Chapter 3

ULTRAMETRIC CANTOR SET

3.1 Introduction

As already mentioned, a main objective of the present work is to develop a non-archimedean framework [20] of a scale invariant analysis which will be naturally relevant on a Cantor set [24, 25, 26]. Since a Cantor set is compact, perfect, totally disconnected set, the conventional framework of real analysis is known to break down. For definiteness, we consider the Cantor set C to be a subset of the unit interval $I = [0, 1]$. We introduce a non-archimedean absolute value on C exploiting a concept of relative infinitesimals which correspond to the arbitrarily small elements $\tilde{x}$ of $I \backslash C$ satisfying $0 < \tilde{x} < \epsilon < x$, $\epsilon \to 0^+$ (together with an inversion rule) relative to the scale ϵ for a given $x \in C$ close to 0. As a consequence, increments among infinitesimals as well as between an infinitesimal and a (real) point of the Cantor set are accomplished by *inversions*, rather than by *translations* that is generally considered in standard real analysis. One of the main results in this chapter is the relationship of the nontrivial valuation with a Cantor function. Indeed, it is proved that the valuation is indeed given by an appropriate Cantor function. In ref.[24], we presented the details of the construction in the light of the middle third Cantor set. In ref.[25], the analysis is extended to a more general class of homogenous

Cantor sets.

We first introduce the definitions of relative infinitesimals, scale invariant infinitesimals and the associated class of inequivalent ultrametrics [24, 25, 26] and discuss the salient properties, mainly, in the context of a class of homogeneous Cantor sets. Next, we discuss and expose the relation of valuation with a Cantor function that arise naturally in the present context. At the next step, we present the arguments extending the ultrametric structure on the scale invariant infinitesimals over the whole Cantor set C. In the next subsection, we present the results on *valued measure* that can be defined on C. We show the valued measure of C gives rise to directly the finite Hausdorff measure of C. The nontrivial convergence of sequences of the form ϵ^{n-nl}, $0 < \epsilon,\ l < 1$ are treated subsequently. The usual limit 0 is replaced by the constant l in the present ultrametrics. This establishes the *metric as well as the topological inequivalence of these scale invariant ultrametrics.* The final subsection contains a discussion of differentiability that could be defined rigorously in the present ultrametric framework.

3.2 Non-archimedean Analysis: Ultrametrics

3.2.1 Absolute Value

Definition 5. *[26] Given an arbitrarily small $x \in C - \{0\}$ (in the sense that $x \to 0^+$ on $C - \{0\}$), $\exists\ \epsilon \in I$ and $\epsilon < x$ and an open interval $\tilde{I} \subset (0, \epsilon)$ such that $\tilde{I} \cap C = \Phi$, the null set. This follows from the total disconnectedness of C. An element $\tilde{x}$ in $\tilde{I}$ satisfying $0 < \tilde{x} < \epsilon < x$ and* *the* inversion *rule*

$$\frac{\tilde{x}}{\epsilon} = \lambda(\epsilon)\frac{\epsilon}{x} \qquad (3.1)$$

for a real constant $\lambda = \lambda(\epsilon)$ $(0 \ll \lambda(\epsilon) \leq 1)$ *is called a* relative infinitesimal *relative to the* scale ϵ. *The infinitesimal gap* $O_{\text{inf}} \subset \tilde{I}$ *is, by definition, the set of these relative infinitesimals satisfying the inversion rule (3.1), as* $\epsilon \to 0^+$, *in an asymptotic sense.*

Definition 6. *The non-empty set* $\tilde{O}_{\text{inf}} = \lim_{\epsilon \to 0^+} \{ \frac{\tilde{x}}{\epsilon} \}$, *is called the set of (positive) scale free (invariant) infinitesimals.*

