CHAPTER - II

Note on the large deflection of a circular plate with clamped edge under symmetrical load.*

PAPER - I

Nomenclature :

The following nomenclature are used in this paper.

w = deflection, normal to the middle plane,

u = radial displacement,

a = radius of the plate,

D = flexural rigidity of the plate = $\frac{Eh^3}{12(1-\sigma^2)}$,

h = thickness of the plate,

σ = Poisson's ratio,

E = Young's modulus.

Introduction :

An approximate method for investigating the large deflection of initially flat isotropic plates has been proposed by Berger (1955). Essentially, the method is based upon neglecting the second invariant of the middle surface strains in the expression of the total potential energy of the system. The application of variational technique on the simplified energy expression yields approximate equations of equilibrium of the plate. For

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the several cases of static loading of initially flat plates investigated by Berger, the approximate equilibrium equations are in an uncoupled form. Although no complete explanation of this method is set forth, the stresses and deflections obtained for both rectangular and circular plates agree well with those found from more precise analysis. An application of this technique to the case of orthotropic plates has been considered by Iwinski and Nowinski (1957) and further boundary value problems associated with circular and rectangular plates have been investigated by Nowinski (1958). Basuli(1962) has shown that the large deflection of a cylindrical shell panel can also be obtained quite elegantly following Berger.

Nash and Modeer (1959) have shown that the method will be accurate for the problems for which there is a symmetry about the axis and for which radial membrane stress is approximately uniform. However problems without symmetry have also been treated by Berger (1955), Nash and Modeer (1957) and Sinha (1963). They are found to be in good agreement with the results known earlier.

The present author's endeavour is to find the large deflection of a clamped circular plate under symmetrical load. The corresponding linear problem was due to Sen (1935).

Analysis :

Following Berger's (1955) method, the differential equation for deflection takes the form

$$\nabla^2(\nabla^2-\alpha^2)W = \frac{\phi(r)}{D} = f(r) \quad \text{(say)} \ldots (1)$$

Where $f(r)$ is the load function,

α is a constant given by the equation

$$\frac{du}{dr} + \frac{u}{r} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 = \frac{\alpha^2 h^2}{12} \qquad \ldots(2)$$

Let us assume $\quad W = \sum_{s=1}^{\infty} A_s \left[J_0(p_s r) - J_0(p_s a)\right] \qquad \ldots(3)$

$p_s a$ being the sth root of $J_1(pa) = 0$

It is evident that the following boundary conditions are satisfied by the above equations.

$$W = \frac{dw}{dr} = 0 \quad at\ r = a.$$

Substituting equation (3) in equation (1) we have $\quad \sum A_s p_s^2 (\alpha^2 + p_s^2) J_0(p_s r) = f(r)$

If it is possible to expand $f(r)$ in a series of Bessel functions, we get $\int_0^a \sum_{s=1}^{\infty} A_s p_s^2 (p_s^2 + \alpha^2) J_0^2(p_s r) r\,dr = \int_0^a f(r) J_0(p_s r) r\,dr$

on integretion which leads to

$$A_s a^2 p_s^2 (p_s^2 + \alpha^2) \cdot \frac{J_0^2(p_s a)}{2} = \int_0^a f(r) J_0(p_s r) r\,dr$$

Hence $\quad A_s = \dfrac{2}{a^2 p_s^2 (p_s^2 + \alpha^2) \cdot J_0^2(p_s a)} \cdot \displaystyle\int_0^a f(r) J_0(p_s r) r\,dr \qquad \ldots(4)$

As an example let us suppose that the load varies as $(b^2 - r^2)^{1/2}$ over a concentric circular area of radius $b < a$.

In this case $\quad f(r) = C(b^2 - r^2)^{1/2}$ when $r < b < a$

$\qquad\qquad\qquad\quad = 0$ when $b < r < a$

where C is a constant.

Now equation (4) becomes $\quad A_s = \dfrac{2C}{a^2 p_s^2 (p_s^2 + \alpha^2) \cdot J_0^2(p_s a)} \cdot \displaystyle\int_0^a J_0(p_s r)(b^2 - r^2)^{1/2} r\,dr$

Putting $r = b \sin\theta$ we have

$$\int_0^a J_0(p_s r)(b^2 - r^2)^{1/2} r\,dr = b^3 \int_0^{\pi/2} J_0(p_s \cdot b \cdot \sin\theta) \cos^2\theta \cdot \sin\theta\, d\theta$$

Using the expansion for $J_0(\beta_s \cdot b\sin\theta)$ and integrating term by term we have the right hand side as $b^3 P(\beta_s b)$

where $$P(\beta_s b) = \frac{1}{3}\left[1 - \frac{\beta_s^2 b^2}{2.5} + \frac{\beta_s^4 b^4}{2.4.5.7} - \cdots\right]$$

Hence $$A_s = \frac{2b^3 c\, P(\beta_s b)}{\alpha^2 \beta_s^2 (\beta_s^2+\alpha^2) J_0^2(\beta_s a)} \qquad \ldots(5)$$

Combining equations (3) and (5) we have

$$W = \sum_{s=1}^{\infty} A_s \left[J_0(\beta_s r) - J_0(\beta_s a)\right]$$

$$= \frac{2b^3 c}{a^2} \sum_{s=1}^{\infty} \frac{P(\beta_s b)}{\beta_s^2 (\beta_s^2+\alpha^2) J_0^2(\beta_s a)} \cdot \left[J_0(\beta_s r) - J_0(\beta_s a)\right] \qquad \ldots(6)$$

which is convergent.

To determine the displacement u we have from (2) and (3)

$$\frac{du}{dr} + \frac{u}{r} = \frac{\alpha^2 h^2}{12} - \frac{1}{2}\left(\frac{dw}{dr}\right)^2$$

$$= \frac{\alpha^2 h^2}{12} - \frac{1}{2}\sum_{s=1}^{\infty} A_s^2 \cdot \beta_s^2 \cdot J_1^2(\beta_s r) - \frac{1}{2}\sum_{s=1}^{\infty}\sum_{m=1,\ s\neq m}^{\infty} A_s \cdot A_m \cdot \beta_s \cdot \beta_m \cdot J_1(\beta_s r)\cdot J_1(\beta_m r)$$

Integrating with respect to r, one gets

$$ur = \frac{\alpha^2 h^2 r^2}{24} - \frac{1}{2}\sum_{s=1}^{\infty} A_s^2 \beta_s^2 \left[\frac{r^2}{2}\left\{\left(1 - \frac{1}{\beta_s^2 r^2}\right) J_1^2(\beta_s r) + J_1'^2(\beta_s r)\right\}\right]$$

$$- \frac{1}{2}\sum_{s=1}^{\infty}\sum_{m=1,\ s\neq m}^{\infty} A_s A_m \beta_s \beta_m \left[r\left\{\frac{\beta_s \cdot J_2(\beta_s r)\cdot J_1(\beta_m r) - \beta_m J_1(\beta_s r)\cdot J_2(\beta_m r)}{\beta_s^2 - \beta_m^2}\right\}\right]$$

$$+ K \qquad \ldots(7)$$

K being the constant of integration.

Using $u = 0$, at $r = a$

$$K = \frac{1}{4}\sum_{s=1}^{\infty} A_s^2 p_s^2 a^2 J_0^2(p_s a) - \frac{\alpha^2 h^2 a^2}{24} \qquad \text{since } J_1(p_s a) = 0$$

To determine α we know that as $r \to 0, u \to 0$ from symmetry, and then equation (7) leads to

$$\frac{\alpha^2 h^2 a^2}{24} = \frac{1}{4}\sum_{s=1}^{\infty} A_s^2 p_s^2 a^2 J_0^2(p_s a) \qquad \ldots(8)$$

Putting $\alpha = 0$ the differential equation (1) corresponds to that of small deflection equation. Now as α tends to zero, equation (6) leads to

$$W = \frac{2b^3 c}{a^2}\sum_{s=1}^{\infty} \frac{P(p_s b)\left[J_0(p_s r) - J_0(p_s a)\right]}{p_s^4 \cdot J_0^2(p_s a)} \qquad \ldots(9)$$

as obtained by Sen (1935),

where $p_s a$ is the sth root of $J_1(pa) = 0$

The deflection will be maximum at the centre of the plate. From equation (6) maximum deflection can be obtained by putting $r = 0$ as

$$W_0 = \frac{2b^3 c}{a^2}\sum_{s=1}^{\infty} \frac{P(p_s b)}{p_s^2(p_s^2 + \alpha^2) J_0^2(p_s a)}\left[1 - J_0(p_s a)\right] \qquad \ldots(10)$$

whereas for small deflection W_0 will be given by

$$W_0 = \frac{2b^3 c}{a^2}\sum_{s=1}^{\infty} \frac{P(p_s b)}{p_s^4 J_0^2(p_s a)}\left[1 - J_0(p_s a)\right] \qquad \ldots(11)$$

from either equation (9) or equation (10).

In figure (4) W_0/h has been plotted against $b^5 c/h$ for both large and small deflection assuming $a = 2b$.

Discussion :

Small deflection theory of a plate which assumes the deflections small as compared with the thickness of the plate is based on the neglect of middle surface strains. In cases in which the deflections are no longer small in comparison with the thickness of the plate but are still small as compared with the other dimensions, the analysis of the problem must be extended to include the strain of the middle plane of the plate. For such problems, strain displacement relations are non-linear. In the present problem the latter theory is investigated.

The graph is plotted against cb^5/h for central deflection w_0/h. In calculating the deflection one has to start from equation (8) with an assumed value of αa leading to a particular value for the load function cb^5/h .

These values of αa and cb^5/h determine w_0/h from equation (10), Here αa has been assumed 1, 2, 3, etc.

It is clear that as αa increases, the load increases, also the central deflection.

For small deflection, w_0/h has been calculated from equation (11) for different values of cb^5/h and has been plotted side by side for comparison.

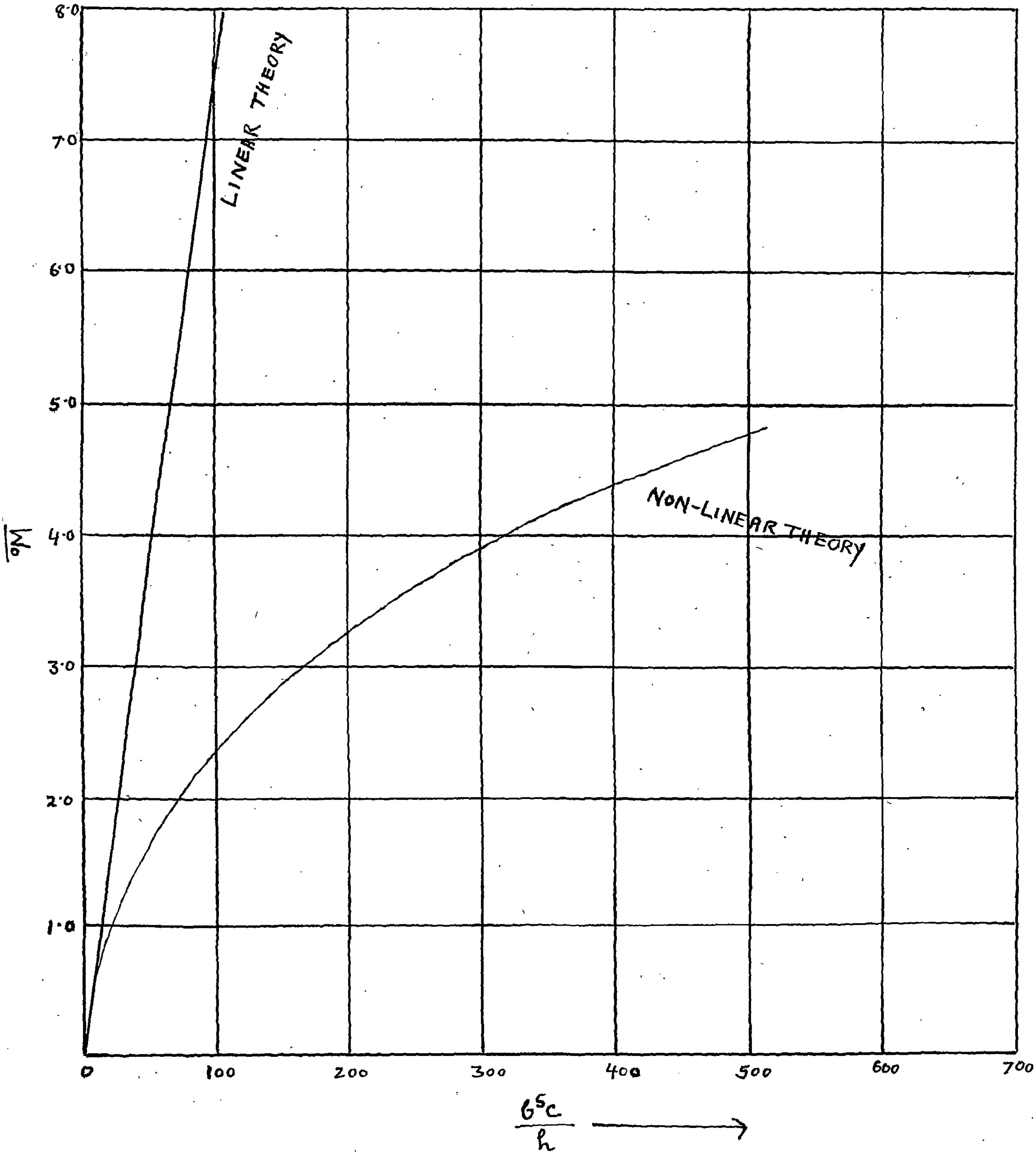

FIG. 4

Graph showing central deflections for various values of the load function $\frac{b^5 c}{h}$

