CHAPTER 2

MEROMORPHIC FUNCTIONS AND THEIR RELATIVE VALIRON DEFECTS

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2.1 Introduction, Definitions and Notations.

Let f be a non-constant meromorphic function defined in the open complex plane $\mathbb{C}$. For $\alpha \in \mathbb{C} \cup \{\infty\}$ let $n(t, \alpha; f)$ denote the number of roots of $f = \alpha$ in $|z| \leq t$, the multiple roots being counted according to their multiplicities and $N(t, \alpha; f)$ is defined in the usual way in terms of $n(t, \alpha; f)$. Similarly, $\bar{n}(t, \alpha; f)$ denotes the number of distinct roots of $f = \alpha$ in $|z| \leq t$ and $\bar{N}(t, \alpha; f)$ is also defined in the usual way in terms of $\bar{n}(t, \alpha; f)$.

The Nevanlinna defect $\delta(\alpha; f)$ and the Valiron defect $\Delta(\alpha; f)$ of 'a' are respectively defined in the following manner :

$$\delta(\alpha; f) = 1 - \limsup_{r\to\infty} \frac{N(r, \alpha; f)}{T(r, f)} = \liminf_{r\to\infty} \frac{m(r, \alpha; f)}{T(r, f)}$$

$$\text{and } \Delta(\alpha; f) = 1 - \liminf_{r\to\infty} \frac{N(r, \alpha; f)}{T(r, f)} = \limsup_{r\to\infty} \frac{m(r, \alpha; f)}{T(r, f)}.$$

Milloux [30] introduced the concept of absolute defect of 'a' with respect to the derivative f'. Later Xiong [46] extended this definition. He introduced the term

$$\delta_R^{(k)}(\alpha; f) = 1 - \limsup_{r\to\infty} \frac{N\left(r, \alpha; f^{(k)}\right)}{T(r, f)}$$

The results of this chapter have been published in **Journal of Mathematics**, see [8].

for $k = 1, 2, 3, ...$ and called it the relative Nevanlinna defect of 'a' with respect to $f^{(k)}$. Xiong [**46**] has shown various relations between the usual defects and the relative defects. Singh [**36**] introduced the term relative defect for distinct zeros and poles and established various relations among it, relative defects and the usual defects.

In this chapter we call the term

$$\Delta_R^{(k)}(\alpha; f) = 1 - \liminf_{r\to\infty} \frac{N\left(r, \alpha; f^{(k)}\right)}{T(r, f)},$$

the relative Valiron defect of 'a' with respect to $f^{(k)}$ for $k = 1, 2, 3, ...$ and prove various relations between it and the relative Nevanlinna defect.

The term $S(r, f)$ denotes any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as r$\rightarrow \infty$ through all values of r if f is of finite order and except possibly for a set of r of finite linear measure otherwise.

The following two definitions are well known.

Definition 2.1.1 *The order ρ_f of a meromorphic function f is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna's characteristic function of f. When $\rho_f < \infty$, then f is of finite order.

Definition 2.1.2 *The term $\Delta_R^{(k,n)}(\alpha; f)$ is defined as follows:*

$$\Delta_R^{(k,n)}(\alpha; f) = 1 - \liminf_{r\to\infty} \frac{N\left(r, \alpha; f^{(k)}\right)}{T\left(r, f^{(n)}\right)}$$

where $k = 1, 2, 3, ...$ and $n = 1, 2, 3, ...$
For $n = 1$ the above definition coincides with the relative Valiron defect of 'a' with respect to $f^{(k)}$ i.e.,

$$\Delta_R^{(k,1)}(\alpha; f) \equiv \Delta_R^{(k)}(\alpha; f).$$

2.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.2.1 *[38] Let f be a meromorphic function of finite order such that $\sum_{a\neq\infty}\delta(a;f)=1$ and $\delta(\infty;f)=1$. Then for any non-negative integer k,*

$$\lim_{r\to\infty}\frac{T\left(r,f^{(k)}\right)}{T(r,f)}=1.$$

Lemma 2.2.2 *[38] Let f be a meromorphic function of finite order with $\sum_{a\neq\infty}\delta(a;f)=\delta(\infty;f)=1$. Then for any α,*

$$\delta_R^{(k)}(\alpha;f)=\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{T(r,f)}.$$

Lemma 2.2.3 *Let f be a meromorphic function of finite order such that $\sum_{a\neq\infty}\delta(a;f)=\delta(\infty;f)=1$. Then for any α,*

$$\Delta_R^{(k,n)}(\alpha;f)=\limsup_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(n)}\right)}$$

where $k=1,2,3,\ldots$ and $n=1,2,3,\ldots$
For $n=1$

$$\Delta_R^{(k,1)}(\alpha;f)\equiv\Delta_R^{(k)}(\alpha;f)=\limsup_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{T(r,f)}.$$

Proof. In view of Lemma 2.2.1 we obtain that

$$\begin{aligned}\Delta_R^{(k,n)}(\alpha;f)&=1-\liminf_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(n)}\right)}\\&=1-\liminf_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}.\lim_{r\to\infty}\frac{T\left(r,f^{(k)}\right)}{T\left(r,f^{(n)}\right)}\\&=1-\liminf_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\\&=\limsup_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\\&=\limsup_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(n)}\right)}.\lim_{r\to\infty}\frac{T\left(r,f^{(n)}\right)}{T\left(r,f^{(k)}\right)}\\&=\limsup_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(n)}\right)}.\end{aligned}$$

This proves the lemma. ■

2.3 Theorems.

In this section we present the main results of the chapter.

