Chapter 9

Radiating Spherical Collapse with Heat Flow

9.1 Introduction

The theory of gravitional collapse has many interesting applications in astrophysics where the formation of cold compact stellar objects are usually preceded by a period of radiative collapse. The problem of gravitional collapse was first investigated by Oppenheimer and Snyder [78], who considered the contraction of a spherically symmetric dust cloud. In that case the exterior spacetime was described by the Schwarzschild metric and the interior space-time was represented by a Friedman-like solution. Later on, Vaidya [79] derived the line element which describes correctly the exterior gravitational field of a spherically symmetric radiating mass. This has enabled many investigators to model the interior of a radiating star by matching the interior solution to the exterior space-time given by Vaidya [79], e.g., [80], [81], [82], [83] & [84]. The junction conditions for a spherically symmetric shear-free radiating star was completely derived by Santos [85]. The crucial result that follows from Santos is that the pressure on the boundary of a radiating sphere is nonvanishing in general. Subsequently, many models of radiative gravitational collapse with heat flow were found by utilising these junction conditions. In particular, special attention was given to models in which an initial static stellar configuration started collapsing by dissipating energy in the form of a radial heat flux (Bonnor *et al* [87]). The initial static configuration was taken to be an exact solution of the Einstein field equations. In a recent paper Schäfer and Goenner [86] presented a model of a highly idealized spherically object radiating away its mass with constant luminosity. The body starts collapsing at time $t = -\infty$ with both infinite mass and radius and contracts to a point at time $t = 0$ without forming an event horizon.

Our aim here is to consider the evolution of a star undergoing a radiative gravitional collapse with its final state being that of a superdense star. We want to study this evolutionary process starting from the final non radiating state and interpolating to earlier times when it was emitting radiation.

In Section 9.2 we present the relevant background material and the line element for the interior space time. Einstein equations for an energy momentum tensor with heat flux are solved without assuming a particular form for the final static configuration. The Vaidya solution is introduced in Section 9.3 and the matching condition is utilized to determine approximately the temporal evolution of the model. In Section 9.4 we use the solution of Mukherjee *et al* [67] to describe the final state of the star and find out the evolution of the star making use of the knowledge of the final static configuration. Although we are using a simple model, the results obtained analytically are expected to provide the general trends of results for realistic stars. In Section 9.5 we summarize our results.

9.2 Interior spacetime

Let us assume that a general shear-free metric of the form

$$ds^2 = -A^2(t,r)dt^2 + B^2(t,r)(dr^2 + r^2 d\Omega^2) \tag{9.1}$$

represents the interior spacetime of a star. The choice of a shear-free metric is motivated by the simplicity of the resulting equations, which remain tractable throughout. Moreover, as observed by Bonnor *et al* [87], it is possible to show through Raychaudhuri's equation that the slowest possible collapse is for shear-free fluids. Thus our results will be appropriate in any case if the collapse is not very fast. Also, the choice of a shear-free metric will later on help us to apply the same junction conditions derived by Santos [85] in our model.

The energy momentum tensor for the interior matter distribution is taken to be

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu \tag{9.2}$$

where the flow vector satisfies

$$q^\mu u_\nu = 0,$$

u^μ being a timelike four velocity vector. Following Bonnor *et al* [87], we choose the metric functions as follows:

$$A(r,t) = A_0(r) + \epsilon a(r)T(t) \tag{9.3}$$

$$B(r,t) = B_0(r) + \epsilon b(r)T(t) \tag{9.4}$$

and the energy density ρ and the isotropic pressure p as

$$\rho(r,t) = \rho_0(r) + \epsilon\bar{\rho}(r,t) \tag{9.5}$$

$$p(r,t) = p_0(r) + \epsilon\bar{p}(r,t) \tag{9.6}$$

The radial heat flux is of the order of $\epsilon(0 < \epsilon \ll 1)$. However, unlike the case studied by Bonnor *et al* [87], A_0 and B_0 describe here the final static solutions of the cold star. Einstein's field equations for the static configuration give the relations:

$$\rho_0 = -\frac{1}{B_0^2}\left[2\left(\frac{B_0''}{B_0}\right) - \left(\frac{B_0'}{B_0}\right)^2 + \frac{4}{r}\left(\frac{B_0'}{B_0}\right)\right] \tag{9.7}$$

$$p_0 = \frac{1}{B_0^2}\left[\left(\frac{B_0'}{B_0}\right)^2 + 2\frac{A_0'}{A_0}\frac{B_0'}{B_0} + \frac{2}{r}\left(\frac{A_0'}{A_0} + \frac{B_0'}{B_0}\right)\right] \tag{9.8}$$

where prime($\prime$) denotes differentiation with respect to r.

