



## CHAPTER 2

ON THE GROWTH ESTIMATES OF  
DIFFERENTIAL MONOMIALS AND  
DIFFERENTIAL POLYNOMIALS  
GENERATED BY MEROMORPHIC  
FUNCTIONS

235179



## Chapter 2

# ON THE GROWTH ESTIMATES OF DIFFERENTIAL MONOMIALS AND DIFFERENTIAL POLYNOMIALS GENERATED BY MEROMORPHIC FUNCTIONS

### 2.1 Introduction, Definitions and Notations.

For any two transcendental entire functions  $f$  and  $g$  defined in the open complex plane  $\mathbb{C}$ , Clunie [6] proved that

$$\lim_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, f)} = \infty \quad \text{and} \quad \lim_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, g)} = \infty.$$

Singh [42] proved some comparative growth properties of  $\log T(r, f \circ g)$  and  $T(r, f)$ . He also raised the problem of investigating the comparative growth of  $\log T(r, f \circ g)$  and  $T(r, g)$  which he was unable to solve. However, some results on the comparative growth of  $\log T(r, f \circ g)$  and  $T(r, g)$  are proved in [26].

Let  $f$  be a non-constant meromorphic function defined in the open complex plane  $\mathbb{C}$ . Also let  $n_{0j}, n_{1j}, \dots, n_{kj} (k \geq 1)$  be non-negative integers such that for each  $j$ ,  $\sum_{i=0}^k n_{ij} \geq 1$ . We call

$$M_j[f] = A_j(f)^{n_{0j}} \left(f^{(1)}\right)^{n_{1j}} \dots \left(f^{(k)}\right)^{n_{kj}}$$

---

Communicated

where  $T(r, A_j) = S(r, f)$  to be a differential monomial generated by  $f$ . The numbers

$$\gamma_{M_j} = \sum_{i=0}^k n_{ij} \quad \text{and} \quad \Gamma_{M_j} = \sum_{i=0}^k (i+1)n_{ij}$$

are called respectively the degree and weight of  $M_j[f]$  {[8],[38]}. The expression  $P[f] = \sum_{j=1}^s M_j[f]$  is called a differential polynomial generated by  $f$ . The numbers

$$\gamma_P = \max_{1 \leq j \leq s} \gamma_{M_j} \quad \text{and} \quad \Gamma_P = \max_{1 \leq j \leq s} \Gamma_{M_j}$$

are called respectively the degree and weight of  $P[f]$  {[8],[38]}. Also we call the numbers  $\gamma_P = \min_{1 \leq j \leq s} \gamma_{M_j}$  and  $k$  (the order of the highest derivative of  $f$ ) the lower degree and the order of  $P[f]$  respectively. If  $\gamma_P = \gamma_P$ ,  $P[f]$  is called a homogeneous differential polynomial. In this chapter we further investigate the question of Singh [42] mentioned earlier and prove some new results relating to the comparative growths of composite entire or meromorphic functions and differential monomials, differential polynomials generated by one of the factors. Throughout the chapter we consider only the non-constant differential polynomials and we denote by  $P_0[f]$  a differential polynomial not containing  $f$  i.e., for which  $n_{0j} = 0$  for  $j = 1, 2, \dots, s$ . We consider only those  $P[f]$ ,  $P_0[f]$  singularities of whose individual terms do not cancel each other. We also denote by  $M[f]$  a differential monomial generated by a transcendental meromorphic function  $f$ .

The following definitions are well known.

**Definition 2.1.1** *The order  $\rho_f$  and lower order  $\lambda_f$  of a meromorphic function  $f$  are defined as*

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}.$$

*If  $f$  is entire, one can easily verify that*

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log r},$$

*where  $\log^{[k]} x = \log(\log^{[k-1]} x)$  for  $k = 1, 2, 3, \dots$  and  $\log^{[0]} x = x$ .*

**Definition 2.1.2** The hyper order  $\bar{\rho}_f$  and hyper lower order  $\bar{\lambda}_f$  of a meromorphic function  $f$  are defined as follows

$$\bar{\rho}_f = \limsup_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f)}{\log r} \quad \text{and} \quad \bar{\lambda}_f = \liminf_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f)}{\log r}.$$

If  $f$  is entire, then

$$\bar{\rho}_f = \limsup_{r \rightarrow \infty} \frac{\log^{[3]} M(r, f)}{\log r} \quad \text{and} \quad \bar{\lambda}_f = \liminf_{r \rightarrow \infty} \frac{\log^{[3]} M(r, f)}{\log r}.$$

**Definition 2.1.3** [29] Let  $f$  be a meromorphic function of order zero. Then the quantities  $\rho_f^*$  and  $\lambda_f^*$  are defined in the following way :

$$\rho_f^* = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log^{[2]} r}, \quad \lambda_f^* = \liminf_{r \rightarrow \infty} \frac{\log T(r, f)}{\log^{[2]} r}$$

If  $f$  is entire then clearly

$$\rho_f^* = \limsup_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log^{[2]} r}, \quad \lambda_f^* = \liminf_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log^{[2]} r}$$

**Definition 2.1.4** The type  $\sigma_f$  and lower type  $\eta_f$  of a meromorphic function  $f$  is defined as

$$\sigma_f = \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f}} \quad \text{and} \quad \eta_f = \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

When  $f$  is entire, then

$$\sigma_f = \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\rho_f}} \quad \text{and} \quad \eta_f = \liminf_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

**Definition 2.1.5** [10] The weak type  $\tau_f$  of a meromorphic function  $f$  of finite positive lower order  $\lambda_f$  is defined by

$$\tau_f = \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda_f}}$$

For entire  $f$ ,

$$\tau_f = \liminf_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\lambda_f}}, \quad 0 < \lambda_f < \infty.$$

**Definition 2.1.6** Let 'a' be a complex number, finite or infinite. The Nevanlinna deficiency and the Valiron deficiency of 'a' with respect to a meromorphic function  $f$  are defined as

$$\delta(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \rightarrow \infty} \frac{m(r, a; f)}{T(r, f)}$$

and

$$\Delta(a; f) = 1 - \liminf_{r \rightarrow \infty} \frac{N(r, a; f)}{T(r, f)} = \limsup_{r \rightarrow \infty} \frac{m(r, a; f)}{T(r, f)}.$$

From the second fundamental theorem it follows that the set of values of  $a \in \mathbb{C} \cup \{\infty\}$  for which  $\delta(a; f) > 0$  is countable and  $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) \leq 2$  (cf.[21], p.43). If in particular,  $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$ , we say that  $f$  has the maximum deficiency sum.

**Definition 2.1.7** The quantity  $\Theta(a; f)$  of a meromorphic function  $f$  is defined as follows

$$\Theta(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, a; f)}{T(r, f)}.$$

**Definition 2.1.8 [46]** For  $a \in \mathbb{C} \cup \{\infty\}$ , we denote by  $n(r, a; f | = 1)$ , the number of simple zeros of  $f - a$  in  $|z| \leq r$ .  $N(r, a; f | = 1)$  is defined in terms of  $n(r, a; f | = 1)$  in the usual way. We put

$$\delta_1(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f | = 1)}{T(r, f)},$$

the deficiency of 'a' corresponding to the simple  $a$ -points of  $f$  i.e., simple zeros of  $f - a$ .

Yang [45] proved that there exists at most a denumerable number of complex numbers  $a \in \mathbb{C} \cup \{\infty\}$  for which  $\delta_1(a; f) > 0$  and  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) \leq 4$ .