Theorem 2.3.1 *Let f be a meromorphic function of finite order. Then for any two positive integers n and k with $n > k$,*

$$\Delta_R^{(k)}(\infty; f) + \Delta_R^{(n)}(0; f) \geqslant \delta_R^{(k)}(0; f) + \delta_R^{(k)}(a; f) + \Delta_R^{(n)}(\infty; f),$$

where 'a' is any non-zero finite complex number.

Proof. Let us consider the identity

$$\frac{a}{f^{(k)}} = 1 - \frac{f^{(k)} - a}{f^{(n)}} . \frac{f^{(n)}}{f^{(k)}} \text{ where } n > k.$$

Since $m\left(r, \frac{a}{f^{(k)}}\right) \leq m\left(r, \frac{1}{f^{(k)}}\right) + O(1)$, we get from the above identity

$$m\left(r, \frac{1}{f^{(k)}}\right) \leq m\left(r, \frac{f^{(k)} - a}{f^{(n)}}\right) + S(r, f). \tag{2.1}$$

Now by the relation $T(r, \frac{1}{f}) = T(r, f) + O(1)$ and Milloux's theorem {p.55, [**19**]} it follows from (2.1) that

$$m\left(r, \frac{1}{f^{(k)}}\right) \leq T\left(r, \frac{f^{(k)} - a}{f^{(n)}}\right) - N\left(r, \frac{f^{(k)} - a}{f^{(n)}}\right) + S(r, f)$$

$$i.e., \ m\left(r, \frac{1}{f^{(k)}}\right) \leq T\left(r, \frac{f^{(n)}}{f^{(k)} - a}\right) - N\left(r, \frac{f^{(k)} - a}{f^{(n)}}\right) + S(r, f) + O(1)$$

$$i.e., \ m\left(r, \frac{1}{f^{(k)}}\right)$$

$$\leq N\left(r, \frac{f^{(n)}}{f^{(k)} - a}\right) - N\left(r, \frac{f^{(k)} - a}{f^{(n)}}\right) + S(r, f) + O(1). \tag{2.2}$$

In view of the relation $N\left(r, \frac{f'}{f}\right) - N\left(r, \frac{f}{f'}\right) = N\left(r, \frac{1}{f}\right) - N(r, f) - N\left(r, \frac{1}{f'}\right) + N(r, f')$ {p.34, [**19**]} it follows from (2.2) that

$$m\left(r, \frac{1}{f^{(k)}}\right)$$

$$\leq N\left(r, f^{(n)}\right) + N\left(r, \frac{1}{f^{(k)} - a}\right) - N\left(r, f^{(k)} - a\right) - N\left(r, \frac{1}{f^{(n)}}\right)$$

$$+ S(r, f) + O(1)$$

$$i.e., \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f^{(k)}}\right)}{T(r,f)}$$

$$\leq \liminf_{r\to\infty}\left\{\frac{N\left(r,f^{(n)}\right)}{T(r,f)}-\frac{N\left(r,f^{(k)}\right)}{T(r,f)}-\frac{N\left(r,\frac{1}{f^{(n)}}\right)}{T(r,f)}\right\}+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(k)}-a}\right)}{T(r,f)}$$

$$\leq \liminf_{r\to\infty}\frac{N\left(r,f^{(n)}\right)}{T(r,f)}-\liminf_{r\to\infty}\frac{N\left(r,f^{(k)}\right)}{T(r,f)}-\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(n)}}\right)}{T(r,f)}$$
$$+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(k)}-a}\right)}{T(r,f)}$$

In view of Lemma 2.2.2 we obtain from above

$$\delta_R^{(k)}(0;f)\leq\left\{1-\Delta_R^{(n)}(\infty;f)\right\}-\left\{1-\Delta_R^{(k)}(\infty;f)\right\}-\left\{1-\Delta_R^{(n)}(0;f)\right\}$$
$$+\left\{1-\delta_R^{(k)}(a;f)\right\}$$

$$i.e., \Delta_R^{(k)}(\infty;f)+\Delta_R^{(n)}(0;f)\geqslant\delta_R^{(k)}(0;f)+\delta_R^{(k)}(a;f)+\Delta_R^{(n)}(\infty;f).$$

This proves the theorem. ■

Remark 2.3.1 *The sign '* $\geq$ *' in Theorem* 2.3.1 *cannot be replaced by '* $>$ *' only which is evident from the following example.*

Example 2.3.1

Let $f=\exp z.$

Then $\Delta_R^{(k)}(\infty;f)=\Delta_R^{(n)}(0;f)=\Delta_R^{(n)}(\infty;f)=1$

and $\delta_R^{(k)}(0;f)=\delta_R^{(k)}(\infty;f)=1.$

So $\delta_R^{(k)}(a;f)=0.$

Then $\Delta_R^{(k)}(\infty;f)+\Delta_R^{(n)}(0;f)=2=\delta_R^{(k)}(0;f)+\delta_R^{(k)}(a;f)+\Delta_R^{(n)}(\infty;f).$