The pressure isotropy equation is given by

$$\left(\frac{A_0'}{A_0} + \frac{B_0'}{B_0}\right)' - \left(\frac{A_0'}{A_0} + \frac{B_0'}{B_0}\right)^2 - \frac{1}{r}\left(\frac{A_0'}{A_0} + \frac{B_0'}{B_0}\right) + 2\left(\frac{A_0'}{A_0}\right)^2 = 0 \tag{9.9}$$

We assume that the presure isotropy condition is satisfied for a known static solution (A_0, B_0). The perturbed field equations up to first order in ϵ can be written as

$$\bar{\rho} = -3\rho_0\frac{b}{B_0}T + \frac{1}{B_0^3}\left[-\left(\frac{B_0'}{B_0}\right)^2 b + 2\left(\frac{B_0'}{B_0} - \frac{2}{r}\right)b' - 2b''\right]T \tag{9.10}$$

$$\bar{p} = -2p_0\frac{b}{B_0}T + \frac{2}{B_0^2}\left[\left(\frac{B_0'}{B_0} + \frac{1}{r} + \frac{A_0'}{A_0}\right)\left(\frac{b}{B_0}\right)' + \left(\frac{B_0'}{B_0} + \frac{1}{r}\right)\left(\frac{a}{A_0}\right)'\right]T - 2\frac{b}{A_0^2 B_0}\ddot{T} \tag{9.11}$$

$$q = \frac{2\epsilon}{B_0^2}\left(\frac{b}{A_0 B_0}\right)'\dot{T} \tag{9.12}$$

where an overhead dot denotes differentiation with respect to t.

The condition of pressure isotropy for the perturbed matter distribution can be written as

$$\left[\left(\frac{a}{A_0}\right)' + \left(\frac{b}{B_0}\right)'\right]' - 2\left[\left(\frac{a}{A_0}\right)' + \left(\frac{b}{B_0}\right)'\right]\left(\frac{A_0'}{A_0} + \frac{B_0'}{B_0}\right) - \frac{1}{r}\left[\left(\frac{a}{A_0}\right)' + \left(\frac{b}{B_0}\right)'\right] + 4\frac{A_0'}{A_0}\left(\frac{a}{A_0}\right)' = 0 \tag{9.13}$$

The general solution to this equation is

$$\frac{b}{B_0} = C_0\int rA_0^2B_0^2dr + C_1 - \int rA_0^2B_0^2\int A_0^{-4}\left(\frac{(\frac{a}{A_0})'A_0^2}{rB_0^2}\right)' drdr \tag{9.14}$$

or, equivalently,

$$\frac{a}{A_0} = C_0\int\frac{rB_0^2}{A_0^2}dr + C_1 - \int\frac{rB_0^2}{A_0^2}\int A_0^4\left(\frac{(\frac{b}{B_0})'}{rA_0^2B_0^2}\right)' drdr \tag{9.15}$$

where C_0 and C_1 are constants.

Equations (9.14) or (9.15) gives a relation between $a(r)$ and $b(r)$. For specific calculations, however, it will be helpful to follow an alternative approach. We write $\left(\frac{a}{A_0}\right)' = X(r)$ and $\left(\frac{b}{B_0}\right)' = Y(r)$ and rewrite equation (9.13) as

$$\frac{(X+Y)'}{X+Y} - \left(\frac{2A_0'}{A_0} + \frac{2B_0'}{B_0} + \frac{1}{r}\right) + 4\frac{A_0'}{A_0}\frac{X}{X+Y} = 0. \tag{9.16}$$

Equation (9.16) can be integrated easily if we assume

$$\frac{X}{X+Y} = A_0\frac{dg(A_0)}{dA_0} \tag{9.17}$$

which gives

$$X = k_1 r A_0^3 B_0^2 y e^{-4g(A_0)} \tag{9.18}$$

and

$$Y = \left(\frac{1}{yA_0} - 1\right) X \tag{9.19}$$

where k_1 is an integration constant and $y = \frac{dg(A_0)}{dA_0}$.

