CHAPTER - I

INTRODUCTION

1.1 Preliminaries

The *intersection graph* of a family of sets $\mathcal{F} = \{ S_v \}$ is the graph with vertices corresponding to the sets such that two vertices are adjacent if and only if the corresponding sets intersect. Many classes of intersection graphs in which the sets are restricted in some way have been studied during the last thirty years; the best known class is the *interval graphs*. These are the graphs with intersection representations in which $\mathcal{F}$ is a family of intervals on the real line.

During the seventies, mathematicians started thinking of generating graphs by ways related to intersection model. In a containment representtation, edges correspond to containment of sets $S_u \subset S_v$ or $S_u \supset S_v$. In an overlap representation, edges correspond to pairs of sets that intersect without either set containing the other. A special issue of *Discrete Mathematics* [1985] is devoted to the deduction of significant new results on interval graphs and other related classes of graphs. An excellent survey by Trotter [1988] summarizes a variety of recent results and open problems.

The area to study directed graphs (digraphs) from the view point of intersection representation and its exploration has started only during the last decade. Since undirected graphs are special types of directed graphs, the problem of finding a natural translation of the theory of intersection graphs alongwith its abundant literature is very well worth investigating. Previously, intersection digraphs of a family of ordered pairs of intervals on the real line and of arcs of a circle, respectively called interval digraphs and circular arc digraphs were studied and their characterizations obtained [Sen et al, 1989 a, b] and these results provide the background materials for this dissertation.

In the second chapter, we introduce *indifference digraphs*, a generalization of indifference graphs. In our work, indifference digraphs are shown to be equivalent to two restricted classes of interval digraphs, *unit interval digraphs* and *proper interval digraphs* also introduced in this chapter. These digraphs are characterized in terms of their adjacency matrices and in terms of a generalized concept of semiorder.

In chapter III, we study and obtain some more results on interval digraphs. In particular, we make an important observation that an interval digraph is actually a generalized concept of an interval graph in the sense that a graph is an interval graph if and only if the corresponding symmetric digraph with loops is an interval digraph.

A model of containment digraph is introduced in Chapter IV. The interval containment digraphs are precisely the digraphs of

Ferrers dimension 2 and the circular arc containment digraphs are the complements of circular arc intersection digraphs. The Ferrers dimension of a digraph is the minimum number of Ferrers digraphs whose intersection is the given digraph.

A model of overlap digraphs for pairs of intervals is also introduced in the same chapter. The unit overlap digraphs are the indifference digraphs and the adjacency matrices of all overlap digraphs have a simple structural characterizations making their Ferrers dimension at most 3.

From the study of different model representation of digraphs it is observed that their adjacency matrices have 0's rearranged in a definite pattern such as moving along the right and/ or below and so on. D. B. West is an explorer of this field of study and he posed the question of characterizing a binary matrix whose 0's are such that corresponding to any 0 in the matrix, all its positions along at least one of the four directions are also 0's. In the last chapter of this thesis we have called such a matrix a 4-directable matrix and have characterized it in terms of the adjacency matrix of a digraph having an overlap base interval representation. A base interval is an ordered pair (S_v, p_v) where S_v is an interval on the real line and p_v is a point of S_v.

In conclusion, we pose some of the problems that come naturally as we proceed along.

1.1.1 Basic definitions

The basic terminology about graphs, directed graphs and relations that we have used throughout this dissertation are given below.

Given a graph $G(V,E)$, $V = V(G)$ will denote its vertex set and $E = E(G)$ will denote its edge set. A graph H is a *subgraph* of a graph $G(V,E)$ if $V(H) \subset V(G)$ and $E(G) \subset E(G)$. A subgraph H will be said to be a *generated subgraph* or an *induced subgraph* of G if $V(H) \subset V(G)$ and two vertices x and y are adjacent in H iff they are adjacent in G. The *complement* $\overline{G}(V,E)$ of a graph $G(V,E)$ has the same vertex set V and two vertices are adjacent iff they are not adjacent in G.

A set of vertices in a graph G is said to be an *independent* set (or a *stable* set) of vertices if no two vertices in the set are adjacent. A *bipartite graph* is a graph $G(V,E)$ whose vertex set V can be partitioned into two stable sets. A *complete bipartite graph* is a bipartite graph G such that G contains every possible edge between two stable sets. A *complete bipartite subgraph* of a graph G is a subgraph B of G such that B is itself a complete bipartite graph.

The *union* of two graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ is another graph $G_3(V_3,E_3)$ whose vertex set $V_3 = V_1 \cup V_2$ and edge set

$E_3 = E_1 \cup E_2$. Likewise the *intersection* of graphs G_1 and G_2 is a graph G_3 consisting only those vertices and edges that are in both G_1 and G_2.

The *n*-cube Q_n is a graph having 2^n points which have coordinates $(a_1, a_2 \ldots . a_n)$ where each a_i is either 0 or 1 and two point of Q_n are adjacent if their coordinates differ at exactly one place.

For any graph $G = (V,E)$ and integer $K \geq 1$, the graph G^K has vertex set V and edges joining vertices $u, v \in V$ whenever $d(u.v) \leq K$ where $d(u.v)$ is the distance (length of the shortest path) in G between u and v.

A *directed graph* (digraph) $D(V,E)$ consists of a nonempty set V of points together with a collection E of ordered pairs of points. For a digraph $D(V,E)$ an edge (arc) $uv \in E$ is represented by a line segment between u and v with arrow directed from u to v. Sometimes we write $u \longrightarrow v$ to mean $uv \in E$ and $u \not\longrightarrow v$ to mean $uv \notin E$. All throughout we will assume that a digraph may have loops but no multiple edges.

The *successor set* of a vertex V is the set of vertices u such that vu is an edge of the digraph $D(V,E)$. The *predecessor set* of a vertex V is the set of vertices u such that uv is an edge of the digraph $D(V,E)$.

The digraph *D(V,E)* is an *acyclic digraph* (with no isolates) when it contains at least one vertex with out-degree zero (receiver) and at least one vertex with in-degree zero (transmitter).

The concept of union and intersection of two digraphs is the same as that of undirected graphs. The union of two digraphs D_1 and D_2 is *complete* if every ordered pair of vertices (including $u = v$) forms an edge in at least one of them .

The adjacency matrix *A(G)* of a graph *G* (*A(D)* of a digraph *D*) on vertex set $V_i = \left\{v_1 .... v_x\right\}$ is the 0,1 matrix with a 1 in row and column *j* if and only if (v_i, v_j) is an edge in the graph (digraph).

Other definitions on graph theory will be given as and when needed throughout the chapters. For all undefined terms one is referred to Harary [1969], Roberts [1976] and Golumbic [1980].

Graphs, digraphs and partial orders are all binary relations. A binary relation *R* on a set *A* is a subset of $A \times A$. A binary relation *(A,R)* is *irreflexive* if $\sim a\ R\ a$ for all $a \in A$. A binary relation *(A,R)* is *symmetric* if $a\ R\ b \Rightarrow b\ R\ a$ for all *a* and *b* in *A*. An undirected graph is a symmetric related set. A binary relation *(A,R)* is *transitive* if for all $a, b, c \in A$, $a\ R\ b$

and $b\ R\ c$ implies $a\ R\ c$. A binary relation R on A is a *partial order* if R is irreflexive and transitive. We refer to (A,R) as a poset (partially ordered set). For more background materials on relations see Fishburn [1985].

1.1.2 Notations

In general any graph theoretic notation is that of Harary [1969]. For convenience, the most frequently used notations are listed here.

$G(V,E)$	A graph whise vertex set is V and edge set is E
$D(V,E)$	A digraph whose vertex set is V and edge set is E
$\bar{D}$	Complement of D
Z_n	Cycle of length n
K_n	A complete graph with n vertices
$K_{m,n}$	Complete bipartite graph, wher size of the stable sets are m and n.
G^k	kth power of graph G.
$G_1 \cup G_2$	Union of G_1 and G_2
$\lfloor x \rfloor$	Greatest integer $\leq x$
$\lceil x \rceil$	Least integer $\geq x$
$\lvert S \rvert$	Cardinality of set S.