Theorem 2.3.2 *Let* f *be a meromorphic function of finite order such that* $\sum_{a\neq\infty}\delta(a;f)=\delta(\infty;f)=1$. *Then for any two positive integers* n *and* k *with* $n>k$

$$\Delta_R^{(k)}(\infty;f)+\Delta_R^{(n)}(0;f)\geqslant\delta_R^{(k)}(0;f)+\delta_R^{(k)}(\infty;f).$$

Proof. Since $f^{(k)} = f^{(n)}.\frac{f^{(k)}}{f^{(n)}}$ for $n > k$ we get that

$$m\left(r, f^{(k)}\right) \le m\left(r, f^{(n)}\right) + m\left(r, \frac{f^{(k)}}{f^{(n)}}\right). \tag{2.3}$$

Now by the relation $T(r, \frac{1}{f}) = T(r, f) + O(1)$ and Milloux's theorem $\{p.55, [\mathbf{19}]\}$ we obtain from (2.3),

$$m\left(r, f^{(k)}\right) \le m\left(r, f^{(n)}\right) + T\left(r, \frac{f^{(k)}}{f^{(n)}}\right) - N\left(r, \frac{f^{(k)}}{f^{(n)}}\right)$$

$$i.e.,\ m\left(r, f^{(k)}\right) \le m\left(r, f^{(n)}\right) + T\left(r, \frac{f^{(n)}}{f^{(k)}}\right) - N\left(r, \frac{f^{(k)}}{f^{(n)}}\right) + O(1)$$

$$i.e.,\ m\left(r, f^{(k)}\right)$$
$$\le m\left(r, f^{(n)}\right) + N\left(r, \frac{f^{(n)}}{f^{(k)}}\right) - N\left(r, \frac{f^{(k)}}{f^{(n)}}\right) + S(r, f) + O(1). \tag{2.4}$$

In view of the relation $N\left(r, \frac{f'}{f}\right) - N\left(r, \frac{f}{f'}\right) = N\left(r, \frac{1}{f}\right) - N(r, f) - N\left(r, \frac{1}{f'}\right) + N\left(r, f'\right)$ $\{p.34, [\mathbf{19}]\}$ it follows from (2.4) that

$$m\left(r, f^{(k)}\right)$$
$$\le m\left(r, f^{(n)}\right) + N\left(r, f^{(n)}\right) + N\left(r, \frac{1}{f^{(k)}}\right) - N\left(r, f^{(k)}\right)$$
$$- N\left(r, \frac{1}{f^{(n)}}\right) + S(r, f) + O(1)$$

$$i.e.,\ \liminf_{r\to\infty} \frac{m\left(r, f^{(k)}\right)}{T(r, f)}$$
$$\le \liminf_{r\to\infty} \left\{ \frac{N\left(r, f^{(n)}\right)}{T(r, f)} - \frac{N\left(r, f^{(k)}\right)}{T(r, f)} - \frac{N\left(r, \frac{1}{f^{(n)}}\right)}{T(r, f)} \right\}$$
$$+ \limsup_{r\to\infty} \left\{ \frac{N\left(r, \frac{1}{f^{(k)}}\right)}{T(r, f)} + \frac{m\left(r, f^{(n)}\right)}{T(r, f)} \right\}$$

$$i.e., \liminf_{r\to\infty}\frac{m\left(r,f^{(k)}\right)}{T(r,f)}$$

$$\leq \liminf_{r\to\infty}\frac{N\left(r,f^{(n)}\right)}{T(r,f)} - \liminf_{r\to\infty}\frac{N\left(r,f^{(k)}\right)}{T(r,f)} - \liminf_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(n)}}\right)}{T(r,f)}$$

$$- \liminf_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(n)}}\right)}{T(r,f)} + \limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(k)}}\right)}{T(r,f)} + \limsup_{r\to\infty}\frac{m\left(r,f^{(n)}\right)}{T(r,f)}. \tag{2.5}$$

By Lemma 2.2.2 and Lemma 2.2.3 we obtain from (2.5) that

$$\delta_R^{(k)}(\infty;f) \leq \left\{1-\Delta_R^{(n)}(\infty,f)\right\} - \left\{1-\Delta_R^{(k)}(\infty,f)\right\} - \left\{1-\Delta_R^{(n)}(0,f)\right\}$$
$$+\left\{1-\delta_R^{(k)}(0,f)\right\} + \Delta_R^{(n)}(\infty,f)$$

$$i.e., \ \Delta_R^{(k)}(\infty;f) + \Delta_R^{(n)}(0;f) \geqslant \delta_R^{(k)}(0;f) + \delta_R^{(k)}(\infty;f).$$

Thus the theorem is established. ■

Remark 2.3.2 *Considering* $f = \exp z$ *one can easily verify that '*$\geq$*' cannot be replaced by '*$>$*' only in Theorem* 2.3.2 .