Thus we get

$$\frac{a}{A_0} = k_1 \int r A_0^3 B_0^2 y e^{-4g(A_0)} dr + k_2 \tag{9.20}$$

where k_2 is another integration constant.

The right hand side of equation (9.20) can be integrated for different choices of the function $g(A_0)$. It is instructive to consider a simple case:

$$g(A_0) = \frac{1}{2} Ln A_0 \tag{9.21}$$

This gives

$$\frac{a}{A_0} = \frac{b}{B_0} = \frac{k_1}{2} \int r B_0^2 dr + k_2. \tag{9.22}$$

It may be useful to calculate the total energy entrapped within the surface Σ of the star. Up to first order in ϵ, this is given by

$$m(r_\Sigma, t) = m_0(r_\Sigma) + \epsilon \bar{m}(r_\Sigma, t) \tag{9.23}$$

where,

$$m_0(r_\Sigma) = -\left(r^2 B_0' + r^3 \tfrac{B_0'^2}{2B_0}\right)_\Sigma \tag{9.24}$$

$$\bar{m}(r_\Sigma, t) = \left[\left(-r^2 b' - r^3 \tfrac{B_0'^2}{2B_0}\left(2\tfrac{b'}{B_0'} - \tfrac{b}{B_0}\right)\right) T(t)\right]_\Sigma \tag{9.25}$$

where r_Σ corresponds to the boundary. The evaluation of $m_0(r_\Sigma)$ and $\bar{m}(r_\Sigma, t)$ will be taken up in section 9.4.

9.3 Junction conditions

The boundary of the collapsing star divides spacetime into two distinct regions, the interior spacetime described by the metric (9.1) and the exterior spacetime. Since the collapsing star is radiating energy, the exterior spacetime is described by Vaidya's outgoing metric

$$ds^2 = -\left(1 - \tfrac{2m(v)}{r}\right) dv^2 - 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{9.26}$$

$m(v)$ being an arbitrary function of the retarded time v. The solution (9.26) is the unique spherically symmetric solution of the Einstein field equations for radiation in the form of a null fluid. The Vaidya solution is often used to describe the exterior gravitational field of a radiating star, e.g. de Oliveira *et al* [80, 81, 82], Kolassis *et al* [88] and Kramer [83].

In order to facilitate the smooth matching of the interior space-time to the Vaidya exterior, we utilise the junction conditions deduced by Santos [85]. We can rewrite equation (9.11) and (9.12) as

$$\bar{p} = -2p_0 \tfrac{b}{B_0} T + 2\tfrac{b}{A_0^2 B_0}\left(\alpha T - \ddot{T}\right) \tag{9.27}$$

$$q = \tfrac{4b\epsilon}{A_0^2 B_0^2} \beta \dot{T} \tag{9.28}$$

where

$$\alpha = \frac{A_0^2}{bB_0}\left[\left(\frac{B_0'}{B_0}+\frac{1}{r}+\frac{A_0'}{A_0}\right)\left(\frac{b}{B_0}\right)'+\left(\frac{B_0'}{B_0}+\frac{1}{r}\right)\left(\frac{a}{A_0}\right)'\right] \tag{9.29}$$

$$\beta = \frac{A_0^2}{2b}\left(\frac{b}{A_0B_0}\right)'. \tag{9.30}$$

To find $T(t)$ we make use of the junction condition $p_\Sigma = (qB)_\Sigma$ together with $(p_0)_\Sigma = 0$ which gives

$$\alpha_\Sigma T - \ddot{T} = 2\beta_\Sigma \dot{T} \tag{9.31}$$

where Σ represents the boundary of the star. The solution of (9.31) is given by

$$T(t) = T_0 Exp\left[-\left(\beta_\Sigma + \sqrt{\alpha_\Sigma + \beta_\Sigma{}^2}\right)t\right] \tag{9.32}$$

which satisfies the boundary conditions

$$T(t)|_{t=\infty} = 0 \text{ and } T(t)|_{t=0} = T_0 \tag{9.33}$$

where T_0 is a constant. Since we expect $T(t)$ to decrease as t increases, we must have $\alpha_\Sigma > 0$. This will be taken up in the next section.

