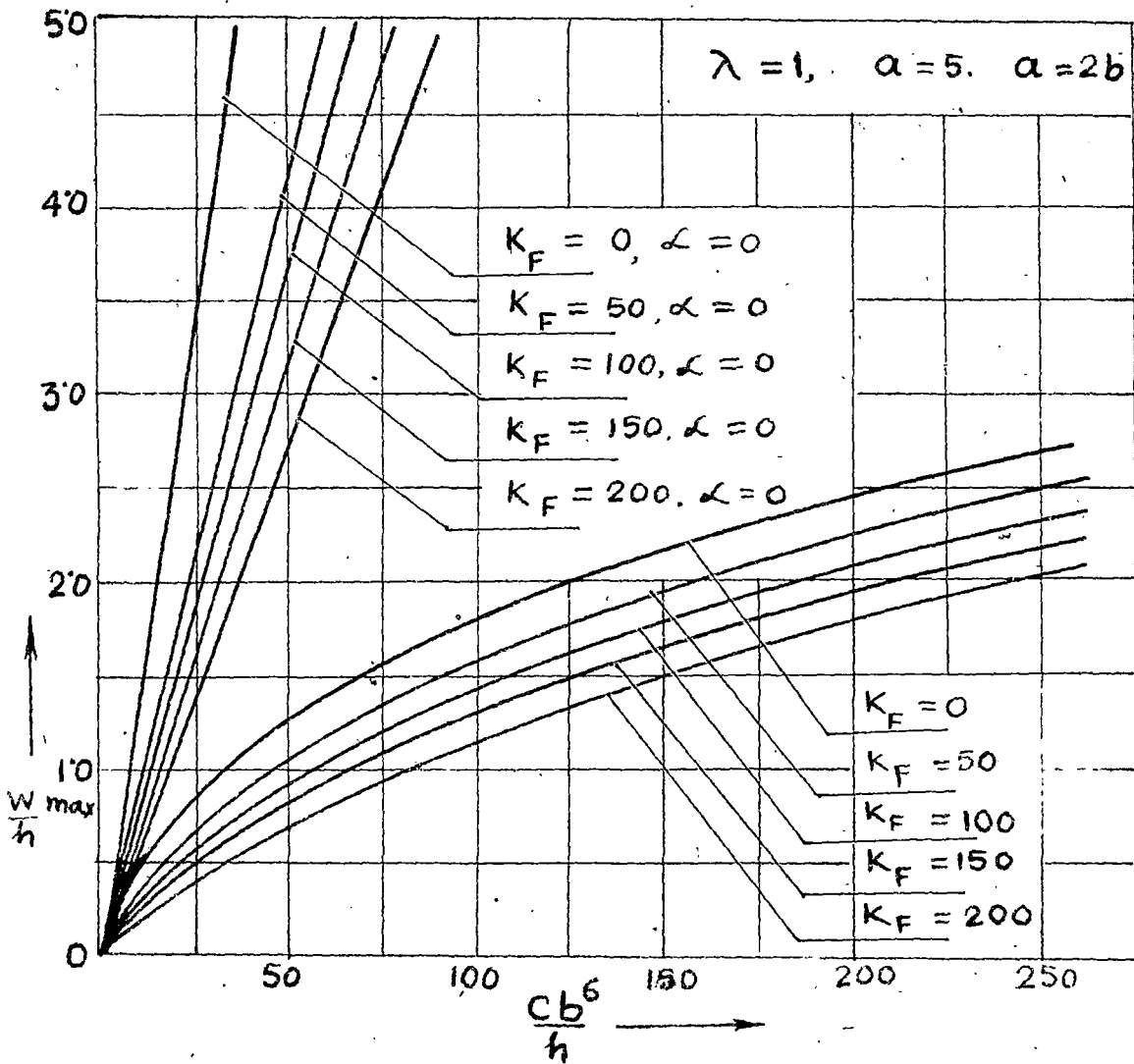
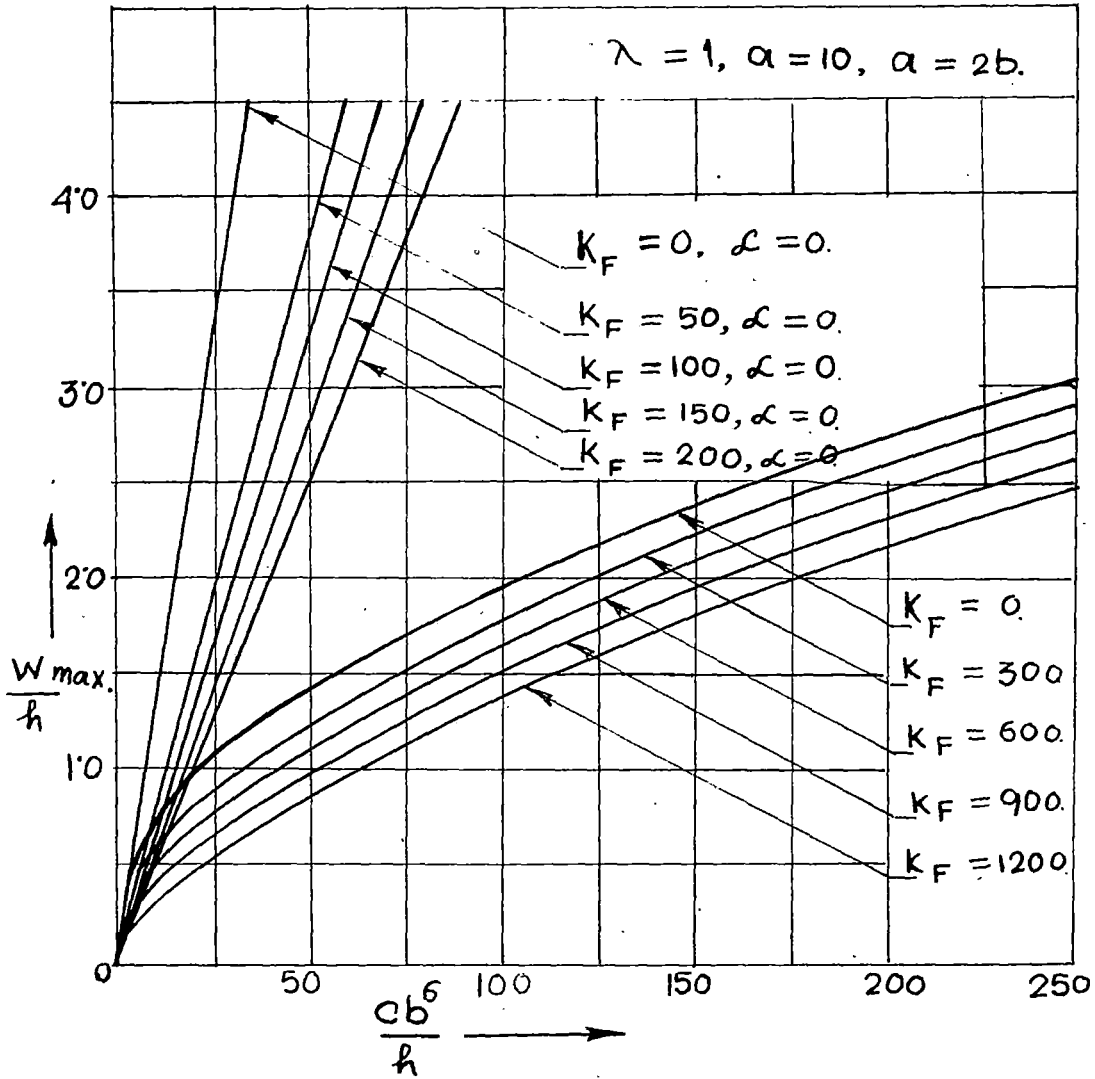


FIG. 3'6 CURVES FOR DEFLECTION.



R
 FIG. 37 CURVES FOR DEFLECTION.
 A

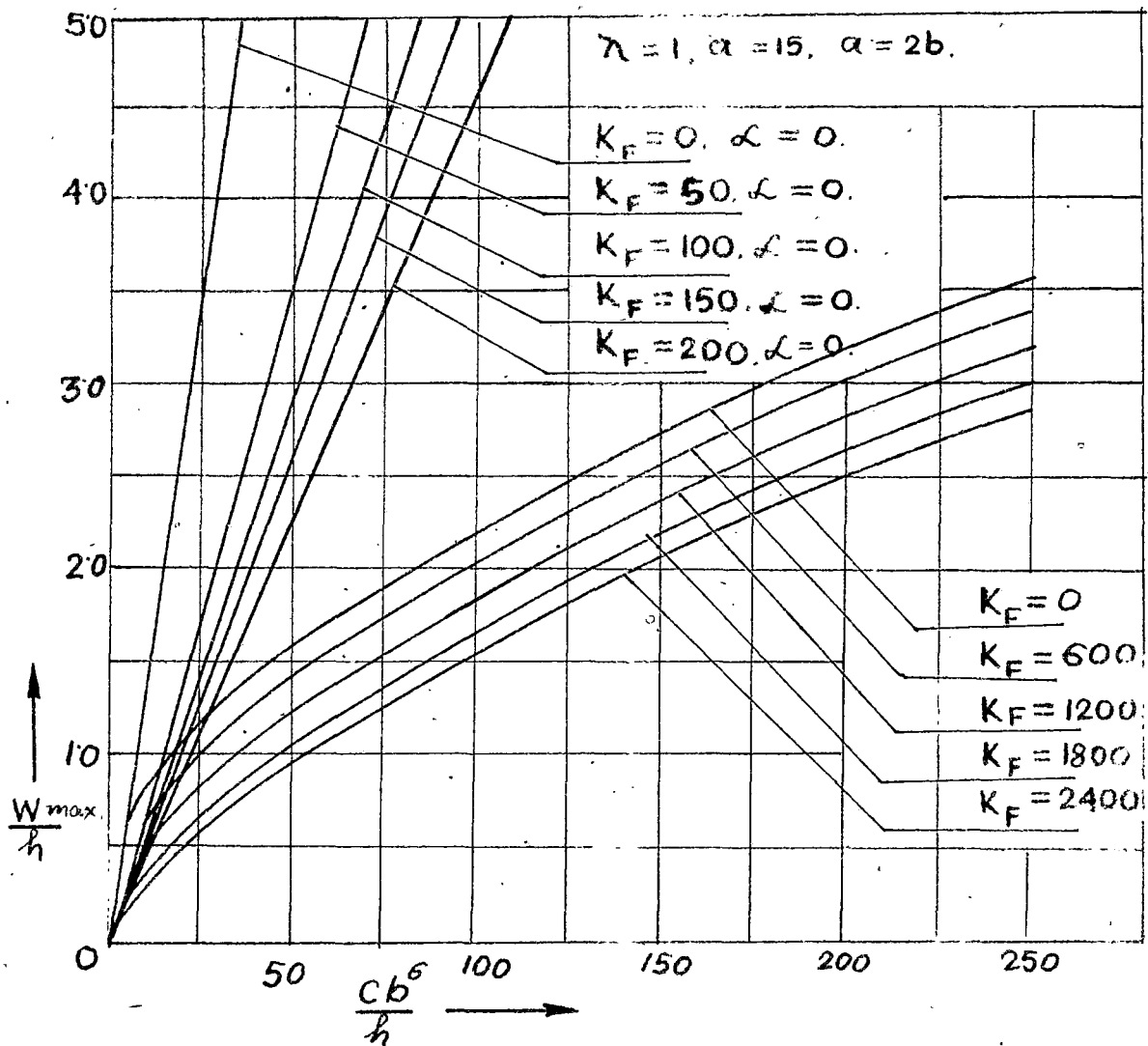


FIG. 38 CURVES FOR DEFLECTION

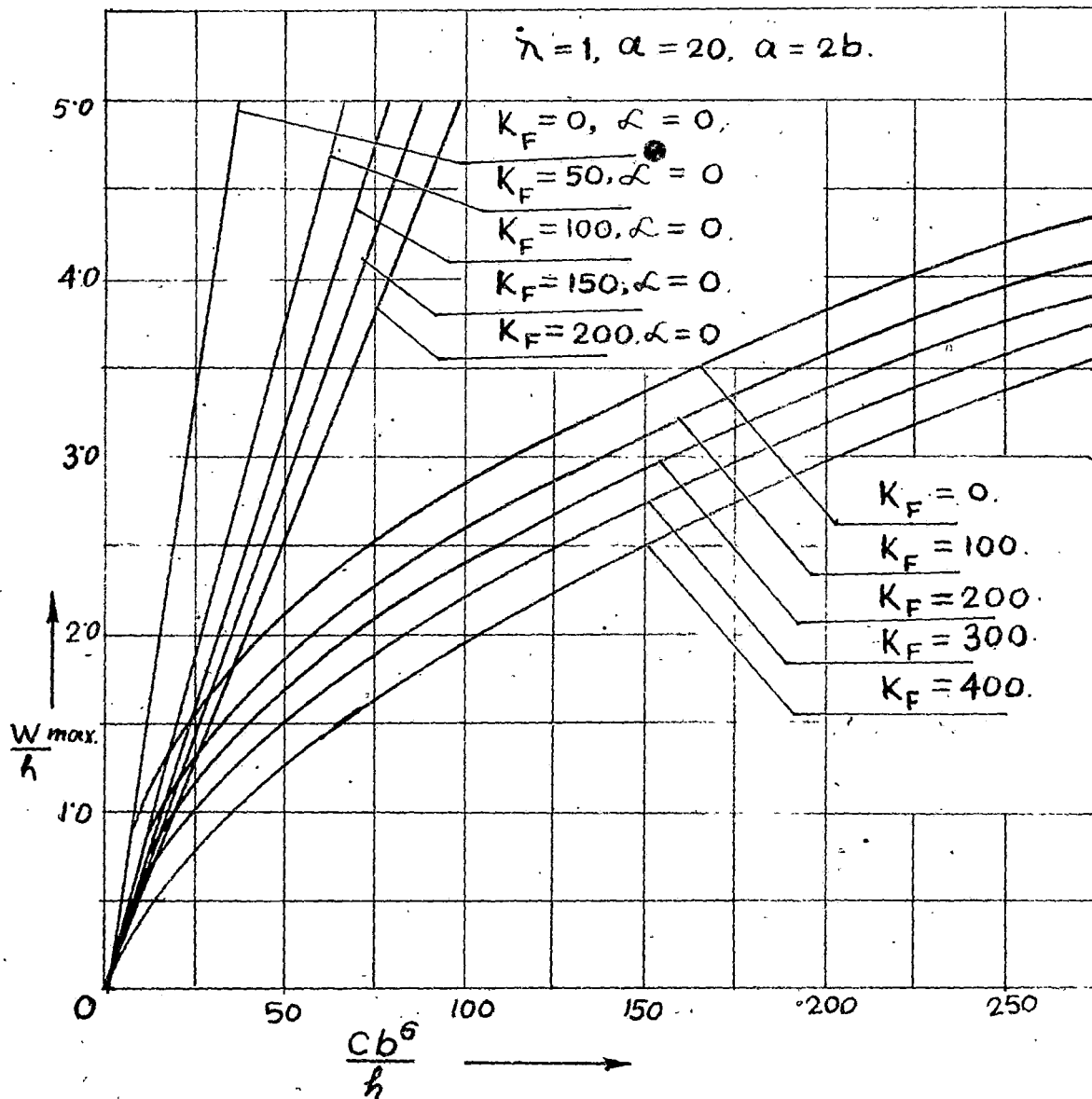


FIG. 3.9 CURVES FOR DEFLECTION.

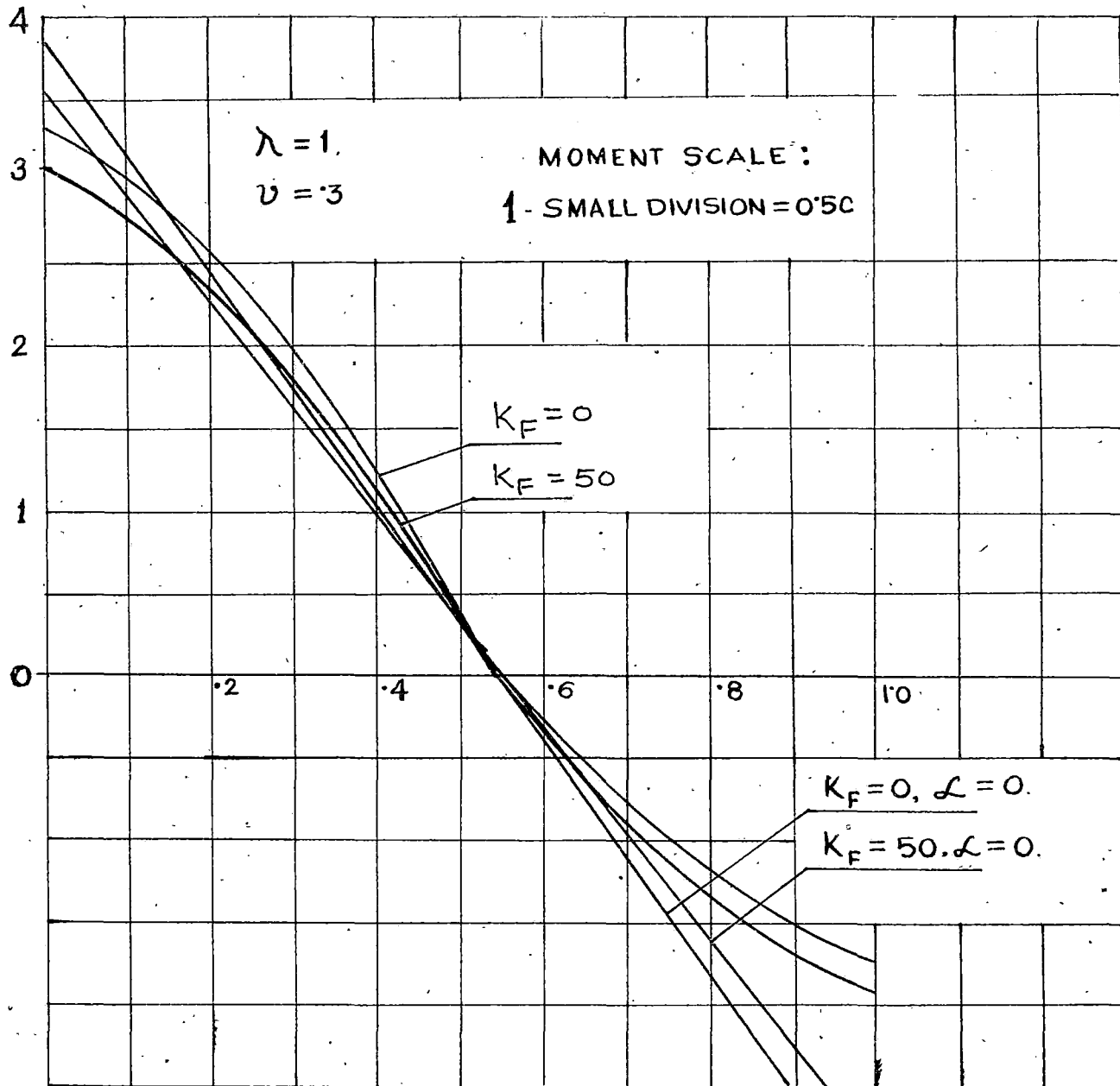


FIG. 3.10 CURVES FOR BENDING MOMENT.

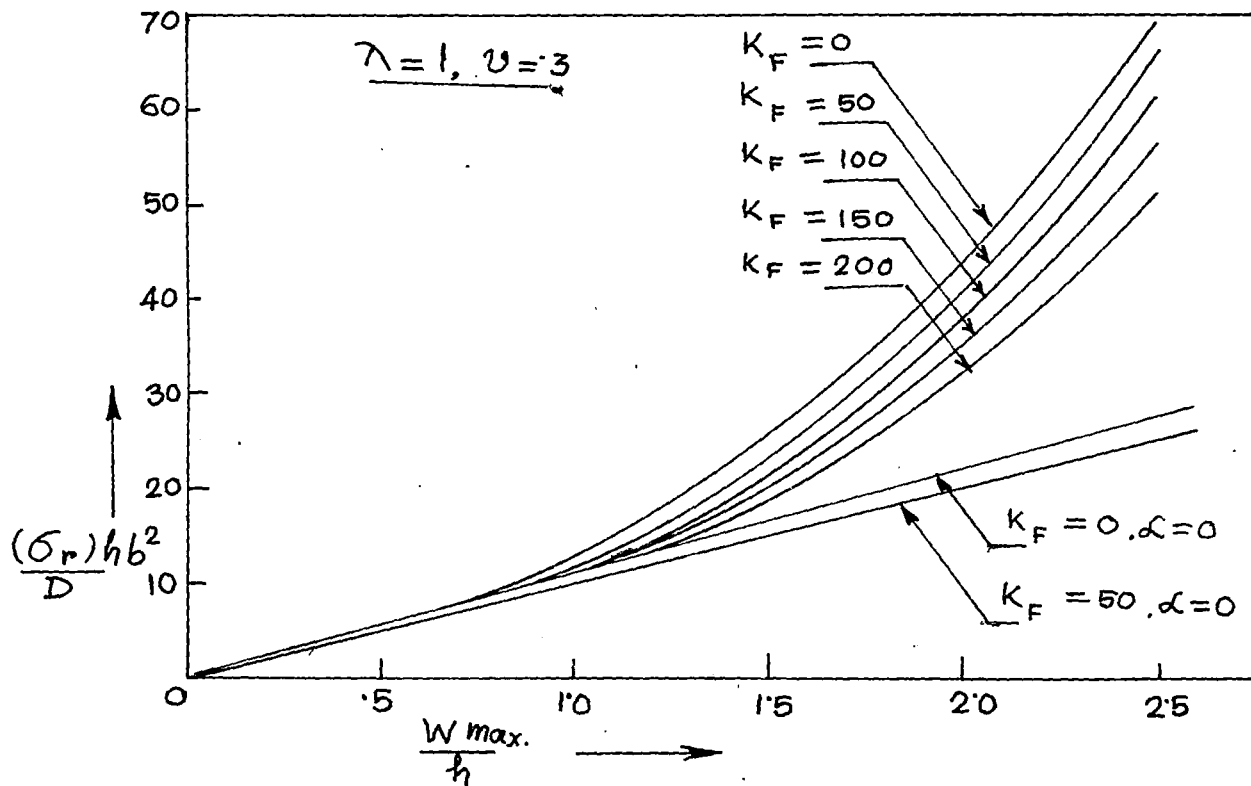


FIG. 3'11 CURVES FOR STRESSES.

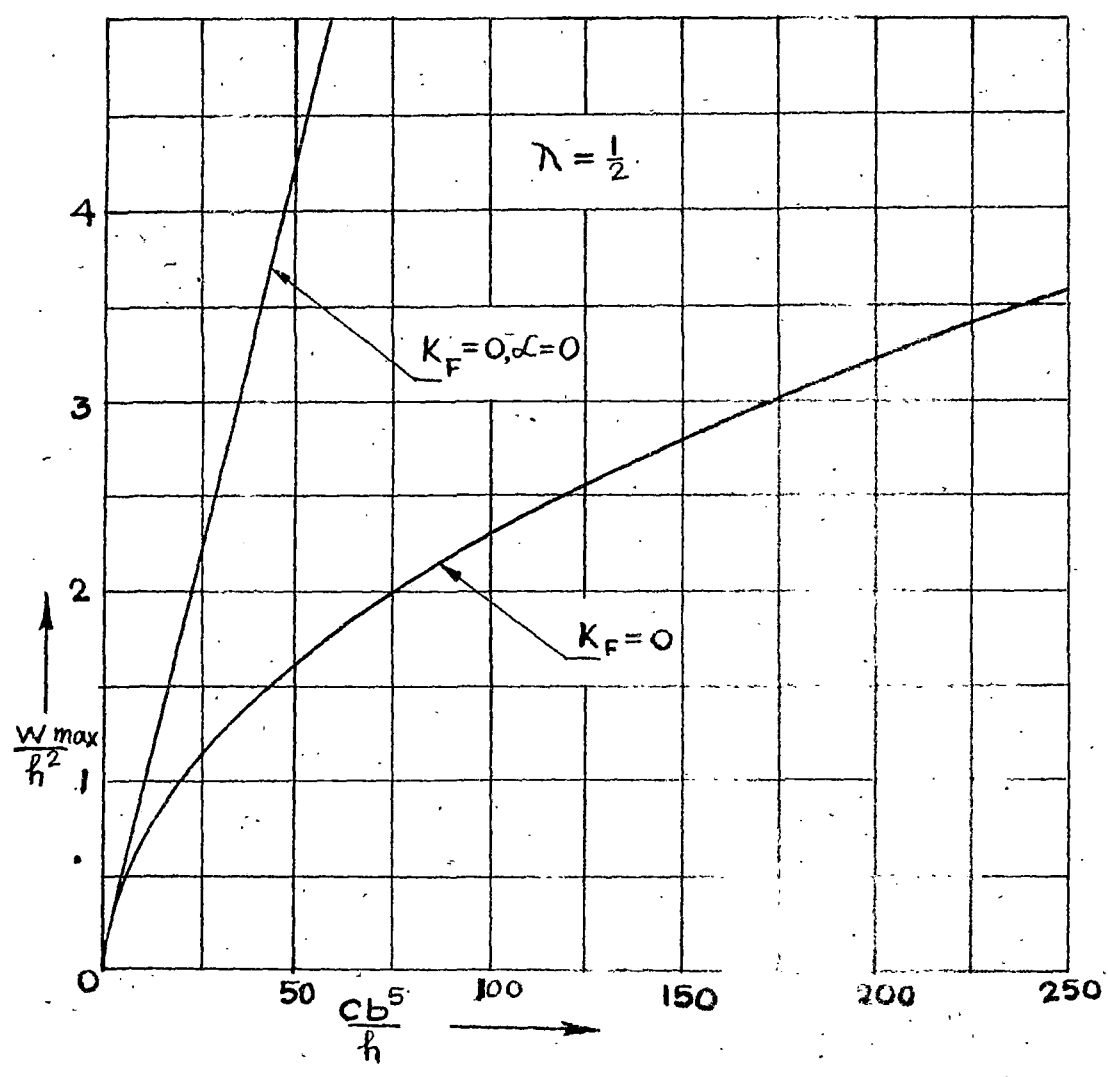


FIG. 3'12 CURVE FOR DEFLECTION.

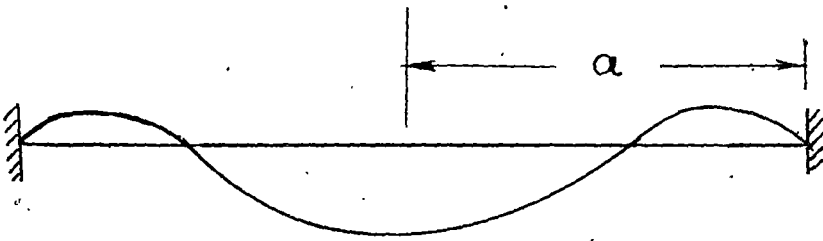


FIG. 3'13 DEFLECTED PLATE SHAPE.

**LARGE DEFLECTION OF A SEMI-CIRCULAR PLATE ON ELASTIC
FOUNDATION UNDER A UNIFORM LOAD***

PAPER - IV

INTRODUCTION

Large deflection of a simply supported semicircular plate placed on elastic foundation of Winkler type and subjected to a uniform load has been investigated following Berger's approximate method. Expressions for the deflections and bending moments are obtained and the theoretical results have been presented in the form of graphs.

SOLUTION OF PROBLEM

Let us take a plate in the form of a semicircle, Fig.3.14 and let it be simply supported and placed on an elastic foundation having the reaction K per unit area per unit deflection. Let the centre be the origin, the bounding diameter be the initial line and the plate be uniformly loaded. The governing differential equations in polar coordinates are

$$\nabla^4 w - \alpha^2 \nabla^2 w + \frac{K}{D} w = \frac{q}{D} \quad \dots (3.99)$$

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where α is a constant given by

$$\frac{\alpha^2 h^2}{12} = \frac{\partial u}{\partial r} + \frac{1}{r} \left(\frac{\partial w}{\partial r} \right)^2 + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{1}{2r^2} \left(\frac{\partial w}{\partial \theta} \right)^2 \quad \dots (3.100)$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

To solve Eq.(3.99) let us put it in the following form

$$\left(\nabla^2 - p_1^2 \right) \left(\nabla^2 - p_2^2 \right) w = \frac{q}{D} \quad \dots (3.101)$$

$$p_1^2 + p_2^2 = \alpha^2 \quad \dots (3.102)$$

$$p_1^2 p_2^2 = \frac{k}{D} \quad \dots (3.103)$$

Expanding the load into the appropriate Fourier series,

$$q = \frac{4q}{\pi} \sum_{m=1,3,5}^{\infty} \frac{\sin m\theta}{m} \quad \dots (3.104)$$

and assuming

$$w = \sum_{m=1,3,5}^{\infty} R_m \sin m\theta \quad \dots (3.105)$$

where R_m is a function of r only, and substituting Eq.(3.104) and (3.105) in (3.101) one gets

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - P_1^2 \right) \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - P_2^2 \right) R_m = \frac{4q}{\lambda m D} \quad \dots (3.106)$$

The appropriate solution of Eq.(3.101) is given by

$$W = \sum_{m=1,3,5}^{\infty} \left[A_m I_m(P_1 r) + B_m I_m(P_2 r) + \frac{4q}{\lambda m D} x \right]$$

$$x \sum_{s=0}^{\infty} \frac{\lambda_s}{(1/P_2)^{4+2s}} S_{3+2s,m}(1/P_2 r) \int \sin m\theta \quad \dots (3.107)$$

$$\text{where } \lambda_s = \frac{(-1)^s (1/P_1)^{2s}}{(2^2 - m^2)(4^2 - m^2) \dots \{(2 + 2s)^2 - m^2\}}$$

$$\text{and } S_{3+2s,m}(1/P_2 r) = \sum_{n=0}^{\infty} \frac{(-1)^n (1/P_2 r)^{4+2s+2n}}{\{(4+2s)^2 - m^2\} \dots \{(4+2s+2n)^2 - m^2\}}$$

is the Lommel's function which is uniformly convergent.

The required boundary conditions are

$$(W)_{r=a} = 0 \quad \dots (3.108)$$

$$(u)_{r=a} = 0 \quad \dots (3.108a)$$

$$\left[\frac{\partial^2 w}{\partial r^2} + \nu \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right]_{r=a} = 0 \quad \dots (3.109)$$

Considering eqs.(3.107),(3.108) and (3.109) and solving for the constants A_m and B_m one gets

$$\mathcal{L}^{-1} \left\{ a P_2^2 I_m''(P_2 a) + \nu P_2 I_m'(P_2 a) \right\} \sum_{s=0}^{\infty} \frac{\lambda_s}{(i P_2)^{4+2s}} S_{3+2s,m}(i P_2 a) -$$

$$- I_m(P_2 a) \left\{ a \sum_{s=0}^{\infty} \frac{\lambda_s}{(i P_2)^{4+2s}} S_{3+2s,m}''(i P_2 a) + \right.$$

$$\left. + \nu \sum_{s=0}^{\infty} \frac{\lambda_s}{(i P_2)^{4+2s}} S_{3+2s,m}'(i P_2 a) \right\} \quad \text{---}$$

$$A_m = \frac{4q}{\pi D_m} \frac{\mathcal{L}^{-1} I_m(P_2 a) \left\{ a P_1^2 I_m''(P_1 a) + \nu P_1 I_m'(P_1 a) \right\} -$$

$$- I_m(P_1 a) \left\{ a P_2^2 I_m''(P_2 a) + \nu P_2 I_m'(P_2 a) \right\}}{\quad} \quad \dots (3.110)$$

$$\mathcal{L}^{-1} \left\{ a P_1^2 I_m''(P_1 a) + P_1 I_m'(P_1 a) \right\} \sum_{s=0}^{\infty} \frac{\lambda_s}{(i P_2)^{4+2s}} \cdot S_{3+2s,m}(i P_2 a) -$$

$$- I_m(P_1 a) \left\{ a \sum_{s=0}^{\infty} \frac{\lambda_s}{(i P_2)^{4+2s}} S_{3+2s,m}''(i P_2 a) + \right.$$

$$\left. + \nu \sum_{s=0}^{\infty} \frac{\lambda_s}{(i P_2)^{4+2s}} S_{3+2s,m}'(i P_2 a) \right\} \quad \text{---}$$

$$B_m = - \frac{4q}{\pi D_m} \frac{\mathcal{L}^{-1} I_m(P_2 a) \left\{ a P_1^2 I_m''(P_1 a) + \nu P_1 I_m'(P_1 a) \right\} -$$

$$- I_m(P_1 a) \left\{ a P_2^2 I_m''(P_2 a) + \nu P_2 I_m'(P_2 a) \right\}}{\quad} \quad \dots (3.111)$$

where dashes represent differentiations with respect to r .

Since we are interested only in the lateral displacement w , let us determine α by eliminating u and v from Eq.(3.100)

Let

$$u = \sum U(r) \cos m\theta \quad \dots (3.112)$$

$$v = \sum V(r) \sin m\theta \quad \dots (3.113)$$

Multiplying Eq.(3.100) by $r d\theta dr$ and integrating within the limits 0 to a and 0 to π one gets

$$\begin{aligned} & \int_0^a \int_0^\pi r \sum U'(r) \cos m\theta \, d\theta \, dr + \frac{1}{2} \int_0^a \int_0^\pi r \left(\frac{\partial w}{\partial r} \right)^2 \, d\theta \, dr + \\ & + \int_0^a \int_0^\pi \sum U(r) \cos m\theta \, d\theta \, dr + \int_0^a \int_0^\pi \sum m V(r) \cos m\theta \, d\theta \, dr + \\ & + \frac{1}{2} \int_0^a \int_0^\pi \frac{1}{r} \left(\frac{\partial w}{\partial \theta} \right)^2 \, d\theta \, dr = \frac{\alpha^2 h^2}{12} \int_0^a \int_0^\pi r \, d\theta \, dr \end{aligned}$$

After evaluating the integrals the following equation leading to α is obtained.

$$\begin{aligned}
& \sum_{m=1,3,5}^{\infty} \left[A_m^2 P_1^2 \left(-\frac{a^2}{8} \left\{ \frac{1}{2} \left[I_{m-2}(P_1 a) + I_m(P_1 a) \right]^2 - \left[1 + \frac{(m-1)^2}{P_1^2 a^2} \right] x \right. \right. \right. \\
& \times I_{m-2}^2(P_1 a) \left. \left. \right\} - \frac{a^2}{8} \left\{ \frac{1}{2} \left[I_m(P_1 a) + I_{m+2}(P_1 a) \right]^2 - \left[1 + \frac{(m+1)^2}{P_1^2 a^2} \right] x \right. \right. \\
& \times I_{m+1}^2(P_1 a) \left. \left. \right\} + \frac{1}{2} \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \phi \left(\frac{P_1}{2} \right)^{2n+2t+2t} \right) + B_m^2 P_2^2 x \\
& \times \left(\frac{1}{2} \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \phi \left(\frac{P_2}{2} \right)^{2n+2t+2t} - \frac{a^2}{8} \left\{ \frac{1}{2} \left[I_{m-2}(P_2 a) + I_m(P_2 a) \right]^2 + \right. \right. \\
& + \left. \left[1 + \frac{(m-1)^2}{P_2^2 a^2} \right] I_{m-1}^2(P_2 a) \right\} - \frac{a^2}{8} \left\{ \frac{1}{2} \left[I_m(P_2 a) + I_{m+2}(P_2 a) \right]^2 + \right. \\
& + \left. \left[1 + \frac{(m+1)^2}{P_2^2 a^2} \right] I_{m+1}^2(P_2 a) \right\} \left. \right) + \frac{16 a^2}{\lambda^2 D^2 m^2} \left\{ \sum_{s=0}^{\infty} \frac{\lambda^2 s}{(1 P_2)^{8+4s}} x \right. \\
& \times \left[\sum_{n=0}^{\infty} \frac{\lambda^2 n (4+2s+2n)^2 a^{8+4s+4n}}{8+4s+4n} + \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \right. \\
& \left. \left. \frac{\lambda_n \lambda_t (4+2s+2n) (4+2s+2t) a^{8+4s+2n+2t}}{8+4s+2n+2t} \right] \right\} + \frac{1}{2} A_n B_m P_1 P_2 x
\end{aligned}$$

$$\times \left\{ \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \left[\frac{\left(\frac{P_1}{2}\right)^{m-1+2n} \left(\frac{P_2}{2}\right)^{m-1+2t} a^{2n+2n+2t}}{(2n+2n+2t) \Gamma(n) \Gamma(t) \Gamma(m+n) \Gamma(m+t)} \right. \right. \\
 \left. \left. + \frac{\left(\frac{P_1}{2}\right)^{m+1+2n} \left(\frac{P_2}{2}\right)^{m+1+2t} a^{2n+2n+2t+4}}{(2n+2n+2t+4) \Gamma(n) \Gamma(t) \Gamma(m+n+2) \Gamma(m+t+2)} \right] \right.$$

$$+ \phi \cdot \left(\frac{P_1}{2}\right)^{m-1+2n} \left(\frac{P_2}{2}\right)^{m+1+2t} +$$

$$+ \phi \cdot \left(\frac{P_1}{2}\right)^{m+1+2n} \left(\frac{P_2}{2}\right)^{m-1+2t} \left. \right\} +$$

$$+ 4 A_n P_1 \frac{q}{\pi D_n} \sum_{s=0}^{\infty} \frac{\lambda_s}{(1P_2)^{4+2s}} \left\{ \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \left[\mu_n \left(\frac{P_1}{2}\right)^{m-1+2t} \psi \right. \right.$$

$$+ \mu_n \psi_1 \left(\frac{P_1}{2}\right)^{m+1+2t} \left. \right\} +$$

$$+ 4 B_n P_2 \frac{q}{\pi D_n} \sum_{s=0}^{\infty} \frac{\lambda_s}{(1P_2)^{4+2s}} \left\{ \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \mu_n \psi \left(\frac{P_2}{2}\right)^{m-1+2t} \right.$$

$$+ \mu_n \psi_1 \left(\frac{P_2}{2}\right)^{m+1+2t} \left. \right\} + m^2 \left(A_n^2 \sum_{n=0}^{\infty} \psi_2 \left(\frac{P_1}{2}\right)^{2n+4n} \right.$$

$$+ A_m^2 \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \psi_3 \left(\frac{P_1}{2} \right)^{2n+2t+2t} + B_m^2 \sum_{n=0}^{\infty} \psi_2 \left(\frac{P_2}{2} \right)^{2n+4n} +$$

$$+ B_m^2 \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \psi_3 \left(\frac{P_2}{2} \right)^{2n+2t+2t} + \frac{16^3}{\lambda D^2 m^2} \sum_{s=0}^{\infty} \frac{\lambda_s^2}{(1P_2)^{3+4s}} \times$$

$$\times \left\{ \sum_{n=0}^{\infty} \frac{\mu_n^2 a^{3+4s+4n}}{8+4s+4n} + \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \frac{\mu_n \mu_t a^{3+4s+2n+2t}}{8+4s+2n+2t} \right\} +$$

$$+ 2A_m B_m \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \frac{\left(\frac{P_1}{2} \right)^{n+2n} \left(\frac{P_2}{2} \right)^{m+2t} a^{2n+2n+2t}}{(2n+2n+2t) \sqrt{n} \sqrt{t} \sqrt{(m+n+1)} \sqrt{(m+t+1)}} +$$

$$+ 8 A_m \frac{q}{\lambda D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(1P_2)^{4+2s}} \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \mu_n \phi_1 \left(\frac{P_1}{2} \right)^{m+2t} +$$

$$+ 8 B_m \frac{q}{\lambda D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(1P_2)^{4+2s}} \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \mu_n \cdot \phi_1 \left(\frac{P_2}{2} \right)^{m+2t}$$

$$= \frac{\alpha^{2,2,2} h^2 a^2}{6}$$

... (3.114)

where

$$\mu_n = \frac{(-1)^n (iP_2)^{4+2s+2n}}{\{(4+2s)^2 - m^2\} \dots \{(4+2s+2n)^2 - m^2\}}$$

$$\mu_t = \frac{(-1)^n (iP_2)^{4+2s+2t}}{\{(4+2s)^2 - m^2\} \dots \{(4+2s+2t)^2 - m^2\}}$$

$$\phi = \frac{a^{2n+2n+2t+2}}{(2n+2n+2t+2) \angle n \angle t \sqrt{m+n} \sqrt{m+t+2}}$$

$$\phi_1 = \frac{a^{4+2s+2n+2t+m}}{(4+2s+2n+2t+m) \angle t \sqrt{m+t+1}}$$

$$\psi = \frac{(4+2s+2n) a^{4+2s+2n+2t+m}}{(4+2s+2n+2t+m) \angle t \sqrt{m+t}}$$

$$\psi_1 = \frac{(4+2s+2n) a^{6+2s+2n+2t+m}}{(6+2s+2n+2t+m) \angle t \sqrt{m+t+2}}$$

$$\psi_2 = \frac{a^{2n+4n}}{(2n+4n) \left\{ \angle n \sqrt{m+n+1} \right\}^2}$$

$$\psi_3 = \frac{a^{2m+2n+2t}}{(2m+2n+2t) \int^n \int^t \sqrt{m+n+1} \sqrt{m+t+1}}$$

As $P_1 \rightarrow 0$, $P_2 \rightarrow 0$, Eq.(3.107) reduces to

$$W = \frac{qa^4}{D} \sum_{m=1,3,5}^{\infty} \left\{ \frac{4r^4}{a^4} \frac{1}{m \pi (16-m^2) (4-m^2)} \right. \\ \left. + \frac{r^m}{a^m} \frac{m+5+\nu}{m \pi (16-m^2) (2-m) \left[m + \frac{1}{2} (1+\nu) \right]} \right. \\ \left. - \frac{r^{m+2}}{a^{m+2}} \frac{m+3+\nu}{m \pi (4+m)(4-m^2) \left[m + \frac{1}{2} (1+\nu) \right]} \right\} \sin m\theta$$

as obtained by Timoshenko (1959) for the corresponding problem of small deflection without any elastic foundation.

RESULTS

To obtain deflection for a given value of plate radius 'a' and foundation modulus 'Kp' one has to start from Eq.(3.114) with an assumed value of ' α ' in order to obtain the corresponding value of the load function $\left(\frac{qa^4}{Dh}\right)$. Once this relationship is obtained the corresponding

deflection ($\frac{W}{h}$) can be calculated from Eq.(3.107) with the help of Eq.(3.110) and (3.111). For $a=30\text{mm}$, $\nu = .3$, and $K_p = 350$ deflections have been plotted in Fig.3.15 for various values of load function ($\frac{qa^4}{D}$).

On examination of the Eq.(3.107), it is clear that the radius of symmetry of the plate undergoes the maximum deflection with respect to other radii. The expression for the deflection at a given point on the radius of symmetry can be expressed in the form $W = \beta \frac{qa^4}{D}$, where β is a numerical factor. Deflections at various points on the radius of symmetry are plotted in Fig.3.16 for a given value of load function. From Fig.3.16 it is observed that maximum deflection occurs at the centre of gravity of the plate.

The plate is subjected to bending moments in radial and tangential directions, as well as to a twisting moment. The moments can be easily computed, because the deflection w is known. The expressions for bending and twisting moments are

$$\begin{aligned}
 M_r = & \sum_{m=1,3,5}^{\infty} -D \frac{A_m P_1^2}{4} \left\{ I_{m+2}(P_1 r) + 2I_m(P_1 r) + I_{m-2}(P_1 r) \right\} + \\
 & + \frac{B_m P_2^2}{4} \left\{ I_{m+2}(P_2 r) + 2I_m(P_2 r) + I_{m-2}(P_2 r) \right\} + \\
 & + \nu \left\{ \frac{A_m P_1}{2r} \left[I_{m-1}(P_1 r) + I_{m+1}(P_1 r) \right] + \frac{B_m P_2}{2r} \right. \\
 & \left. \times \left[I_{m-1}(P_2 r) + I_{m+1}(P_2 r) \right] - \frac{m^2}{r^2} \left[A_m I_m(P_1 r) + \right. \right.
 \end{aligned}$$

$$\begin{aligned}
& + C_m I_m(P_2 r) \int \left. + \frac{4q}{\lambda D_m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(1P_2)^{4+2s}} \right. x \\
& x \left\{ s_{3+2s,m}^n (1P_2 r) + \frac{\nu}{r} s_{3+2s,m}^i (1P_2 r) - \frac{m^2 \nu}{r^2} s_{3+2s,m}^v (1P_2 r) \right\} \int \sin m\theta \\
& \dots (3.115)
\end{aligned}$$

$$\begin{aligned}
M_0 = & \sum_{m=1,3,5}^{\infty} - D \int \frac{A_m P_1^2 \nu}{4} \left\{ I_{m+2}(P_1 r) + 2 I_m(P_1 r) + I_{m-2}(P_1 r) \right\} + \\
& + \frac{B_m P_2^2 \nu}{4} \left\{ I_{m+2}(P_2 r) + 2 I_m(P_2 r) + I_{m-2}(P_2 r) \right\} + \\
& + \frac{A_m P_1}{2r} \left\{ I_{m-1}(P_1 r) + I_{m+1}(P_1 r) \right\} + \frac{B_m P_2}{2r} \left\{ I_{m-1}(P_2 r) + \right. \\
& + I_{m+1}(P_2 r) \left. \right\} - \frac{m^2}{r^2} \left\{ A_m I_m(P_1 r) + B_m I_m(P_2 r) \right\} + \\
& + \frac{4q}{\lambda D_m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(1P_2)^{4+2s}} \cdot \left\{ \nu s_{3+2s,m}^n (1P_2 r) + \right.
\end{aligned}$$

$$+ \frac{1}{r} S'_{3+2s,m}(1P_2r) - \frac{m^2}{r^2} S_{3+2s,m}(1P_2r) \} \int \sin m\theta \quad \dots (3.116)$$

$$M_{r\theta} = (1-\nu) D \sum_{m=1,3,5}^{\infty} m \int \frac{A_m P_1}{2r} \{ I_{m-1}(P_1r) + I_{m+1}(P_1r) \} +$$

$$+ \frac{B_m P_2}{2r} \{ I_{m-1}(P_2r) + I_{m+1}(P_2r) \} - \frac{1}{r^2} \{ A_m I_m(P_1r) + B_m I_m(P_2r) \} +$$

$$+ \frac{4q}{\lambda D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(1P_2)^{4+2s}} \cdot \left\{ \frac{1}{r} S'_{3+2s,m}(1P_2r) -$$

$$- \frac{1}{r^2} S_{3+2s,m}(1P_2r) \} \int \cos m\theta \quad \dots (3.117)$$

Eqs. (3.115), (3.116) and (3.117) show that the bending moments M_r and M_θ are both maximum on the radius of symmetry, the twisting moment $M_{r\theta}$ is maximum on the bounding diameter. The bending moments can be expressed in the form

$$M_r = \beta_1 qa^2 \quad M_\theta = \beta_2 qa^2 \quad \dots (3.118)$$

The stresses can be calculated from the expressions

$$(\sigma_r)_{\max.} = \frac{6 (M_r)_{\max.}}{h^2}, \quad (\sigma_\theta)_{\max.} = \frac{6 (M_\theta)_{\max.}}{h^2} \quad \dots (3.119)$$

Expressions for shearing forces can be obtained with the help of Eq.(3.107) from the expressions

$$Q_r = -D \frac{\partial}{\partial r} (\nabla^2 w - \alpha^2 w), \quad Q_\theta = -\frac{D}{r} \frac{\partial}{\partial \theta} (\nabla^2 w - \alpha^2 w) \quad \dots (3.120)$$

For the circular plate Eq.(3.120) can be expressed in the form

$$\begin{aligned} Q_r = & -D \sum_{m=1,3,5}^{\infty} \left\{ \left[A_m P_1^3 I_m^{\alpha_1} (P_1 r) + B_m P_2^3 I_m^{\alpha_1} (P_2 r) + \frac{4q}{\lambda Dm} r \right. \right. \\ & \times \sum_{s=0}^{\infty} \frac{\lambda_s}{(1P_2)^{4+2s}} \cdot S_{3+2s,m}^{\alpha_1} (1P_2 r) \left. \right] + \frac{1}{r} \left[A_m P_1^2 I_m^{\alpha_1} (P_1 r) + B_m P_2^2 I_m^{\alpha_1} (P_2 r) + \right. \\ & + \frac{4q}{\lambda Dm} \sum_{s=0}^{\infty} \frac{\lambda_s}{(1P_2)^{4+2s}} \cdot S_{3+2s,m}^{\alpha_1} (1P_2 r) \left. \right] - \left(\alpha^2 + \frac{m^2}{r^2} \right) \left[A_m P_1 I_m^{\alpha_1} \right. \\ & \left. \left. (P_1 r) + B_m P_2 I_m^{\alpha_1} (P_2 r) + \frac{4q}{\lambda Dm} \sum_{s=0}^{\infty} \frac{\lambda_s}{(1P_2)^{4+2s}} \cdot S_{3+2s,m}^{\alpha_1} (1P_2 r) \right] \right\} \times \\ & r \sin m\theta \quad \dots (3.121) \end{aligned}$$

$$Q_0 = -\frac{D}{r} \sum_{m=1,3,5}^{\infty} m \left[A_m P_1^2 I_m''(P_1 r) + B_m P_2^2 I_m''(P_2 r) + \frac{4q}{\lambda D m} \right] \times$$

$$\times \sum_{s=0}^{\infty} \frac{\lambda_s}{(1P_2)^{4+2s}} \cdot S_{3+2s,m}''(1P_2 r) + \frac{1}{r} \left\{ A_m P_1 I_m'(P_1 r) + B_m P_2 I_m'(P_2 r) + \right.$$

$$\left. + \frac{4q}{\lambda D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(1P_2)^{4+2s}} \cdot S_{3+2s,m}'(1P_2 r) \right\} - \left(\alpha^2 + \frac{m^2}{r^2} \right) \times$$

$$\times \left\{ A_m I_m(P_1 r) + B_m I_m(P_2 r) + \frac{4q}{\lambda D m} \cdot \sum_{s=0}^{\infty} \frac{\lambda_s}{(1P_2)^{4+2s}} \cdot S_{3+2s,m}(1P_2 r) \right\} \times$$

$\times \cos m\theta$

... (3.122)

The shearing stresses can be calculated from the expressions

$$\tau_{r\theta} = \frac{6 M_{r\theta}}{h^2}, \quad \tau_{rz} = \frac{3}{2} \frac{Q_r}{h}, \quad \tau_{\theta z} = \frac{3}{2} \frac{Q_\theta}{h}$$

... (3.123)

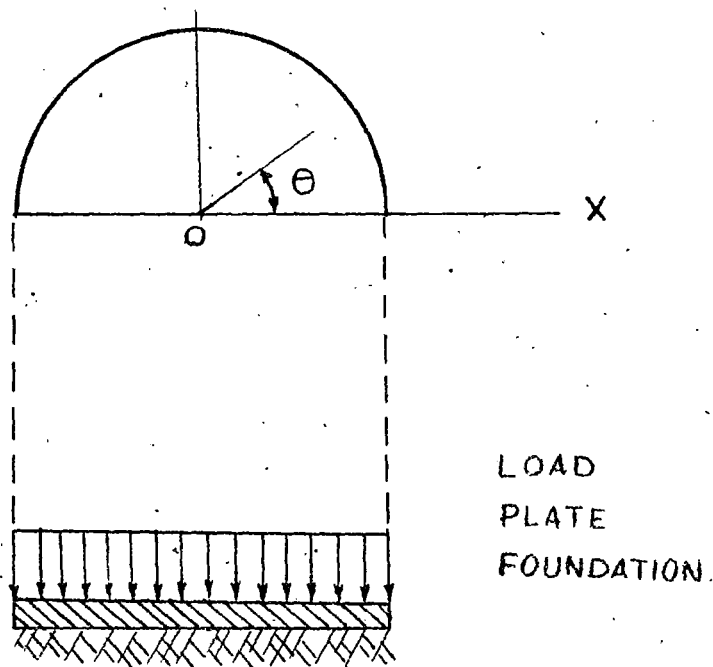


FIG. 314 SEMICIRCULAR PLATE
ON FOUNDATION.