Theorem 2.3.3 *Let* f *be a meromorphic function of finite order and* a, b *be any two distinct finite complex numbers. Then for any two positive integers* n *and* k *with* $n > k$,

$$\Delta_R^{(n)}(0;f) + \Delta_R^{(k)}(\infty;f) \geqslant \Delta_R^{(n)}(\infty;f) + \delta_R^{(k)}(a;f) + \frac{1}{2}\delta_R^{(k)}(b;f).$$

Proof. Considering the identity

$$\frac{b-a}{f^{(k)}-a} = \frac{f^{(n)}}{f^{(k)}-a}\left\{\frac{f^{(k)}-a}{f^{(n)}} - \frac{f^{(k)}-b}{f^{(n)}}\right\}$$

with $n > k$ we obtain in view of Milloux's theorem{p.55, **[19]**}

$$m\left(r,\frac{b-a}{f^{(k)}-a}\right) \leq m\left(r,\frac{f^{(k)}-a}{f^{(n)}}\right) + m\left(r,\frac{f^{(k)}-b}{f^{(n)}}\right) + S(r,f)$$

$$i.e., \ m\left(r,\frac{b-a}{f^{(k)}-a}\right) \leq T\left(r,\frac{f^{(k)}-a}{f^{(n)}}\right) - N\left(r,\frac{f^{(k)}-a}{f^{(n)}}\right)$$
$$+T\left(r,\frac{f^{(k)}-b}{f^{(n)}}\right) - N\left(r,\frac{f^{(k)}-b}{f^{(n)}}\right) + S(r,f). \tag{2.6}$$

Since $m\left(r, \frac{1}{f^{(k)}-a}\right) \leq m\left(r, \frac{b-a}{f^{(k)}-a}\right) + O(1)$ and $T(r,f) = T\left(r, \frac{1}{f}\right) + O(1)$, it follows from (2.6) that

$$m\left(r, \frac{1}{f^{(k)}-a}\right) \leq N\left(r, \frac{f^{(n)}}{f^{(k)}-a}\right) - N\left(r, \frac{f^{(k)}-a}{f^{(n)}}\right) + N\left(r, \frac{f^{(n)}}{f^{(k)}-b}\right) - N\left(r, \frac{f^{(k)}-b}{f^{(n)}}\right) + S(r,f) + O(1). \tag{2.7}$$

In view of the relation $N\left(r, \frac{f'}{f}\right) - N\left(r, \frac{f}{f'}\right) = N\left(r, \frac{1}{f}\right) - N(r,f) - N\left(r, \frac{1}{f'}\right) + N\left(r, f'\right)$ {p.34, [**19**]} we get from (2.7),

$$m\left(r, \frac{1}{f^{(k)}-a}\right) \leq N\left(r, f^{(n)}\right) + N\left(r, \frac{1}{f^{(k)}-a}\right) - N\left(r, f^{(k)}-a\right) - N\left(r, \frac{1}{f^{(n)}}\right) + N\left(r, f^{(n)}\right) + N\left(r, \frac{1}{f^{(k)}-b}\right) - N\left(r, f^{(k)}-b\right) - N\left(r, \frac{1}{f^{(n)}}\right) + S(r,f) + O(1).$$

$$\text{i.e., } \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f^{(k)}-a}\right)}{T(r,f)}$$

$$\leq 2\liminf_{r\to\infty} \left\{ \frac{N\left(r, f^{(n)}\right)}{T(r,f)} - \frac{N\left(r, \frac{1}{f^{(n)}}\right)}{T(r,f)} - \frac{N\left(r, f^{(k)}\right)}{T(r,f)} \right\} + \limsup_{r\to\infty} \left\{ \frac{N\left(r, \frac{1}{f^{(k)}-a}\right)}{T(r,f)} + \frac{N\left(r, \frac{1}{f^{(k)}-b}\right)}{T(r,f)} \right\}$$

$$\leq 2\left\{ \liminf_{r\to\infty} \frac{N\left(r, f^{(n)}\right)}{T(r,f)} - \liminf_{r\to\infty} \frac{N\left(r, \frac{1}{f^{(n)}}\right)}{T(r,f)} - \liminf_{r\to\infty} \frac{N\left(r, f^{(k)}\right)}{T(r,f)} \right\} + \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f^{(k)}-a}\right)}{T(r,f)} + \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f^{(k)}-b}\right)}{T(r,f)}.$$

By Lemma 2.2.2 we obtain from above that

$\delta_R^{(k)}(a;f) \leq 2\left\{1 - \Delta_R^{(n)}(\infty;f)\right\} - 2\left\{1 - \Delta_R^{(n)}(0;f)\right\}$

$$-2\left\{1-\Delta_R^{(k)}(\infty;f)\right\}+\left\{1-\delta_R^{(k)}(a;f)\right\}+\left\{1-\delta_R^{(k)}(b;f)\right\}$$

i.e., $2\delta_R^{(k)}(a;f)\leq 2\Delta_R^{(n)}(0;f)+2\Delta_R^{(k)}(\infty;f)-2\Delta_R^{(n)}(\infty;f)-\delta_R^{(k)}(b;f)$

i.e., $\Delta_R^{(n)}(0;f)+\Delta_R^{(k)}(\infty;f)\geqslant\Delta_R^{(n)}(\infty;f)+\delta_R^{(k)}(a;f)+\frac{1}{2}\delta_R^{(k)}(b;f)$.