Note on the large deflection of a circular plate under concentrated load *

P A P E R - II

Nomenclature :

The following nomenclature are used in this paper.

a = radius of the plate,

w = lateral displacement,

u, v = radial and cross-radial displacements,

h = thickness of the plate,

D = flexural rigidity of the plate $= Eh^3/12(1-\sigma^2)$,

E = Young's modulus,

σ = Poisson's ratio,

P = concentrated load at a distance b from the centre.

Introduction :

Following Berger (1955), many problems on the large deflection of initially flat isotropic plates have been investigated. In this paper the large deflection of a clamped circular plate under a concentrated load at a distance from the centre has been investigated.

The corresponding problem with the load at the centre was obtained by Basuli (1961).

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Fundamental Equations and Solution of the Problem :

Following Berger [1955], the deflection w of the plate (except at the load) satisfies the equation

$$\nabla_1^2(\nabla_1^2-\alpha^2)w=0 \qquad \ldots (1)$$

where

$$\frac{\alpha^2h^2}{12}=\frac{\partial u}{\partial r}+\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \qquad \ldots (2)$$

u, v being radial and crossradial displacements and α being supposed to be constant.

Let the concentrated load P be placed at a distance b from the centre of the plate and let the radius of the plate be a. To solve the problem let us divide the plate by a concentric cylindrical surface of radius b passing through the load. Taking the line joining the centre of the plate and the load as the initial line and centre of the plate as pole, the equation (1) can be written as

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}-\alpha^2 w\right)=0 \ldots (3)$$

As solutions of (3) we assume

$$w=w_1=R_0+\sum_{m=1}^{\infty}R_m\cos m\theta, \qquad \text{for } r>b \quad \ldots (4)$$

$$w=w_2=R_0'+\sum_{m=1}^{\infty}R_m'\cos m\theta, \qquad \text{for } r<b \quad \ldots (5)$$

where R_0, R_0', R_m, R_m' are functions of r only.

Now substituting (4) in (3) and considering the contributions of R_0 and R'_0 only, we see that they satisfy the equation of the form

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}-\alpha^2R\right)=0 \qquad \text{.... (6)}$$

The solutions of (6) may be put in the form

$$R=R_0=A_0I_0(\alpha r)+B_0K_0(\alpha r)+C_0+D_0\log r\,, \quad \text{for} \quad r>b \quad \text{... (7)}$$

$$R=R'_0=A'_0I_0(\alpha r)+C'_0\,, \qquad \text{for} \quad r<b \quad \text{... (8)}$$

where $I_0(\alpha r)$, $K_0(\alpha r)$ are the Modified Bessel functions of the 1st and 2nd kind of order zero.

If the boundary be clamped,

$$W_1=\frac{\partial W_1}{\partial r}=0\,, \qquad \text{on} \quad r=a \qquad \text{... (9)}$$

As the deflections, slope and bending moment will be continuous on the dividing circle, we get,

$$W_1=W_2\,;\ \frac{\partial W_1}{\partial r}=\frac{\partial W_2}{\partial r}\,;\ \frac{\partial^2 W_1}{\partial r^2}=\frac{\partial^2 W_2}{\partial r^2} \qquad \text{on} \quad r=b \quad \text{... (10)}$$

These continuity conditions ensure the same α for W_1 and W_2. The equations (7) and (8) contain altogether six constants and relations in (9) and

(10) are five in number. To get the sixth relation we shall have to consider the shearing force on the dividing circle, and this is continuous at every point of that circle except at the concentrated load.

Representing the load in the form of an infinite series

$$\frac{P}{\pi b}\left\{\frac{1}{2}+\sum_{m=1}^{\infty}\cos m\theta\right\}$$

the discontinuity in the shearing force is given by

$$\left[D\frac{\partial}{\partial r}\left\{\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\alpha^2\right\}w_1\right]_{r=b}$$

$$-\left[D\frac{\partial}{\partial r}\left\{\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\alpha^2\right\}w_2\right]_{r=b}$$

$$=\frac{P}{\pi b}\left\{\frac{1}{2}+\sum_{m=1}^{\infty}\cos m\theta\right\} \qquad \ldots (11)$$

where D is flexural rigidity of the plate and $=Eh^3/12(1-\sigma^2)$, σ is Poisson's ratio, E is Young's modulus, h is thickness of the plate. Substituting equations (7) and (8) in (9), (10) and (11) and solving for A_0, B_0, C_0, D_0, C_0', A_0' we get,

$$R_0=\frac{P}{2\pi D\alpha^3 a I_1(\alpha a)}\Big[I_0(\alpha r)+I_0(\alpha b)+\alpha a I_1(\alpha a)\log a/r$$

$$-I_0(\alpha r)I_0(\alpha b)K_1(\alpha a)\alpha a-I_0(\alpha a)-I_1(\alpha a)K_0(\alpha r)I_0(\alpha b)\alpha a\Big]$$

$$R_0' = \frac{P}{2\pi D\alpha^3 a I_1(\alpha a)}\Big[I_0(\alpha r) + I_0(\alpha b) - I_0(\alpha a) - \alpha a I_1(\alpha a)\operatorname{Log} b/a$$

$$- \alpha a I_0(\alpha r) I_1(\alpha a) K_0(\alpha b) - I_0(\alpha r)\alpha a I_0(\alpha b) K_1(\alpha a)\Big]$$

Now taking the equations (3), (4) and (5) and contributions of R_m, R_m' only we see that they satisfy the equation of the form

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{m^2}{r^2}R - \alpha^2 R\right) = 0$$

This equation is satisfied if

$$R = R_m = A_m I_m(\alpha r) + B_m K_m(\alpha r) + C_m r^m + D_m r^{-m}, \quad r > b \qquad \ldots (12)$$

$$R = R_m' = A_m' I_m(\alpha r) + C_m' r^m, \qquad r < b \qquad \ldots (13)$$

where $I_m(\alpha r)$; $K_m(\alpha r)$ **are Modified Bessel functions of order** m. **Considering the equations (9),(10),(11) (12) and (13) and solving for the constants** A_m, B_m, C_m, D_m, A_m', C_m', **we get**

$$R_m = \frac{P}{\pi D\alpha^3 a I_{m+1}(\alpha a)}\Big[(b/a)^m I_m(\alpha r) - \alpha a I_m(\alpha b) K_{m+1}(\alpha a) I_m(\alpha r)$$

$$- I_m(\alpha b) K_m(\alpha r)\alpha a I_{m+1}(\alpha a) + \left(\frac{r}{a}\right)^m \left\{I_m(\alpha b) - \left(\frac{b}{a}\right)^m I_{m-1}(\alpha a)\frac{\alpha a}{2m}\right\}$$

$$+ \left(\frac{b}{r}\right)^m \frac{\alpha a}{2m} I_{m+1}(\alpha a)\Big]$$

$$R_m' = \frac{P}{\pi D\alpha^3 a I_{m+1}(\alpha a)}\Big[I_m(\alpha r)\left(\frac{b}{a}\right)^m - I_m(\alpha r) K_m(\alpha b)\alpha a I_{m+1}(\alpha a)$$

$$- I_m(\alpha r) I_m(\alpha b) K_{m+1}(\alpha a)\alpha a + \left(\frac{r}{a}\right)^m \Big\{I_m(\alpha b) + \left(\frac{a}{b}\right)^m \frac{\alpha a}{2m} I_{m+1}(\alpha a)$$

$$- \left(\frac{b}{a}\right)^m \frac{\alpha a}{2m} I_{m-1}(\alpha a)\Big\}\Big]$$

Therefore

$$w_1 = R_0 + \sum_{m=1}^{\infty} R_m \cos m\theta \,, \qquad r > b$$
$$w_2 = R_0' + \sum R_m' \cos m\theta \,, \qquad r < b$$

are determined.

To find the constant α , we have the following equation for α

$$\frac{\alpha^2 h^2}{12} = \frac{\partial U}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{U}{r} + \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 .$$

Let U_1 , V_1 be the radial, cross-radial displacements when $r > b$ and U_2, V_2 be those displacements for $r < b$.

Let

$$\left.\begin{aligned} U_1 &= U_0(r) + \sum_{m=1}^{\infty} U_m(r) \cos m\theta \\ V_1 &= V_0(r) + \sum_{m=1}^{\infty} V_m(r) \sin m\theta \end{aligned}\right\} \quad r > b$$

$$\left.\begin{aligned} U_2 &= U_0'(r) + \sum_{m=1}^{\infty} U_m'(r) \cos m\theta \\ V_2 &= V_0'(r) + \sum_{m=1}^{\infty} V_m'(r) \sin m\theta \end{aligned}\right\} \quad r < b$$

As we have no interest in the radial and cross-radial displacements U, V we eliminate them by multiplying the last equation by $r\,d\theta\,dr$ and integrating between the limits b to a and 0 to 2π. For the outer portion we have,

$$\frac{\alpha^2 h^2}{12}\int_0^{2\pi}\int_b^a r\,dr\,d\theta = \int_0^{2\pi}\int_b^a \frac{\partial U_0(r)}{\partial r} r\,dr\,d\theta + \sum_{m=1}^{\infty}\int_0^{2\pi}\int_b^a \frac{\partial U_m(r)}{\partial r} r\cos m\theta\,dr\,d\theta$$

$$+\int_0^{2\pi}\int_b^a U_0(r)\,dr\,d\theta + \sum_{m=1}^{\infty}\int_0^{2\pi}\int_b^a U_m(r)\cos m\theta\,dr\,d\theta$$

$$+\frac{1}{2}\int_0^{2\pi}\int_b^a \left(\frac{\partial w_1}{\partial r}\right)^2 r\,dr\,d\theta + \frac{1}{2}\int_0^{2\pi}\int_b^a \frac{1}{r}\left(\frac{\partial w_1}{\partial \theta}\right)^2 dr\,d\theta$$

$$+\sum_{m=1}^{\infty} m\int_0^{2\pi}\int_b^a V_m(r)\cos m\theta\,dr\,d\theta$$

which leads to

$$\frac{\alpha^2 h^2 \pi(a^2-b^2)}{12} = 2\pi\left[aU_0(a) - bU_0(b)\right]$$

$$+\frac{1}{2}\int_0^{2\pi}\int_b^a \left(\frac{\partial w_1}{\partial r}\right)^2 r\,dr\,d\theta + \frac{1}{2}\int_0^{2\pi}\int_b^a \frac{1}{r}\left(\frac{\partial w_1}{\partial \theta}\right)^2 dr\,d\theta \qquad \ldots (14)$$

Similarly for the inner portion we have,

$$\frac{\alpha^2 h^2}{12}\int_0^{2\pi}\int_0^b r\,d\theta\,dr = \frac{\pi\alpha^2 h^2 b^2}{12} = 2\pi b U_0'(b) + \frac{1}{2}\int_0^{2\pi}\int_0^b \left(\frac{\partial w_2}{\partial r}\right)^2 r\,dr\,d\theta$$

$$+\frac{1}{2}\int_0^{2\pi}\int_0^b \frac{1}{r}\left(\frac{\partial w_2}{\partial \theta}\right)^2 dr\,d\theta \qquad \ldots (15)$$