**Definition 2.1.9 [28]** For  $a \in \mathbb{C} \cup \{\infty\}$ , let  $n_p(r, a; f)$  denotes the number of zeros of  $f - a$  in  $|z| \leq r$ , where a zero of multiplicity  $< p$  is counted according to its multiplicity and a zero of multiplicity  $\geq p$  is counted exactly  $p$  times;

and  $N_p(r, a; f)$  is defined in terms of  $n_p(r, a; f)$  in the usual way. We define the quantity  $\delta_p(a; f)$  as follows :

$$\delta_p(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N_p(r, a; f)}{T(r, f)}.$$

**Definition 2.1.10** [3]  $P[f]$  is said to be admissible if

- (i)  $P[f]$  is homogeneous, or
- (ii)  $P[f]$  is non homogeneous and  $m(r, f) = S(r, f)$ .

Since the natural extension of a derivative is a differential polynomial, in this chapter we also prove our results for special type of linear differential polynomials viz., the wronskians. In this chapter we prove some new results depending on the comparative growth properties of composite entire or meromorphic functions and wronskians generated by one of the factors which improve some earlier theorems.

To this end we give the following two definitions :

**Definition 2.1.11** A meromorphic function  $a = a(z)$  is called small with respect to  $f$  if  $T(r, a) = S(r, f)$ .

**Definition 2.1.12** Let  $a_1, a_2, \dots, a_k$  be linearly independent meromorphic functions and small with respect to  $f$ . We denote by  $L(f) = W(a_1, a_2, \dots, a_k, f)$  the Wronskian determinant of  $a_1, a_2, \dots, a_k, f$  i.e.,

$$L(f) = \begin{vmatrix} a_1 & a_2 & \dots & \dots & a_k & f \\ a_1' & a_2' & \dots & \dots & a_k' & f' \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1^{(k)} & a_2^{(k)} & \dots & \dots & a_k^{(k)} & f^{(k)} \end{vmatrix}.$$

## 2.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 2.2.1** [1] If  $f$  is meromorphic and  $g$  is entire then for all sufficiently large values of  $r$ ,

$$T(r, f \circ g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).$$

**Lemma 2.2.2 [2]** Let  $f$  be meromorphic and  $g$  be entire and suppose that  $0 < \mu < \rho_g \leq \infty$ . Then for a sequence of values of  $r$  tending to infinity,

$$T(r, f \circ g) \geq T(\exp(r^\mu), f).$$

**Lemma 2.2.3 [3]** Let  $P_0[f]$  be admissible. If  $f$  is of finite order or of non-zero lower order and  $\sum_{a \neq \infty} \Theta(a; f) = 2$  then

$$\lim_{r \rightarrow \infty} \frac{T(r, P_0[f])}{T(r, f)} = \Gamma_{P_0}.$$

**Lemma 2.2.4 [3]** Let  $f$  be either of finite order or of non-zero lower order such that  $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$  or  $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ . Then for homogeneous  $P_0[f]$ ,

$$\lim_{r \rightarrow \infty} \frac{T(r, P_0[f])}{T(r, f)} = \gamma_{P_0}.$$

**Lemma 2.2.5 [10]** If  $f$  be a meromorphic function of regular growth i.e.,  $\lambda_f = \rho_f$ , then the following quantities

$$\begin{aligned} (i) \quad \sigma_f &= \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f}}, & (ii) \quad \tau_f &= \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda_f}} \\ (i) \quad \eta_f &= \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f}}, & (ii) \quad \bar{\tau}_f &= \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda_f}} \end{aligned}$$

are all equivalent.

**Lemma 2.2.6** Let  $f$  be a meromorphic function of finite order or of non zero lower order. If  $\sum_{a \neq \infty} \Theta(a; f) = 2$ , then the order (lower order) of homogeneous  $P_0[f]$  is same as that of  $f$ . Also the type and lower type of  $P_0[f]$  is  $\Gamma_{P_0}$  times that of  $f$  if  $f$  is of positive finite order.

**Proof.** By Lemma 2.2.3,  $\lim_{r \rightarrow \infty} \frac{\log T(r, P_0[f])}{\log T(r, f)}$  exists and is equal to 1.

$$\begin{aligned} \rho_{P_0[f]} &= \limsup_{r \rightarrow \infty} \frac{\log T(r, P_0[f])}{\log r} \\ &= \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} \cdot \lim_{r \rightarrow \infty} \frac{\log T(r, P_0[f])}{\log T(r, f)} \\ &= \rho_f \cdot 1 = \rho_f. \end{aligned}$$

In a similar manner,  $\lambda_{P_0[f]} = \lambda_f$ .

Again by Lemma 2.2.3,

$$\begin{aligned}\sigma_{P_0[f]} &= \limsup_{r \rightarrow \infty} \frac{T(r, P_0[f])}{r^{\rho_{P_0[f]}}} \\ &= \lim_{r \rightarrow \infty} \frac{T(r, P_0[f])}{T(r, f)} \cdot \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f}} \\ &= \Gamma_{P_0} \sigma_f.\end{aligned}$$

Similarly  $\eta_{P_0[f]} = \Gamma_{P_0} \eta_f$ .

This proves the lemma. ■

**Lemma 2.2.7** *Let  $f$  be a meromorphic function of finite order or of non-zero lower order such that  $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$  or  $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) =$*

*1. Then the order (lower order) of homogeneous  $P_0[f]$  and  $f$  are same. Also the type and lower type of  $P_0[f]$  is  $\gamma_{P_0}$  times that of  $f$  when  $f$  is of finite positive order.*

We omit the proof of the lemma because it can be carried out in the line of Lemma 2.2.6 and with the help of Lemma 2.2.4.

**Lemma 2.2.8** *Let  $f$  be a meromorphic function of finite order or of non zero lower order. If  $\sum_{a \neq \infty} \Theta(a; f) = 2$ , then the weak type of  $P_0[f]$  is  $\Gamma_{P_0}$  times that of  $f$  if  $f$  is of finite positive lower order.*

**Proof.** By Lemma 2.2.3

$$\lambda_{P_0[f]} = \lambda_f.$$

Again by Lemma 2.2.3

$$\begin{aligned}\tau_{P_0[f]} &= \liminf_{r \rightarrow \infty} \frac{T(r, P_0[f])}{r^{\lambda_{P_0[f]}}} \\ &= \lim_{r \rightarrow \infty} \frac{T(r, P_0[f])}{T(r, f)} \cdot \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda_f}} \\ &= \Gamma_{P_0} \tau_f.\end{aligned}$$

This proves the lemma ■

**Remark 2.2.1** If  $f$  be a meromorphic function of regular growth i.e.,  $\lambda_f = \rho_f$  and of finite order or of non zero lower order with  $\sum_{a \neq \infty} \Theta(a; f) = 2$ , then

the quantity  $\bar{\tau}_{P_0[f]}$  is  $\Gamma_{P_0}$  times that of  $f$  where  $f$  is of finite positive lower order.

We omit the proof because it can be carried out in the line of Lemma 2.2.8 and with the help of Lemma 2.2.5.

**Lemma 2.2.9** Let  $f$  be a meromorphic function of finite order or of non-zero lower order such that  $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$  or  $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) =$

1. Then the weak type of  $P_0[f]$  is  $\gamma_{P_0}$  times that of  $f$  when  $f$  is of finite positive lower order.

We omit the proof of the lemma because it can be carried out in the line of Lemma 2.2.8 and with the help of Lemma 2.2.4.

**Remark 2.2.2** If  $f$  be a meromorphic function of regular growth i.e.,  $\lambda_f = \rho_f$  and of finite order or of non zero lower order with  $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) =$

1 or  $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ , then the quantity  $\bar{\tau}_{P_0[f]}$  is  $\gamma_{P_0}$  times that of  $f$  where  $f$  is of finite positive lower order.

**Lemma 2.2.10** Let  $f$  be a meromorphic function of finite order or of non-zero lower order and  $\sum_{a \neq \infty} \Theta(a; f) = 2$ . Then the hyper order (hyper lower order) of  $P_0[f]$  and  $f$  are equal.

We omit the proof of Lemma 2.2.10 because it can be carried out in the line of Lemma 2.2.6.