This proves the theorem. ■

Theorem 2.3.4 *Let f be any meromorphic function of finite order and n,p be any two positive integers. Then for every integer k with $0\leq k<\min\{n,p\}$*

$$\Delta_R^{(p)}(\infty;f)+\Delta_R^{(n)}(0;f)\geqslant\Delta_R^{(n)}(\infty;f)+\delta_R^{(p)}(a;f)+\delta_R^{(k)}(a;f)$$

where 'a' is any finite non zero complex number.

Proof. From the identity

$$\frac{1}{f^{(k)}-a}=\frac{1}{a}\left\{\frac{f^{(p)}}{f^{(k)}-a}-\frac{f^{(p)}-a}{f^{(n)}}.\frac{f^{(n)}}{f^{(k)}-a}\right\}$$

and by Milloux's theorem $\{p.55,$ **[19]**$\}$ we get,

$$m\left(r,\frac{1}{f^{(k)}-a}\right)\leq m\left(r,\frac{f^{(p)}-a}{f^{(n)}}\right)+S(r,f). \tag{2.8}$$

Now by the relation $T(r,\frac{1}{f})=T(r,f)+O(1)$ it follows from (2.8) that

$$m\left(r,\frac{1}{f^{(k)}-a}\right)\leq T\left(r,\frac{f^{(p)}-a}{f^{(n)}}\right)-N\left(r,\frac{f^{(p)}-a}{f^{(n)}}\right)+S(r,f)$$

$$\text{i.e., } m\left(r,\frac{1}{f^{(k)}-a}\right)\leq T\left(r,\frac{f^{(n)}}{f^{(p)}-a}\right)-N\left(r,\frac{f^{(p)}-a}{f^{(n)}}\right)+S(r,f)+O(1)$$

$$\text{i.e., } m\left(r,\frac{1}{f^{(k)}-a}\right)\leq N\left(r,\frac{f^{(n)}}{f^{(p)}-a}\right)-N\left(r,\frac{f^{(p)}-a}{f^{(n)}}\right)+S(r,f)+O(1). \tag{2.9}$$

In view of the relation $N\left(r,\frac{f'}{f}\right)-N\left(r,\frac{f}{f'}\right)=N\left(r,\frac{1}{f}\right)-N\left(r,f\right)-N\left(r,\frac{1}{f'}\right)+N\left(r,f'\right)$ {p.34, [**19**]} we obtain from (2.9) that

$$m\left(r,\frac{1}{f^{(k)}-a}\right)$$
$$\leq N\left(r,f^{(n)}\right)+N\left(r,\frac{1}{f^{(p)}-a}\right)-N\left(r,f^{(p)}-a\right)-N\left(r,\frac{1}{f^{(n)}}\right)$$
$$+S\left(r,f\right)+O(1)$$

$$i.e., \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f^{(k)}-a}\right)}{T\left(r,f\right)}$$
$$\leq \liminf_{r\to\infty}\left\{\frac{N\left(r,f^{(n)}\right)}{T\left(r,f\right)}-\frac{N\left(r,f^{(p)}\right)}{T\left(r,f\right)}-\frac{N\left(r,\frac{1}{f^{(n)}}\right)}{T\left(r,f\right)}\right\}$$
$$+\limsup_{r\to\infty}\left\{\frac{N\left(r,\frac{1}{f^{(p)}-a}\right)}{T\left(r,f\right)}\right\}$$

$$i.e., \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f^{(k)}-a}\right)}{T\left(r,f\right)}$$
$$\leq \liminf_{r\to\infty}\frac{N\left(r,f^{(n)}\right)}{T\left(r,f\right)}-\liminf_{r\to\infty}\frac{N\left(r,f^{(p)}\right)}{T\left(r,f\right)}-\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(n)}}\right)}{T\left(r,f\right)}$$
$$+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(p)}-a}\right)}{T\left(r,f\right)}$$

In view of Lemma 2.2.2 we get from above that

$$\delta_R^{(k)}\left(a;f\right)$$
$$\leq\left\{1-\Delta_R^{(n)}\left(\infty;f\right)\right\}-\left\{1-\Delta_R^{(p)}\left(\infty;f\right)\right\}-\left\{1-\Delta_R^{(n)}\left(0;f\right)\right\}$$
$$+\left\{1-\delta_R^{(p)}\left(a;f\right)\right\}$$

$$i.e., \ \Delta_R^{(p)}\left(\infty;f\right)+\Delta_R^{(n)}\left(0;f\right)\geqslant\Delta_R^{(n)}\left(\infty;f\right)+\delta_R^{(p)}\left(a;f\right)+\delta_R^{(k)}\left(a;f\right).$$