9.4 Final static solution

We assume that the line element (9.1) represents a static solution at large time i.e. at $t \to \infty$, and the metric coefficients are then represented by A_0 and B_0, the static part of equations (9.3) and (9.4), respectively.

For the static solution we take the soutions of Mukherjee *et al* [67] of the Vaidya-Tikekar [63] model, viz.

$$ds^2 = -e^{2\gamma(\bar{r})}dt^2 + e^{2\mu(\bar{r})}d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{9.34}$$

where

$$e^{2\mu(\bar{r})} = \frac{1+\lambda\bar{r}^2/R^2}{1-\bar{r}^2/R^2} \tag{9.35}$$

and

$$e^{\gamma(\bar{r})} = A\left[\frac{\cos[(n+1)\zeta+\delta]}{n+1} - \frac{\cos[(n-1)\zeta+\delta]}{n-1}\right] \tag{9.36}$$

In equations (9.35) and (9.36), A, δ and R are constants, $\zeta = \cos^{-1} z$, $n^2 = \lambda + 2$ and $z^2 = (\frac{\lambda}{\lambda+1})(1 - \frac{\bar{r}^2}{R^2})$.

The solution satisfies strong and weak energy conditions as well as causality condition and is vaid for any value of $\lambda \geq \frac{3}{17}$ [67]. In earlier chapters we have shown that this model can describe cold compact stars like Her X-1 and SAX J1808.4-3658. The model also has a scaling property which allows it to describe a class of stars with the same compactness [133]. Moreover, the stability of the stellar configurations for various λ can be verified easily in this model, as shown in chapter 6. Thus although the solution provides a simple model of a star, its simple analytic form is found to be very useful in studying the gross features of a star with an EOS specified by the parameter λ.

However, to make use of this solution, we need to transform the metric (9.1) to the form given by (9.34), i.e. change the radial coordinate r to $\bar{r}$. Comparing equations (9.1) and (9.34), we get,

$$\bar{r} = rB_0(r) \text{ and } e^{\mu(\bar{r})}d\bar{r} = B_0(r)dr \tag{9.37}$$

Using equations (9.35) and (9.37), we get the inverse transformation relation as

$$r = k_3 exp\left[-tanh^{-1}\left(\frac{sin\chi}{\sqrt{1+\lambda cos^2\chi}}\right) - \sqrt{\lambda}sin^{-1}\left(\sqrt{\frac{\lambda}{\lambda+1}}sin\chi\right)\right] \tag{9.38}$$

where $\chi = cos^{-1}\left(\frac{\bar{r}}{R}\right)$ and k_3 is an integration constant.

The integration in equation (9.22) is over r, but using equation (9.37) we can integrate it over $\bar{r}$ so that the final results are expressed in terms of $\bar{r}$. We get,

$$\frac{a}{A_0} = \frac{b}{B_0} = -\frac{k_1}{4l^2}R^2\sqrt{\lambda}\left[sin^{-1}(lx) + lx\sqrt{1-l^2x^2}\right] + k_2 \tag{9.39}$$

where $l = \sqrt{\frac{\lambda}{\lambda+1}}$, $x = \sqrt{1 - \frac{\bar{r}^2}{R^2}}$.

This gives us an analytic solution for the early evolutionary stages of the star. However, to get an idea about the behaviour of the solution, we need to take specific cases and do some numerical work. This will be taken up in the next section.