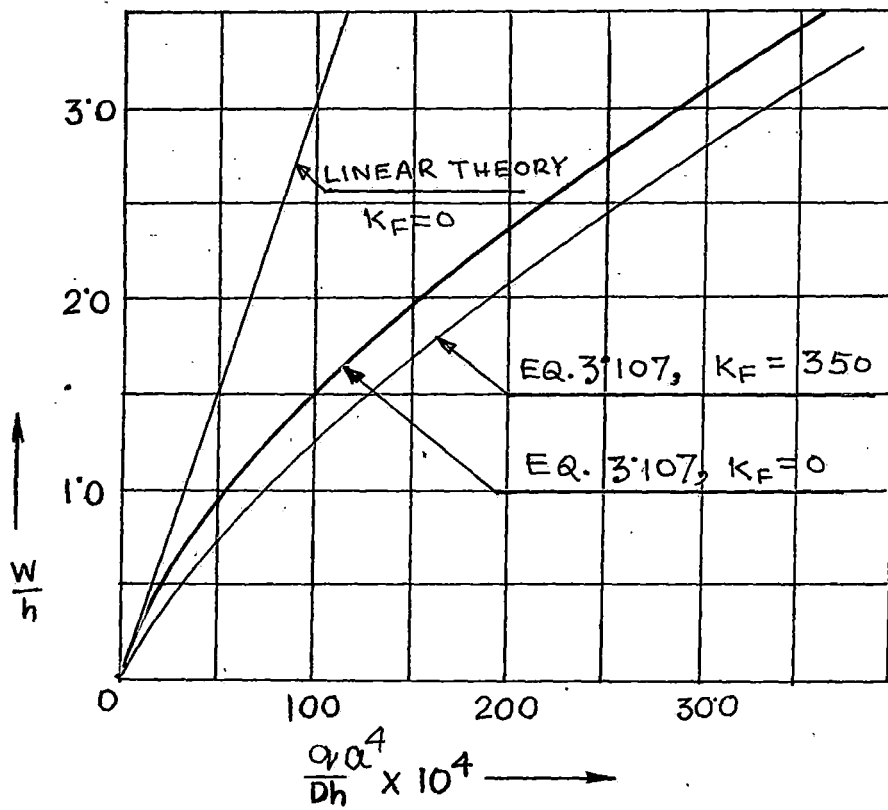


FIG. 3.15 DEFLECTION CURVE.

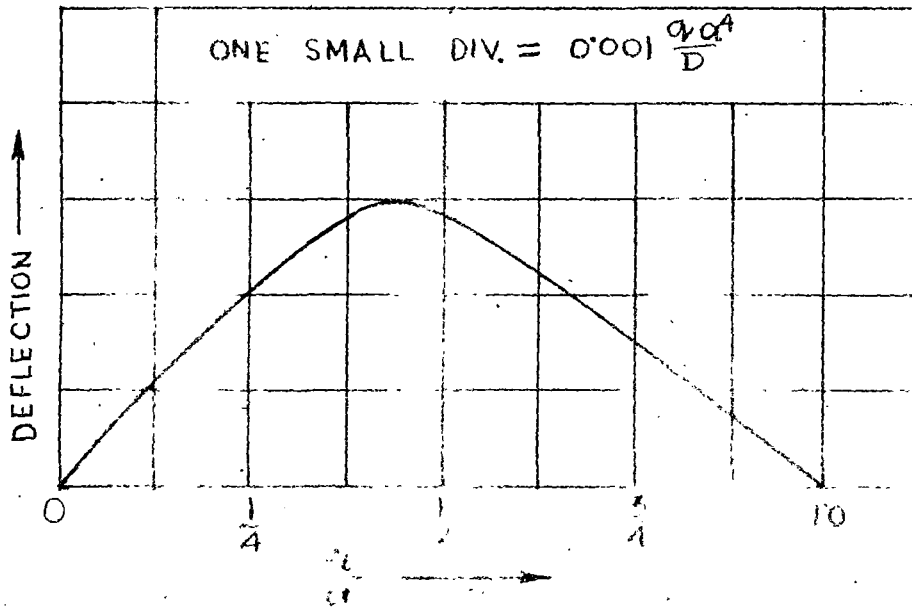


FIG. 376 DEFLECTION CURVE

LARGE DEFLECTION OF A TRIANGULAR ORTHOTROPIC
PLATE ON ELASTIC FOUNDATION*

PAPER - V

INTRODUCTION

Triangular reinforced concrete slabs are sometimes used as bottom slabs of bunkers. Thus the design of this type of structure is of practical interest for Defence. These slabs may rest freely on soil or sand and generally are subjected to a uniform load. If the thickness of the slab is small compared to the other dimensions, then it may be regarded as a thin orthotropic plate resting on elastic foundation and subjected to a uniform load. Following Berger's method numerous isotropic plate problems have been solved with ease and accuracy. Iwinski and Nowinski (1957) generalised the procedure of Berger to Orthotropic plates and found the deflections of circular and rectangular plates under uniform load ^{and} under various boundary conditions.

In this Paper large deflection of an equilateral triangular orthotropic plate, such as reinforced concrete, resting on elastic foundation has been solved following Berger's method. The plate is under uniform load and the foundation is assumed to be such that its reaction is proportional to the deflection of the plate.

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New Delhi, Vol.25, No.3, 1975.

FORMULATION OF PROBLEM

For moderately large deflections, the strain displacement relationships are

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2$$

and

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y}$$

Neglecting the second middle surface strain invariant, the strain energy due to bending and stretching of the middle surface of the plate of thickness, h , can be written as

$$V_1 = \frac{1}{2} \iint \left[D_x \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2D_1 \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + D_y \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 4D_{xy} \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right. \\ \left. + D_x \frac{12}{h^2} \epsilon_1^2 \right] dx dy \quad \dots (3.124)$$

in which

$$D_x = \frac{E'_x h^3}{12}, \quad D_y = \frac{E'_y h^3}{12}, \quad D_1 = \frac{E'' h^3}{12}, \quad D_{xy} = \frac{G h^3}{12} \quad \dots (3.124a)$$

$$\epsilon_1 = \frac{\partial u}{\partial x} + K_1 \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{K_1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \quad \dots (3.124b)$$

$$K_1^2 = \frac{D_y}{D_x} \quad \dots (3.124c)$$

and E'_x , E'_y , E'' , and G are constants to characterise the elastic properties of the material. By adding the potential energy of the uniform normal load, q , and of the foundation reaction, K , to the energy expression of Eq.(3.124) the modified energy expression is obtained as follows :

$$V = \frac{1}{2} \iint \left[D_x \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2 D_1 \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + D_y \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 4 D_{xy} \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 + D_x \frac{12}{h^2} e_1^2 \int dx dy - \iint q w dx dy + \frac{1}{2} \iint K w^2 dx dy \right] \dots (3.125)$$

According to the principle of minimum potential energy, the displacements satisfying the equilibrium conditions make the potential energy, V , minimum. In order for the integral of Eq.(3.125) to be an extremum, its integrand, F , must satisfy the following Euler's variational principle :

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial w_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial w_y} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial w_{xx}} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial F}{\partial w_{yy}} \right) + \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial F}{\partial w_{xy}} \right) = 0 \dots (3.126)$$

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) = 0 \dots (3.127)$$

and

$$\frac{\partial F}{\partial v} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial v_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial v_y} \right) = 0 \quad \dots (3.128)$$

Application of Eqs.(3.127) and (3.128) to Eq.(3.125) yields

$$\frac{\partial}{\partial x} (e_1) = 0 \quad \dots (3.129)$$

$$\frac{\partial}{\partial y} (e_1) = 0 \quad \dots (3.130)$$

Thus

$$e_1 = C \quad \dots (3.131)$$

a normalised constant of integration to be determined. Applying Eq.(3.126) to Eq.(3.125) and considering Eq.(3.131) one gets

$$\frac{\partial^4 v}{\partial x^4} + K_1^2 \frac{\partial^4 v}{\partial y^4} + \frac{2(D_1 + 2 D_{xy})}{D_x} \frac{\partial^4 v}{\partial x^2 \partial y^2} - \frac{12C}{h^2} \left(\frac{\partial^2 v}{\partial x^2} + K_1 \frac{\partial^2 v}{\partial y^2} \right) + \frac{K}{D_x} = \frac{q}{D_x} \quad \dots (3.132)$$

Introducing the notation

$$H = D_1 + 2 D_{xy}$$

Eq.(3.132) can be written as

$$\frac{\partial^4 w}{\partial x^4} + K_1^2 \frac{\partial^4 w}{\partial y^4} + 2 \frac{H}{D_x} \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} - \frac{12C}{h^2} \left(\frac{\partial^2 w}{\partial x^2} + K_1 \frac{\partial^2 w}{\partial y^2} \right) + \frac{K}{D_x} = \frac{q}{D_x} \quad \dots (3.133)$$

For a slab with two way reinforcement in the directions x and y , H can be taken as [Timoshenko (1959), P.366]

$$H = \sqrt{D_x D_y}$$

Introducing now

$$x_1 = x$$

$$y_1 = y \left(\frac{D_x}{D_y} \right)^{\frac{1}{2}} \quad \dots (3.134)$$

Eq.(3.133) is reduced to the form

$$\left(\nabla^2 - \alpha^2 \right) \nabla^2 w + \frac{K}{D_x} w = \frac{q}{D_x} \quad \dots (3.135)$$

in which

$$\alpha^2 = \frac{12C}{h^2}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}$$

SOLUTION OF PROBLEM

To solve equation (3.135) tri-linear co-ordinates as shown by Sen, B. (1968) has been used. Let the plate be in the form of an equilateral triangle, ABC (Fig.3.17) having each side of length, $2a$. Let the centroid, O, be the origin, X - axis and Y - axis perpendicular and parallel to the base BC, respectively. If x_1, y_1 be the cartesian co-ordinates of any point, P, within the triangle, P_1, P_2, P_3 be the three perpendiculars from P on CA, AB, and BC respectively, and r , the radius of the inscribed circle, then

$$P_1 = r + \frac{x_1}{2} - \frac{y_1 \sqrt{3}}{2},$$

$$P_2 = r + \frac{x_1}{2} + \frac{y_1 \sqrt{3}}{2},$$

$$P_3 = r - x_1,$$

$$P_1 + P_2 + P_3 = \sqrt{3}a = K_2 = \text{constant},$$

and

$$\begin{aligned} \nabla^2 &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} \\ &= \frac{\partial^2}{\partial P_1^2} + \frac{\partial^2}{\partial P_2^2} + \frac{\partial^2}{\partial P_3^2} - \frac{\partial^2}{\partial P_1 \partial P_2} - \frac{\partial^2}{\partial P_2 \partial P_3} - \frac{\partial^2}{\partial P_3 \partial P_1} \end{aligned}$$

Using the trilinear co-ordinates (P_1, P_2, P_3) the deflection, w can be taken in the form

$$W = \sum_{n=1}^{\infty} A_n \left[\sin \frac{2n\pi P_1}{K_2} + \sin \frac{2n\pi P_2}{K_2} + \sin \frac{2n\pi P_3}{K_2} \right] \quad \dots (3.136)$$

where

$$A_n = \text{a constant.}$$

The above form of W satisfies the following boundary conditions of simply supported edges :

$$\left. \begin{array}{l} W = 0 \\ \nabla^2 W = 0 \end{array} \right\} \text{at } P_1 = 0, P_2 = 0, P_3 = 0$$

Expanding the transverse uniform load, q , into Fourier sine series

$$q = \sum_{n=1}^{\infty} \frac{2q}{n\pi} \left[\sin \frac{2n\pi P_1}{K_2} + \sin \frac{2n\pi P_2}{K_2} + \sin \frac{2n\pi P_3}{K_2} \right] \quad \dots (3.137)$$

and substituting Eqs.(3.136) and (3.137) into Eq.(3.135) one gets

$$A_n = \sum_{n=1}^{\infty} \frac{2q}{n\pi D_x} \cdot \frac{1}{\left[\left(\frac{2n\pi}{K_2} \right)^4 + \alpha^2 \left(\frac{2n\pi}{K_2} \right)^2 + \frac{K}{D_x} \right]} \quad \dots (3.138)$$

Thus w is determined.

To determine α , Eq.(3.124b) is transformed into x_1, y_1 co-ordinates in the following form

$$\frac{\alpha^2 h^2}{12} = \frac{\partial u}{\partial x_1} + \sqrt{K_1} \frac{\partial v}{\partial y_1} + \frac{1}{2} \left(\frac{\partial w}{\partial x_1} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y_1} \right)^2 \quad \dots (3.139)$$

The boundary conditions on u and v are

$$u = 0 \quad \text{at } P_3 = 0 \quad \dots (3.140)$$

$$\sqrt{3} v + u = 0 \quad \text{at } P_2 = 0 \quad \dots (3.141)$$

$$\sqrt{3} v - u = 0 \quad \text{at } P_1 = 0 \quad \dots (3.142)$$

The following forms of u and v satisfy the above boundary conditions.

$$u = \sum_{m=1}^{\infty} \sqrt{3} B_m \left[\sin \frac{2m\pi (P_2 + P_3)}{K_2} + \sin \frac{2m\pi (P_1 + P_3)}{K_2} \right] \quad \dots (3.143)$$

$$v = \sum_{m=1}^{\infty} \frac{1}{\sqrt{K_1}} B_m \left[\sin \frac{2m\pi (P_1 + P_3)}{K_2} - \sin \frac{2m\pi (P_2 + P_3)}{K_2} \right] \quad \dots (3.144)$$

in which B_m is a constant.

Substituting the expressions for u, v and w into Eq.(3.139) and integrating over the whole area of the plate, the following equation determining α is obtained.

$$\frac{\alpha^2 h^2}{12} = \sum_{n=1}^{\infty} \frac{3 A_n^2 n^2 \pi^2}{K_2^2} \quad \dots (3.145)$$

Thus W is completely determined in the following form in x, y co-ordinates

$$W = A_n \left[3 \sin 2n\pi \left(\frac{1}{3} + \frac{x}{2\sqrt{3}a} \right) \cos \frac{2n\pi y}{2\sqrt{K_1} a} + \sin 2n\pi \left(\frac{1}{3} - \frac{x}{\sqrt{3}a} \right) \right] \quad \dots (3.146)$$

If $D_x = D_y = D$, $\alpha \rightarrow 0$, and $K = 0$, Eqs.(3.136) and (3.138) give the small deflection result for an isotropic plate not resting on the elastic foundation in the following form :

$$W = \sum_{n=1}^{\infty} \frac{q K_2^4}{8 n^5 \pi^5 D} \left[\sin \frac{2n\pi P_1}{K_2} + \sin \frac{2n\pi P_2}{K_2} + \sin \frac{2n\pi P_3}{K_2} \right] \quad \dots (3.147)$$

The corresponding equation as obtained by S. Woinowsky - Krieger [1959, P.313] for a plate having each side of length $\frac{2a}{\sqrt{3}}$ is

$$W = \frac{q}{64aD} \left[x^3 - 3y^2x - a(x^2 + y^2) + \frac{4}{27} a^3 \right] \left(\frac{4}{9} a^2 - x^2 - y^2 \right) \quad \dots (3.147a)$$

At the origin ($P_1 = P_2 = P_3$), w is given by Eq.(3.147)

$$W = \frac{27 q a^4}{8 \pi^5 D} \sum_{n=1}^{\infty} \frac{1}{n^5} \sin \frac{2n\pi}{3} = 0.009 \frac{qa^4}{D} \quad \dots (3.147b)$$

which is numerically equal to that obtained from Eq.(3.147a) for the plate having each side of length $2a$ as

$$(W)_{x=y=0} = \frac{qa^4}{103D} = 0.009 \frac{qa^4}{D}$$

RESULTS

To calculate deflection at any point within the plate, one has to start from Eq.(3.145) with an assumed value of (αa) leading to the corresponding value of the load function $\frac{qa^4}{D_x h}$. Once this relationship is obtained, the corresponding deflection can be obtained from the Eq.(3.136) and with the help of Eq.(3.133).

At the origin maximum deflection is obtained and is given by

$$\frac{w_{\max.}}{h} = \frac{6}{\pi} \left(\frac{qa^4}{D_x h} \right) \sum_{n=1}^{\infty} \frac{\sin \frac{2n\pi}{3}}{\sqrt{\frac{16\lambda^4 n^4}{9} + \frac{4 \pi^2 n^2 \alpha^2 a^2}{3}}} + K_F \quad \dots (3.148)$$

in which the nondimensional foundation modulus

$$K_F = \frac{Ka^4}{D_x}$$

For $K_f = 0$ and $K_f = 100$ graphs are plotted in Fig.3.18 showing the deflection $\frac{W}{h}$ at the centroid of the plate against the load. Fig.3.18 also contains a graph plotted according to the linear theory.

From Fig.3.18 it is clear that design calculations should be made according to the nonlinear theory, because deflections calculated according to small deflection theory will be far from the actual values for higher values of load function. The effect of the foundation is to reduce the deflection for a given value of load function.

Because the deflection, W , has been determined, bending moments and stresses can be computed easily. The bending moments M_x , and M_y at the centroid of the plate are obtained as

$$M_x = 4(1 + \nu_c)qa^2 \sum_{n=1}^{\infty} \frac{n \sin \frac{2n\pi}{3}}{\left[\frac{16 \pi^4 n^4}{9} + \frac{4 \pi^2 n^2 \alpha^2 a^2}{3} + K_f \right]} \dots (3.149)$$

$$M_y = K_1 M_x \dots (3.150)$$

in which ν_c is the Poisson's ratio for concrete.

For isotropic plate without elastic foundation and undergoing small deflection $\nu_c = \nu$, $K_1 = 1$, $K_f = 0$, $\alpha \rightarrow 0$ and for a plate having each side of length $\frac{2a}{\sqrt{3}}$, Eq.(3.149) and (3.150) lead to

$$M_x = M_y = (1 + \nu) \frac{qa^2}{54} \dots (3.151)$$

which is the same result obtained by Timoshenko [1959, P. 314].

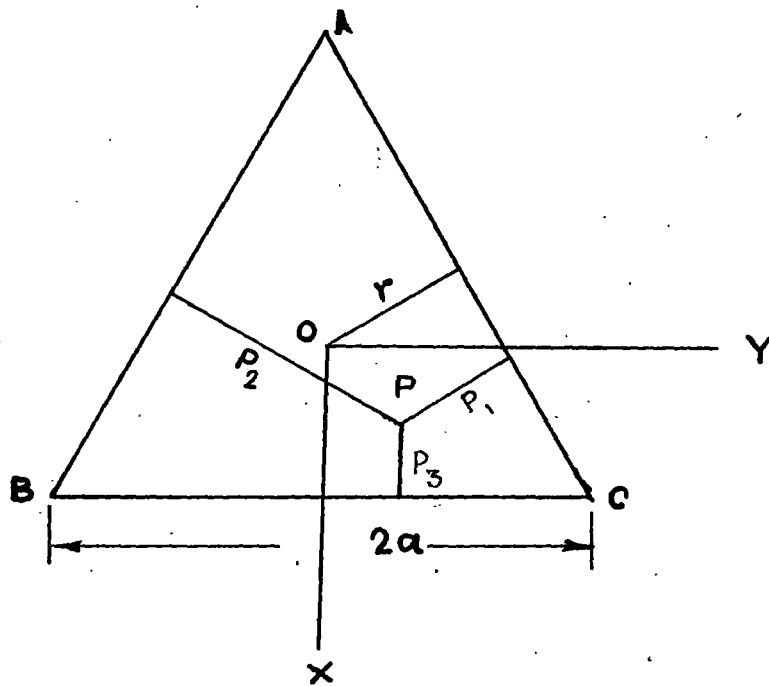


FIG. 3'17

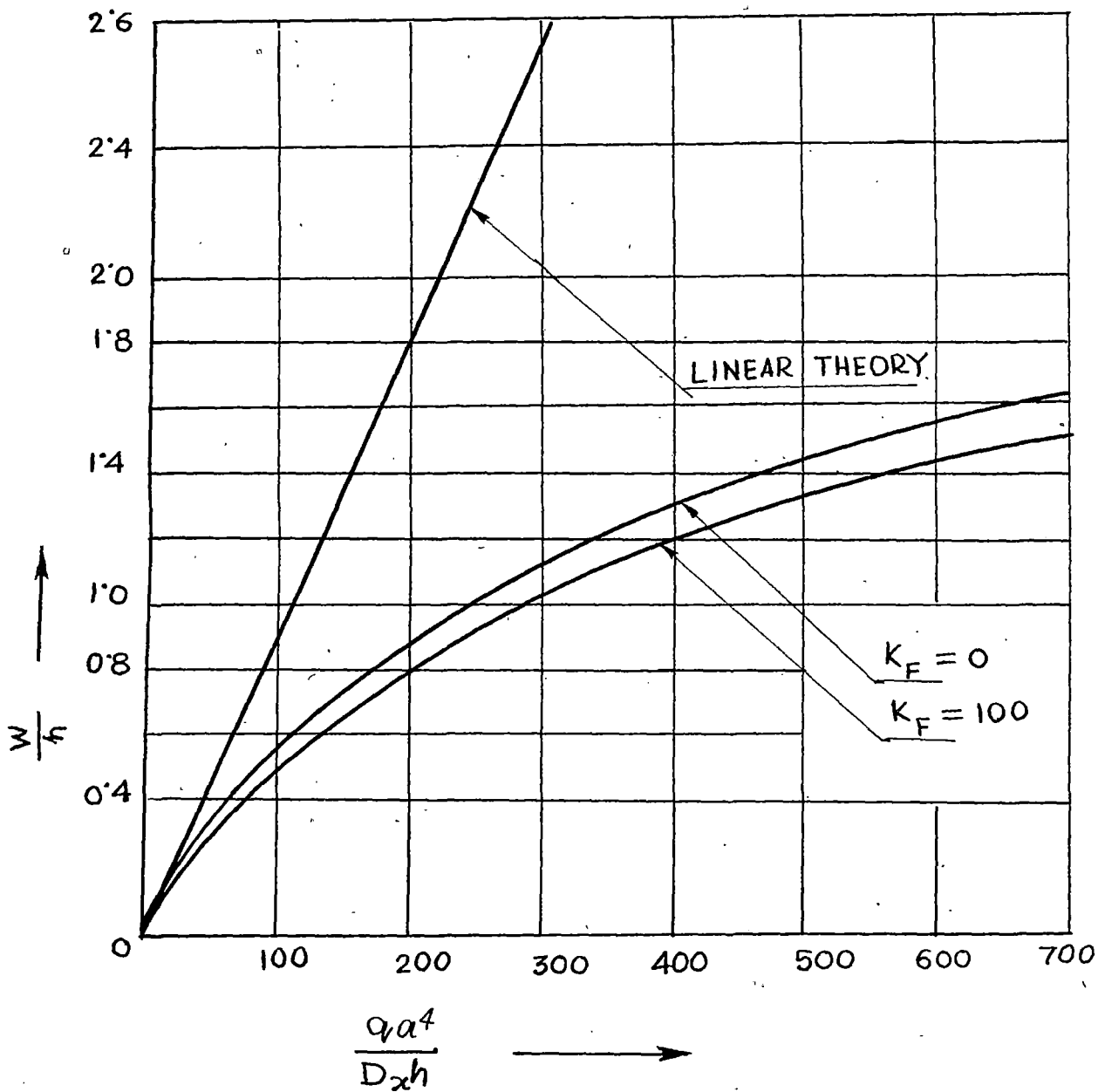


FIG. 3'18 DEFLECTION CURVE,

LARGE DEFLECTION OF A HEATED ELLIPTIC PLATE
ON ELASTIC FOUNDATION *

PAPER - VI

INTRODUCTION

In recent years there has been a rapid development of thermoelasticity stimulated by various engineering sciences. In the field of machine structures, mainly with air-craft, steam and gasturbines and in chemical and nuclear engineering, thermal stresses play an important and frequently even a primary role. Determination of thermal deflections of plates, espacially of thin plates, is of vital importance in the design of machine structures, because excessive deflections may cause heavy undesirable thermal stresses.

Following Berger many non-linear plate problems have been solved under various edge conditions and different types of loads. Berger's technique of neglecting the second invariant of the middle surface strains has been extended by Basuli (1963) to the large deflection problems of heated plates to obtain the large deflections of heated rectangular, circular and right-angled triangular plates without any elastic foundation and under uniform load and ^{stationary} stationary temperature

* Published in the Journal of the Indian Institute of Science, Bangalore, Vol. 58, No.5, 1976.

distribution. Sinha [1963] has extended this method of Berger to investigate the large deflections of circular and rectangular plates on elastic foundation of Winkler type.

In this paper the author has applied the method of Basuli and Sinha to investigate the large deflection of an elliptic plate placed on elastic foundation and heated under stationary temperature distribution. The deflection is obtained in terms of Mathieu function of the first kind and of zero order. Numerical results have been presented in the form of graphs.

FORMULATION OF PROBLEM

The strain energy due to bending and stretching of the middle surface of the plate is given by

$$V_1 = \frac{D}{2} \iint \left[(\nabla^2 w)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu) \left\{ \frac{12}{h^2} e_2 + \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} \right] dx dy \quad \dots (3.152)$$

Combining the potential energy of the foundation reaction and also the potential energy due to heating with Eq.(3.152) and neglecting eq, the modified energy expression for the total energy becomes

$$\begin{aligned}
 V = & \frac{D}{2} \iint \left[(\nabla^2 w)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu) \left\{ \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} + \right. \\
 & \left. + \frac{K}{D} w^2 \right] dx dy - \iiint_{-h/2}^{h/2} \frac{E\alpha T'}{1-\nu} (e_1 - z \nabla^2 w) dx dy dz \\
 & \dots (3.153)
 \end{aligned}$$

in which T' is the temperature distribution at any point given by [Basuli, (1963)]

$$T'(x,y,z) = T_0(x,y) + g(z) T(x,y) \quad \dots (3.154)$$

and

$$\int_{-h/2}^{h/2} z g(z) dz = f(h) \quad ; \quad \int_{-h/2}^{h/2} g(z) dz = 0 \quad \dots (3.155)$$

Combining Eqs. (3.153), (3.154) and (3.155) one gets

$$\begin{aligned}
 V = & \frac{D}{2} \iint \left[(\nabla^2 w)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu) \left\{ \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} + \right. \\
 & \left. + \frac{K}{D} w^2 \right] dx dy - \iint \frac{E\alpha}{1-\nu} \left\{ T_0 e_1 h - f(h) T \nabla^2 w \right\} dx dy \quad \dots (3.156)
 \end{aligned}$$

According to the principle of minimum potential energy, the displacements that satisfy the equilibrium conditions make the potential energy, V , minimum. In order for the integral of Eq.(3.156) to be an extremum, the integrand, F , must satisfy the following Euler's equations of the calculus of variation

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) = 0 \quad \dots (3.157a)$$

$$\frac{\partial F}{\partial v} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial v_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial v_y} \right) = 0 \quad \dots (3.157b)$$

$$\begin{aligned} & \frac{\partial F}{\partial w} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial w_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial w_y} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial w_{xx}} \right) + \\ & + \frac{\partial^2}{\partial y^2} \left(\frac{\partial F}{\partial w_{yy}} \right) + \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial F}{\partial w_{xy}} \right) = 0 \quad \dots (3.157c) \end{aligned}$$

Application of the Eqs. (3.157a), (3.157b) and (3.157c) to Eq.(3.156) yields

$$\frac{\partial}{\partial x} \left\{ e_1 - (1+\nu) \alpha T_0 \right\} = 0 \quad \dots (3.158a)$$

$$\frac{\partial}{\partial y} \left\{ e_1 - (1+\nu) \alpha T_0 \right\} = 0 \quad \dots (3.158b)$$

$$\nabla^4 w - \frac{12}{h^2} \left\{ e_1 - (1+\nu) \alpha T_0 \right\} \nabla^2 w + \frac{k}{D} w + \frac{E \alpha f(h)}{D(1-\nu)} \nabla^2 T = 0 \quad \dots (3.158c)$$

Eqs.(3.153a) and (3.153b) prove that

$\{e_1 - (1+\nu)\alpha T_0\}$ is independent of x and y and therefore

$$e_1 - (1+\nu)\alpha T_0 = \text{constant} = \frac{\beta^2 h^2}{12} \quad \dots (3.159a)$$

in which β is a normalised constant of integration, and

$$e_1 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \quad \dots (3.159b)$$

Considering Eq.(3.159a), Eq.(3.153a) reduces to

$$\nabla^2 (\nabla^2 - \beta^2) w + \frac{k}{D} w = - \frac{E \alpha f(h)}{D(1-\nu)} \nabla^2 T \quad \dots (3.160)$$

SOLUTION OF PROBLEM

Let us take an elliptic plate of thickness, h . The centre of the plate in the middle surface is taken as the origin and the Z - axis downwards.

If there is no source of heat inside the plate the following differential equations must be satisfied for stationary temperature distribution [Nowacki (1962)]

$$\nabla^2 T_0 - \epsilon T_0 = - \frac{\epsilon_0}{2} (\theta_1 + \theta_2) \quad \dots (3.161)$$

$$\nabla^2 T - \frac{12}{h^2} (1 + \epsilon) T = - \frac{12\epsilon}{h^3} (\theta_1 - \theta_2) \quad \dots (3.162)$$

in which θ_1 and θ_2 denote temperatures at the upper and lower media of the plate respectively.

If $\theta_1 = \theta_2$, Eq.(3.162) becomes

$$\nabla^2 T - \beta_1^2 T = 0 \quad \dots (3.163)$$

in which

$$\beta_1^2 = (1 + \epsilon) \frac{12}{h^2} \quad \dots (3.164)$$

Transferring to elliptic co-ordinates (ξ, η) defined by

$x + iy = d \cosh(\xi + i\eta)$, where $2d$ is the interfocal distance of the ellipse, Eq.(3.163) reduces to

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2} - \frac{\beta_1^2 d^2}{2} (\cosh 2\xi - \cos 2\eta) T = 0 \quad \dots (3.165)$$

Solution of Eq.(3.165) can be taken in the following form

$$T = \sum_{m=0}^{\infty} C_{2m} C_{e_{2m}}(\xi, -q) \mathcal{L}_{e_{2m}}(\eta, -q) \quad \dots (3.166)$$

in which $C_{e_{2m}}(\xi, -q)$ and $\mathcal{L}_{e_{2m}}(\eta, -q)$ are modified Mathieu function and ordinary Mathieu function of the first kind and of order $2m$ respectively, and

$$q = \frac{\beta_1^2 d^2}{4} \quad \dots (3.167)$$

while solving a problem of bending of a plate with an elliptic hole, by taking a single Mathieu function of the second order instead of taking Mathieu functions of all orders, Naghdi (1955) has shown that the results are satisfactory for larger elliptic holes. In this paper also similar approximation is made by taking Mathieu function of zero order and on this assumption Eq.(3.166) reduces to

$$T = C_0 C_{e_0}(\xi, -\nu) \mathcal{L}_{e_0}(\eta, -\nu) \quad \dots (3.168)$$

The following boundary condition is imposed on T

$$T = \text{Constant} = K_1 \text{ on } \xi = \xi_0$$

with the above boundary condition Eq.(3.168) yields

$$K_1 = C_0 C_{e_0}(\xi_0, -\nu) \mathcal{L}_{e_0}(\eta, -\nu) \quad \dots (3.169)$$

Multiplying Eq.(3.169) by $\mathcal{L}_{e_0}(\eta, -\nu)$ and integrating with respect to η from 0 to 2π and using the orthogonality relation and normalisation [Melachian] one gets

$$C_0 = \frac{2 A_0^{(0)} K_1}{C_{e_0}(\xi_0, -\nu)} \quad \dots (3.170)$$

in which $A_0^{(0)}$ is the first Fourier coefficient in the expansion of $\mathcal{L}_{e_0}(\eta, -\nu)$.

Therefore

$$T = \frac{2A_0^{(0)} K_1}{C_{e_0}(\xi_0, -\eta)} C_{e_0}(\xi, -\eta) \mathcal{L}_{e_0}(\eta, -\eta) \quad \dots (3.171)$$

is determined.

Changing Eq.(3.160) to elliptic co-ordinates and substituting the expression for $\nabla^2 T$ one gets

$$(\nabla^2 - \rho_1^2)(\nabla^2 - \rho_2^2)W = \lambda C_{e_0}(\xi, -\eta) \mathcal{L}_{e_0}(\eta, -\eta) \quad \dots (3.172)$$

in which

$$\rho_1^2 + \rho_2^2 = \beta^2 \quad \dots (3.173)$$

$$\rho_1^2 \rho_2^2 = \frac{K}{D} \quad \dots (3.174)$$

$$\lambda = - \frac{E\alpha f(h)}{D(1-\nu)} \cdot \frac{2\beta_1^2 A_0^{(0)} K_1}{C_{e_0}(\xi_0, -\eta)} \quad \dots (3.175)$$

$$\nabla^2 = \frac{2}{d^2(\cosh 2\xi - \cos 2\eta)} \left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right] \quad \dots (3.176)$$

Complimentary function of Eq.(3.172) is given by

$$W = B_0 C_{e_0}(\xi, -\eta) \mathcal{L}_{e_0}(\eta, -\eta) + D_0 C_{e_0}(\xi, -\eta/2) \mathcal{L}_{e_0}(\eta, -\eta/2) \quad \dots (3.177)$$

in which

$$q_1 = \frac{p_1^2 d^2}{4} ; \quad q_2 = \frac{p_2^2 d^2}{4} \quad \dots (3.178)$$

Clearly the particular integral of Eq.(3.172) is

$$\frac{\lambda}{(\beta_1^2 - p_2^2)(\beta_1^2 - p_1^2)} C_{e_0}(\xi, -\eta) \mathcal{L}_{e_0}(\eta, -\eta) \quad \dots (3.179)$$

Thus the complete solution of Eq.(3.172) is

$$W = B_0 C_{e_0}(\xi, -\eta_1) \mathcal{L}_{e_0}(\eta, -\eta_1) + D_0 C_{e_0}(\xi, -\eta_2) \mathcal{L}_{e_0}(\eta, -\eta_2) + \\ + \frac{\lambda}{(\beta_1^2 - p_2^2)(\beta_1^2 - p_1^2)} C_{e_0}(\xi, -\eta) \mathcal{L}_{e_0}(\eta, -\eta) \quad \dots (3.180)$$

If the outer boundary of the plate $\xi = \xi_0$ be clamped, the boundary conditions are

$$(W)_{\xi=\xi_0} = \left(\frac{\partial W}{\partial \xi} \right)_{\xi=\xi_0} = 0 \quad \dots (3.181)$$

Using Eq.(3.181) in Eq.(3.180) one gets the following two conditional equations

$$B_0 C_{e_0}(\xi_0, -\eta_1) \mathcal{L}_{e_0}(\eta, -\eta_1) + D_0 C_{e_0}(\xi_0, -\eta_2) \mathcal{L}_{e_0}(\eta, -\eta_2) + \\ + \frac{\lambda}{(\beta_1^2 - p_2^2)(\beta_1^2 - p_1^2)} C_{e_0}(\xi_0, -\eta) \mathcal{L}_{e_0}(\eta, -\eta) = 0 \quad \dots (3.182a)$$

$$B_0 c'_{e_0}(\xi_0, -\eta_1) \ell_{e_0}(\eta_1, -\eta_1) + D_0 c'_{e_0}(\xi_0, -\eta_2) \ell_{e_0}(\eta_1, -\eta_2) + \\ + \frac{\lambda}{(\beta_1^2 - \beta_2^2)(\beta_1^2 - \beta_2^2)} c'_{e_0}(\xi_0, -\eta) \ell_{e_0}(\eta, -\eta) = 0$$

... (3.182b)

Multiplying Eqs.(3.182a) and (3.182b) by $\ell_{e_0}(\eta, -\eta_1)$ and integrating with respect to η from 0 to 2π and using the orthogonality relation and normalisation one gets

$$B_0 = - \frac{\Psi \Phi_2}{\Psi_1 \pi} [c_{e_0}(\xi_0, -\eta_2) c'_{e_0}(\xi_0, -\eta) - c_{e_0}(\xi_0, -\eta) c'_{e_0}(\xi_0, -\eta_2)]$$

... (3.183)

$$D_0 = \frac{\Psi \Phi_2}{\Psi_1 \Phi_1} [c_{e_0}(\xi_0, -\eta_1) c'_{e_0}(\xi_0, -\eta) - c_{e_0}(\xi_0, -\eta) c'_{e_0}(\xi_0, -\eta_1)]$$

... (3.184)

in which

$$\Psi = \frac{\lambda}{(\beta_1^2 - \beta_2^2)(\beta_1^2 - \beta_2^2)}$$

$$\Psi_1 = c_{e_0}(\xi_0, -\eta_2) c'_{e_0}(\xi_0, -\eta_1) - c_{e_0}(\xi_0, -\eta_1) c'_{e_0}(\xi_0, -\eta_2)$$

$$\Phi_1 = 2 \bar{A}_0^{(0)} \bar{A}_0^{(0)} + \sum_{\lambda=1}^{\infty} \bar{A}_{2\lambda}^{(0)} \bar{A}_{2\lambda}^{(0)}$$

$$\Phi_2 = 2 A_0^{(0)} \bar{A}_0^{(0)} + \sum_{\lambda=1}^{\infty} \bar{A}_{2\lambda}^{(0)} A_{2\lambda}^{(0)}$$

and $\bar{A}_{2\lambda}^{(0)}$,

$A_{2\lambda}^{(0)}$ and $A_{2\lambda}^{(0)}$ are the Fourier coefficients in the expression

of $\mathcal{L}_{e_0}(\eta, -\eta_1)$, $\mathcal{L}_{e_0}(\eta, -\eta_2)$ and $\mathcal{L}_{e_0}(\eta, -\eta)$ respectively.

The constants B_0 and D_0 thus being determined, the deflection W is known.

Let $T_0 =$ constant which is clearly solution of the differential equation (3.161).

To determine the constant β^2 , Eq.(3.159) is transformed into elliptic co-ordinates in the form

$$\begin{aligned} h_1 h_2 \left\{ \frac{\partial}{\partial \xi} \left(\frac{u_\xi}{h_2} \right) + \frac{\partial}{\partial \eta} \left(\frac{u_\eta}{h_1} \right) \right\} + \frac{1}{2} h_1 h_2 \left\{ \left(\frac{\partial w}{\partial \xi} \right)^2 + \left(\frac{\partial w}{\partial \eta} \right)^2 \right\} \\ = \frac{\beta^2 h^2}{12} + (1+\nu) \alpha T_0 \end{aligned} \quad \dots (3.185)$$

in which

$$h_1 = h_2 = \frac{1}{d \sqrt{\sinh^2 \xi + \sin^2 \eta}}$$

The boundary conditions for u_ξ and u_η are

$$u_\xi = 0 = u_\eta \text{ at } \xi = \xi_0 \quad \dots (3.186)$$

Let

$$u_\xi = \sum_{n=0}^{\infty} P(\xi) \cos 2n\eta \quad \dots (3.187)$$

$$u_\eta = \sum_{n=1}^{\infty} Q(\xi) \sin 2n\eta \quad \dots (3.188)$$

subject to the conditions

$$P(\xi_0) = Q(\xi_0) = 0$$

Substituting Eqs.(3.180), (3.187) and (3.188) in Eq.(3.185) and integrating over the surface of the plate one gets

$$\begin{aligned} & \int_0^{2\pi} \int_0^{\xi_0} \left\{ \left(\frac{\partial w}{\partial \xi} \right)^2 + \left(\frac{\partial w}{\partial \eta} \right)^2 \right\} d\xi d\eta \\ &= d^2 \left\{ \frac{\beta^2 h^2}{6} + 2(1+\nu) \alpha T_0 \right\} \int_0^{2\pi} \int_0^{\xi_0} (\sinh^2 \xi + \sin^2 \eta) d\xi d\eta \end{aligned}$$

... (3.189)

After evaluating the integrals the following equation leading to β is obtained.

$$\begin{aligned} & B_0 \left[\left(2 \{ A_0^{-(0)} \}^2 + \sum_{n=1}^{\infty} \{ \bar{A}_{2n}^{(0)} \}^2 \right) \left\{ \sum_{n=1}^{\infty} 4n^2 \{ A_{2n}^{(0)} \}^2 \Psi_2 + \right. \right. \\ & + \left. \sum_{n=1}^{\infty} \sum_{\substack{s=1 \\ n \neq s}}^{\infty} 2n s (-1)^n (-1)^s A_{2n}^{(0)} A_{2s}^{(0)} \Psi_3 \right\} + \\ & + \left(\sum_{n=1}^{\infty} 4n^2 \{ A_{2n}^{(0)} \}^2 \right) \left\{ (A_0^{(0)})^2 \xi_0 + A_0^{(0)} \sum_{n=1}^{\infty} A_{2n}^{(0)} (-1)^n \frac{\sinh 2n \xi_0}{n} + \right. \\ & \left. + \sum_{n=1}^{\infty} A_{2n}^{(0)} \Psi_4 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{\substack{s=1 \\ n \neq s}}^{\infty} (-1)^n (-1)^s A_{2n}^{(0)} A_{2s}^{(0)} \Psi_5 \right\} \left. \right] + \end{aligned}$$

$$\begin{aligned}
& + D_0^2 \left[\left(2 \{ \bar{A}_0^{(0)} \}^2 + \sum_{n=1}^{\infty} \{ \bar{A}_{2n}^{(0)} \}^2 \right) \left\{ \sum_{n=1}^{\infty} 4n^2 \{ A_{2n}^{(0)} \}^2 \Psi_2 + \right. \\
& + \sum_{\substack{n=1 \\ n \neq s}}^{\infty} \sum_{s=1}^{\infty} 2n s (-1)^n (-1)^s A_{2n}^{(0)} A_{2s}^{(0)} \Psi_3 \left. \right\} + \\
& + \left(\sum_{n=1}^{\infty} 4n^2 \{ \bar{A}_{2n}^{(0)} \}^2 \right) \left\{ \left(A_0^{(0)} \right)_{10}^2 + A_0^{(0)} \sum_{n=1}^{\infty} A_{2n}^{(0)} (-1)^n \frac{\text{Sinh} 2n \eta_0}{2n} + \right. \\
& + \sum_{n=1}^{\infty} A_{2n}^{(0)} \Psi_4 + \frac{1}{2} \sum_{\substack{n=1 \\ n \neq s}}^{\infty} \sum_{s=1}^{\infty} (-1)^n (-1)^s A_{2n}^{(0)} A_{2s}^{(0)} \Psi_5 \left. \right\} + \\
& + 2B_0 D_0 \left[\left(2 \bar{A}_0^{(0)} \bar{A}_0^{(0)} + \sum_{n=1}^{\infty} \bar{A}_{2n}^{(0)} \bar{A}_{2n}^{(0)} \right) \left\{ \sum_{n=1}^{\infty} 4n^2 A_{2n}^{(0)} A_{2n}^{(0)} \Psi_2 + \right. \\
& + \sum_{\substack{n=1 \\ n \neq s}}^{\infty} \sum_{s=1}^{\infty} 2n s (-1)^n (-1)^s A_{2n}^{(0)} A_{2s}^{(0)} \Psi_3 \left. \right\} + \\
& + \left(\sum_{n=1}^{\infty} 4n^2 \bar{A}_{2n}^{(0)} \bar{A}_{2n}^{(0)} \right) \left\{ A_0^{(0)} A_0^{(0)} \eta_0^2 + A_0^{(0)} \sum_{n=1}^{\infty} (-1)^n A_{2n}^{(0)} \frac{\text{Sinh} 2n \eta_0}{2n} + \right. \\
& + A_0^{(0)} \sum_{n=1}^{\infty} (-1)^n A_{2n}^{(0)} \frac{\text{Sinh} 2n \eta_0}{2n} + \sum_{n=1}^{\infty} A_{2n}^{(0)} A_{2n}^{(0)} \Psi_4 + \\
& + \frac{1}{2} \sum_{\substack{n=1 \\ n \neq s}}^{\infty} \sum_{s=1}^{\infty} (-1)^n (-1)^s A_{2n}^{(0)} A_{2s}^{(0)} \Psi_5 \left. \right\} + \\
& + 2B_0 \Psi \left[\left(2 A_0^{(0)} \bar{A}_0^{(0)} + \sum_{n=1}^{\infty} A_{2n}^{(0)} \bar{A}_{2n}^{(0)} \right) \left\{ \sum_{n=1}^{\infty} 4n^2 A_{2n}^{(0)} A_{2n}^{(0)} \Psi_2 + \right.
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{\pi=1 \\ \pi \neq \Delta}}^{\infty} \sum_{\Delta=1}^{\infty} 2\pi\Delta (-1)^{\pi} (-1)^{\Delta} A_{2\Delta}^{(0)'} a_{2\pi}^{(0)} \Psi_3 \} + \\
& + \left(\sum_{\pi=1}^{\infty} 4\pi^2 A_{2\pi}^{(0)} \bar{A}_{2\pi}^{(0)} \right) \left\{ a_0^{(0)} A_0^{(0)'} \xi_0 + a_0^{(0)} \sum_{\pi=1}^{\infty} (-1)^{\pi} A_{2\pi}^{(0)} \frac{\text{Sinh} 2\pi \xi_0}{2\pi} + \right. \\
& + A_0^{(0)'} \sum_{\pi=1}^{\infty} (-1)^{\pi} a_{2\pi}^{(0)} \frac{\text{Sinh} 2\pi \xi_0}{2\pi} + \sum_{\pi=1}^{\infty} a_{2\pi}^{(0)} A_{2\pi}^{(0)'} \Psi_4 + \\
& \left. + \frac{1}{2} \sum_{\substack{\pi=1 \\ \pi \neq \Delta}}^{\infty} \sum_{\Delta=1}^{\infty} (-1)^{\pi} (-1)^{\Delta} a_{2\pi}^{(0)} A_{2\Delta}^{(0)} \Psi_5 \right\} + \\
& + 2D_0 \Psi \left[\left(2A_0^{(0)} \bar{A}_0^{(0)} + \sum_{\pi=1}^{\infty} A_{2\pi}^{(0)} \bar{A}_{2\pi}^{(0)} \right) \left\{ \sum_{\pi=1}^{\infty} 4\pi^2 A_{2\pi}^{(0)''} a_{2\pi}^{(0)} \Psi_2 + \right. \right. \\
& + \sum_{\substack{\pi=1 \\ \pi \neq \Delta}}^{\infty} \sum_{\Delta=1}^{\infty} 2\pi\Delta (-1)^{\pi} (-1)^{\Delta} A_{2\Delta}^{(0)''} a_{2\pi}^{(0)} \Psi_3 \} + \\
& \left. + \left(\sum_{\pi=1}^{\infty} 4\pi^2 A_{2\pi}^{(0)} \bar{A}_{2\pi}^{(0)} \right) \left\{ a_0^{(0)} A_0^{(0)''} \xi_0 + a_0^{(0)} \sum_{\pi=1}^{\infty} (-1)^{\pi} A_{2\pi}^{(0)''} \frac{\text{Sinh} 2\pi \xi_0}{2\pi} + \right. \right. \\
& \left. + A_0^{(0)''} \sum_{\pi=1}^{\infty} (-1)^{\pi} a_{2\pi}^{(0)} \frac{\text{Sinh} 2\pi \xi_0}{2\pi} + \sum_{\pi=1}^{\infty} a_{2\pi}^{(0)} A_{2\pi}^{(0)''} \Psi_4 + \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{\substack{s=1 \\ n \neq s}}^{\infty} (-1)^n (-1)^s a_{2n}^{(0)} A_{2s}^{(0)} \Psi_5 \Big] + \\
& + \Psi^2 \left[\left(2 \{A_0^{(0)}\}^2 + \sum_{n=1}^{\infty} \{A_{2n}^{(0)}\}^2 \right) \left\{ \sum_{n=1}^{\infty} 4n^2 \{a_{2n}^{(0)}\}^2 \Psi_2 + \right. \right. \\
& + \sum_{\substack{n=1 \\ n \neq s}}^{\infty} \sum_{s=1}^{\infty} 2ns (-1)^n (-1)^s a_{2n}^{(0)} a_{2s}^{(0)} \Psi_3 + \\
& + \left. \left(\sum_{n=1}^{\infty} 4n^2 \{A_{2n}^{(0)}\}^2 \right) \left\{ (a_0^{(0)})^2 \xi_0 + a_0^{(0)} \sum_{n=1}^{\infty} a_{2n}^{(0)} (-1)^n \frac{\text{Sinh} 2n \xi_0}{n} + \right. \right. \\
& + \sum_{n=1}^{\infty} a_{2n}^{(0)} \Psi_4 + \left. \left. \frac{1}{2} \sum_{n=1}^{\infty} \sum_{\substack{s=1 \\ n \neq s}}^{\infty} (-1)^n (-1)^s a_{2n}^{(0)} a_{2s}^{(0)} \Psi_5 \right\} \right] \\
& = \frac{d^2}{2} \left\{ \frac{\beta^2 \hbar^2}{6} + 2(1+\nu) \alpha T_0 \right\} \text{Sinh} 2\xi_0
\end{aligned}$$

... (3.190)

in which

$A_{2n}^{(0)}$, $A_{2n}^{(1)}$ and $A_{2n}^{(2)}$ are the Fourier coefficients in the expansions of $C_{e_0}(\xi, -\eta)$, $C_{e_0}(\xi, -\eta_1)$ and $C_{e_0}(\xi, -\eta_2)$ respectively, and

$$\psi_2 = \frac{\text{Si}h 4n \xi_0}{8n} - \frac{\xi_0}{2},$$

$$\psi_3 = \frac{\text{Si}h (2n+2s) \xi_0}{2n+2s} - \frac{\text{Si}h (2n-2s) \xi_0}{2n-2s},$$

$$\psi_4 = \frac{\text{Si}h 4n \xi_0}{8n} + \frac{\xi_0}{2},$$

$$\psi_5 = \frac{\text{Si}h (2n+2s) \xi_0}{2n+2s} + \frac{\text{Si}h (2n-2s) \xi_0}{2n-2s}$$

Since β is determined, W is determined completely.