1.2 Intersection graphs

The study of intersection representations of graphs is one of the rich areas in graph theory. Given a finite family of sets $\mathcal{F}$, a graph $G(V,E)$ is an *intersection graph* of the family $\mathcal{F}$ if there exists a function that assigns to each vertex $v \in V(G)$ a set $S_v \in \mathcal{F}$ such that for all $u, v \in V(G)$,

$$uv \in E \Leftrightarrow S_u \cap S_v \neq \phi$$

Note that S_u and S_v may be equal even if $u \neq v$. If $G(V,E)$ is a graph that is (isomorphic to) the intersection graph of $\mathcal{F}$, then $\mathcal{F}$ is called a representation of G. It was shown by Marczewski [1945] that every graph is an intersection graph of some collection. This can be done by taking S_v to be the set of all edges in G incident to v.

Interesting problems arise only when the sets of the family $\mathcal{F}$ are restricted to some specifically defined family of sets. A substantial part of this vast topic of intersection graphs is devoted to answering this question when the sets in the family $\mathcal{F}$ are restricted to some collection such as subsets of a particular host set. An excellent introduction to the theory and problems of intersection graphs is given in Golumbic [1980].

1.3 Interval graphs

1.3.1 Interval graphs and their applications

Of all types of intersection graphs the first one which engaged attention was interval graph. The idea was first introduced by Hajös [1957]. This particular type of intersection graph is best known and well studied. An *interval graph* is the intersection graph of a family of intervals on the real line. The intervals may be open, closed or half-open. It can easily be seen that every graph is not an interval graph. Here one may ask, "Under what conditions each vertex $v \in V$ of a graph $G(V,E)$ can be associated with an interval I_v on the real line such that

$$uv \in E \Leftrightarrow I_u \cap I_v \neq \phi \text{ ? "}$$

This area of interval graphs having a long history has flourished due to intrinsic mathematical interests and a wide variety of applications. Benzer [1959], the renowned molecular biologist, during his investigation discovered the underlying linear arrangements in the fine structure of genes through overlaps caused by mutations. If the substructures studied in the tests be considered as vertices and edges correspond to pairs of overlapping substructures then the graph obtained is an interval graph. Since then interval graphs have been used in different real world problems.

Interval graphs have their applications in seriation by Kendall [1971], Hubert [1974], in archaeology by Skrien [1984b], in developmental psychology by Coombs and Smith [1973]. In arriving at solutions to general traffic phasing problems Stoffers [1968] and Roberts [1976, 1978, 1979c] used interval graphs. Another general type of problem whose solutions have incorporated interval graph is that of fleet maintenance (Golumbic [1980], Opsut and Roberts [1980]). Frequency assignment problem which is another application of interval graphs was studied by Gilbert [1972] and Pennotti [1976]. In the introduction to a special issue of Discrete Mathematics devoted to interval graphs and related topics Golumbic [1985] cites ten applications of interval graphs.

Nicholson [1992] applied interval graphs to computing a protein model. The augmented ribbon model for 3-dimensional protein structure is a labeled graph. An interval graph naturally arises in this model and provides an algorithm for effective solution.

1.3.2 Some characterizations of interval graphs

Interval graphs have several well-known characterizations. Hajös [1957] showed that the interval graphs are necessarily triangulated graphs. A graph is a *triangulated* graph (or chordal graph) if every cycle of length strictly greater than 3 possesses a chord. It is easy to verify that the simple cycle of length 4 (or the complete bipartite graph $K_{2,2}$) is the only four point graph that is not an interval graph. Lekkerkerker and Boland [1962] first characterized interval graphs with the help of an asteroidal triple. An *asteroidal triple* is a collection of three distinct vertices such that each pair is joined by some path having no vertex adjacent to the third.

Theorem 1.1 (Lekkerkerker and Boland [1962])

A graph G is an interval graph if and only if G is triangulated and contains no asteroidal triple.

In that paper they again provided a complete set of forbidden subgraphs for interval graphs.

The result of Lekkerkerker and Boland was generalized by R. Halin [1982] to infinite graphs.

Another characterization of interval graphs was given by Gilmore and Hoffman [1964] in which they relate interval graph to what is called a comparability graph. A digraph is *transitive* if whenever there is an arc from u to v and an arc from v to w, $u \neq w$, there is an arc from u to w. An *orientation* of a graph G is an assignment of a direction to each of the edges of the graph to get a digraph. A graph has a *transitive orientation* if there is an orientation of its edges so that the resulting digraph is transitive. Graphs which have transitive orientations are called *comparability graphs*. They are also known as *partially orderable graphs*. These graphs have been characterized by Ghouila-Houri [1962].

Theorem 1.2 (Gilmore-Hoffman [1964])

G is an interval graph if and only if it has no chordless cycle and its complement is a comparability graph.

A *clique* of a graph is a set of vertices which forms a complete induced subgraph. A clique is called *maximal* if it is not contained in any larger clique of the graph. An ordering $k_1, k_2, \ldots k_n$ of maximal cliques is consecutive if for $p<q<r$ and for any vertex v that belong to k_p and k_r, v belong to k_q.

Theorem 1.3 (Fulkerson and Gross [1965])

G is an interval graph if and only if the maximal cliques of G can be linearly ordered such that for every vertex v of G, the cliques containing v occur consecutively.

A (0,1) matrix is said to have a *consecutive 1's property* for rows if it is possible to permute the columns of the matrix so that ones in each row appear consecutively. Ryser [1969] studied consecutive ones property and certain generalizations. Tucker [1972] characterized the consecutive ones problem in terms of forbidden configuration.

The vertex-maximal clique incidence matrix $M = (m_{ij})$ of a graph is a (0,1) matrix whose rows and columns corresponds to the vertices and maximal cliques respectively of the graph and

$$m_{ij} = \begin{cases} 1, & \text{if the } i \text{ th vertex belongs to the } j \text{ th column} \\ 0, & \text{otherwise.} \end{cases}$$

In termes of this matrix M, the result of Fulkerson and gross [1965] can be stated as follows: *a graph G is an interval graph if and only if its vertex-maximal clique incidenced matrix has the consecutive ones property for rows.*

A matrix is said to be *quasi-diagonalizable* if for simultaneous row and column permutations, consecutive ones appear in each row, starting at the main diagonal. Mirkin [1972] showed that a graph is an interval graph if and only if its augmented adjacency matrix is quasi-diagonalizable.

Booth and Lueker [1976] proved that interval graphs can be recognized in linear time (for linear time algorithm see sec.1.9). An extensive survey of algorithmic aspects of interval graphs will be found in Golumbic [1980] and Möhring [1985]

Smadici [1987] defined a set N(G) corresponding to a graph G as the set of all edges of G with the property that for each end vertex of such an edge there exists a vertex adjacent to it and non-adjacent to other end vertex. He then used this notion to obtain yet another characterization of an interval graph.

1.3.3. Notions related to interval graphs

When the basic structural problems for interval graphs were solved, researchers began considering sets other than intervals for assignment to vertices. The sets used include rectangles, boxes, multiple-intervals, spheres, subtrees of a tree, convex sets etc. The use of more general sets in intersection representation has led to the introduction of parameters to measure the difficulty in representing a graph, such as by minimising the dimension of the sets used in the representation.

There are several graphs which are related to interval graphs in a natural way. One such well-known generalization of interval graphs arises when we associate to each of the vertices not a single interval but a finite set of intervals. This was introduced independently by Trotter and Harary [1979] and Griggs and West [1980]. A graph is called a *t-interval graph* if it is the intersection graph of, at most, t-intervals (t is a positive integer) on the real line. Any graph is a t-interval graph for some t. The *interval number* of a given graph G, denoted by $i(G)$, is the least t for which the given graph G is a t-interval graph. Note that a graph is an interval graph if and only if $i(G) = 1$. Griggs [1979] determined the sharp upper bound $\left\lceil \frac{n+1}{4} \right\rceil$ on the interval number of a graph on n vertices. Multiple interval graphs and interval numbers have also been studied by Harary and Kabell [1980], Hopkins and Trotter [1981], Scheinerman and West [1983], Hopkins, Trotter and West[1984], Scheinerman [1985a,b,1988] and Erdös and West [1985].