Remark 1. Any finite (nonzero) $x \in C$ may be said to carry the trivial scale of 1 (unit, say). Nontrivial scales ϵ are said to emerge when $x \to 0^+$. Idea is that as x gets smaller and smaller in the sense that x becomes smaller than any preassigend positive number δ, and so becomes indistinguishable from 0 from the point of view of the unit scale 1, the Definition 5 now gives us a mechanism of zooming out the classical $\delta-$neighbourhood of 0 and then identifying it as an ultrametric neighbourhood (see below). The consideration of asymptotic limit as $\epsilon \to 0^+$ for a fixed but, nevertheless, arbitrarily small $x > 0$ allows one to erase traces of possible spurious (trivial) scales and also to probe and analyze nontrivial effects, if any, over and above the standard classical analytic results (cf, Remark 4(1)). Note that the framework of the classical analysis (calculus) does not naturally accommodate a scale. Nontrivial scales, however, arise in the context of a Cantor set. The new class of nontrivial scales now provides a new tool to explore the rich analytic and geometric structures of such a Cantor set.

Definition 7. *To each $x \in C$, $\exists$ an arbitrarily small $\epsilon > 0$ and a (relative) infinitesimal neighbourhood $\mathbf{I}_\epsilon(x) = (x - \epsilon, x + \epsilon) \subset I$, $x \neq 0, 1$ such that $C \cap \mathbf{I}_\epsilon(x) = \{x\}$. Points in $\mathbf{I}_\epsilon(x)$ are called the relative infinitesimal neighbours in I of $x \in C$.*

Remark 2. For each choice of x and ϵ, we have a unique $\tilde{x}$ for a given $\lambda \in (0, 1)$. Consequently, by varying λ in an open subinterval of (0,1), we get an open interval of relative infinitesimals in the interval $(0, \epsilon)$, all of which are related to x by the inversion formula. In the limit $\epsilon \to 0^+$, $O_{\text{inf}} = \Phi$, in the usual topology. However, the corresponding set of *scale invariant infinitesimals* $\tilde{O}_{\text{inf}} = \lim_{\epsilon \to 0^+} \{\tilde{X} \mid \tilde{X} = \frac{\tilde{x}}{\epsilon} \approx \mu \epsilon^\alpha \pm o(\epsilon^\beta)\}$ where μ is a constant and $1 > \beta > \alpha \geq 0$, may be a non-null subset of (0,1) (for instance, when $\alpha = 0$, in particular) (for an explanation of the asymptotic expansion of $\tilde{X}$ see Remarks 4.1 and 4.3). Notice that constants α, β and μ may, however, depend on λ. Notice also that the infinitesimal gap $O_{\text{inf}} = O(x, \epsilon, \lambda)$ depends on ϵ, but apparently also on the arbitrarily small element x of the Cantor set along with the parameter λ appearing in the inversion law. But x and ϵ are very closely related, so that O_{inf} essentially depends only on ϵ and $\lambda(\epsilon)$.

For a point x from a Cantor set C, it is natural to assume that the scale ϵ is determined by the privileged scale of the Cantor set. Two relative infinitesimals $\tilde{x}$ and $\tilde{y}$ must necessarily satisfy the condition $0 < \tilde{x} \leq \tilde{y} < \tilde{x} + \tilde{y} < \epsilon$. As indicated already, the inversion rule maps an open interval of (relative) infinitesimals of size determined by the parameter λ to an arbitrarily small element x of C.

Lemma 2. $\mathbf{I}_\epsilon(x) = x + \mathbf{I}_0$, $\mathbf{I}_0 = \mathbf{I}_0^+ \cup \mathbf{I}_0^-$, $\mathbf{I}_0^- = \{ -\tilde{x} \mid \tilde{x} \in \mathbf{I}_o^+ \}$ *and* $\mathbf{I}_0^+ \risingdotseq O_{\text{inf}}$. *Further* $\exists$ *a bijection between* $\mathbf{I}_0^+$ *and* $(0,1)$ *for a given* ϵ.

Proof. Let $y \in \mathbf{I}_\epsilon(x)$. Then $y = x \pm \tilde{x}$, $0 < \tilde{x} < \epsilon < z$, so that $\tilde{x} = \lambda \, \frac{\epsilon^2}{z}$ for a fixed z and a variable λ. Thus $y \in x + \mathbf{I}_0$. The other inclusion also follows similarly. Finally, the bijection is given by the mapping $\tilde{x} \to \frac{\tilde{x}}{\epsilon}$. ■

Definition 8. *[26] Given* $\tilde{x} \in \mathbf{I}_o$, *we define a scale free absolute value of* $\tilde{x}$ *by* $v : \mathbf{I}_0 \to [0,1]$ *where*

$$v(\tilde{x}) = \lim_{\epsilon \to 0} \log_{\epsilon^{-1}} \frac{\epsilon}{|\tilde{x}|} \tag{3.2}$$

and $v(0) = 0$.