RESULTS

To find the deflection at a given point, one has to start from Eq.(3.190) with an assumed value of β leading to the corresponding value of $\frac{E \alpha f(h) k_1}{D(1-\nu)}$. With this value of $\frac{E \alpha f(h) k_1}{D(1-\nu)}$ and considering Eqs.(3.183) and (3.184) the deflection will be obtained from Eq.(3.180).

For numerical calculation the following values have been assumed.

$$\xi = 0, \quad \eta = \frac{\pi}{2}, \quad \xi_0 = 3, \quad d^2 = 2.5, \quad h = 1, \quad f(h) = h,$$

$$K_F = \frac{K}{D} \xi_0^4 = 100, \quad \epsilon = 0.03, \quad \nu = 0.3, \quad \alpha T_0 = 2.5 \times 10^{-3}$$

The interfocal distance being assumed and the values of β^2 , P_1^2 and P_2^2 being known, the values of q , q_1 and q_2 are determined.

q , q_1 , and q_2 being known the corresponding values of the Fourier coefficients as well as those of Mathieu functions are determined.

The maximum deflection W_0 is obtained at the centre of the plate.

These deflections are graphically presented in Fig. 3.19 in which $\frac{W_0}{h}$ for $K_F = 0$ and $K_F = 100$ are plotted against the nondimensional load function λ . By setting $\beta \rightarrow 0$ the deflections according to the linear theory is obtained. For comparison Fig. 3.19 also includes a straightline which represents small deflections for $K_F = 0$. The results obtained in this study could not be compared in absence of any known results.

From Fig. 3.19 it is observed that the error according to the linear theory increases progressively with the increase in load function. The solution proposed in this study is rapidly convergent and no computational difficulty other than computational effort is involved. The parameter q for the series $Le(\xi, q)$ may be real or imaginary and the corresponding coefficients can be computed with accuracy. The numerical results presented in this study are obtained by taking the first two terms of the series and sufficient for practical purposes. Since the deflection at any point is known the corresponding stresses can now be easily estimated.

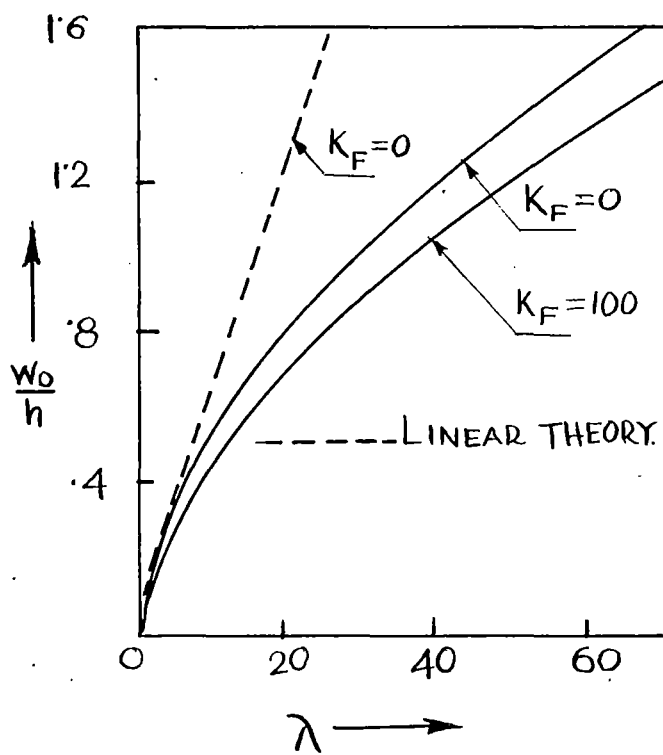


FIG. 319 LOAD-DEFLECTION CURVE.

CHAPTER-IV**LARGE DEFLECTIONS OF ELLIPTIC PLATES EXHIBITING
RECTILINEAR ORTHOTROPY ***INTRODUCTION

Treatments of the large deflections of orthotropic elliptic plates placed on elastic foundations are not found in literature. Only a few treatments of the large deflections of isotropic elliptic plates are found in literature and they are due to Perry [1950], Weill and Newmark [1956] and Nash and Cooley [1959]. These investigators employed various numerical methods for solution of the problem without ascertaining the appropriate stress function.

This paper deals with the large amplitude deflections and induced stresses in a clamped edge thin orthotropic elliptic plate under uniform load and placed on an elastic foundation of the Winkler type. The investigation is based on Von Kármán's equations generalised to the rectilinearly orthotropic case. After postulating the shape of the deflection surface, the stress function is found from the compatibility equation. The final solution for deflection is obtained by applying

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Galerkin's method. Results for isotropic elliptic and circular plates have been deduced from those obtained for orthotropic elliptic plates. These results are presented in the form of graphs and compared with other known results. The results obtained for the isotropic case are more accurate than the other known results and are in excellent agreement with the practical values given by Nash and Cooley [1959]. The results obtained for the orthotropic case could not be compared in absence of any known result.

ANALYSIS

Consider a flat elliptic rectilinearly orthotropic elastic plate with major axis $2a$ and minor axis $2b$. Let the origin O of a cartesian rectangular co-ordinate system x, y, z be located at the centre of the plate. Consider that the plate is clamped along the boundary and subjected to a uniform load of intensity q . The plate is placed on an elastic foundation of the Winkler type having the foundation reaction k_1 per unit area per unit deflection. Transverse deflections of the plate are considered to be large, that is of the order of magnitude of the plate thickness, h . Let the flexural and torsional rigidities of the plate be denoted by D_1, D_2 , and D_3 respectively,

$$D_1 = \frac{E_1 I}{1 - \nu_1 \nu_2}, \quad D_2 = \frac{E_2 I}{1 - \nu_1 \nu_2}, \quad D_3 = \frac{1}{2} (D_1 \nu_2 + D_2 \nu_1) + 2D_k \quad \dots (4.1)$$

where $I = \frac{h^3}{12}$, $D_k = G_{12} I$, E_1 and E_2 denote the Young's moduli in the x and y directions respectively, G_{12} denotes the shear modulus,

ν_1 represents the contraction in the y direction influenced by the tension in the x direction and ν_2 is given by the familiar relation $E_1 \nu_2 = E_2 \nu_1 = E^*$. In what follows the following symbolism will be utilised :

$$k^2 = \frac{D_2}{D_1} = \frac{E_2}{E_1}, \quad \nu^2 = \frac{D_3}{D_1} = 2 \frac{G_{12}}{E_1} (1 - k^2 \nu_1^2), \quad p^2 = \frac{E_2}{G_{12}} - 2 \nu_2$$

... (4.2)

With the above notations, the familiar von Karman equations generalised to rectilinearly orthotropic case take the following form :

$$\frac{\partial^4 w}{\partial x^4} + 2 \nu^2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + k^2 \frac{\partial^4 w}{\partial y^4} + \frac{k_1 w}{D_1} = \frac{h}{D_1} \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} - 2 \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 \phi}{\partial x^2} \right)$$

... (4.3)

$$\frac{\partial^4 \phi}{\partial x^4} + p^2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + k^2 \frac{\partial^4 \phi}{\partial y^4} = E_2 \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right]$$

... (4.4)

where w is the transverse deflection of the middle plane of the plate, ϕ is the stress function by means of which the membrane forces are represented in the customary form :

$$N_x = h \frac{\partial^2 \phi}{\partial y^2}, \quad N_y = h \frac{\partial^2 \phi}{\partial x^2}, \quad N_{xy} = -h \frac{\partial^2 \phi}{\partial x \partial y}$$

Let the deflection function $w(x,y)$ be taken in the following form satisfying the clamped edge boundary conditions

$$w(x,y) = w_0 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^2 \quad \dots (4.5)$$

Substitution of Eq.(4.5) into Eq.(4.4) yields

$$\frac{\partial^4 \phi}{\partial x^4} + p^2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + k^2 \frac{\partial^4 \phi}{\partial y^4} = \frac{16E_2 w_0^2}{a^2 b^2} \left[\frac{4x^2}{a^2} + \frac{4y^2}{b^2} - \frac{3x^4}{a^4} - \frac{3y^4}{b^4} - \frac{6x^2 y^2}{a^2 b^2} - 1 \right] \quad \dots (4.6)$$

As a solution of Eq.(4.6), the stress function, $\phi(x,y)$ is taken in the following form

$$\begin{aligned} \phi(x,y) = & \frac{1}{30.56} (A_1 x^3 + A_2 y^3) + \frac{1}{12.30} (A_3 a^2 x^6 + A_4 b^2 y^6) + \\ & + \frac{1}{24} (A_5 a^4 x^4 + A_6 b^4 y^4) + \frac{1}{2} (F_1 a^6 x^2 + F_2 b^6 y^2) + \frac{1}{360} x \\ & \times (B_1 x^6 y^2 + B_2 x^2 y^6) + \frac{1}{24} (H_1 a^2 x^4 y^2 + H_2 b^2 x^2 y^4) + \frac{1}{24} x \\ & \times B_3 x^4 y^4 + \frac{1}{4} B_4 a^2 b^2 x^2 y^2 \quad \dots (4.7) \end{aligned}$$

The 14 co-efficients, A_1, A_2, A_3 etc. in Eq.(4.7) are to be determined from Eq.(4.6) and the prescribed boundary conditions. By virtue of Eq.(4.6) the following six relations exist :

$$-\frac{3\lambda_1}{a^4} = A_1 + k^2 B_3 + \frac{p^2 B_1}{6}$$

$$-\frac{3\lambda_1}{b^4} = B_3 + A_2 k^2 + \frac{B_2 p^2}{6}$$

$$\frac{4\lambda_1}{a^2} = A_3 a^2 + H_2 b^2 k^2 + H_1 a^2 p^2$$

... (4.8)

$$\frac{4\lambda_1}{b^2} = H_1 a^2 + k^2 A_4 b^2 + H_2 p^2 b^2$$

$$-\frac{6\lambda_1}{a^2 b^2} = B_1 + B_2 k^2 + 6B_3 p^2$$

$$-\lambda_1 = A_5 a^4 + k^2 A_6 b^4 + B_4 p^2 a^2 b^2$$

where

$$\lambda_1 = \frac{16E_2 v_0^2}{a^2 b^2}$$

For the immovable edges, the inplane displacements u and v vanish on the boundary i.e.

$$u = \int_0^x \left\{ \left[\frac{E_2 \frac{\partial^2 \phi}{\partial y^2} - E'' \frac{\partial^2 \phi}{\partial x^2}}{E_1 E_2 - (E'')^2} \right] - \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right\} dx = 0$$

$$v = \int_0^y \left\{ \left[\frac{E_1 \frac{\partial^2 \phi}{\partial x^2} - E'' \frac{\partial^2 \phi}{\partial y^2}}{E_1 E_2 - (E'')^2} \right] - \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right\} dy = 0$$

on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$... (4.9)

Eq.(4.9) will yield the following eight additional conditions :

$$\frac{8 E_2 A_2}{525} + \frac{2 E_2 A_4}{45} + \frac{A_3 E_2}{3} + F_2 E_2 + B_1 \left[\frac{a^6}{b^6} \frac{E_2}{1260} - \frac{a^4}{b^4} \frac{E''}{210} \right] +$$

$$+ B_2 \left[\frac{2 a^2}{b^2} \frac{E_2}{315} - \frac{4 E''}{1575} \right] + H_1 \left[\frac{a^6}{b^6} \frac{E_2}{60} - \frac{a^4}{b^4} \frac{E''}{15} \right] + H_2 \left[\frac{E_2 a^2 / b^2}{15} - \frac{2 E''}{45} \right]$$

$$+ B_3 \left[\frac{a^4}{b^4} \frac{E_2}{35} - \frac{a^2}{b^2} \frac{4 E''}{105} \right] + B_4 \left[\frac{a^4}{b^4} \frac{E_2}{6} - \frac{a^2}{b^2} \frac{E''}{3} \right] - \frac{A_1 E''}{210} \frac{a^6}{b^6} -$$

$$- \frac{A_3 E''}{60} \frac{a^6}{b^6} - \frac{A_5 E''}{6} \frac{a^6}{b^6} - F_1 E'' \frac{a^6}{b^6} = 0 \quad \dots (4.10a)$$

$$- \frac{A_2 E_2}{210} + B_1 \left[\frac{a^6}{b^6} \frac{E_2}{1260} + \frac{E''}{84} \frac{a^4}{b^4} \right] + B_2 \left[\frac{E_2 a^2}{84 b^2} + \frac{E''}{1260} \right] -$$

$$- \frac{A_1 E''}{210} \frac{a^6}{b^6} - B_3 \left[\frac{E_2}{14} \frac{a^4}{b^4} + \frac{E''}{14} \frac{a^2}{b^2} \right] = 0 \quad \dots (4.10b)$$

$$\frac{A_2 E_2}{175} + \frac{A_4 E_2}{60} + B_2 \left[\frac{a^3}{b^2} \frac{E_2}{420} - \frac{E''}{1050} \right] + B_1 \left[\frac{E_2 a^6}{420 b^6} + \frac{2E'' a^4}{105 b^4} \right] -$$

$$- B_3 \left[\frac{4 E_2 a^4}{35 b^4} + \frac{E'' a^2}{70 b^2} \right] - \frac{A_1 E'' a^6}{70 b^6} - \frac{A_3 E'' a^6}{60 b^6} + H_1 x$$

$$x \left[\frac{a^6}{b^6} \frac{E_2}{60} + \frac{E'' a^4}{10 b^4} \right] - H_2 \left[\frac{E_2 a^2}{10 b^2} + \frac{E''}{60} \right] = 0 \quad \dots (4.10c)$$

$$- \frac{4 A_2 E_2}{525} - A_4 \frac{E_2}{45} - A_6 \frac{E_2}{6} + B_2 \left[\frac{2 E''}{1575} - \frac{E_2 a^2}{315 b^2} \right] +$$

$$+ H_2 \left[\frac{E''}{45} - \frac{E_2 a^2}{30 b^2} \right] +$$

$$+ B_4 \left[\frac{E_2 a^4}{6 b^4} + \frac{E'' a^2}{6 b^2} \right] - A_5 \frac{E'' a^6}{6 b^6} + H_1 \left[\frac{E_2 a^6}{30 b^6} + \frac{E'' a^4}{30 b^4} \right] +$$

$$+ B_3 \left[\frac{2 E'' a^2}{105 b^2} - \frac{1}{70} E_2 \frac{a^4}{b^4} \right] -$$

$$- \frac{A_1 E'' a^6}{70 b^6} - A_3 \frac{E'' a^6}{30 b^6} + B_1 \left[\frac{E_2 a^6}{420 b^6} + \frac{E'' a^4}{420 b^4} \right] = 0$$

... (4.10d)

$$\frac{8 A_1 E_1}{525} + B_1 \left[\frac{2 E_1 b^2}{315 a^2} - \frac{4 E''}{1575} \right] + \frac{2 A_3 E_1}{45} + H_1 \left[\frac{E_1 b^2}{15 a^2} - \frac{2 E''}{45} \right] +$$

$$+ A_5 \frac{E_1}{3} + E_1 E_1 -$$

$$- \frac{A_2 E''}{210} \frac{b^6}{a^6} - \frac{A_6 E''}{6} \frac{b^6}{a^6} - F_2 E'' \frac{b^6}{a^6} + B_3 \left[\frac{E_1 b^4}{35 a^4} - \frac{4 E''}{105} \frac{b^2}{a^2} \right] +$$

$$+ B_4 \left[\frac{E_1 b^4}{6 a^4} - \frac{E''}{3} \frac{b^2}{a^2} \right]$$

$$+ B_2 \left[\frac{E_1 b^6}{1260 a^6} - \frac{E'' b^4}{210 a^4} \right] + H_2 \left[\frac{E_1 b^6}{60 a^6} - \frac{E'' b^4}{15 a^4} \right] -$$

$$- \frac{A_4 E''}{60} \frac{b^6}{a^6} = 0$$

... (4.10e)

$$- \frac{A_1 E_1}{210} + B_1 \left[\frac{E''}{1260} + \frac{E_1 b^2}{84 a^2} \right] - B_3 \left[\frac{E'' b^2}{14 a^2} + \frac{E_1 b^4}{14 a^4} \right] +$$

$$+ B_2 \left[\frac{E'' b^4}{84 a^4} + \frac{E_1 b^6}{1260 a^6} \right] - A_2 \frac{E'' b^6}{210 a^6} = 0 \quad \dots (4.10f)$$

$$\begin{aligned}
& \frac{A_1 E_1}{175} + B_1 \left[-\frac{E_1 b^2}{420 a^2} - \frac{E''}{1050} \right] + B_2 \left[-\frac{E_1 b^6}{420 a^6} + \frac{2E''}{105} \frac{b^4}{a^4} \right] - \\
& - A_2 \frac{E''}{70} \frac{b^6}{a^6} + A_3 \frac{E_1}{60} - \\
& - B_3 \left[-\frac{E''}{70} \frac{b^2}{a^2} + \frac{4E_1}{35} \frac{b^4}{a^4} \right] - A_4 \frac{E''}{60} \frac{b^6}{a^6} - B_1 \left[-\frac{E_1}{10} \frac{b^2}{a^2} + \frac{E''}{60} \right] + \\
& + B_2 \left[-\frac{E''}{10} \frac{b^4}{a^4} + \frac{E_1}{60} \frac{b^6}{a^6} \right] = 0 \quad \dots (4.10g)
\end{aligned}$$

$$\begin{aligned}
& - \frac{4A_1 E_1}{525} - \frac{A_2 E''}{70} \frac{b^6}{a^6} - \frac{A_3 E_1}{45} - \frac{A_5 E_1}{6} - A_6 \frac{E''}{6} \frac{b^6}{a^6} - A_4 \frac{E''}{30} \frac{b^6}{a^6} \\
& + B_1 \left[-\frac{2E''}{1575} - \frac{E_1}{315} \frac{b^2}{a^2} \right] + B_2 \left[-\frac{E''}{420} \frac{b^4}{a^4} + \frac{E_1}{420} \frac{b^6}{a^6} \right] + B_3 \left[-\frac{2E''}{105} \frac{b^2}{a^2} - \frac{E_1}{70} \frac{b^4}{a^4} \right] \\
& + B_4 \left[-\frac{E_1}{6} \frac{b^4}{a^4} + \frac{E''}{6} \frac{b^2}{a^2} \right] + B_1 \left[-\frac{E''}{45} - \frac{E_1}{30} \frac{b^2}{a^2} \right] + B_2 \left[-\frac{E''}{30} \frac{b^4}{a^4} + \frac{E_1}{30} \frac{b^6}{a^6} \right] = 0 \\
& \dots (4.10h)
\end{aligned}$$

The equations in (4.8) and (4.10) are 14 in number. After solving these fourteen simultaneous equations one gets the following values of the fourteen unknown constants :

$$B_1 = \frac{\lambda \left[\left(\frac{2\beta_6}{a^2 b^2} + 6\alpha_2 p^2 \right) (\beta_3 \beta_6 + \beta_4 \beta_5) - (\alpha_1 \beta_6 + \alpha_2 \beta_5) (\beta_6 K^2 + 6\beta_4 p^2) \right]}{\left[(\beta_6 + 6\beta_2 p^2) (\beta_3 \beta_6 + \beta_4 \beta_5) - (\beta_1 \beta_6 + \beta_2 \beta_5) (\beta_6 K^2 + 6\beta_4 p^2) \right]}$$

... (4.11)

$$B_2 = \frac{\lambda \left[(\alpha_1 \beta_6 + \alpha_2 \beta_5) (\beta_6 + 6\beta_2 p^2) - \left(\frac{2\beta_6}{a^2 b^2} + 6\alpha_2 p^2 \right) (\beta_1 \beta_6 + \beta_2 \beta_5) \right]}{\left[(\beta_6 + 6\beta_2 p^2) (\beta_3 \beta_6 + \beta_4 \beta_5) - (\beta_1 \beta_6 + \beta_2 \beta_5) (\beta_6 K^2 + 6\beta_4 p^2) \right]}$$

... (4.12)

where

$$\lambda = -49 \frac{E_2 W_0^2}{a^2 b^2}$$

$$\beta_1 = \frac{E''}{1260} + \frac{E_1}{84} \frac{b^2}{a^2} + \frac{E_1 p^2}{1260}$$

$$\beta_2 = \frac{E_2}{1260} \frac{a^6}{b^6} + \frac{E''}{84} \frac{a^4}{b^4} + \frac{E''}{1260} \frac{p^2 a^6}{b^6}$$

$$\beta_3 = \frac{E''}{84} \frac{b^4}{a^4} + \frac{E_1}{1260} \frac{b^6}{a^6} + \frac{E''}{1260} \frac{p^2}{k^2} \frac{b^6}{a^6}$$

$$\beta_4 = \frac{E_2}{84} \frac{a^2}{b^2} + \frac{E''}{1260} + \frac{E_2}{1260} \frac{p^2}{k^2}$$

$$\beta_5 = \frac{E_1 k^2}{210} - \frac{E'' b^2}{14 a^2} - \frac{E_1 b^4}{14 a^4} + \frac{E'' b^6}{210 k^2 a^6}$$

$$\beta_6 = \frac{E_2 a^4}{14 b^4} + \frac{E'' a^2}{14 b^2} - \frac{E'' k^2 a^6}{210 b^6} - \frac{E_2}{210} k^2$$

$$\alpha_1 = \frac{E_1}{210 a^4} + \frac{E''}{210} \frac{b^2}{k^2 a^6}$$

$$\alpha_2 = \frac{E_2}{210 k^2 b^4} + \frac{E''}{210} \frac{a^2}{b^6}$$

$$B_3 = \frac{\lambda}{3 p^2 a^2 b^2} - B_1 \frac{1}{6 p^2} - B_2 \frac{k^2}{6 p^2} \quad \dots \quad (4.13)$$

$$A_1 = \frac{\lambda}{a^4} - B_1 \frac{p^2}{6} - B_3 k^2 \quad \dots \quad (4.14)$$

$$A_2 = \frac{\lambda}{k^2 b^4} - B_2 \frac{p^2}{6 k^2} - \frac{B_3}{k^2} \quad \dots \quad (4.15)$$

$$H_1 = \frac{\frac{\lambda}{45} (\alpha_3 \Phi_4 - \alpha_4 \Phi_3) - (M \Phi_4 - N \Phi_3)}{\Phi_1 \Phi_4 - \Phi_2 \Phi_3} \quad \dots \quad (4.16)$$

$$H_2 = \frac{\frac{\lambda}{45} (\alpha_4 \Phi_1 - \alpha_3 \Phi_2) - (N \Phi_1 - M \Phi_2)}{\Phi_1 \Phi_4 - \Phi_2 \Phi_3} \quad \dots \quad (4.17)$$

where $\alpha_3 = \frac{E_2}{k^2 b^4} - \frac{E'' a^2}{b^6}$

$$\alpha_4 = \frac{E_1}{a^4} - \frac{E'' b^2}{k^2 a^6}$$

$$\phi_1 = \frac{E_2}{60} \frac{a^6}{b^6} + \frac{E''}{10} \frac{a^4}{b^4} + \frac{E''}{60} \frac{p^2 a^6}{b^6} - \frac{E_2}{60} \frac{a^2}{k^2 b^2}$$

$$\phi_2 = \frac{E''}{60} \frac{b^4}{k^2 a^4} - \frac{E_1}{10} \frac{b^2}{a^2} - \frac{E''}{60} - \frac{E_1}{60} p^2$$

$$\phi_3 = \frac{E'' k^2 a^4 b^2}{60 b^6} - \frac{E_2}{10} \frac{a^2}{b^2} - \frac{E''}{60} - \frac{E_2}{60} \frac{p^2}{k^2}$$

$$\phi_4 = \frac{E''}{60} \frac{p^2 b^6}{k^2 a^6} + \frac{E''}{10} \frac{b^4}{a^4} + \frac{E_1}{60} \frac{b^6}{a^6} - \frac{E_1}{60} \frac{k^2 b^2}{a^2}$$

$$N = -A_1 \frac{E'' a^6}{70 b^6} + A_2 \frac{E_2}{175} + B_1 \left[-\frac{E_2}{420} \frac{a^6}{b^6} + \frac{2 E''}{105} \frac{a^4}{b^4} \right] +$$

$$+ B_2 \left[-\frac{E_2}{420} \frac{a^2}{b^2} - \frac{E''}{1050} \right] - B_3 \left[-\frac{4 E_2}{35} \frac{a^4}{b^4} + \frac{E''}{70} \frac{a^2}{b^2} \right] +$$

$$+ A_1 \frac{E_1}{175} - A_2 \frac{E'' b^6}{70 a^6} + B_1 \left[-\frac{E_1}{420} \frac{b^2}{a^2} - \frac{E''}{1050} \right] +$$

$$+ B_2 \left[-\frac{E_1}{420} \frac{b^6}{a^6} + \frac{2 E^*}{105} \frac{b^4}{a^4} \right] - B_3 \left[-\frac{E^*}{70} \frac{b^2}{a^2} + \frac{4 E_2}{35} \frac{b^4}{a^4} \right]$$

$$A_3 = - \left(\frac{4}{3} \frac{\lambda}{a^4} + H_1 \frac{b^2}{k^2} + H_2 k^2 \frac{b^2}{a^2} \right) \quad \dots (4.18)$$

$$A_4 = - \left(\frac{4}{3} \frac{\lambda}{k^2 b^4} + H_1 \frac{a^2}{k^2 b^2} + H_2 \frac{b^2}{k^2} \right) \quad \dots (4.19)$$

$$A_5 = \frac{(P\phi_8 - Q\phi_7) - \frac{\lambda}{3} (\alpha_6\phi_7 - \alpha_5\phi_8)}{\phi_5\phi_8 - \phi_6\phi_7} \quad \dots (4.20)$$

$$A_6 = \frac{(Q\phi_5 - P\phi_6) - \frac{\lambda}{3} (\alpha_5\phi_6 - \alpha_6\phi_5)}{\phi_5\phi_8 - \phi_6\phi_7} \quad \dots (4.21)$$

where

$$P = -A_1 \frac{E^* a^6}{70 b^6} - A_2 \frac{4E_2}{525} - A_3 \frac{E^* a^6}{30 b^6} - A_4 \frac{E_2}{45} + B_1 \left[\frac{E_2 a^6}{420 b^6} + \frac{E^* a^4}{420 b^4} \right]$$

$$+ B_2 \left[-\frac{2E^*}{1575} - \frac{E_2}{315} \frac{a^2}{b^2} \right] + B_3 \left[-\frac{2E^*}{105} \frac{a^2}{b^2} - \frac{E_2}{70} \frac{a^4}{b^4} \right] +$$

$$+ H_1 \left[-\frac{E_2}{30} \frac{a^6}{b^6} + \frac{E^*}{30} \frac{a^4}{b^4} \right] + H_2 \left[-\frac{E^*}{45} - \frac{E_2}{30} \frac{a^2}{b^2} \right]$$

$$\begin{aligned}
 \phi = & - A_1 \frac{4E_2}{525} - A_2 \frac{E'' b^6}{70 a^6} - A_3 \frac{E_1}{45} - A_4 \frac{E'' b^6}{30 a^6} + B_1 \left[\frac{2E''}{1575} - \frac{E_1 b^2}{315 a^2} \right] + \\
 & + B_2 \left[\frac{E'' b^4}{420 a^4} + \frac{E_1 b^6}{420 a^6} \right] + B_3 \left[\frac{2E'' b^2}{105 a^2} - \frac{E_1 b^4}{70 a^4} \right] + B_4 \left[\frac{E'' E_1 b^2}{45 \cdot 30 a^2} \right] + \\
 & + B_5 \left[\frac{E'' b^4}{30 a^4} + \frac{E_1 b^6}{30 a^6} \right]
 \end{aligned}$$

$$\phi_5 = \frac{E'' a^6}{6 b^6} + \frac{E_2 a^6}{6 p^2 b^6} + \frac{E'' a^4}{6 p^2 b^4}$$

$$\phi_6 = \frac{E_1}{6} + \frac{E_1 b^2}{6 p^2 a^2} + \frac{E''}{6 p^2}$$

$$\phi_7 = \frac{E_2}{6} + \frac{E_2 k^2 a^2}{6 p^2 b^2} + \frac{E'' k^2}{6 p^2}$$

$$\phi_8 = \frac{E'' b^6}{6 a^6} + \frac{E_1 k^2 b^6}{6 p^2 a^6} + \frac{E'' k^2 b^4}{6 p^2 a^4}$$

$$\alpha_5 = \frac{E_2}{6 p^2} \frac{a^2}{b^6} + \frac{E''}{6 p^2 b^4}$$

$$\alpha_6 = \frac{E_1}{6 p^2} \frac{b^2}{a^6} + \frac{E''}{6 p^2 a^4}$$

$$B_4 = \frac{\lambda}{3p^2 a^2 b^2} - A_5 \frac{a^2}{p^2 b^2} - A_6 \frac{E^* a^2}{p^2 a^2} \quad \dots (4.22)$$

$$F_1 = \frac{R E^* \frac{b^6}{a^6} + T E_2}{(E^*)^2 - E_1 E_2} \quad \dots (4.23)$$

$$F_2 = \frac{T E^* \frac{a^6}{b^6} + R E_1}{(E^*)^2 - E_1 E_2} \quad \dots (4.24)$$

where

$$R = -A_1 \frac{E^* a^6}{210 b^6} + A_2 \frac{8 E_2}{525} - A_3 \frac{E^* a^6}{60 b^6} + A_4 \frac{2E_2}{45} - A_5 \frac{E^* a^6}{6 b^6} + A_6 \frac{E_2}{3}$$

$$+ B_1 \left[\frac{E_2 a^6}{1260 b^6} - \frac{E^* a^4}{210 b^4} \right] + B_2 \left[\frac{252 a^2}{315 b^2} - \frac{4E^*}{1675} \right] + B_3 \left[\frac{E_2 a^4}{35 b^4} - \frac{4E^* a^2}{105 b^2} \right]$$

$$+ B_4 \left[\frac{E_2 a^4}{6 b^4} - \frac{E^* a^2}{3 b^2} \right] + B_5 \left[\frac{E_2 a^6}{60 b^6} - \frac{E^* a^4}{15 b^4} \right] + B_6 \left[\frac{E_2 a^2}{15 b^2} - \frac{2E^*}{45} \right]$$

$$T = A_1 \frac{8E_1}{525} - A_2 \frac{E^* b^6}{210 a^6} + A_3 \frac{2E_1}{45} - A_4 \frac{E^* b^6}{60 a^6} + A_5 \frac{E_1}{3} - A_6 \frac{E^* b^6}{6 a^6} +$$

$$\begin{aligned}
& + B_1 \left[\frac{25E_1}{315} \frac{b^2}{a^2} - \frac{4E''}{1575} \right] + B_2 \left[\frac{E_1}{1260} \frac{b^6}{a^6} - \frac{E''}{210} \frac{b^4}{a^4} \right] + \\
& + B_3 \left[\frac{E_1}{35} \frac{b^4}{a^4} - \frac{4E''}{105} \frac{b^2}{a^2} \right] + B_4 \left[\frac{E_1}{6} \frac{b^4}{a^4} - \frac{E''}{3} \frac{b^2}{a^2} \right] + \\
& + H_1 \left[\frac{E_1}{15} \frac{b^2}{a^2} - \frac{2E''}{45} \right] + H_2 \left[\frac{E_1}{60} \frac{b^6}{a^6} - \frac{E''}{15} \frac{b^4}{a^4} \right]
\end{aligned}$$

This completes the determination of the stress function $\phi(x, y)$. We now apply the procedure of Galerkin to obtain the deflection function $w(x, y)$. As a final result of a lengthy calculation we obtain the following cubic equation determining the central deflection w_0

$$\left(\frac{24}{a^4} + \frac{16}{a^2 b^2} + \frac{24 k^2}{b^4} + 0.634 \frac{k_1}{D_1} \right) w_0 D_1 + 24 \times 10^{-4} \frac{w_0 h}{ab}$$

$$\begin{aligned}
& \left[0.3 \left(A_1 \frac{a^7}{b} + A_2 \frac{b^7}{a} \right) + 15 \left(A_3 \frac{a^7}{b} + A_4 \frac{b^7}{a} \right) - 70 \left(A_5 \frac{a^7}{b} + A_6 \frac{b^7}{a} \right) - \right. \\
& - 9 \left(B_1 a^5 b + B_2 a b^5 \right) + 70 \left(H_1 a^5 b + H_2 a b^5 \right) - 5.7 B_3 a^3 b^3 + \\
& \left. + 250 B_4 a^3 b^3 + 415 \left(F_1 \frac{a^7}{b} + F_2 \frac{b^7}{a} \right) \right] = \gamma \quad \dots (4.25)
\end{aligned}$$

Thus the deflection function is completely determined.

RESULTS

Numerical results are presented for two real wooden materials (plywood and delta product) having roughly the following elastic properties [Nowinski (1963)] and taking $a = 2b$ for the ellipse.

| Case | E_1 | E_2 | G_{12} | ν_1 | ν_2 | k^2 | l^2 | p^2 |
|------|-----------------|--------------------|--------------------|---------|---------|-------|-------|-------|
| 1 | 1×10^6 | 0.5×10^5 | 0.1×10^5 | 0.05 | 0.025 | 0.5 | 0.223 | 5 |
| 2 | 1×10^5 | 0.05×10^5 | 0.05×10^5 | 0.2 | 0.01 | 0.05 | 0.103 | 1 |
| 3 | E | E | G | 0.3 | 0.3 | 1 | 1 | 2 |

For a strong anisotropy ($E_2 : E_1 = \frac{1}{20}$) Eq.(4.25) reduces to

$$(0.26 + 0.057 K_F) \left(\frac{w_0}{h} \right) + 0.027 \left(\frac{w_0}{h} \right)^3 = \left(\frac{q b^4}{E_1 h^4} \right) \quad \dots (4.26)$$

where the nondimensional foundation modulus $K_F = \frac{k_1 b^4}{D_1}$.

For a weaker anisotropy ($E_2 : E_1 = \frac{1}{2}$) Eq.(4.25) reduces to

$$(1.2 + 0.057 K_F) \left(\frac{w_0}{h} \right) + 0.03 \left(\frac{w_0}{h} \right)^3 = \left(\frac{q b^4}{E_1 h^4} \right) \quad \dots (4.27)$$

and for isotropy,

$$(2.71 + 0.059 K_F) \left(\frac{w_0}{h} \right) + 1.53 \left(\frac{w_0}{h} \right)^3 = \left(\frac{q b^4}{E h^4} \right) \quad \dots (4.28)$$

Equations (4.26) and (4.27) are presented in Fig.4.1. Equation (4.28) is also presented in the same figure for comparison. The stress function and the deflection function being known, membrane and bending stresses

at any point in the plate can be easily computed. The peak stress occurs in the outer fibers at the ends of the minor axis of the ellipse in the direction of that axis, and the sum of the dimensionless membrane and bending stresses at that point by the theory of the present writer is shown in Fig.4.2.

As a check on the accuracy of the data for deflections and stresses for the elliptic plate, the deflections and the total nondimensional membrane and bending stresses, $(\frac{\sigma_r b^2}{E h^2})$ obtained for isotropic case are presented in Fig.4.3 and 4.4 respectively for comparison with the corresponding results obtained by [Weil and Newmark (1956) and Nash and Cooley (1959)]. It is observed that the deflections obtained by the present writer are in excellent agreement with the experimental values. But the stresses are at variance with those obtained by [Nash and Cooley].

As a further check on the accuracy of the data for deflection with foundation Eq.(4.25) is reduced for circular boundary and isotropy to the following equation

$$(5.86 + 0.055 K_f) \left(\frac{w_0}{h} \right) + 2.73 \left(\frac{w_0}{h} \right)^3 = \left(\frac{q a^4}{E h^4} \right) \quad \dots (4.29)$$

Equation (4.29) is presented in Fig.4.5 for comparison with the corresponding results obtained by Way [1934] and Bolton [1972]. It is observed from Fig.4.5 that the deflections obtained for $K_f = 0$ for circular boundary are in excellent agreement with an exact solution obtained by Way [1934] and agree very well with those results obtained by Bolton [1972] even for higher values of load function. For $K_f \rightarrow 0$ the results of Fig.4.5 agree very well with those obtained by Bolton

and Sinha [1963]. Nondimensional membrane stresses, $(\frac{\sigma_m a^2}{E h^2})$ for circular boundary are also calculated and they are presented in Fig.4.6. It is observed that the stresses obtained are in excellent agreement with the exact solution obtained by Way [1934].

The results obtained for isotropic elliptic and circular plates are more accurate than the results obtained by other investigators. This is due to the fact that the stress functions for the elliptic plate have been determined very accurately. But the results obtained are still approximate because the deflection functions have been determined with the help Galerkin's procedure.

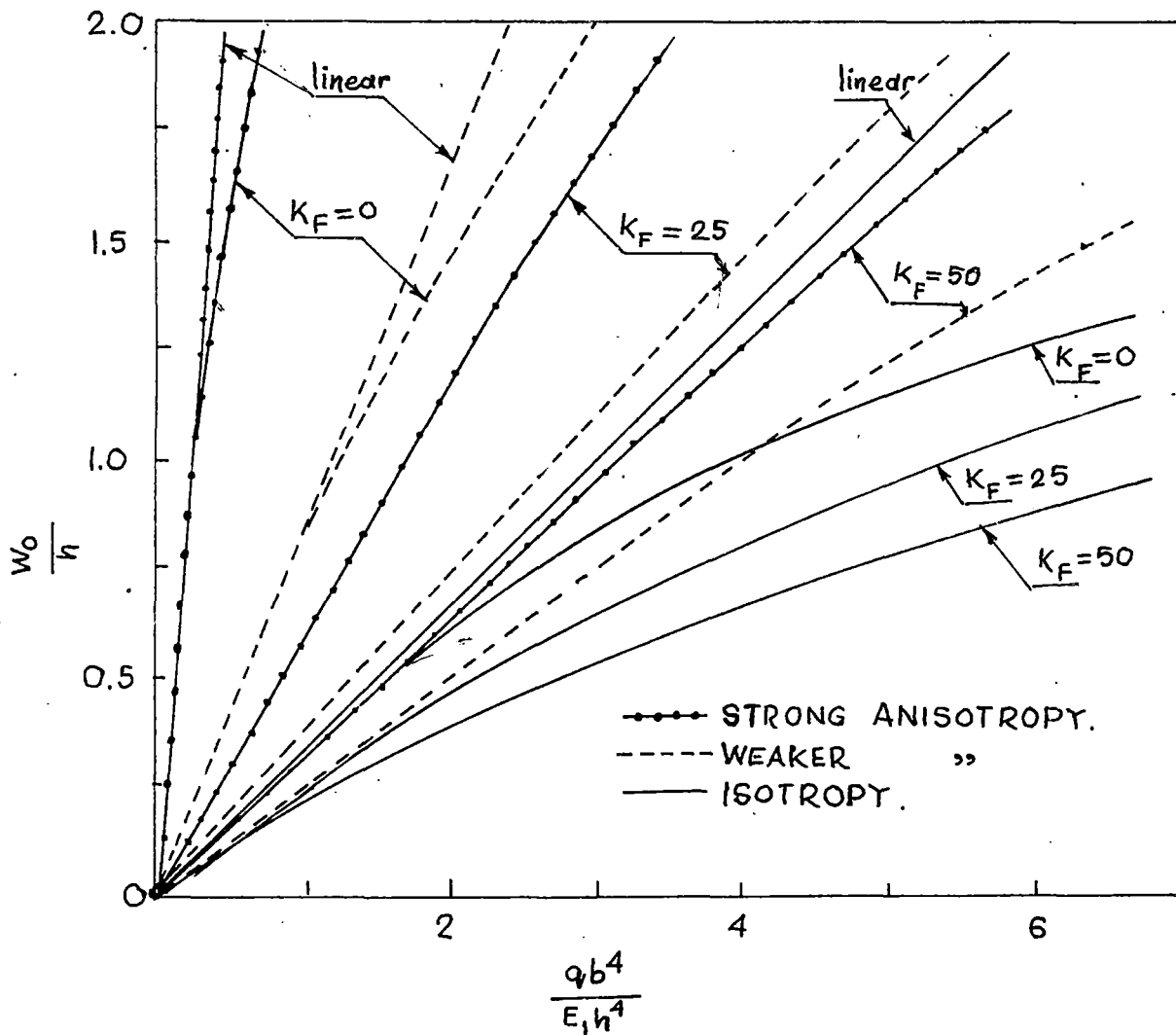


FIG.41 CENTRAL DEFLECTIONS IN ORTHOTROPIC ELLIPTIC PLATES, $\alpha=2b$.

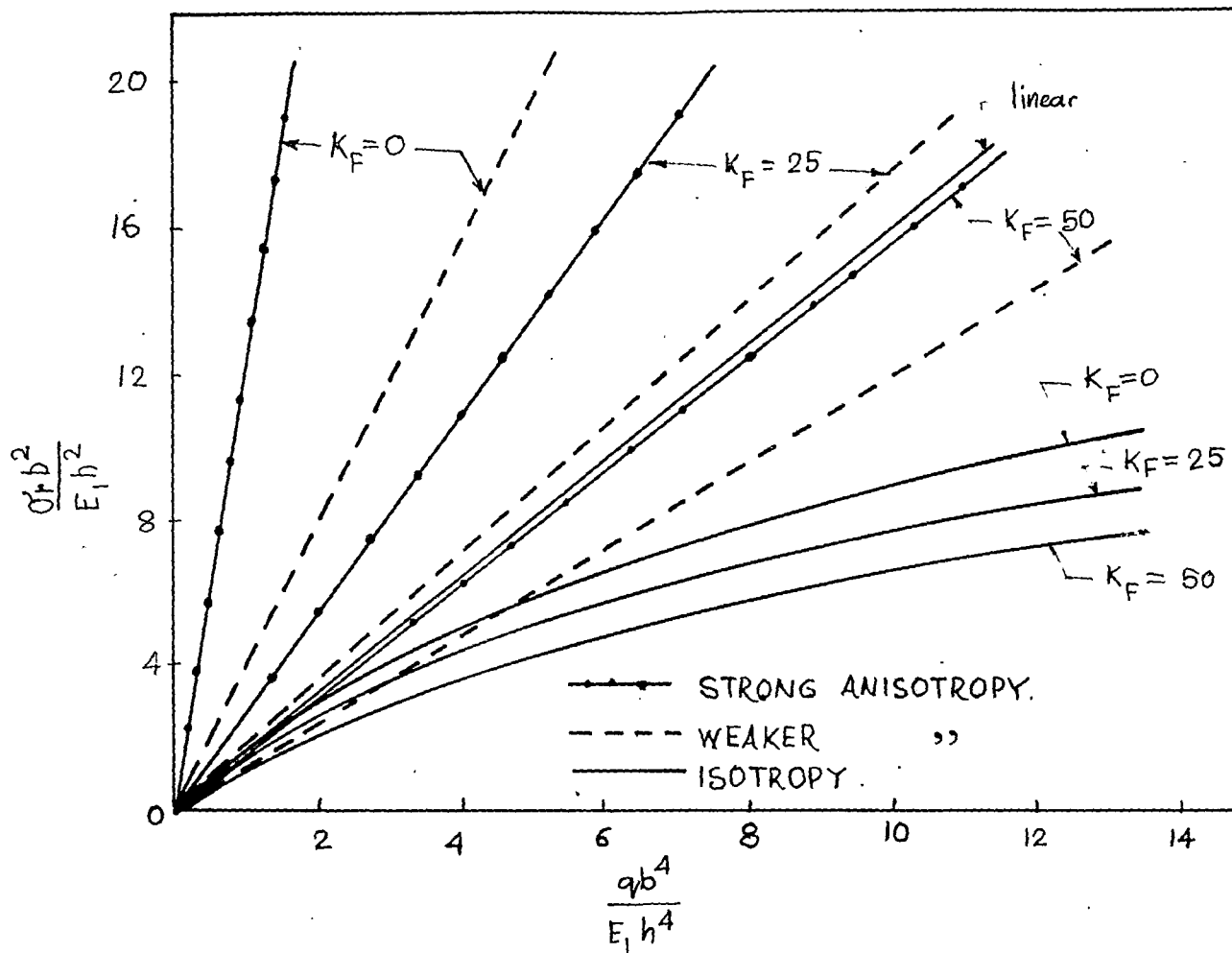


FIG.4.2 STRESSES IN ORTHOTROPIC ELLIPTIC PLATES, $a=2b$

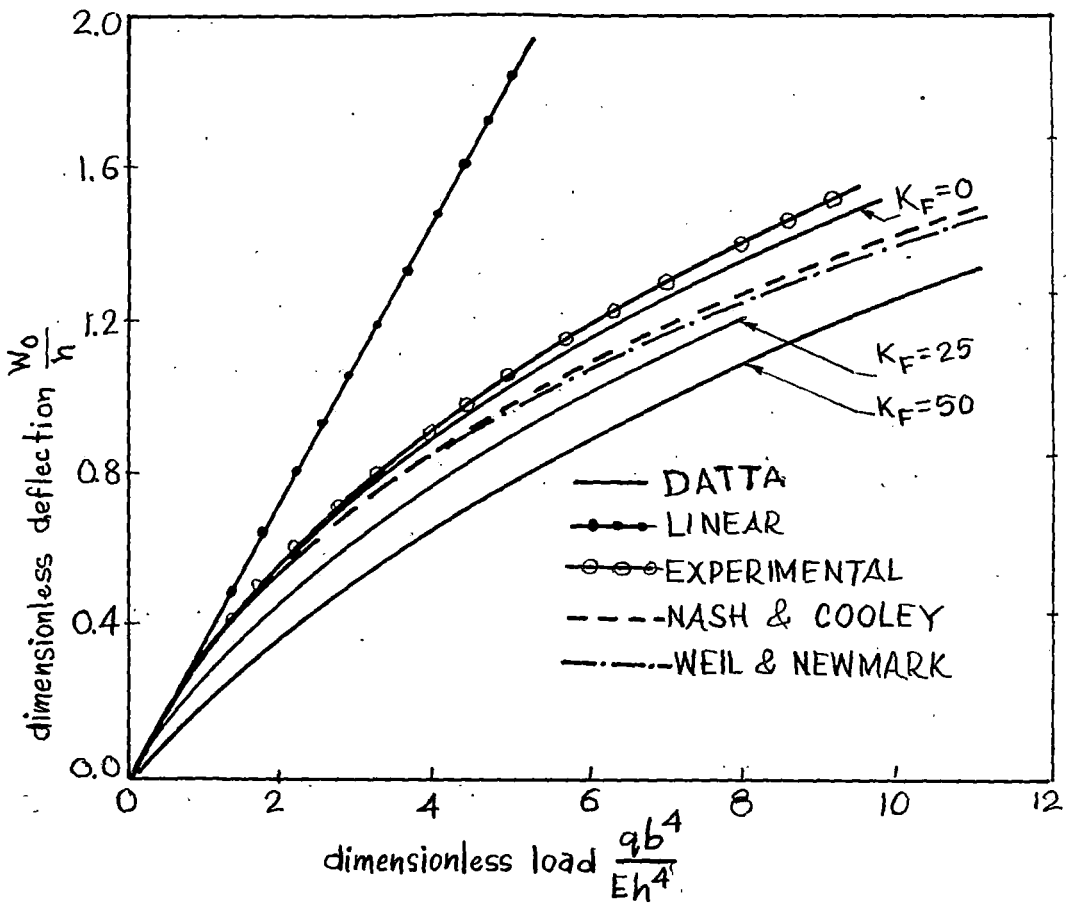


FIG. 43 CENTRAL DEFLECTION CURVE FOR ISOTROPIC ELLIPTIC PLATE, $a=2b$.