Cozzens [1981] defined a cointerval graph as the complement of an interval graph and gave a list of forbidden subgraphs characterizing cointerval graphs. The *line graph* $L(G)$ of a graph G is the intersection graph of the family of edges of G. Skrien [1984] characterized those graphs whose line graphs are interval graphs. Benzaken *et al.* [1985] characterized the class of interval graphs whose complements are also interval graphs.

Harary and McMorris [1987] again studied graphs and bigraphs having interval and co-interval properties. They showed that both a graph G and its complement $\overline{G}$ are interval graphs if and only if G has no induced subgraph isomorphic to any of alist of seven graphs. They also extended the question to bipartite graphs.

A more general family of interval graphs is that of chordal graphs, which are the graphs with no chordless cycle. Dirac [1961] characterized chordal graphs. Buneman [1974], Gavril [1974] characterized chordal graphs as precisely those graphs having an intersection representation in which each vertex is assigned a subtree of a host tree; this is called subtree representation. Rose *et.al* [1976] obtained an efficient algorithm for these graphs. Scheinerman [1988b] obtained interval number of a chordal graph. In-Jen Lin and West [1992] defined the leafage *L(G)* of a chordal graph *G* to be the minimum number of leaves (pendant vertices) in the host tree in any subtree representation of G. The interval graphs are chordal graphs with leafage 2, so this parameter is a measure of how far a chordal graph is from being an interval graph. They gave an algorithm to find the leafage of a chordal graph.

Wegner [1967] showed that every finite graph is representable as the intersection graph of a family of convex sets in 3-dimensional *Euclidean* space R^3. Roberts generalized interval graphs to higher dimensional interval graphs by considering boxes in an n-dimensional *Euclidean* space R^n. A box is a cartesian product of n intervals for some fixed n. Hence this is an intersection graph of a family of boxes with sides parallel to the coordinate axes in R^n.

A familiar real world illustration of box is what is frequently called in ecology the *ecological niche* of a species. For example, the normal healthy environment is determined by a range of values of temperature, of light, of pH, of moisture and so on. If there are n-factors in all, and each defines an interval of values then the box corresponds to ecological niche in n-space.

Any graph G with n vertices is an n-dimensional interval graph. This leads to consider the parameter *boxicity* of a graph G (denoted by $b(G)$). *Boxicity* of G is the smallest integer k such that G is representable as an intersection graph of boxes in R^k. Note that $b(G) \leq 1$ iff G is an interval graph. The graph Z_n is not an interval graph and hence its boxicity must be larger than 1. It is easy to show that $b(Z_n) = 2$, for all $n \geq 4$.

Roberts [1969b] showed that a graph G with n vertices has boxicity at most $\lfloor \frac{n}{2} \rfloor$. This result was rediscovered by Wittenshausen [1980]. Gabai [1974] presented a number of bounds on boxicity. Trotter [1979] characterized those graphs G on n vertices such that $b(G) = \lfloor \frac{n}{2} \rfloor$. Cozzens and Roberts [1983] showed that the problem of computing boxicity is NP-complete. (see section 1.9) Scheinerman [1984] proved that every planar graph has an intersection representation by sets, each of which is the union of two boxes in the plane. Thomassen [1986] showed that every planar graph is the intersection graph of a collection of three dimensional boxes, with intersections occurring only in the boundaries of the boxes. Quest and Wagner [1990] gave a characterization of graphs with $b(G) \leq 2$.

Cohen *et al.* [1979] using exact analysis, asymptotic theory and Monte Carlo simulation, estimated the probability of a random graph to be an interval graph.

The *interval count* of an interval graph is the minimum number of distinct lengths of intervals in the interval representation. Interval graphs with interval count 2 were studied by Leibowitz *et al.* [1982], Skrien [1984].

Scheinerman [1988a] introduced two equivalent models of random interval graphs. Several results about the number of edges, degrees, chromatic number and other indices of almost all interval graphs were also established.

For each graph theoretic property, Mckee [1991] defined a corresponding "intersection property", motivated by the natural relationship of paths with interval graphs and of trees with chordal graphs. He then devoloped a simple formal language, based on vertices and paths, which supports transfer of selected information about the original property to its intersection property. For instance, a simple description of paths produces the asteroidal triple characterization of interval graphs.

1.4 Indifference Graphs

Within the category of interval graphs there has been considerable study of several classes meeting additional restrictions; indifference graphs form one such class. This notion was introduced by Roberts [1969a] as a model for nontransitive indifference. An undirected graph $G(V,E)$ is an *indifference graph* if given $\delta > 0$, there exists a real valued function f on the vertices of G such that the vertices u and v are adjacent if and only if $| f(u) - f(v) | \leq \delta$ for all $u, v \in V$.

Roberts [1969a] characterized indifference graphs as equivalent to unit interval graphs and proper interval graphs. A graph G is a *unit interval graph* if it is the intersection graph of a set of closed intervals of unit length (like interval graphs, closedness and openness is not arbitrary for unit interval graphs and closed intervals are specified). A graph is a *proper interval graph* if it can be represented by a family of intervals on the real line such that no interval properly contains another.

Theorem 1.3 (Roberts [1969a]). *Suppose G is a graph. Then the following statements are equivalent.*