Now, on $r=b$, $U_0(a)=0$, $U_0(b)=U_0'(b)$. Using these in equations (14) and (15) and adding together we get on substitution the expressions for W_1 and W_2 ;

$$
\begin{aligned}
\frac{\alpha^2 h^2 a^2}{6} = {} & 2A_0^2\left[\frac{\alpha^2a^2}{2}\left\{I_1^2(\alpha a) - I_0^2(\alpha a)\right\} + \alpha a I_1(\alpha a) I_0(\alpha a)\right. \\
& \left. - \frac{\alpha^2b^2}{2}\left\{I_1^2(\alpha b) - I_0^2(\alpha b)\right\} - \alpha b I_1(\alpha b) I_0(\alpha b)\right] \\
& + 2B_0^2\left[\frac{\alpha^2a^2}{2}\left\{K_1^2(\alpha a) - K_0^2(\alpha a)\right\} - \alpha a K_1(\alpha a) K_0(\alpha a)\right. \\
& \left. - \frac{\alpha^2b^2}{2}\left\{K_1^2(\alpha b) - K_0^2(\alpha b)\right\} + \alpha b K_1(\alpha b) K_0(\alpha b)\right] \\
& + 2D_0^2 \log\frac{a}{b} - 4A_0B_0\left[\frac{1}{2}\alpha^2a^2\left\{I_1(\alpha a) K_1(\alpha a) + I_0(\alpha a) K_0(\alpha a)\right\}\right. \\
& - \frac{1}{2}(\alpha a)\left\{I_1(\alpha a) K_0(\alpha a) - I_0(\alpha a) K_1(\alpha a)\right\} \\
& - \frac{1}{2}\alpha^2b^2\left\{I_1(\alpha b) K_1(\alpha b) + I_0(\alpha b) K_0(\alpha b)\right\} + \frac{1}{2}\alpha b\left\{I_1(\alpha b) K_0(\alpha b)\right. \\
& \left.\left. - I_0(\alpha b) K_1(\alpha b)\right\}\right] + 4A_0D_0\left[I_0(\alpha a) - I_0(\alpha b)\right] \\
& + 4B_0D_0\left[K_0(\alpha a) - K_0(\alpha b)\right] \\
& + 2A_0'^2\left[\frac{\alpha^2b^2}{2}\left\{I_1^2(\alpha b) - I_0^2(\alpha b)\right\} + \alpha b I_1(\alpha b) I_0(\alpha b)\right] \\
& + \sum_{m=1}^{\infty} 2m A_m C_m\left[a^m I_m(\alpha a) - b^m I_m(\alpha b)\right] \\
& + \sum_{m=1}^{\infty} 2m A_m' C_m' b^m I_m(\alpha b) \\
& + \sum_{m=1}^{\infty} 2m B_m C_m\left[a^m K_m(\alpha a) - b^m K_m(\alpha b)\right] \\
& + \sum_{m=1}^{\infty} 2m B_m D_m\left[K_m(\alpha b) b^{-m} - K_m(\alpha a) a^{-m}\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{m=1}^{\infty} 2m A_m D_m \left[I_m(\alpha a)\, a^{-m} - I_m(\alpha b)\, b^{-m} \right] \\
&+\sum_{m=1}^{\infty} A_m^2 \Big[\tfrac{1}{2}\alpha^2 a^2 I_{m+1}^2(\alpha a) + (m+1)\alpha a I_m(\alpha a) I_{m+1}(\alpha a) \\
&\qquad - \tfrac{1}{2}\alpha^2 a^2 I_m^2(\alpha a) - \tfrac{1}{2}\alpha^2 b^2 I_{m+1}^2(\alpha b) - (m+1)\alpha b I_m(\alpha b) I_{m+1}(\alpha b) + \tfrac{1}{2}\alpha^2 b^2 I_m^2(\alpha b) \Big] \\
&+\sum_{m=1}^{\infty} B_m^2 \Big[\tfrac{1}{2}\alpha^2 a^2 K_{m+1}^2(\alpha a) - (m+1)\alpha a K_m(\alpha a) K_{m+1}(\alpha a) - \tfrac{1}{2}\alpha^2 a^2 K_m^2(\alpha a) \\
&\qquad - \tfrac{1}{2}\alpha^2 b^2 K_{m+1}^2(\alpha b) + (m+1)\alpha b K_m(\alpha b) K_{m+1}(\alpha b) + \tfrac{1}{2}\alpha^2 b^2 K_m^2(\alpha b) \Big] \\
&+\sum_{m=1}^{\infty} A_m'^2 \Big[\tfrac{1}{2}\alpha^2 b^2 I_{m+1}^2(\alpha b) + (m+1)\alpha b I_m(\alpha b) I_{m+1}(\alpha b) - \tfrac{1}{2}\alpha^2 b^2 I_m^2(\alpha b) \Big] \\
&+\sum_{m=1}^{\infty} m A_m^2 \left[I_m^2(\alpha a) - I_m^2(\alpha b) \right] + \sum_{m=1}^{\infty} m A_m'^2 I_m^2(\alpha b) \\
&\qquad + \sum_{m=1}^{\infty} m B_m^2 \left[K_m^2(\alpha a) - K_m^2(\alpha b) \right] \\
&+\sum_{m=1}^{\infty} 2m A_m B_m \left[I_m(\alpha a) K_m(\alpha a) - I_m(\alpha b) K_m(\alpha b) \right] \\
&-\sum_{m=1}^{\infty} 2 A_m B_m \Big[\tfrac{1}{2}\alpha^2 a^2 \left\{ I_{m+1}(\alpha a) K_{m+1}(\alpha a) + I_m(\alpha a) K_m(\alpha a) \right\} \\
&\qquad - (m+1)\frac{\alpha a}{2} \left\{ I_{m+1}(\alpha a) K_m(\alpha a) - I_m(\alpha a) K_{m+1}(\alpha a) \right\} \\
&\qquad - \tfrac{1}{2}\alpha^2 b^2 \left\{ I_{m+1}(\alpha b) K_{m+1}(\alpha b) + I_m(\alpha b) K_m(\alpha b) \right\} \\
&\qquad + (m+1)\frac{\alpha b}{2} \left\{ I_{m+1}(\alpha b) K_m(\alpha b) - I_m(\alpha b) K_{m+1}(\alpha b) \right\} \Big] \\
&+\sum_{m=1}^{\infty} m D_m^2 \left[b^{-2m} - a^{-2m} \right] + \sum_{m=1}^{\infty} m C_m^2 \left[a^{2m} - b^{2m} \right] \\
&+\sum_{m=1}^{\infty} \left[m C_m'^2 b^{2m} + 2m^2 C_m D_m \operatorname{Log} \frac{b}{a} \right]
\end{aligned}
$$

$\cdots (16)$

where

$$A_0 = \frac{P}{2\pi D\alpha^3 a I_1(\alpha a)}\left[1 - I_0(\alpha b)K_1(\alpha a)\alpha a\right]$$

$$B_0 = -\frac{P I_0(\alpha b)}{2\pi D\alpha^2}$$

$$C_0 = \frac{P}{2\pi D\alpha^3 a I_1(\alpha a)}\left[I_0(\alpha b) - I_0(\alpha a) + \alpha a I_1(\alpha a)\operatorname{Log} a\right]$$

$$D_0 = -\frac{P}{2\pi D\alpha^2}$$

$$A_0' = \frac{P}{2\pi D\alpha^3 a I_1(\alpha a)}\left[1 - \alpha a I_1(\alpha a)K_0(\alpha b) - \alpha a I_0(\alpha a)K_1(\alpha a)\right]$$

$$C_0' = \frac{P}{2\pi D\alpha^3 a I_1(\alpha a)}\left[I_0(\alpha b) - I_0(\alpha a) - \alpha a I_0(\alpha a)\operatorname{Log}\frac{b}{a}\right]$$

$$A_m = \frac{P}{\pi D\alpha^3 a I_{m+1}(\alpha a)}\left[\left(\frac{b}{a}\right)^m - \alpha a I_m(\alpha b)K_{m+1}(\alpha a)\right]$$

$$B_m = -\frac{P I_m(\alpha b)}{\pi D\alpha^2}$$

$$C_m = \frac{P}{\pi D\alpha^3 a^{m+1} I_{m+1}(\alpha a)}\left[I_m(\alpha b) - \left(\frac{b}{a}\right)^m I_{m-1}(\alpha a)\frac{\alpha a}{2m}\right]$$

$$D_m = \frac{P b^m}{\pi D\alpha^2 2m}$$

$$A_m' = \frac{P}{\pi D\alpha^3 a I_{m+1}(\alpha a)}\Big[\left(\frac{b}{a}\right)^m - K_m(\alpha b)\alpha a I_{m+1}(\alpha a) - I_m(\alpha b)K_{m+1}(\alpha a)\alpha a\Big]$$

$$C_m' = \frac{P}{\pi D\alpha^3 a^{m+1} I_{m+1}(\alpha a)}\Big[I_m(\alpha b) + \left(\frac{a}{b}\right)^m \frac{\alpha a}{2m} I_{m+1}(\alpha a) - \left(\frac{b}{a}\right)^m \frac{\alpha a}{2m} I_{m-1}(\alpha a)\Big]$$

Now the large deflection of the plate under a concentrated load at the centre can be obtained from W_1 by making b tend to zero everywhere in the form [Basuli (1961)]

$$W = -\frac{P}{2\pi D\alpha^3 a I_1(\alpha a)}\Big[\alpha a\Big\{K_1(\alpha a) I_0(\alpha r) + K_0(\alpha r) I_1(\alpha a)\Big\}$$

$$+ \alpha a I_1(\alpha a) \operatorname{Log}\frac{r}{a} - I_0(\alpha r) + I_0(\alpha a) - 1\Big].$$

Also the value of the constant α under a concentrated load at the centre can also be obtained from (16) taking limit as $b \to o$ in the form

$$\left(\frac{Pa^2}{\pi D h}\right)^2 = \frac{\frac{1}{3}(\alpha a)^6}{\gamma + \operatorname{Log}\frac{\alpha a}{2} - \frac{I_0(\alpha a) + \alpha a K_1(\alpha a) - 2}{\alpha a I_1(\alpha a)} - \frac{1}{2}\left\{\frac{I_0(\alpha a) - 1}{I_1(\alpha a)}\right\}^2}.$$

where γ = Euler's constant.

The deflection is obtained for a plate with $\frac{a}{b} = 2$ for various loads. The figure shows the deflection $\left(\frac{W_2}{h}\right)_{r=b}$ against $\frac{Pa^2}{\pi D h}$. In calculating the deflection one has to start from equation (16) with assumed values of αa and αb leading to the particular value for the load function $\frac{Pa^2}{\pi D h}$. These values of αa and αb together with $\frac{Pa^2}{\pi D h}$ determine corresponding $\left(\frac{W_2}{h}\right)_{r=b}$.