Lemma 3. v *is a non-archimedean semi-norm over* $\mathbf{I}_0$.

Notation 1. *By semi-norm we mean (i)* $v(\tilde{x}) > 0$, $\tilde{x} \neq 0$. *(ii)* $v(-\tilde{x}) = v(\tilde{x})$. *(iii)* $v(\tilde{x} + \tilde{y}) \leq \max\{ v(\tilde{x}), v(\tilde{y}) \}$. *Property (iii) is called the strong (ultrametric) triangle inequality [20]. Note that this definition of seminorm on a set differs from the seminorm on a vector space. However, this suffices our purpose here.*

Proof. The case (i) and (ii) follow from the definition. To prove (iii) let $0 < \tilde{x} \leq \tilde{y} < \tilde{x} + \tilde{y} < \epsilon$. Then $v(\tilde{y}) \leq v(\tilde{x})$ and hence $v(\tilde{x} + \tilde{y}) = \lim_{\epsilon \to 0} \log_{\epsilon^{-1}} \frac{\epsilon}{\tilde{x}+\tilde{y}} \leq \lim_{\epsilon \to 0} \log_{\epsilon^{-1}} \frac{\epsilon}{\tilde{x}} = v(\tilde{x}) = \max\{ v(\tilde{x}), v(\tilde{y}) \}$. Moreover, $v(\tilde{x} - \tilde{y}) = v(\tilde{x} + (-\tilde{y})) \leq \max\{ v(\tilde{x}), v(\tilde{y}) \}$. ■

Remark 3. As remarked already, the set of infinitesimals $O_{\text{inf}} = \Phi$ when $\epsilon \to 0$. However, the corresponding asymptotic expression for the scale free (invariant) infinitesimals is nontrivial, in the sense that the associated valuations (3.2) can be shown to exist as finite real numbers. This also gives an explicit construction of infinitesimals and the associated absolute value.

Choosing $\epsilon = \beta^r$, the Cantor set scale factor, the scale free infinitesimal gaps can be identified as $\tilde{O}^m_{inf} = (0, \beta^m)$ when $\epsilon \to 0$ is realized as $n \to \infty$, $r = n + m$, $m = 1, 2, \ldots$. Assign nonzero constant valuation $v(\tilde{x}_m) = \alpha_m$ $\forall\ \tilde{X}_m = \tilde{x}_m/\epsilon \in \tilde{O}^m_{\mathrm{inf}}$. The set of all possible scale free infinitesimals $\cup\tilde{O}^m_{inf} \subset (0, 1)$ is now realized as nested clopen circles $S_m : \{\tilde{x}_m : \mathrm{v}(\tilde{\mathrm{x}}_\mathrm{m}) = \alpha_m\}$. The ordinary 0 of C is replaced by this set of scale free infinitesimals $0 \to\ \mathbf{0} = O_{\mathrm{inf}}/\sim = \{0, \cup S_m\}$, $\mathbf{0}$ being the equivalence class under the equivalence relation $\sim$, where $x \sim y$ means $v(x) = \mathrm{v(y)}$. The existence of $\tilde{x}$ could also be conceived dynamically as a computational model [24, 25, 29], in which a number, for instance, 0 is identified as an interval $[-\epsilon, \epsilon]$ at an accuracy level determined by $\epsilon = \beta^n$.