**Remark 2.2.3** The conclusion of Lemma 2.2.11 can also be deduced under the hypothesis  $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$  or  $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$  instead of  $\sum_{a \neq \infty} \Theta(a; f) = 2$ .

**Lemma 2.2.11 [31]** Let  $f$  be a transcendental meromorphic function of finite order or of non-zero lower order and  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$ , then

$$\lim_{r \rightarrow \infty} \frac{T(r, M[f])}{T(r, f)} = \Gamma_M - (\Gamma_M - \gamma_M) \Theta(\infty; f),$$

where

$$\Theta(\infty; f) = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, f)}{T(r, f)} .$$

**Lemma 2.2.12** *If  $f$  be a transcendental meromorphic function of finite order or of non-zero lower order and  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$ , then the order and lower*

*order of  $M[f]$  are same as those of  $f$ . Also the type, lower type and weak type of  $M[f]$  is  $\{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; f)\}$  times that of  $f$ , i.e.,  $\rho_{M[f]} = \rho_f$ ,  $\lambda_{M[f]} = \lambda_f$ ,  $\sigma_{M[f]} = \{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; f)\}\sigma_f$ ,  $\eta_{M[f]} = \{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; f)\}\eta_f$  when  $f$  is of finite positive order and  $\tau_{M[f]} = \{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; f)\}\tau_f$  when  $f$  is of finite positive lower order.*

**Proof.** By Lemma 2.2.3,  $\lim_{r \rightarrow \infty} \frac{\log T(r, M[f])}{\log T(r, f)}$  exists and is equal to 1. So

$$\begin{aligned} \rho_{M[f]} &= \limsup_{r \rightarrow \infty} \frac{\log T(r, M[f])}{\log r} \cdot \lim_{r \rightarrow \infty} \frac{\log T(r, M[f])}{\log T(r, f)} \\ &= \rho_f \cdot 1 = \rho_f . \end{aligned}$$

In a similar manner,  $\lambda_{M[f]} = \lambda_f$ .

Again

$$\begin{aligned} \sigma_{M[f]} &= \limsup_{r \rightarrow \infty} \frac{T(r, M[f])}{r^{\rho_{M[f]}}} \\ &= \lim_{r \rightarrow \infty} \frac{T(r, M[f])}{T(r, f)} \cdot \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f}} \\ &= \{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; f)\}\sigma_f . \end{aligned}$$

In a similar manner,  $\eta_{M[f]} = \{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; f)\}\eta_f$ .

Also

$$\begin{aligned} \tau_{M[f]} &= \liminf_{r \rightarrow \infty} \frac{T(r, M[f])}{r^{\lambda_{M[f]}}} \\ &= \lim_{r \rightarrow \infty} \frac{T(r, M[f])}{T(r, f)} \cdot \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda_f}} \\ &= \{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; f)\}\tau_f . \end{aligned}$$

This proves the lemma. ■

**Remark 2.2.4** *In the line of Lemma 2.2.12 one can easily verify that for a transcendental meromorphic function  $f$  of regular growth i.e.,  $\lambda_f = \rho_f$  and of finite order or of non zero lower order with  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$ , the quantity*

$\bar{\tau}_{M[f]}$  *is  $\{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; f)\}$  times that of  $f$  where  $f$  is of finite positive lower order.*

**Lemma 2.2.13** *Let  $f$  be a transcendental meromorphic function of finite order or of non-zero lower order and  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$ . Then the hyper order (hyper lower order) of  $P[f]$  and  $f$  are equal.*

*We omit the proof of Lemma 2.2.13 because it can be carried out in the line of Lemma 2.2.12.*

**Lemma 2.2.14** [32] *Let  $f$  be a transcendental meromorphic function having the maximum deficiency sum . Then*

$$\lim_{r \rightarrow \infty} \frac{T(r, L(f))}{T(r, f)} = 1 + k - k\delta(\infty; f).$$

**Lemma 2.2.15** *If  $f$  be a transcendental meromorphic function with the maximum deficiency sum, then the order and lower order of  $L(f)$  are same as those of  $f$ . Also the type, lower type and the weak type of  $L[f]$  is  $\{1 + k - k\delta(\infty; f)\}$  times that of  $f$ , i.e.,  $\rho_{L[f]} = \rho_f$ ,  $\lambda_{L[f]} = \lambda_f$ ,  $\sigma_{L[f]} = \{1 + k - k\delta(\infty; f)\}\sigma_f$ ,  $\eta_{L[f]} = \{1 + k - k\delta(\infty; f)\}\eta_f$  when  $f$  is of finite positive order and  $\tau_{L[f]} = \{1 + k - k\delta(\infty; f)\}\tau_f$  when  $f$  is of finite positive lower order.*

**Proof.** By Lemma 2.2.14,

$$\lim_{r \rightarrow \infty} \frac{\log T(r, L(f))}{\log T(r, f)}$$

exists and is equal to 1. So

$$\begin{aligned}
\rho_{L[f]} &= \limsup_{r \rightarrow \infty} \frac{\log T(r, L(f))}{\log r} \\
&= \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} \cdot \lim_{r \rightarrow \infty} \frac{\log T(r, L(f))}{\log T(r, f)} \\
&= \rho_f \cdot 1 \\
&= \rho_f.
\end{aligned}$$

In a similar manner,  $\lambda_{L[f]} = \lambda_f$ .

Again

$$\begin{aligned}
\sigma_{L[f]} &= \limsup_{r \rightarrow \infty} \frac{T(r, M[f])}{r^{\rho_{L[f]}}} \\
&= \lim_{r \rightarrow \infty} \frac{T(r, M[f])}{T(r, f)} \cdot \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f}} \\
&= \{1 + k - k\delta(\infty; f)\} \sigma_f.
\end{aligned}$$

Similarly  $\eta_{L[f]} = \{1 + k - k\delta(\infty; f)\} \eta_f$

Also

$$\begin{aligned}
\tau_{L[f]} &= \liminf_{r \rightarrow \infty} \frac{T(r, M[f])}{r^{\lambda_{L[f]}}} \\
&= \lim_{r \rightarrow \infty} \frac{T(r, M[f])}{T(r, f)} \cdot \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda_f}} \\
&= \{1 + k - k\delta(\infty; f)\} \tau_f.
\end{aligned}$$

This proves the lemma. ■

**Remark 2.2.5** *In the line of Lemma 2.2.15 one can easily verify that for a transcendental meromorphic function  $f$  of regular growth i.e.,  $\lambda_f = \rho_f$  and of finite order or of non zero lower order with the maximum deficiency sum, the quantity  $\bar{\tau}_{L[f]}$  is  $\{1 + k - k\delta(\infty; f)\}$  times that of  $f$ .*

**Lemma 2.2.16** *If  $f$  be a transcendental meromorphic function with the maximum deficiency sum. Then the hyper order (hyper lower order) of  $L[f]$  and  $f$  are equal.*

We omit the proof of Lemma 2.2.16 because it can be carried out in the line of Lemma 2.2.15.

**Lemma 2.2.17 [35]** *Let  $f$  and  $g$  be two transcendental entire functions with  $\rho_g < \infty$ ,  $\eta$  be a constant satisfying  $0 < \eta < 1$  and  $\alpha$  be a positive number. Then*

$$\begin{aligned} T(r, f \circ g) + O(1) &\geq N(r, 0; f \circ g) \\ &\geq \log \left( \frac{1}{\eta} \right) \left[ \frac{N(M((\eta r)^{\frac{1}{1+\alpha}}, g), 0, f)}{\log M((\eta r)^{\frac{1}{1+\alpha}}, g)} - O(1) \right] \end{aligned}$$

as  $r \rightarrow \infty$  through all values.

**Lemma 2.2.18 [11]** *If  $f$  be any meromorphic function or entire function of order zero. Then (i)  $\rho_f^* = 1$  and (ii)  $\lambda_f^* = 1$ .*

## 2.3 Theorems.

In this section we present the main results of the chapter.