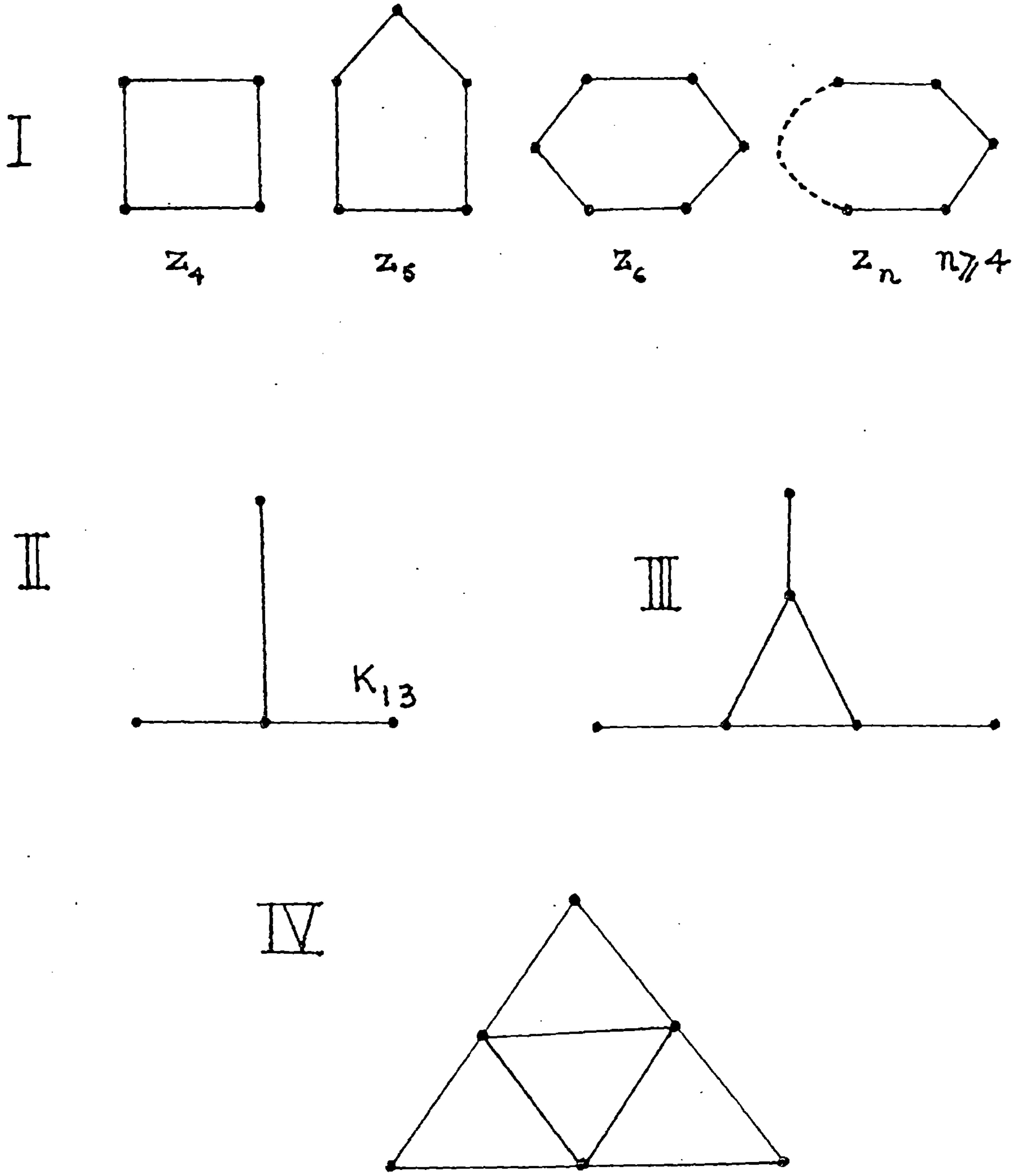
(a) *G is an indifference graph.*

(b) *G is a unit interval graph.*

(c) *G is a proper interval graph.*

(d) *G is an interval graph and does not contain $K_{1,3}$ as a generated subgraph.*

The class of graphs which do not contain subgraphs of types I - IV in fig 1.1 are called structured indifference graphs. Wegner [1967] proved that unit interval graphs are the same as the structured indifference graphs. Consequently Roberts [1969a]



FORBIDDEN SUBGRAPHS FOR INDIFFERENCE GRAPHS

FIG. 1.1

gave the following forbidden subgraph characterization of indifference graphs.

Theorem 1.4 (Roberts [1969a]) *A graph G is an indifference graph if and only if G does not contain any of the graphs depicted in fig 1.1 as a generated subgraph.*

Roberts introduced the notion of extreme point and characterized indifference graphs in terms of this. Some other characterizations of indifference graphs are similar to the results for interval graphs, viz., a reflexive graph is an indifference graph if and only if there is a proper consecutive linear ordering of its maximum cliques (Fishburn [1985]).

An ordering $x_1, x_2 \dots x_n$ of $V(G)$ is a compatible vertex ordering iff $i < j < k$ and $x_i x_k \in E(G) \Rightarrow x_i x_j \in E(G)$ and $x_j x_k \in E(G)$.

Theorem 1.5 (Roberts [1971]) *G is an indifference graph iff it admits a compatible vertex ordering.*

Maehara [1980] also did some work on this class of graphs under the name of time graphs.

Roychaudhury [1987] defined the *Kth* power of a graph as the graph G^k with the vertex set $V(G^k) = \{ v_1, v_2, \dots\dots v_n \}$ and also the edge set $E(G^k)$ such that $v_i v_j \in E(G^k)$ iff the distance $d_O(v_i, v_j) \leq k$. She then proved that it G^{k-1} is an interval (respectively unit interval) graph then G^k is an interval (respectively unit interval) graph.

Recollect that Lekkerkerker and Boland [1962] characterized interval graphs as chordal graphs which have no asteroidal triple. Proper interval graphs can also be characterized in an equivalent way. Three vertices x,y and z form an *astral triple* if between any two vertices there exists a path P such that the third vertex does not belong to P and no two consecutive vertices on P are both adjacent to it. Jackowski [1992] characterized proper interval graphs as graphs which have no astral triple.

A graph is a $\tilde{n}$- graph if we can asssign a set of n consecutive integers to each vertex so that edges correspond to pairs of sets which overlap. Every unit interval graph is an $\tilde{n}$-graph for some positive integer n and conversely every $\tilde{n}$ graph is unit interval. Sakai [1992] studied the problem of finding the minimum n such that a given unit interval graph is an $\tilde{n}$-graph. He also gave a linear time algorithm to compute this number in a particular case.

There are plenty of applications of indifference graphs. One of them is in the channel assignment problem in communication theory. A survey of this can be found in Hale [1980]. Hubert [1974] applied indifference graphs in solving problems in seriation. Indifference graphs also have applications in analyzing judgements of similarity or matching in psychology (Roberts [1970], Fishburn [1970]).

In chapter II, we study indifference representation of digraphs by using ordered pairs of real numbers and show how this concept of indifference digraphs generalizes that of indifference graphs.

1.5 Relation and Graphs

There exists intimate connections between interval graphs and interval orders, and between indifference graphs and semiorders. These connections are made explicit in the following subsections.

1.5.1 Interval order

Fishburn [1970] and Mirkin [1970] introduced independently the concept of interval orders, with the help of which they obtained several characterizations of interval graphs. A binary relation (A,P) is called an interval order it for all $a,b,c,d \in A$

the following conditions hold :

(S_1) P is irreflexive.

(S_2) aPb and $cPd \Rightarrow aPd$ or cPb.

Every interval order is transitive and is therefore a poset.

Fishburn characterized interval order and explored the close relationship between interval graphs and interval orders.

Theorem 1.6 (Fishburn [1970]) *A partial order (V,P) is an interval order if and only if the elements $v \in V$ can be represented by intervals I_v on the real line such that $uPv \Leftrightarrow I_u$ lies entirely to the left of I_v ($I_u < I_v$)*.

Theorem 1.7 (Fishburn [1970]) *A graph G (V,E) is an interval graph if and only if there exists a binary relation R on $V(G)$ such that (V,R) is an interval order and $\bar{E} = R \cup R^{-1}$ where $\bar{G}$ $(V,\bar{E})$ is the complement at the graph G.*

A relation (V,R) is known as Ferrers relation it the sets of successors $S(v) = \{ u \in V \,/\, v\,u \in R \}$, $v \in V$ (or equivalently the sets of predecessors $P(v) = \{ u \in V \,/\, u\,v \in R \}$, $v \in v)$ are linearly ordered by inclusion. Mirkin [1970] characterized interval orders by Ferrers relation as follows:-

Theorem 1.8 [Mirkin 1970] *An irreflexive relation is an interval order if and only if it is a Ferrers relation.*

The digraphs corresponding to Ferrers relation are known as *Ferrers digraphs*. As our work is very much related to the theory of Ferrers digraphs we shall discuss these digraphs in details in section 1.8.

1.5.2 Semiorders

Semiorders, originally introduced by Luce [1956], are special type of interval orders. Scott and Suppes [1958] defined semiorder in the following way: a binary relation (A,P) is called a *semiorder* if for all $a,b,c,d \in A$ (not necessarily distinct), the following conditions hold:

(S_1) P is irreflexive;

(S_2) aPb and $cPd \Rightarrow aPd$ or cPb

(S_3) aPb and $bPc \Rightarrow aPd$ or dPc

Note that by dropping the condition (S_3) we get an interval order. It must be mentioned here that Luce [1956] treated this concept of semiorder in somewhat different way. Scott and Suppes [1958], Scott [1964], Rabinovitch [1978] and Roberts [1979b] each have different proofs of the following celebrated representation theorem, which is called the Scott-Suppes Theorem.

Theorem 1.9 (Scott-Suppes [1958]) *Suppose P is a binary relation on a finite set A and δ is a positive number. Then there is a real valued function f on A such that*

$$xPy \Leftrightarrow f(x) > f(y) + \delta \qquad \ldots(1)$$

is satisfied if and only if (A,P) is a semiorder.

If a binary relation P on a finite set A is in the above form (1) for some positive number δ, then it is representable in the form (1) for any positive number δ. In particular it is representable with $\delta = 1$,

The relation (1) expresses P as a transitive orientation of the complement of an indifference graph, and so f is called a co-indifference representation.

As an immediate consequence of the Scott-Suppes Theorem [1958] on semiorder, Roberts rephrased the following characterization of indifference graphs.