Remark 4. 1. The concept of infinitesimals and the associated absolute value considered here become significant only in a limiting problem (or process), which is reflected in the explicit presence of "$\lim\limits_{\epsilon\to 0}$" in the relevant definitions. Recall that for a continuous real valued function $f(x)$, the statement $\lim\limits_{x\to 0} f(x) = f(0)$, means that $x \to 0$ essentially is $x = 0$. This may be considered to be a *passive* evaluation (interpretation) of limit. The present approach is *dynamic*, in the sense that it offers not only a more refined evaluation of the limit, but also provides a clue how one may induce new (nonlinear) structures (ingredients) in the limiting (asymptotic) process. The inversion rule (3.1) is one such nonlinear structure which may act nontrivially as one investigates more carefully the *motion* of a real variable x (and hence of the associated scale $\epsilon < x$) as it goes to 0 more and more accurately. Notice that at any "instant", elements defined by inequalities $0 < \tilde{x} < \epsilon < x$ in the limiting process, are well defined; relative infinitesimals are *meaningful* only in that *dynamic* sense (classically, these are all zero, as x itself *is* zero). Scale invariant infinitesimals $\tilde{X}$, however, may or may not be zero classically. $\tilde{X} = \mu \ (\neq 0)$, a constant, for instance, is nonzero even when x and ϵ go to zero. On the other hand, $\tilde{X} = \epsilon^{\alpha}, \ 0 < \alpha < 1$, of course, vanish classically, but as shown below, are nontrivial in the present formalism. As a consequence, relative (and scale invariant) infinitesimals may be said to *exist* even as real numbers in this dynamic sense. The accompanying metric $|.|_u$, however, is an ultrametric. Notice that the 'limit' above refers to the standard limit on R with usual metric.

2. However, a genuine (nontrivial) scale free infinitesimal $\tilde{X}$ can not be a constant. Let $\tilde{x}_0 = \mu\epsilon, \ 0 < \mu < 1$, μ being a constant. Then $v(x_0) = \lim_{\epsilon \to 0} \log_\epsilon \mu = 0$, so that $\tilde{x}_0$ is essentially the trivial infinitesimal 0 (more precisely, such a relative infinitesimal belongs to the equivalence class of 0).

3. The scale free infinitesimals of the form $\tilde{X}_m \approx \epsilon^{\alpha_m}$ go to 0 at a slower rate compared to the linear motion of the scale ϵ. The associated nontrivial absolute value $v\left(\tilde{x}_m\right) = \alpha_m$ essentially quantifies this decelerated motion.

Definition 9. *The set* $B_r(a) = \{ \ x \mid v(x-a) < r \ \}$ *is called an open ball in* $\mathbf{I}_0$. *The set* $\bar{B}_r(a) = \{ \ x \mid v(x-a) \leq r \ \}$ *is a closed ball in* $\mathbf{I}_0$.

Lemma 4. (*i*) *Every open ball is closed and vice-versa (clopen ball)* (*ii*) *every point $b \in B_r(a)$ is a centre of $B_r(a)$. (iii) Any two balls in $\mathbf{I}_0$ are either disjoint or one is contained in another. (iv) $\mathbf{I}_0$ is the union of at most a countable family of clopen balls.*

Proof follows directly from the ultrametric inequality and the fact that $\mathbf{I}_0$ is an open set. It also follows that in the topology determined by the semi-norm, O_{inf} is a totally disconnected set. We next show that a closed ball in O_{inf} is compact.

Lemma 5. *[24] A closed (clopen) ball in O_{inf} is both complete and compact.*

Proof. The proof follows from the following observation. Given $\epsilon > 0$, consider a closed interval $[a, b] \subset O_{\text{inf}}$ (in the usual topology) such that $0 < a < b < \epsilon$. The valuation v realizes this closed interval as an ultrametric (sub)space U of O_{inf} which is an union of atmost a countable family of disjoint clopen balls (Lemma 4). Completeness now follows from the standard ultrametric properties: a sequence $\{x_n\} \subset U$ is Cauchy $\Leftrightarrow$ $v(x_n - x_m) \to 0 \Leftrightarrow v(x_{n+1} - x_n) \to 0 \Rightarrow \exists\ N > 0$ such that $v(x_{n+1}) = v(x_n)$ for $n \geq N$. Noting that for a nonzero infinitesimal x_n, the associated valuation is nonzero, it follows that $x_n \to x_N \in U$ in the ultrametric in the sense that $v(x_n) = v(x_N)$ as $n \to \infty$. Compactness is a consequence of the fact that any sequence in U has a convergent subsequence. Indeed, a sequence $\{x_n\} \subset U$ can not diverge (and can atmost be oscillating) in the given ultrametric since $0 \leq v(x_n) \leq 1$. ■