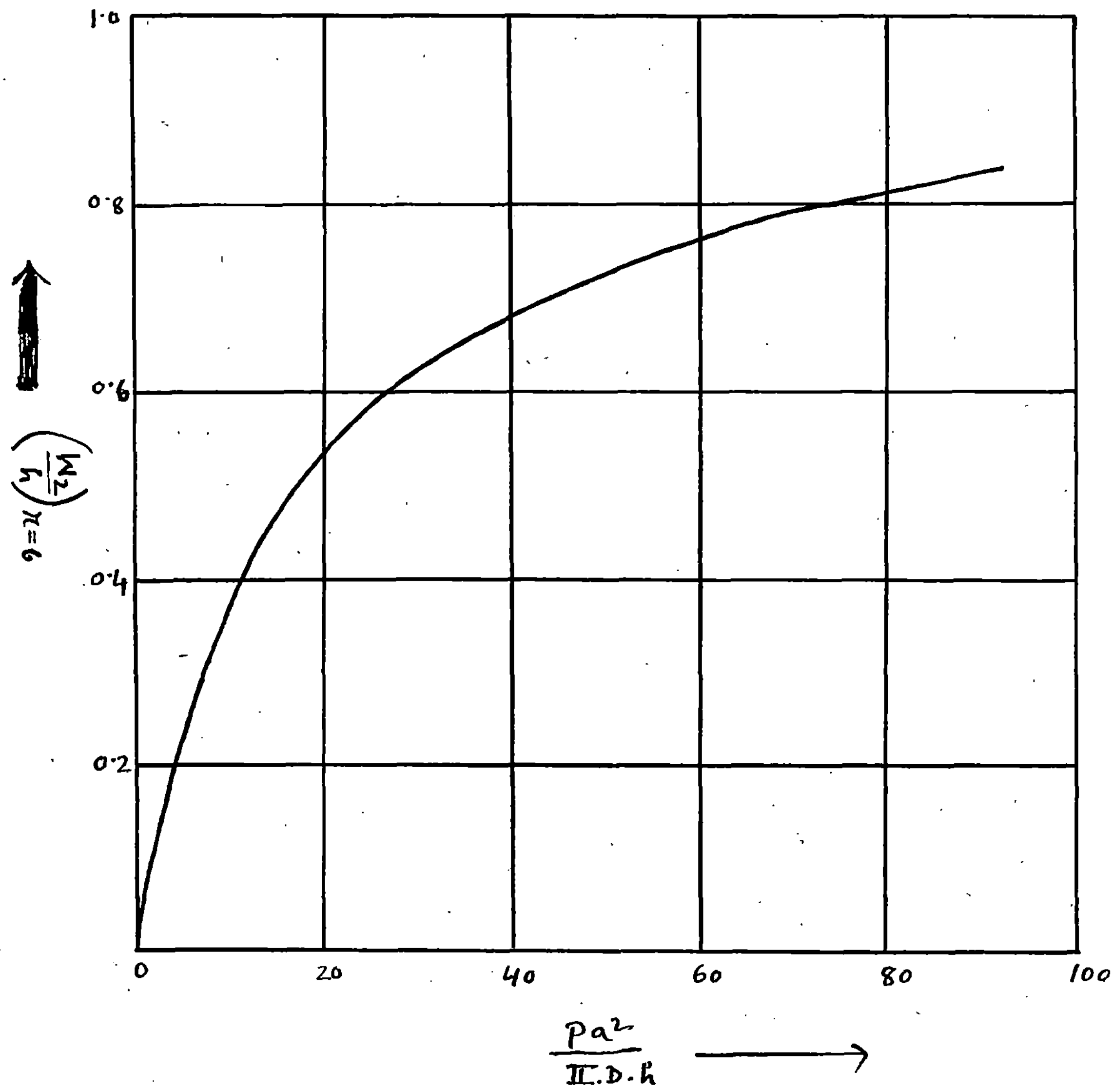

FIG. 5

Graph showing deflections for various values of $\frac{Pa^2}{\Pi Dh}$

Note on the large deflection of an orthotropic circular plate under a concentrated load.*

PAPER - III

Nomenclature :

The following nomenclature are used in this paper.

P = concentrated load at the centre,

u = radial displacement,

w = deflection, normal to the plane,

a = radius of the plate,

h = thickness of the plate,

D_r = average flexural rigidity of the plate,

σ_r, σ_t = Poisson's ratios corresponding to radial and cross - radial directions.

$K^2 = \frac{\sigma_t}{\sigma_r}$

Introduction :

Following Berger's (1955) approximate method, numerous problems have been solved with remarkable ease and satisfactory results.

Iwinski and Nowinski (1957) generalized the procedure of Berger to orthotropic plates and found out the deflections of circular and rectangular plates under uniform load with different boundary conditions. In this

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paper the above method has been applied in the case of an orthotropic circular plate under a concentrated load at the centre.

Analysis :

In the case of circular symmetry if h is the thickness of the plate, w the displacement perpendicular to the middle plane, u the radial displacement in the middle plane under a concentrated load at the centre, then the differential equation for w and u will be (Iwinski and Nowinski, 1957)

$$\frac{d^4 w}{dr^4} + \frac{2}{r}\frac{d^3 w}{dr^3} - \frac{K^2}{r^2}\left(\frac{d^2 w}{dr^2} - \frac{1}{r}\frac{dw}{dr}\right) - \frac{12}{h^2 r}\frac{d}{dr}\left(e_1^* r \frac{dw}{dr}\right) = 0 \qquad \ldots(1)$$

except at the load.

and

$$\frac{de_1^*}{dr} + \frac{1-K}{r} e_1^* = 0 \qquad \ldots(2)$$

where $$e_1^* = \frac{du}{dr} + K\frac{u}{r} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \qquad \ldots(3)$$

Again considering the radial stress and shearing stress on a concentric circular ring of radius r, the concentrated load P at the centre, and since u and $\frac{dw}{dr}$ are both zero at the centre, we have,

$$D_r \cdot \lim_{r \to 0} r\left[\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{K^2}{r^2}\frac{dw}{dr} - \frac{12}{h^2}e_1^* \cdot \frac{dw}{dr}\right] = \frac{P}{2\pi} \qquad \text{...(4)}$$

Solving (2) we have

$$e_1^* = c r^{K-1} \qquad \text{...(5)}$$

Hence we have the following differential equation for w

$$r^3\frac{d^4w}{dr^4} + 2r^2\frac{d^3w}{dr^3} - r\left(K^2+\chi^2 r^{K+1}\right)\frac{d^2w}{dr^2} + \left(K^2-\chi^2 K r^{K+1}\right)\frac{dw}{dr} = 0 \qquad \text{...(6)}$$

where $$\chi^2 = \frac{12c}{h^2}$$

After changing the variables the equation takes the form

$$\frac{d^3z}{dr^3} + 2r^{-1}\frac{d^2z}{dr^2} - r^{-2}\left(K^2+\chi^2 r^{K+1}\right)\frac{dz}{dr} + r^{-3}\left(K^2-\chi^2 K r^{K+1}\right)z = 0 \qquad \text{...(7)}$$

The above equation can be put in the form

$$\left(\frac{d}{dr} + \frac{1}{r}\right)\left[\frac{d^2z}{dr^2} + \frac{1}{r}\frac{dz}{dr} - \left(K^2 r^{-2} + \chi^2 r^{K-1}\right)z\right] = 0 \qquad \text{...(8)}$$

This equation can be represented by a system of two differential equations,

$$\frac{d^2z}{d\eta^2} + \frac{1}{\eta}\frac{dz}{d\eta} - \left(K^2\eta^{-2} + \lambda^2\eta^{K-1}\right)z = P_\cdot'(\eta) \qquad \ldots(9)$$

$$\frac{dP'}{d\eta} + P'\eta^{-1} = 0 \qquad \ldots(10)$$

Solving (10) we get

$$P' = \frac{C_3'}{\eta} \qquad \ldots(11)$$

Hence equation (9) is equivalent to

$$\eta^2\frac{d^2z}{d\eta^2} + \eta\frac{dz}{d\eta} - \left(K^2 + \lambda^2\eta^{K+1}\right)z = C_3'\eta \qquad \ldots(12)$$

Solving (12) the deflection w can be put in the form

$$w = C_1\eta^{\frac{1-K}{2}} I_{\frac{K-1}{K+1}}\left(\frac{2\lambda\eta^{\frac{1+K}{2}}}{1+K}\right) + C_2\left[\eta^{\frac{1-K}{2}} K_{\frac{K-1}{K+1}}\left(\frac{2\lambda\eta^{\frac{1+K}{2}}}{1+K}\right) - \right.$$

$$\left. - \mu\left\{\frac{a^{1-K}}{1-K} - \frac{\eta^{1-K}}{1-K}\right\}\right] + C_3 \qquad \ldots(13)$$

where $\mu = \frac{1}{2}\Gamma\left(\frac{2K}{1+K}\right)(\lambda)^{1-\frac{2K}{1+K}}(1+K)^{\frac{2K}{1+K}}$

and I and K represent Modified

Bessel functions of first and second kind.

Boundary conditions on w **are**

$$w = \frac{dw}{dr} = 0 \quad \text{at } r = a \qquad \ldots(14)$$

Considering equations (4) and (13) we have

$$C_2 = -\frac{P}{\mu\, 2\pi\, \lambda^2 D_r} \qquad \ldots(15)$$

Combining equations(13), (14) and (15) we have

$$C_1 = C_2 \left[\frac{\lambda a^{K} K_{\frac{2K}{1+K}}\left(\frac{2\lambda a^{\frac{1+K}{2}}}{1+K}\right) - \mu}{\lambda a^{K} I_{\frac{2K}{1+K}}\left(\frac{2\lambda a^{\frac{1+K}{2}}}{1+K}\right)} \right]$$

$$C_3 = C_2 \cdot \frac{1}{\lambda a^{K} I_{\frac{2K}{1+K}}\left(\frac{2\lambda a^{\frac{1+K}{2}}}{1+K}\right)} \left[\mu a^{\frac{1-K}{2}} I_{\frac{K-1}{K+1}}\left(\frac{2\lambda a^{\frac{1+K}{2}}}{1+K}\right) - \frac{1+K}{2} \right]$$

To determine the displacement u **we have from equation (2)**

$$\frac{du}{dr} + \frac{K}{r} u = C r^{K-1} - \frac{1}{2}\left(\frac{dw}{dr}\right)^2$$

Substituting the expression for w from (13) and solving for u one gets,

$$r^K u = \frac{\lambda^2 h^2}{24K} r^{2K} - \frac{1}{2}\int r^K \Big[c_1^2 \lambda^2 I^2_{\frac{2K}{1+K}}(z_1)$$

$$+ c_2^2 \Big\{ \lambda^2 K^2_{\frac{2K}{1+K}}(z_1) + r^{-2K}\mu^2 - 2\lambda\mu K_{\frac{2K}{1+K}}(z_1) r^{-K} \Big\}$$

$$+ 2c_1c_2 \Big\{ \mu\lambda r^{-K} I_{\frac{2K}{1+K}}(z_1) - \lambda^2 I_{\frac{2K}{1+K}}(z_1) K_{\frac{2K}{1+K}}(z_1) \Big\}\Big] dr$$

$$+ K_1$$

where $z_1 = \dfrac{2\lambda r^{\frac{1+K}{2}}}{1+K}$

After evaluating the integrals we have,

$$r^K u = \frac{\lambda^2 h^2}{24K} r^{2K} - \frac{1}{2} c_1^2 \Big[\frac{\lambda^2 r^{1+K}}{1+K} \Big\{ \Big[I_{\frac{2K}{1+K}}(z_1) \Big]^2 \Big[1 + \frac{K^2}{\lambda^2} r^{-1-K} \Big]$$

$$- \Big[I'_{\frac{2K}{1+K}}(z_1) \Big]^2 \Big\}\Big] -$$

$$- \frac{C_2^2}{2}\left[\frac{\lambda^2 r^{1+K}}{1+K}\left\{\left[K_{\frac{2K}{1+K}}(z_1)\right]^2\left[1+\frac{K^2}{\lambda^2}r^{-1-K}\right]-\left[K'_{\frac{2K}{1+K}}(z_1)\right]^2\right\}\right]$$

$$- \frac{C_2^2}{2}\frac{r^{1-K}}{1-K}\mu^2 - \mu C_2^2 r^{\frac{1-K}{2}} K_{\frac{K-1}{K+1}}(z_1) - C_1 C_2 \mu r^{\frac{1-K}{2}} I_{\frac{K-1}{K+1}}(z_1)$$

$$+ C_1 C_2 \frac{\lambda^2 r^{1+K}}{1+K}\left[I_{\frac{2K}{1+K}}(z_1) K_{\frac{2K}{1+K}}(z_1) + I_{\frac{2K}{1+K}-1}(z_1) K_{\frac{2K}{1+K}-1}(z_1)\right.$$

$$\left. + \frac{K}{\lambda} r^{\frac{-1-K}{2}}\left\{K_{\frac{2K}{1+K}}(z_1) I_{\frac{2K}{1+K}-1}(z_1) - I_{\frac{2K}{1+K}}(z_1) K_{\frac{2K}{1+K}-1}(z_1)\right\}\right]$$

$$+ K_1 \qquad \ldots(16)$$

Using the boundary condition $r \to a,\ u \to 0$ **the integration constant** K_1 **can be evaluated as**