Theorem 1.10 (Roberts[1969a]) *A graph with edge set E is an indifference graph if and only if there is a semiorder P such that $\bar{E} = P \cup P^{-1}$ where $\bar{E}$ is the complement of E and P^{-1} is the digraph obtained by reversing the edges in P.*

Interest in interval orders and semiorders in behavioral sciences arose from several sources such as preference comparisons in consumer economics. The name "Semiorder" was first used and precisely axiomatized by Luce [1956]. Nontransitive indifference arises from "the imperfect power of discrimination of the human mind whereby the inequalities become recognizable only when of sufficient magnitude." To use an example of Luce's, it is reasonable to suppose that a person will be indifferent between x and x + 1 grains of sugar in his coffee for x = 0,1..., yet have a definite preference between 0 grains and 5000 grains.

Ducamp and Falmagne [1969] introduced a generalization of semiorder, called bisemiorder. A *bisemiorder* is a quadruple (S,E,R,T) where S and E are two disjoint sets and R and T are two binary relations on $S \cup E$, both $R \subseteq S \times E$ and $T \subseteq S \times E$ satisfying certain axioms; it was shown that it is equivalent to finding functions $s : S \longrightarrow R$ and $e : E \longrightarrow R$ and two positive numbers δ_1, δ_2 ($\delta_1 > \delta_2$) such that for all $a \in S$, $b \in E$

$$a\ R\ b \iff S(a) > e(b) + \delta_1$$
$$a\ T\ b \iff S(a) > e(b) + \delta_2$$

The functions s and e provide a generalization of the guttman scale on two relations (Guttman [1944]). Ducamp [1978] using a method developed by Scott [1964] gave an alternate proof of the representation theorem for bisemiorders.

Cozzens and Roberts [1982] discussed *double semiorders* as a pair of binary relation P_1 and P_2 on the same set A and obtained necessary and sufficient conditions for the existence of a real valued function f on A and two positive numbers δ_1, δ_2(δ_1, δ_2) so that for all x, $y \in A$.

$$x\ P_1 y \iff f(x) > f(y) + \delta_1$$
$$x\ P_2 y \iff f(x) > f(y) + \delta_2$$

hold. For a graph theoretical version of the above problem, they again introduced the notion of double indifference graph, a multigraph (a pair of graphs with the same vertex set) of certain type. Translating the results of double semiorder they characterized double indifference graphs and obtained results analogous to Roberts [1969].

In Chapter II we will introduce generalized semiorder and obtain semiorder (as defined by Scott and Suppes [1958]) as its particular case. Then we will characterize indifference digraphs in terms of this generalized semiorder. In doing so we will generalize the celebrated Scott and Suppes theorem [1958].

1.6 Notions related to Unit interval graphs

The analogue to unit intervals in higher dimensional space is the unit boxes with sides parallel to co-ordinate axes. Precisely, a *unit box in* R^n is the n-certesian product $\overset{n}{\underset{i=1}{\times}}(J_i)$, where J_i's are unit intervals (open, closed or half-open) on the real line. The *cubicity* of a graph *G*, denoted by *C(G)*, is the least integer *k* such that *G* is the intersection graph of unit boxes in R^k. Note that *G* is the intersection graph of unit boxes in R^k iff *G* is the intersection of *K* unit interval graphs. Roberts [1969b] first introduced the notion of cubicity and proved that cubicity is well-defined for any graph *G*. In fact, he showed that if *C(G)* denote the cubicity of a graph *G*, then $C(G) \leq \lfloor 2n/3 \rfloor$, where *n* is the number of vertices in *G*. It can be seen that b(G) ≤ *c(G)* where *b(G)* denotes the boxicity of *G* (see section 1.3.3). Cozzens [1981] gave methods of calculating the cubicity of some classes of graphs.

Again one could extend unit intervals to unit spheres in Euclidean space R^n. Havel and Kuntz [1980] defined the (unit) sphericity of a graph *G* to be the least integer *K* such that *G* is the intersection graph of closed unit spheres in *K* dimensional space. Havel [1982] showed that there are graphs of sphericity 2 with arbitrary large cubicity whereas Fishburn [1983] showed that there are graphs of cubicity 2 and 3 with sphericity larger than cubicity. Maehara, [1984a, 1984b] used the term *space graph* to descirbe intersection graphs of spheres and derived certain bounds on the sphericity of several classes of graphs.

Sen [1984] studied intersection graphs of subcubes of a unit cube Q_n and proved that any graph G is the intersection graph of a family of subcubes of a unit cube Q_k. He introduced the concept of *n-index* of G analogous to boxicity, which is the minimum k such that the graph G is the intersection graph of subcubes of Q_k. He also proved that n-index of a graph G is equal to the minimum number of complete bipartite subgraphs of $\bar{G}$ whose union is $\bar{G}$. This is analogous to the works on boxicity by Cozzens and Roberts [1983] who proved that the boxicity of a graph G is equal to the minimum number of interval graphs whose intersection is G. In terms of the complement, this means that boxicity of a graph G is equal to the minimum number of cointerval graphs whose union is $\bar{G}$. Sen [1984] has replaced these cointerval graphs by complete bipartite subgraphs in his work and this provides a setting in which n-index of a graph is perhaps more natural than boxicity.

Cozzens and Roberts [1989] introduced the notion of a dimensional property of graphs. A *dimensional property* of graphs is a property P such that every graph G is the intersection graph having property *P*. For a dimensional property *P*, they described a general method for computing the least integer *K* so that *G* is the intersection of *K* graphs having property *P*. They then gave simple applications of the method to compute the boxicity, the cubicity and a number of such other dimensions of a graph.

Analogous to the work done by Griggs [1979] on interval number of multiple graphs, Andreae [1990] obtained results on unit-interval number. The *unit interval number* denotes the minimum number t such that one can assign to each vertex of a simple graph *G* a collection of at most *t*-unit intervals on the real line where two vertices *v* and *w* in *G* are adjacent if and only if some interval for *v* intersects some interval for *w*. Andreae obtained the extremal graphs by devising an elaborate construction using fewer intervals for non-extremal graphs. For $n = 2k \geq 6$, the unique extremal graph is the star $K_{1,2k-1}$ while for $n = 2k + 1 \geq 7$, the extremal graphs are those that contain an induced $K_{1,2k-1}$.

1.7 Circular arc graphs

Geometrical interests extend the class of interval graphs to another useful class of intersection graphs which is known as circular arc graphs. A graph is a *circular arc graph* if it is the intersection graph of a family of arcs on a circle. Tucker [1971, 1974, 1978, 1980] did extensive work on circular arc graphs to solve the problems of characterization and recognition algorithm. He also obtained a structure theorem for some special types of circular arc graphs, viz, *unit circular arc graphs* in which the arcs are of unit length and *proper circular arc graphs* in which no arc properly contains another. It is to be noted that these two concepts do not coincide here as they would for interval graphs.

Gavril [1974] defined some subclasses of circular arc graphs and succeeded in characterizing these classes, viz, Δ-Circular arc graphs and Θ circular arc graphs. A graph is a *Δ-circular arc graph* if it is the intersection graph of a family of arcs on a circle, so that for three arcs, if every pair intersects then the intersection of the three arcs is non-empty. A graph is a *Θ-circular arc graph* if for every clique, the intersection of the arcs corresponding to the vertices of the clique is non-empty. Note that a Θ-circular arc graph is a Δ-circular arc graph but the converse is not true. Moreover, she provided efficient algorithms for recognizing these two classes and for finding a maximum clique, a maximum independent set and a minimum covering by cliques of circular arc graphs.

As in the case of interval graphs, circular-arc graphs were also generalized in n-dimensional space R^n. Feinberg [1979] showed that any graph G is the interrsection graph of the products of arcs on a sphere, called 'patches' on a sphere. He then used the notion of *Circular dimension* of a graph G as the least integer K such that G is the intersection graph of patches on K-spheres.

Maehara [1990] considered the asymptotic behaviour of the intersection graph G of n arcs as $n \longrightarrow \infty$ and finds the threshold functions for connected subgraph, for complete subgraph, for isolated vertices and for connectivity.

Circular arc graphs have important applications in testing for circular arrangement of genetic molecules (Stahl [1967]), in designing traffic signals (Stoffers [1968]) in circular indifference situations such as colour wheels and musical tones (Luce [1971], Hubert [1974]).