**Theorem 2.3.1** *Let  $f$ ,  $g$  and  $h$  be three entire functions such that  $\rho_h < \lambda_f \leq \rho_f < \infty$  and  $\rho_h < \lambda_g$ . Also let  $\sum_{a \neq \infty} \Theta(a; f) = \sum_{a \neq \infty} \Theta(a; g) = 2$ . Then*

$$\lim_{r \rightarrow \infty} \frac{\{\log T(r, f \circ h)\}^2}{T(r, P_0[f])T(r, P_0[g])} = 0.$$

**Proof.** It is well known that for an entire function  $h$ ,  $T(r, h) \leq \log^+ M(r, h)$ .

So in view of Lemma 2.2.1 we get for all sufficiently large values of  $r$ ,

$$\begin{aligned} T(r, f \circ h) &\leq \{1 + o(1)\}T(M(r, h), f) \\ \text{i.e., } \log T(r, f \circ h) &\leq \log\{1 + o(1)\} + \log T(M(r, h), f) \\ \text{i.e., } \log T(r, f \circ h) &\leq o(1) + (\rho_f + \epsilon) \log M(r, h) \\ \text{i.e., } \log T(r, f \circ h) &\leq o(1) + (\rho_f + \epsilon)r^{\rho_h + \epsilon}. \end{aligned} \tag{2.1}$$

Also by Lemma 2.2.6 we obtain for all sufficiently large values of  $r$ ,

$$T(r, P_0[g]) \geq r^{(\lambda_{P_0[g]} - \epsilon)} = r^{\lambda_g - \epsilon}. \tag{2.2}$$

Now combining (2.1) and (2.2) it follows for all sufficiently large values of  $r$ ,

$$\begin{aligned} \frac{\log T(r, f \circ h)}{T(r, P_0[g])} &\leq \frac{o(1) + (\rho_f + \epsilon)r^{\rho_h + \epsilon}}{r^{\lambda_g - \epsilon}} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, P_0[g])} &\leq \limsup_{r \rightarrow \infty} \frac{o(1) + (\rho_f + \epsilon)r^{\rho_h + \epsilon}}{r^{\lambda_g - \epsilon}}. \end{aligned} \tag{2.3}$$

As  $\rho_h < \lambda_g$ , we can choose  $\epsilon (> 0)$  in such a way that

$$\rho_h + \epsilon < \lambda_g - \epsilon.$$

So in view of (2.3) we get from above that

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, P_0[g])} &= 0 \\ \text{i.e., } \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, P_0[g])} &= 0. \end{aligned} \quad (2.4)$$

Again in view of Lemma 2.2.6 we get for all sufficiently large values of  $r$ ,

$$\begin{aligned} \log T(r, P_0[f]) &\geq (\lambda_{P_0[f]} - \epsilon) \log r \\ \text{i.e., } T(r, P_0[f]) &\geq r^{\lambda_f - \epsilon}. \end{aligned} \quad (2.5)$$

Since  $\rho_h < \lambda_f$ , we can choose  $\epsilon (> 0)$  in such a way that

$$\rho_h + \epsilon < \lambda_f - \epsilon.$$

Now combining (2.1) and (2.5) it follows from above for all sufficiently large values of  $r$ ,

$$\begin{aligned} \frac{\log T(r, f \circ h)}{T(r, P_0[f])} &\leq \frac{o(1) + (\rho_f + \epsilon)r^{\rho_h + \epsilon}}{r^{\lambda_f - \epsilon}} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, P_0[f])} &= 0 \\ \text{i.e., } \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, P_0[f])} &= 0. \end{aligned} \quad (2.6)$$

Therefore in view of (2.4) and (2.6), we obtain that

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{\{\log T(r, f \circ h)\}^2}{T(r, P_0[f])T(r, P_0[g])} &= \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, P_0[f])} \cdot \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, P_0[g])} \\ \text{i.e., } \lim_{r \rightarrow \infty} \frac{\{\log T(r, f \circ h)\}^2}{T(r, P_0[f])T(r, P_0[g])} &= 0. \end{aligned}$$

This proves the theorem. ■

**Remark 2.3.1** *The conclusion of Theorem 1 can also be deduced under the hypothesis  $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$  or  $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$  and*

$\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$  or  $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$  instead of  $\sum_{a \neq \infty} \Theta(a; f) = 2$  and  $\sum_{a \neq \infty} \Theta(a; g) = 2$  respectively.

In the line of Theorem 2.3.1 and with the help of Lemma 2.2.1 and Lemma 2.2.12 we may proof the following theorem.

**Theorem 2.3.2** *Let  $f, g$  be two transcendental entire functions and  $h$  be any entire function such that  $\rho_h < \lambda_f \leq \rho_f < \infty$  and  $\rho_h < \lambda_g$ . Also let  $f, g$  be transcendental with  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = \sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$ . Then*

$$\lim_{r \rightarrow \infty} \frac{\{\log T(r, f \circ h)\}^2}{T(r, M[f]) T(r, M[g])} = 0.$$

**Proof.** In view of Lemma 2.2.12 we obtain for all sufficiently large values of  $r$ ,

$$T(r, M[g]) \geq r^{(\lambda_{M[g]} - \epsilon)} = r^{\lambda_g - \epsilon}. \quad (2.7)$$

Now combining (2.1) and (2.7) it follows for all sufficiently large values of  $r$ ,

$$\begin{aligned} \frac{\log T(r, f \circ h)}{T(r, M[g])} &\leq \frac{o(1) + (\rho_f + \epsilon)r^{\rho_h + \epsilon}}{r^{\lambda_g - \epsilon}} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, M[g])} &\leq \limsup_{r \rightarrow \infty} \frac{o(1) + (\rho_f + \epsilon)r^{\rho_h + \epsilon}}{r^{\lambda_g - \epsilon}}. \end{aligned} \quad (2.8)$$

As  $\rho_h < \lambda_g$ , we can choose  $\epsilon (> 0)$  in such a way that

$$\rho_h + \epsilon < \lambda_g - \epsilon.$$

So in view of (2.8) we get from above that

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, M[g])} &= 0 \\ \text{i.e., } \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, M[g])} &= 0. \end{aligned} \quad (2.9)$$

Again in view of Lemma 2.2.12 we get for all sufficiently large values of  $r$ ,

$$\begin{aligned} \log T(r, M[f]) &\geq (\lambda_{M[f]} - \epsilon) \log r \\ \text{i.e., } T(r, M[f]) &\geq r^{\lambda_f - \epsilon}. \end{aligned} \quad (2.10)$$

Since  $\rho_h < \lambda_f$ , we can choose  $\epsilon (> 0)$  in such a way that

$$\rho_h + \epsilon < \lambda_f - \epsilon.$$

Now combining (2.1) and (2.10) it follows from above for all sufficiently large values of  $r$ ,

$$\begin{aligned} \frac{\log T(r, f \circ h)}{T(r, M[f])} &\leq \frac{o(1) + (\rho_f + \epsilon)r^{\rho_h + \epsilon}}{r^{\lambda_f - \epsilon}} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, M[f])} &= 0 \\ \text{i.e., } \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, M[f])} &= 0. \end{aligned} \quad (2.11)$$

Therefore in view of (2.9) and (2.11), we obtain that

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{\{\log T(r, f \circ h)\}^2}{T(r, M[f])T(r, M[g])} &= \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, M[f])} \cdot \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, M[g])} \\ &= 0 \\ \text{i.e., } \lim_{r \rightarrow \infty} \frac{\{\log T(r, f \circ h)\}^2}{T(r, M[f])T(r, M[g])} &= 0. \end{aligned}$$

This proves the theorem. ■

In the next theorem we establish a result parallel to Theorem 2.3.1 considering  $f$  and  $g$  to be two transcendental entire functions having the maximum deficiency sum.