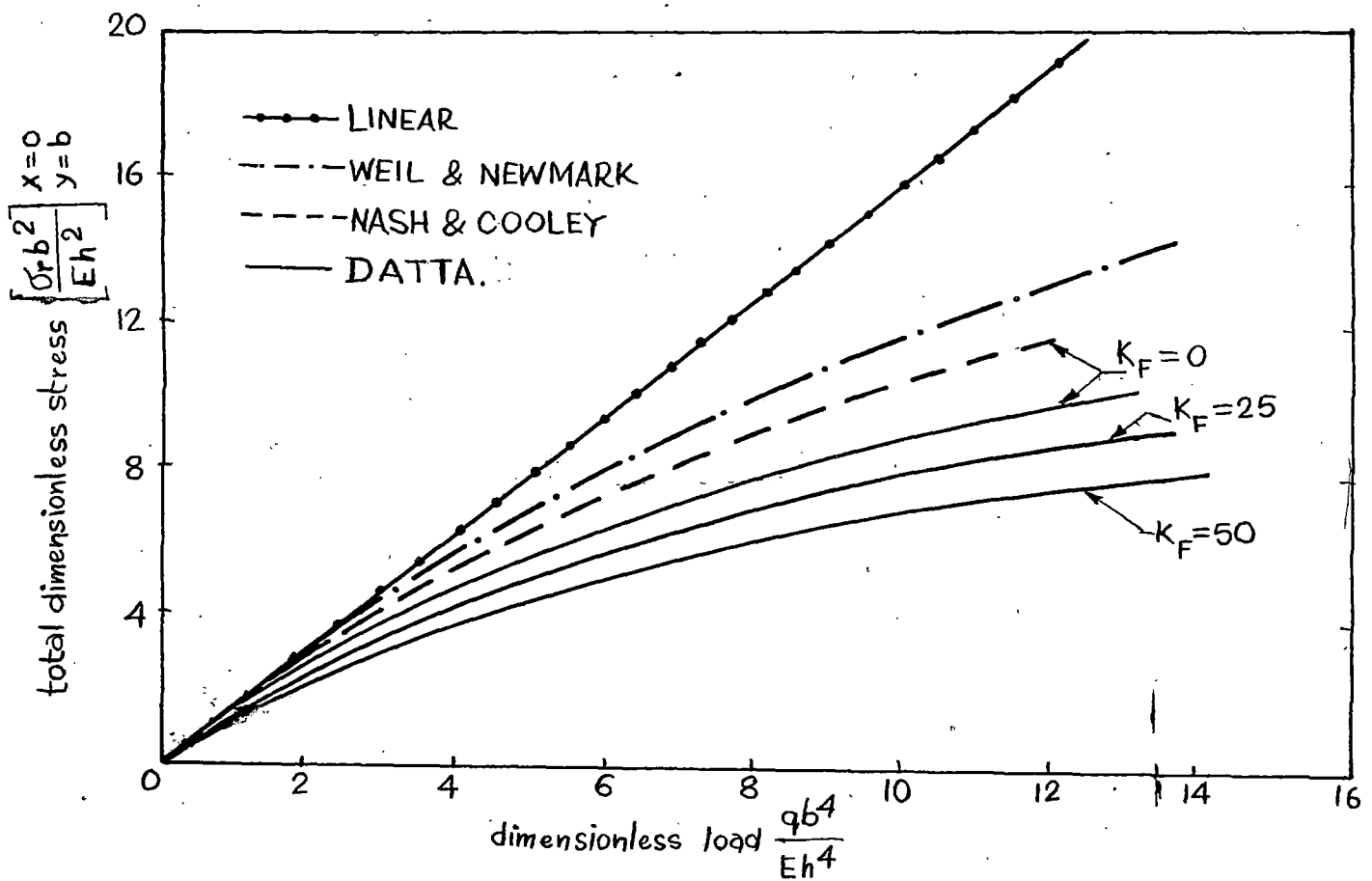


FIG. 44 STRESSES IN ISOTROPIC ELLIPTIC PLATES, $\alpha = 2b$.

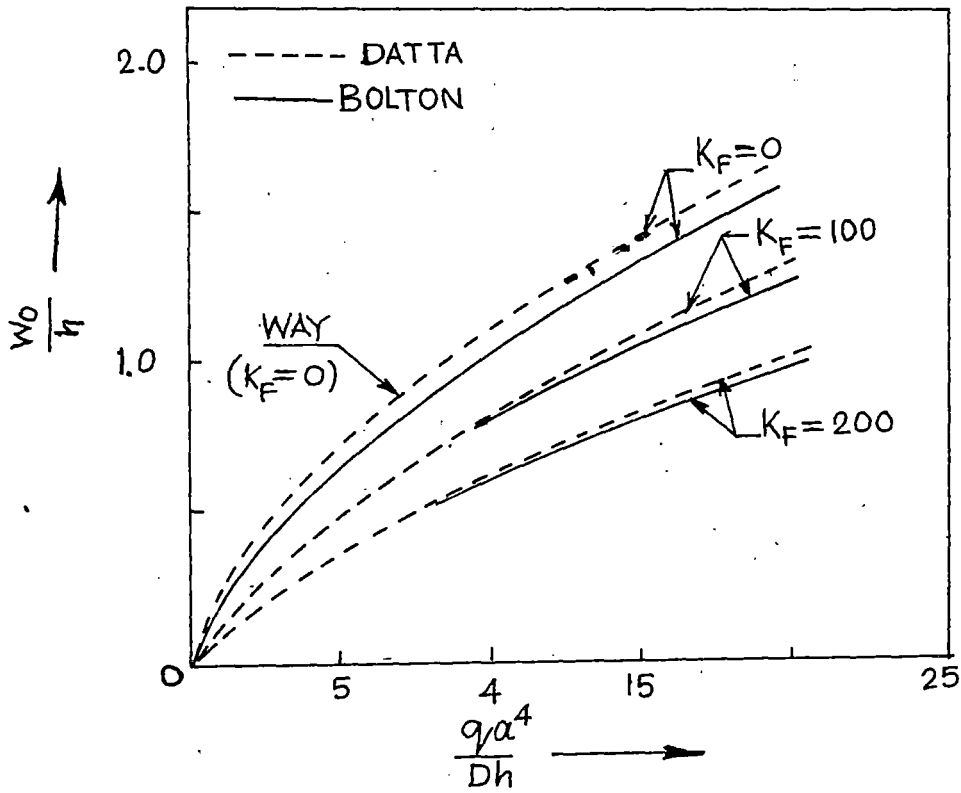


FIG.45 CENTRAL DEFLECTIONS IN CIRCULAR ISOTROPIC PLATES.

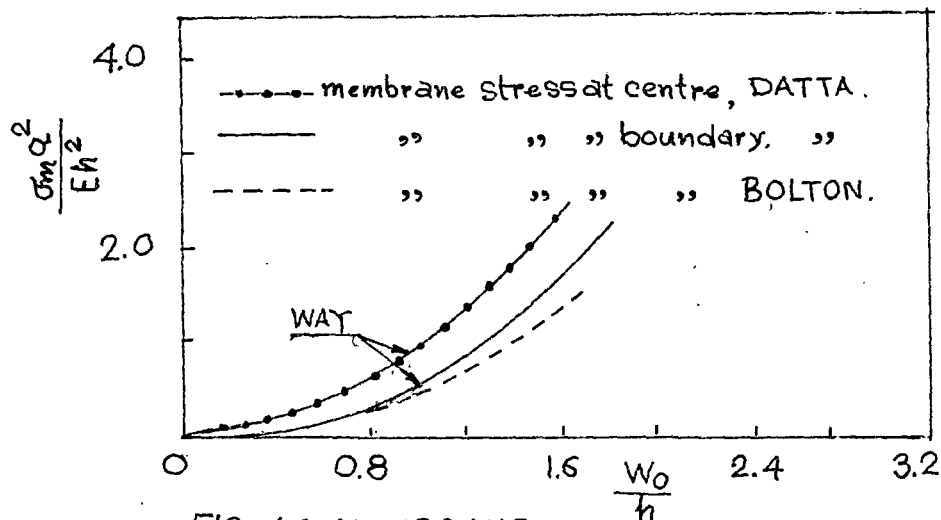


FIG. 4.6 MEMBRANE STRESSES IN CIRCULAR PLATE.

CHAPTER-V

LARGE AMPLITUDE FREE VIBRATIONS OF IRREGULAR PLATES
PLACED ON ELASTIC FOUNDATION *INTRODUCTION

An approximate method for investigating the large deflection of initially flat isotropic plates has been proposed by Berger (1955). An application of this technique has been made to the vibration problems by Nash and Modeer (1960) who found the large amplitude free vibrations of rectangular and circular plates.

In this paper a unified method for determining the lower natural frequency of large amplitude free vibrations of thin elastic plates of any shape and placed on elastic foundation is given. Following Berger's method a simple fourth order differential equation coupled with a second order nonlinear equation is obtained. If the boundary of the plate is a curve natural to any of the common coordinate system, the solution of the differential equation can be expressed in terms of known functions. For more "exotic" boundaries, the natural co-ordinate must first be determined and after this is done, the solution would inevitably involve some unfamiliar functions. The determination of natural frequencies in this case will then be very complicated. Therefore a common co-ordinate system and its associated function is used for the case of plates with complicated boundaries.

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In order to satisfy the prescribed boundary conditions, the domain is conformally transformed onto a unit circle. Once the transformation function is known, the problem is reduced to the solution of the transformed differential system. In this paper Galerkin's method is used to solve the transformed equation.

The ratio of time periods for circular, square and cornered plates placed on elastic foundation have been determined under simply supported and clamped edge boundary conditions. The foundation is assumed to be of the Winkler type. Experimental values are also obtained for circular and square plates under both the boundary conditions. The results are presented in the form of graphs and they are compared with other known results.

T H E O R Y

Let us consider the large amplitude free vibrations of a thin elastic plate placed on an elastic foundation having the reaction, k^* per unit area per unit deflection.

By adding the potential energy of the foundation reaction to the energy expression, using Hamilton's principle and Euler's variational equations one gets the following two differential equations after neglecting e_2 and the inertia effects in the plane of the plate [Kash and Moddeer (1960)]

$$\nabla^4 W - \alpha^2 F^2(t) \nabla^2 W + \frac{12}{h^2 c_p^2} \frac{\partial^2 W}{\partial t^2} + \frac{K'}{D} W = 0 \quad \dots (5.1)$$

$$e_1 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 = \frac{\alpha^2 h^2}{12} f(t) \quad \dots (5.2)$$

in which $c_p^{-2} = \frac{\rho h^3}{12D}$... (5.3)

$f(t)$ being equal to $F^2(t)$... (5.4)

Let $w = W(x, y) F(t)$... (5.5)

Combining Eqs.(5.1) and (5.5) one gets

$$F(t) \nabla^4 W - \alpha^2 F^3(t) \nabla^2 W + \frac{12}{h^2 c_p^2} \cdot \frac{d^2 F}{dt^2} \cdot W + \frac{K'}{D} W F(t) = 0$$

... (5.6)

Eq.(5.6) may be written as

$$\left(\frac{\nabla^4 W}{W} + \frac{K'}{D} \right) F(t) - \alpha^2 F^3(t) \frac{\nabla^2 W}{W} + \frac{12}{h^2 c_p^2} \cdot \frac{d^2 F}{dt^2} = 0$$

... (5.7)

A solution of Eq.(5.7) is possible if

$$\frac{\nabla^4 W}{W} = K^4$$

... (5.8)

and

$$\frac{\nabla^2 W}{W} = -K^2$$

... (5.9)

in which k is a constant.

From Eq.(5.8)

$$(\nabla^2 - k^2)(\nabla^2 + k^2) W = 0$$

... (5.10a)

From Eq.(5.9)

$$(\nabla^2 + k^2) W = 0$$

... (5.10b)

Therefore a solution of Eq.(5.7) can be obtained by satisfying Eq.(5.10b). To satisfy the prescribed boundary conditions for plates of any irregular shape let the domain be conformally transformed onto a unit circle. If $z = x + iy$, $\bar{z} = x - iy$, Eq.(5.10b) changes into

$$4 \frac{\partial^2 w}{\partial z \partial \bar{z}} + k^2 w = 0 \quad \dots (5.11)$$

Let $z = f(\xi)$ be the analytic function which maps the boundary under consideration in the ξ - plane onto a unit circle. Thus Eq.(5.11) transforms into complex co-ordinates as

$$\left[\nabla^2 + k^2 \left(\frac{dz}{d\xi} \right)^2 \right] w(\xi \bar{\xi}) = 0 \quad \dots (5.12)$$

in which

$$\xi = \pi e^{i\theta}, \quad \bar{\xi} = \pi e^{-i\theta}$$

The solution of Eq.(5.12) can be expressed in the form

$$W \approx \sum_{n=1}^{\infty} B_n \left[1 - (\xi \bar{\xi})^n \right] \quad \dots (5.13a)$$

or

$$W \approx \sum_{n=1}^{\infty} B_n \left[1 - (\xi \bar{\xi})^n \right]^2 \quad \dots (5.13b)$$

According to the prescribed boundary conditions, Eq.(5.13a) is an admissible function for the simply supported edge condition in the sense that this satisfies the kinematic boundary condition $w = 0$ at $r = 1$, but does not satisfy the force boundary condition $M_r = 0$.

The form of w in Eq.(5.13b) satisfies $w = 0 = \frac{dw}{dr}$ at $r = 1$ and can be taken as an admissible function for the clamped edge condition. Substituting Eq.(5.13a) or Eq.(5.13b) into Eq.(5.12) results the error function, $\epsilon_{n,0}$, which does not vanish, in general, since Eq.(5.13a) or Eq.(5.13b) is not an exact solution. Galerkin's procedure requires that the error function, $\epsilon_{n,0}$ to be orthogonal over the domain under consideration : i.e.,

$$\int_C \epsilon_{n,0}(\xi\bar{\xi}) w(\xi\bar{\xi}) d\xi = 0 \quad (n = 1, 2, 3, \dots, N) \quad \dots (5.14)$$

From Eq.(5.14) a homogeneous system of linear equations is obtained. Such a system can have nontrivial solutions only if the determinant of the coefficients of the unknowns vanishes identically. From this determinantal equation, the values of $k_1^2, k_2^2, \dots, k_N^2$ can be solved. For the fundamental frequency the lowest value of k^2 is to be taken. Combining Eqs.(5.7), (5.8) and (5.9) the following differential equation for determining $F(t)$ is obtained.

$$\ddot{F}(t) + \lambda_1 F(t) + \mu F^3(t) = 0 \quad \dots (5.15)$$

in which

$$\lambda_1 = \frac{1}{12} \left(K^4 + \frac{K'}{D} \right) h^2 c_p^2 \quad \dots (5.16)$$

$$\mu = \frac{1}{12} \left(\alpha^2 K^2 h^2 c_p^2 \right) \quad \dots (5.17)$$

Eq.(5.15) is to be solved subject to the initial conditions

$$F(0) = 1, \quad \dot{F}(0) = 0 \quad \dots (5.18)$$

Solution of Eq.(5.15) can be taken in the form

$$F(t) = C_n(\omega, t, \lambda_2) \quad \dots (5.19)$$

in which ω_1 and λ_2 are positive constants given by

$$\omega_1^2 = \frac{1}{12} \left(1 + \frac{\alpha^2}{k^2} + \frac{k'}{Dk^4} \right) h^2 c^2 k^4 \quad \dots (5.20)$$

$$\lambda_2^2 = \frac{1}{2 \left(1 + \frac{k^2}{\alpha^2} + \frac{k'}{D\alpha^2 k^2} \right)} \quad \dots (5.21)$$

and C_n is Jacobi's elliptic function.

To determine α Eq.(5.8) is transformed into complex co-ordinates by the transformation $z = x + iy$, $\bar{z} = x - iy$. Thus one gets

$$\frac{\alpha^2 h^2}{12} f(t) = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right) u + i \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right) v + 2 \frac{\partial w}{\partial z} \frac{\partial w}{\partial \bar{z}} \quad \dots (5.22)$$

If the mapping function $z = f(\xi)$ be introduced, Eq.(5.22) reduces to

$$\begin{aligned} \frac{\alpha^2 h^2}{12} \frac{dz}{d\xi} \frac{d\bar{z}}{d\bar{\xi}} f(t) &= \frac{\partial u}{\partial \xi} \frac{d\bar{z}}{d\bar{\xi}} + \frac{\partial u}{\partial \bar{\xi}} \frac{dz}{d\xi} + \\ &+ \left\{ \frac{\partial v}{\partial z} \frac{d\bar{z}}{d\bar{\xi}} - \frac{\partial v}{\partial \bar{z}} \frac{dz}{d\xi} \right\} i + 2 \frac{\partial w}{\partial z} \frac{\partial w}{\partial \bar{z}} \quad \dots (5.23) \end{aligned}$$

Now the normalised constant α can be determined from Eq.(5.13a) or (5.13b) and (5.23) by integrating Eq.(5.23) over the cycle 2π . The terms involving u and v can be easily eliminated (since u and v are of little importance in the case of large amplitude vibration)

by considering suitable expressions for u and v compatible with the boundary conditions. Finally the following integral will determine α

$$\int_S \int \frac{\alpha^2 h^2}{12} \frac{dz}{d\eta} \frac{d\bar{z}}{d\bar{\eta}} ds = 2 \int_S \int \frac{\partial w}{\partial \eta} \frac{\partial w}{\partial \bar{\eta}} ds \quad \dots (5.24)$$

Thus having determined k and α , the nonlinear frequency, ω_1 , is completely determined. The nonlinear period, T_1 , is given by

$$T_1 = \frac{4K}{\omega_1} \quad \dots (5.25)$$

K being the complete elliptic integral of the first kind. The linear period, T_2 , is given by

$$T_2 = \frac{2\pi}{\omega_2} \quad \dots (5.26)$$

in which ω_2 is to be determined from the equation

$$\ddot{F}(t) + \lambda_1 F(t) = 0 \quad \dots (5.27)$$

in the form $\omega_2^2 = \lambda_1$

Thus the ratio of the periods, $\frac{T_1}{T_2}$, is obtained as

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \frac{1}{\sqrt{\left\{1 + \frac{K^2 \alpha^2}{K^4 + \frac{K'}{D}}\right\}}} \quad \dots (5.28)$$

APPLICATIONS

a. Let us apply the procedure explained above to the case of a clamped cornered plate. The mapping function is given by

$$z = \frac{25}{49} a \left(\xi - \frac{1}{25} \xi^5 \right) \quad \dots (5.29)$$

Using Eq.(5.13b) with $n = 1$ an approximate value of k^2 is obtained from Eq.(5.14)

$$k^2 = \frac{24 \cdot 55}{a^2} \quad \dots (5.30)$$

With $n = 2$, an improved lower value of k^2 is obtained

$$k^2 = \frac{21 \cdot 71}{a^2} \quad \dots (5.31)$$

To determine α the following functions for u and v are taken,

$$u = \sum_{m=1,3,5,\dots}^{\infty} U_m(\eta) \cos m\theta F^2(t) \quad \dots (5.32)$$

$$v = \sum_{m=1,3,5,\dots}^{\infty} V_m(\eta) \sin m\theta F^2(t) \quad \dots (5.33)$$

Substituting Eqs.(5.32) and (5.33) in Eq.(5.23) one gets Eq.(5.24)

determining α . To determine the value of α for the fundamental frequency, the value of n in Eq.(5.13b) is taken to be 1. Substituting Eq.(5.13b) with $n = 1$, and Eq.(5.29) in Eq.(5.24) the following value of α corresponding to the lowest frequency is obtained,

$$\alpha^2 = 29 \cdot 23 \frac{B_1^2}{a^2 h^2} \quad \dots (5.34)$$

Thus $\frac{T_1}{T_2}$ is obtained from Eq.(5.23)

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + \frac{B_1^2}{h^2} \left(\frac{636}{471 + K_F} \right) \right]^{\frac{1}{2}}} \quad \dots (5.35)$$

in which the nondimensional foundation modulus, K_F , is given by

$$K_F = \frac{k'}{D} a^4$$

The mapping function of a square plate is given by

$$z = 1.03a \left[\zeta - \frac{1}{10} \zeta^5 + \dots \right] \quad \dots (5.36)$$

Using Eq.(5.13b) with $n = 1$ and proceeding in the same manner as before one gets for a clamped square plate

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + \frac{B_1^2}{h^2} \left(\frac{457}{400 + 12 \cdot 3 K_F} \right) \right]^{\frac{1}{2}}} \quad \dots (5.37)$$

The mapping function for a circular plate is given by

$$z = a \zeta \quad \dots (5.38)$$

and for a clamped circular plate one gets

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + \frac{B_1^2}{h^2} \left(\frac{480}{400 + 9 K_F} \right) \right]^{\frac{1}{2}}} \quad \dots (5.39)$$

b. Let us consider the case of a simply supported circular plate. Using Eq.(5.13a) with $n = 1$ and proceeding in the same manner as before one gets the ratio $\frac{T_1}{T_2}$,

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + \frac{B_1^2}{h^2} \left(\frac{72}{35 + K_F} \right) \right]^{\frac{1}{2}}} \quad \dots (5.40)$$

For $K_F = 0$, Eq. (5.40) becomes

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + 2 \frac{B_1^2}{h^2} \right]^{\frac{1}{2}}} \quad \dots (5.41)$$

The corresponding result for the circular plate obtained by Nash and Norder (1960) is

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + 3 \frac{A^2}{h^2} J_1^2(KR) \right]^{\frac{1}{2}}} \quad \dots (5.42)$$

where $J_0(KR) = 0$, R being the radius of the circle.

For a simply supported square plate one gets

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + \frac{B_1^2}{h^2} \left(\frac{67.5}{35 + 1.37 K_F} \right) \right]^{\frac{1}{2}}} \quad \dots (5.43)$$

EXPERIMENTAL VERIFICATION

Experimental verifications were made with circular and square plates under simply supported and clamped edge boundary conditions. The circular plates were 150 mm diameter and the square plates with 150 mm side. Plate material was mild steel of 0.75 mm thickness. Free transverse vibrations of different amplitudes and frequencies were initiated in the apparatus shown in Fig.5.1. The test piece, T, was statically deflected by the load spindle, L and the central deflection was measured by the dial indicator, D. After giving a predetermined central deflection the spindle, L, was lifted quickly by the release spring, R, and the corresponding frequency was measured in a vibration meter, M, with the help of a noncontact type of vibration pick up, P. Simply supported edge conditions were realised by placing the edges of the plates over a knife edge placed around the periphery of the cavity, C, the shape of which conformed the shape of the plates used. Clamped edge conditions were achieved by clamping the edges of the plates rigidly by means of eight bolts, B, with the base of the apparatus. Experiments were carried out first with the cavity, empty and next by placing the plates over eight free helical springs, S, each spring being located at the centre of eight equal areas of the plates. The combined reaction of the springs used was determined experimentally to be $K_g = 6^{\circ}2$. Care was taken in selecting the stiffness of the spring, R, so that the spindle, L was released quickly from the plate without obstructing the upward motion of the plates.

RESULTS

Numerical as well as experimental results for the case of simply supported circular and square plates without foundation have been presented in Fig.5.2 and 5.3 respectively. The corresponding results obtained by Nash and Modeer (1960) for the circular and square plates and the results obtained by Chu and Herrmann (1956) for the square plates have also been presented for comparison. Numerical and experimental results for clamped circular and square plates both with and without foundation have been presented in Fig.5.4 and 5.5 respectively.

Lowest natural frequencies of large amplitude free vibrations of thin plates of any shape can readily be calculated by the conformal mapping technique used in this study, if the mapping functions are known. From Figs.5.4 and 5.5 it is observed that the results obtained with one term approximation of the trial function Eq.(5.13b) for the clamped edge boundary conditions are in excellent agreement with the practical values. For the simply supported edge conditions the theoretical results given in Fig.5.2 and 5.3 are in somewhat poorer agreement with the values obtained experimentally. By using higher term approximations of the trial functions Eq.(5.13a) and (5.13b) and with smoothed mapping functions the results for both simply supported and clamped edge boundary conditions will be refined. But this will involve additional numerical computations.

The periods for rectangular plates obtained by Chu and Herrmann is dependent on the aspect ratio of the plate, whereas the corresponding results obtained by Nash and Modeer are independent of that ratio. The mapping functions for rectangular plates with different aspect ratio

will be different and therefore the present study indicates that the periods will depend on the aspect ratio. It should be pointed out that the theory used in this study allows the solution of the eigenvalue problem under consideration from a unified point of view since the trial functions used are the same for all shapes. Considering the fact that one term approximation reduces considerably the computational effort, the results obtained in this study are considered excellent for practical purposes.

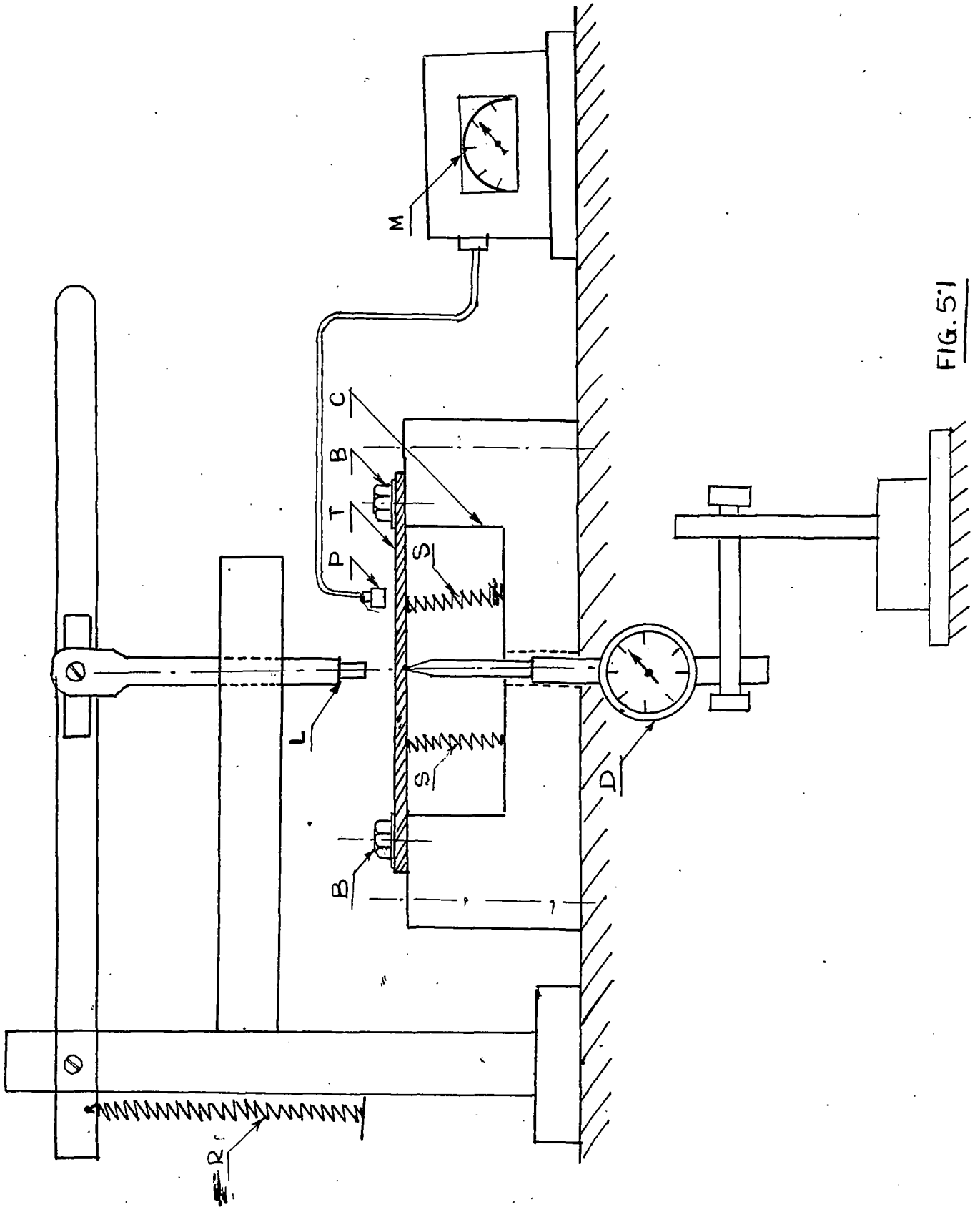


FIG. 51

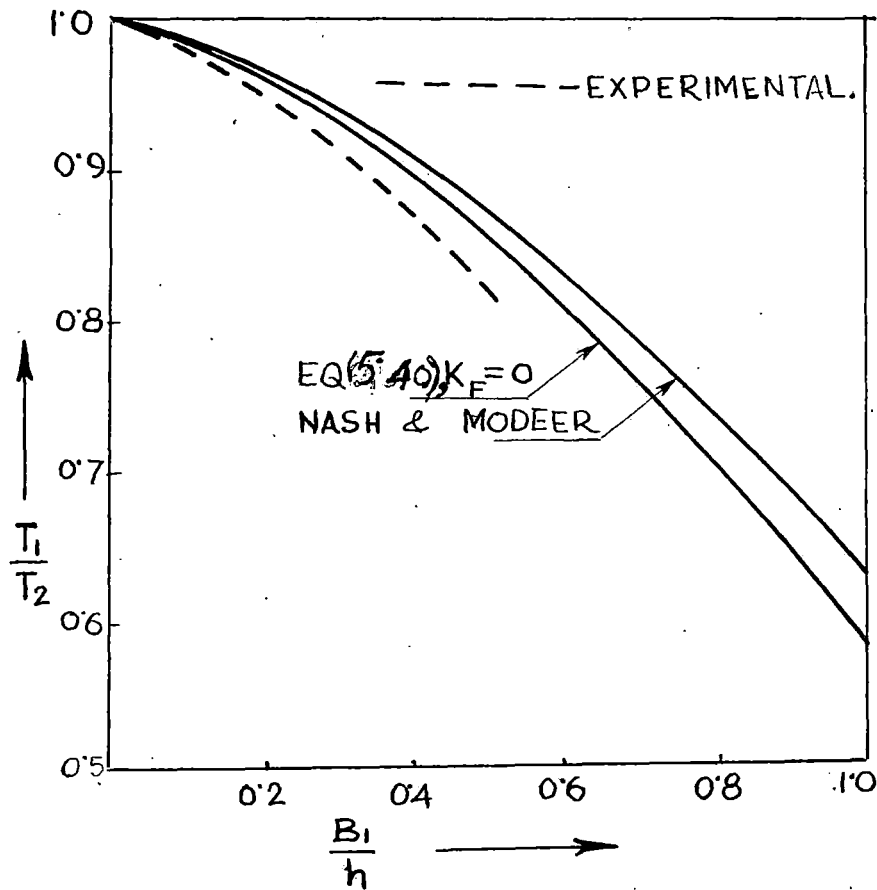


FIG. 5.2 SIMPLY SUPPORTED CIRCULAR PLATE.

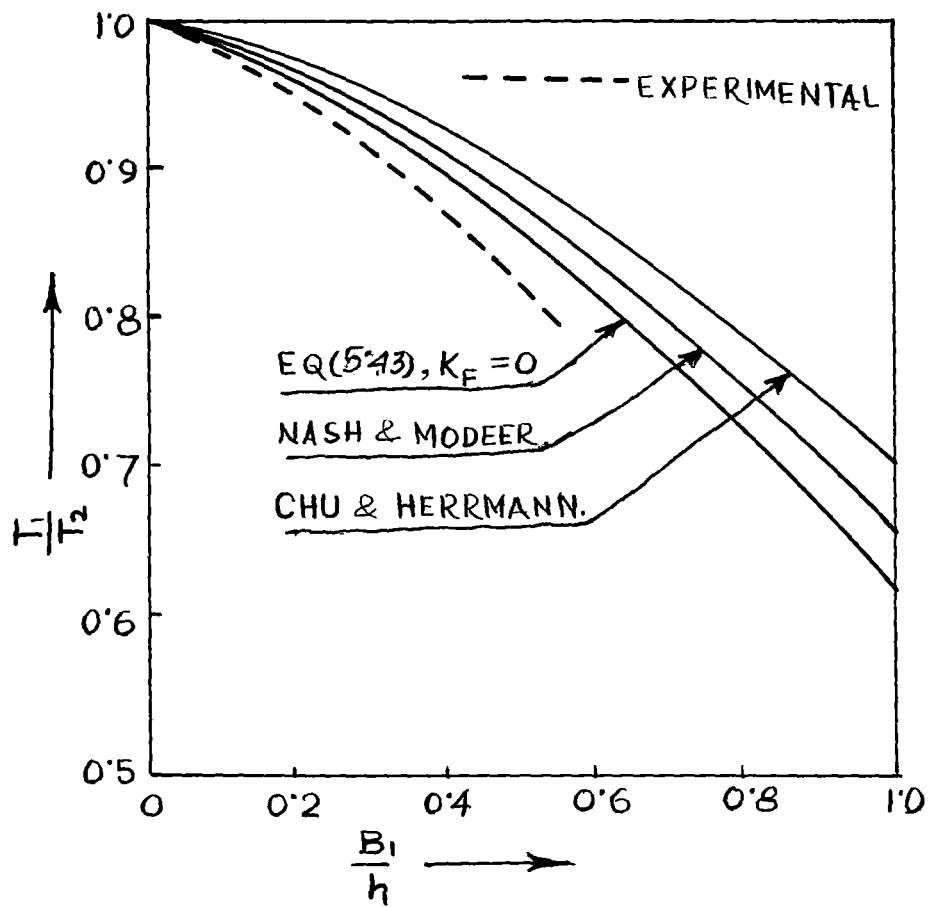


FIG. 5.3 SIMPLY SUPPORTED SQUARE PLATE.

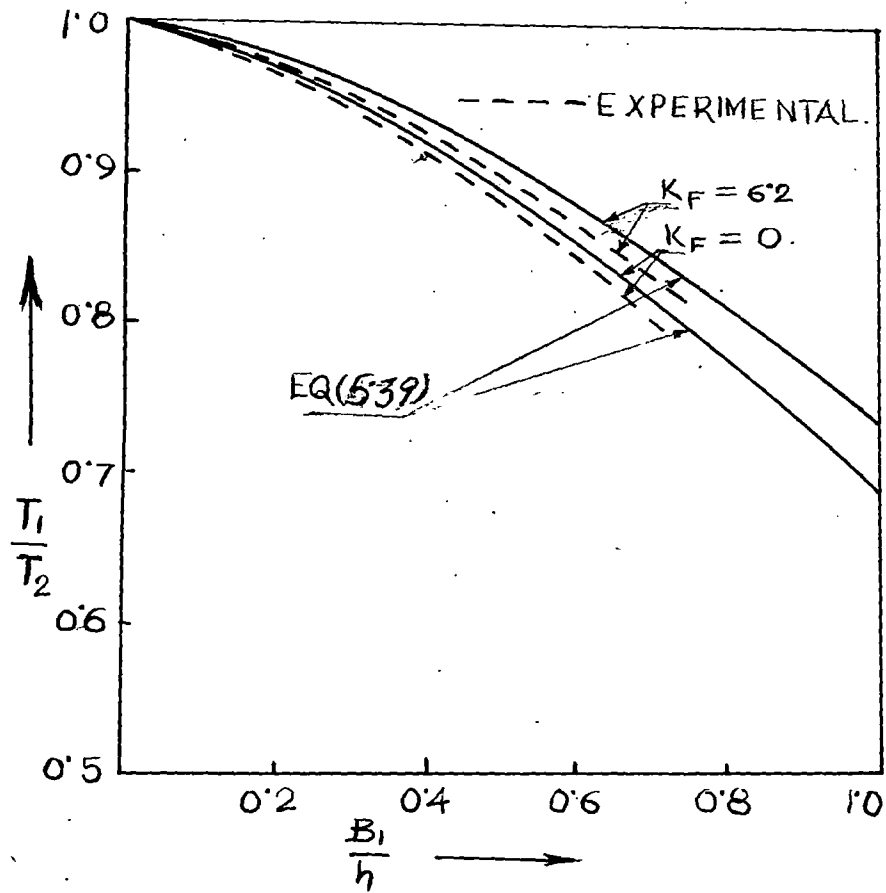


FIG.54 CLAMPED CIRCULAR PLATE.

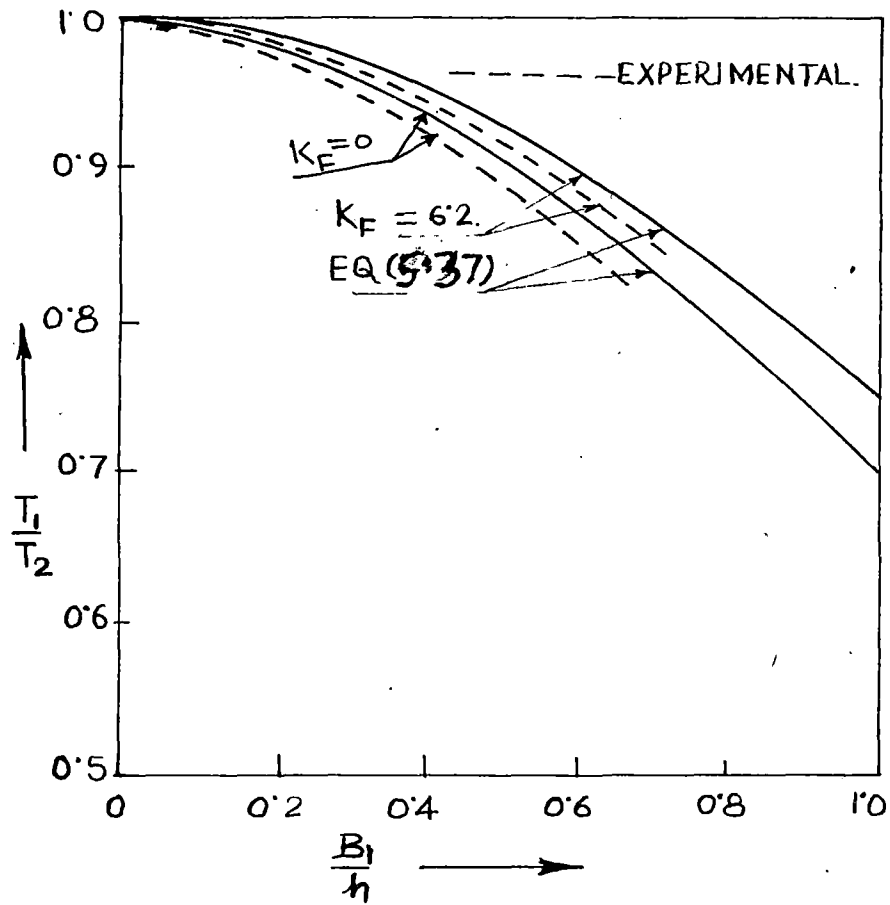


FIG.5.5 CLAMPED SQUARE PLATE.

N A T A T I O N

The following symbols are used in this thesis :

$$D = \text{flexural rigidity of a plate} = \frac{Eh^3}{12(1-\nu^2)} ;$$

E = modulus of elasticity in tension and compression ;

e_1 = first invariant of middle surface strains

$$= \epsilon_x + \epsilon_y \text{ in rectangular coordinates}$$

$$= \epsilon_r + \epsilon_\theta \text{ in cylindrical coordinates ;}$$

e_2 = second invariant of middle surface strains

$$= \epsilon_x \epsilon_y - \frac{1}{2} \gamma_{xy}^2 \text{ in rectangular coordinates}$$

$$= \epsilon_r \epsilon_\theta \text{ in polar coordinates in case of circular symmetry ;}$$

G = modulus of elasticity in shear ;

h = thickness of a plate ;

I = modified Bessel's function ;

J = Bessel function of first kind ;

K_p = nondimensional foundation modulus ;

M_r, M_θ, M_τ = radial, tangential and twisting moments per unit length of sections of a plate in polar coordinates,

N_{cr} = critical value of middle plane force ;

$$N_T = \text{buckling temperature} = \alpha E \int_{-\frac{h}{2}}^{\frac{h}{2}} T dz$$

- N_x, N_y, N_{xy} = normal force resultants per unit length in middle plane of a plate ;
- P = single concentrated load ;
- Q = shearing force per unit length of a section of a plate ;
- q = intensity of a continuously distributed load ;
- r, θ = polar co-ordinates ;
- S = Lommel's function ;
- T = temperature ;
- u, v, w = displacement in $x, y,$ and z directions respectively ;
- x, y, z = rectangular coordinates ;
- α = coefficient of linear thermal expansion ;
- σ = direct stress ;
- $(\sigma)_{ch}$ = critical buckling stress ;
- τ = shear stress ;
- ϵ = direct strain ;
- ϵ_n = error function in Galerkin's method ;
- γ = shear strain ;
- ρ = density of plate material ;
- ν = Poisson's ratio ;
- Γ = Gamma function ;
- $\xi = r e^{i\theta}$ = a complex quantity.

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LARGE DEFLECTION OF A SEMI-CIRCULAR PLATE
ON ELASTIC FOUNDATION UNDER A UNIFORM
LOAD

BY
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Large deflection of a semi-circular plate on elastic foundation under a uniform load

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ABSTRACT

Large deflection of a simply supported semi-circular plate placed on elastic foundation and subjected to a uniform load has been investigated following Berger's approximate method. Expressions for the deflections and bending moments are obtained and the theoretical results have been presented in the form of graphs.

1. INTRODUCTION

SMALL deflections of thin plates on elastic foundation have been examined by Timoshenko and Woinowsky-Krieger¹ and several other authors on the assumption that the strains of the middle plane of the plate can be neglected. But when the deflection is moderately large, that is, of the order of the thickness of the plate, then the strain of the middle plane should be considered. Thus for moderately large deflections, solutions of differential equations for deflections and displacements become difficult because of their non-linear character. Way² and many other authors have examined moderately large deflections of plates not resting on elastic foundations and the methods used by them involve considerable computation. Berger³ suggested that the strain energy due to the second strain invariant of the middle surface strains may be neglected in analysing moderately large deflection of plates having axisymmetric deformation. Berger's technique of neglecting the second strain invariant in the expression corresponding to the total potential energy of the system reduces computation and although no complete explanation of this method is offered, the stresses and deflections obtained for both rectangular and circular plates are in good agreement with those found in practical analysis. Berger's method has been applied successfully by Nowinski⁴ to his different plate problems; Nash and Modeer⁵ investigated the problems having no axial symmetry using Berger's technique.

Berger's technique of neglecting the second strain invariant in the middle plane has been applied by Sinha⁶ to determine large deflection of circular and rectangular plates placed on elastic foundation and under uniform lateral load.

In this paper large static deflection of a simply supported semi-circular isotropic plate placed on elastic foundation and under a uniform load has been solved. Foundation is assumed to be such that its reaction is proportional to the deflection of the plate.

2. FORMULATION OF PROBLEM

For moderately large deflections, the strain displacement relationships and the strain energy of the middle plane of the plate are:

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \quad (1)$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \quad (2)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \quad (3)$$

$$V_1 = \frac{D}{2} \iint \left[(\nabla^2 w)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu) \left\{ \frac{12}{h^2} e_2 + \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} \right] dx dy \quad (4)$$

Neglecting e_2 and by adding the potential energy of the transverse load and of the foundation reaction the modified energy equation becomes

$$V = \frac{D}{2} \iint \left[(\nabla^2 w)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu) \left\{ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} + \frac{K}{D} w^2 - \frac{2qw}{D} \right] dx dy \quad (5)$$

Applying Euler's variational method to (5) the following differential equations in polar co-ordinates are obtained:

$$\nabla^4 w - \alpha^2 \nabla^2 w + \frac{K}{D} w = \frac{q}{D} \quad (6)$$

where α is a constant given by

$$\frac{\alpha^2 h^2}{12} = \frac{\partial u}{\partial r} + \frac{1}{2} \left(\frac{\partial w}{\partial r} \right)^2 + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{1}{2r^2} \left(\frac{\partial w}{\partial \theta} \right)^2 \quad (7)$$

and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

3. SOLUTION OF PROBLEM

Let the plate be in the form of a semicircle (figure 1) and simply supported. Let the centre be the origin, the bounding diameter be the initial line and the plate be uniformly loaded. To solve equation (6) let us put it in following form:

$$(\nabla^2 - P_1^2)(\nabla^2 - P_2^2)w = \frac{q}{D} \quad (8)$$

where

$$P_1^2 + P_2^2 = \alpha^2 \quad (9)$$

$$P_1^2 P_2^2 = \frac{K}{D} \quad (10)$$

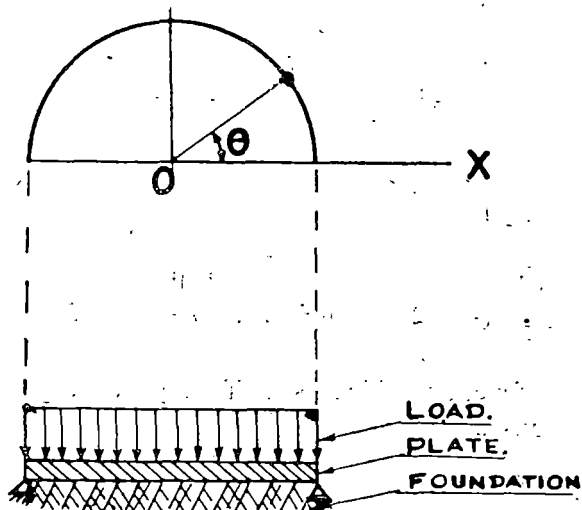


Figure 1. Semi-circular plate on foundation.

Expanding the load into the appropriate Fourier series,

$$q = \frac{4q}{\pi} \sum_{m=1, 3, 5}^{\infty} \frac{\sin m\theta}{m} \quad (11)$$

and assuming

$$w = \sum_{m=1, 3, 5}^{\infty} R_m \sin m\theta \quad (12)$$

where R_m is a function of r only, and substituting (11) and (12) in (8) one gets

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - P_1^2 \right) \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - P_2^2 \right) R_m = \frac{4q}{\pi D m} \quad (13)$$

The appropriate solution of (8) is given by

$$w = \sum_{m=1, 3, 5}^{\infty} \left[A_m I_m(P_1 r) + B_m I_m(P_2 r) + \frac{4q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} S_{3+2s, m}(iP_2 r) \right] \sin m\theta \quad (14)$$

where

$$\lambda_s = \frac{(-1)^s (iP_1)^{2s}}{(2^2 - m^2)(4^2 - m^2) \dots \{(2 + 2s)^2 - m^2\}}$$

and

$$S_{3+2s, m}(iP_2 r) = \sum_{n=0}^{\infty} \frac{(-1)^n (iP_2 r)^{4+2s+2n}}{\{(4 + 2s)^2 - m^2\} \dots \{(4 + 2s + 2n)^2 - m^2\}}$$

is the Lommel's function which is uniformly convergent. The required boundary conditions are

$$(w)_{r=a} = 0 \quad (15)$$

$$\left[\frac{\partial^2 w}{\partial r^2} + \nu \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right]_{r=a} = 0. \quad (16)$$

Considering equations (14), (15) and (16) and solving for the constants A_m and B_m one gets

$$\begin{aligned}
 A_m = \frac{4q}{\pi Dm} \cdot \frac{
 \left[
 \{aP_2^2 I_m''(P_2a) + \nu P_2 I_m'(P_2a)\} \sum_{s=1}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} S_{3+2s, m}(iP_2a)
 \right. \\
 - I_m(P_2a) \left\{ a \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} S''_{3+2s, m}(iP_2a)
 \right. \\
 \left. \left. + \nu \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} S'_{3+2s, m}(iP_2a) \right\}
 \right]
 }{
 [I_m(P_2a) \{aP_1^2 I_m''(P_1a) + \nu P_1 I_m'(P_1a)\} \\
 - I_m(P_1a) \{aP_2^2 I_m''(P_2a) + \nu P_2 I_m'(P_2a)\}]
 }
 \end{aligned} \quad (17)$$

$$\begin{aligned}
 B_m = -\frac{4q}{\pi Dm} \cdot \frac{
 \left[
 \{aP_1^2 I_m''(P_1a) + \nu P_1 I_m'(P_1a)\} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} S_{3+2s, m}(iP_2a)
 \right. \\
 - I_m(P_1a) \left\{ a \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} S''_{3+2s, m}(iP_2a)
 \right. \\
 \left. \left. + \nu \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} S'_{3+2s, m}(iP_2a) \right\}
 \right]
 }{
 [I_m(P_2a) \{aP_1^2 I_m''(P_1a) + \nu P_1 I_m'(P_1a)\} \\
 - I_m(P_1a) \{aP_2^2 I_m''(P_2a) + \nu P_2 I_m'(P_2a)\}]
 }
 \end{aligned} \quad (18)$$

To determine a , let

$$u = \sum U(r) \cos m\theta \quad (19)$$

$$v = \sum V(r) \sin m\theta \quad (20)$$

subject to the boundary conditions $U(a) = V(a) = 0$.