$$K_1 = \frac{C_1^2}{2}\left[\frac{\lambda^2 a^{1+K}}{1+K}\left\{\left[I_{\frac{2K}{1+K}}(z_2)\right]^2\left[1+\frac{K^2}{\lambda^2}a^{-1-K}\right]-\left[I'_{\frac{2K}{1+K}}(z_2)\right]^2\right\}\right] +$$

$$+\frac{c_2^2}{2}\left[\frac{\lambda^2 a^{1+K}}{1+K}\left\{\left[K_{\frac{2K}{1+K}}(z_2)\right]^2\left[1+\frac{K^2}{\lambda^2}a^{-1-K}\right]-\left[K'_{\frac{2K}{1+K}}(z_2)\right]^2\right\}\right]$$

$$+\frac{c_2^2}{2}\frac{a^{1-K}}{1-K}\mu^2+\mu c_2^2 a^{\frac{1-K}{2}}K_{\frac{K-1}{K+1}}(z_2)+c_1c_2\mu a^{\frac{1-K}{2}}I_{\frac{K-1}{K+1}}(z_2)$$

$$-c_1c_2\frac{\lambda^2 a^{1+K}}{1+K}\left[I_{\frac{2K}{1+K}}(z_2)K_{\frac{2K}{1+K}}(z_2)+I_{\frac{2K}{1+K}-1}(z_2)K_{\frac{2K}{1+K}-1}(z_2)\right.$$

$$\left.+\frac{K}{\lambda}a^{\frac{-1-K}{2}}\left\{K_{\frac{2K}{1+K}}(z_2)I_{\frac{2K}{1+K}-1}(z_2)-I_{\frac{2K}{1+K}}(z_2)K_{\frac{2K}{1+K}-1}(z_2)\right\}\right]$$

$$-\frac{\lambda^2 h^2}{24K}a^{2K} \qquad \ldots(17)$$

where $\quad z_2=\dfrac{2\lambda a^{\frac{1+K}{2}}}{1+K}$

To determine the constant λ we shall use the condition that $u \to 0$ as $r \to 0$. Thus we have

$$\frac{c_1c_2K}{2}-\frac{\mu c_1c_2\left(\frac{\lambda}{1+K}\right)^{\frac{K-1}{K+1}}}{\Gamma\left(\frac{2K}{1+K}\right)}+\frac{\mu c_2^2\pi\left(\frac{\lambda}{1+K}\right)^{\frac{K-1}{K+1}}}{2\sin\left(\frac{K-1}{K+1}\pi\right)\Gamma\left(1+\frac{K-1}{K+1}\right)}$$

$$-\frac{c_2^2\pi K\,\Gamma\left(\frac{2K}{1+K}\right)}{4\sin\left(\frac{K-1}{K+1}\pi\right)}\left[\frac{\lambda^{\frac{2-2K}{1+K}}(1+K)^{\frac{2K-2}{1+K}}}{\Gamma\left(1+\frac{K-1}{K+1}\right)}\right]+K_1=0 \qquad \ldots(18)$$

As $K \to 1$, equation (13) reduces to the corresponding deflection for isotropic plate under a concentrated load at the centre as obtained by Basuli (1961) in the form

$$W = -\frac{P}{2\pi D\alpha^3 a I_1(\alpha a)}\Big\{\alpha a\left[K_1(\alpha a)\, I_0(\alpha r) + K_0(\alpha r)\, I_1(\alpha a)\right]$$

$$+ \alpha a I_1(\alpha a) \operatorname{Log}\frac{r}{a} - I_0(\alpha r) + I_0(\alpha a) - 1\Big\}$$

replacing λ by α

Also in the above case, equation (18) to determine α reduces to (Basuli 1961)

$$\left(\frac{Pa^2}{\pi D h}\right)^2 = \frac{\frac{1}{3}(\alpha a)^6}{\gamma + \operatorname{Log}\frac{\alpha a}{2} - \frac{I_0(\alpha a) + \alpha a K_1(\alpha a) - 2}{\alpha a I_1(\alpha a)} - \frac{1}{2}\left(\frac{I_0(\alpha a) - 1}{I_1(\alpha a)}\right)^2}$$

γ = Euler's constant.

<u>Numerical calculation :</u>

Let us take $\lambda = 1.5$, $a = 10$, $K = 1/3$

Putting all these valus in (18) we get the load function in the form

$$\frac{P.10^4}{2\pi D_r h} = 76.4$$

For this value of the load function the maximum deflection (deflection at the centre) is given from (13) in the form,

$$\frac{w_0}{h} = 2.14$$

Large deflection of a semi-circular plate under a uniform load.*

PAPER IV

Nomenclature :

The following nomenclature are used throught this paper :

q = uniform lateral load,

u, v = radial and cross-radial displacements,

h = thickness of the plate,

D = flexural rigidity of the plate = $\frac{Eh^3}{12(1-\sigma^2)}$,

E = Young's modulus,

σ = Poisson's ratio,

a = radius of the plate,

W = lateral displacement.

Introduction :

Approximate equations governing the non-linear behaviour of the plates (flat) have been given first by Berger (1955). Following Berger, a large number of non-linear problems have been solved by different authors. The present author's attempt is to apply this method to a semi-circular plate, simply-supported along the bounding diameter.

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Analysis :

Let us consider a plate in the form of a semicircle, simply-supported along the boundary.

Let us take the centre as pole and the bounding diameter as initial line. Following Berger, the differential equation satisfying the lateral displacement W is

$$\nabla^4 W - \alpha^2 \nabla^2 W = \frac{q}{D} \qquad \ldots(1)$$

where α is a constant given by

$$\frac{\alpha^2 h^2}{12} = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \qquad \ldots(2)$$

Expanding the load into the appropriate Fourier series we have

$$q = \frac{4q}{\pi} \sum_{m=1,3,\ldots} \frac{\sin m\theta}{m} \qquad \ldots(3)$$

Now, assuming

$$W = \sum R_m \sin m\theta \qquad \ldots(4)$$

where R_m is a function of r only and substituting the expressions for q and W (Equations (3) and (4)) into Equation (1) we get

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)\left(\frac{d^2 R_m}{dr^2} + \frac{1}{r}\frac{dR_m}{dr} - \frac{m^2}{r^2}R_m - \alpha^2 R_m\right) = \frac{4q}{m\pi D} \qquad \ldots(5)$$

The solution of the above equation can be written in the form

$$R_m = A_m r^m + B_m r^{-m} + C_m J_m(i\alpha r) + D Y_m(i\alpha r) + \frac{4q\, S_{3,m}(i\alpha r)}{m\pi D\alpha^4(2^2-m^2)} \quad \ldots(6)$$

where J_m and Y_m are the Bessel functions of 1st and 2nd kind of order m and

$$S_{3,m}(i\alpha r) = \sum_{n=0}^{\infty} \frac{(-1)^n (i\alpha r)^{3+1+2n}}{\{(3+1)^2-m^2\}\cdots\{(3+1+2n)^2-m^2\}}$$

is the Lommel function.

The solution satisfying the boundary condition along the diameter is

$$R_m = A_m r^m + C_m J_m(i\alpha r) + \frac{4q\, S_{3,m}(i\alpha r)}{m\pi D\alpha^4(2^2-m^2)} \quad \ldots(7)$$

Hence,

$$W = \sum_{m=1,3,\ldots}^{\infty} \left[A_m r^m + C_m J_m(i\alpha r) + \frac{4q\, S_{3,m}(i\alpha r)}{m\pi D\alpha^4(2^2-m^2)} \right] \sin m\theta \quad \ldots(8)$$

In the case of a simply-supported plate, boundary conditions are as follows :

$$(u)_{r=a} = (W)_{r=a} = 0 \quad \ldots(9)$$

$$\left[\frac{\partial^2 w}{\partial r^2} + \sigma\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \right) \right]_{r=a} = 0 \quad \ldots(10)$$

Combining equations (8), (9) and (10) and solving for the constants, we get

$$A_m = \frac{\dfrac{4q\, J_m(i\alpha a)}{m\pi D(2^2-m^2)}\left[\dfrac{\sigma i S'_{3,m}(i\alpha a)}{a\alpha^3} - \dfrac{S''_{3,m}(i\alpha a)}{\alpha^2}\right] - \dfrac{4q\, S_{3,m}(i\alpha a)}{m\pi D\alpha^4(2^2-m^2)}\left[\dfrac{\sigma i\alpha}{a} J'_m(i\alpha a) - \alpha^2 J''_m(i\alpha a)\right]}{a^m\left[\dfrac{\sigma i\alpha}{a} J'_m(i\alpha a) - \alpha^2 J''_m(i\alpha a)\right] - J_m(i\alpha a)\, a^{m-2}(m^2-m+m\sigma)} \qquad \ldots(11)$$

$$C_m = \frac{\dfrac{4q\, S_{3,m}(i\alpha a)}{m\pi D\alpha^4(2^2-m^2)}\left[a^{m-2}(m^2-m+m\sigma)\right] - \dfrac{4q\, a^m}{m\pi D(2^2-m^2)}\left[\dfrac{\sigma i S'_{3,m}(i\alpha a)}{a\alpha^3} - \dfrac{S''_{3,m}(i\alpha a)}{\alpha^2}\right]}{a^m\left[\dfrac{\sigma i\alpha}{a} J'_m(i\alpha a) - \alpha^2 J''_m(i\alpha a)\right] - J_m(i\alpha a)\, a^{m-2}(m^2-m+m\sigma)} \qquad \ldots(12)$$

To determine α , let us assume :

$$u = \sum U(r) \cos m\theta \qquad \ldots(13)$$

$$v = \sum V(r) \sin m\theta \qquad \ldots(14)$$

Multiplying equation (2) by $r\,d\theta\,dr$ and integrating within the limits 0 to a and 0 to π we have,

$$\int_0^a\int_0^\pi r\sum U'(r)\cos m\theta\,d\theta\,dr + \int_0^a\int_0^\pi \sum U(r)\cos m\theta\,d\theta\,dr$$

$$+\int_0^a\int_0^\pi \sum m\,V(r)\cos m\theta\,d\theta\,dr + \frac{1}{2}\int_0^a\int_0^\pi \left(\frac{\partial w}{\partial r}\right)^2 r\,d\theta\,dr$$

$$+\frac{1}{2}\int_0^a\int_0^\pi \frac{1}{r}\left(\frac{\partial w}{\partial \theta}\right)^2 d\theta\,dr = \frac{\alpha^2 h^2}{12}\int_0^a\int_0^\pi r\,d\theta\,dr$$

After evaluating the integrals, we obtain the following equation determining α

$$A_m^2 m a^{2m} + C_m^2\left[\frac{\alpha^2 a^2}{2}J_m^2(i\alpha a) + mJ_m^2(i\alpha a) - \frac{\alpha^2 a^2}{2}J_{m+1}^2(i\alpha a) +\right.$$

$$\left. + i\alpha(m+1)J_{m+1}(i\alpha a)J_m(i\alpha a)\right] + 2mA_mC_m a^m J_m(i\alpha a) +$$

$$+\frac{8A_m q a^m S_{3,m}(i\alpha a)}{\pi D\alpha^4(2^2-m^2)} - \frac{16q^2}{m^2\pi^2 D^2\alpha^6(2^2-m^2)^2}\times$$

$$\times\left[\sum_{\substack{n=0\\ s=0\\ n\neq s}}^{\infty}\left\{\frac{(i\alpha)^{6+4n}\cdot(4+2n)^2\cdot a^{8+4n}}{(8+4n)\left[(4^2-m^2)\cdots\{(4+2n)^2-m^2\}\right]^2}\right.\right. +$$

$$+\frac{(-1)^n(-1)^s\cdot(i\alpha)^{6+2n+2s}\cdot(4+2n)(4+2s)\cdot a^{8+2n+2s}}{(8+2n+2s)\left[(4^2-m^2)\cdots\{(4+2n)^2-m^2\}\right]\left[(4^2-m^2)\cdots\{(4+2s)^2-m^2\}\right]}\Bigg\}\Bigg]$$

$$+\frac{16q^2}{\pi^2D^2\alpha^8(2^2-m^2)^2}\left[\sum_{\substack{n=0\\s=0\\n\neq s}}^{\infty}\left\{\frac{(i\alpha)^{8+4n}\cdot a^{8+4n}}{(8+4n)\left[(4^2-m^2)\cdots\{(4+2n)^2-m^2\}\right]}+\right.\right.$$