9.5 Results and discussions

As an example, we consider a star whose final state has a mass $m = 0.88\ M_\odot$ and radius $b = 7.7$ km. Note that, these values fall well within the estimated mass and radius of the well known pulsar Her X-1 [40]. Moreover, with a particular choice of the parameter λ ($\lambda = 100$), it has been shown in chapter 5 that the equation of state obtained by using the static solution, agrees accurately with the equation of state obtained by Horvath and Pacheco [72] for a quark-diquark mixture. Matching the static solution to the Schwarzchild exterior solution and using the boundary condition $p_0 = 0$ at $\bar{r} = b$ (note that in the evolutionary stage pressure does not vanish at the boundary), we calculate the values of the constants as $R = 108.779$ km, $\delta = 2.58577$ and $A = 26.6039$.

We, now choose the constants k_1, k_2 & k_3 in such a way that our model describes the expected early evolutionary stages of the star. The time dependence of $T(t)$, heat flux $q(t)$ and mass $m(r,t)$ are shown in Fig.9.1, Fig.9.2 and Fig.9.3, respectively, where we considered two exemplary cases: (1) $k_1 = 1$, $k_2 = 10^5$ & $k_3 = 1$ (solid line) and (2) $k_1 = 1$, $k_2 = 10^{5.2}$ and $k_3 = 1$ (dashed line). Arbitrary choices of these values may not give realistic results. In Fig.9.1 & 9.2, we find that both $T(t)$ and $q(t)$ decreases with time, as expected. Also, in Fig.9.3, the mass of the star decreases with increasing time and as time goes to infinity the mass saturates to it final static value of $0.88\ M_\odot$.

Thus, our model gives a description of the evolution of a radiating star. For a known static configuration of a star, this model generates solutions for the earlier stages of the star by a perturbative approach. This may be looked upon as complementary to the approach of Bonnor *et al* [87]. Possibly, the two methods, when combined carefully, will be able to give a total picture of the radiative collapse of a star, which ends up as a cold compact star. In our method, we have made use of the general solution given by Mukherjee *et al* [67]. Although we chose a simple form of $g(A_0)$ more general cases may also be studied.