As a result, O_{inf} is the union of countable family of disjoint closed (clopen) balls, in each of which $v(\tilde{x})$ can assume a constant value. With this assumption, $v : O_{\text{inf}} \to [0,1]$ is discretely valued. Next, to restore the product rule viz : $v(\tilde{x}\tilde{y}) = v(\tilde{x}).v(\tilde{y})$, we note that given $\tilde{x}$ and ϵ, $0 < \tilde{x} < \epsilon$, there exist $0 < \sigma(\epsilon) < 1$ and $a : O_{\text{inf}} \to R$ such that

$$\frac{\tilde{x}}{\epsilon} = \epsilon^{\sigma^{a(\tilde{x})}}.\epsilon^{t(\tilde{x},\epsilon)} \tag{3.3}$$

so that $v(\tilde{x}) = \sigma^{a(\tilde{x})}$ for an indeterminate vanishingly small $t : O_{\text{inf}} \to R$ i.e. $t(\tilde{x},\epsilon) \to 0$ as $\epsilon \to 0^+$. For the given Cantor set C there is a unique (natural) choice of σ dictated by the scale factors of C viz : $\sigma = p^{-n} = r^{-ns}$, $s = \frac{\log p}{\log r}$, for some natural number n.

The mapping $a(\tilde{x})$ is a valuation and satisfies (i) $a(\tilde{x}\tilde{y}) = a(\tilde{x}) + a(\tilde{y})$, (ii) $a(\tilde{x}+\tilde{y}) \geq \min\{\, a(\tilde{x}), a(\tilde{y})\,\}$. Now discreteness of $v(\tilde{x})$ implies range $\{\, a(\tilde{x})\,\} = \{\, a_n \mid n \in Z^+\,\}$. Again for a given scale ϵ, O_{inf} is identified with a copy of $(0,1)$ (by Lemma 1) which is clopen in the semi-norm. Thus O_{inf} is covered by a finite number of disjoint clopen balls $B(\tilde{x}_n)$ (say), $\tilde{x}_n \in O_{\text{inf}}$. Because of finiteness, values of $a(\tilde{x})$ on each of the balls can be ordered $0 = a_0 < a_1 < \cdots < a_n = s_0$ (say). Let $v_0 = v(B(\tilde{x}_n)) = \sigma^{s_0}$. Then we can write $v_i = v(B(\tilde{x}_i)) = \alpha_i v_0 = \alpha_i\ \sigma^{s_0}$ for an ascending sequence $\alpha_i > 0$, $i = 0, 1, \ldots, n$. We also note that $a_0 = 0$ corresponding to the unit $\tilde{x}_u$ so that $v\,(\tilde{x}_u) = 1$.

From equation (3.3) we have $\frac{\tilde{x}_u}{\epsilon} = \epsilon^{1+t(\tilde{x},\epsilon)}$ and so it follows that $\tilde{x} \in O_{\text{inf}}$ will admit a factorization

$$\frac{\tilde{x}}{\epsilon} = \frac{\tilde{x}_i}{\epsilon}.\frac{\tilde{x}_u}{\epsilon^2} \tag{3.4}$$

since $\tilde{x} \in B(\tilde{x}_i)$ for some i.

Thus

$$\tilde{x} = \tilde{x}_i\ (1 + \tilde{x}_\epsilon) \tag{3.5}$$

where $\tilde{x}_u = \epsilon^2(1 + \tilde{x}_\epsilon)$, $\tilde{x}_\epsilon \in O_{\mathrm{inf}}$, so that $v(\tilde{x}) = v(\tilde{x}_i)$, as $v(\tilde{x}_\epsilon) < 1$.