$$\left.\left.+\frac{(-1)^n\cdot(-1)^s(i\alpha)^{8+2n+2s}\cdot a^{8+2n+2s}}{(8+2n+2s)\left[(4^2-m^2)\cdots\{(4+2n)^2-m^2\}\right]\left[(4^2-m^2)\cdots\{(4+2s)^2-m^2\}\right]}\right\}\right]$$

$$-\frac{8C_mq}{m\pi D\alpha^2(2^2-m^2)}\left[\sum_{n=0}^{\infty}\left\{\frac{(-1)^n(i\alpha)^{2+2n}\cdot(4+2n)\cdot a^{4+2n}}{(4^2-m^2)\cdots\{(4+2n)^2-m^2\}}\times J_m(i\alpha a)-\right.\right.$$

$$\left.\left.-\frac{(-1)^n\cdot(4+2n)^2\cdot\frac{a}{i\alpha}\left[(2+m+2n)\cdot J_m(i\alpha a)\cdot S_{2+2n,m-1}(i\alpha a)-J_{m-1}(i\alpha a)\cdot S_{3+2n,m}(i\alpha a)\right]}{(4^2-m^2)\cdots\{(4+2n)^2-m^2\}}\right\}\right]$$

$$+\frac{8C_mq}{m\pi D\alpha^4(2^2-m^2)}\left[\sum_{n=0}^{\infty}\frac{(-1)^n(i\alpha a)\left[(2+2n+m)J_m(i\alpha a)\cdot S_{2+2n,m-1}(i\alpha a)-J_{m-1}(i\alpha a)\cdot S_{3+2n,m}(i\alpha a)\right]}{(4^2-m^2)\cdots\{(4+2n)^2-m^2\}}\right]$$

$$=\frac{\alpha^2a^2h^2}{6}\qquad\cdots(15)$$

As $\alpha\longrightarrow 0$ equation (8) reduces to

$$W=\frac{qa^4}{D}\cdot\sum_{m=1,3,\ldots}^{\infty}\left[\frac{4r^4}{a^4}\cdot\frac{1}{m\pi(4-m^2)(16-m^2)}+\frac{r^m}{a^m}\cdot\frac{m+5+\sigma}{m\pi(2+m)(16-m^2)\left(m+\frac{1}{2}+\frac{\sigma}{2}\right)}\right.$$

$$\left.-\frac{r^{m+2}}{a^{m+2}}\cdot\frac{m+3+\sigma}{m\pi(4+m)(4-m^2)\left(m+\frac{1}{2}+\frac{\sigma}{2}\right)}\right]\sin m\theta$$

as obtained by Timoshenko and Woinowsky-Krieger (1959) for the corresponding problem of small deflections.

The deflection is obtained for a plate with $a = 10$, $\sigma = 0{\cdot}25$ at $\theta = \frac{\pi}{4}$, $r = \frac{a}{2}$ for various loads.

The figure shows the deflection w/h against $q \cdot 10^4/\pi D h$. In calculating the deflection one has to start from equation (15) with an assumed value of $i\alpha$ leading to a particular value for the load function $q \cdot 10^4/\pi D h$. These values of $i\alpha$ and $q \cdot 10^4/\pi D h$ determine the corresponding w/h from equation (8). Here the values of $i\alpha$ have been assumed to be equal to $0{\cdot}1$, $0{\cdot}2$, etc.

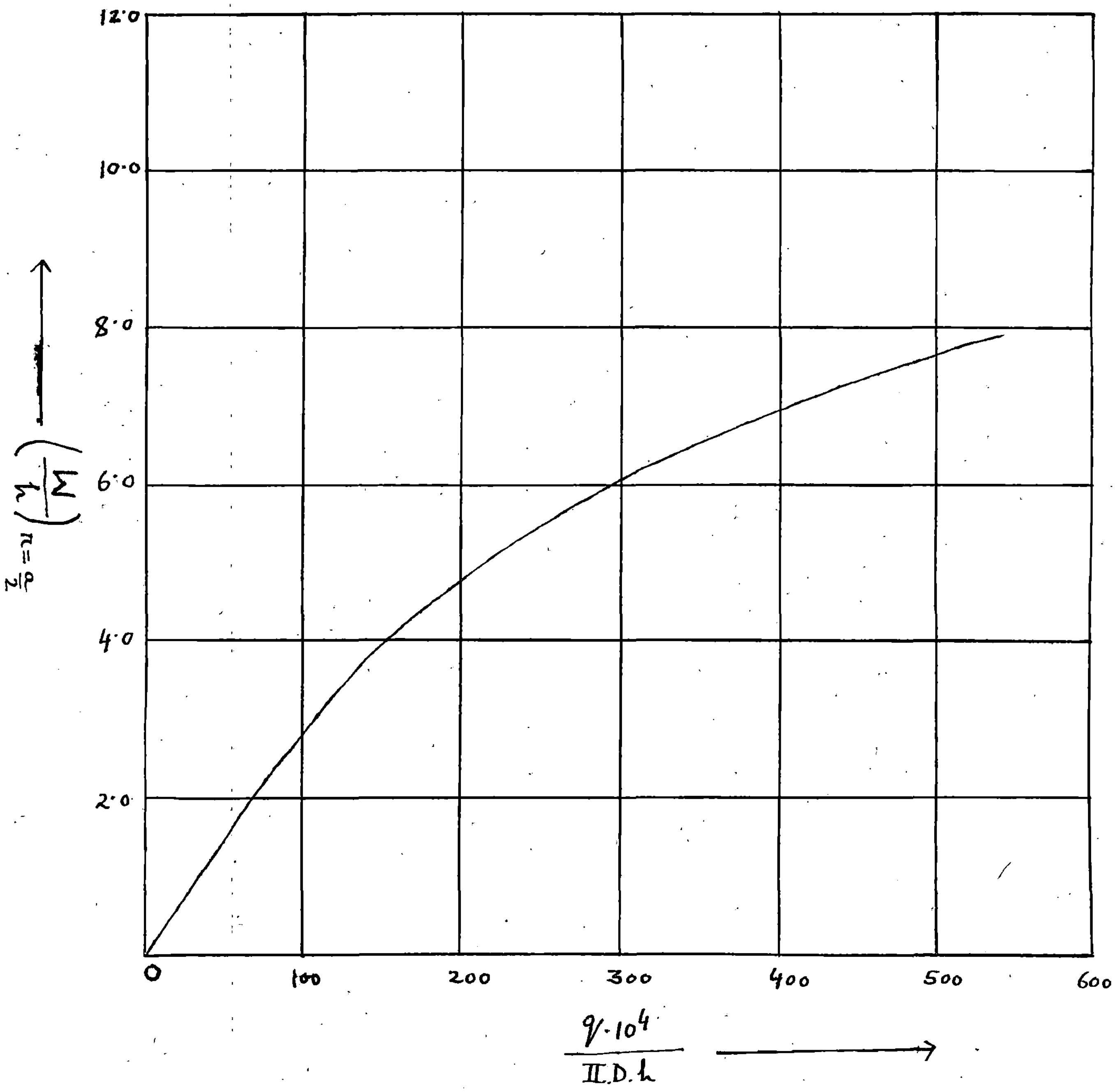

FIG. 6

Graph showing deflections for various values of the load function $\frac{q \cdot 10^4}{\pi \cdot D \cdot h}$

Note on the large deflection of elliptic plates.*

PAPER - V

Nomenclature :

The following nomenclature are used in this paper.

$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$,

$D =$ flexural rigidity of the plate $= \frac{Eh^3}{12(1-\sigma^2)}$,

$h =$ thickness of the plate,

$E =$ Young's modulus,

$\sigma =$ Poisson's ratio,

$q =$ uniform load, normal to the plane,

$w =$ deflection, normal to the plane,

$u, v =$ displacements corresponding to x and y axes.

Introduction :

Following Berger's (1955) approximate method for large deflection, an attempt has been made to investigate the large deflection of elliptic plates with clamped edges. The general solution is obtained in terms of Mathieu functions of zero and even orders. Retaining only zero order, the deflection is obtained and with usual limiting process the known results for corresponding circular plates have also been deduced.

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Analysis :

Following Berger (1955) the deflection W of an elastic plate satisfies the differential equation

$$\nabla_1^2(\nabla_1^2 - \alpha^2)W = \frac{q}{D} \qquad \ldots(1)$$

where α is a constant given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{\alpha^2 h^2}{12} \qquad \ldots(2)$$

A particular integral W_0 of (1) is given by

$$W_0 = -\frac{q}{4D\alpha^2}(x^2+y^2) \qquad \ldots(3)$$

Transferring to elliptic co-ordinates (ξ, η) defined by $x+iy = d\cosh(\xi+i\eta)$, where $2d$ is the interfocal distance of the ellipse,
the particular integral becomes

$$W_0 = -\frac{qd^2}{8D\alpha^2}(\cosh 2\xi + \cos 2\eta) \qquad \ldots(4)$$

For the complementary function let us assume $W = W_1 + W_2$

such that $\nabla_1^2 W_1 = 0$ and $\nabla_1^2 W_2 - \alpha^2 W_2 = 0$ $\qquad \ldots(5)$

Changing to elliptic co-ordinates we have

$$\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)W_1 = 0 \qquad \ldots(6)$$

and $$\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)W_2 - \frac{\alpha^2 d^2}{2}\left(\cosh 2\xi - \cos 2\eta\right) W_2 = 0 \qquad \ldots(7)$$

Periodic solutions of (6) and (7) which are symmetric about the centre can be represented by

$$W_1 = \sum_{m=0}^{\infty} C_{2m} \cosh 2m\xi \cdot \cos 2m\eta \qquad \ldots(8)$$

$$W_2 = \sum_{m=0}^{\infty} \bar{C}_{2m} \mathbf{Ce}_{2m}(\xi,-q') \, ce_{2m}(\eta,-q') \qquad \ldots(9)$$

where $ce_{2m}(\eta,-q')$ and $\mathbf{Ce}_{2m}(\xi,-q')$ are Mathieu function and Modified Mathieu function of the first kind of order $2m$ and

$$q' = \frac{\alpha^2 d^2}{4}$$

Combining equations (4), (8) and (9), the general solution can be written as

$$W = \sum_{m=0}^{\infty} C_{2m} \cosh 2m\xi \cdot \cos 2m\eta$$

$$+ \sum_{m=0}^{\infty} \bar{C}_{2m} \mathbf{Ce}_{2m}(\xi,-q') \, ce_{2m}(\eta,-q') - \frac{q d^2}{8D\alpha^2}\left(\cosh 2\xi + \cos 2\eta\right) \qquad \ldots(10)$$

While solving a problem of bending of a plate with elliptic hole, instead of taking Mathieu function of all orders, taking a single Mathieu function of second order, Naghdi (1955) has shown that the results obtained are satisfactory for larger elliptic hole. In our present problem we also make similar approximation by taking a single Mathieu function of order zero.