Thus the theorem is established. ■

Theorem 2.3.5 *Let f be a meromorphic function of finite order such that $\sum_{\alpha\neq\infty}\delta(\alpha;f)=\delta(\infty;f)=1$, 'a' be a finite complex number and 'b', 'c' be two distinct non zero complex numbers. Then for any two positive integers n and k with $n>k$,*

$$\Delta_R^{(k)}(a;f)+\delta_R^{(n)}(b;f)+\delta_R^{(n)}(c;f)\leq 2.$$

Proof. Since $\frac{1}{f^{(k)}-a}=\frac{f^{(n)}}{f^{(k)}-a}\cdot\frac{1}{f^{(n)}}$ for $n>k$, by Milloux's theorem{p.55, **[19]**} we obtain that

$$m\left(r,\frac{1}{f^{(k)}-a}\right)\leq m\left(r,\frac{1}{f^{(n)}}\right)+S(r,f). \tag{2.10}$$

Now by the relation $T(r,\frac{1}{f})=T(r,f)+O(1)$ we get from(2.10) that

$$m\left(r,\frac{1}{f^{(k)}-a}\right)\leq T\left(r,f^{(n)}\right)-N\left(r,\frac{1}{f^{(n)}}\right)+S(r,f)+O(1). \tag{2.11}$$

Now by Nevanlinna's second fundamental theorem it follows from (2.11) that

$$\begin{aligned}&m\left(r,\frac{1}{f^{(k)}-a}\right)\\ &\leq \bar{N}\left(r,\frac{1}{f^{(n)}}\right)+\bar{N}\left(r,\frac{1}{f^{(n)}-b}\right)+\bar{N}\left(r,\frac{1}{f^{(n)}-c}\right)\\ &\quad-N\left(r,\frac{1}{f^{(n)}}\right)+S(r,f)+O(1).\end{aligned} \tag{2.12}$$

Since $\bar{N}\left(r,\frac{1}{f^{(n)}}\right)-N\left(r,\frac{1}{f^{(n)}}\right)\leq 0$, we obtain from (2.12),

$$m\left(r,\frac{1}{f^{(k)}-a}\right)\leq N\left(r,\frac{1}{f^{(n)}-b}\right)+N\left(r,\frac{1}{f^{(n)}-c}\right)+S(r,f)+O(1)$$

$$\begin{aligned}&i.e.,\ m\left(r,\frac{1}{f^{(k)}-a}\right)\\ &\leq T\left(r,\frac{1}{f^{(n)}-b}\right)-m\left(r,\frac{1}{f^{(n)}-b}\right)+T\left(r,\frac{1}{f^{(n)}-c}\right)\\ &\quad-m\left(r,\frac{1}{f^{(n)}-c}\right)+S(r,f)+O(1)\end{aligned}$$

$$i.e.,\ m\left(r,\frac{1}{f^{(k)}-a}\right)\le 2T\left(r,f^{(n)}\right)-m\left(r,\frac{1}{f^{(n)}-b}\right)$$
$$-m\left(r,\frac{1}{f^{(n)}-c}\right)+S\left(r,f\right)+O(1)$$

$$i.e., \limsup_{r\to\infty}\frac{m\left(r,\frac{1}{f^{(k)}-a}\right)}{T\left(r,f\right)}\le 2\limsup_{r\to\infty}\frac{T\left(r,f^{(n)}\right)}{T\left(r,f\right)}-\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f^{(n)}-b}\right)}{T\left(r,f\right)}$$
$$-\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f^{(n)}-c}\right)}{T\left(r,f\right)}. \tag{2.13}$$

Now by Lemma 2.2.1 , Lemma 2.2.2 and Lemma 2.2.3 we get from(2.13),

$$\Delta_R^{(k)}\left(a;f\right)\le 2-\delta_R^{(n)}\left(b;f\right)-\delta_R^{(n)}\left(c;f\right)$$

$$i.e.,\ \Delta_R^{(k)}\left(a;f\right)+\delta_R^{(n)}\left(b;f\right)+\delta_R^{(n)}\left(c;f\right)\le 2.$$

This proves the theorem. ■

Theorem 2.3.6 *Let f be a meromorphic function of finite order satisfying $\sum\limits_{\alpha\neq\infty}\delta\left(a;f\right)=1$ and $\delta\left(\infty;f\right)=1$. Then for any two positive integers p and k with $p>k$*

$$\delta_R^{(k)}\left(0;f\right)+\Delta_R^{(p)}\left(\alpha;f\right)\le 1.$$

Proof. Considering the identity

$$\frac{\alpha}{f^{(k)}}=\frac{f^{(p)}}{f^{(k)}}-\frac{f^{(p)}-\alpha}{f^{(n)}}.\frac{f^{(n)}}{f^{(k)}}\text{ for }n>p>k$$

we get in view of Milloux's theorem{p.55, [**19**]} and by the relation $T(r,\frac{1}{f}) = T(r,f) + O(1)$ that

$$m\left(r,\frac{1}{f^{(k)}}\right) \leq m\left(r,\frac{f^{(p)}-\alpha}{f^{(n)}}\right) + S(r,f)$$

$$i.e., m\left(r,\frac{1}{f^{(k)}}\right) \leq T\left(r,\frac{f^{(p)}-\alpha}{f^{(n)}}\right) - N\left(r,\frac{f^{(p)}-\alpha}{f^{(n)}}\right) + S(r,f)$$

$$i.e., m\left(r,\frac{1}{f^{(k)}}\right) \leq T\left(r,\frac{f^{(n)}}{f^{(p)}-\alpha}\right) - N\left(r,\frac{f^{(p)}-\alpha}{f^{(n)}}\right) + S(r,f) + O(1)$$

$$i.e., m\left(r,\frac{1}{f^{(k)}}\right) \leq N\left(r,\frac{f^{(n)}}{f^{(p)}-\alpha}\right) - N\left(r,\frac{f^{(p)}-\alpha}{f^{(n)}}\right) + S(r,f) + O(1). \tag{2.14}$$