**Theorem 2.3.3** *Let  $f$  and  $g$  be any two transcendental entire functions having maximum deficiency sum and  $h$  be any entire function such that  $\rho_h < \lambda_f \leq \rho_f < \infty$  and  $\rho_h < \lambda_g$ . Then*

$$\lim_{r \rightarrow \infty} \frac{\{\log T(r, f \circ h)\}^2}{T(r, L[f])T(r, L[g])} = 0.$$

**Proof.** In view of Lemma 2.2.15 we obtain for all sufficiently large values of  $r$ ,

$$T(r, L[g]) \geq r^{(\lambda_{L[g]} - \epsilon)} = r^{\lambda_g - \epsilon} \quad (2.12)$$

and

$$T(r, L[f]) \geq r^{(\lambda_{L[f]} - \epsilon)} = r^{\lambda_f - \epsilon} \quad (2.13)$$

As  $\rho_h < \lambda_g$ , we can choose  $\epsilon (> 0)$  in such a way that

$$\rho_h + \epsilon < \lambda_g - \epsilon.$$

Now combining (2.1) and (2.12) it follows from above for all sufficiently large values of  $r$ ,

$$\begin{aligned} \frac{\log T(r, f \circ h)}{T(r, L[g])} &\leq \frac{o(1) + (\rho_f + \epsilon)r^{\rho_h + \epsilon}}{r^{\lambda_g - \epsilon}} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, L[g])} &\leq \limsup_{r \rightarrow \infty} \frac{o(1) + (\rho_f + \epsilon)r^{\rho_h + \epsilon}}{r^{\lambda_g - \epsilon}} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, L[g])} &= 0 \\ \text{i.e., } \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, L[g])} &= 0. \end{aligned} \quad (2.14)$$

Since  $\rho_h < \lambda_f$ , we can choose  $\epsilon (> 0)$  in such a way that

$$\rho_h + \epsilon < \lambda_f - \epsilon.$$

Now combining (2.1) and (2.13) it follows from above for all sufficiently large values of  $r$ ,

$$\begin{aligned} \frac{\log T(r, f \circ h)}{T(r, L[f])} &\leq \frac{o(1) + (\rho_f + \epsilon)r^{\rho_h + \epsilon}}{r^{\lambda_f - \epsilon}} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, L[f])} &= 0 \\ \text{i.e., } \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, L[f])} &= 0. \end{aligned} \quad (2.15)$$

Therefore in view of (2.14) and (2.15), we obtain that

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{\{\log T(r, f \circ h)\}^2}{T(r, L[f])T(r, L[g])} &= \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, L[f])} \cdot \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ h)}{T(r, L[g])} \\ \text{i.e., } \lim_{r \rightarrow \infty} \frac{\{\log T(r, f \circ h)\}^2}{T(r, L[f])T(r, L[g])} &= 0. \end{aligned}$$

This proves the theorem. ■

**Theorem 2.3.4** *Let  $f$  be meromorphic and  $g$  be entire transcendental such that  $\rho_f = 0$  and  $\sum_{a \neq \infty} \Theta(a; f \circ g) = 2$ . Then  $\rho_{P_0[f \circ g]} \geq \mu$  where  $0 < \mu < \rho_g$ .*

**Proof.** In view of Lemma 2.2.6, and for  $0 < \mu < \rho_g$  we get by Lemma 2.2.2 that

$$\begin{aligned}
\rho_{P_0[f \circ g]} &= \rho_{f \circ g} = \limsup_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{\log r} \\
&\geq \liminf_{r \rightarrow \infty} \frac{\log T(\exp(r^\mu), f)}{\log r} \\
&\geq \liminf_{r \rightarrow \infty} \frac{\log^{[2]}(\exp(r^\mu))}{\log r} \cdot \liminf_{r \rightarrow \infty} \frac{\log T(\exp(r^\mu), f)}{\log^{[2]}(\exp(r^\mu))} \\
&= \liminf_{r \rightarrow \infty} \frac{\log r^\mu}{\log r} \cdot \lambda_f^* \\
&= \mu \cdot \lambda_f^* .
\end{aligned}$$

Thus in view of Lemma 2.2.18, the theorem follows from above. ■

**Remark 2.3.2** *The condition  $\mu < \rho_g$  in Theorem 2.3.4 is essential as we see in the following example.*

**Example 2.3.1** *Let*

$$f = z \text{ and } g = \exp z .$$

*Also let  $s = 1$ ,  $A_1 = 1$  and*

$$\begin{aligned}
n_{i1} &= 1, \text{ for } i = 1 \\
&= 0, \text{ for } i \neq 1.
\end{aligned}$$

*Then*

$$P_0[f \circ g] = \exp z$$

*and*

$$\sum_{a \neq \infty} \Theta(a; f \circ g) = 2, \quad \rho_f = 0, \quad \lambda_f^* = 1 \text{ and } \rho_g = \rho_{f \circ g} = 1 .$$

*Also let*

$$\mu = 2 .$$

*Therefore*

$$\mu \lambda_f^* = 2,$$

*which is contrary to Theorem 2.3.4.*

**Remark 2.3.3** The conclusion of Theorem 2.3.4 can also be deduced under the hypothesis  $\Theta(\infty; f \circ g) = \sum_{a \neq \infty} \delta_p(a; f \circ g) = 1$  or  $\delta(\infty; f \circ g) = \sum_{a \neq \infty} \delta(a; f \circ g) = 1$  instead of  $\sum_{a \neq \infty} \Theta(a; f \circ g) = 2$ .

**Theorem 2.3.5** Let  $f$  be meromorphic and  $g$  be transcendental entire such that  $\rho_f = 0$  and  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f \circ g) = 4$ . Then  $\rho_{M[f \circ g]} \geq \mu$  where  $0 < \mu < \rho_g$ .

The proof is omitted because it can be carried out in the line of Theorem 2.3.4 and with the help of Lemma 2.2.2 , Lemma 2.2.12 and Lemma 2.2.18.

**Theorem 2.3.6** Let  $f \circ g$  be transcendental meromorphic function having maximum deficiency sum such that  $\rho_f = 0$  . Then  $\rho_{L[f \circ g]} \geq \mu$  where  $0 < \mu < \rho_g$ .

The proof of the above theorem is also omitted because it can be carried out in the line of Theorem 2.3.4 and with the help of Lemma 2.2.2 , Lemma 2.2.15 and Lemma 2.2.18.

**Remark 2.3.4** The condition  $\mu < \rho_g$  in Theorem 2.3.6 is essential as we see in the following example.

**Example 2.3.2** Let

$$f = z \text{ and } g = \exp z .$$

So

$$f \circ g = \exp z .$$

Also let

$$\mu = 3 .$$

Then

$$\rho_f = 0 , \lambda_f^* = 1, \rho_g = \rho_{f \circ g} = 1 \text{ and } \sum_{a \neq \infty} \delta(a; f \circ g) + \delta(\infty; f \circ g) = 2.$$

Taking  $a_1 = 1$  ,  $a_2 = \dots = a_k = 0$  and  $k = 1$  in Definition 2.1.12 we get that

$$L(f \circ g) = \begin{vmatrix} 1 & \exp z \\ 0 & \exp z \end{vmatrix} = \exp z .$$

Therefore

$$\mu \lambda_f^* = 3,$$

which is contrary to Theorem 2.3.6.