Hence on this approximation equation (10) reduces to

$$W = c_1 Ce_0(\xi, -q')\, ce_0(\eta, -q') - \frac{q d^2}{8D\alpha^2}\left(\cosh 2\xi + \cos 2\eta\right) + c_2 \qquad \ldots(11)$$

If the outer boundary of the plate $\xi = \xi_0$ be clamped,

we have $W = \frac{\partial W}{\partial \xi} = 0$, when $\xi = \xi_0$ . ...(12)

Using the above boundary conditions, the equations to determine the contstants will be

$$c_1 Ce_0(\xi_0, -q')\, ce_0(\eta, -q') - \frac{q d^2}{8D\alpha^2}\left(\cosh 2\xi_0 + \cos 2\eta\right) + c_2 = 0 \qquad \ldots(13)$$

$$c_1 Ce_0'(\xi_0, -q')\, ce_0(\eta, -q') - \frac{q d^2}{4D\alpha^2}\sinh 2\xi_0 = 0 \qquad \ldots(14)$$

Multiplying these equations by $ce_0(\eta, -q')$ and integrating w.r.t. η from 0 to 2π and using the orthogonality relations and normalization (Mclachlan, 1947, P-24), we get

$$\left.\begin{aligned} c_1 &= \frac{q d^2}{D\alpha^2}\,\frac{A_0(0)\sinh 2\xi_0}{Ce_0'(\xi_0, -q')} \\ c_2 &= -\frac{q d^2}{4D\alpha^2}\left[\frac{\sinh 2\xi_0\, Ce_0(\xi_0, -q')}{Ce_0'(\xi_0, -q')} - \frac{1}{2}\cosh 2\xi_0 + \frac{1}{4}\frac{A_2(0)}{A_0(0)}\right] \end{aligned}\right\} \qquad \ldots(15)$$

$A_0(0)$ and $A_2(0)$ being the first two Fourier co-efficients in the expression of $ce_0(\eta, -q')$. Hence deflection is given by

$$W = \frac{qd^2}{4D\alpha^2}\left[\frac{2A_0(0)\sinh 2\xi_0}{Ce_0'(\xi_0,-q')}\, Ce_0(\xi,-q')\, ce_0(\eta,-q')\right.$$

$$- \frac{\sinh 2\xi_0\, Ce_0(\xi_0,-q')}{Ce_0'(\xi_0,-q')} + \frac{1}{2}\cosh 2\xi_0 -$$

$$\left. - \frac{1}{4}\frac{A_2(0)}{A_0(0)} - \frac{1}{2}\left(\cosh 2\xi + \cos 2\eta\right)\right] \qquad \ldots(16)$$

To determine α we know that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{\alpha^2 h^2}{12}$$

In elliptic co-ordinates, the above equation reduces to

$$h_1 h_2\left[\frac{\partial}{\partial \xi}\left(\frac{u_\xi}{h_2}\right) + \frac{\partial}{\partial \eta}\left(\frac{u_\eta}{h_1}\right)\right] + \frac{1}{2}h_1 h_2\left[\left(\frac{\partial w}{\partial \xi}\right)^2 + \left(\frac{\partial w}{\partial \eta}\right)^2\right]$$

$$= \frac{\alpha^2 h^2}{12} \qquad \ldots(17)$$

where $h_1 = h_2 = \dfrac{1}{d\sqrt{\sinh^2\xi + \sin^2\eta}}$

Boundary conditions for u_ξ and u_η are $u_\xi = 0 = u_\eta$ at $\xi = \xi_0$.

Let us assume that

$$\left.\begin{aligned} u_\xi &= \sum_{\eta=0}^{\infty} P(\xi)\cos 2n\eta \\ u_\eta &= \sum_{\eta=1}^{\infty} G(\xi)\sin 2\eta\eta \end{aligned}\right\} \qquad \ldots(18)$$

subject to the conditions $P(\xi_0) = G(\xi_0) = 0$

Integrating equation (17) over the surface of the plate we have

$$\int_0^{2\pi}\int_0^{\xi_0}\left[\left(\frac{\partial w}{\partial \xi}\right)^2+\left(\frac{\partial w}{\partial \eta}\right)^2\right]d\xi\,d\eta = \frac{\alpha^2 h^2 d^2}{6}\int_0^{2\pi}\int_0^{\xi_0}\left(\sin^2 h\,\xi+\sin^2\eta\right)d\xi\,d\eta$$

OR,

$$\frac{c_1^2}{2}\int_0^{\xi_0}\int_0^{2\pi}\left[\mathbf{c}^2e_0'(\xi,-q')\,ce_0(\eta,-q')+\mathbf{c}^2e_0(\xi,-q')\,ce_0'(\eta,-q')\right]d\xi\,d\eta$$

$$+\frac{1}{2}\int_0^{\xi_0}\int_0^{2\pi}\frac{q^2d^4\sinh^2 2\xi}{16D^2\alpha^4}\,d\xi\,d\eta+\frac{1}{2}\int_0^{\xi_0}\int_0^{2\pi}\frac{q^2d^4\sin^2 2\eta}{16D^2\alpha^4}\,d\xi\,d\eta$$

$$-c_1\int_0^{\xi_0}\int_0^{2\pi}\mathbf{c}e_0'(\xi,-q')\,ce_0(\eta,-q')\frac{qd^2}{4D\alpha^2}\sinh 2\xi\,d\xi\,d\eta+$$

$$+ C_1 \int_0^{\xi_0}\int_0^{2\pi} \mathbf{Ce}_0(\xi,-q')\, ce_0'(\eta,-q')\, \frac{qd^2}{4D\alpha^2} \sin 2\eta \, d\xi\, d\eta$$

$$= \frac{\alpha^2 h^2 d^2}{12} \int_0^{\xi_0}\int_0^{2\pi} \left(\sin^2\eta + \sin^2 h\xi\right) d\xi\, d\eta \qquad \ldots(19)$$

After evaluating the integrals, we get the equations to determine α in the form

$$\frac{C_1 q d^2}{D\alpha^2}\Bigg[\frac{1}{2}\left\{A_0^{(0)}\right\}^2 \sinh 2\xi_0 - A_0^{(0)} \sinh 2\xi_0 \mathbf{Ce}_0(\xi_0,-q')$$

$$+ \frac{1}{2} A_0^{(0)} \sum_{r=1}^{\infty} \frac{(-1)^r}{r}\left(A_{2r+2}^{(0)} + A_{2r-2}^{(0)}\right) \sinh 2r\xi_0 - A_0^{(0)} A_2^{(0)}(\xi_0)\Bigg]$$

$$+ C_1^2 \Bigg[\sum_{r=1}^{\infty} \left\{A_{2r}^{(0)}\right\}^2 r \sinh 4r\xi_0 + \sum_{r=1}^{\infty}\sum_{\substack{s=1 \\ r\neq s}}^{\infty} \frac{(-1)^r(-1)^s A_{2r}^{(0)} A_{2s}^{(0)}\, 4rs}{r^2-s^2}$$

$$\times \left\{ r \sinh 2r\xi_0 \cosh 2s\xi_0 - s \sinh 2s\xi_0 \cosh 2r\xi_0 \right\}$$

$$+ \sum_{r=1}^{\infty} 4r^2 \left(A_{2r}^{(0)}\right)^2 \Bigg\{ \sum_{r=1}^{\infty} \left(A_{2r}^{(0)}\right)^2 \left(\xi_0 + \frac{\sinh 4r\xi_0}{4r}\right)$$

$$+ 2A_0^{(0)} \sum_{r=1}^{\infty} (-1)^r A_{2r}^{(0)} \frac{\sinh 2r\xi_0}{r} + \sum_{r=1}^{\infty}\sum_{\substack{s=1 \\ r\neq s}}^{\infty} \frac{(-1)^r(-1)^s A_{2r}^{(0)} A_{2s}^{(0)}}{r^2-s^2} \times$$

$$\times\left(r \sinh 2s\xi_0 \cosh 2r\xi_0 - s \sinh 2r\xi_0 \cosh 2s\xi_0\right)\Big\}\Big]$$

$$+ \frac{q^2 d^4 \sinh 4\xi_0}{64 D^2 \alpha^4}$$

$$= \frac{\alpha^2 h^2 d^2 \sinh 2\xi_0}{12} \qquad \ldots(20)$$

In the limiting case, when an elliptic plate of semi-major axis a tends to a circulate plate of radius a, $\xi \to \infty$, $d \to 0$.

Hence

$$\frac{Ce_0(\xi_0,-q')}{Ce_0'(\xi_0,-q')} \to \frac{I_0(\alpha a)}{\alpha a\, I_0'(\alpha a)}, \quad \frac{Ce_0(\xi,-q')}{Ce_0'(\xi,-q')} \longrightarrow \frac{I_0(\alpha r)}{\alpha r\, I_0'(\alpha r)}$$

and $ce_0(\eta,-q') \to \frac{1}{\sqrt{2}}$, $A_0^{(0)} \to \frac{1}{\sqrt{2}}$, $A_2(0) \to 0$, $d^2 \sinh 2\xi_0 \to 2a^2$,
$A_{2r}^{(0)} \to 0$ and $\cosh 2\xi d\xi \longrightarrow \frac{2r\,dr}{d^2}$

Then the equation (16) reduces to

$$W = \frac{q a^2}{2D\alpha^3 a I_1(\alpha a)}\left[I_0(\alpha r) - I_0(\alpha a)\right] + \frac{q}{4D\alpha^2}\left(a^2 - r^2\right) \qquad \ldots(21)$$

which gives the large deflection of a uniformly loaded circular plate of radius a.

Also the equation to determine α reduces to in the limiting case

$$\frac{q^2 a I_1(\alpha a) I_0(\alpha a)}{2D^2\alpha I_1^2(\alpha a)} + \frac{q^2\int_0^a r I_0^2(\alpha r)\,dr}{2D^2 I_1^2(\alpha a)} + \frac{q^2 a^2}{8D^2}$$

$$- \frac{q^2}{aD^2 I_1(\alpha a)}\int_0^a r^2 I_1(\alpha r)\,dr = \frac{\alpha^6 h^2}{6}$$

OR,

$$\frac{\alpha^2 h^2 a^2}{24} = \frac{q^2 a^2}{8D^2\alpha^6 I_1^2(\alpha a)}\left[\frac{\alpha^2 a^2}{2}\left\{I_1^2(\alpha a) - I_0^2(\alpha a)\right\} + \alpha a I_0(\alpha a) I_1(\alpha a)\right]$$

$$+ \frac{q^2 a^4}{32D^2\alpha^4} - \frac{q^2 a^3 I_2(\alpha a)}{4D^2\alpha^5 I_1(\alpha a)}$$

which is the equation for α in the case of uniformly loaded circular plate.

Numerical Calculation :

Putting $d = 2$, $\alpha = 2\sqrt{2}$, $\xi_0 = 3$, $\xi = 2{\cdot}2$ and $\eta = \pi/4$

in equation (20), the value of the load function is found in the form

$$\frac{10^4 q}{Dh} = 110{\cdot}12$$

Putting this value of the load function in (16) we get the deflection in the form

$$\frac{W}{h} = 2{\cdot}13$$

Large deflection of an isoceles right-angled triangular plate. *

PAPER - VI

Nomenclature :

The following nomenclature are used in this paper.

q = uniform load,

u, v = displacements along x and y axes,

w = deflection, normal to the middle plane,

a = equal sides of the plate,

E_1, E_2 = Young's modulus corresponding to the directions of x and y,

σ_1, σ_2 = Poisson's ratios corresponding to the directions of x and y,

h = thickness of the plate,

D_1 = average flexural rigidity $= \frac{(EI)_1}{1-\sigma_1\sigma_2}$,

D_2 = average flexural rigidity $= \frac{(EI)_2}{1-\sigma_1\sigma_2}$,

$D_3 = \frac{1}{2}(\sigma_1 D_2 + \sigma_2 D_1) + 2D_K$,

D_K = average torsional rigidity,

$l^2 = D_3/D_1$,

$K^2 = D_2/D_1$.

Introduction :

Following Berger's approximate method for large deflection, a good number of problems have been solved.

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Iwinski and Nowinski (1957) extended this method to the case of orthotropic plates and arrived at the satisfactory results. The present author's attempt is to apply this method to the case of orthotropic isoceles right - angled triangular plate. The corresponding deflection of isotropic triangular plate has also been deduced.

Analysis :

Let us consider an isoceles right - angled triangular plate of equal sides a . The equation governing the deflection of an orthotropic plate in cartesian co-ordinates can be written as (Iwinski and Nowinski, 1957)

$$\frac{\partial^4 w}{\partial x^4} + 2L^2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + K^2 \frac{\partial^4 w}{\partial y^4} - \frac{12c}{h^2}\left(\frac{\partial^2 w}{\partial x^2} + K\frac{\partial^2 w}{\partial y^2}\right) = \frac{q}{D_1} \quad \ldots (1)$$

where $e_1^* = \bar{\epsilon}_x + K\bar{\epsilon}_y = \frac{\partial u}{\partial x} + K\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{K}{2}\left(\frac{\partial w}{\partial y}\right)^2 = c \quad \ldots (2)$

and $e_1 = \epsilon_x + \epsilon_y$

Let the boundary be simply - supported with the following edge conditions.

$$\left.\begin{aligned} u = w = \frac{\partial^2 w}{\partial x^2} &= 0 \quad \text{at} \quad x = 0 \\ v = w = \frac{\partial^2 w}{\partial y^2} &= 0 \quad \text{at} \quad y = 0 \\ u + v = w = \frac{\partial^2 w}{\partial \nu^2} &= 0 \quad \text{at} \quad x + y = a \end{aligned}\right\} \quad \ldots (3)$$

where $\frac{\partial}{\partial \nu} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$

Assuming the deflection in the form

$$W = \sum A_m \left(\sin\frac{2m\pi x}{a} \sin\frac{m\pi y}{a} + \sin\frac{2m\pi y}{a} \cdot \sin\frac{m\pi x}{a} \right) \qquad \ldots \quad (4)$$

where m is an odd integer, A_m being constant, the boundary conditions on w can be satisfied identically.