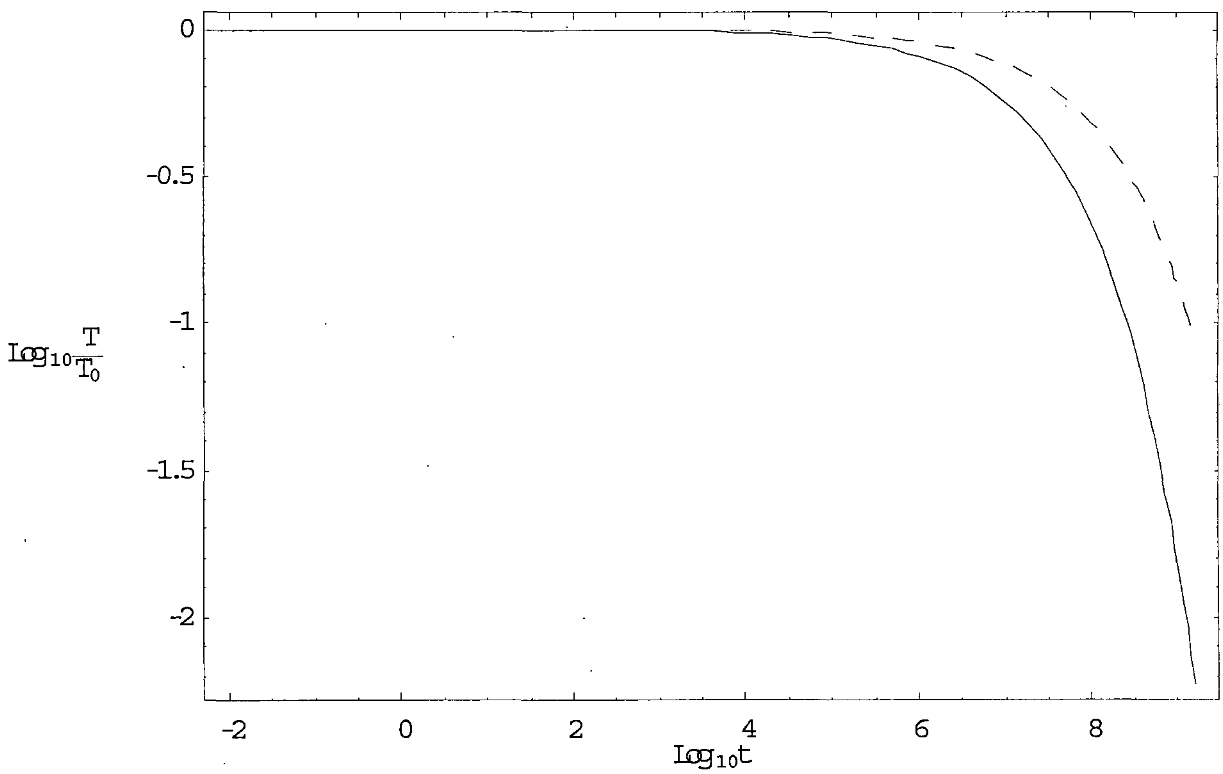

Figure 9.1: Evolution of T(t). The solid line is for $k_1 = 1$, $k_2 = 10^5$ & $k_3 = 1$, while the dashed line corresponds to $k_1 = 1$, $k_2 = 10^{5.2}$ & $k_3 = 1$.

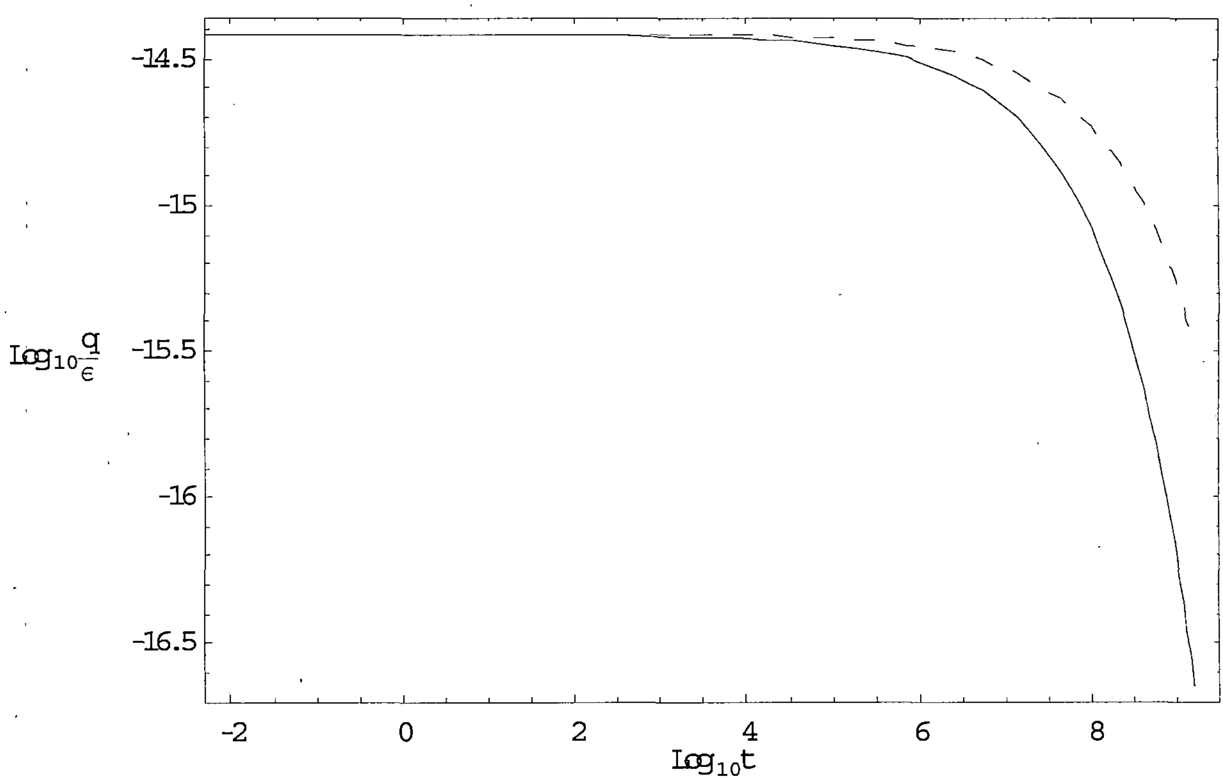

Figure 9.2: Variation of heat flux q as a function of t. The solid line is for $k_1 = 1$, $k_2 = 10^5$ & $k_3 = 1$, while the dashed line corresponds to $k_1 = 1$, $k_2 = 10^{5.2}$ & $k_3 = 1$.