In view of the relation $N\left(r,\frac{f'}{f}\right) - N\left(r,\frac{f}{f'}\right) = N\left(r,\frac{1}{f}\right) - N(r,f) - N\left(r,\frac{1}{f'}\right) + N\left(r,f'\right)$ {p.34, [**19**]} it follows from (2.14) that

$$m\left(r,\frac{1}{f^{(k)}}\right) \leq N\left(r,f^{(n)}\right) + N\left(r,\frac{1}{f^{(p)}-\alpha}\right) - N\left(r,f^{(p)}-\alpha\right) - N\left(r,\frac{1}{f^{(n)}}\right) + S(r,f) + O(1)$$

$$i.e., m\left(r,\frac{1}{f^{(k)}}\right) \leq \left\{N\left(r,\frac{1}{f^{(p)}-\alpha}\right) - N\left(r,\frac{1}{f^{(n)}}\right)\right\} + \left\{N\left(r,f^{(n)}\right) - N\left(r,f^{(p)}\right)\right\} + S(r,f) + O(1)$$

$$i.e., m\left(r,\frac{1}{f^{(k)}}\right) \leq N\left(r,\alpha;f^{(p)}\right) + (n-p)\,\bar{N}(r,f) + S(r,f) + O(1).$$

Since $\delta(\infty;f) = 1$, it follows that

$$\lim_{r\to\infty}\frac{\bar{N}(r,f)}{T(r,f)} = 0.$$

So from above we get that

$$\liminf_{r\to\infty}\frac{m\left(r,0;f^{(k)}\right)}{T\left(r,f\right)}\leq\liminf_{r\to\infty}\frac{N\left(r,\alpha;f^{(p)}\right)}{T\left(r,f\right)}$$

$$i.e., \delta_R^k\left(0;f\right)\leq 1-\Delta_R^{(p)}\left(\alpha;f\right)$$

$$i.e., \delta_R^k\left(0,f\right)+\Delta_R^{(p)}\left(\alpha;f\right)\leq 1.$$

Since $\sum_{a\neq\infty}\delta\left(a;f\right) = \delta\left(\infty;f\right) = 1$, by Lemma 2.2.3 we note that $\Delta_R^{(p)}\left(\alpha;f\right)\geqslant 0$.
This proves the theorem. ■

Remark 2.3.3 *The inequality in Theorem* 2.3.6 *is best possible in the sense that '$\leq$' cannot be replaced by '$<$' only which is evident from the following example.*

Example 2.3.2

$$\textit{Let}\quad f=\exp z.$$

$$\textit{So}\quad \delta_R^{(k)}\left(0;f\right)=1$$

$$\textit{and}\quad \delta_R^{(p)}\left(0;f\right)=\delta_R^{(p)}\left(\infty;f\right)=1.$$

Now by Nevanlinna's second fundamental theorem and in view of above we get

$$T\left(r,f^{(p)}\right)\leq N\left(r,\alpha;f^{(p)}\right)+S\left(r,f^{(p)}\right)\leq T\left(r,f^{(p)}\right)+S\left(r,f^{(p)}\right)$$

$$\begin{aligned} i.e., \frac{T\left(r,f^{(p)}\right)}{T\left(r,f\right)} &\leq \frac{N\left(r,\alpha;f^{(p)}\right)}{T\left(r,f\right)}+\frac{S\left(r,f^{(p)}\right)}{T\left(r,f^{(p)}\right)}.\frac{T\left(r,f^{(p)}\right)}{T\left(r,f\right)} \\ &\leq \frac{T\left(r,f^{(p)}\right)}{T\left(r,f\right)}+\frac{S\left(r,f^{(p)}\right)}{T\left(r,f^{(p)}\right)}.\frac{T\left(r,f^{(p)}\right)}{T\left(r,f\right)}\end{aligned}$$

By Lemma 2.2.1 *it follows from above that*

$$\lim_{r\to\infty}\frac{N\left(r,\alpha;f^{(p)}\right)}{T\left(r,f\right)}=1.$$

$$\textit{Therefore}\quad \Delta_R^{(p)}\left(\alpha;f\right)=0.$$

$$\textit{Thus}\quad \delta_R^{(k)}\left(0;f\right)+\Delta_R^{(p)}\left(\alpha;f\right)=1.$$