Substituting (4) in (1) we have,

$$\sum A_m \left(C_m \sin\frac{2m\pi x}{a} \cdot \sin\frac{m\pi y}{a} + D_m \sin\frac{2m\pi y}{a} \cdot \sin\frac{m\pi x}{a} \right) = \frac{q}{D_1} \qquad \ldots \quad (5)$$

where

$$\left. \begin{aligned} C_m &= \frac{m^4\pi^4}{a^4}\left(K^2 + 8l^2 + 16\right) + \frac{12C}{h^2}\,\frac{m^2\pi^2}{a^2}\left(4+K\right) \\ D_m &= \frac{m^4\pi^4}{a^4}\left(16K^2 + 8l^2 + 1\right) + \frac{12C}{h^2}\,\frac{m^2\pi^2}{a^2}\left(4K+1\right) \end{aligned} \right\} \qquad \ldots \quad (6)$$

Now the load can be expanded in the form,

$$q = \sum q_m \left(C_m \sin\frac{2m\pi x}{a} \cdot \sin\frac{m\pi y}{a} + D_m \sin\frac{2m\pi y}{a} \cdot \sin\frac{m\pi x}{a} \right) \qquad \ldots \quad (7)$$

Multiplying both sides of equation (7) by $\left[C_m \sin\frac{2m\pi x}{a} \cdot \sin\frac{m\pi y}{a} + D_m \sin\frac{2m\pi y}{a} \cdot \sin\frac{m\pi x}{a} \right]$ and integrating over the surface of the plate we have,

$$q_m = \frac{32q\left(C_m + D_m\right)}{3m^2\pi^2\left(C_m^2 + D_m^2\right)} \qquad \ldots \quad (8)$$

Substituting (8) and (7) in (5) we have

$$A_m = \frac{32q\,(C_m + D_m)}{3m^2\pi^2\,(C_m^2 + D_m^2)\,D_1} \qquad \dots (9)$$

Hence $W = \sum A_m \left(\sin\frac{2m\pi x}{a} \cdot \sin\frac{m\pi y}{a} + \sin\frac{2m\pi y}{a} \cdot \sin\frac{m\pi x}{a} \right) \qquad \dots (10)$

is known.

To determine the constant c, we know from (2)

$$\frac{\partial u}{\partial x} + K\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{K}{2}\left(\frac{\partial w}{\partial y}\right)^2 = c$$

The boundary conditions $u=0$ at $x=0$, $v=0$ at $y=0$ and $u+v=0$ at $x+y=a$ are satisfied by the functions

$$u = \sum_{n=1,3,5,\dots}^{\infty} B_n \sin\frac{n\pi x}{a}\left(\cos\frac{n\pi y}{a} + \sin\frac{n\pi x}{a} - \frac{n\pi}{4}\right) \qquad \dots (11)$$

$$v = \sum_{n=1,3,5,\dots}^{\infty} B_n \sin\frac{n\pi y}{a}\left(\cos\frac{n\pi x}{a} - \sin\frac{n\pi y}{a} + \frac{n\pi}{4}\right) \qquad \dots (12)$$

where B_n is a constant.

Integrating (2) with respect to x and y over the surface of the plate we have

$$\int_0^a\int_0^{a-y} \sum B_n \frac{n\pi}{a}\left[\cos\frac{n\pi x}{a}\cdot\cos\frac{n\pi y}{a} + \sin\frac{2n\pi x}{a} - \frac{n\pi}{4}\cos\frac{n\pi x}{a}\right] dx\,dy$$

$$+ K\int_0^a\int_0^{a-y} B_n \frac{n\pi}{a}\left[\cos\frac{n\pi x}{a}\cdot\cos\frac{n\pi y}{a} - \sin\frac{2n\pi y}{a} + \frac{n\pi}{4}\cos\frac{n\pi y}{a}\right] dx\,dy$$

$$+ \frac{1}{2}\int_0^a\int_0^{a-y}\left[\sum A_m\left(\cos\frac{2m\pi x}{a}\cdot\sin\frac{m\pi y}{a}\cdot\frac{2m\pi}{a} + \frac{m\pi}{a}\sin\frac{2m\pi y}{a}\cos\frac{m\pi x}{a}\right)\right]^2 dx\,dy$$

$$+ \frac{K}{2}\int_0^a\int_0^{a-y}\left[\sum A_m\left(\sin\frac{2m\pi x}{a}\cos\frac{m\pi y}{a}\cdot\frac{m\pi}{a} + \frac{2m\pi}{a}\cos\frac{2m\pi y}{a}\sin\frac{m\pi x}{a}\right)\right]^2 dx\,dy$$

$$= c\int_0^a\int_0^{a-y} dx\,dy \qquad \ldots (13)$$

After evaluating the integrals we have

$$\sum A_m^2 (1+K) m^2\pi^2 = \frac{8ca^2}{5} \qquad \ldots (14)$$

If the plate be isotropic, we have

$$K = L = 1, \quad D_1 = D_2 = D_3 = D,$$

$$\sigma_1 = \sigma_2 = \sigma, \quad E_1 = E_2 = E, \quad C = \frac{\alpha^2 h^2}{12}$$

In that case the differential equation (I) reduces to

$$\nabla^2(\nabla^2-\alpha^2)W = \frac{q}{D} \quad \cdots (15), \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Corresponding deflection is given by

$$W = \sum B_m \left(\sin\frac{2m\pi x}{a} \cdot \sin\frac{m\pi y}{a} + \sin\frac{2m\pi y}{a} \cdot \sin\frac{m\pi x}{a} \right)$$

$$= \frac{32 q a^2}{15 D \pi^4} \sum \frac{\left(\sin\frac{2m\pi x}{a} \cdot \sin\frac{m\pi y}{a} + \sin\frac{2m\pi y}{a} \cdot \sin\frac{m\pi x}{a} \right)}{m^4 \left(\frac{5m^2\pi^2}{a^2} + \alpha^2 \right)} \quad \ldots (16)$$

where

$$B_m = \frac{32 q a^2}{15 D \pi^4} \cdot \frac{1}{m^4 \left(\frac{5m^2\pi^2}{a^2} + \alpha^2 \right)}$$

The corresponding equation to determine α reduces to

$$\sum B_m^2 m^2 \pi^2 = \frac{\alpha^2 h^2 a^2}{15} \quad \ldots (17)$$

If $\alpha \to 0$, we get the corresponding small deflection for isotropic right - angled isoceles triangular plate with simply - supported edges in the form,

$$W = \frac{32 q a^4}{75 D \pi^6} \sum \frac{1}{m^6} \left(\sin\frac{2m\pi x}{a} \cdot \sin\frac{m\pi y}{a} + \sin\frac{2m\pi y}{a} \cdot \sin\frac{m\pi x}{a} \right) \quad \ldots (18)$$

Which is numerically equal to that obtained by Timoshenko. S and S. Woinowsky-Krieger (1959) in the form

$$W = \frac{16 q a^4}{\pi^6 D} \left[\sum_{m=1,3,\ldots}^{\infty} \sum_{n=2,4,\ldots}^{\infty} \frac{n \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{a}}{m(n^2-m^2)(m^2+n^2)^2} + \right.$$

$$+\sum_{m=2,4,\ldots}^{\alpha}\ \sum_{n=1,3,\ldots}^{\alpha} \frac{m \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{a}}{n(m^2-n^2)(m^2+n^2)^2}\Bigg] \quad \ldots \quad (19)$$

The deflection is obtained for a plate at the point

$$\frac{x}{a} = \frac{y}{a} = {}^{\cdot}25$$

The graph is ploted showing the deflection $\frac{w}{h}$ of the isotropic plate against $\frac{qa^4}{\pi^4 Dh}$. In calculating the deflection one has to start from equation (17) with an assumed value of αa leading to a particular value for the load function $\frac{qa^4}{\pi^4 Dh}$. These values of αa and $\frac{qa^4}{\pi^4 Dh}$ determine corresponding $\frac{w}{h}$ from equation (16). Here αa has been assumed 1, 3, 5, 7, 9 etc.

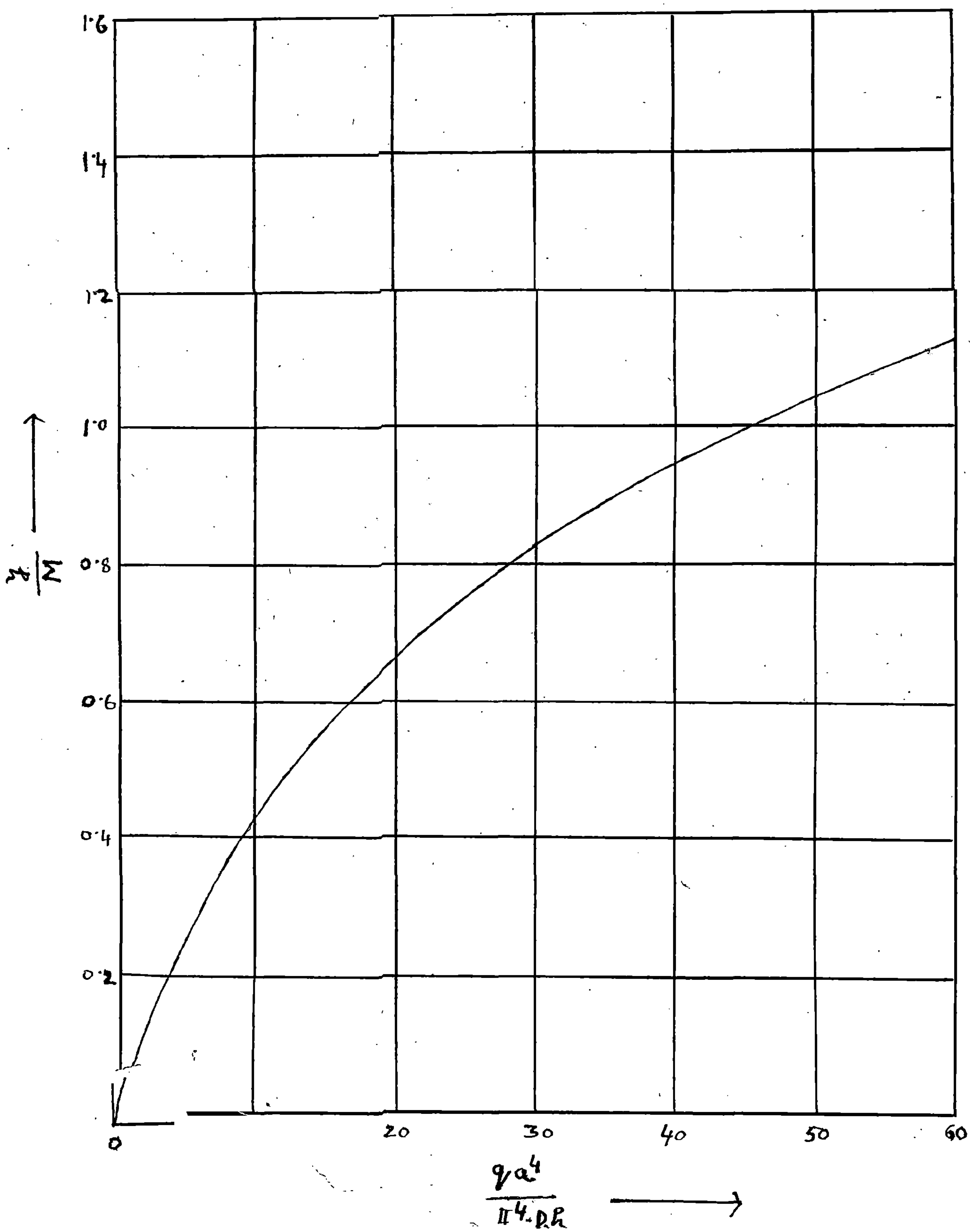

FIG.7

Graph showing deflections for various values of the load function $\frac{qa^4}{\pi^4 . D h}$