We thus have,

Lemma 6. *[25] v is a discretely valued non-archimedean absolute value on O_{inf} Any infinitesimal $\tilde{x} \in O_{\mathrm{inf}}$ have the decomposition given by equation* (3.5) *so that v has the canonical form*

$$v(\tilde{x}) = \alpha_i \ \sigma^{s_0}, \ \tilde{x} \in B(x_i). \tag{3.6}$$

Definition 10. *The infinitesimals given by equation (3.5) and having absolute value* (3.6) *are called valued infinitesimals.*

Theorem 1. *[26] The non-archimedean infinitesimal absolute value (3.2) is given by a Cantor function associated with the Cantor set containing the relative infinitesimals. Conversely, given a Cantor function, there exists a class of infinitesimals, determined by the Cantor function, that live in an extended ultrametric neighbourhood of 0, denoted* **0**.

Proof. The (infinitesimal) absolute value (valuation)* $v(\tilde{x})$, as a mapping from O_{inf} to $I \subset R$, is continuous. The equation (3.6), however, defines $v(\cdot)$ only for points in the clopen balls $B(a_i)$, $i = 1, 2, \cdots$. The definition can be extended continuously over the entire set O_{inf} for points outside the clopen balls. Indeed, let for a given primary scale ϵ, σ_i be the secondary scale. Let also that $y \in O_{\mathrm{inf}} / \cup B(a_i)$. Then there exist $y_i \in B(a_i)$, $y_{i+1} \in B(a_{i+1})$ such that $y_i < y < y_{i+1}$, and $v(y_{i+1}) - v(y_i) = (\alpha_{i+1} - \alpha_i)\sigma_i$ (to be precise, the selection of the sequence y_i actually requires one to invoke the axiom of choice). Clearly, the sequence

*In algebraic number theory valuation means usually the exponent $a(\tilde{x})$ in (3.3). We however often use the word valuation to denote non-archimedean absolute value as well.

$v(y_{i+1})$ is increasing and $v(y_i)$ is decreasing. Thus, $v(y) := \lim v(y_i)$, as $i \to \infty$. We have thus proved that the scale invariant valuation $v(\tilde{x})$ is indeed given by a Cantor function. Conversely, given a Cantor function $\phi(x)$, $x \in [0,1]$, one can define a set of infinitesimals by the asymptotic formula $\tilde{x} \approx \epsilon\epsilon^{\phi(\tilde{x}/\epsilon)}$ as $\epsilon \to 0$ that is assumed to live in a nontrivial neighbourhood of 0. ■

With this class of valuations, the seminorm now extends to a non-archimedean absolute value, satisfying also the product rule (iv) $v(\tilde{x}\ \tilde{y})$ $= v(\tilde{x}).v(\ \tilde{y})$.

We now make use these valued infinitesimals to define a non-trivial absolute value on C in the following steps [24, 25].

(i) Given $x \in C$ define a set of multiplicative neighbours of x which are induced by the valued infinitesimals $\tilde{x}_i \in \mathbf{I}_0^+$ by

$$X_{\pm}^i = x.\ x^{\mp v(\tilde{x}_i)} \tag{3.7}$$

where $v(\tilde{x}_i) = \alpha_i \sigma^{s_0}$ and $\alpha_i = \alpha_i(x)$ may now depend on x. We note that the non-archimedean topology induced by v makes the infinitesimal neighbourhood of 0^+ in I totally disconnected. Equation (3.7) thus introduces a finer infinitesimal subdivisions in the neighbourhood of x $\in C$.

(ii) We define the new absolute value of $x \in C$ by

$$\| x \| = \inf \log_{x^{-1}} \frac{X_+^i}{x} = \inf \log_{x^{-1}} \frac{x}{X_-^i} \tag{3.8}$$

so that $\| x \| = \sigma^s$ where $\sigma^s = \inf \alpha_i \sigma^{s_0}$ and the infimum is over all i. It thus follows that

Corollary 1. *$\|\,.\,\| : C \to R_+$ is a non-archimedean absolute value on a Cantor set C.*

We now define [24, 25] the valued measure $\mu_v : C \to R_+$ by

(a) $\mu_v(\Phi) = 0$, Φ the null set.

(b) $\mu_v[(0,x)] = ||x||$, when $x \in C$.