Theorem 2.3.7 *Let n and k be any two positive integers such that $n > k$ and 'a' be a finite complex number. Then for any meromorphic function f of finite order satisfying $\sum_{\alpha\neq\infty} \delta(\alpha; f) = \delta(\infty; f) = 1$,*

$$\Delta_R^{(n)}(a; f) \geqslant \delta_R^{(n)}(a; f) + \delta_R^{(k)}(0; f).$$

Proof. Let $b \neq a$ be a finite complex number.
Since

$$\frac{a-b}{f^{(n)}-a} = \frac{f^{(k)}}{f^{(n)}-a}\left\{\frac{f^{(n)}-b}{f^{(k)}} - \frac{f^{(n)}-a}{f^{(k)}}\right\} \text{ for } n > k$$

we obtain in view of Milloux's theorem{$p.55$, [**19**]} and by the relation $T(r, \frac{1}{f}) = T(r, f) + O(1)$,

$$m\left(r, \frac{a-b}{f^{(n)}-a}\right) \leq m\left(r, \frac{f^{(k)}}{f^{(n)}-a}\right) + S(r, f)$$

$$\text{i.e., } m\left(r, \frac{1}{f^{(n)}-a}\right) \leq T\left(r, \frac{f^{(k)}}{f^{(n)}-a}\right) - N\left(r, \frac{f^{(k)}}{f^{(n)}-a}\right) + S(r, f)$$

$$\text{i.e., } m\left(r, \frac{1}{f^{(n)}-a}\right) \leq T\left(r, \frac{f^{(n)}-a}{f^{(k)}}\right) - N\left(r, \frac{f^{(k)}}{f^{(n)}-a}\right) + S(r, f) + O(1)$$

$$\text{i.e., } m\left(r, \frac{1}{f^{(n)}-a}\right) \leq N\left(r, \frac{f^{(n)}-a}{f^{(k)}}\right) - N\left(r, \frac{f^{(k)}}{f^{(n)}-a}\right) + S(r, f) + O(1). \quad (2.15)$$

In view of the relation $N\left(r, \frac{f'}{f}\right) - N\left(r, \frac{f}{f'}\right) = N\left(r, \frac{1}{f}\right) - N(r, f) - N\left(r, \frac{1}{f'}\right) + N\left(r, f'\right)$ {$p.34$, [**19**]} it follows from (2.15) that

$$m\left(r, \frac{1}{f^{(n)}-a}\right)$$
$$\leq N\left(r, f^{(n)}-a\right) + N\left(r, \frac{1}{f^{(k)}}\right) - N\left(r, f^{(k)}\right) - N\left(r, \frac{1}{f^{(n)}-a}\right)$$
$$+ S(r, f) + O(1)$$

$$i.e., \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f^{(n)}-a}\right)}{T(r,f)}$$
$$\leq \liminf_{r\to\infty} \left\{ \frac{N\left(r, f^{(n)}\right)}{T(r,f)} - \frac{N\left(r, f^{(k)}\right)}{T(r,f)} - \frac{N\left(r, \frac{1}{f^{(n)}-a}\right)}{T(r,f)} \right\} + \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f^{(k)}}\right)}{T(r,f)}.$$

$$i.e., \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f^{(n)}-a}\right)}{T(r,f)}$$
$$\leq \liminf_{r\to\infty} \frac{N\left(r, f^{(n)}\right)}{T(r,f)} - \liminf_{r\to\infty} \frac{N\left(r, f^{(k)}\right)}{T(r,f)} - \liminf_{r\to\infty} \frac{N\left(r, \frac{1}{f^{(n)}-a}\right)}{T(r,f)}$$
$$+ \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f^{(k)}}\right)}{T(r,f)}. \tag{2.16}$$

Since $\delta(\infty; f) = 1$,

$$\lim_{r\to\infty} \frac{N(r,f)}{T(r,f)} = \lim_{r\to\infty} \frac{\bar{N}(r,f)}{T(r,f)} = 0$$

and so

$$\lim_{r\to\infty} \frac{N\left(r, f^{(k)}\right)}{T(r,f)} = \lim_{r\to\infty} \frac{N(r,f)}{T(r,f)} + k \lim_{r\to\infty} \frac{\bar{N}(r,f)}{T(r,f)} = 0.$$

Similarly,

$$\lim_{r\to\infty} \frac{N\left(r, f^{(n)}\right)}{T(r,f)} = 0.$$

Thus by Lemma 2.2.2 it follows from (2.16) that

$$\delta_R^{(n)}(a;f) \leq \Delta_R^{(n)}(a;f) - 1 + \left\{1 - \delta_R^{(k)}(0;f)\right\}$$

$$i.e., \Delta_R^{(n)}(a;f) \geqslant \delta_R^{(n)}(a;f) + \delta_R^{(k)}(0;f).$$

Thus the theorem is established. ∎

Remark 2.3.4 *The condition that 'a' is a finite complex number in Theorem* 2.3.7 *is essential as we see in the following example.*

Example 2.3.3

$$\text{Let } f = \exp z \text{ , } n = 2 \text{ , } k = 1 \text{ and } a = \infty \text{ .}$$

$$\text{Then } \sum_{\alpha \neq \infty} \delta(\alpha;f) = \delta(\infty;f) = 1 \text{ .}$$

$$\text{Also } \Delta_R^{(n)}(a;f) = \delta_R^{(n)}(a;f) = \delta_R^{(k)}(0;f) = 1.$$

$$\text{Therefore } \Delta_R^{(n)}(a;f) = 1 \neq 2 = \delta_R^{(n)}(a;f) + \delta_R^{(k)}(0;f) \text{ ,}$$

which is a contrary to Theorem 2.3.7 .

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