Multiplying (7) by $r d\theta dr$ and integrating within the limits 0 to a and 0 to π one gets

$$\begin{aligned}
 \int_0^a \int_0^\pi r \sum U'(r) \cos m\theta \, d\theta dr + \frac{1}{2} \int_0^a \int_0^\pi r \left(\frac{\partial w}{\partial r} \right)^2 \, d\theta dr \\
 + \int_0^a \int_0^\pi \sum U(r) \cos m\theta \, d\theta dr + \int_0^a \int_0^\pi \sum mV(r) \cos m\theta \, d\theta dr \\
 + \frac{1}{2} \int_0^a \int_0^\pi \frac{1}{r} \left(\frac{\partial w}{\partial \theta} \right)^2 \, d\theta dr = \frac{\alpha^2 h^2}{12} \int_0^a \int_0^\pi r \, d\theta dr.
 \end{aligned}$$

After evaluating the integrals the following equation leading to α is obtained:

$$\begin{aligned}
 & \sum_{m=1, 3, 5, \dots}^{\infty} \left[A_m^2 P_1^2 \left(-\frac{a^2}{8} \left\{ \frac{1}{4} [I_{m-2}(P_1 a) + I_m(P_1 a)]^2 \right. \right. \right. \\
 & \quad - \left. \left. \left[1 + \frac{(m-1)^2}{P_1^2 a^2} \right] I_{m-1}^2(P_1 a) \right\} - \frac{a^2}{8} \left\{ \frac{1}{4} [I_m(P_1 a) + I_{m+2}(P_1 a)]^2 \right. \right. \\
 & \quad - \left. \left. \left[1 + \frac{(m+1)^2}{P_1^2 a^2} \right] I_{m+1}^2(P_1 a) \right\} + \frac{1}{2} \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \phi \left(\frac{P_1}{2} \right)^{2m+2n+2t} \right) \\
 & \quad + B_m^2 P_2^2 \left(\frac{1}{2} \sum_{n=0}^{\infty} \sum_{\substack{t=1 \\ n \neq t}}^{\infty} \phi \left(\frac{P_2}{2} \right)^{2m+2n+2t} - \frac{a^2}{8} \left\{ \frac{1}{4} [I_{m-2}(P_2 a) \right. \right. \\
 & \quad + I_m(P_2 a)]^2 + \left. \left. \left[1 + \frac{(m-1)^2}{P_2^2 a^2} \right] I_{m-1}^2(P_2 a) \right\} \right. \\
 & \quad - \frac{a^2}{2} \left\{ [I_m(P_2 a) + I_{m+2}(P_2 a)]^2 + \left[1 + \frac{(m+1)^2}{P_2^2 a^2} \right] I_{m+1}^2(P_2 a) \right\} \right) \\
 & \quad + \frac{16q^2}{\pi^2 D^2 m^2} \left\{ \sum_{s=0}^{\infty} \frac{\lambda_s^2}{(iP_2)^{8+4s}} \left[\sum_{n=0}^{\infty} \frac{\mu_n^2 (4+2s+2n)^2 a^{8+4s+4n}}{8+4s+4n} \right. \right. \\
 & \quad \left. \left. + \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \frac{\mu_n \mu_t (4+2s+2n)(4+2s+2t) a^{8+4s+2n+2t}}{8+4s+2n+2t} \right] \right\} \\
 & \quad + \frac{1}{2} A_m B_m P_1 P_2 \left\{ \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \left[\frac{\left(\frac{P_1}{2} \right)^{m-1+2n} \left(\frac{P_2}{2} \right)^{m-1+2t} a^{2m+2n+2t}}{(2m+2n+2t) |n| |t| \Gamma(m+n) \Gamma(m+t)} \right. \right. \\
 & \quad + \frac{\left(\frac{P_1}{2} \right)^{m+1+2n} \left(\frac{P_2}{2} \right)^{m+1+2t} a^{2m+2n+2t+4}}{(2m+2n+2t+4) |n| |t| \Gamma(m+n+2) \Gamma(m+t+2)} \\
 & \quad \left. \left. + \phi \left(\frac{P_1}{2} \right)^{m-1+2n} \left(\frac{P_2}{2} \right)^{m+1+2t} + \phi \left(\frac{P_1}{2} \right)^{m+1+2n} \left(\frac{P_2}{2} \right)^{m-1+2t} \right] \right\} \\
 & \quad + 4A_m P_1 \frac{q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \left\{ \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \left[\mu_n \psi \left(\frac{P_1}{2} \right)^{m-1+2t} \right. \right. \\
 & \quad \left. \left. + \mu_n \psi_1 \left(\frac{P_1}{2} \right)^{m+1+2t} \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + 4B_m P_2 \frac{q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \left\{ \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \left[\mu_n \psi \left(\frac{P_2}{2} \right)^{m-1+2t} \right. \right. \\
 & \left. \left. + \mu_n \psi_1 \left(\frac{P_2}{2} \right)^{m+1+2t} \right] \right\} + m^2 \left(A_m^2 \sum_{n=0}^{\infty} \psi_2 \left(\frac{P_1}{2} \right)^{2m+4n} \right. \\
 & \left. + A_m^2 \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \psi_3 \left(\frac{P_1}{2} \right)^{2m+2n+2t} + B_m^2 \sum_{n=0}^{\infty} \psi_2 \left(\frac{P_2}{2} \right)^{2m+4n} \right. \\
 & \left. + B_m^2 \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \psi_3 \left(\frac{P_2}{2} \right)^{2m+2n+2t} + \frac{16q^2}{\pi^2 D^2 m^2} \sum_{s=0}^{\infty} \frac{\lambda_s^2}{(iP_2)^{8+4s}} \right. \\
 & \left. \times \left\{ \sum_{n=0}^{\infty} \frac{\mu_n^2 a^{8+4s+4n}}{8+4s+4n} + \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \frac{\mu_n \mu_t a^{8+4s+2n+2t}}{8+4s+2n+2t} \right\} \right. \\
 & \left. + 2A_m B_m \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \frac{\left(\frac{P_1}{2} \right)^{m+2n} \left(\frac{P_2}{2} \right)^{m+2t} a^{2m+2n+2t}}{(2m+2n+2t) |n| |t| \Gamma(m+n+1) \Gamma(m+t+1)} \right. \\
 & \left. + 8A_m \frac{q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \mu_n \phi_1 \left(\frac{P_1}{2} \right)^{m+2t} \right. \\
 & \left. + 8B_m \frac{q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \mu_n \phi_1 \left(\frac{P_2}{2} \right)^{m+2t} \right] = \frac{\alpha^2 h^2 a^2}{6} \quad (21)
 \end{aligned}$$

where

$$\begin{aligned}
 \mu_n &= \frac{(-1)^n (iP_2)^{4+2s+2n}}{\{(4+2s)^2 - m^2\} \cdots \{(4+2s+2n)^2 - m^2\}} \\
 \mu_t &= \frac{(-1)^n (iP_2)^{4+2s+2t}}{\{(4+2s)^2 - m^2\} \cdots \{(4+2s+2t)^2 - m^2\}} \\
 \phi &= \frac{\alpha^{2m+2n+2t+2}}{(2m+2n+2t+2) |n| |t| \Gamma(m+n) \Gamma(m+t+2)}
 \end{aligned}$$

$$\phi_1 = \frac{a^{4+2s+2n+2t+m}}{(4+2s+2n+2t+m) \Gamma(m+t+1)}$$

$$\psi = \frac{(4+2s+2n) a^{4+2s+2n+2t+m}}{(4+2s+2n+2t+m) \Gamma(m+t)}$$

$$\psi_1 = \frac{(4+2s+2n) a^{6+2s+2n+2t+m}}{(6+2s+2n+2t+m) \Gamma(m+t+2)}$$

$$\psi_2 = \frac{a^{2m+4n}}{(2m+4n) \{ \Gamma(m+n+1) \}^2}$$

$$\psi_3 = \frac{a^{2m+2n+2t}}{(2m+2n+2t) \Gamma(m+n+1) \Gamma(m+t+1)}$$

As

$$P_1 \rightarrow 0, \quad P_2 \rightarrow 0$$

Equation (14) reduces to

$$w = \frac{qa^4}{D} \sum_{m=1, 3, 5, \dots}^{\infty} \left\{ \frac{4r^4}{a^4 m\pi (16-m^2)(4-m^2)} + \frac{r^m}{a^m m\pi (16-m^2)(2+m) [m + \frac{1}{2}(1+\nu)]} - \frac{r^{m+2}}{a^{m+2} m\pi (4+m)(4-m^2) [m + \frac{1}{2}(1+\nu)]} \right\} \sin m\theta$$

as obtained by Timoshenko¹ for the corresponding problem of small deflection without any elastic foundation.

4. NUMERICAL CALCULATION

To obtain deflection for a given value of plate radius 'a' and foundation modulus 'K_F' one has to start from (21) with an assumed value of 'a' in order to obtain the corresponding value of the load function (qa⁴/Dh). Once this relationship is obtained the corresponding deflection w/h can be calculated by (14) with the help of (17) and (18). For a = 80 mm, ν = 0.3, and K_F = 350 deflections have been plotted in figure 2 for various values of load function (qa⁴/Dh).

5. DISCUSSION

On examination of eq. (14), it is clear that the radius of symmetry of the plate undergoes the maximum deflection with respect to other radii.

The expression for the deflection at a given point on the radius of symmetry can be expressed in the form $w = \beta (qa^4/D)$, where β is a numerical factor. Deflections at various points on the radius of symmetry are plotted in figure 3 for a given value of load function. From figure 3 it is observed that maximum deflection occurs at the centre of gravity of the plate.

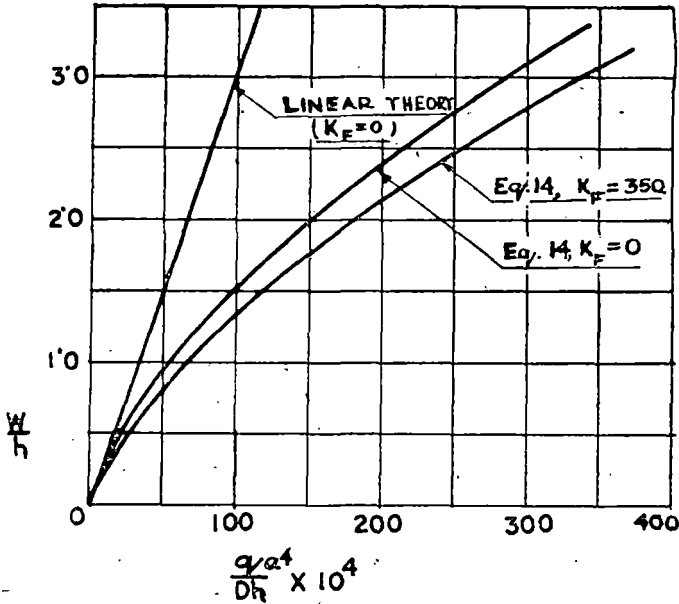


Figure 2. Deflection curve.

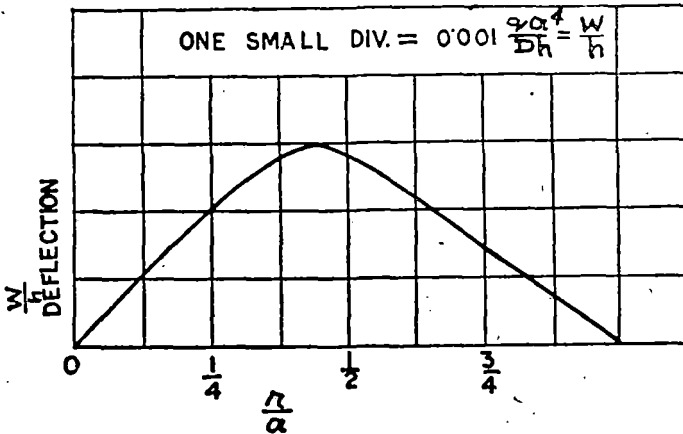


Figure 3. Deflection curve.

The plate is subjected to bending moments in radial and tangential directions as well as to a twisting moment. The expressions for bending and twisting moments are

$$\begin{aligned}
 M_r = & \sum_{m=1, 3, 5, \dots}^{\infty} -D \left[\frac{A_m P_1^2}{4} \{I_{m+2}(P_1 r) + 2I_m(P_1 r) + I_{m-2}(P_1 r)\} \right. \\
 & + \frac{B_m P_2^2}{4} \{I_{m+2}(P_2 r) + 2I_m(P_2 r) + I_{m-2}(P_2 r)\} \\
 & + \nu \left\{ \frac{A_m P_1}{2r} [I_{m-1}(P_1 r) + I_{m+1}(P_1 r)] + \frac{B_m P_2}{2r} [I_{m-1}(P_2 r) \right. \\
 & \left. + I_{m+1}(P_2 r)] - \frac{m^2}{r^2} [A_m I_m(P_1 r) + B_m I_m(P_2 r)] \right\} \\
 & + \frac{4q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \left\{ S''_{3+2s, m}(iP_2 r) + \frac{\nu}{r} S'_{3+2s, m}(iP_2 r) \right. \\
 & \left. - \frac{m^2 \nu}{r^2} S_{3+2s, m}(iP_2 r) \right\} \Big] \sin m\theta \quad (22)
 \end{aligned}$$

$$\begin{aligned}
 M_\theta = & \sum_{m=1, 3, 5, \dots}^{\infty} -D \left[\frac{A_m P_1^2 \nu}{4} \{I_{m+2}(P_1 r) + 2I_m(P_1 r) + I_{m-2}(P_1 r)\} \right. \\
 & + \frac{B_m P_2^2 \nu}{4} \{I_{m+2}(P_2 r) + 2I_m(P_2 r) + I_{m-2}(P_2 r)\} \\
 & + \frac{A_m P_1}{2r} \{I_{m-1}(P_1 r) + I_{m+1}(P_1 r)\} \\
 & + \frac{B_m P_2}{2r} \{I_{m-1}(P_2 r) + I_{m+1}(P_2 r)\} \\
 & - \frac{m^2}{r^2} \{A_m I_m(P_1 r) + B_m I_m(P_2 r)\} + \frac{4q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \\
 & \times \left\{ \nu S''_{3+2s, m}(iP_2 r) + \frac{1}{r} S'_{3+2s, m}(iP_2 r) \right. \\
 & \left. - \frac{m^2}{r^2} S_{3+2s, m}(iP_2 r) \right\} \Big] \sin m\theta \quad (23)
 \end{aligned}$$

$$\begin{aligned}
 M_{r\theta} = & (1 - \nu) D \sum_{m=1,3,5,\dots}^{\infty} m \left[\frac{A_m P_1}{2r} \{I_{m-1}(P_1 r) + I_{m+1}(P_1 r)\} \right. \\
 & + \frac{B_m P_2}{2r} \{I_{m-1}(P_2 r) + I_{m+1}(P_2 r)\} \\
 & - \frac{1}{r^2} \{A_m I_m(P_1 r) + B_m I_m(P_2 r)\} + \frac{4q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \\
 & \left. \times \left\{ \frac{1}{r} S'_{3+2s,m}(iP_2 R) - \frac{1}{r^2} S_{3+2s,m}(iP_2 r) \right\} \right] \cos m\theta. \quad (24)
 \end{aligned}$$

Eqs (22), (23) and (24) show that the bending moments M_r and M_θ are both maximum on the radius of symmetry and the twisting moment $M_{r\theta}$ is maximum on the bounding diameter.

The bending moments can be expressed in the form

$$M_r = \beta_1 q a^2; \quad M_\theta = \beta_2 q a^2 \quad (25)$$

Bending stresses can be calculated from the expressions

$$\sigma_r = \frac{6M_r}{h^2} \quad \sigma_\theta = \frac{6M_\theta}{h^2} \quad (26)$$

Expressions for shearing forces can be obtained with the help of (14)

$$\begin{aligned}
 Q_r = & -D \sum_{m=1,3,5,\dots}^{\infty} \left\{ \left[A_m P_1^3 I_m''(P_1 r) + B_m P_2^3 I_m''(P_2 r) \right. \right. \\
 & + \frac{4q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} S_{3+2s,m}''(iP_2 r) \Big] \\
 & + \frac{1}{r} \left[A_m P_1^2 I_m'(P_1 r) + B_m P_2^2 I_m'(P_2 r) \right. \\
 & + \frac{4q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} S_{3+2s,m}'(iP_2 r) \\
 & - \left(a^2 + \frac{\bar{m}^2}{r^2} \right) \left[A_m P_1 I_m(P_1 r) + B_m P_2 I_m(P_2 r) \right. \\
 & \left. \left. + \frac{4q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \times S'_{3+2s,m}(iP_2 r) \right] \right\} \sin m\theta \quad (27)
 \end{aligned}$$

$$\begin{aligned}
Q_{\theta} = & -\frac{D}{r} \sum_{m=1, 3, 5, \dots}^{\infty} m \left[A_m P_1^2 I_m''(P_1 r) + B_m P_2^2 I_m''(P_2 r) \right. \\
& + \frac{4q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(i P_2)^{4+2s}} S_{3+2s, m}(i P_2 r) \\
& + \frac{1}{r} \left(A_m P_1 I_m'(P_1 r) + B_m P_2 I_m'(P_2 r) \right. \\
& + \left. \frac{4q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(i P_2)^{4+2s}} S_{3+2s, m}(i P_2 r) \right) \\
& - \left(\alpha^2 + \frac{m^2}{r^2} \right) \left(A_m I_m(P_1 r) + B_m I_m(P_2 r) \right. \\
& \left. + \frac{4q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(i P_2)^{4+2s}} S_{3+2s, m}(i P_2 r) \right) \left. \right] \cos m\theta \quad (28)
\end{aligned}$$

The shearing stresses can be calculated from the expressions

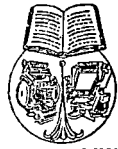
$$\tau_{r\theta} = \frac{6M_{r\theta}}{h^2}; \quad \tau_{rz} = \frac{3}{2} \frac{Q_r}{h}; \quad \tau_{\theta z} = \frac{3}{2} \frac{Q_{\theta}}{h} \quad (29)$$

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LARGE DEFLECTION OF A CIRCULAR PLATE ON ELASTIC FOUNDATION AND SUPPORTED AT SEVERAL POINTS

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ABSTRACT

Large deflections of a uniformly loaded circular plate placed on elastic foundation and supported at several points along the boundary have been analysed following Berger's method. A particular case, where the number of supports is two, has been treated fully. Numerical results have been presented in the form of graphs.

Key words : Large deflections, elastic foundation

INTRODUCTION

Small deflections of thin plates placed on elastic foundations have been examined by S. Timoshenko and S. Woinowsky Krieger [1] and several other authors on the assumption that strain due to stretching of the middle surface of the plate is negligible. When the deflections are moderately large, that is, on the order of thickness of the plate, then the forces in the middle surface of the plate must be taken into account. In the case of such large deflections of plates placed on elastic foundations, three differential equations for displacement and deflection may be written, but it is usually difficult to obtain the solutions of these equations because of their nonlinear character.

On the other hand, various problems of large deflections of plates not resting on elastic foundations have been examined by S. Way [2], S. Levy [3] and many other authors. But the methods used by them involve and require considerable computation. A simple and approximate, yet fairly accurate, method of analysing large deflections of plates was suggested by H. M. Berger [4]. The method uses the technique of neglecting the strain energy due to the second strain invariant of the middle surface strains in

analysing large deflection of plates having axisymmetric deformation. Berger's method reduces computation and although no complete explanation of this method is offered in, the stresses and deflections obtained for both rectangular and circular plates are in good agreement with those found in practical analysis. Berger's method has been applied successfully by Nowinsk [5] to his plate problems and Nash and Modeer [6] investigated the problems having no axial symmetry.

The technique of neglecting the second strain invariant in the expression corresponding to the total potential energy of the system has been successfully applied by Sinha [7] to determine large deflection of circular and rectangular plates placed on elastic foundations and under uniform lateral loads.

In this paper large deflection of a circular plate placed on elastic foundation and supported at several points along the boundary has been solved. The load is assumed to be uniformly distributed and the foundation is of the Winkler type. A complete analysis of a particular case, where the number of supports is two is given.

FORMULATION OF PROBLEM

For moderately large deflections, the strain displacement relationships and the strain energy of the middle plane of the plate are

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \quad (1)$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \quad (2)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \quad (3)$$

$$V_1 = \frac{D}{2} \int \int \left[(\nabla^2 w)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu) \right. \\ \left. \times \left\{ \frac{12}{h^2} e_2 + \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} \right] dx dy \quad (4)$$

in which e_1 and e_2 are the first and second middle surface strain invariants, respectively.

Neglecting e_2 and by adding the potential energy of the transverse load and of the foundation reaction, K , the modified energy equation becomes

$$V = \frac{D}{2} \int \int \left[(\nabla^2 w)^2 + \frac{12}{h^2} e_1^2 - 2(1 - \nu) \right. \\ \left. \times \left\{ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} + \frac{K}{D} w^2 - \frac{2qw}{D} \right] dx dy. \quad (5)$$

Applying Euler's variational method to eq. 5 the following differential equations in polar co-ordinates are obtained [7]

$$\nabla^4 w - \alpha^2 \nabla^2 w + \frac{K}{D} w = \frac{q}{D} \quad (6)$$

where α is a constant given by

$$\frac{\alpha^2 h^2}{12} = \frac{\partial u}{\partial r} + \frac{1}{2} \left(\frac{\partial w}{\partial r} \right)^2 + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{1}{2r^2} \left(\frac{\partial w}{\partial \theta} \right)^2 \\ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}. \quad (7)$$

SOLUTION OF PROBLEM

Let the circular plate (Fig. 1) be of radius a , supported at several points along the boundary and placed on the elastic foundation. Let the centre of the plate be the origin and a diameter as the initial line, $\theta = 0$. The general solution of eq. 6 is

$$w = w_0 + w_1 \quad (8)$$

in which w_0 is the large deflection of a plate placed on elastic foundation and simply supported along the entire boundary and w_1 satisfies the equation

$$\nabla^4 w_1 - \alpha^2 \nabla^2 w_1 + \frac{K}{D} w_1 = 0 \quad (9)$$

Eq. (9) can be written in the form

$$(\nabla^2 - P_1^2)(\nabla^2 - P_2^2) w_1 = 0 \quad (10)$$

where

$$P_1^2 + P_2^2 = \alpha^2 \quad (11)$$

$$P_1^2 P_2^2 = \frac{K}{D}. \quad (12)$$

Considering the number of points of support is i , and denoting the concentrated reactions at these points $N_1, N_2 \dots N_i$, the expression for each reaction N_i is (1, P. 293)

$$\frac{N_i}{\pi a} \left[\frac{1}{2} + \sum_{m=1}^{\infty} \cos m\theta_i \right] \quad (13)$$

where $\theta_i = \theta - \psi_i$, ψ_i is the angle defining the position of the support i .

The intensity of the reactive forces at any point of the boundary is then given by the expression.

$$\sum_{i=1}^i \frac{N_i}{\pi a} \left[\frac{1}{2} + \sum_{m=1}^{\infty} \cos m\theta_i \right] \quad (13 a)$$

in which the summation is extended over all the concentrated reactions. Assuming that the plate is solid and considering that deflections and moments

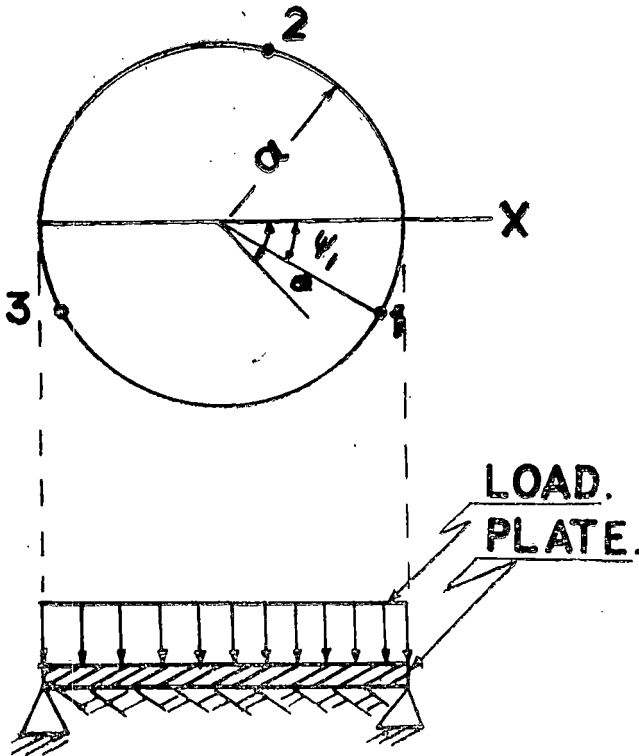


FIG. 1. Circular plate on foundation.

at the centre must be finite, the appropriate solution of eq. 9 can be taken in the form

$$w_1 = A_0 I_0(P_1 r) + B_0 I_0(P_2 r) + \sum_{m=1}^{\infty} [A_m I_m(P_1 r) + B_m I_m(P_2 r)] \times \cos m\theta + \sum_{m=1}^{\infty} [A'_m I_m(P_1 r) + B'_m I_m(P_2 r)] \sin m\theta \quad (14)$$

in which I_0 is the modified Bessel function of the first kind and zero order; and I_m is of the first kind and m th order. For determining the constants we have the following conditions at the boundary:

$$w \Big|_{\substack{r=a \\ \theta=0, \pi}} = 0 \quad (15)$$

$$\left[\frac{\partial^2 w_1}{\partial r^2} + \frac{\nu}{r} \frac{\partial w_1}{\partial r} + \frac{\nu}{r^2} \frac{\partial^2 w_1}{\partial \theta^2} \right]_{r=a} = 0 \quad (16)$$

$$\left[Q_r - \frac{1}{r} \frac{\partial}{\partial \theta} M_{r\theta} \right]_{r=a} = - \sum_{i=1}^i \frac{N_i}{\pi a} \left[\frac{1}{2} + \sum_{m=1}^{\infty} \cos m\theta_i \right] \quad (17)$$

where

$$Q_r = -D \frac{\partial}{\partial r} [(\nabla^2 - \alpha^2) w_1] \quad (17 a)$$

$$M_{r\theta} = (1 - \nu) D \left[\frac{1}{r} \frac{\partial^2 w_1}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial w_1}{\partial \theta} \right]. \quad (17 b)$$

Consider a particular case when the plate is supported at two points which are the two end points of the diameter taken as the initial line from which θ is measured. Then

$$\psi_1 = 0, \quad \psi_2 = \pi.$$

Considering the above boundary conditions one gets after solving for the constants

$$A_0 = \frac{P}{\pi D a} \beta \psi_0(a) \quad (18)$$

$$B_0 = - \frac{P}{\pi D a} \beta \phi_0(a) \quad (19)$$

$$A_m = -\frac{P}{\pi D a} \frac{\mu_m(a)}{\{\beta_m(a) \mu_m(a) - \lambda_m(a) \eta_m(a)\}} \quad (20)$$

$$B_m = \frac{P}{\pi D a} \frac{\lambda_m(a)}{\{\beta_m(a) \mu_m(a) - \lambda_m(a) \eta_m(a)\}} \quad (21)$$

$$A'_m = 0 = B'_m \quad (22)$$

where $P = \pi a^2 q =$ total load on the plate

$$\beta = \frac{[\lambda_m(a) I_m(P_2 a) - \mu_m(a) I_m(P_1 a)]}{[\beta_m(a) \mu_m - ((a) \lambda_m(a) \eta_m a)]} \times \frac{[I_0(P_2 a) \phi_0(a) - I_0(P_1 a) \psi_0(a)]}{\quad} \quad (23)$$

$$\psi_0(a) = P_2^2 I_0''(P_2 a) + \frac{\nu}{a} P_2 I_1(P_2 a) \quad (24)$$

$$\phi_0(a) = P_1^2 I_0''(P_1 a) + \frac{\nu}{a} P_1 I_1(P_1 a) \quad (25)$$

$$\mu_m(a) = P_2^2 I_m''(P_2 a) + \frac{\nu}{a} P_2 I_m'(P_2 a) - \frac{\nu m^2}{a^2} I_m(P_2 a) \quad (26)$$

$$\beta_m(a) = P_2^2 P_1 I_m'(P_1 a) - (1 - \nu) \left\{ \frac{m^2}{a^3} I_m(P_1 a) - \frac{P_1 m^2}{a^2} I_m'(P_1 a) \right\} \quad (27)$$

$$\lambda_m(a) = P_1^2 I_m''(P_1 a) + \frac{\nu}{a} P_1 I_m'(P_1 a) - \frac{\nu m^2}{a^2} I_m(P_1 a) \quad (28)$$

$$\eta_m(a) = P_1^2 P_2 I_m'(P_2 a) - (1 - \nu) \left\{ \frac{m^2}{a^3} I_m(P_2 a) - \frac{P_2 m^2}{a^2} I_m'(P_2 a) \right\} \quad (29)$$

Thus the complete solution of eq. 6 is obtained in the following form

$$w = w_0 + A_0 I_0(P_1 r) + B_0 I_0(P_2 r) + \sum_{m=2, 4, 6, \dots}^{\infty} [A_m I_m(P_1 r) + B_m I_m(P_2 r)] \cos m\theta \quad (30)$$

where

$$w_0 = \frac{q}{K} + A_0' I_0(P_1 r) + B_0' I_0(P_2 r) \quad (31)$$

$$A_0 = -\frac{q}{K} \left[\frac{P_2^2 I_0''(P_2 a) + P_2 \frac{\nu}{a} I_1(P_2 a)}{\phi(P a)} \right] \quad (31 a)$$

$$B_0' = \frac{q}{K} \left[\frac{P_1^2 I_0''(P_1a) + P_1 \frac{\nu}{a} I_1(P_1a)}{\phi(Pa)} \right] \tag{31 b}$$

$$\begin{aligned} \phi(Pa) = & \{I_0(P_1a) P_2^2 I_0''(P_2a) - I_0(P_2a) P_1^2 I_0''(P_1a)\} \\ & + \frac{\nu}{a} \{P_2 I_1(P_2a) I_0(P_1a) - P_1 I_1(P_1a) I_0(P_2a)\} \end{aligned} \tag{31 c}$$

Substitution of the values of the constants A_0' , B_0' , A_0 , B_0 , A_m and B_m into eq. 30 yields

$$\begin{aligned} \frac{w}{h} = & \left(\frac{qa^4}{Dh} \right) \left[\frac{1}{K_F} \left\{ 1 + \frac{[P_1^2 I_0''(P_1a) + P_1 \frac{\nu}{a} I_1(P_1a)] I_0(P_2r)}{\phi(Pa)} \right. \right. \\ & \left. \left. - \frac{[P_2^2 I_0''(P_2a) + P_2 \frac{\nu}{a} I_1(P_2a)] I_0(P_1r)}{\phi(Pa)} \right\} \right. \\ & + \frac{1}{a^3} \left\{ \beta \psi_0(a) I_0(P_1r) - \beta \phi_0(a) I_0(P_2r) \right. \\ & \left. \left. - \sum_{m=2, 4, 6, \dots}^{\infty} \left[\frac{\mu_m(a) I_m(P_1r) - \lambda_m(a) I_m(P_2r)}{\beta_m(a) \mu_m(a) - \lambda_m(a) \eta_m(a)} \right] \cos m\theta \right\} \right]. \end{aligned} \tag{32}$$

As $P_1 \rightarrow 0$ and $P_2 \rightarrow 0$, eq. 32 reduces to

$$\begin{aligned} w = w_0 + & \frac{Pa^2}{2\pi D} \frac{1}{(3 + \nu)} \left\{ 2 \log 2 - 1 + \frac{1 + \nu}{1 - \nu} \left(2 \log 2 - \frac{\pi^2}{12} \right) \right. \\ & - \sum_{m=2, 4, 6, \dots}^{\infty} \left[\frac{1}{m(m-1)} + \frac{2(1 + \nu)}{m^2(m-1)(1 - \nu)} \right. \\ & \left. \left. - \frac{(r/a)^2}{m(m+1)} \right] \left(\frac{r}{a} \right)^m \cos m\theta \right\} \end{aligned} \tag{33}$$

as obtained by Timoshenko [1] in the corresponding small deflection problem for a plate supported at two points on the boundary.

The normalised constant α can be determined from Eqs. 7 and 30. Since we are interested only in the lateral displacement w , the radial and cross-radial displacements u and v have been eliminated by choosing suitable expressions for u and v , compatible with their boundary conditions and

integrating over the whole area of the plate. The radial and cross-radial displacements have been assumed in the forms

$$u = \Sigma U(r) \cos m\theta \quad (34)$$

$$v = \Sigma V(r) \sin m\theta \quad (35)$$

subject to the boundary conditions $U(a) = V(a) = 0$. Multiplying both sides of the equation 7 by $rdrd\theta$ and integrating between the limits 0 to a and 0 to 2π , one gets

$$\begin{aligned} & \int_0^a \int_0^{2\pi} rU'(r) \cos m\theta \, drd\theta + \int_0^a \int_0^{2\pi} U(r) \cos m\theta \, drd\theta \\ & + \int_0^a \int_0^{2\pi} mV(r) \cos m\theta \, drd\theta + \frac{1}{2} \int_0^a \int_0^{2\pi} r \left(\frac{\partial W}{\partial r} \right)^2 \, drd\theta \\ & + \frac{1}{2} \int_0^a \int_0^{2\pi} \frac{1}{r} \left(\frac{\partial W}{\partial \theta} \right)^2 \, drd\theta = \int_0^a \int_0^{2\pi} \frac{\alpha^2 h^2}{12} r \, drd\theta. \end{aligned}$$

After evaluating the integrals the following equation leading to α is obtained.

$$\begin{aligned} \frac{\alpha^2 h^2 a^2}{12} &= -\frac{1}{2} A_0'^2 P_1^2 a^2 \left\{ \frac{1}{4} [I_0(P_1 a) + I_2(P_1 a)]^2 \right. \\ & - \left[1 + \frac{1}{P_1^2 a^2} \right] I_1^2(P_1 a) \left. \right\} - \frac{1}{2} B_0'^2 P_2^2 a^2 \\ & \times \left\{ \frac{1}{4} [I_0(P_2 a) + I_2(P_2 a)]^2 - \left[1 + \frac{1}{P_2^2 a^2} \right] I_1^2(P_2 a) \right\} \\ & + 2A_0' B_0' P_1 P_2 \frac{a}{P_2^2 - P_1^2} \left[-\frac{1}{2} P_1 I_1(P_2 a) \{I_0(P_1 a) \right. \\ & + I_2(P_1 a)\} + \frac{1}{2} P_2 I_1(P_1 a) \{I_0(P_2 a) + I_2(P_2 a)\} \right] \\ & + \sum_{m=2, 4, 6, \dots}^{\infty} \left[A^2 m P_1^2 \left\{ -\frac{1}{8} a^2 \left[\frac{1}{4} \{I_{m-2}(P_1 a) + I_m(P_1 a)\}^2 \right. \right. \right. \\ & \left. \left. \left\{ 1 + \frac{(m-1)^2}{P_1^2 a^2} \right\} I_m^2(P_1 a) \right] \right. \\ & \left. - \frac{1}{8} a^2 \left[\frac{1}{4} \{I_m(P_1 a) + I_{m+2}(P_1 a)\}^2 - \left\{ 1 + \frac{(m+1)^2}{P_1^2 a^2} \right\} \right. \right. \end{aligned}$$

$$\begin{aligned}
 & \times I_{m+1}^2 (P_1 a) \Big] + \frac{1}{2} \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \left(\frac{P_1}{2}\right)^{2m+2n+2t} \cdot \phi \Big\} \cdot \frac{1}{2} \\
 & + B_m^2 P_2^2 \left\{ -\frac{1}{8} a^2 \left[\frac{1}{4} \{ I_{m-2} (P_2 a) + I_m (P_2 a) \}^2 \right. \right. \\
 & + \left. \left. \left\{ 1 + \frac{(m-1)^2}{P_2^2 a^2} \right\} I_{m-1}^2 (P_2 a) \right] - \frac{1}{8} a^2 \left[\frac{1}{4} \{ I_m (P_2 a) \right. \right. \right. \\
 & + \left. \left. \left. I_{m+2} (P_2 a) \right\}^2 + \left\{ 1 + \frac{(m+1)^2}{P_2^2 a^2} \right\} I_{m+1}^2 (P_2 a) \right] \right. \\
 & + \left. \frac{1}{2} \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \left(\frac{P_2}{2}\right)^{2m+2n+2t} \cdot \phi \Big\} \frac{1}{2} + \frac{1}{2} A_m B_m P_1 P_2 \\
 & \times \left\{ \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \left[\frac{\left(\frac{P_1}{2}\right)^{m+2n-1} \left(\frac{P_2}{2}\right)^{m+2t-1} a^{2m+2n+2t}}{(2m+2n+2t) |n| |t| \Gamma(m+n) \Gamma(m+t)} \right. \right. \\
 & + \left. \left. \left(\frac{P_2}{2}\right)^{m+2n-1} \left(\frac{P_2}{2}\right)^{m+2t+1} \phi \right. \right. \\
 & + \left. \left. \frac{\left(\frac{P_1}{2}\right)^{m+2n+1} \left(\frac{P_2}{2}\right)^{m+2t-1} a^{2m+2n+2t+2}}{(2m+2n+2t+2) |n| |t| \Gamma(m+t) \Gamma(m+n+2)} \right. \right. \\
 & + \left. \left. \frac{\left(\frac{P_1}{2}\right)^{m+2n+1} \left(\frac{P_2}{2}\right)^{m+2t+1} a^{2m+2n+2t+4}}{(2m+2n+2t+4) |n| |t| \Gamma(m+n+2) \Gamma(m+t+2)} \right] \right\} \frac{1}{2} \\
 & + A_m^2 \left\{ \sum_{n=0}^{\infty} \left(\frac{P_1}{2}\right)^{2m+4n} \phi_1 + \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \left(\frac{P_1}{2}\right)^{2m+2n+2t} \psi \right\} \frac{m^2}{2} \\
 & + B_m^2 \left\{ \sum_{n=0}^{\infty} \left(\frac{P_2}{2}\right)^{2m+4n} \phi_1 + \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \left(\frac{P_2}{2}\right)^{2m+2n+2t} \psi \right\} \frac{m^2}{2} \\
 & + 2A_m B_m \left\{ \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \left(\frac{P_1}{2}\right)^{m+2n} \left(\frac{P_2}{2}\right)^{m+2t} \psi \right\} \frac{m^2}{2} \quad (36)
 \end{aligned}$$

where

$$\phi = \frac{a^{2m+2n+2t+2}}{(2m+2n+2t+2) \underline{n} \underline{t} \Gamma(m+n) \Gamma(m+t+2)}$$

$$\phi_1 = \frac{a^{2m+4n}}{(2m+4n) \{\underline{n} \Gamma(m+n+1)\}^2}$$

$$\psi = \frac{a^{2m+2n+2t}}{(2m+2n+2t) \underline{n} \underline{t} \Gamma(m+n+1) \Gamma(m+t+1)}$$

Thus the deflection, w is completely determined. The expressions for the bending and twisting moment can now be determined.

$$\begin{aligned} M_r = & -D \left[P_1^2 (A_0' + A_0) I_0'' (P_1 r) + P_2^2 (B_0' + B_0) I_0'' (P_2 r) \right. \\ & + \sum_{m=2, 4, 6, \dots}^{\infty} \{P_1^2 A_m I''_m (P_1 r) + P_2^2 B_m I''_m (P_2 r)\} \cos m\theta \\ & + \nu \left\{ \frac{P_1}{r} (A_0' + A_0) I_1' (P_1 r) + \frac{P_2}{r} (B_0' + B_0) I_1' (P_2 r) \right. \\ & + \frac{1}{r} \sum_{m=2, 4, 6, \dots}^{\infty} [P_1 A_m I'_m (P_1 r) + P_2 B_m I'_m (P_2 r)] \cos m\theta \\ & \left. \left. - \frac{1}{r^2} \sum_{m=2, 4, 6, \dots}^{\infty} m^2 [A_m I_m (P_1 r) + B_m I_m (P_2 r)] \cos m\theta \right\} \right] \end{aligned} \quad (37)$$

$$\begin{aligned} M_\theta = & -D \left[\frac{P_1}{r} (A_0' + A_0) I_1' (P_1 r) + \frac{P_2}{r} (B_0' + B_0) I_1' (P_2 r) \right. \\ & + \frac{1}{r} \sum_{m=2, 4, 6, \dots}^{\infty} \{P_1 A_m I'_m (P_1 r) + P_2 B_m I'_m (P_2 r)\} \cos m\theta \\ & - \frac{1}{r^2} \sum_{m=2, 4, 6, \dots}^{\infty} m^2 \{A_m I_m (P_1 r) + B_m I_m (P_2 r)\} \cos m\theta \\ & \left. + \nu \left\{ P_1^2 (A_0' + A_0) I_0'' (P_1 r) + P_2^2 (B_0' + B_0) I_0'' (P_2 r) \right. \right. \end{aligned}$$

$$+ \sum_{m=2, 4, 6, \dots}^{\infty} [P_1^2 A_m I_m'' (P_1 r) + P_2^2 B_m I_m'' (P_2 r)] \cos m\theta \Big\} \tag{38}$$

$$M_{r\theta} = (1 - \nu) D \left[-\frac{1}{r} \sum_{m=2, 4, 6, \dots}^{\infty} m \{P_1 A_m I_m' (P_1 r) + P_2 B_m I_m' (P_2 r)\} \right. \\ \times \sin m\theta + \frac{1}{r^2} \sum_{m=2, 4, 6, \dots}^{\infty} m \{A_m I_m (P_1 r) + B_m I_m (P_2 r)\} \\ \left. \times \sin m\theta \right]. \tag{39}$$

The stresses can be calculated from the expressions

$$\sigma_r = \frac{6M_r}{h^2}; \quad \sigma_\theta = \frac{6M_\theta}{h^2}; \quad \tau_{r\theta} = \frac{6M_{r\theta}}{h^2} \tag{40}$$

NUMERICAL CALCULATION

To obtain deflection for a given value of plate radius ‘a’ and foundation modulus ‘K_F’ one has to start from the equation (36) with an assumed value of ‘a’ in order to obtain the corresponding value of the load function qa⁴/Dh. Once this relationship is obtained the corresponding deflection w/h can be calculated by eq. 32. For a = 50 mm, h = 0.75 mm, ν = 0.3 and K_F = 80 deflections have been presented in Fig 2.

CONCLUDING REMARKS

An examination of the eq. 32 will reveal that the deflection (w/h) depends on K_F, the plate radius ‘a’ and on the value of the angle, θ. For a given value of the load function eq. 32 can be written as

$$\left(\frac{w}{h}\right)_{\substack{r=0 \\ \theta=0}} = K_1 \left(\frac{qa^4}{Dh}\right); \quad \left(\frac{w}{h}\right)_{\substack{r=a \\ \theta=\pi/2}} = K_2 \left(\frac{qa^4}{Dh}\right) \tag{41}$$

where K₁ and K₂ are two numerical constants, K₂ being greater than K₁. Because of the reactive forces at the two points of support, deflections on the diameter at θ = 0 will be less than those on the diameter at θ = π/2. Maximum deflection will occur at the boundary at θ = ± π/2. Deflections according to the linear theory have also been plotted in Fig. 2 and it is

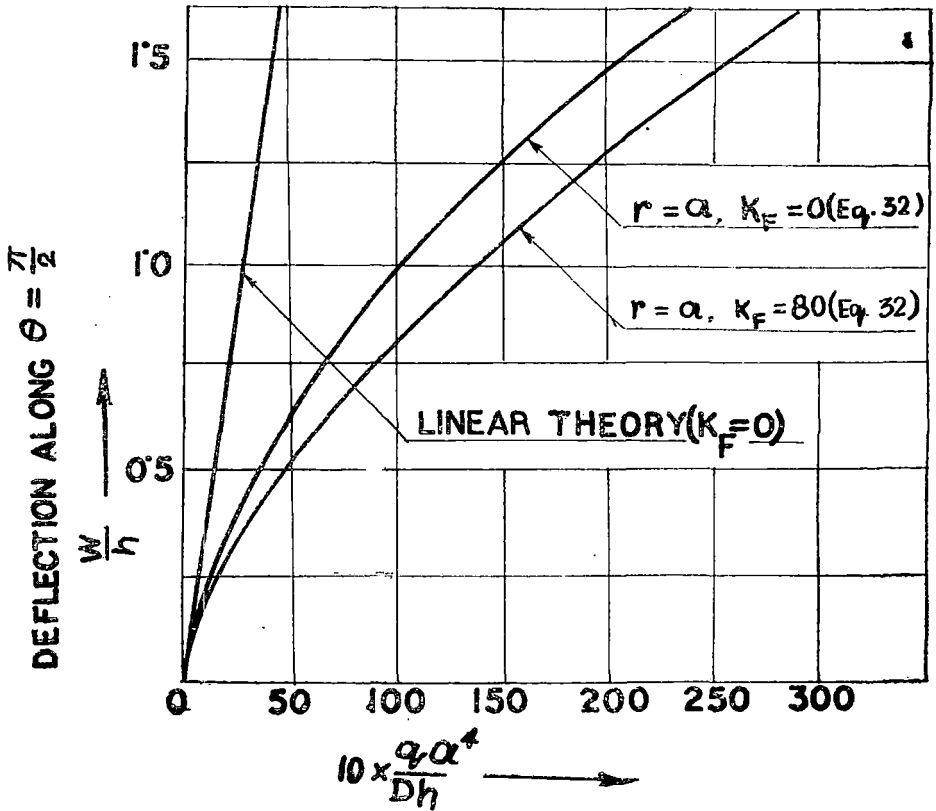


FIG. 2. Load deflection curve.

clear that the errors of the linear theory increases as the load increases. In order to study the variation of moments, eqs. 37, 38 and 39 are plotted in Fig. 3 for various values of (r/a) and for the angles at which they become maximum. It is observed that the maximum bending moments, their magnitudes being unequal, are developed at $r = 3a/4$, $\theta = \pm \pi/2$ and the twisting moment is maximum at $r = a$, $\theta = \pm \pi/4$, $\pm 3\pi/4$.

As the plate must be in equilibrium on the supports, the foregoing analysis for two simple supports represents the worst condition when the deflections and stresses are maximum for a given load function. With the increase in the number of supports, w_1 in eq. 8 decreases. For an infinitely large number of supports, w in eq. 8 will approach to w_0 in the limit and the point of maximum bending moments will shift to the centre of the plate, $(M_r)_{\max}$ being equal to $(M_\theta)_{\max}$ in that case,

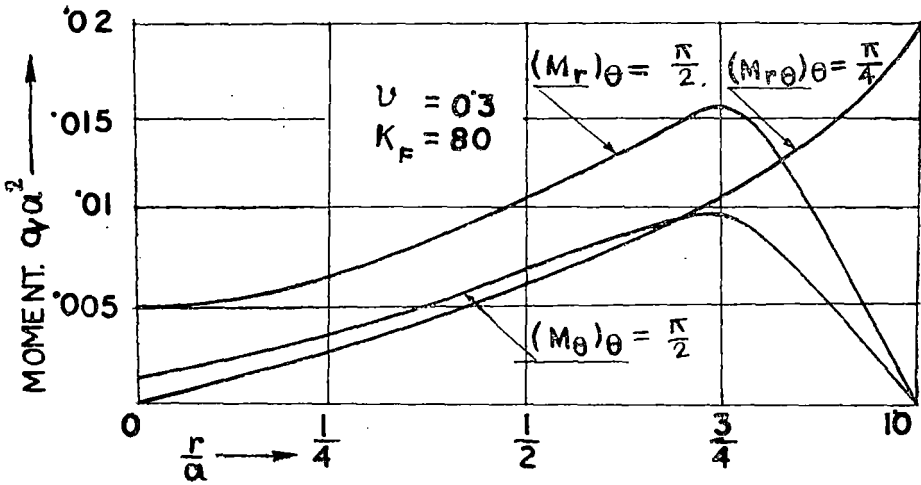


FIG. 3. Moment curve.

The present study can be extended to any number of supports, provided the supports are so chosen as not to disturb the equilibrium of the plate. For example, if three equidistant supports are chosen, $\psi_1 = 0$, $\psi_2 = 2\pi/3$, $\psi_3 = 4\pi/3$, the differential equations together with the boundary condition remaining unchanged. If the plate is clamped on the supports, the boundary conditions and the concentrated reactions at the supports will change totally demanding a separate investigation.