Also these graphs have found important applications in computer science for designing a compiler or any other basic software system (Tucker [1978]).

1.8 Ferrers digraphs and Ferrers dimension

A particular class of digraphs, known as Ferrers digraphs, was introduced independently by Guttman [1944] and Riguet [1951]. Riguet defined *Ferrers digraphs* to be a digraph $D(V,E)$ in which for all x, y, z and $w \in V$,

$xy,\ zw \in E \Leftrightarrow xw \in E$ or $zy \in E$ (inclusive) (vertices need not be distinct). Riguet [1951] characterized Ferrers digraphs as those whose successor sets (equivalently predecessor sets) are linearly ordered by inclusion. In other words, if the rows of the adjacency matrix are indexed by vertices in decreasing order of out degree and the columns by vertices in decreasing order of indegree, then the rearranged adjacency matrix takes the form

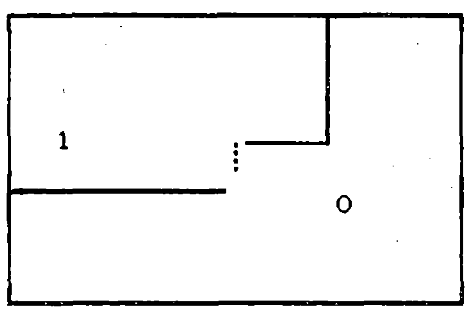


Fig. 1.2

The representation of 1'S (or 0's) in the above form will be referred to as a Ferrers diagram. Riguet showed that D is a Ferrers digraph if and only if its adjacency matrix *A(D)* has no 2 by 2 sub matrix that is a permutation matrix.

Threshold graphs were introduced by Chvatal and Hammer [1977]. A graph *G(V,E)* is a *threshold graph* if there exists a real mapping $f: X \longrightarrow R$ and a threshold $t \in R$ such that for every $S \subseteq X$

$$S \text{ is stable} \Leftrightarrow \sum_{x \in S} V(X) \leq t$$

For an excellent survey of these two topics of Ferrers digraphs and threshold graphs see Cogis [1982a]. These two concepts look completely different, still they are related to each other in a peculiar way. Cogis [1982a] again discovered that the underlying graphs of symmetric Ferrers digraphs are precisely the threshold graphs. For each characterization of Ferrers digraphs, there is a corresponding characterization of threshold graphs and vice-versa.

Intersection of Ferrers digraphs was studied by Bouchet [1971] and he showed that any digraph *D* is the intersection of a family of Ferrers digraphs containing *D*. This induces one to introduce the important notion of Ferrers dimension.The *Ferrers dimension* (F.D) of a digraph *D* is the minimum number of Ferrers digraphs whose intersection is *D*. Bouchet [1971, 1984] also obtained several interesting results on Ferrers dimension.

The digraphs with F.D.2 were characterized independently by Cogis [1979, 1982b] and also by Doignon *et al* [1984]. Another characterization of digraphs with F.D.2 was given by Sen *et al.* [1989 a] in terms of their adjacency matrices.

Theorem 1.11 (Cogis [1979], Sen et al [1989a])

The following conditions are equivalent :

(A) *D has Ferrers dimension at most 2.*

(B) *The rows and columns of A(D) can be (independently) permuted so that no 0 has a 1 both below it and to its right.*

(C) *The graph H(D) of couples in D is bipartite.*

The *order dimension* of a partial order was introduced by Dushnik and Miller [1941]. It is the minimum number of linear orders whose intersection is *G*. Bouchet [1971] showed that the notion of Ferrers dimension of a Ferrers digraph is an extension of the order dimension of partial order. Cogis [1982a] gave another proof of this result for finite graphs. Doignon *et al* [1984] extended this result to infinite graphs. Yannakakis [1982] showed that the problem of designing efficient algorithms for order dimension exceeding 2 is NP-Complete, although 2-dimensional posets are polynomially recognizable.

A very good summary of the notion of order dimension and analogous parameters for graphs and directed graphs such as boxicity, threshold dimension and Ferrers dimension to name only a few is to be found in a review paper by West [1985].

In Chapte IV we will introduce containment digraphs and show that this class of digraphs is again equivalent to digraphs of Ferrers dimension 2.

1.9. Efficiency of algorithms

The concept of an algorithm is one of the basic concepts in Mathematics. An *algorithm* is a finite set of rules, which gives a sequence of operations for solving a specific type of problem. The *time complexity* is a measure of time required to execute an algorithm. For a given graph theory problem it would be desirable to have an algorithm which guarantees a solution in an execution time proportional to some constant power of n or e (as usual, n and e are the number of vertices and edges respectively in a given graph). If an algorithm processes inputs of size n in time $cf(n)$ for some constant c, then the *time complexity* of that algorithm is said to be $O(f(n))$. An algorithm whose computation time is bounded by a polynomial in n or e is called a *polynomial bounded* algorithm. Note that the bounds in terms of e and n are convertible into each other. The class of problems solvable by polynomial algorithm is usually denoted by P. There are graph theoretic problems for which it is simply not possible to have polynomial bounded algorithm. Again there is another category of problems for which so far no polynomial bounded algorithms have been discovered nor has it been possible to show that polynomial bounded algorithms do not exists for these problems. The class of these problems is denoted by NP.

There is a subclass of NP called NP-*complete problems.* The theory of NP - completeness was initiated by Stephen Cook [1971] and after that Richard Karp [1972] presented a large collection of NP - complete problems. These problems have the property that all known algorithms for solving them require exponential time. NP - Complete problems are important because these problems have the property that if a polynomial - bounded algorithm exists for one, polynomial - bounded algorithm can be found for the others, whereas if it can be proved that no polynomial algorithm is possible for any of them, then the same will be true for all of them. Indeed the collection of NP - Complete problems is growing regularly with time. A detailed discussion on complexity theory may be found in Garey and Johnson [1979], Stinson [1985].

Below we indicate only a few recent results in this area of graph algorithm and complexity.

The isomorphism problem for chordal graphs is as hard as that for simple undirected graph. However, it can be solved efficiently when restricted to the following classes among several others : trees, planar graphs and interval graphs. Veeraraghavulu *et al*. [1991] introduced a new sub - class of chordal graphs and developed a linear time isomorphism testing algorithm.

Very recently, Damaschke [1993] provided a foundation for obtaining polynomial algorithms for several problems concerning paths in interval graphs and interval models, such as finding hamiltonian paths and circuits and partitions into paths. As a main result, he created an algorithm for finding Hamiltonian paths in circular arc graphs which runs in time $O(n^5)$.

Shih *et al.* [1992] however presented a more efficient algorithm for Hamiltonian cycle problem on circular arc graphs running in $O(n^2 \log n)$ time.

1.10 Intersection representations for digraphs

1.10.1 Interval Catch digraphs

The concepts of intersection graph and interval graph have been well studied for undirected graphs. Now the family of sets in intersection graphs are replaced by a family of pointed sets $\{(S_i, b_i)\}$ with a distingushed base point $b_i \in S_i$ for each $1 \leq i \leq n$. Then we get the family of catch graphs which also have $\{v_1, v_2, \ldots, v_n\}$ as vertex set, but now v_i and v_j are adjacent if and only if either $b_i \in S_j$ or $b_j \in S_i$. But it seems more natural to study catch digraphs by a suitable restriction of the above condition.