**Theorem 2.3.7** Let  $f$  be a meromorphic function and  $g$  be an entire function with  $0 < \lambda_f \leq \rho_f < \infty$ ,  $\rho_g < \infty$  and  $\sum_{a \neq \infty} \Theta(a; f) = 2$ . Then for any  $\alpha \in (-\infty, \infty)$ ,

$$\lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, P_0[f])} = 0 .$$

**Proof.** If  $1 + \alpha \leq 0$ , the theorem is obvious. So we suppose that  $1 + \alpha > 0$ . In view of Lemma 2.1.1, we have for all sufficiently large values of  $r$ ,

$$\begin{aligned} \log\{T(r, f \circ g) \log M(r, g)\} &\leq \log T(r, g) + \log T(M(r, g), f) + \log\{1 + o(1)\} \\ \text{i.e., } \log\{T(r, f \circ g) \log M(r, g)\} &\leq (\rho_g + \epsilon) \log r + (\rho_f + \epsilon)r^{\rho_g + \epsilon} + o(1) \end{aligned}$$

$$\text{i.e., } \log\{T(r, f \circ g) \log M(r, g)\}$$

$$\leq r^{\rho_g + \epsilon} \left\{ (\rho_f + \epsilon) + \frac{(\rho_g + \epsilon) \log r + o(1)}{r^{\rho_g + \epsilon}} \right\} . \quad (2.16)$$

Again in view of Lemma 2.2.6, we get for all sufficiently large values of  $r$ ,

$$\begin{aligned} \log T(\exp r, P_0[f]) &\geq (\lambda_{P_0[f]} - \epsilon) \log(\exp r) \\ \text{i.e., } T(\exp r, P_0[f]) &\geq \exp\{(\lambda_f - \epsilon)r\} . \end{aligned} \quad (2.17)$$

Now combining (2.16) and (2.17) it follows for all sufficiently large values of  $r$ ,

$$\begin{aligned} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, P_0[f])} &\leq \frac{r^{(\rho_g + \epsilon)(1+\alpha)} \left\{ (\rho_f + \epsilon) + \frac{(\rho_g + \epsilon) \log r + o(1)}{r^{\rho_g + \epsilon}} \right\}^{1+\alpha}}{\exp\{(\lambda_f - \epsilon)r\}} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, P_0[f])} &= 0 , \end{aligned}$$

from which the theorem follows. ■

**Remark 2.3.5** If we replace  $\sum_{a \neq \infty} \Theta(a; f) = 2$  by  $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$  or  $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$  and the other conditions remain the same then also Theorem 2.3.7 is valid.

**Remark 2.3.6** The condition  $\rho_g < \infty$  in Theorem 2.3.7 is necessary as we see in the following example.

**Example 2.3.3** Let

$$f = \exp z, \quad g = \exp^{[3]} z \quad \text{and} \quad \alpha = 0.$$

Also let  $s = 1$ ,  $A_1 = 1$  and

$$\begin{aligned} n_{i1} &= 1, \quad \text{for } i = 1 \\ &= 0, \quad \text{for } i \neq 1. \end{aligned}$$

Then  $P_0[f] = \exp z$

and

$$\sum_{a \neq \infty} \Theta(a; f) = 2, \quad \lambda_f = \rho_f = 1, \quad \rho_g = \infty.$$

Now

$$\begin{aligned} T(r, f \circ g) &\geq \frac{1}{3} \log M\left(\frac{r}{2}, f \circ g\right) \\ \text{i.e., } T(r, f \circ g) &\geq \frac{1}{3} \log \exp^{[4]}\left(\frac{r}{2}\right) \\ \text{i.e., } T(r, f \circ g) &\geq \frac{1}{3} \exp^{[3]}\left(\frac{r}{2}\right) \end{aligned}$$

and

$$\log M(r, g) = \log M(r, \exp^{[3]} z) = \exp^{[2]} r.$$

Therefore,

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, P_0[f])} &= \liminf_{r \rightarrow \infty} \frac{\log\{T(r, f \circ g) \log M(r, g)\}}{T(\exp r, \exp z)} \\ &\geq \liminf_{r \rightarrow \infty} \frac{\log\{\frac{1}{3} \exp^{[3]}\left(\frac{r}{2}\right) \cdot \exp^{[2]} r\}}{\left\{\frac{\exp r}{\pi}\right\}} \\ &= \infty, \end{aligned}$$

which is contrary to Theorem 2.3.7.

**Theorem 2.3.8** Let  $f$  be meromorphic and  $g$  be entire such that  $0 < \lambda_g \leq \rho_g < \infty$ ,  $\rho_f < \infty$  and  $\sum_{a \neq \infty} \Theta(a; g) = 2$ . Then for any  $\alpha \in (-\infty, \infty)$ ,

$$\lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, P_0[g])} = 0.$$

**Proof.** If  $1 + \alpha \leq 0$ , the theorem is obvious. So we suppose that  $1 + \alpha > 0$ . In view of Lemma 2.1.6, we have for all sufficiently large values of  $r$ ,

$$\begin{aligned} \log T(\exp r, P_0[g]) &\geq (\lambda_{P_0[g]} - \epsilon) \log(\exp r) \\ \text{i.e., } T(\exp r, P_0[g]) &\geq \exp\{(\lambda_g - \epsilon)r\} . \end{aligned} \quad (2.18)$$

Now combining (2.16) and (2.18) it follows for all sufficiently large values of  $r$ ,

$$\begin{aligned} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, P_0[g])} &\leq \frac{r^{(\rho_g + \epsilon)(1+\alpha)} \left\{ (\rho_f + \epsilon) + \frac{(\rho_g + \epsilon) \log r + o(1)}{r^{\rho_g + \epsilon}} \right\}^{1+\alpha}}{\exp\{(\lambda_g - \epsilon)r\}} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, P_0[g])} &= 0 , \end{aligned}$$

from which the theorem follows. ■

**Remark 2.3.7** If we replace  $\sum_{a \neq \infty} \Theta(a; f) = 2$  by  $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$  or  $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$  and the other conditions remain the same then also Theorem 2.3.8 is valid.

**Remark 2.3.8** Considering

$$\begin{aligned} f &= \exp^{[3]} z, \quad g = \exp z, \\ s &= 1, \quad A_1 = 1 \text{ and} \\ n_{i1} &= 1, \quad \text{for } i = 1 \\ &= 0, \quad \text{for } i \neq 1 \end{aligned}$$

one can easily verify that the condition  $\rho_f < \infty$  in Theorem 2.3.8 is essential.

**Theorem 2.3.9** Let  $f$  be a transcendental meromorphic function and  $g$  be an entire function with  $0 < \lambda_f \leq \rho_f < \infty$ .  $\rho_g < \infty$  and  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$ .

Then for any  $\alpha \in (-\infty, \infty)$ ,

$$\lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, M[f])} = 0 .$$

**Proof.** If  $1 + \alpha \leq 0$ , the theorem is obvious. So we suppose that  $1 + \alpha > 0$ . In view of Lemma 2.1.12, we have for all sufficiently large values of  $r$ ,

$$\begin{aligned} \log T(\exp r, M[f]) &\geq (\lambda_{M[f]} - \epsilon) \log(\exp r) \\ \text{i.e., } T(\exp r, M[f]) &\geq \exp\{(\lambda_f - \epsilon)r\} . \end{aligned} \quad (2.19)$$

Now combining (2.16) and (2.19) it follows for all sufficiently large values of  $r$ ,

$$\begin{aligned} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, M[f])} &\leq \frac{r^{(\rho_g + \epsilon)(1+\alpha)} \left\{ (\rho_f + \epsilon) + \frac{(\rho_g + \epsilon) \log r + o(1)}{r^{\rho_g + \epsilon}} \right\}^{1+\alpha}}{\exp\{(\lambda_f - \epsilon)r\}} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, M[f])} &= 0 , \end{aligned}$$

from which the theorem follows. ■

**Theorem 2.3.10** *Let  $f$  be meromorphic and  $g$  be transcendental entire such that  $0 < \lambda_g \leq \rho_g < \infty$ ,  $\rho_f < \infty$  and  $\sum_{a \in \mathbb{C} \cup \infty} \delta_1(a; g) = 4$ . Then for any  $\alpha \in (-\infty, \infty)$ ,*

$$\lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, M[g])} = 0 .$$

**Proof.** If  $1 + \alpha \leq 0$ , the theorem is obvious. So we suppose that  $1 + \alpha > 0$ . In view of Lemma 2.1.12, we have for all sufficiently large values of  $r$ ,

$$\begin{aligned} \log T(\exp r, M[g]) &\geq (\lambda_{P_0[g]} - \epsilon) \log(\exp r) \\ \text{i.e., } T(\exp r, M[g]) &\geq \exp\{(\lambda_g - \epsilon)r\} . \end{aligned} \quad (2.20)$$