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NOTATION

The following symbols have been used in this paper:

- a = plate radius
- $A_0', A_0, B_0', B_0, Am, Bm$ = Constants
- D = flexural rigidity of the plate = $\frac{Eh^3}{12(1-\nu^2)}$
- E = Young's modulus
- e_1 = first invariant of middle surface strains
 = $\epsilon_x + \epsilon_y$ in rectangular co-ordinates
 = $\epsilon_r + \epsilon_\theta$ in cylindrical co-ordinates
- e_2 = second invariant of middle surface strains
 = $\epsilon_x\epsilon_y - \frac{1}{4}\gamma_{xy}^2$ in rectangular co-ordinates
 = $\epsilon_r\epsilon_\theta$ in cylindrical co-ordinates in case of circular symmetry
- h = plate thickness
- I_0, I_m = Modified Bessel's function of the first kind and of the zero order and m th order respectively.
- K = foundation reaction per unit area per unit deflection

| | |
|-------------|----------------------------------------------------------|
| K_F | = dimensionless foundation modulus = $\frac{K}{D} a^4$. |
| M | = moment |
| q | = uniform lateral load |
| r, θ | = polar co-ordinates |
| u, v | = radial and crossradial displacements |
| V_1 | = strain energy |
| w | = deflection in z -direction |
| σ | = direct stress |
| τ | = shear stress |
| ϵ | = direct strain |
| γ | = shear strain |
| ν | = Poisson's ratio |
| Γ | = Gamma function. |

for a bonded composite material, it should be assumed that interactions between constituents occur between congruent particles. Since the constituents can undergo individual motions, this assumption implies that during any motion of the material, interactions between constituents occur between material particles which can occupy different spatial positions. This is in contrast to the usual continuum theories of mixtures, in which interactions between constituents are assumed to occur between material particles which are spatially superimposed [3].

In [2], the interaction force vector \mathbf{b} between constituents was assumed to be a function of the relative displacement $\mathbf{u}_{(pm)} = \mathbf{x}_{(p)} - \mathbf{x}_{(m)}$ between congruent particles. Assuming that it also depends upon the relative velocity $\dot{\mathbf{u}}_{(pm)} = \dot{\mathbf{x}}_{(p)} - \dot{\mathbf{x}}_{(m)}$ introduces an interesting difficulty in that the component of the relative velocity normal to the line joining the congruent particles is not invariant under imposed rigid body rotations and thus does not satisfy the principle of material-frame indifference (PMI) [4].

Formally, consider motions of the constituents

$$\bar{\mathbf{x}}_{(m)} = \bar{\mathbf{x}}_{(m)}(X_{(m)}, t), \quad \bar{\mathbf{x}}_{(p)} = \bar{\mathbf{x}}_{(p)}(X_{(p)}, t) \quad (4)$$

related to the motions (3) by

$$\bar{\mathbf{x}}_{(m)} = \mathbf{Q}\mathbf{x}_{(m)} + \mathbf{c}, \quad \bar{\mathbf{x}}_{(p)} = \mathbf{Q}\mathbf{x}_{(p)} + \mathbf{c} \quad (5)$$

where \mathbf{Q} is an arbitrary orthogonal time-dependent linear transformation and \mathbf{c} is an arbitrary time-dependent vector. The PMI requires that a vector-valued constitutive variable \mathbf{v} associated with the motion given by equations (3) be related to its value $\bar{\mathbf{v}}$ associated with the motion given by equations (4) by $\bar{\mathbf{v}} = \mathbf{Q}\mathbf{v}$. If this is satisfied, \mathbf{v} is said to be indifferent.

The relative displacement between congruent particles is indifferent,

$$\bar{\mathbf{u}}_{(pm)} = \mathbf{Q}\mathbf{u}_{(pm)}, \quad (6)$$

but the relative velocity is not.

$$\bar{\dot{\mathbf{u}}}_{(pm)} = \mathbf{Q}\dot{\mathbf{u}}_{(pm)} + \dot{\mathbf{Q}}\mathbf{u}_{(pm)}. \quad (7)$$

Let a decomposition of $\dot{\mathbf{u}}_{(pm)}$ into indifferent ($\dot{\mathbf{u}}_{(pm)}^T$) and nonindifferent ($\dot{\mathbf{u}}_{(pm)}^N$) parts be sought,³ $\dot{\mathbf{u}}_{(pm)} = \dot{\mathbf{u}}_{(pm)}^T + \dot{\mathbf{u}}_{(pm)}^N$, such that

$$\bar{\dot{\mathbf{u}}}_{(pm)}^T = \mathbf{Q}\dot{\mathbf{u}}_{(pm)}^T, \quad \bar{\dot{\mathbf{u}}}_{(pm)}^N = \mathbf{Q}\dot{\mathbf{u}}_{(pm)}^N + \dot{\mathbf{Q}}\mathbf{u}_{(pm)} \quad (8)$$

It is easy to show that equations (8) are satisfied if $\dot{\mathbf{u}}_{(pm)}^T$ and $\dot{\mathbf{u}}_{(pm)}^N$ are the components of $\dot{\mathbf{u}}_{(pm)}$ tangential and normal to $\mathbf{u}_{(pm)}$. For, defining

$$\dot{\mathbf{u}}_{(pm)}^T = \begin{cases} \left[\frac{\dot{\mathbf{u}}_{(pm)} \cdot \mathbf{u}_{(pm)}}{\mathbf{u}_{(pm)} \cdot \mathbf{u}_{(pm)}} \right] \mathbf{u}_{(pm)} & \text{if } \mathbf{u}_{(pm)} \neq 0 \\ \dot{\mathbf{u}}_{(pm)} & \text{if } \mathbf{u}_{(pm)} = 0 \end{cases} \quad (9)$$

the transformation equation is

$$\bar{\dot{\mathbf{u}}}_{(pm)}^T = \left[\frac{\bar{\dot{\mathbf{u}}}_{(pm)} \cdot \bar{\mathbf{u}}_{(pm)}}{\bar{\mathbf{u}}_{(pm)} \cdot \bar{\mathbf{u}}_{(pm)}} \right] \bar{\mathbf{u}}_{(pm)} = \mathbf{Q}\dot{\mathbf{u}}_{(pm)}^T \quad (10)$$

and then

$$\bar{\dot{\mathbf{u}}}_{(pm)}^N = \bar{\dot{\mathbf{u}}}_{(pm)} - \bar{\dot{\mathbf{u}}}_{(pm)}^T = \mathbf{Q}\dot{\mathbf{u}}_{(pm)}^N + \dot{\mathbf{Q}}\mathbf{u}_{(pm)}. \quad (11)$$

It is also easy to see that the decomposition is unique. Thus interactions between congruent particles can depend only on the component $\dot{\mathbf{u}}_{(pm)}^T$ of $\dot{\mathbf{u}}_{(pm)}$.

A physical picture which helps to clarify this result is to imagine

³ This is similar to the familiar decomposition of the velocity gradient into indifferent (deformation rate) and nonindifferent (vorticity) parts.

the congruent particles of the two constituents to be connected by springs and dashpots. The dashpot (relative velocity dependent) forces do not depend on the relative velocity components normal to the lines joining the particles, but depend only on the tangential components.

Assuming that the interaction force vector \mathbf{b} and the stress tensor in the elastic material $t_{(m)}$ depend upon $\dot{\mathbf{u}}_{(pm)}$, $\dot{\mathbf{u}}_{(pm)}^T$ and the strain of the elastic material, the linearized isotropic three-dimensional forms of equations (1) and (2) are, in Cartesian tensor form,

$$\rho_{(m)} \frac{\partial^2 u_{(m)k}}{\partial t^2} = t_{(m)kj,j} + b_k, \quad \rho_{(m)} \frac{\partial^2 u_{(p)k}}{\partial t^2} = -b_k, \quad (12)$$

where

$$t_{(m)kj} = \lambda_{(m)} \delta_{kj} e_{(m)nn} + 2\mu_{(m)} e_{(m)kj}, \quad (13)$$

$$b_k = \nu u_{(pm)k} + \sigma \dot{u}_{(pm)k}^T,$$

and

$$e_{(m)kj} = \frac{1}{2} (u_{(m)k,j} + u_{(m)j,k}). \quad (14)$$

For one-dimensional compressional or shear wave motion, equations (12)–(14) reduce to the form of equations (1) and (2). However, for more complicated motions, including combined compression and shear, it is interesting to note that equations (12)–(14) are intrinsically nonlinear due to the term $\dot{u}_{(pm)k}^T$, constituting what might be called a geometric material nonlinearity.

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Large Deflection of a Circular Plate on Elastic Foundation Under a Concentrated Load at the Center

S. Datta¹

Introduction

For moderately large deflections of thin plates, solutions of the differential equations for deflections and displacements become difficult because of their nonlinear character. Neglecting the second strain invariant of the middle surface strains, Berger [1]² solved the large deflection of circular and rectangular plates under uniform load with ease and sufficient accuracy. Following Berger's method many investigators [2–6] have solved various large deflection problems and have obtained satisfactory results.

Following Berger's method the large deflection of a clamped cir-

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² Numbers in brackets designate References at end of Note.

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BRIEF NOTES

cular isotropic plate resting on elastic foundation of the Winkler type has been investigated in this study for a concentrated load applied at the center. The results have been experimentally verified.

Analysis

Using polar coordinates the governing differential equation for a circular plate of radius a , with a concentrated load P , at the center is given by

$$\nabla^2(\nabla^2 - \alpha^2)w + \frac{K}{D}w = 0 \text{ (except at the load point)} \quad (1)$$

where α is a constant determined from

$$\frac{u}{r} + \frac{du}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 = \frac{\alpha^2 h^2}{12} \quad (2)$$

and

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}; \quad w = \text{displacement}$$

along z -direction; K = foundation reaction per unit area per unit deflection; $D = (Eh^3/12(1 - \nu^2))$, E being the modulus of elasticity; h is the plate thickness, and ν is Poisson's ratio.

Considering the radial stress and shearing stress on a concentric circular area of radius r , the concentrated load P at the center, and since u and dw/dr are both zero at the center one gets

$$\lim_{r \rightarrow 0} Dr \frac{d}{dr} [(\nabla^2 - \alpha^2)w] = \frac{P}{2\pi} \quad (3)$$

Solution of equation (1) can be taken in the following convenient form

$$w = AI_0(P_1 r) + BI_0(P_2 r) + C[K_0(P_1 r) - K_0(P_2 r)] \quad (4)$$

in which I_0 is the modified Bessel function of the first kind and zero order and K_0 is the modified Bessel function of the second kind and zero order

$$P_1^2 + P_2^2 = \alpha^2; \quad P_1^2 P_2^2 = \frac{K}{D}$$

A , B , and C are constants to be determined from the boundary conditions. Clamped edge boundary conditions are

$$(w)_{r=a} = 0 = \left(\frac{dw}{dr} \right)_{r=a} \quad (5)$$

Considering equations (3) and (4) one gets

$$C = \frac{P}{2\pi D(P_2^2 - P_1^2)}$$

Considering equations (4) and (5) one gets

$$A = C \left[\frac{1 - \psi}{\phi} \right]; \quad B = C \left[\frac{1 - \psi_1}{\phi} \right]$$

where

$$\begin{aligned} \psi &= K_1(P_1 a) P_1 I_0(P_2 a) - K_0(P_1 a) P_2 I_1(P_2 a) \\ \psi &= P_2 I_1(P_2 a) I_0(P_1 a) - P_1 I_1(P_1 a) I_0(P_2 a) \\ \psi_1 &= K_0(P_2 a) P_1 I_1(P_1 a) - K_1(P_2 a) P_2 I_0(P_1 a) \end{aligned}$$

Thus

$$w = C \left[\left\{ \frac{1 - \psi}{\phi} \right\} I_0(P_1 r) + \left\{ \frac{1 - \psi_1}{\phi} \right\} I_0(P_2 r) + \{K_0(P_1 r) - K_0(P_2 r)\} \right] \quad (6)$$

is determined completely.

Setting $r \rightarrow 0$ in equation (4) one gets the maximum deflection w_0 , at the center of the plate,

$$w_0 = A + B + C \log_e \left(\frac{P_2}{P_1} \right) \quad (7)$$

To determine the displacement u , one gets from equation (2)

$$ur = \frac{\alpha^2 h^2 r^2}{24} - \frac{1}{2} \int \left(\frac{dw}{dr} \right)^2 r dr + K' \quad (8)$$

where K' is the constant of integration. After evaluating the integrals and using the boundary condition $u \rightarrow 0$ as $r \rightarrow a$, the equation determining u may be obtained. Also as $r \rightarrow 0$, $u \rightarrow 0$ from symmetry. Thus the equation for α is obtained as

$$\begin{aligned} & \frac{\alpha^2 h^2 a^2}{12} \\ &= A^2 P_1^2 \left[\frac{a I_0(P_1 a) I_1(P_1 a)}{P_1} + \frac{1}{2} I_1^2(P_1 a) a^2 - \frac{1}{2} I_0^2(P_1 a) a^2 \right] \\ &+ B^2 P_2^2 \left[\frac{a I_0(P_2 a) I_1(P_2 a)}{P_2} + \frac{1}{2} I_1^2(P_2 a) a^2 - \frac{1}{2} I_0^2(P_2 a) a^2 \right] \\ &+ C^2 P_1^2 \left[\frac{1}{2} a^2 K_1^2(P_1 a) - \frac{1}{2} K_0^2(P_1 a) a^2 - \frac{a}{P_1} K_0(P_1 a) K_1(P_1 a) \right] \\ &+ C^2 P_2^2 \left[\frac{1}{2} a^2 K_1^2(P_2 a) - \frac{1}{2} K_0^2(P_2 a) a^2 - \frac{a}{P_2} K_0(P_2 a) K_1(P_2 a) \right] \\ &+ \frac{2ABP_1 P_2 a}{P_2^2 - P_1^2} [P_2 I_1(P_1 a) I_0(P_2 a) - P_1 I_1(P_2 a) I_0(P_1 a)] \\ &- \frac{2C^2 P_1 P_2 a}{P_2^2 - P_1^2} [P_1 K_1(P_2 a) K_0(P_1 a) - P_2 K_1(P_1 a) K_0(P_2 a)] \\ &- \frac{2ACP_1 P_2 a}{P_2^2 - P_1^2} [P_2 I_1(P_1 a) K_0(P_2 a) + P_1 K_1(P_2 a) I_0(P_1 a)] \\ &- \frac{2BCP_1 P_2 a}{P_2^2 - P_1^2} [P_1 I_1(P_2 a) K_0(P_1 a) + P_2 K_1(P_1 a) I_0(P_2 a)] \\ &+ BC [P_2^2 a^2 \{I_1(P_2 a) K_1(P_2 a) + I_0(P_2 a) K_0(P_2 a)\} \\ &- P_2 a \{I_1(P_2 a) K_0(P_2 a) - I_0(P_2 a) K_1(P_2 a)\}] \\ &- AC [P_1^2 a^2 \{I_1(P_1 a) K_1(P_1 a) + I_0(P_1 a) K_0(P_1 a)\} \\ &- P_1 a \{I_1(P_1 a) K_0(P_1 a) - I_0(P_1 a) K_1(P_1 a)\}] \\ &+ \frac{2ACP_1^2}{P_2^2 - P_1^2} + \frac{2BCP_2^2}{P_2^2 - P_1^2} + AC - BC \\ &- C^2 \left[I + \frac{P_2^2 + P_1^2}{P_2^2 - P_1^2} \log_e \frac{P_1}{P_2} \right] \quad (9) \end{aligned}$$

If $P_1 \rightarrow 0$, $P_2 \rightarrow \alpha$ or $P_1 \rightarrow \alpha$, $P_2 \rightarrow 0$, one gets from equation (6)

$$\begin{aligned} w &= -\frac{P}{\pi 2D \alpha^3 a I_1(\alpha a)} \left[I_0(\alpha a) - 1 - I_0(\alpha r) \right. \\ &+ I_0(\alpha r) \alpha a K_1(\alpha a) + \alpha a I_1(\alpha a) \log_e \left(\frac{r}{a} \right) \\ &\left. + K_0(\alpha r) \alpha a I_1(\alpha a) \right] \quad (10) \end{aligned}$$

and equation (9) also reduces to

$$\left[\frac{Pa^2}{\pi Dh} \right]^2 = \left[\frac{\frac{1}{3} (\alpha a)^6}{\gamma + \log_e \frac{\alpha a}{2} - I_0(\alpha a) + \alpha a K_1(\alpha a) - 2 \alpha a I_1(\alpha a)} - \frac{1}{2} \left\{ \frac{I_0(\alpha a) - 1}{I_1(\alpha a)} \right\}^2 \right] \quad (11)$$

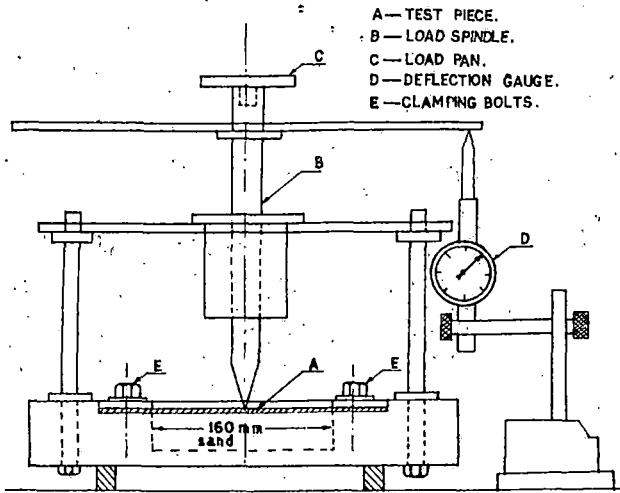


Fig. 1 Experimental apparatus for load deflection test

Equations (10) and (11) are the results obtained by Basuli [5].

Experimental and Theoretical Results

Deflections of a 0.75-mm thick mild steel plate were measured experimentally using sand as foundation material. The sketch of the apparatus used is shown in Fig. 1 and is self-explanatory. The value of the nondimensional foundation modulus, $K_F = (K/D)a^4$ for sand used was determined experimentally to be 430.

The theoretical results according to Berger's approximate method and experimental results both for $K_F = 0$ and $K_F = 430$ have been presented in Fig. 2. Results according to Timoshenko and Krieger [7] and Schmidt [8] corresponding to $K_F = 0$ have also been presented for comparison in the same graph. It is observed that results obtained from Berger's method approach more closely toward the practical values and the error increases progressively with the increase of load function.

Acknowledgment

The author wishes to thank Dr. B. Banerjee, Jalpaiguri Government Engineering College for his help and guidance in preparation of this Note.

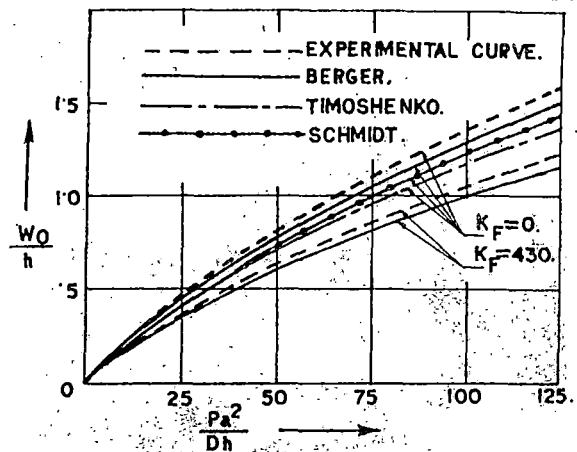


Fig. 2 Deflection curve

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Motion of a Stretched String Loaded by an Accelerating Force¹

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The use of explosives to simulate distributed impulse loads on various structural shapes has recently motivated several analytical solutions for the response of structures to moving forces traveling at a constant velocity. Solutions are available for stretched strings [1-3],³ beams [4, 5], and circular rings [6]. A recently developed rocket-propelled trolley facility at Sandia Laboratories, Albuquerque motivated this analysis. Unlike the problems which have forces moving at constant velocity, the trolley is propelled at nearly constant acceleration along a steel cable suspended between two

mountain peaks. In order to estimate the cable motion, a wave solution for the response of a pinned end, semi-infinite stretched string loaded by a force moving at constant acceleration is derived.

Analysis

A wave solution is obtained by employing Laplace and Fourier transform techniques, and the definitions for the integral transforms given in [7] are adopted. For purposes of analysis the string is considered infinite, and an image load is included in the equation of motion to satisfy the pinned end boundary condition. The equation of motion is

$$\rho \partial^2 y / \partial t^2 - T \partial^2 y / \partial x^2 = P \delta(x - at^2/2) - P \delta(x + at^2/2) \quad (1)$$

where ρ is mass per unit length, T is tension, y is deflection, t is time, x is the axial coordinate measured positive to the right, P is the force magnitude, a is the constant acceleration, and δ is the Dirac delta function. The response is evaluated for positive x , and the second term on the right-hand side of equation (1) is the image load. The transformed solution is

$$\bar{y}^*(\xi, s) = \frac{P}{\rho} \int_0^\infty \frac{\exp[-s(2\alpha/a)](e^{-t\alpha} - e^{t\alpha})}{(2\alpha/a)^{1/2}(s^2 + c^2\xi^2)} d\alpha \quad (2)$$

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³ Numbers in brackets designate References at end of Note.

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THERMAL BUCKLING OF SOME HEATED PLATES PLACED ON ELASTIC FOUNDATION

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Thermal buckling of a heated equilateral triangular plate and a clamped elliptic plate placed on elastic foundation has been investigated. The boundary of the plate is transformed conformally onto the unit circle. The critical buckling temperature is obtained with the help of error function.

Thermal buckling of thin elastic plates is of much practical importance in modern engineering. Nowacki¹ has discussed the thermal buckling of a rectangular plate under different boundary conditions. Mansfield² has investigated the buckling and curling of a heated thin circular plate of constant thickness. Klosner & Forray³ have studied the thermal buckling of simply supported plates under symmetrical temperature distribution.

Stability of thin elastic plates having exotic boundaries subjected to hydrostatic in-plane loading can easily be investigated with the help of approximate techniques such as collocation, finite difference, finite elements, etc. Laura & Shahady⁴ have shown that it is convenient to conformally transform the given domain onto a simpler one, i.e., the unit circle and the boundary conditions can then be satisfied identically.

In this paper thermal buckling of a heated equilateral triangular plate and a clamped elliptic plate placed on elastic foundation has been investigated. The foundation is assumed to be of the Winkler type. The boundary has been transformed conformally onto the unit circle and solution has been obtained with the help of error function.

NOTATIONS

The following notations have been used in this paper :

B_n, B_m = constants ;

D = flexural rigidity of the plate = $\frac{Eh^3}{12(1-\nu^2)}$,

E = Young's modulus ;

h = plate thickness ;

$$N_T = \alpha E \int_{-\frac{h}{2}}^{\frac{h}{2}} T dz ;$$

T = temperature ;

u, v = displacement in x and y direction respectively ;

W = deflection normal to the middle plane of the plate ;

K_1 = foundation reaction per unit area per unit deflection ;

α = coefficient of linear thermal expansion.

THEORY

Let us consider a plate of thickness, h , subjected to a temperature distribution T which is independent of x and y , but varies arbitrarily through the thickness, i.e.,

$$\bar{T} = T(z)$$

The plate is subjected to no external load and motion of all supports in the plane of the plate is prevented. It justifies then, that under the above condition there are no displacements in the plane of the plate, i.e.,

$$u = v = 0$$

On the above propositions the differential equation for the displacement⁵ is

$$D \nabla^4 W + \frac{N_T}{1-\nu} \nabla^2 W = 0 \tag{1}$$

For a plate placed on elastic foundation having the foundation reaction, K_1 , (1) becomes

$$D \nabla^4 W + \frac{N_T}{1-\nu} \nabla^2 W + K_1 W = 0 \tag{2}$$

Eq. (2) may be written as

$$(\nabla^2 + P_1^2)(\nabla^2 + P_2^2)W = 0 \tag{3}$$

in which

$$P_1^2 P_2^2 = \frac{K_1}{D} \tag{4}$$

$$P_1^2 + P_2^2 = \frac{N_T}{D(1-\nu)} \tag{5}$$

If $z = x + iy, \bar{z} = x - iy$ Eq. (3) changes into

$$\left(4 \frac{\partial^2}{\partial z \partial \bar{z}} + P_1^2\right) \left(4 \frac{\partial^2}{\partial z \partial \bar{z}} + P_2^2\right) W = 0 \tag{6}$$

Let $z = f(\xi)$ be the analytic function which maps the given shape in the ξ -plane onto a unit circle. Thus (6) transforms into complex co-ordinates as

$$\left(\nabla^2 + P_1^2 \left[\frac{dz}{d\xi}\right]^2\right) \left(\nabla^2 + P_2^2 \left[\frac{dz}{d\xi}\right]^2\right) W(\xi, \bar{\xi}) = 0 \tag{7}$$

Eq. (7) is written as

$$(\nabla^2 + \lambda_1^2)(\nabla^2 + \lambda_2^2)W(\xi, \bar{\xi}) = 0 \tag{8}$$

in which

$$\lambda_1^2 = P_1^2 (dz/d\xi)^2, \lambda_2^2 = P_2^2 (dz/d\xi)^2$$

Let

$$W = \sum_{n=1}^{\infty} B_n \left[1 - (\xi \bar{\xi})^n\right] \tag{9}$$

Clearly the above form of W satisfies the edge condition $W = 0$ at $r = 1$. Putting (9) in (8) one gets the error function, $\epsilon_{r,\theta}$. Galerkin's procedure requires that the error function to be orthogonal over the domain, i.e.,

$$\int_C \epsilon_{r,\theta}(\xi, \bar{\xi}) W(\xi, \bar{\xi}) dC = 0 \quad (n = 1, 2, \dots, K) \tag{10}$$

This generates $(K \times K)$ determinantal equation. The lowest root of this gives the critical buckling temperature.

APPLICATIONS

(I) Let us consider an equilateral triangular plate of side, $2a$, and placed on an elastic foundation.

To solve the differential (8) let us put

$$W = W_1 + W_2 \tag{11}$$

From (8) one gets

$$(\nabla^2 + \lambda_1^2) W_1 = 0 \tag{12}$$

$$(\nabla^2 + \lambda_2^2) W_2 = 0 \tag{13}$$

For the edge condition $W = 0$ along the boundary, let

$$W_2 \approx \sum_{n=1}^K B_n \left[1 - (\xi \bar{\xi})^n \right] = \sum_{n=1}^K B_n (1 - r^{2n}) \tag{14}$$

It is sufficient to solve either (12) or (13). The mapping function

$$z = 1.1352 a \left[\xi + \frac{1}{6} \xi^4 + \frac{5}{63} \xi^7 + \frac{4}{81} \xi^{10} \right] \tag{15}$$

maps an equilateral triangular plate a unit circle in the ξ -plane.

With this mapping function putting (14) in (13) and remembering $\xi = r e^{i\theta}$ one gets the required error function. After evaluating the integral given by (10) and taking $K=2$, the following determinant is obtained.

$$\begin{vmatrix} \frac{\lambda_2^2}{6} - 1 & \frac{5\lambda_2^2}{24} - \frac{4}{3} \\ \frac{5\lambda_2^2}{24} - \frac{4}{3} & \frac{4\lambda_2^2}{15} - 2 \end{vmatrix} = 0 \tag{16}$$

Solving (16) for the lowest root, one gets the critical buckling temperature.

$$(N_T)_{cr} = D (1 - \nu) \left[\frac{5.8}{(1.1352 a)^2} - \frac{K_1 (1.1352 a)^2}{5.8 D} \right] \tag{17}$$

(II) Let us consider an elliptic plate having centre at the origin. Let h be the thickness of the plate. For clamped edge boundary condition let us take W in the following form.

$$W = \sum_{n=1}^K B_n \left[1 - (\xi \bar{\xi})^n \right]^2 \tag{18}$$

Clearly

$$W = \frac{\partial W}{\partial r} = 0 \quad \text{at } r = 1$$

For the ellipse

$$\left(\frac{x^2}{4/3} + \frac{y^2}{4/5} \right) = 1$$

mapping function

$$z = 0.99 b (\xi + 0.12 \xi^3 + 0.03 \xi^5 + 0.01 \xi^7) \tag{19}$$

maps the above ellipse a unit circle in the ξ -plane. With this mapping function putting (18) in (8) and remembering $\xi = r e^{i\theta}$ one gets the required error function. After evaluating the integral given by (10) and taking $K = 2$, the following determinant is obtained.

$$\begin{vmatrix} \frac{32}{3} - \frac{2}{3} (\lambda_1^2 + \lambda_2^2) + \frac{\lambda_1^2 \lambda_2^2}{10} & \frac{256}{15} - \frac{4}{5} (\lambda_1^2 + \lambda_2^2) + \frac{33}{70} \lambda_1^2 \lambda_2^2 \\ \frac{256}{15} - \frac{4}{5} (\lambda_1^2 + \lambda_2^2) + \frac{29}{576} \lambda_1^2 \lambda_2^2 & \frac{5632}{105} - \frac{4}{3} (\lambda_1^2 + \lambda_2^2) + \frac{127}{315} \lambda_1^2 \lambda_2^2 \end{vmatrix} = 0 \tag{20}$$

Solving (20) for the lowest root, the critical buckling temperature is obtained as

$$(N_T)_{Cr} = D(1 - \nu) \left[\frac{44.6}{b^2} + 0.094 b^2 \frac{K_1}{D} - \left\{ \left(\frac{29.2}{b^2} + 0.094 b^2 \frac{K_1}{D} \right)^2 + 0.106 b^4 \left(\frac{K_1}{D} \right)^2 \right\}^{\frac{1}{2}} \right] \quad (21)$$

CONCLUSION

Solutions obtained in this study are only approximate, because only the first term of the mapping function is considered and K is taken to be 2. More accurate results are obtained by considering the remaining terms of the mapping function and taking K more than 2. Solution of the eigenvalue problem governing the stability of the thin elastic plates having various configurations, such as regular polygonal shape, circular boundary with flat sides, epitrochoidal boundary etc. is easily accomplished with the help of the complex variable theory applied in this study.

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LARGE DEFLECTION OF A TRIANGULAR ORTHOTROPIC PLATE ON ELASTIC FOUNDATION

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Large deflection of an equilateral triangular orthotropic plate, resting on elastic foundation has been solved for a uniform load throughout the plate. General expressions for deflection and bending moment at a particular point have been obtained and the limiting values of the theoretical results have been verified with the known results for small deflection and without any elastic foundation of the corresponding isotropic plate. Theoretical results have also been presented in the form of graphs.

Triangular reinforced concrete slabs are sometimes used as bottom slabs of bunkers. Thus the design of this type of structure is of practical interest for Defence. These slabs may rest freely on soil or sand and generally are subjected to a uniform load. If the thickness of the slab is small compared to the other dimensions, then it may be regarded as a thin orthotropic plate resting on elastic foundation and subjected to a uniform load. Within the elastic limit, the deflection of such plates may be large, i.e., the deflection is on the order of the thickness of the plate. When a plate undergoes large deflection, three differential equations for displacement and deflection may be written, but it is usually difficult to obtain solutions of these equations because of their non-linear character.

Various problems of large deflections of thin plates not resting on elastic foundation have been examined by Way¹, Levy² and many other authors. But the methods used by them involve and require considerable computation. Berger³ suggested that the strain energy due to the second strain invariant of the middle surface strains may be neglected in analysing large deflection of plates having axis symmetric deformation. Berger's method reduces computation and although no complete explanation of this method is offered in, Berger has shown that the deflections and stresses obtained for circular plates under uniform load are in good agreement with those found in practical analysis. Since then numerous problems have been solved with remarkable ease and satisfactory approximation by using this method. Iwinski and Nowinski⁴ generalised the procedure of Berger to orthotropic plates and found out the deflections of circular and rectangular plates under uniform load and various boundary conditions. By using this approximate method Banerjee⁵ obtained deflections of a circular orthotropic plate under a concentrated load at the centre.

Berger's technique of neglecting the second strain invariant in the middle plane has been applied by Sinha⁶ to determine large deflection of circular and rectangular plates under uniform load and resting on elastic foundation.

In this paper large deflection of an equilateral triangular orthotropic plate, such as reinforced concrete, resting on elastic foundation has been solved for a uniform load throughout the plate. Foundation is assumed to be such that its reaction is proportional to the deflection of the plate.

NOTATIONS

- a = one-half of the length of each side of the plate
- e_1 = first invariant of middle surface strains
- = $\epsilon_x + E_y$ in rectangular coordinates
- h = plate thickness
- K = foundation reaction per unit area per unit deflection
- K_F = non-dimensional foundation modulus = $\frac{K}{D} a^4$
- q = uniform lateral load
- u, v = displacement along x and y direction respectively
- V, V_1 = strain energy
- w = deflection in z -direction
- ϵ = direct strain
- γ = shear strain

FORMULATION OF PROBLEM

For moderately large deflections, the strain displacement relationships are

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \tag{1}$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \tag{2}$$

and

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \tag{3}$$

Neglecting the second middle surface strain invariant, the strain energy due to bending and stretching of the middle surface of the plate of thickness, h , can be written as

$$V_1 = \frac{1}{2} \iint \left[D_x \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2D_1 \frac{\partial^2 w}{\partial x^2} + D_y \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 4D_{xy} \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 + D_x \frac{12}{h^2} e_1^2 \right] dx dy \tag{4}$$

in which

$$D_x = \frac{E'_x h^3}{12}, D_y = \frac{E'_y h^3}{12}, D_1 = \frac{E'' h^3}{12}, D_{xy} = \frac{G h^3}{12} \tag{5}$$

$$e_1 = \frac{\partial u}{\partial x} + K_1 \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{K_1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \tag{6}$$

$$K_1^2 = \frac{D_y}{D_x} \tag{7}$$

and $E'_x, E'_y, E'',$ and G are constants to characterise the elastic properties of the material.

By adding the potential energy of the uniform normal load, ' q ' and of the foundation reaction, K to the energy expression (4), the modified energy expression is obtained as follows:

$$V = \frac{1}{2} \iint \left[D_x \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2D_1 \frac{\partial^2 w}{\partial x^2} + D_y \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 4D_{xy} \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 + D_x \frac{12}{h^2} e_1^2 \right] dx dy - \iint q w dx dy + \frac{1}{2} \iint K w^2 dx dy \tag{8}$$

According to the principle of minimum potential energy, the displacements satisfying the equilibrium conditions make the potential energy V minimum. In order for the integral of equation (8) to be an extremum, its integrand F , must satisfy the following Euler's variational principle:

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial w_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial w_y} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial w_{xx}} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial F}{\partial w_{yy}} \right) + \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial F}{\partial w_{xy}} \right) = 0 \tag{9}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) = 0 \tag{10}$$

and

$$\frac{\partial}{\partial y} \left(\frac{\partial F}{\partial v_y} \right) = 0 \tag{11}$$

Applying (10) and (11) to (8) respectively, we get

$$\frac{\partial}{\partial x} (e_1) = 0 \tag{12}$$

$$\frac{\partial}{\partial y} (e_1) = 0 \quad (13)$$

Thus

$$e_1 = C \quad (14)$$

a normalised constant of integration to be determined. Applying (9) to (8) and considering (14), we get

$$\frac{\partial^4 w}{\partial x^4} + K_1^2 \frac{\partial^4 w}{\partial y^4} + \frac{2(D_1 + 2D_{xy})}{D_x} \frac{\partial^4 w}{\partial x^2 \partial y^2} - \frac{12C}{h^2} \left(\frac{\partial^2 w}{\partial x^2} + K_1 \frac{\partial^2 w}{\partial y^2} \right) + \frac{K}{D_x} = \frac{q}{D_x} \quad (15)$$

Introducing the notation

$$H = D_1 + 2 D_{xy}$$

Equation (15) can be written as

$$\frac{\partial^4 w}{\partial x^4} + K_1^2 \frac{\partial^4 w}{\partial y^4} + 2 \frac{H}{D_x} \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} - \frac{12 C}{h^2} \left(\frac{\partial^2 w}{\partial x^2} + K_1 \frac{\partial^2 w}{\partial y^2} \right) + \frac{K}{D_x} = \frac{q}{D_x} \quad (16)$$

For a slab with two-way reinforcement in the directions x and y , H can be taken as⁷

$$H = (D_x D_y)^{\frac{1}{2}}$$

Introducing now

$$x_1 = x$$

$$y_1 = y \left(\frac{D_x}{D_y} \right)^{\frac{1}{2}} \quad (17)$$

Equation (16) is reduced to the form

$$(\nabla^2 - \alpha^2) \nabla^2 w + \frac{K}{D_x} w = \frac{q}{D_x} \quad (18)$$

in which

$$\alpha^2 = \frac{12 C}{h^2}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}$$

SOLUTION OF PROBLEM

Let the plate be in the form of an equilateral triangle, ABC (Fig. 1) having each side of length $2a$. Let the centroid O be the origin, X -axis and Y -axis perpendicular and parallel to the base BC respectively. If x_1, y_1 be the cartesian coordinates of any point, p , within the triangle $p_1 p_2 p_3$ be the three perpendiculars from P on CA, AB and BC respectively, and r be the radius of the inscribed circle, then

$$P_1 = r + \frac{x_1}{2} - \frac{y_1 \sqrt{3}}{2},$$

$$P_2 = r + \frac{x_1}{2} + \frac{y_1 \sqrt{3}}{2},$$

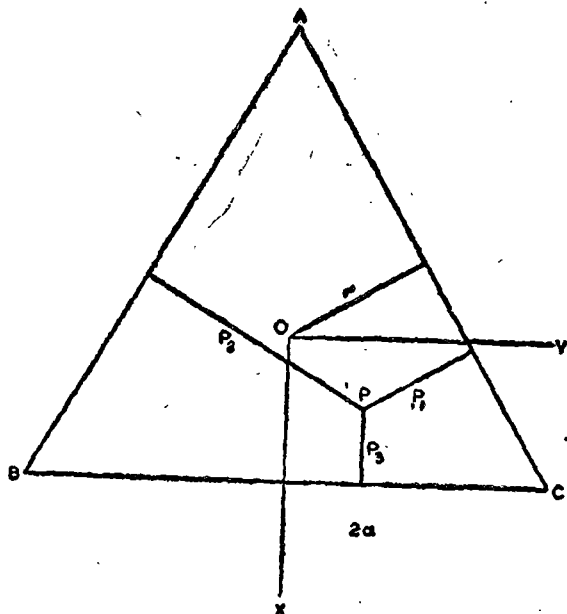


Fig. 1- Equilateral triangular orthotropic plate.

$$P_3 = r - x_1,$$

$$P_1 + P_2 + P_3 = 3r = \sqrt{3}a = K_2 = \text{constant},$$

and

$$\begin{aligned} \nabla^2 &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} \\ &= \frac{\partial^2}{\partial P_1^2} + \frac{\partial^2}{\partial P_2^2} + \frac{\partial^2}{\partial P_3^2} - \frac{\partial^2}{\partial P_1 \partial P_2} - \frac{\partial^2}{\partial P_2 \partial P_3} - \frac{\partial^2}{\partial P_3 \partial P_1} \end{aligned}$$

Using the trilinear Coordinates⁸ p_1, p_2, p_3 the deflection w can be taken in the form

$$w = \sum_{n=1}^{\infty} A_n \left[\sin \frac{2n\pi P_1}{K_2} + \sin \frac{2n\pi P_2}{K_2} + \sin \frac{2n\pi P_3}{K_2} \right] \quad (19)$$

where $A_n =$ a constant.

The above form of w satisfies the following boundary conditions of simply supported edges :

$$w = 0 \text{ at } P_1 = 0, P_2 = 0, P_3 = 0$$

Expanding the transverse uniform load q , into Fourier Sine series

$$q = \sum_{n=1}^{\infty} \frac{2q}{n\pi} \left[\sin \frac{2n\pi P_1}{K_2} + \sin \frac{2n\pi P_2}{K_2} + \sin \frac{2n\pi P_3}{K_2} \right] \quad (20)$$

and substituting (19) and (20) into (18), we get

$$A_n = \sum_{n=1}^{\infty} \frac{2q}{n\pi D_x} \cdot \frac{1}{\left[\left(\frac{2n\pi}{K_2} \right)^2 + \alpha^2 \left(\frac{2n\pi}{K_2} \right) + \frac{K}{D_x} \right]} \quad (21)$$

To determine α , Equation (6) is transformed into x_1, y_1 coordinates in the following form

$$\frac{\alpha^2 h^2}{12} = \frac{\partial u}{\partial x_1} + (K_1)^{\frac{1}{2}} \frac{\partial v}{\partial y_1} + \frac{1}{2} \left(\frac{\partial w}{\partial x_1} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y_1} \right)^2 \quad (22)$$

The boundary conditions on u and v are

$$u = 0 \text{ at } P_3 = 0 \quad (23)$$

$$\sqrt{3} v + u = 0 \text{ at } P_2 = 0 \quad (24)$$

$$\sqrt{3} v - u = 0 \text{ at } P_1 = 0 \quad (25)$$

The following forms of u and v satisfy the above boundary conditions.

$$u = \sum_{m=1}^{\infty} \sqrt{3} B_m \left[\sin \frac{2m\pi (P_2 + P_3)}{K_2} + \sin \frac{2m\pi (P_1 + P_3)}{K_2} \right] \quad (26)$$

$$v = \sum_{m=1}^{\infty} \frac{1}{\sqrt{K_1}} B_m \left[\sin \frac{2m\pi (P_1 + P_3)}{K_2} - \sin \frac{2m\pi (P_2 + P_3)}{K_2} \right] \quad (27)$$

in which B_m is a constant.

Substituting the expressions for u , v and w into (22) and integrating over the whole area of the plate, the following equation determining α is obtained:

$$\frac{\alpha^2 h^2}{12} = \sum_{n=1}^{\infty} \frac{3 A_n^2 n^2 \pi^2}{K_2^2} \quad (28)$$

Thus w is completely determined in the following form in x, y coordinates

$$w = A_n \left[2 \sin 2n \pi \left(\frac{1}{3} + \frac{x}{2\sqrt{3}a} \right) \cos \frac{2n \pi y}{2\sqrt{K_1} a} + \sin 2n \pi \left(\frac{1}{3} - \frac{x}{\sqrt{3}a} \right) \right] \quad (29)$$

If $D_x = D_y = D$, $\alpha \rightarrow 0$, and $K = 0$, (19) and (21) give the deflection equation for an isotropic plate not resting on the elastic foundation in the following form :

$$w = \sum_{n=1}^{\infty} \frac{q K_2^4}{8 n^5 \pi^5 D} \left[\sin \frac{2n \pi P_1}{K_2} + \sin \frac{2n \pi P_2}{K_2} + \sin \frac{2n \pi P_3}{K_2} \right] \quad (30)$$

The corresponding equation as obtained by S. Woinowsky-Krieger for a plate having each side of length $\frac{2a}{\sqrt{3}}$ is

$$w = \frac{q}{64aD} \left[x^3 - 3y^2x - a(x^2 + y^2) + \frac{4}{27} a^3 \right] \left(\frac{4}{9} a^2 - x^2 - y^2 \right) \quad (31)$$

At the origin ($P_1 = P_2 = P_3$), w is given by (30) as

$$w = \frac{27 q a^4}{8 \pi^5 D} \sum_{n=1}^{\infty} \frac{1}{n^5} \sin \frac{2n \pi}{3} = 0.09 \frac{qa^4}{D} \quad (32)$$

which is numerically equal to that obtained from (31) for the plate having each side of length $2a$ as

$$(w)_{x=y=0} = \frac{qa^4}{108D} = 0.009 \frac{qa^4}{D}$$

NUMERICAL CALCULATION

To calculate deflection at any point within the plate, we have to start from (28) with an assumed value of (αa) leading to the corresponding value of the load function $\frac{qa^4}{D_x h}$. Once this relationship is obtained, the corresponding deflection can be obtained from (19) with the help of (21).

At the origin maximum deflection is obtained and is given by

$$\frac{w_{max}}{h} = \frac{6}{\pi} \left(\frac{qa^4}{D_x h} \right) \sum_{n=1}^{\infty} \frac{\sin \frac{2n \pi}{3}}{n \left[\frac{16 \pi^4 n^4}{9} + \frac{4 \pi^2 n^2 \alpha^2 a^2}{3} + K_F \right]} \quad (33)$$

in which the non-dimensional foundation modulus

$$K = \frac{K a^4}{D_x} \quad (34)$$

For $K_F = 0$ and $K_F = 100$ graphs are plotted in Fig. 2 showing the deflection $\frac{w}{h}$ at the centroid of the plate against the loads. Fig. 2. also contains a graph plotted according to the linear theory.

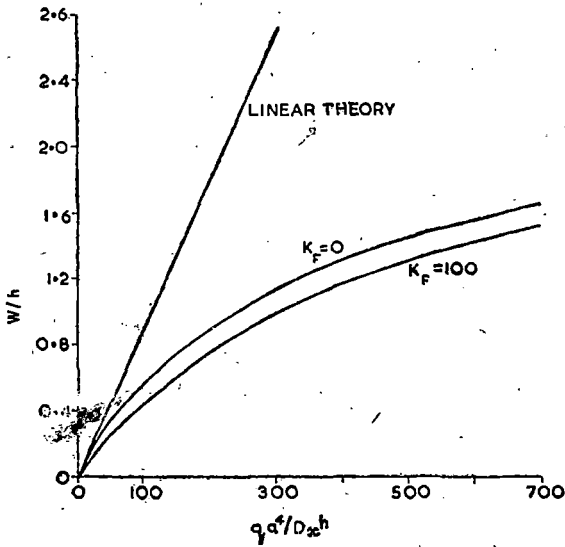


Fig. 2- Deflection curve.

CONCLUSION

From Fig. 2 it is clear that design calculations should be made according to the non-linear theory, because deflections calculated according to small deflection theory will be far from the actual values for higher values of load function. The effect of the foundation is to reduce the deflection for a given value of load function.

Because the deflection, w , has been determined, bending moments and stresses can be computed easily. The bending moments M_x and M_y at the centroid of the plate are obtained as

$$M_x = 4(1 + \nu_c)qa^2 \sum_{n=1}^{\infty} \frac{n \sin \frac{2n\pi}{3}}{\left[\frac{16\pi^4 n^4}{9} + \frac{4\pi^2 n^2 \alpha^2 a^2}{3} + K \right]} \tag{35}$$

$$M_y = K_1 M_x \tag{36}$$

ν_c is the Poisson's ratio for concrete.

For isotropic plate without elastic foundation and undergoing small deflection $\nu_c = \nu$, $K_1 = 1$, $K_F = 0$, $\alpha \rightarrow 0$ and for a plate having each side of length $\frac{2a}{\sqrt{3}}$, (35) and (36) lead to

$$M_x = M_y = (1 + \nu) \frac{qa^2}{54} \tag{37}$$

which is the same result obtained by Timoshenko⁷.

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Buckling of a non-homogeneous rectangular plate on elastic foundation

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Critical buckling conditions of a non-homogeneous simply supported rectangular plate under the action of combined bending and compression and placed on an elastic foundation are investigated with the help of the error function. Results obtained are presented in the form of graphs.

Critical buckling conditions of homogeneous thin rectangular plates subjected to combined bending and compression were investigated by Timoshenko and Gere¹, Johnson and Noel², and many others. The object of this paper is to use error function to obtain the approximate solutions in the case of buckling of a non-homogeneous thin rectangular plate under the action of combined bending and compression in the middle plane of the plate. The plate is placed on an elastic foundation and is simply supported. Bradley³ used finite difference approximations to the governing differential equations to investigate stability of equilateral triangular plates. There are other numerical methods for the solutions of these types of buckling problems. But these methods are time consuming.

Since the governing differential equation obtained in this paper cannot be exactly solved, approximate solutions have been sought with the help of error function and by applying Galerkin's principle. It is observed that the results obtained from this method for the homogeneous plate not resting on foundation are in good agreement with the known results obtained by strain energy method. Flexural rigidity of the

plate is assumed to vary exponentially and the foundation is taken of the Winkler⁴ type. Results obtained have been presented in the form of graphs.

ANALYSIS

Consider a simply supported rectangular plate of varying flexural rigidity, and along whose sides $x = 0$ and $x = a$ (Fig. 1) distributed forces, acting in the middle plane of the plate are applied. The intensity of the forces are given by the equation

$$N_x = N_0 \left(1 - \alpha \frac{y}{b} \right) \quad (1)$$

where N_0 is the intensity of compressive force at edge $y = 0$ and α is a numerical factor. The plate is placed on an elastic foundation having the reaction, K_1 per unit area per unit deflection and is subjected to a uniform transverse load, q .

The governing differential equation of equilibrium of an element of the plate not resting on foundation is (Timoshenko and Krieger⁵)

$$\frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = - \left(+qN_x \frac{\partial^2 w}{\partial x^2} \right) \quad (2)$$

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$$\left. \begin{aligned} \text{where } M_x &= -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \\ M_y &= -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \\ M_{xy} &= D(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \end{aligned} \right\} \quad (3)$$

D being the flexural rigidity, ν the Poisson's ratio, and w the deflection in z -direction. Substituting Eq. (3) in Eq. (2) and observing that the flexural rigidity is a function of the co-ordinates x and y , one gets the differential equation of equilibrium for a plate resting on elastic foundation

$$\nabla^2 (D \nabla^2 w) - (1-\nu) \left\{ \frac{\partial^2 D}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 D}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 D}{\partial y^2} \frac{\partial^2 w}{\partial x^2} \right\} + K_1 w = q + N_x \frac{\partial^2 w}{\partial x^2} \quad (4)$$

For simply supported edges the deflection can be represented by the double series

$$W = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (5)$$

As the flexural rigidity is variable, let

$$D = D_0 e^{-2\alpha_1 x/a} \quad (6)$$

where D_0 and α_1 are constants.