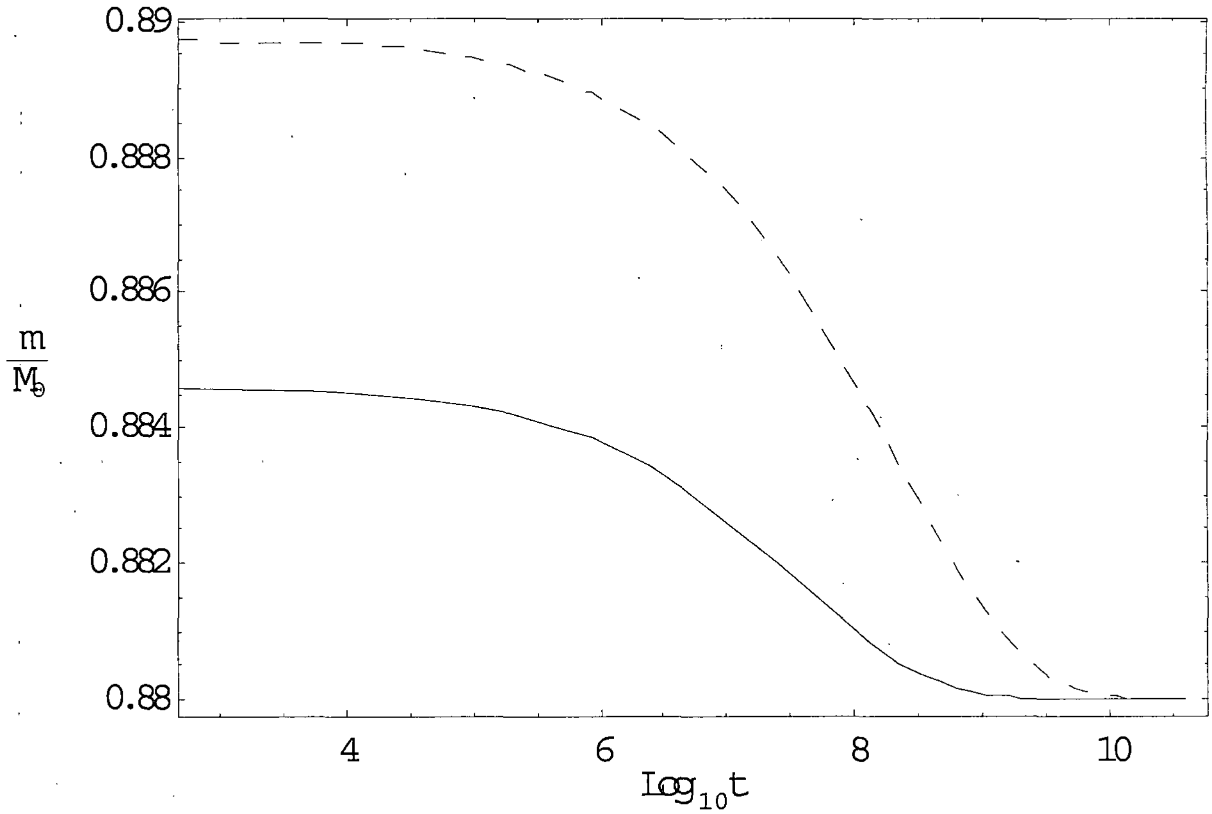

Figure 9.3: Evolution of mass of a star whose final static configuration has mass $m = 0.88\ M_\odot$. The solid line is for $k_1 = 1$, $k_2 = 10^5$ & $k_3 = 1$, while the dashed line corresponds to $k_1 = 1$, $k_2 = 10^{5.2}$ & $k_3 = 1$.