Catch digraphs were studied by Maehara [1984]. Let $\mathcal{F} = \{(S_x, p_x), x \in V\}$ be a family of pointed sets. The *catch digraph* of $\mathcal{F}$ is the digraph where two distinct vertices x and y are joined by an edge iff $p_y \in S_x$. When the sets are replaced by intervals on the real line we get *interval catch digraph.* He also studied the catch digraphs of pointed convex sets, pointed boxes and pointed (solid) spheres in Euclidean space. Prisner [1989] characterized interval catch digraphs in a way which is quite analogous to Lekkerkerker - Boland [1962] characterization of interval graphs. As x-y chain is called z- avoiding if no initial end point of an arc of the chain precedes z. Three vertices a_1, a_2, a_3 belonging to V form a *diasteroidal triple* if for every permutation Π of (1,2,3) there is $a_{\pi(1)}$ - avoiding $a_{\pi(2)} - a_{\pi(3)}$ chain in D. With this definition of diasteriodal triple he obtained the following result :

Theorem 1.12 (Prisner [1989]) *A digraph is an interval catch digraph if and only if it has no diasteriodal triple.*

For any digraph $D = (V, A)$ one can obtain the underlying graph of D denoted by $U(D)$ where

$U(D) = (V, \{xy / xy$ or $yx \in A)$. Ogden and Roberts [1970] showed that a symmetric finite digraph D is an interval catch digraph if and only if its underlying graph $U(D)$ is an indifference graph.

1.10.2 Interval digraphs.

Beineke and Zamfirescu [1982] and Sen *et al.* [1989a] studied in different contexts a natural analogue of intersection models for digraphs. In case of digraphs the distinction between heads and tails of edges is crucial. Thus they considered a family of ordered pairs of sets and to each ordered pair assigned a vertex. An *intersection representation* of a digraph $D = (V,E)$ is a family $\{ (S_u, T_u): u \in V \}$ of ordered pairs of sets, such that $(u,v) \in E$ if and only it $S_u \cap T_v \neq \phi$. S_u is called the source set and T_u the sink (terminal) set for $u \in V$ and D is the intersection digraph of the family. Here loops are allowed but not multiple arcs. Every finite directed graph is an intersection digraph of finite sets. Recall that analogous result also exists for intersection graph, (Marczewski [1945]). Also the intersection number of digraphs has an analogous characterization obtained earlier by Erdös *et al.* [1966] for intersection graphs.

An *interval digraph* is an intersection digraph of a family of ordered pairs of intervals on the real line. These digraphs were characterized by Sen *et al* [1989b] A *generalized complete bipartite subdigraph* (abbreviated GBS) is a subdigraph generated by vertex sets X,Y whose edges are all xy such that $x \in X$, $y \in Y$. It is called generalized because X,Y need not be disjoint, which means that loops may arise. Let $\mathcal{B} = \{ (X_k, Y_k) \}$ be a collection of GBS's whose union is *D*. The *vertex-source incidence matrix* for $\mathcal{B}$ (abbreviated *V*,*X*-matrix) is the incidence matrix between the vertices and the source sets $\{X_k\}$. Similarly, the vertex-terminus incidence matrix for $\mathcal{B}$ (abbreviated *V*,*Y*-matrix) is the incidence matrix between the vertices and the terminal sets $\{ Y_k \}$. A characterization for interval digraph which is analogous to the Fulkenson and Gross [1965] characterization for interval graph is given below.

Theorem 1.13 (Sen *et al.*[1989a]. *D is an interval digraph if and only if there is a numbering of the GBS's in some covering* $\mathcal{B}$ *of D such that the ones in rows appear consecutively for both the V,X-matrix and V,Y-matrix of D.*

As a particular case of interval digraphs they introduced *interval-point digraph. An interval-point digraph* is an interval digraph where terminal intervals are singleton points. A digraph is an interval-point digraph if and only if its adjacency matrix has the consecutive ones property for rows. This result extends the corresponding result for interval catch digraphs by Maehara [1984] where the terminal point has been restricted to be a member of the source interval.

Sen *et al.* [1989a] again gave a characterization of an interval digraph as follows:

Theorem 1.14 (Sen *et al* [1989a]) *For a digraph D, the following statements are equivalent:*

1) D is an interval digraph;

2) $\bar{D}$ is the union of two disjoint Ferrers digraphs where $\bar{D}$ is the complement of D.

3) the rows and columns of A(D) can be permuted independently so that each 0 is labelled by an R or C in such a way that every position to the right of an R is an R and every position below a C is a C.

An adjacency matrix satisfying condition 3) above is called *Zero-partitionable.*

Recall that the Ferrers dimension of *D* is the minimum number of Ferrers digraphs whose intersection is *D*. The above theorem implies that an interval digraph is necessarily of F.D. at most 2. But this is not a sufficient condition. Sen *et al* in the same paper presented a 7-vertex digraph of Ferrers dimension 2 that is not an interval digraph, and thereby showed that interval digraphs are properly contained in the set of digraphs with Ferrers dimension at most 2.

In order to characterize a digraph *D* of Ferrers dimension 2, Cogis [1979] associated an undirected graph *H(D)* with *D* in a suitable way, the vertices of *H(D)* corresponding to the zeroes of the adjacency matrix of D. He proved that D has Ferrers dimension at most 2 if and only if $H(D)$ is bipartite.

Das and Sen [1993] introduced the notion of *interior edges* of the two Ferrers digraphs whose union is $\bar{D}$. Depending on the characterization of Cogis they used this concept of interior edge to obtain some properties of a digraph of F.D 2 and then showed how the notion of interior edges is related to an interval digraph.

In-Jen Lin *et al* [preprint] proved that any digraph has a subtree representation. This induced them to define and work on a new parameter, $L(D)$ the leafage at D. The *leafage* of a digraph is the minimum number of leaves (*i.e.*pendant vertices) in a host tree T in any subtree representation of a digraph. To be noted that , the interval digraphs are precisely the digraphs with leafage 2.

A subclass of interval digraphs is interval nest digraph. An *interval nest digraph* is an interval digraph where each terminal interval is contained in the corresponding source interval. Note that when all the terminal intervals are singleton points, an interval nest digraph becomes an interval catch digraph.

A subset K of a vertex set V of a digraph D is called *independent* it for any two vertices $x,y \in K$ $(x \neq y)$, neither xy

nor yx is an edge. An independent set K is called a *Kernel* of D it for any other vertex $Z \in V \setminus K$, there is some edge from Z to some vertex of K. D is called *Kernel-perfect* if any induced subdigraph has a Kernel. Galeuna-Sanchez and Neumann-Lara [1984] had some results on Kernels. Prisner [preprint] proved that finite interval nest digraphs and reversals of finite interval nest digraphs are Kernel-perfect and Kernels can be found in $O(n^2)$ time if a representation is given. This result is no more true for interval point digraphs.

Based on powers of digraphs, Prisner(personal communication) obtained a theorem which holds good for interval nest digraphs and interval catch digraphs.

Theorem 1.15 (prisner [preprint]) *for any integer* $K \geq 2$ *and every digraph* $D = (V,A)$, *if* D^{k-1} *is an interval nest digraph (interval catch digraph) then* D^k *is again an interval nest digraph (interval catch digraph).*

1.10.3 Circular arc digraph

Circular arc graph is a natural topological extension of interval graph. Sen *et al* [1989b] introduced and characterized the digraph representation of circular arc which can also be viewed as a normal extension of interval digraph. A digraph is a *circular arc digraph* if it is the intersection digraph of a family of ordered pairs of arcs on a circle.

A 0,1-matrix has the *circular ones property* for rows [columns] if its rows [columns] can be permuted so that the 1's in each row [column] are circular. By 1's being "circular" it means that their positions are consecutive when one views the positions as a cycle, with the first row (or column) following the last. Stair partition and generalized circular ones property are additional concepts required to characterize circular arc digraphs. A *stair partition* of a matrix is a partition of its positions into two sets (L,U) by a polygonal path from the upper left to the lower right, such that the set L is closed under leftward or downward movement, and the set U is closed under rightward or upward movement. Next let a 0,1-matrix A and a stair partition (L,U) is given. Also let V_i $[W_j]$ be the 1's in row i [column j] that begin at the stair and continue rightward [downward] (around if possible) until the first 0 is reached. Then A has the generalized *circular ones* property if it has a stair partition (L,U) such that the V_i's and W_j's together cover all 1's of A. The following result for circular arc digraph is analogous to Tucker's characterization of circular arc graph using quasi-circular ones property.