Eq. (5) is an approximate solution of Eq. (4) and therefore substitution of Eq. (5) into Eq. (4) results in the following error function, $E(x,y)$

$$\begin{aligned} E(x,y) &= C_{mn} D_0 \left[\left\{ \left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right\}^2 \right. \\ &\quad \left. - \frac{4\alpha_1^2}{a^2} \left\{ \left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right\} \right] e^{-2\alpha_1 x/a} \\ &\quad \times \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} + C_{mn} K_1 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} - q \\ &\quad - C_{mn} N_0 \left(\frac{m\pi}{a} \right)^2 \left(1 - \frac{\alpha}{b} y \right) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \end{aligned} \quad (7)$$

According to Galerkin's principle, the following condition is imposed on the error function, $E(x,y)$

$$\int_0^a \int_0^b E(x,y) W(x,y) dx dy = 0 \quad (8)$$

Substituting Eq. (5) into Eq. (8) and observing that

$$\int_0^b y \sin \frac{i\pi y}{b} \sin \frac{j\pi y}{b} dy = \frac{b^2}{4} \text{ for } i = j = 0 \text{ for}$$

$$i \neq j \text{ and } i \pm j \text{ an even number}$$

$$= -\frac{4b^2}{\pi^2} \frac{ij}{(i^2 - j^2)^2} \text{ for } i \neq j$$

$$\text{and } i \pm j \text{ an odd number}$$

one gets the following

$$\begin{aligned} &\frac{\pi^6}{2\alpha_1} (1 - e^{-2\alpha_1}) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \left[\pi^2 \left(m^2 + \frac{n^2 a^2}{b^2} \right)^2 \right. \\ &\quad \left. - 4\alpha_1^2 \left(m^2 + \nu \frac{n^2 a^2}{b^2} \right) \right] \frac{m^2}{(\pi^2 m^2 + \alpha_1^2)} \\ &\quad + \pi^2 K_F \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} - \frac{16qa^4}{D_0} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} \\ &\quad - \frac{N_0}{D_0} \pi^4 a^2 \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 C_{mn} - \frac{\alpha}{2} \sum_{m=1}^{\infty} m^2 \left\{ \sum_{n=1}^{\infty} C_{mn} \right. \right. \\ &\quad \left. \left. - \frac{16}{\pi^2} \sum_{n=1}^{\infty} \sum_i C_{mi} \frac{ni}{(n^2 - i^2)^2} \right\} \right] = 0 \end{aligned} \quad (9)$$

Where the nondimensional foundation modulus,

$$K_F = \frac{K_1 a^4}{D_0} \text{ and } n \pm i \text{ is always odd.}$$

Taking $n = i$, the deflection, W is obtained from Eq. (9)

$$W = \frac{16qa^4}{D_0} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \times \frac{1}{\lambda_{mn}}$$

where,

$$\begin{aligned} \lambda_{mn} &= \left[\frac{\pi^6 (1 - e^{-2\alpha_1})}{2\alpha_1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\pi^2 \left(m^2 + \frac{n^2 a^2}{b^2} \right)^2 \right. \right. \\ &\quad \left. \left. - 4\alpha_1^2 \left(m^2 + \nu \frac{n^2 a^2}{b^2} \right) \right] \frac{m^2}{(\pi^2 m^2 + \alpha_1^2)} + \pi^2 K_F \right. \\ &\quad \left. - \frac{N_0 \pi^4 a^2}{D_0} \left\{ \sum_{m=1}^{\infty} m^2 \left(1 - \frac{\alpha}{2} \right) \right\} \right] \end{aligned} \quad (10)$$

From Eq. (9) the critical buckling condition is obtained when

$$\begin{aligned} & \frac{N_0 a^2}{D_0} \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 C_{mn} - \frac{\alpha}{2} \sum_{m=1}^{\infty} m^2 \left\{ \sum_{n=1}^{\infty} C_{mn} \right. \right. \\ & \left. \left. - \frac{16}{\pi^2} \sum_{n=1}^{\infty} \sum_i C_{ni} \frac{ni}{(n^2-i^2)^2} \right\} \right] \\ & = \frac{\pi^2(1-e^{-2\alpha_1})}{2\alpha_1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \left[\pi^2 \left(m^2 + \frac{n^2 a^2}{b^2} \right)^2 \right. \\ & \left. - 4\alpha_1^2 \left(m^2 + \nu \frac{n^2 a^2}{b^2} \right) \right] \frac{m^2}{(\pi^2 m^2 + \alpha_1^2)} \\ & + \frac{1}{\pi^2} K_F \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \end{aligned} \quad (11)$$

The plate may buckle in such a way that there can be several half-waves in the direction of compression but only one half-wave in the perpendicular direction. For one half wave buckling $m = 1$, for two half-waves buckling $m = 2$ and so on.

If the plate buckles in one half-wave, one gets from Eq. (11) by taking $m=1$ a system of equations of the following kind

$$\begin{aligned} & C_{1n} \left[\frac{(1-e^{-2\alpha_1})}{2\alpha_1(\pi^2 + \alpha_1^2)} \left\{ \pi^2 \left(1 + \frac{n^2 a^2}{b^2} \right)^2 - 4\alpha_1^2 \right. \right. \\ & \left. \left. \times \left(1 + \nu \frac{n^2 a^2}{b^2} \right) \right\} + \frac{K_F}{\pi^4} - \frac{\sigma_{cr} a^2 h}{\pi^2 D_0} \left(1 - \frac{\alpha}{2} \right) \right] \\ & - 8 \alpha \sigma_{cr} \frac{a^2 h}{\pi^4 D_0} \sum_i C_{1i} \frac{ni}{(n^2-i^2)^2} = 0 \end{aligned} \quad (12)$$

where $\sigma_{cr} = (N_0)_{cr}/h$, h being the plate thickness.

The lowest root of the determinantal equation thus formed will yield the critical buckling load. From the first approximate lowest root, one gets by taking $n = 1$

$$\sigma_{cr} = K \frac{\pi^2 D_0}{b^2 h} \quad (13)$$

where

$$\begin{aligned} K & = \frac{1}{1-\alpha/2} \left[\frac{1-e^{-2\alpha_1}}{2\alpha_1(\pi^2 + \alpha_1^2)} \left\{ \pi^2 \left(\frac{b}{a} + \frac{a}{b} \right)^2 \right. \right. \\ & \left. \left. - 4\alpha_1^2 \left(\frac{b^2}{a^2} + \nu \right) \right\} + \frac{K_F}{\pi^4} \frac{b^2}{a^2} \right] \end{aligned}$$

Thus the buckling load is a function of a/b and the foundation modulus, K_F .

For $\alpha = 0$, the critical buckling load, σ_{cr} is obtained from Eq. (11) by taking $n = 1$

$$\sigma_{cr} = K \frac{\pi^2 D_0}{b^2 h} \quad (14)$$

where

$$\begin{aligned} K & = \frac{1-e^{-2\alpha_1}}{2\alpha_1(\pi^2 m^2 + \alpha_1^2)} \left\{ \pi^2 \left(m^2 \frac{b}{a} + \frac{a}{b} \right)^2 \right. \\ & \left. - 4\alpha_1^2 \left(m^2 \frac{b^2}{a^2} + \nu \right) \right\} + \frac{K_F}{\pi^4 m^2} \frac{b^2}{a^2} \\ & (m = 1, 2, 3, \dots) \end{aligned} \quad (15)$$

For homogeneous material, $D_0 \rightarrow D$ when $\alpha_1 \rightarrow 0$. Setting $\alpha_1 \rightarrow 0$ in Eq. (13) one gets the critical buckling load for a homogeneous plate on elastic foundation for one half-wave buckling

$$\sigma_{cr} = K \frac{\pi^2 D}{b^2 h} \quad (16)$$

$$\text{where } K = \frac{1}{1-\alpha/2} \left[\left(\frac{b}{a} + \frac{a}{b} \right)^2 + \frac{K_F}{\pi^4} \frac{b^2}{a^2} \right]$$

For $K_F=0$, Eq. (16) is the result obtained by Timoshenko and Gere¹. For m half-waves buckling, K in Eq. (16) can be expressed

$$\text{as } K = \frac{1}{m^2} \left(m^2 \frac{b}{a} + \frac{a}{b} \right)^2 + \frac{K_F}{\pi^4} \frac{1}{m^2} \frac{b^2}{a^2} \quad (17)$$

The ratio a/b for which σ_{cr} becomes a minimum for uniform compression is obtained from Eq. (15) and denoting this ratio by $(a/b)_{cr}$, one gets for a homogeneous plate

$$\left(\frac{a}{b} \right)_{cr} = \frac{1}{\pi} (K_F + \pi^4 m^4)^{1/4} \quad (18)$$

and for a nonhomogeneous plate

$$\left(\frac{a}{b} \right)_{cr} = \left[m^4 + \frac{K_F}{\pi^6 m^2} \frac{2\alpha_1(\pi^2 m^2 + \alpha_1^2)}{(1-e^{-2\alpha_1})} - \frac{4\alpha_1^2 m^2}{\pi^2} \right]^{1/4} \quad (19)$$

The ratio a/b at which the transition from m to $m+1$ half-waves buckling occur can also be computed from Eq. (15). For homogeneous plate under uniform compression, transition from one to two half-waves occurs when

$$\frac{a}{b} = \left(4 - \frac{K_F}{\pi^4} \right)^{1/4}$$

and transition from two to three half-waves occurs when

$$\frac{a}{b} = \left(36 - \frac{K_F}{\pi^4}\right)^{1/4}$$

Eq. (13) gives satisfactory results for small values of α . An improved result is obtained by taking two equations of the system Eq. (12) with coefficients C_{11} and C_{12} and setting the determinant equal to zero. Thus for one half-wave buckling

$$\sigma_{cr} = K \frac{\pi^2 D_0}{b^2 h} \quad (20)$$

where

$$K = \left[\frac{1}{\{0.065 \alpha^2 - 2(1-\alpha/2)^2\}} \times \left\{ 0.1 \left(1 - \frac{\alpha}{2}\right)^2 \right. \right. \\ \times \left(B + \frac{2K_F}{\pi^4} \right)^2 + \left. \left. \left[0.013 \alpha^2 - 0.041 \left(1 - \frac{\alpha}{2}\right)^2 \right] \right. \right. \\ \times \left. \left. \left[A + B \frac{K_F}{\pi^4} + \frac{K_F^2}{\pi^8} \right] \right\}^{1/2} \right. \\ \left. - \frac{(1-\alpha/2)(B + 2K_F/\pi^4)}{0.064 \alpha^2 - 2(1-\alpha/2)^2} \right] \times \frac{b^2}{a^2}$$

and

$$A = \left\{ \frac{1 - e^{-2\alpha_1}}{2\alpha_1(\pi^2 + \alpha_1^2)} \right\}^2 \left\{ \pi^2 \left(1 + \frac{a^2}{b^2}\right)^2 \right. \\ \left. - 4\alpha_1^2 \left(1 + \nu \frac{a^2}{b^2}\right) \right\} \left\{ \pi^2 \left(1 + \frac{4a^2}{b^2}\right)^2 \right. \\ \left. - 4\alpha_1^2 \left(1 + 4\nu \frac{a^2}{b^2}\right) \right\} \\ B = \frac{1 - e^{-2\alpha_1}}{2\alpha_1(\pi^2 + \alpha_1^2)} \left\{ \pi^2 \left(2 + 10 \frac{a^2}{b^2} + 17 \frac{a^4}{b^4}\right) \right. \\ \left. - 4\alpha_1^2 \left(2 + 5\nu \frac{a^2}{b^2}\right) \right\}$$

For pure bending when $\alpha = 2$ Eq. (20) reduces to

$$\sigma_{cr} = \frac{\pi^2 D_0}{b^2 h} \left[2.77 \frac{b^2}{a^2} \left(A + B \frac{K_F}{\pi^4} + \frac{K_F^2}{\pi^8} \right)^{1/2} \right] \quad (21)$$

Setting $\alpha_1 \rightarrow 0$ in Eq. 21 one gets the buckling load under pure bending for a homogeneous plate for one half-wave buckling

$$\sigma_{cr} = K \frac{\pi^2 D}{b^2 h} \quad (22)$$

where

$$K = 2.77 \frac{b^2}{a^2} \left[\left(1 + \frac{a^2}{b^2}\right)^2 \left(1 + \frac{4a^2}{b^2}\right)^2 \right. \\ \left. + \left(2 + 10 \frac{a^2}{b^2} + 17 \frac{a^4}{b^4}\right) \frac{K_F}{\pi^4} + \frac{K_F^2}{\pi^8} \right]^{1/2}$$

For m half-wave buckling K in Eq. (22) can be written as

$$K = \frac{1}{m^2} \times 2.77 \frac{b^2}{a^2} \left[\left(m^2 + \frac{a^2}{b^2}\right)^2 \left(m^2 + \frac{4a^2}{b^2}\right)^2 \right. \\ \left. + \left(2m^4 + 10m^2 \frac{a^2}{b^2} + 17 \frac{a^4}{b^4}\right) \frac{K_F}{\pi^4} + \frac{K_F^2}{\pi^8} \right]^{1/2} \quad (23)$$

The presence of the foundation modulus, K_F in Eq. (10) reduces the deflection, w and hence the bending stress. Therefore Eqs. (20), (21) and (22) resulting from two term approximation of Eq. (12) may be taken as fairly accurate.

RESULTS

Numerical results are obtained for two cases (a) when the plate is under uniform compression, and (b) when it is under pure bending. For uniform compression of a homogeneous plate, the values of K are calculated for different values of a/b with the help of Eq. (16) for one half-wave buckling taking $K_F = \pi^4$, and $\alpha = 0$. For two half-waves and three half-waves buckling Eq. (17) is used with the same value of K_F . These results are presented in the form of graphs in Fig. (2). For a non homogeneous plate under uniform compression, the values of K are calculated for one half-wave, two, and three half-waves buckling by taking the same value of K_F and $\alpha_1 = 0.1$ in Eq. (15). These results are presented in Fig. 3. The values of K when the plates are not on the foundation are also presented in Figs. 2 and 3 for comparison.

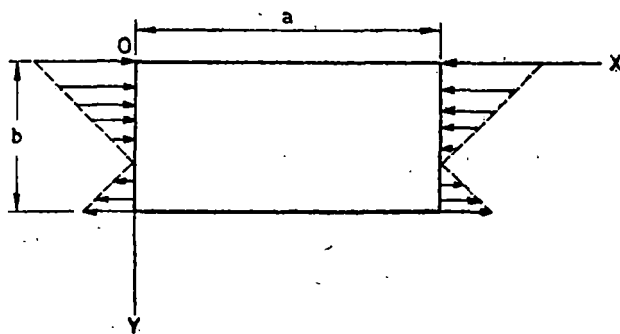


FIG. 1 RECTANGULAR PLATE

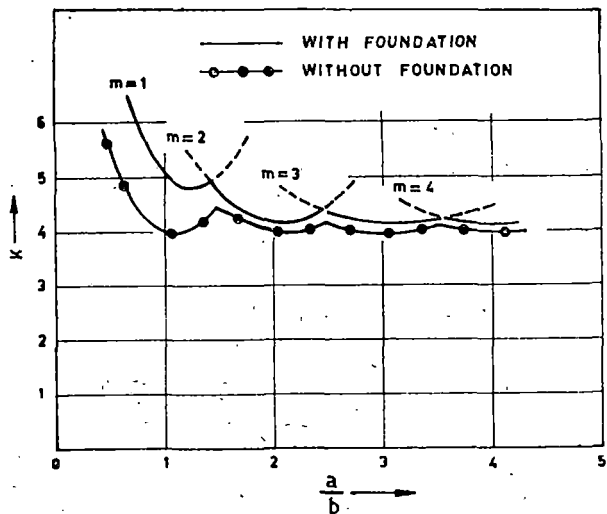


FIG. 2 HOMOGENEOUS PLATE UNDER UNIFORM COMPRESSION

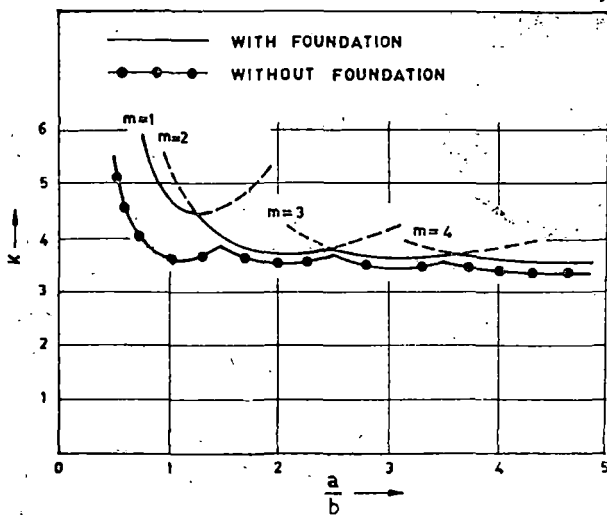


FIG. 3 NON-HOMOGENEOUS PLATE UNDER UNIFORM COMPRESSION

For pure bending of a homogeneous plate Eq. (23) is used for calculation of K for different values of a/b taking $K_F = \pi^4$ and the results are presented in Fig. 4. In the same figure the corresponding results for K without foundation are also presented for comparison.

CONCLUSIONS

From the foregoing analysis and from Figs. 2, 3 and 4 the following conclusions may be drawn:

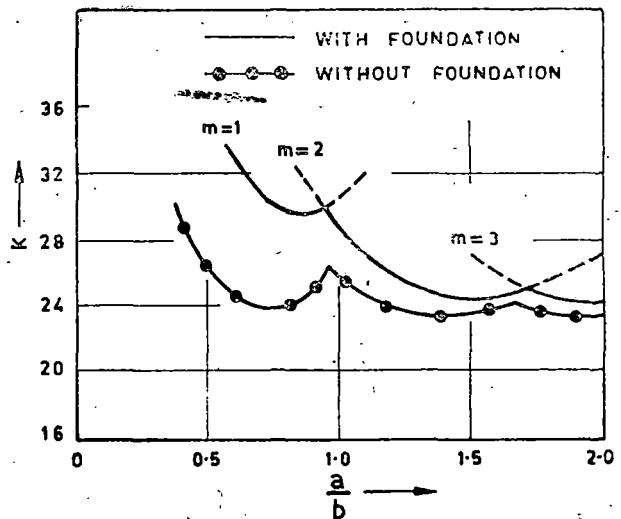


FIG. 4 HOMOGENEOUS PLATE UNDER PURE BENDING

- (i) Foundation increases the buckling load
- (ii) Resistance offered by the foundation is more for one half-wave buckling compared to multiple half-waves buckling. When buckling is in more than one half-wave, the foundation resistance remains practically constant.
- (iii) Foundation increases the $(a/b)_{cr}$ ratio and reduces the a/b ratio at which transition takes place as compared to a plate not resting on foundation.
- (iv) A non-homogeneous plate will have lower buckling load as compared to a homogeneous plate.

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NOTATION

- a, b Length and breadth of the plate respectively
- D Flexural rigidity of the plate = $\frac{Eh^3}{12(1-\nu^2)}$
- E Modulus of elasticity
- h Plate thickness
- K A numerical factor
- q Lateral distributed load
- W Deflection in z-direction

x, y Cartesian co-ordinates
 D_0 Flexural rigidity of the plate at the edge $x=0$
 K_1 Foundation reaction per unit area per unit deflection
 K_F Non-dimensional foundation modulus $= \frac{K_1}{D_0} a^4$
 N_x Normal force resultants per unit length in middle plane of the plate
 N_0 Intensity of compressive force at the edge $y=0$
 α, α_1 Numerical constants
 $(N_0)_{cr}$ Critical buckling load
 σ_{cr} Critical buckling stress
 ν Poisson's ratio

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Large Deflection of a Circular Plate on Elastic Foundation under Symmetrical Load

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ABSTRACT

The large deflection of a clamped circular plate on elastic foundation under nonuniform but symmetrical loads has been investigated following Berger's approximate method. The deflections are obtained in the form of an infinite series involving Bessel functions. Graphs are plotted for deflections, bending moments, and bending stresses for various values of foundation modulus and load functions.

INTRODUCTION

Timoshenko and Woinowsky-Krieger [1] and several other authors have examined small deflections of thin plates on elastic foundations on the assumption that the strains of the middle plane of the plate can be neglected. When the deflection is moderately large, that is, on the order of the thickness

of the plate, then the strain of the middle plane of the plate must be considered. In that case the analytical solution of the differential equations becomes difficult because of their nonlinear character. Way [2] and many other authors have examined moderately large deflections of plates not resting on elastic foundations, and the methods used by them involve considerable computation.

Berger [3] has suggested that the strain energy due to the second strain invariant of the middle surface strains may be neglected in analyzing moderately large deflection of plates having axisymmetric deformation. Berger's technique reduces the computational effort considerably, yet the stresses and deflections obtained for both rectangular and circular plates are in good agreement with those found by exact analysis. Berger's method has been extended by Nowinski [4] to the case of orthotropic plates. Nash and Modeer [5] have investigated problems without axial symmetry by using Berger's technique. The same approximate method has also been applied by Sinha [6] to determine the moderately large static deflections of circular and rectangular plates resting on elastic foundations and under uniform load distribution.

In this paper we study moderately large static deflections of circular plates on elastic foundation and subjected to special classes of symmetrical transverse loads, which are distributed over a concentric circular portion of the plate. Deflections, bending moments, and bending stresses are calculated for different values of foundation modulus, and these are presented in the form of graphs.

FORMULATION OF THE PROBLEM

For moderately large deflections the total potential energy of the system is given by

$$V = \frac{D}{2} \iint \left[(\nabla^2 W)^2 + \frac{12}{h^2} e_1^2 - 2(1 - \nu) \left(W_{xx} W_{yy} - W_{xy}^2 + \frac{12}{h^2} e_2 \right) + \frac{K}{D} W^2 - \frac{2qW}{D} \right] dx dy \quad (1)$$

in which the last two terms in the integrand represent the potential energy of the foundation and of the applied load, respectively, and e_1 and e_2 are the first and second invariants of the membrane strains. If, following Berger [3], e_2 is neglected, then the variation of V with respect to the in-plane displacements leads to the drastic simplification that e_1 is constant.

In polar coordinates, and under the assumption of circular symmetry, the governing equations become

$$e_1 \equiv \frac{du}{dr} + \frac{u}{r} + \frac{1}{2} \left(\frac{dW}{dr} \right)^2 = \frac{\alpha^2 h^2}{12} = \text{const} \quad (2)$$

$$\nabla^4 W - \alpha^2 \nabla^2 W + \frac{K}{D} W = \frac{q}{D} \quad (3)$$

in which

$$\nabla^2 \equiv \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \quad (4)$$

To these equations must be added a suitable set of boundary conditions. As has been observed in a recent note by Nowinsky and Ohnabe [10], the present simplified method leads to acceptable results if these boundary conditions involve fixity against in-plane displacements. This assumption has been adopted in the following examples.

SOLUTION OF PROBLEM

Let us consider a clamped circular plate of radius a , with the center of the plate taken as the origin. Let there be a symmetrical distribution of transverse load varying as $(b^2 - r^2)^\lambda$, ($\lambda > -1$), over a concentric circular area of radius $b < a$. Hence

$$\frac{q}{D} = \begin{cases} f(r) = C(b^2 - r^2)^\lambda & (r < b < a) \\ 0 & (b < r < a) \end{cases} \quad (5)$$

and Eq. (3) now becomes

$$(\nabla^2 - \alpha^2) \nabla^2 W + \frac{K}{D} W = f(r) \quad (6)$$

The boundary conditions for clamped edges are

$$(W)_{r=a} = 0 = \left(\frac{dW}{dr} \right)_{r=a} \quad (7)$$

Let us now assume the deflection W in the form

$$W = \sum_{s=1}^{\infty} A_s [J_0(P_s r) - J_0(P_s a)] \quad (8)$$

where J_0 is the Bessel function of the first kind and zero order and P_s is the S th root of $J_1(Pa) = 0$, J_1 being the Bessel function of the first kind and first order.

This automatically satisfies the boundary conditions for clamped edges. Since

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right] J_0(P_s r) = -P_s^2 J_0(P_s r)$$

substitution of Eq. (8) in Eq. (6) leads to

$$\sum_{s=1}^{\infty} A_s \left[P_s^4 J_0(P_s r) + \alpha^2 P_s^2 J_0(P_s r) + \frac{K}{D} \{J_0(P_s r) - J_0(P_s a)\} \right] = f(r) \quad (9)$$

or, by expanding $f(r)$ in a series of Bessel functions,

$$A_s \frac{a^2}{2} \left[P_s^2 (P_s^2 + \alpha^2) + \frac{K}{D} \right] J_0^2(P_s a) = \int_0^a f(r) J_0(P_s r) r dr \quad (10)$$

Setting $r = b \sin \theta$ and $f(r) = C(b^2 - r^2)^\lambda$ in the integral of (10) one obtains

$$\begin{aligned} \int_0^a f(r) J_0(P_s r) r dr &= \int_0^b C r (b^2 - r^2)^\lambda J_0(P_s r) dr \\ &= C b^{2(\lambda+1)} \int_0^{\pi/2} \sin \theta \cos^{2\lambda+1} \theta J_0(P_s b \sin \theta) d\theta \\ &= \frac{C b^{2(\lambda+1)} J_{\lambda+1}(P_s b) 2^\lambda (\lambda+1)}{(P_s b)^{\lambda+1}} \end{aligned} \quad (11)$$

This is a special form of Sonine's first definite integral containing Bessel function [7], where $\lambda > -1$. Finally, with the value obtained from Eq. (11) used in Eq. (10) one gets, after simplification,

$$A_s = \frac{C(2b)^{\lambda+1} J_{\lambda+1}(P_s b) \Gamma(\lambda+1)}{a^2 \left[P_s^{\lambda+3} (P_s^2 + \alpha^2) + P_s^{\lambda+1} \frac{K}{D} \right] J_0^2(P_s a)} \quad (12)$$

and hence

$$W = \frac{C(2b)^{\lambda+1}\Gamma(\lambda + 1)}{a^2} \sum_{s=1}^{\infty} \frac{J_{\lambda+1}(P_s b)[J_0(P_s r) - J_0(P_s a)]}{\left[P_s^{\lambda+3}(P_s^2 + \alpha^2) + P_s^{\lambda+1} \frac{K}{D} \right] J_0^2(P_s a)} \quad (13)$$

Except for the as yet unknown value of α this determines the deflection curve $W(r)$, including the maximum deflection

$$W_{\max} = W(0) = \frac{C(2b)^{\lambda+1}\Gamma(\lambda + 1)}{a^2} \times \sum_{s=1}^{\infty} \frac{J_{\lambda+1}(P_s b)[1 - J_0(P_s a)]}{\left[P_s^{\lambda+3}(P_s^2 + \alpha^2) + P_s^{\lambda+1} \frac{K}{D} \right] J_0^2(P_s a)} \quad (14)$$

To determine the displacement u we obtain, from Eqs. (2) and (8),

$$\begin{aligned} \frac{du}{dr} + \frac{u}{r} &= \frac{\alpha^2 h^2}{12} - \frac{1}{2} \left(\frac{dw}{dr} \right)^2 \\ &= \frac{\alpha^2 h^2}{12} - \frac{1}{2} \sum_{s=1}^{\infty} A_s^2 P_s^2 J_1^2(P_s r) \\ &\quad - \frac{1}{2} \sum_{s=1}^{\infty} \sum_{\substack{m=1, \\ s \neq m}}^{\infty} A_s A_m P_s P_m J_1(P_s r) J_1(P_m r) \end{aligned} \quad (15)$$

and, by multiplying Eq. (15) by r and integrating with respect to r ,

$$\begin{aligned} ru &= \frac{\alpha^2 h^2 r^2}{24} - \frac{1}{2} \sum_{s=1}^{\infty} A_s^2 P_s^2 \left[\frac{r^2}{2} \left\{ \left(1 - \frac{1}{P_s^2 r^2} \right) J_1'^2(P_s r) + J_1'^2(P_s r) \right\} \right] \\ &\quad - \frac{1}{2} \sum_{s=1}^{\infty} \sum_{\substack{m=1 \\ s \neq m}}^{\infty} A_s A_m P_s P_m \\ &\quad \times \left[r \left\{ \frac{P_s J_2(P_s r) J_1(P_m r) - P_m J_1(P_s r) J_2(P_m r)}{P_s^2 - P_m^2} \right\} \right] + K' \end{aligned} \quad (16)$$

where K' is the constant of integration, whose value is determined from the boundary condition

$$(u)_{r=a} = 0 \quad (17)$$

Thus

$$K' = \frac{1}{4} \sum_{s=1}^{\infty} A_s^2 P_s^2 a^2 J_0^2(P_s a) - \frac{\alpha^2 h^2 a^2}{24} = 0 \quad (18)$$

where the second equality follows from the regularity condition

$$(u)_{r=0} = 0 \quad (19)$$

Hence

$$\frac{\alpha^2 h^2}{6} = \sum_{s=1}^{\infty} A_s^2 P_s^2 J_0^2(P_s a) \quad (20)$$

determines the value of α . For example, let

$$\lambda = \frac{1}{2}, \quad K = 0 \quad (21)$$

then the deflection W is given by

$$\begin{aligned} W &= \sum_{s=1}^{\infty} A_s [J_0(P_s r) - J_0(P_s a)] \\ &= \frac{2b^3 C}{a^2} \sum_{s=1}^{\infty} \frac{Q(P_s b)}{P_s^2 (P_s^2 + \alpha^2) J_0^2(P_s a)} [J_0(P_s r) - J_0(P_s a)] \end{aligned} \quad (22)$$

in which

$$Q(P_s b) = \frac{1}{3} \left[1 - \frac{P_s^2 b^2}{2 \times 5} + \frac{P_s^4 b^4}{2 \times 4 \times 5 \times 7} - \dots \right]$$

As is common in Berger's approximation the large deflection effect is contained entirely in the value of α . With $\alpha = 0$, Eq. (22) agrees with the result obtained by Sen [8] for the corresponding small deflection problem.

Another type of transverse load function to be considered is given by

$$f(r) = \begin{cases} C(r^4 - b^4) & (0 \leq r \leq b < a) \\ 0 & (b \leq r \leq a) \end{cases} \quad (23)$$

Expanding $f(r)$ in a series of Bessel functions and proceeding in the same

manner we find

$$A_s = \frac{32bC(4 - P_s^2b^2)J_1(P_sb)}{a^2 \left[P_s^7(P_s^2 + \alpha^2) + P_s^5 \frac{K}{D} \right] J_0^2(P_sa)} \quad (24)$$

or

$$W = \frac{32bC}{a^2} \sum_{s=1}^{\infty} \frac{(4 - P_s^2b^2)J_1(P_sb)[J_0(P_sr) - J_0(P_sa)]}{\left[P_s^7(P_s^2 + \alpha^2) + P_s^5 \frac{K}{D} \right] J_0^2(P_sa)} \quad (25)$$

The central deflection is obtained putting $r = 0$, that is,

$$W_{\max} = \frac{32bC}{a^2} \sum_{s=1}^{\infty} \frac{(4 - P_s^2b^2)J_1(P_sb)[1 - J_0(P_sa)]}{\left[P_s^7(P_s^2 + \alpha^2) + P_s^5 \frac{K}{D} \right] J_0^2(P_sa)} \quad (26)$$

Once again, $u(r)$ and α are found by substituting Eq. (24) in Eqs. (16), (18), and (20).

With W a function of r only, the radial bending moment is

$$M_r = -D \left[\frac{d^2W}{dr^2} + \nu \left(\frac{1}{r} \frac{dW}{dr} \right) \right] \quad (27)$$

Considering Eqs. (13) and (27) the value for the bending moment for the type of loading in Eq. (5) then becomes

$$M_r = \frac{DC(2b)^{\lambda+1} \Gamma(\lambda+1)}{a^2} \sum_{s=1}^{\infty} \frac{P_s J_{\lambda+1}(P_sb)}{\left[P_s^{\lambda+3}(P_s^2 + \alpha^2) + P_s^{\lambda+1} \frac{K}{D} \right] J_0^2(P_sa)} \\ \times \left[P_s J_0(P_sr) + (\nu - 1) \frac{1}{r} J_1(P_sr) \right] \quad (28)$$

For clamped edges the bending moment is a maximum at the center, that is,

$$(M_r)_{\max} = \frac{DC(2b)^{\lambda+1} \Gamma(\lambda+1)}{a^2} \times \frac{1 + \nu}{2} \\ \times \sum_{s=1}^{\infty} \frac{P_s^2 J_{\lambda+1}(P_sb)}{\left[P_s^{\lambda+3}(P_s^2 + \alpha^2) + P_s^{\lambda+1} \frac{K}{D} \right] J_0^2(P_sa)} \quad (29)$$

The maximum bending stress is given by

$$(\sigma_r)_{\max} = -\frac{6}{h^2}(M_r)_{\max} \quad (30)$$

Hence, in summary, the maximum deflection, bending moment, and bending stress for $\lambda = 1$ are as follows:

$$W_{\max} = \frac{4b^2C}{a^2} \sum_{s=1}^{\infty} \frac{J_2(P_s b)[1 - J_0(P_s a)]}{\left[P_s^4(P_s^2 + \alpha^2) + P_s^2 \frac{K}{D} \right] J_0^2(P_s a)} \quad (31)$$

$$(M_r)_{\max} = \frac{2DCb^2}{a^2} (1 + \nu) \sum_{s=1}^{\infty} \frac{J_2(P_s b)}{\left[P_s^2(P_s^2 + \alpha^2) + \frac{K}{D} \right] J_0^2(P_s a)} \quad (32)$$

$$(\sigma_r)_{\max} = -\frac{12DCb^2}{a^2 h^2} (1 + \nu) \sum_{s=1}^{\infty} \frac{J_2(P_s b)}{\left[P_s^2(P_s^2 + \alpha^2) + \frac{K}{D} \right] J_0^2(P_s a)} \quad (33)$$

For small deflections ($\alpha = 0$) and for $K = 0$:

$$W_{\max} = \frac{4b^2C}{a^2} \sum_{s=1}^{\infty} \frac{J_2(P_s b)[1 - J_0(P_s a)]}{P_s^6 J_0^2(P_s a)} \quad (34)$$

$$(M_r)_{\max} = \frac{2DCb^2}{a^2} (1 + \nu) \sum_{s=1}^{\infty} \frac{J_2(P_s b)}{P_s^4 J_0^2(P_s a)} \quad (35)$$

$$(\sigma_r)_{\max} = -\frac{12DCb^2}{a^2 h^2} (1 + \nu) \sum_{s=1}^{\infty} \frac{J_2(P_s b)}{P_s^4 J_0^2(P_s a)} \quad (36)$$

The results are now used for the numerical computations and evaluation.

NUMERICAL RESULTS

Numerical results are presented here for the case of the circular plate with clamped edge. The type of load function considered is as in Eq. (5), with $\lambda = 1$ and $a = 2b$. The maximum deflections and bending stresses are calculated for various values of the load and for various values of the foundation modulus. These are presented in the form of graphs. Central deflection and

maximum bending stresses are also calculated for small deflections, and these are also presented in the form of graphs for comparison. Variation of the bending moment along the radius is also calculated both for small deflection and large deflection.

In calculating the central deflection we start from Eq. (20), with an assumed value of (αa) leading to a particular value of the load. Once this relationship is obtained the maximum value of the deflection can be obtained from Eq. (31) for various values of the foundation modulus. These results are presented in Fig. 2. An examination of Eq. (31) reveals that as the radius of the plate in-

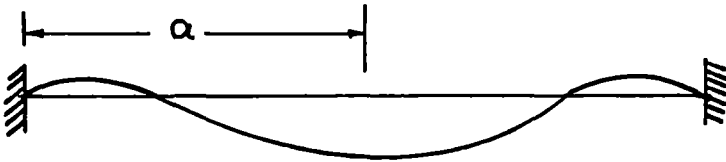


Fig. 1 Deflected plate shape.

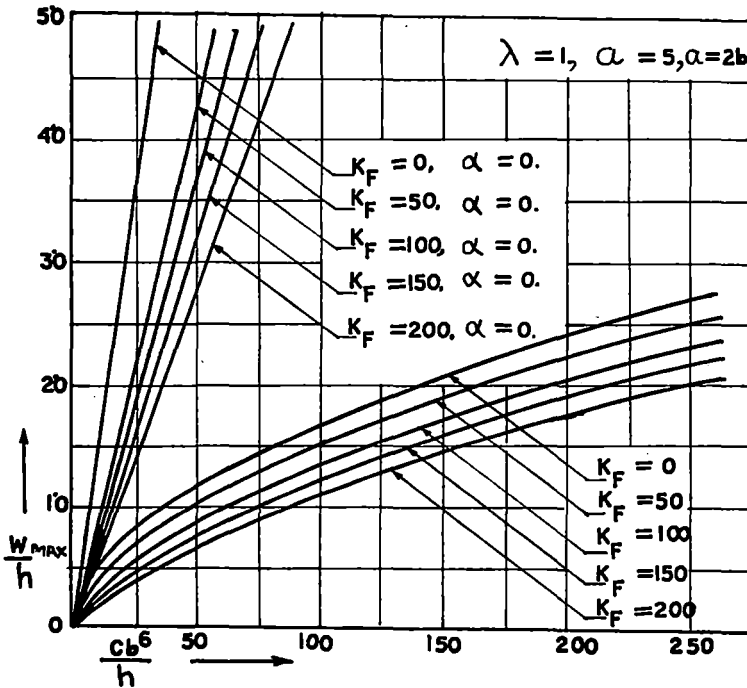


Fig. 2 Deflection.

creases, the central deflection also increases for given load. For small deflection Eq. (34) is to be used for the calculation of the central deflection.

In calculating the bending moment for various values of (r/a) , Eq. (28) is used, with $\lambda = 1$. The variation of the bending moment along the radius of the plate is presented in Fig. 3. Variation of the bending moment along the radius according to the linear theory can be calculated with the help of Eq. (28) by letting $\alpha = 0$ and $\lambda = 1$. The maximum bending stresses both for large and small deflection and for various values of foundation modulus can be calculated with the help of Eqs. (33) and (36). These values are presented in Fig. 4.

For the type of loading in Eq. (23) the central deflection for various values of the load and foundation modulus are calculated with the help of Eq. (26) in conjunction with the corresponding equation for α . Values of the bending

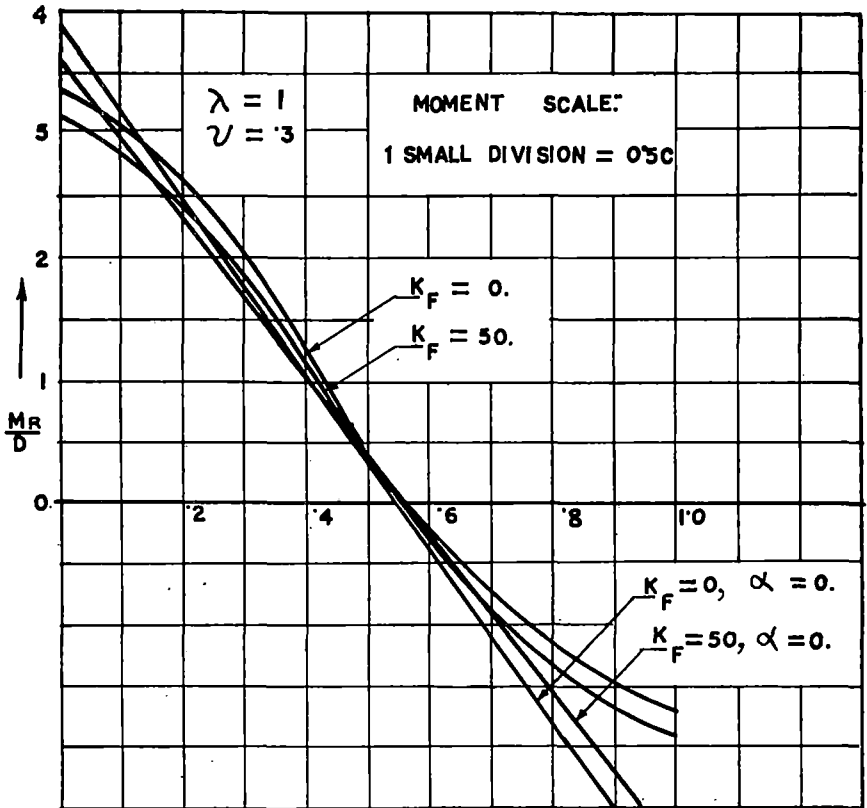


Fig. 3 Bending moment.

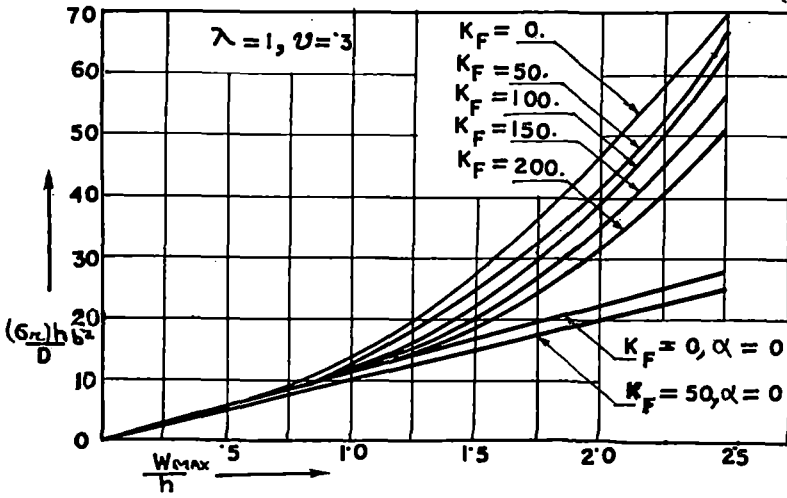


Fig. 4 Stresses.

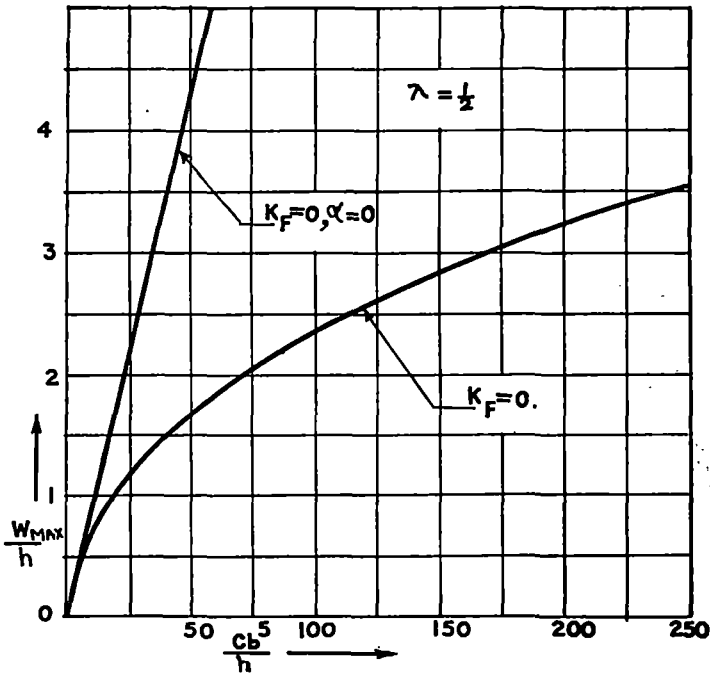


Fig. 5 Deflection.

moment and bending stresses can also be calculated from equations similar to (28), (32), and (33).

The results obtained are in good agreement with those found by other authors (where applicable). This includes the case of $\lambda = 1/2$ (see Eq. (5)) and $K_F = 0$ as shown in Fig. 5, which was previously treated by Banerjee [9].

APPENDIX—NOTATION

The following symbols have been adopted:

- a radius of the plate
- b a constant less than a
- C a constant
- D flexural rigidity of the plate = $\frac{Eh^3}{12(1 - \nu^2)}$
- E Young's modulus
- e_1 first invariant of middle surface strains
- e_2 second invariant of middle surface strains
- h thickness of plate
- J_0 Bessel function of the first kind and zero order
- J_1 Bessel function of the first kind and first order
- J_2 Bessel function of the first kind and second order
- K foundation reaction per unit area per unit deflection
- q load
- r, θ polar coordinates
- u radial displacement
- V strain energy
- W deflection of plate in Z -direction
- x, y rectangular coordinates
- ϵ strain in middle surface
- K_F foundation modulus = $\frac{K}{D} a^4$
- ν Poisson's ratio
- Γ gamma function

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LARGE AMPLITUDE FREE VIBRATIONS OF IRREGULAR PLATES PLACED ON AN ELASTIC FOUNDATION

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Abstract—A unified method for determining the lowest natural frequency of large amplitude free vibrations of thin elastic plates of any shape and placed on elastic foundation is given. The conformal mapping technique is introduced and Galerkin's method is used to calculate approximate values of the lowest natural frequency. Time periods for circular, square and cornered plates placed on elastic foundation have been determined for simply supported and clamped edge boundary conditions. Practical values have also been determined experimentally. The results are presented in the form of graphs and they are compared with other known results.

INTRODUCTION

An approximate method for investigating the large deflection of initially flat isotropic plates has been proposed by Berger [1]. Essentially, this method is based on the neglect of the second invariant of the middle surface strains in the expression corresponding to the total potential energy of the system. An application of this technique to the case of orthotropic plates has been offered by Iwinski and Nowinski [2] and further boundary value problems associated with circular and rectangular plates have been investigated by Nowinski [3]. Sinha [4] applied this method to investigate large deflections of circular and rectangular plates placed on elastic foundation. Nash and Modeer [5] found the large amplitude free vibrations of rectangular and circular plates by applying the technique offered by Berger.

In this paper a unified method for determining the lowest natural frequency of large amplitude free vibrations of thin elastic plates of any shape and placed on an elastic foundation is given. Following Berger's method a simple fourth-order differential equation coupled with a second-order non-linear equation is obtained. If the boundary of the plate is a curve natural to any of the common coordinate systems, the solution of the differential equation can be expressed in terms of known functions. For more unusual boundaries, the natural coordinates must first be determined and after this is done, the solution would inevitably involve some unfamiliar functions. The determination of natural frequencies in this case will then be very complicated. Therefore a common coordinate system and its associated functions is used for the case of plates with complicated boundaries.

In order to satisfy the prescribed boundary conditions, the domain is conformally transformed on to a unit circle. Once the transformation function is known, the problem is reduced to the solution of the transformed differential system. In this paper Galerkin's method is used to solve the transformed equation.

The ratio of time periods for circular, square and cornered plates placed on an elastic foundation have been determined for simply supported and clamped edge boundary conditions. The foundation is assumed to be of the Winkler type. Experimental values are also obtained for circular and square plates under both boundary conditions. The results are presented in the form of graphs and they are compared with other known results.

THEORY

Let us consider the large amplitude free vibrations of a thin elastic plate placed on an elastic foundation having the reaction k' per unit area per unit deflection.

The strain energy V_1 , due to bending and stretching of the middle surface of the deflected plate, may be written in the Cartesian coordinates in the form [6]

$$V_1 = \frac{D}{2} \iint \left[\left\{ (\nabla^2 w)^2 + \frac{12}{h^2} e_1^2 \right\} - 2(1-\nu) \left\{ \frac{12}{h^2} e_2 + \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} \right] dx dy \quad (1)$$

in which D is the flexural rigidity of the plate given by $Eh^3/12(1-\nu^2)$, E being the modulus of elasticity, h the thickness of the plate, ν Poisson's ratio, and w the deflection in the direction normal to the middle plane. Also, e_1 , the first invariant of the middle surface strains, is defined by the relation

$$e_1 = \varepsilon_x + \varepsilon_y = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \quad (2)$$

$$\left. \begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \\ \varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \end{aligned} \right\} \quad (3)$$

e_2 , the second invariant, is defined by

$$e_2 = \varepsilon_x \varepsilon_y - \frac{1}{4} \gamma_{xy}^2 \quad (4)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \quad (5)$$

and u, v are the displacements in the x and y directions, respectively.

By adding the potential energy of the foundation reaction to equation (1) and neglecting e_2 one gets

$$V = \frac{D}{2} \iint \left[\left\{ (\nabla^2 w)^2 + \frac{12}{h^2} e_1^2 \right\} - 2(1-\nu) \left\{ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} + \frac{k'}{D} w^2 \right] dx dy \quad (6)$$

The kinetic energy, T , of the vibrating plate is

$$T = \frac{\rho h}{2} \iint (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) dx dy \quad (7)$$

in which ρ is the density of the plate material and $\dot{u}, \dot{v}, \dot{w}$ are derivations with respect to time.