Now combining (2.16) and (2.20) it follows for all sufficiently large values of  $r$ ,

$$\begin{aligned} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, M[g])} &\leq \frac{r^{(\rho_g + \epsilon)(1+\alpha)} \left\{ (\rho_f + \epsilon) + \frac{(\rho_g + \epsilon) \log r + o(1)}{r^{\rho_g + \epsilon}} \right\}^{1+\alpha}}{\exp\{(\lambda_g - \epsilon)r\}} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, M[g])} &= 0 , \end{aligned}$$

from which the theorem follows. ■

**Theorem 2.3.11** *Let  $f$  be a transcendental meromorphic function having maximum deficiency sum and  $g$  be an entire function with  $0 < \lambda_f \leq \rho_f < \infty$  and  $\rho_g < \infty$ . Then for any  $\alpha \in (-\infty, \infty)$ ,*

$$\lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, L[f])} = 0 .$$

**Proof.** If  $1 + \alpha \leq 0$ , the theorem is obvious. So we suppose that  $1 + \alpha > 0$ . In view of Lemma 2.1.15, we have for all sufficiently large values of  $r$ ,

$$\begin{aligned} \log T(\exp r, L[f]) &\geq (\lambda_{L[f]} - \epsilon) \log(\exp r) \\ \text{i.e., } T(\exp r, L[f]) &\geq \exp\{(\lambda_f - \epsilon)r\} . \end{aligned} \quad (2.21)$$

Now combining (2.16) and (2.21) it follows for all sufficiently large values of  $r$ ,

$$\begin{aligned} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, L[f])} &\leq \frac{r^{(\rho_g + \epsilon)(1+\alpha)} \left\{ (\rho_f + \epsilon) + \frac{(\rho_g + \epsilon) \log r + o(1)}{r^{\rho_g + \epsilon}} \right\}^{1+\alpha}}{\exp\{(\lambda_f - \epsilon)r\}} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, L[f])} &= 0 , \end{aligned}$$

from which the theorem follows. ■

**Remark 2.3.9** *The condition  $\rho_g < \infty$  in Theorem 2.3.11 is necessary as we see in the following example.*

**Example 2.3.4** *Let*

$$f = \exp z , \quad g = \exp^{[4]} z \quad \text{and} \quad \alpha = 0 .$$

*Then*

$$\lambda_f = \rho_f = 1 , \quad \rho_g = \infty \quad \text{and} \quad \sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2 .$$

*Taking  $a_1 = 1$ ,  $a_2 = \dots = a_k = 0$  and  $k = 1$  in Definition 2.1.12 we get that*

$$L(f) = \begin{vmatrix} 1 & \exp z \\ 0 & \exp z \end{vmatrix} = \exp z .$$

Now

$$\begin{aligned} T(r, f \circ g) &\geq \frac{1}{3} \log M\left(\frac{r}{2}, f \circ g\right) \\ \text{i.e., } T(r, f \circ g) &\geq \frac{1}{3} \log \exp^{[5]}\left(\frac{r}{2}\right) \\ \text{i.e., } T(r, f \circ g) &\geq \frac{1}{3} \exp^{[4]}\left(\frac{r}{2}\right) \end{aligned}$$

and

$$\log M(r, g) = \log M(r, \exp^{[4]} z) = \exp^{[3]} r.$$

Therefore,

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, L[f])} &= \lim_{r \rightarrow \infty} \frac{\log\{T(r, f \circ g) \log M(r, g)\}}{T(\exp r, \exp z)} \\ &\geq \lim_{r \rightarrow \infty} \frac{\log\{\frac{1}{3} \exp^{[4]}\left(\frac{r}{2}\right) \cdot \exp^{[3]} r\}}{\left\{\frac{\exp r}{\pi}\right\}} \\ &= \infty, \end{aligned}$$

which is contrary to Theorem 2.3.11 .

**Theorem 2.3.12** Let  $f$  be a meromorphic function and  $g$  be a transcendental entire function having maximum deficiency sum with  $0 < \lambda_g \leq \rho_g < \infty$  and  $\rho_f < \infty$  . Then for any  $\alpha \in (-\infty, \infty)$ ,

$$\lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, L[g])} = 0 .$$

**Proof.** If  $1 + \alpha \leq 0$ , the theorem is obvious. So we suppose that  $1 + \alpha > 0$ . In view of Lemma 2.1.15, we have for all sufficiently large values of  $r$ ,

$$\begin{aligned} \log T(\exp r, L[g]) &\geq (\lambda_{L[g]} - \epsilon) \log(\exp r) \\ \text{i.e., } T(\exp r, L[g]) &\geq \exp\{(\lambda_g - \epsilon)r\} . \end{aligned} \tag{2.22}$$

Now combining (2.16) and (2.22) it follows for all sufficiently large values of  $r$ ,

$$\begin{aligned} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, L[g])} &\leq \frac{r^{(\rho_g + \epsilon)(1+\alpha)} \left\{ (\rho_f + \epsilon) + \frac{(\rho_g + \epsilon) \log r + o(1)}{r^{\rho_g + \epsilon}} \right\}^{1+\alpha}}{\exp\{(\lambda_g - \epsilon)r\}} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, L[g])} &= 0 , \end{aligned}$$

from which the theorem follows. ■

**Remark 2.3.10** The condition  $\rho_f < \infty$  in Theorem 2.3.12 is essential as we see in the following example.

**Example 2.3.5** Let

$$f = \exp^{[3]} z, \quad g = \exp z \quad \text{and} \quad \alpha = 0.$$

Then

$$\lambda_f = \rho_f = \infty, \quad \rho_g = 1 \quad \text{and} \quad \sum_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2.$$

Taking  $a_1 = 1, a_2 = \dots = a_k = 0$  and  $k = 1$  in Definition 2.1.12 we get that

$$L(g) = \begin{vmatrix} 1 & \exp z \\ 0 & \exp z \end{vmatrix} = \exp z.$$

Now

$$\begin{aligned} T(r, f \circ g) &\geq \frac{1}{3} \log M\left(\frac{r}{2}, f \circ g\right) \\ \text{i.e., } T(r, f \circ g) &\geq \frac{1}{3} \log \exp^{[4]}\left(\frac{r}{2}\right) \\ \text{i.e., } T(r, f \circ g) &\geq \frac{1}{3} \exp^{[3]}\left(\frac{r}{2}\right) \end{aligned}$$

and

$$\log M(r, g) = \log M(r, \exp z) = r.$$

Therefore,

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{T(\exp r, L[g])} &= \lim_{r \rightarrow \infty} \frac{\log\{T(r, f \circ g) \log M(r, g)\}}{T(\exp r, \exp z)} \\ &\geq \lim_{r \rightarrow \infty} \frac{\log\{\frac{1}{3} \exp^{[3]}\left(\frac{r}{2}\right) \cdot r\}}{\left\{\frac{\exp r}{\pi}\right\}} \\ &= \infty, \end{aligned}$$

which is contrary to Theorem 2.3.12.