(c) For any $E \subset C$, on the other hand, we have $\mu_v(E) = \lim_{\delta \to 0} \inf \sum_i \{d_{\text{na}}(I_i)\}$, where $I_i \in \tilde{I}_\delta$ and the infimum is over all countable δ- covers $\tilde{I}_\delta$ of E by clopen balls. Moreover, $d_{\text{na}}(I_i)$=the non-archimedean diameter of $I_i = \sup\{||x-y|| : x,\ y \in I_i\}$. Denoting the diameter in the usual (Euclidean) sense by $d(I_i)$, one notes that $d_{\text{na}}(I_i) \leq \{d(I_i)\}^s$, since $x,\ y \in C$ and $|x-y| = d$, imply $||x-y|| = \epsilon^s \leq d^s$, as the scale ϵ satisfies, by definition, $\epsilon \leq d \leq \delta$.

Thus μ_v is a metric (Lebesgue outer) measure on C realized as a non-archimedean ultrametric space. Now to compare this with the Hausdorff s measure, we first note that $\mu_v[E] \leq \mu_s[E]$ since $d_{\text{na}}(I_i) \leq \{d(I_i)\}^s$ for a given cover of (Euclidean) size ϵ. Next, for a cover of clopen balls of sizes ϵ_i, we have $\sum_i \{d_{\text{na}}(I_i)\} = \sum_i \epsilon_i^s$. For the Hausdorff measure, on the other hand, covers by any arbitrary sets are considered. Using the monotonicity of measures it follows that

$$\inf \sum_i \{(d(I_i))^s\} \leq \inf \sum_i \{d_{\text{na}}(I_i)\} \tag{3.9}$$

so that letting $\epsilon \to 0$ we have $\mu_v[E] \geq \mu_s[E]$. Hence

$$\mu_v[E] = \mu_s[E] \tag{3.10}$$

for any subset E of C. Finally, for s = dimension of C, $\mu_s[C]$ is finite and hence the valued measure of C is also finite. Notice that the valued measure selects *naturally* the finite Hausdorff measure of the Cantor set, when s is it's Hausdorff dimension.

3.2.2 Topological Inequivalence

The metric properties of the present ultrametric are indeed distinct from the natural ultrametric (c.f., [24, 25]), since the Lebesgue measure of C in the natural ultrametric is zero, but in the present case, the corresponding valued measure equals the Hausdorff measure. More importantly, topologies induced by the two ultrametrics are also different, as it is shown in the following example [26].

Example 4. *The sequence* $\epsilon_n = \epsilon^{n-nl},\ 0 < \epsilon < 1,\ 0 < l < 1$ *converges to 0 in the usual metric (ultrametric), but converges to* l *in the present ultrametric. For a sufficiently large* n, *choose* ϵ^n *as non-trivial scale factor and then relative infinitesimals are* $\tilde{\epsilon}_n = \lambda^{-1}\epsilon^{n+nl},\ 0 \ll \lambda < 1$. *Then, letting the secondary scale* $\epsilon \to 0$, *we have* $v(\tilde{\epsilon}_n) = \lim \log_{\epsilon^{-n}}(\epsilon^n/\tilde{\epsilon}_n) = l$ *and hence* $||\epsilon_n|| = l$, *by equation (3.8), for a sufficiently large* n. *Thus,* $\{\epsilon_n\} \to l$ *in the ultrametric* $||.||$.

Letting $\epsilon = \tilde{\epsilon}^m$, *the sequence* ϵ^{n-nl} *is replaced by* $\tilde{\epsilon}^{N-Nl},\ N = nm$, *so that the limit* $\epsilon \to 0$ *of the secondary scale is well defined, since it is realized as* $m \to \infty$.

Note, however, that the sequence $\{\epsilon^n\}$ *converges to 0, even in* $||.||$. *For a sufficiently large but fixed* n, *we choose* ϵ^{n+1} *as the scale factor, so that* $\tilde{\epsilon}_n = \lambda^{-1}\epsilon^{n+2}$, *are relative infinitesimals and* $v(\tilde{\epsilon}_n) = \frac{1}{n+1}$. *More generally,* *for scales* ϵ^{n+r}, *r being a nonnegative real, we have* $v(\tilde{\epsilon}_n) = \frac{r}{n+r}$. *Thus,* $||\epsilon^n|| = \inf_r \frac{r}{n+r} = 0$.