Theorem 1.16 [Sen *et al* 1989b] *A digraph is a circular arc digraph it and only if the rows and columns of its adjacency matrix can be permuted independently so that the resulting matrix has the generalized circular ones property.*

Recall the definition of Ferrers digraphs and Ferrers dimension given in section 1.8. It has been already stated that all interval digraphs have Ferrers dimension at most 2. However there is no such bounds for circular arc digraph. In fact there exists circular arc digraphs of arbitrarily high Ferrers dimension. On the other hand there exists digraph of Ferrers dimension 2 that is not circular arc digraph. They proved that the complement of any digraph with Ferrers dimension at most 2 is a circular arc digraph but the complement of a circular arc digraph need not be a circular arc digraph.

In chapter IV we define containment digraph and show that complement of a circular arc digraph is in fact a circular arc containment digraph and this answers the question raised in the earlier paper.

1.10.4 Interval acyclic digraph

Harary, Kabell and McMorris [1990] introduced another model for obtaining digraph from sets. Their definition yields only the acyclic digraphs but allows for very simple characterizations when representing sets are intervals. Let $S_1, \ldots, S_n$ be the subsets of a poset $P(X,<)$ for which $Inf(S_i)$ exists for all $i = 1,\ldots,n$. The *intersection acyclic digraph* is the digraph $D = (V,A)$ where $V = \{v_1,\ldots,v_n\}$ and $v_i v_j \in A$ it and only if $S_i \cap S_j \neq \phi$ and $Inf(S_i) < Inf(S_j)$. They showed that a digraph is acyclic if and only if it is an intersection acyclic digraph. When the partially ordered set is the set of real numbers with the usual ordering and each set S_i is an open interval, then the corresponding intersection acyclic digraph is an *interval acyclic digraph.* They proved that a digraph is an interval acyclic digraph if and only if there exists a labelling of the vertices $v_1 \ldots v_n$ such that $v_i v_j \in A$ then $i < j$ for all k with $i < k < j, v_i v_k \in A$. If all the intervals in an interval acyclic digraph are of unit length then it is a *unit-interval acyclic digraph.* They studied these digraphs and obtained forbidden subdigraph characterization.

In a separate paper they again characterized those digraphs that are the acyclic intersection digraphs of subtrees of a directed tree. Instead of taking directed trees, McMorris and Mülder [preprint] started with subpaths of a directed tree and proceeding similarly, they characterized digraphs that are called subpath acyclic digraphs.

1.11 Overlap graphs and Containment graphs.

It is not very difficult to conceive that intersection is only one type of interaction between several objects of similar type (from the real world). So several graph theorists started thinking of other type of interactions which might produce different classes of graphs and their corresponding theories. In this section graphs obtained by interactions other than intersection will be discussed. It is to be noted that some of these graphs eventually become intersection graphs of certain families.

Even and Itai [1971] introduced the concept of overlap graphs. A graph $G(V,E)$ is an *overlap graph* if its vertices can be put into a one-one correspondence with a collection of intervals on the real line in such a way that two vertices are adjacent if the corresponding intervals intersect but neither contains the other. Overlap graphs were studied by Gavril [1973], Fournier [1978] and Buckingham [1980]. It turns out, nevertheless, that this class of graphs is exactly the same as the class of *circle graphs*, the intersection graphs of a finite collection of chords on a circle.

Golumbic [1984,1985] studied another type of graphs called containment graphs. Let $\mathcal{F}$ be a family of non-empty sets. A simple finite graph $G = (V,E)$ is a containment graph provided one can assign to each vertex $v_i \in V$ a set $S_i \in \mathcal{F}$ such that $v_i v_j \in E$ if and only if $S_i \subset S_j$ or $S_i \supset S_j$. The function f: $v \longrightarrow \mathcal{F}$ which assigns sets of $\mathcal{F}$ to elements of V by $f(v_i) = S_i$ is called a $\mathcal{F}$ containment representation for G. Golumbic and Scheinerman [1985] defined containment graphs in terms of partially ordered sets and showed that the class of containment graphs is equivalent to the class of all comparability graphs.

As in the case of intersection graph one can restrict the family of sets $\mathcal{F}$ to a certain type (such as intervals on a line, circles in a plane, paths in a tree etc) and then ask for characterizing the nature of graphs so obtained. Dusknik and Miller [1941] characterized interval containment graph and obtained the following result.

Theorem 1.17 (Dusknik and Miller [1941]) *The following conditions are equivalent*

(i) *A graph G is a containment graph of intervals on a line.*

(ii) *G is a comparability graph Of a poset of dimension at most 2.*

(iii) *G and its complement $\overline{G}$ are both comparability graphs.*

The graphs which satisfy the preceeding theorem are equivalent to the class of permutation graphs (Even *et al* [1972]). For permutation graphs see section 1.12.

A *box containment graph* in d-space is a containment graph of rectilinear boxes (with sides parallel to the axes) in d-dimentional Euclidean space. Golumbic and Scheinerman [1985] proved the analogue of above theorem and showed that G is a containment graph of boxes in d-space if and only if G is the intersection of $2d-1$ interval containment graphs. As a corollary they showed that every comparability graph on n vertices is the containment graph of boxes in $\lceil \frac{n}{4} \rceil$ dimensional Euclidean space.

Mckee [1992a] showed that large parts of graph theory can be devoloped in terms of intersection graphs or other families of graphs (containment graphs among others) which have properties resembling those of intersection graphs. He introduced a notion called "connection graphs" and examined how much of the resemblance can be captured by this general notion.

In chapter IV , we study the nature of digraphs by choosing models other than intersection. We introduce the notion of containment digraphs and overlap digraphs and characterize them considering the sets to be intervals on a line or arcs of a circle.

In Chapter V , we use the notion of a base interval; it is an ordered pair (S_v , ρ_v) where S_v is an interval on the real line and p_v is a point of S_v. We associate with every vertex an ordered pair of base intervals. Relying heavily on this notion we obtain an analogous concept of intersection and overlap representation and characterize these digraphs.

1.12. Miscellaneous.

An undirected graph $G = (V,E)$ on $V = \{v_1 ....v_n\}$ is called a *permutation graph* if there is a labelling $l:v \longrightarrow \{1,2......n\}$ and a permutation π of $\{1,2......n\}$ such that $v_i v_j \in E$ if and only if $l(v_i) < l(v_j)$ and $l(v_j)$ preceeds $l(v_i)$ in π. The function l is called a permutation labelling of G and π is called the defining permutation. Permutation graphs are comparability graphs of posets of dimension at most two. Permutation graphs have many known characterizations and efficient recognition algorithms (Spinrad [1985]).

Golumbic and Monma [1982] introduced the concept of measured interaction and defined tolerance graphs to generalize both interval graphs and permutation graphs. A graph $G(V,E)$ is a *tolerance graph* if there exists assignment of intervals I_v and tolerance t_v to each vertex v such that

$$uv \in E \Leftrightarrow | I_u \cap I_v | \geq \min \{ t_u, t_v \},$$

where $| I |$ is the length of the interval I. They noted that the requirement of equal tolerances for all vertices yields, precisely, the interval graphs, while requiring $t_x = | I_x |$

yields the permutation graphs (Golumbic *et al.*[1984b]). Algorithms to compute the stability number, the chromatic number and the clique cover number of a tolerance graph were presented by Narasimhan and Manber [1992].

The question of "unit" and "proper" intersection representations also arise with *(interval) tolerance representations.* By analogy with interval graphs, *proper tolerance* graphs have tolerance representations in which no interval contains another and the *unit tolerance graphs* have tolerance representations with intervals of equal length. Any unit tolerance graph is a proper tolerance graph and any proper tolerance graph is a bounded tolerance graph.

Bogart *et al.*[1992] proved that the class of proper interval tolerance graphs is not equivalent to the class of unit interval tolerance graphs. In fact they presented a proper interval tolerance graph that is not a unit tolerance graph.

Competition graphs were introduced by Cohen [1968]. The *competition graph* G of a digraph $D = (V,A)$ has $V(G) = V(D)$ and $xy \in E(G)$, $(x \neq y)$ iff for some z in $V(D)$, xy and $yz \in A(D)$. Lundgren *et al.* [1993] obtained a necessary and sufficient condition for the competition graph of a loopless symmetric digraph to be an interval graph or unit interval graph.