Neglecting the inertia effects in the plane of the plate and applying Euler's variational equations to equations (6) and (7), one gets the following differential equation for w [5]

$$\nabla^4 w - \alpha^2 F^2(t) \nabla^2 w + \frac{12}{h^2 c_p^2} \frac{\partial^2 w}{\partial t^2} + \frac{k'}{D} w = 0 \quad (8)$$

in which

$$c_p^{-2} = \frac{\rho h^3}{12D}; \quad \frac{\alpha^2 h^2}{12} f(t) = e_1, \quad \text{and} \quad f(t) = F^2(t). \quad (9)$$

Let

$$w = w(x, y)F(t). \quad (10)$$

Combining equations (8) and (10) one finds

$$F(t) \nabla^4 w - \alpha^2 F^3(t) \nabla^2 w + \frac{12}{h^2 c_p^2} \frac{d^2 F}{dt^2} w + \frac{k'}{D} w F(t) = 0. \quad (11)$$

Equation (11) may be written as

$$\left(\frac{\nabla^4 w}{w} + \frac{k'}{D} \right) F(t) - \alpha^2 F^3(t) \frac{\nabla^2 w}{w} + \frac{12}{h^2 c_p^2} \frac{d^2 F}{dt^2} = 0. \quad (12)$$

A solution of equation (12) is possible if

$$\frac{\nabla^4 w}{w} = k^4 \tag{13a}$$

and

$$\frac{\nabla^2 w}{w} = -k^2 \tag{13b}$$

in which k is a constant. From equation (13a)

$$(\nabla^2 - k^2)(\nabla^2 + k^2)w = 0. \tag{14a}$$

From equation (13b)

$$(\nabla^2 + k^2)w = 0. \tag{14b}$$

Therefore a solution of equation (12) can be obtained by satisfying equation (14b). To satisfy the prescribed boundary conditions, let the domain be conformally transformed on to a unit circle. If $z = x + iy$, $\bar{z} = x - iy$, equation (14b) becomes

$$\frac{\partial^2 w}{\partial z \partial \bar{z}} + k^2 w = 0. \tag{15}$$

Let $z = f(\xi)$ be the analytic function which maps the boundary under consideration in the ξ -plane on to a unit circle. Thus equation (15) transforms into complex coordinates as

$$\left[\nabla^2 + k^2 \left(\frac{dz}{d\xi} \right)^2 \right] w(\xi \bar{\xi}) = 0 \tag{16}$$

in which $\xi = re^{i\theta}$, $\bar{\xi} = re^{-i\theta}$.

The solution of equation (16) can be expressed in the form

$$w \approx \sum_{n=1}^{\infty} B_n [1 - (\xi \bar{\xi})]^n \tag{17a}$$

or

$$w \approx \sum_{n=1}^{\infty} B_n [1 - (\xi \bar{\xi})^n]^2 \tag{17b}$$

according to the prescribed boundary conditions. Equation (17a) is an admissible function for the simply supported edge condition in the sense that this satisfies the kinematic boundary condition $w = 0$ at $r = 1$, but does not satisfy the force boundary condition $M_n = 0$. The form of w in equation (17b) satisfies $w = 0 = dw/dr$ at $r = 1$ and can be taken as an admissible function for the clamped edge condition. Substituting equation (17a) or equation (17b) into equation (16) yields the error function, $\varepsilon_{n,\theta}$, which does not vanish, in general, since equation (17a) or equation (17b) is not an exact solution. Galerkin's procedure requires that the error function, $\varepsilon_{n,\theta}$, be orthogonal over the domain under consideration, i.e.

$$\int_C \varepsilon_{n,\theta}(\xi \bar{\xi}) w(\xi \bar{\xi}) dC = 0 \quad (n = 1, 2, 3, \dots, N). \tag{18}$$

From equation (18) a homogeneous system of linear equations is obtained. Such a system can have nontrivial solutions only if the determinant of the coefficients of the unknowns vanishes identically. From this equation, the values of $k_1^2, k_2^2 \dots k_N^2$ can be found. For the fundamental frequency the lowest value of k^2 is to be taken.

Combining equations (12), (13a), and (13b) the following differential equation for determining $F(t)$ is obtained:

$$\ddot{F}(t) + \lambda_1 F(t) + \mu F^3(t) = 0 \tag{19}$$

in which

$$\lambda_1 = \frac{1}{12} \left(k^4 + \frac{k'}{D} \right) h^2 c_p^2 \tag{20}$$

$$\mu = \frac{1}{12} \alpha^2 k^2 h^2 c_p^2. \tag{21}$$

Equation (19) is to be solved subject to the initial conditions

$$F(0) = 1, \quad \dot{F}(0) = 0. \tag{22}$$

The solution of equation (19) can be taken in the form

$$F(t) = c_n(\omega_1 t, \lambda_2) \tag{23}$$

in which ω_1 and λ_2 are positive constants given by

$$\omega_1^2 = \frac{1}{12} \left(1 + \frac{\alpha^2}{k^2} + \frac{k'}{Dk^4} \right) h^2 c_p^2 k^4 \tag{24}$$

$$\lambda_2^2 = \frac{1}{2 \left(1 + \frac{k^2}{\alpha^2} + \frac{k'}{D\alpha^2 k^2} \right)} \tag{25}$$

and c_n is Jacobi's elliptic function. To determine α , equation (9) is transformed into complex coordinates by the transformation $z = x + iy, \bar{z} = x - iy$. Thus one finds

$$\frac{\alpha^2 h^2}{12} f(t) = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right) u + i \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right) v + 2 \frac{\partial w}{\partial z} \cdot \frac{\partial w}{\partial \bar{z}}. \tag{26}$$

If the mapping function $z = f(\xi)$ be introduced, equation (26) reduces to

$$\frac{\alpha^2 h^2}{12} \frac{dz}{d\xi} \cdot \frac{d\bar{z}}{d\bar{\xi}} f(t) = \frac{\partial u}{\partial \xi} \cdot \frac{d\bar{z}}{d\bar{\xi}} + \frac{\partial u}{\partial \bar{\xi}} \cdot \frac{dz}{d\xi} + i \left\{ \frac{\partial v}{\partial \xi} \cdot \frac{d\bar{z}}{d\bar{\xi}} - \frac{\partial v}{\partial \bar{\xi}} \cdot \frac{dz}{d\xi} \right\} + 2 \frac{\partial w}{\partial z} \cdot \frac{\partial w}{\partial \bar{z}}. \tag{27}$$

Now the normalised constant α can be determined from equation (17a) or (17b), and (27) by integrating equation (27) over the cycle 2π . The terms involving u and v can be eliminated (since u and v are of little importance in the case of large amplitude vibration) by considering suitable expressions for u and v , compatible with the boundary conditions. Finally the following integral will determine α :

$$\int_S \int \frac{\alpha^2 h^2}{12} \frac{dz}{d\xi} \cdot \frac{d\bar{z}}{d\bar{\xi}} dS = 2 \int_S \int \frac{\partial w}{\partial \xi} \cdot \frac{\partial w}{\partial \bar{\xi}} dS. \tag{28}$$

Thus having determined k and α , the non-linear frequency, ω_1 is completely determined. The non-linear period, T_1 , is given by

$$T_1 = \frac{4K}{\omega_1}, \tag{29}$$

K being the complete elliptic integral of the first kind. The linear period, T_2 , is given by

$$T_2 = \frac{2\pi}{\omega_2} \tag{30}$$

in which ω_2 is to be determined from the equation

$$\ddot{F}(t) + \lambda_1 F(t) = 0 \tag{31}$$

in the form $\omega_2^2 = \lambda_1$. Thus the ratio of the periods, T_1/T_2 , is obtained as

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\sqrt{\left(1 + \frac{k^2 \alpha^2}{k^4 + \frac{k'}{D}} \right)}}. \tag{32}$$

APPLICATIONS

(a) Let us apply the procedure explained above to the case of a clamped cornered plate. The mapping function is given by

$$z = \frac{25}{48} a \left(\xi - \frac{1}{25} \xi^5 + \dots \right). \tag{33}$$

Using equation (17b) with $n = 1$ an approximate value of k^2 is obtained from equation (18), viz.:

$$k^2 = \frac{24.55}{a^2}. \quad (34)$$

With $n = 2$, an improved lower value of k^2 is obtained

$$k^2 = \frac{21.71}{a^2}. \quad (35)$$

To determine α the following functions for u and v are taken,

$$u = \sum_{m=1,3,5,\dots}^{\infty} Um(r) \cos m\theta F^2(t) \quad (36)$$

$$v = \sum_{m=1,3,5,\dots}^{\infty} Vm(r) \sin m\theta F^2(t). \quad (37)$$

Substituting equations (36) and (37) in equation (27) one gets equation (28) for determining α . To determine the value of α for the fundamental frequency the value of n in equation (17b) is taken to be 1. Substituting equation (17b) with $n = 1$, and equation (33) in equation (28) the following value of α corresponding to the lowest frequency is obtained:

$$\alpha^2 = 29.28 \frac{B_1^2}{a^2 h^2}. \quad (38)$$

Thus T_1/T_2 is obtained from equation (32) as

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + \frac{B_1^2}{h^2} \left(\frac{636}{471 + K_F}\right)\right]^{1/2}} \quad (39)$$

in which the nondimensional foundation modulus, K_F , is given by $K_F = (K'/D)a^4$.

The mapping function of a square plate is given by

$$Z = 1.08a \left[\xi - \frac{1}{10} \xi^5 + \dots \right]. \quad (40)$$

Using equation (17b) with $n = 1$ and proceeding in the same manner as before, one gets for a clamped square plate

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + \frac{B_1^2}{h^2} \left(\frac{457}{400 + 12.3K_F}\right)\right]^{1/2}}. \quad (41)$$

The mapping function for a circular plate is given by

$$Z = a\xi \quad (42)$$

and for a clamped circular plate one finds

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + \frac{B_1^2}{h^2} \left(\frac{480}{400 + 9K_F}\right)\right]^{1/2}}. \quad (43)$$

(b) Let us consider the case of a simply supported circular plate. Using equation (17a) with $n = 1$ and proceeding in the same manner as before, one gets the ratio T_1/T_2 ,

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + \frac{B_1^2}{h^2} \left(\frac{72}{36 + K_F}\right)\right]^{1/2}}. \quad (44)$$

For $K_F = 0$, equation (44) becomes

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + 2 \frac{B_1^2}{h^2}\right]^{1/2}}. \quad (45)$$

The corresponding result for the circular plate obtained by Nash and Modeer [5] is

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + 6 \frac{A^2}{h^2} J_1^2(KR)\right]^{1/2}}, \quad (46)$$

where $J_0(KR) = 0$, R being the radius of the circle. For a simply supported square plate one finds

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + \frac{B_1^2}{h^2} \left(\frac{67.5}{36 + 1.37K_F}\right)\right]^{1/2}}. \quad (47)$$

EXPERIMENTAL VERIFICATION

Experimental verifications were made with circular and square plates having either simply supported or clamped boundary conditions. The circular plates were 150 mm dia and the square plates had 150 mm side. The plate material was mild steel 0.75 mm thickness. Free transverse vibrations of different amplitudes and frequencies were initiated by the apparatus shown in Fig. 1. The test piece, T , was statically deflected by the load spindle, L ,

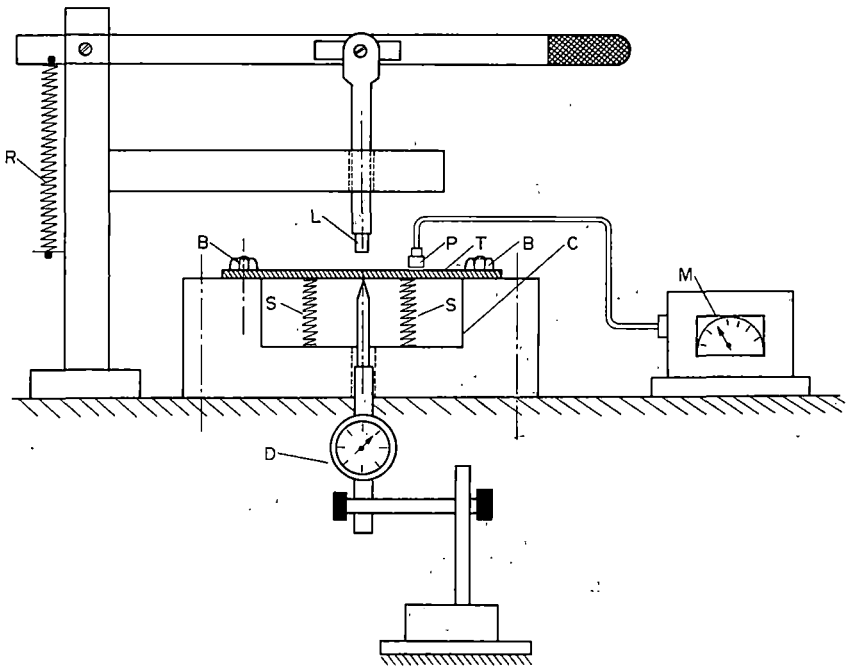


Fig. 1

and the central deflection was measured by the dial indicator, D . After giving a pre-determined central deflection the spindle, L , was lifted quickly by the release spring, R , and the corresponding frequency was measured in a vibration meter, M , with the help of a noncontact type of vibration pick-up, P . Simply supported edge conditions were realised by placing the edges of the plates over a knife edge placed around the periphery of the cavity, C , the shape of which conformed to the shape of the plate used. Clamped edge conditions were achieved by clamping the edges of the plates rigidly by means of eight bolts, B , with the base of the apparatus. Experiments were carried out first with the cavity empty and next by placing the plates over eight free helical springs, S , each spring being located at the centre of eight equal areas of the plates. The combined reaction of the springs used was determined experimentally to be $K_F = 6.2$. Care was taken in selecting the stiffness of the spring, R , so that the spindle, L , was released quickly from the plate without obstructing the upward motion of the plates.

RESULTS

Numerical as well as experimental results for the case of simply supported circular and square plates without any foundation have been presented in Figs. 2 and 3 respectively.

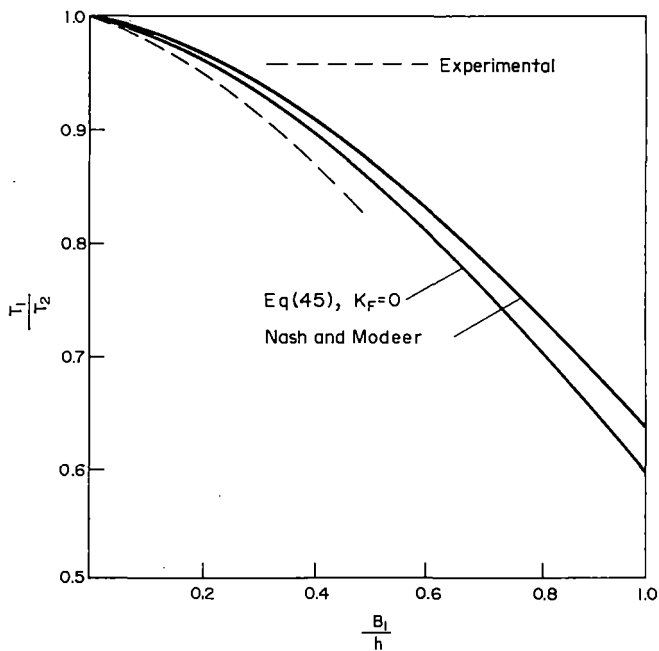


Fig. 2. Simply supported circular plate.

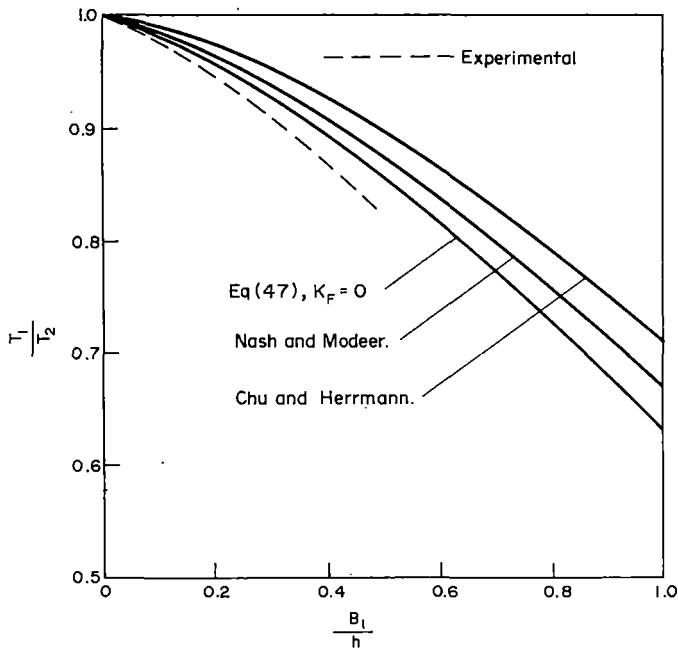


Fig. 3. Simply supported square plate.

The corresponding results obtained by Nash and Modeer [5] for the circular and square plates and the results obtained by Chu and Herrmann [7] for the square plates have also been presented for comparison. Numerical and experimental results for clamped circular and square plates both with and without foundation have been presented in Figs. 4 and 5 respectively.

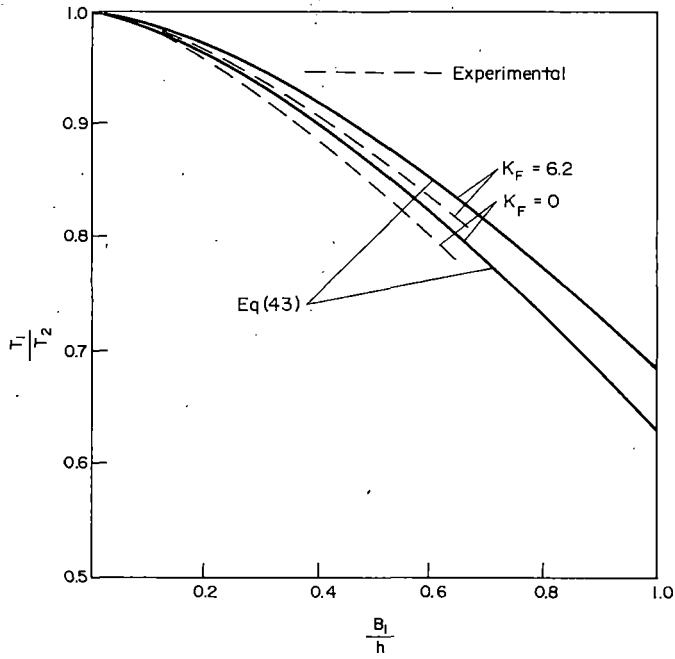


Fig. 4. Clamped circular plate.

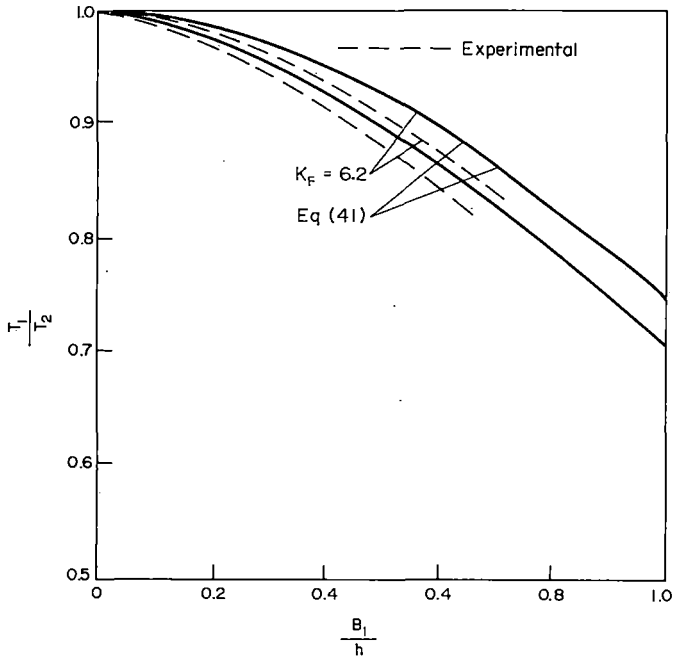


Fig. 5. Clamped square plate.

CONCLUSIONS

Lowest natural frequencies of large amplitude free vibrations of thin plates of any shape can readily be calculated by conformal mapping techniques used in this study if the mapping functions are known. From Figs. 4 and 5 it is observed that the results obtained with a one-term approximation of the trial function, equation (17b), for the clamped edge boundary conditions are in excellent agreement with the practical values. For the simply supported edge conditions the theoretical results given in Figs. 2 and 3 are in somewhat poorer agreement with the values obtained experimentally. By using higher approximations of the trial

functions, equation (17a) and (17b) and with smoothed mapping functions the results for both simply supported and clamped edge boundary conditions will be refined.

The periods for rectangular plates obtained by Chu and Herrmann [7] is dependent on the aspect ratio of the plate, whereas the corresponding results obtained by Nash and Modeer [5] are independent of that ratio. The mapping functions for rectangular plates with different aspect ratios will be different and therefore the present study indicates that the periods will depend on the aspect ratio. It should be pointed out that the theory used in this study allows the solution of the eigenvalue problem under consideration from a unified point of view since the trial functions used are the same for all shapes. For a one-term approximation the results obtained in this study are considered satisfactory for practical purposes.

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RÉSUMÉ

On donne une méthode unifiée pour déterminer la plus petite fréquence propre pour des vibrations libres de grande amplitude de plaques élastiques minces de forme quelconque placées sur un support élastique. On introduit la technique de transformation conforme et on utilise la méthode de Galerkin pour calculer la valeur approchée de la plus petite fréquence propre. On détermine les périodes de plaques circulaires, carrées et en coin placées sur un support élastique pour des conditions aux limites en appui simple et encastées. Les valeurs pratiques ont également été déterminées expérimentalement. On présente les résultats sous forme graphique et on les compare avec d'autres résultats connus.

Zusammenfassung:

Eine einheitliche Methode für die Bestimmung der niedrigsten Eigenfrequenz von frei, mit grosser Amplitude schwingenden dünnen elastischen Scheiben beliebiger Form mit elastischer Lagerung wird gegeben. Das Verfahren der konformen Abbildung wird eingeführt und die Galerkinsche Methode wird zur Berechnung von Näherungswerten der niedrigsten Eigenfrequenz benutzt. Die Schwingungsdauern für kreisförmige, quadratische und eckige Scheiben auf elastischer Lagerung wurden für die Randbedingungen der frei aufliegenden und eingespannten Kanten bestimmt. Praktische Werte wurden experimentell bestimmt. Die Resultate werden in Form von Diagrammen dargestellt und mit anderen bekannten Ergebnissen verglichen.

LARGE DEFLECTION OF A HEATED ELLIPTIC PLATE ON ELASTIC FOUNDATION

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ABSTRACT

Following Berger's method the large deflection of a heated elliptic plate with clamped edges and placed on elastic foundation has been investigated under stationary temperature distribution. The deflection is obtained in terms of Mathieu function of the first kind and of zero order.

Keywords: Berger's method; Mathieu function; Elliptic plate; Elastic foundation.

INTRODUCTION

In recent years there has been a rapid development of thermoelasticity stimulated by various engineering sciences. In the field of machine structures, mainly with aircraft, steam and gas turbines and in chemical and nuclear engineering, thermal stresses play an important and frequently even a primary role. Determination of thermal deflections of plates, especially of thin plates, is of vital importance in the design of machine structures, because excessive deflections may cause heavy undesirable thermal stresses.

The classical large deflection of thin plate problems usually lead to non-linear differential equations which cannot be exactly solved. H. M. Berger [1] has shown that if, in deriving the differential equations from the expressions for strain energy, the strain energy due to second invariant in the middle plane of the plate is neglected, a simple fourth order differential equation coupled with a non-linear second order equation is obtained. Although no complete explanation of the method is set forth, the stresses and deflections obtained by Berger himself for rectangular and circular plates agree well with those found from more precise analysis. This approximate method has been extended to orthotropic plates by Iwinski and Nowinski [2] and further boundary value problems associated with rectangular and circular

plates have been solved by Nowinski [3]. Thein Wah and Robert Schmidt [4] and Nash and Modeer [5] obtained satisfactory results following this method Basuli [6] has extended this approximate method of Berger to problems under uniform load and heating under stationary temperature distribution.

Berger's technique of neglecting the second invariant of the middle surface strains has been applied by Sinha [7] to circular and rectangular plates placed on elastic foundation and under uniform transverse load.

In this paper the author has applied the method of Berger to investigate the large deflection of an elliptic plate placed on elastic foundation and heated under stationary temperature distribution. The foundation is assumed to be such that its reaction is proportional to the deflection. The deflection is obtained in terms of Mathieu function of the first kind and of zero order.

NOTATIONS

The following notations have been used in the paper:

$$D = \text{Flexural rigidity of the plate} = \frac{Eh^3}{12(1-\nu^2)}$$

E, ν, α = Young's modulus, Poisson's ratio and Coefficient of thermal expansion respectively.

h = Thickness of plate.

u, v = Displacement along the x and y axis respectively.

w = Lateral displacement

e_1 = First strain invariant;

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2$$

e_2 = Second strain invariant.

K = Foundation reaction per unit area per unit deflection.

∇ = Laplacian operator.

FORMULATION OF PROBLEM

The strain energy due to bending and stretching of the middle surface of the plate is given by:

$$V_1 = \frac{D}{2} \iint \left[(\nabla^2 W)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu) \left\{ \frac{12}{h^2} e_2 \right. \right.$$

$$+ \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} = \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \Big] dx dy \quad (1)$$

Combining the potential energy of the foundation reaction and also the potential energy due to heating with Eq. 1 and neglecting e_2 , the modified energy expression for the total energy becomes:

$$V = \frac{D}{2} \iint \left[(\nabla^2 w)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu) \left\{ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} + \frac{K}{D} w^2 \right] dx dy - \int \int \int_{-h/2}^{h/2} \frac{E \alpha T'}{1-\nu} \times (e_1 - z \nabla^2 w) dx dy dz \quad (2)$$

in which T' is the temperature distribution at any point given by (Basuli [6])

$$T'(x, y, z) = T_0(x, y) + g(z) T(x, y) \quad (3)$$

and

$$\int_{-h/2}^{h/2} z g(z) dz = f(h); \quad \int_{-h/2}^{h/2} g(z) dz = 0 \quad (4)$$

Combining Equations 2, 3 and 4 one gets

$$V = \frac{D}{2} \iint \left[(\nabla^2 w)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu) \left\{ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} + \frac{K}{D} w^2 \right] dx dy - \int \int \frac{E \alpha}{1-\nu} (T_0 e_1 h - f(h) T \nabla^2 w) dx dy. \quad (5)$$

According to the principle of minimum potential energy, the displacements that satisfy the equilibrium conditions make the potential energy, V , minimum. In order for the integral of Eq. 5 to be an extremum, the integrand, F , must satisfy the following Euler's equations of the calculus of variation:

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) = 0 \quad (6a)$$

$$\frac{\partial F}{\partial v} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial V_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial V_y} \right) = 0 \quad (6b)$$

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial w_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial w_y} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial w_{xx}} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial F}{\partial w_{yy}} \right) + \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial F}{\partial w_{xy}} \right) = 0. \quad (6 c)$$

Application on the Eqs. 6 a, 6 b, and 6 c to Eq. 5 yields:

$$\frac{\partial}{\partial x} \{e_1 - (1 + \nu) \alpha T_0\} = 0 \quad (7 a)$$

$$\frac{\partial}{\partial y} \{e_1 - (1 + \nu) \alpha T_0\} = 0. \quad (7 b)$$

$$\nabla^4 w - \frac{12}{h^2} \{e_1 - (1 + \nu) \alpha T_0\} \nabla^2 w + \frac{K}{D} w + \frac{E \alpha f(h)}{D(1 - \nu)} \nabla^2 T = 0. \quad (7 c)$$

Eqs. 7 a, and 7 b prove that:

$$\{e_1 - (1 + \nu) \alpha T_0\}$$

is independent of x and y and therefore

$$e_1 - (1 + \nu) \alpha T_0 = \text{constant} = \frac{\beta^2 h^2}{12} \quad (8 a)$$

in which β is a normalised constant of integration, and

$$e_1 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \quad (8 b)$$

Considering Eq. 8 a, Eq. 7c reduces to

$$\nabla^2 (\nabla^2 - \beta^2) w + \frac{K}{D} w = - \frac{E \alpha f(h)}{D(1 - \nu)} \nabla^2 T \quad (9)$$

SOLUTION OF PROBLEM

Let us take an elliptic plate of thickness, h . The centre of the plate in the middle surface is taken as the origin and the Z -axis downwards.

If there is no source of heat inside the plate the following differential equations must be satisfied for stationary temperature distribution (Nowacki [8])

$$\nabla^2 T_0 - \epsilon T_0 = -\frac{\epsilon_0}{2} (\theta_1 + \theta_2)$$

$$\nabla^2 T - \frac{12}{h^2} (1 + \epsilon) T = -\frac{12\epsilon}{h^3} (\theta_1 - \theta_2) \quad (11)$$

in which θ_1 and θ_2 denote temperatures at the upper and lower media of the plate respectively.

If $\theta_1 = \theta_2$, Eq. (11) becomes

$$\nabla^2 T - \beta_1^2 T = 0 \quad (12)$$

In which

$$\beta_1^2 = (1 + \epsilon) \frac{12}{h^2} \quad (13)$$

Transferring to elliptic co-ordinates (ξ, η) defined by $x + iy = d \cosh(\xi + i\eta)$, where $2d$ is the interfocal distance of the ellipse, Eq. 12 reduces to

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2} - \frac{\beta_1^2 d^2}{2} (\cosh 2\xi - \cos 2\eta) T = 0. \quad (14)$$

Solution of Eq. 14 can be taken in the following form

$$T = \sum_{m=0}^{\infty} C_{2m} C_{e2m}(\xi, -q) c_{e2m}(\eta, -q) \quad (15)$$

in which $C_{e2m}(\xi, -q)$ and $c_{e2m}(\eta, -q)$ are modified Mathieu function and ordinary Mathieu function of the first kind and of order $2m$ respectively, and

$$q = \frac{\beta_1^2 d^2}{4} \quad (16)$$

While solving a problem of bending of a plate with an elliptic hole, by taking a single Mathieu function of the second order instead of taking Mathieu functions of all orders, Naghdi [9] has shown that the results are satisfactory for larger elliptic holes. In this paper also similar approximation is made by taking Mathieu function of zero order and on this assumption Eq.15 reduces to

$$T = C_0 C_{e0}(\xi, -q) c_{e0}(\eta, -q) \quad (17)$$

The following boundary condition is imposed on T .

$$T = \text{Constant} = K_1 \text{ on } \xi = \xi_0$$

with the above boundary condition Eq. 17 yields

$$K_1 = C_0 C_{eo}(\xi_0, -q) c_{eo}(\xi, -q) \quad (18)$$

Multiplying Eq. 18 by $c_{eo}(\eta, -q)$ and integrating with respect to η from 0 to 2π and using the orthogonality relation and normalisation (Melachlan [10]) one gets

$$C_0 = \frac{2A_0^{(0)} K_1}{C_{eo}(\xi_0, -q)} \quad (19)$$

in which $A_0^{(0)}$ is the first Fourier Coefficient in the expansion of $c_{eo}(\eta, -q)$

Therefore

$$T = \frac{2A_0^{(0)} K_1}{C_{eo}(\xi_0, -q)} C_{eo}(\xi, -q) c_{eo}(\eta, -q) \quad (20)$$

is determined.

Changing Eq. 9 to elliptic Co-ordinates and substituting the expression for $\nabla^2 T$ one gets

$$(\nabla^2 - P_1^2)(\nabla^2 - P_2^2)W = \lambda C_{eo}(\xi, -q) c_{eo}(\eta, -q) \quad (21)$$

in which

$$P_1^2 + P_2^2 = -\beta^2 \quad (22)$$

$$P_1^2 P_2^2 = \frac{K}{D} \quad (23)$$

$$\lambda = -\frac{Ecf(h)}{D(1-\nu)} \frac{2\beta_1^2 A_0^{(0)} K_1}{C_{eo}(\xi_0, -q)} \quad (24)$$

$$\nabla^2 = \frac{2}{d^2(\cosh 2\xi - \cos 2\eta)} \left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right] \quad (25)$$

Complimentary function of Eq. 21 is given by

$$W = B_0 C_{eo}(\xi, -q_1) c_{eo}(\eta, -q_1) + D_0 C_{eo}(\xi, -q_2) c_{eo}(\eta, -q_2) \quad (26)$$

in which

$$q_1 = \frac{P_1^2 d^2}{4}; q_2 = \frac{P_2^2 d^2}{4} \quad (27)$$

Clearly the particular integral of Eq. 21 is

$$\frac{\lambda}{(\beta_1^2 - P_2^2)(\beta_1^2 - P_1^2)} C_{eo}(\xi, -q) c_{eo}(\eta, -q) \quad (28)$$

Thus the complete solution of Eq. 21 is

$$W = B_0 C_{eo} (\xi, -q_1) c_{eo} (\eta, -q_1) + D_0 C_{eo} (\xi, -q_2) c_{eo} (\eta, +q_2) \\ + \frac{\lambda}{(\beta_1^2 - P_2^2)(\beta_1^2 - P_1^2)} C_{eo} (\xi, -q) c_{eo} (\eta, -q) \quad (29)$$

If the outer boundary of the plate $\xi = \xi_0$ be clamped, the boundary conditions are

$$(W)_{\xi=\xi_0} = \left(\frac{\partial W}{\partial \xi} \right)_{\xi=\xi_0} = 0 \quad (30)$$

Using Eq. 30 in Eq. 29 one gets the following two conditional equations

$$B_0 C_{eo} (\xi_0, -q_1) c_{eo} (\eta, -q_1) + D_0 C_{eo} (\xi_0, -q_2) c_{eo} (\eta, -q_2) \\ + \frac{\lambda}{(\beta_1^2 - P_2^2)(\beta_1^2 - P_1^2)} C_{eo} (\xi_0, -q) c_{eo} (\eta, -q) = 0 \quad (31 a)$$

$$B_0 C'_{eo} (\xi_0, -q_1) c_{eo} (\eta, -q_1) + D_0 C'_{eo} (\xi_0, -q_2) c_{eo} (\eta, -q_2) \\ + \frac{\lambda}{(\beta_1^2 - P_2^2)(\beta_1^2 - P_1^2)} C'_{eo} (\xi_0, -q) c_{eo} (\eta, -q) = 0 \quad (31 b)$$

Multiplying Eqs. 31 a and 31 b by $c_{eo} (\eta, -q_1)$ and integrating with respect to η from 0 to 2π and using the orthogonality relation and normalisation one gets

$$B_0 = - \frac{\lambda \phi_2}{\pi \psi \phi_3} \{ C_{eo} (\xi_0, -q_2) C'_{eo} (\xi_0, -q) - C_{eo} (\xi_0, -q) \\ C'_{eo} (\xi_0, -q_2) \} \quad (32)$$

$$D_0 = \frac{\lambda \phi_2}{\psi \phi_3 \phi_1} \{ C_{eo} (\xi_0, -q_1) C'_{eo} (\xi_0, -q) \\ - C_{eo} (\xi_0, -q) C'_{eo} (\xi_0, -q_1) \} \quad (33)$$

in which

$$\psi = (\beta_1^2 - P_2^2)(\beta_1^2 - P_1^2)$$

$$\phi_1 = 2\bar{A}_0^{(0)} \bar{A}_0^{(0)} + \sum_{r=1}^{\infty} \bar{A}_{2r}^{(0)} \bar{A}_{2r}^{(0)}$$

$$\phi_2 = 2\bar{A}_0^{(0)} \bar{A}_0^{(0)} + \sum_{r=1}^{\infty} \bar{A}_{2r}^{(0)} \bar{A}_{2r}^{(0)}$$

$$\phi_3 = C_{eo} (\xi_0, -q_2) C'_{eo} (\xi_0, -q_1) - C_{eo} (\xi_0, -q_1) C'_{eo} (\xi_0, -q_2)$$

$\bar{A}_{2r}^{(0)}$, $\bar{A}_{2r}^{(0)}$ and $A_{2r}^{(0)}$ are the Fourier Coefficients in the expansion of $c_{e0}(\eta, q_1)$, $c_{e0}(\eta, -q_2)$ and $c_{e0}(\eta, -q)$ respectively.

To determine the constant β^2 , Eq. 8 is transformed into elliptic Co-ordinates in the form

$$h_1 h_2 \left\{ \frac{\partial}{\partial \xi} \left(\frac{u_\xi}{h_2} \right) + \frac{\partial}{\partial \eta} \left(\frac{u_\eta}{h_1} \right) \right\} + \frac{1}{2} h_1 h_2 \left\{ \left(\frac{\partial w}{\partial \xi} \right)^2 + \left(\frac{\partial w}{\partial \eta} \right)^2 \right\} \\ = \frac{\beta^2 h^2}{12} + (1 + \nu) \alpha T_0 \quad (34)$$

in which

$$h_1 = h_2 = \frac{1}{d \sqrt{\sinh^2 \xi + \sin^2 \eta}}$$

The boundary conditions for u_ξ and u_η are

$$u_\xi = 0 = u_\eta \text{ at } \xi = \xi_0 \quad (35)$$

Let

$$u_\xi = \sum_{n=0}^{\infty} P(\xi) \cos 2n\eta \quad (36)$$

$$u_\eta = \sum_{n=1}^{\infty} G(\xi) \sin 2n\eta \quad (37)$$

subject to the conditions

$$P(\xi_0) = G(\xi_0) = 0$$

Substituting Eqs. 29, 36, and 37 in Eq. 34 and integrating over the surface of the plate one gets

$$\int_0^{2\pi} \int_0^{\xi_0} \left\{ \left(\frac{\partial w}{\partial \xi} \right)^2 + \left(\frac{\partial w}{\partial \eta} \right)^2 \right\} d\xi d\eta \\ = d^2 \left\{ \frac{\beta^2 h^2}{6} + 2(1 + \nu) \alpha T_0 \right\} \int_0^{2\pi} \int_0^{\xi_0} (\sinh^2 \xi + \sin^2 \eta) d\xi d\eta \quad (38)$$

After evaluating the integrals the following equation leading to β is obtained.

$$E_{01}^{22} \left[(2 \{ \bar{A}_0^{(0)} \}^2 + \sum_{r=1}^{\infty} \{ \bar{A}_{2r}^{(0)} \}^2) \left\{ \sum_{r=1}^{\infty} 4r^2 \{ A_{2r}^{(0)} \}^2 \right\} \psi_1 \right]$$

$$\begin{aligned}
 & + \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} 2rs (-1)^r (-1)^s A'_{2r}{}^{(0)} A'_{2s}{}^{(0)} \psi_2 \\
 & + \left(\sum_{r=1}^{\infty} 4r^2 \{\bar{A}_{2r}{}^{(0)}\}^2 \right) \{ (A_0'{}^{(0)})^2 \xi_0 + A_0'{}^{(0)} \sum_{r=1}^{\infty} A'_{2r}{}^{(0)} (-1)^r \psi_5 \\
 & + \sum_{r=1}^{\infty} A'_{2r}{}^{(0)} \psi_4 + \frac{1}{2} \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} (-1)^r (-1)^s A'_{2r}{}^{(0)} A'_{2s}{}^{(0)} \psi_3 \} \\
 & + D_0^2 [(2\{\bar{A}_0^{(0)}\}^2 + \sum_{r=1}^{\infty} \{\bar{A}_{2r}{}^{(0)}\}^2) \{ \sum_{r=1}^{\infty} 4r^2 \{A''_{2r}{}^{(0)}\}^2 \psi_1 \\
 & + \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} 2rs (-1)^r (-1)^s A''_{2r}{}^{(0)} A''_{2s}{}^{(0)} \psi_2 \\
 & + \left(\sum_{r=0}^{\infty} 4r^2 \{\bar{A}_{2r}{}^{(0)}\}^2 \right) \{ (A_0''{}^{(0)})^2 \xi_0 + A_0''{}^{(0)} \sum_{r=1}^{\infty} A''_{2r}{}^{(0)} (-1)^r \psi_5 \\
 & + \sum_{r=1}^{\infty} A''_{2r}{}^{(0)} \psi_4 + \frac{1}{2} \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} (-1)^r (-1)^s A''_{2r}{}^{(0)} A''_{2s}{}^{(0)} \psi_3 \} \\
 & + 2B_0 D_0 [(2\bar{A}_0^{(0)} \bar{A}_0^{(0)} + \sum_{r=1}^{\infty} \bar{A}_{2r}{}^{(0)} \bar{A}_{2r}{}^{(0)}) \{ \sum_{r=1}^{\infty} 4r^2 A'_{2r}{}^{(0)} A''_{2r}{}^{(0)} \psi_1 \\
 & + \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} 2rs (-1)^r (-1)^s A'_{2r}{}^{(0)} A''_{2s}{}^{(0)} \psi_2 \\
 & + \left(\sum_{r=1}^{\infty} 4r^2 \bar{A}_{2r}{}^{(0)} \bar{A}_{2r}{}^{(0)} \right) \{ A_0'{}^{(0)} A_0'{}^{(0)} \xi_0 + A_0'{}^{(0)} \sum_{r=1}^{\infty} (-1)^r A''_{2r}{}^{(0)} \psi_5 \\
 & + A_0'{}^{(0)} \sum_{r=1}^{\infty} (-1)^r A''_{2r}{}^{(0)} \psi_5 + \sum_{r=1}^{\infty} A''_{2r}{}^{(0)} A'_{2r}{}^{(0)} \psi_4 \\
 & + \frac{1}{2} \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} (-1)^r (-1)^s A''_{2r}{}^{(0)} A'_{2s}{}^{(0)} \psi_3 \} \\
 & + \frac{2B_0 \lambda}{\psi} [(2A_0^{(0)} \bar{A}_0^{(0)} + \sum_{r=1}^{\infty} A_{2r}^{(0)} \bar{A}_{2r}{}^{(0)}) \{ \sum_{r=1}^{\infty} 4r^2 A'_{2r}{}^{(0)} a_{2r}^{(0)} \psi_1 \\
 & + \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} 2rs (-1)^r (-1)^s A'_{2r}{}^{(0)} a_{2s}^{(0)} \psi_2 + \left(\sum_{r=1}^{\infty} 4r^2 A_{2r}^{(0)} \bar{A}_{2r}{}^{(0)} \right) \\
 & \times \{ a_0^{(0)} A_0'{}^{(0)} \xi_0 + a_0^{(0)} \sum_{r=1}^{\infty} (-1)^r A'_{2r}{}^{(0)} \psi_5 + A_0'{}^{(0)} \sum_{r=1}^{\infty} (-1)^r a_{2r}^{(0)} \psi_5 \\
 & + \sum_{r=1}^{\infty} a_{2r}^{(0)} A'_{2r}{}^{(0)} \psi_4 + \frac{1}{2} \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} (-1)^r (-1)^s a_{2r}^{(0)} A'_{2s}{}^{(0)} \psi_3 \} \\
 \end{aligned}$$

$$\begin{aligned}
& + \frac{2D_0\lambda}{\psi} [(2A_0^{(0)} \bar{A}_0^{(0)} + \sum_{r=1}^{\infty} A_{2r}^{(0)} \bar{A}_{2r}^{(0)}) \{ \sum_{r=1}^{\infty} 4r^2 A_{2r}^{''(0)} a_{2r}^{(0)} \psi_1 \\
& + \sum_{\substack{r=1, \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} 2rs (-1)^s (-1)^r A_{2r}^{''(0)} a_{2r}^{(0)} \psi_2 \} + (\sum_{r=1}^{\infty} 4r^2 A_{2r}^{(0)} \bar{A}_{2r}^{(0)}) \\
& \times \{ a_0^{(0)} A_0^{''0} \xi_0 + a_0^{(0)} \sum_{r=1}^{\infty} (-1)^r A_{2r}^{''(0)} \psi_5 + A_0^{''(0)} \sum_{r=1}^{\infty} (-1)^r a_{2r}^{(0)} \psi_5 \\
& + \sum_{r=1}^{\infty} a_{2r}^{(0)} A_{2r}^{''(0)} \psi_4 + \frac{1}{2} \sum_{\substack{r=1, \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} (-1)^r (-1)^s a_{2r}^{(0)} A_{2s}^{''(0)} \psi_3 \}] \\
& + \frac{\lambda^2}{\psi^2} [(2 \{A_0^{(0)}\}^2 + \sum_{r=1}^{\infty} \{A_{2r}^{(0)}\}^2) \{ \sum_{r=1}^{\infty} 4r^2 \{a_{2r}^{(0)}\}^2 \psi_1 \\
& + \sum_{\substack{r=1, \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} 2rs (-1)^r (-1)^s a_{2r}^{(0)} a_{2s}^{(0)} \psi_2 \} + (\sum_{r=1}^{\infty} 4r^2 \{A_{2r}^{(0)}\}^2) \\
& \times \{ (a_0^{(0)})^2 \xi_0 + a_0^{(0)} \sum_{r=1}^{\infty} a_{2r}^{(0)} (-1)^r \psi_5 + \sum_{r=1}^{\infty} a_{2r}^{(0)} \psi_4 \\
& + \frac{1}{2} \sum_{\substack{r=1, \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} (-1)^r (-1)^s a_{2r}^{(0)} a_{2s}^{(0)} \psi_3 \}] \\
& = \frac{d^2}{2} \left\{ \frac{\beta^2 h^2}{6} + 2(1 + \nu) \alpha T_0 \right\} \sinh 2 \xi_0 \tag{39}
\end{aligned}$$

where

$$\begin{aligned}
\psi_1 &= \frac{\text{Sinh } 4r\xi_0}{8r} - \frac{\xi_0}{2} \\
\psi_2 &= \frac{\text{Sinh } \overline{2r+2s} \xi_0}{2r+2s} - \frac{\text{Sinh } \overline{2r-2s} \xi_0}{2r-2s} \\
\psi_3 &= \frac{\text{Sinh } \overline{2r+2s} \xi_0}{2r+2s} + \frac{\text{Sinh } \overline{2r-2s} \xi_0}{2r-2s} \\
\psi_4 &= \frac{\xi_0}{2} + \frac{\text{Sinh } 4r\xi_0}{8r} \\
\psi_5 &= \frac{\text{Sinh } 2r \xi_0}{2r}
\end{aligned}$$

and

$a_{2r}^{(0)}$, $A_{2r}^{''(0)}$, and $A_{2r}^{''(0)}$ are the Fourier Coefficients in the expansions of $C_{e0}(\xi, -q)$, $C_{e0}(\xi, -q_1)$, and $C_{e0}(\xi, -q_2)$ respectively.

Since β is determined, w is determined completely.

NUMERICAL CALCULATION

To find the deflection at a given point, one has to start from Eq. 39 with an assumed value of β leading to the corresponding value of λ . With this value of λ and considering Eqs. 32 and 33 the deflection will be obtained from Eq. (29).

For numerical calculation the following values have been assumed:

$$\xi = 0, \eta = \frac{\pi}{2}, \xi_0 = 3, d^2 = 2.5, h = 1, f(h) = h,$$

$$K_F = \frac{K}{D} \xi_0^4 = 100, \epsilon = 0.03, \nu = 0.3, \alpha T_0 = 2.5 \times 10^{-3}.$$

The interfocal distance $2d$ being assumed and the values of β^2, P_1^2 and P_2^2 being known, the values of q, q_1 and q_2 are determined. q, q_1 and q_2 being

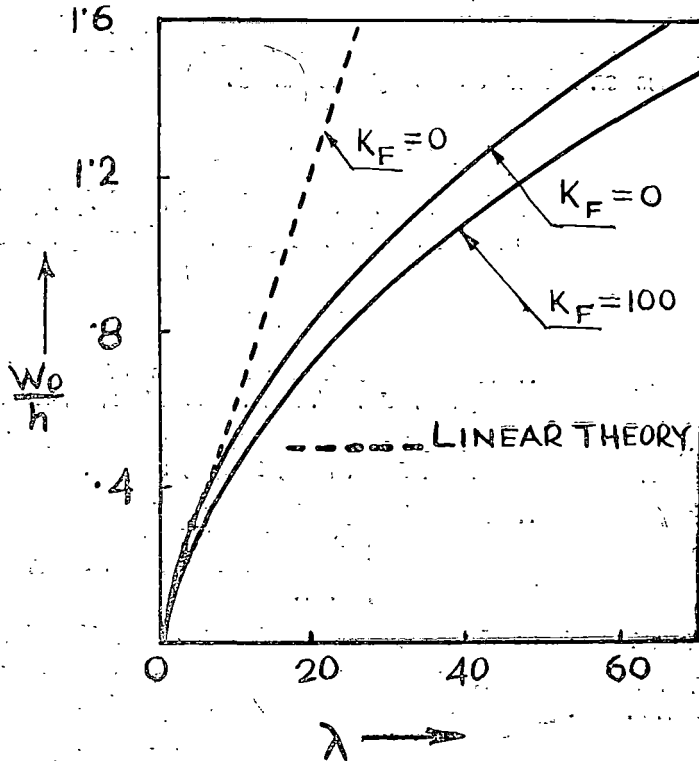


FIG. 1. Load-Deflection Curve

known the corresponding values of the Fourier Coefficients as well as those of Mathieu functions are determined. The maximum deflection W_0 is obtained at the centre of the plate. These deflections are graphically presented in Fig. 1 in which W_0/h for $K_F = 0$ and $K_F = 100$ are plotted against the non-dimensional load function λ . By setting $\beta \rightarrow 0$ the deflections according to the linear theory is obtained. For comparison Fig. 1 also includes a straightline which represents small deflections for $K_F = 0$. The results obtained in this study could not be compared in absence of any known results.

CONCLUSIONS

From Fig. 1 it is observed that the error according to the linear theory increases progressively with the increase in load function. The solution proposed in this study is rapidly convergent and no computational difficulty other than computational effect is involved. The parameter q for the series $c_{eo}(\xi, q)$ may be real or imaginary and the corresponding coefficients can be computed with accuracy. The numerical results presented in this study are obtained by taking the first two terms of the series and sufficient for practical purposes. Since the deflection at any point is known the corresponding stresses can be easily estimated.

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