Incidentally, by letting $\bar{\epsilon} = \epsilon^{1-l}$, *one may like to conclude that* $||\bar{\epsilon}^n|| = 0$, *which would contradict our original claim that* $||\epsilon^{n-nl}|| = l$. *But this demonstrates the basic fact that* $||\epsilon^n|| = 0$ *since the sequence* ϵ^n *does not have any* natural (nontrivial) scale *other than* ϵ^n *itself. The sequence of the form* ϵ^{n-nl} *has, however, access to the natural scale* ϵ^n *and hence affords to have a nontrivial limiting ultrametric value l when the ultrametric absolute value is evaluated using the natural scale. Clearly, the limiting value depends on the choice of a nontrivial scale. For a given choice of scale the limit of course is unique.*

This example also gives an alternative proof that the metric $||.||$ is really an ultrametric. This follows because of the eventual constancy of a converging sequence (to a non zero limit) under an ultrametric (c.f., page 22).

3.2.3 Differentiability

To discuss the formalism of the Calculus on C we change the notations of section 3.2 a little. Let X denote a valued infinitesimal while an arbitrarily small real $x \in I$ denote the scale ϵ. The set of infinitesimals is covered by n clopen balls B_n in each of which v is constant. Let

$$\tilde{v}_n(x) = v(\ X_n(x)\ ) = \log_{x^{-1}} \frac{x}{X_n} = \alpha_n\ x^{s_0} \tag{3.11}$$

so that $X_n = x.\ x^{\tilde{v}_n(x)} \in B_n$. For each x, $\tilde{v}_n$ is constant on B_n.

Definition 11. *A function $f : C \to R$ is said to have the limit $l \in R$ as x approaches $x_0 \in C$ on C if given $\epsilon > 0, \exists\ \delta > 0$ s.t. $|f(x) - l| < \epsilon$ $\forall\ 0 < \|x - x_0\| < \delta$.*

Definition 12. *[25] A function $f : C \to I$ is said to be differentiable at $x_0 \in C$ if given $\epsilon > 0$, $\exists$ a finite l and $\delta > 0$ such that*

$$\left| \frac{| f(x) - f(x_0) |}{\| x - x_0 \|} - l \right| < \epsilon \tag{3.12}$$

when $0 <\| x - x_0 \|< \delta$ and we write $f'(x_0) = l$.

Now $\| x - x_0 \| = \inf \tilde{v}_n(x - x_0) = \log_{x_0^{-1}} \frac{x_0}{X}$, where the valued infinitesimal X ($\propto (x - x_0)$) $\in \tilde{B}$, an open sub-interval of $[0, 1]$ in the usual topology and $\tilde{B}$ is the ball which corresponds to the infimum of $\tilde{v}_n$. Further $f(x) - f(x_0) = (\log x_0)^{-1} \tilde{f}(X)$, since $x = x_0.x_0^{\pm v(x)}$, and $\tilde{f}$ is a differentiable function on $\tilde{B}$ in the usual sense. Thus equation (3.12), viz., the equality $f'(x_0) = l$, extends over $\tilde{B}$ as a scale free differential equation

$$\frac{d\tilde{f}}{d \log X} = l. \tag{3.13}$$

Definition 13. *[25] Let $f : C \to C$ be a mapping on a Cantor set C to itself. Then f is differentiable at $x_0 \in C$ if $\exists\ l$ such that given $\epsilon > 0, \exists$ $\delta > 0$ so that*

$$\left| \frac{\| f(x) - f(x_0) \|}{\| x - x_0 \|} - l \right| < \epsilon \tag{3.14}$$

when $0 <\| x - x_0 \|< \delta$.

As before we write $f'(x_0) = l$ (with an abuse of notation). It follows that the above equality now extends over to a scale free equation of the form

$$\frac{d \log \tilde{f}(X)}{d \log X} = l \tag{3.15}$$

where notations are analogous to the above.

Remark 5. The discrete point like structures of C are replaced by infinitesimal open intervals over which the ordinary continuum calculus is carried over on logarithmic variables via the scale invariant non-archimedean metric. We consider some applications in chapter 4 (Sec.4.4) and chapter 10.