**Theorem 2.3.13** Let  $f$  be a meromorphic function and  $g$  be an entire function with  $0 < \lambda_f \leq \rho_f < \infty, \rho_g < \infty$  and  $\sum_{a \neq \infty} \Theta(a; f) = 2$ . Then

$$\lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{\log T(\exp r, P_0[f])} = 0 \quad \text{if} \quad 0 < 1 + \alpha < \frac{1}{\rho_g}.$$

**Proof.** If  $1 + \alpha \leq 0$ , the theorem is obvious. So we take  $1 + \alpha > 0$ .

In view of Lemma 2.2.6 we obtain for all sufficiently large values of  $r$ ,

$$\begin{aligned} \log T(\exp r, P_0[f]) &\geq (\lambda_{P_0[f]} - \epsilon) \log(\exp r) \\ \text{i.e., } \log T(\exp r, P_0[f]) &\geq (\lambda_f - \epsilon)r. \end{aligned} \quad (2.23)$$

Now from (2.16) and (2.23) it follows for all sufficiently large values of  $r$ ,

$$\frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{\log T(\exp r, P_0[f])} \leq \frac{r^{(\rho_g + \epsilon)(1+\alpha)} \left\{ (\rho_f + \epsilon) + \frac{(\rho_g + \epsilon) \log r + o(1)}{r^{\rho_g + \epsilon}} \right\}^{1+\alpha}}{(\lambda_f - \epsilon)r}. \quad (2.24)$$

Since  $1 + \alpha < \frac{1}{\rho_g}$ , we can choose  $\epsilon (> 0)$  in such a way that

$$(\rho_g + \epsilon)(1 + \alpha) < 1. \quad (2.25)$$

Thus the theorem follows from (2.24) and (2.25). ■

**Remark 2.3.11** *Theorem 2.3.13 is still valid under the condition  $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$  or  $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$  instead of  $\sum_{a \neq \infty} \Theta(a; f) = 2$ .*

**Remark 2.3.12** *The condition  $0 < 1 + \alpha < \frac{1}{\rho_g}$  is essential in Theorem 2.3.13 as we see in the following example.*

**Example 2.3.6** *Let  $f = g = \exp z$  and  $\alpha = 0$ .*

*Then*

$$\lambda_f = \rho_f = \lambda_g = \rho_g = 1, \quad \sum_{a \neq \infty} \Theta(a; f) = 2 \quad \text{and} \quad f \circ g = \exp^{[2]} z.$$

*Also let  $s = 1$ ,  $A_1 = 1$  and*

$$\begin{aligned} n_{i1} &= 1 \quad \text{for } i = 1 \\ &= 0 \quad \text{for } i \neq 1. \end{aligned}$$

*Then  $P_0[f] = \exp z$ .*

*Now*

$$\begin{aligned} \log T(r, f \circ g) &= \log T(r, \exp^{[2]} z) \\ &\sim \log \left\{ \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \right\} \quad (r \rightarrow \infty) \\ &= r - \frac{1}{2} \log r + O(1) \quad (r \rightarrow \infty) \end{aligned}$$

and

$$\log M(r, g) = \log M(r, \exp z) = r.$$

Therefore,

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{\log T(\exp r, P_0[f])} &= \lim_{r \rightarrow \infty} \frac{\log\{T(r, f \circ g) \log M(r, g)\}}{\log T(\exp r, \exp z)} \\ &= \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ g) + \log^{[2]} M(r, g)}{\log T(\exp r, \exp z)} \\ &= \lim_{r \rightarrow \infty} \frac{r - \frac{1}{2} \log r + \log r + O(1)}{\log \left\{ \frac{\exp r}{\pi} \right\}} \\ &= \lim_{r \rightarrow \infty} \frac{r + \frac{1}{2} \log r + O(1)}{r + O(1)} \\ &= 1, \end{aligned}$$

which is contrary to Theorem 2.3.13 .

**Theorem 2.3.14** *Let  $f$  be a transcendental meromorphic function and  $g$  be an entire function with  $0 < \lambda_f \leq \rho_f < \infty$ ,  $\rho_g < \infty$  and  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$ .*

Then

$$\lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{\log T(\exp r, M[f])} = 0 \text{ if } 0 < 1 + \alpha < \frac{1}{\rho_g} .$$

**Proof.** For all large values of  $r$  we get from (2.19) that

$$\log T(\exp r, M[f]) \geq (\lambda_f - \epsilon)r. \quad (2.26)$$

From (2.16) and (2.26) it follows for all sufficiently large values of  $r$ ,

$$\frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{\log T(\exp r, M[f])} \leq \frac{r^{(\rho_g + \epsilon)(1+\alpha)} \left\{ (\rho_f + \epsilon) + \frac{(\rho_g + \epsilon) \log r + o(1)}{r^{\rho_g + \epsilon}} \right\}^{1+\alpha}}{(\lambda_f - \epsilon)r}. \quad (2.27)$$

Thus the theorem follows from (2.25) and (2.27) . ■

**Theorem 2.3.15** *Let  $f$  be a transcendental meromorphic function having the maximum deficiency sum and  $g$  be an entire function with  $0 < \lambda_f \leq \rho_f < \infty$ ,  $\rho_g < \infty$  . Then*

$$\lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{\log T(\exp r, L[f])} = 0 \text{ if } 0 < 1 + \alpha < \frac{1}{\rho_g} .$$

**Proof.** From (2.21) it follows for all large values of  $r$  that

$$\log T(\exp r, L[f]) \geq (\lambda_f - \epsilon)r. \quad (2.28)$$

From (2.16) and (2.28) it follows for all sufficiently large values of  $r$ ,

$$\frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{\log T(\exp r, M[f])} \leq \frac{r^{(\rho_g + \epsilon)(1+\alpha)} \left\{ (\rho_f + \epsilon) + \frac{(\rho_g + \epsilon) \log r + o(1)}{r \rho_g + \epsilon} \right\}^{1+\alpha}}{(\lambda_f - \epsilon)r}. \quad (2.29)$$

Thus the theorem follows from (2.25) and (2.29). ■

**Remark 2.3.13** *The condition  $0 < 1 + \alpha < \frac{1}{\rho_g}$  is essential in Theorem 2.3.15 as we see in the following example.*

**Example 2.3.7** *Let  $f = g = \exp z$  and  $\alpha = 0$ .*

*Then*

$$\lambda_f = \rho_f = \lambda_g = \rho_g = 1, \quad \sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2 \text{ and } f \circ g = \exp^{[2]} z.$$

*Taking  $a_1 = 1$ ,  $a_2 = \dots = a_k = 0$  and  $k = 1$  in Definition 2.1.12 we get that*

$$L(f) = \begin{vmatrix} 1 & \exp z \\ 0 & \exp z \end{vmatrix} = \exp z.$$

*Now*

$$\begin{aligned} \log T(r, f \circ g) &= \log T(r, \exp^{[2]} z) \\ &\sim \log \left\{ \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \right\} \quad (r \rightarrow \infty) \\ &= r - \frac{1}{2} \log r + O(1) \quad (r \rightarrow \infty) \end{aligned}$$

*and*

$$\log M(r, g) = \log M(r, \exp z) = r.$$

Therefore,

$$\begin{aligned}
\lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{\log T(\exp r, P_0[f])} &= \lim_{r \rightarrow \infty} \frac{\log\{T(r, f \circ g) \log M(r, g)\}}{\log T(\exp r, \exp z)} \\
&= \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ g) + \log^{[2]} M(r, g)}{\log T(\exp r, \exp z)} \\
&= \lim_{r \rightarrow \infty} \frac{r - \frac{1}{2} \log r + \log r + O(1)}{\log \left\{ \frac{\exp r}{\pi} \right\}} \\
&= \lim_{r \rightarrow \infty} \frac{r + \frac{1}{2} \log r + O(1)}{r + O(1)} \\
&= 1,
\end{aligned}$$

which is contrary to Theorem 2.3.15 .

**Theorem 2.3.16** *Let  $f$  be meromorphic and  $g$  be entire such that  $0 < \lambda_g \leq \rho_g < \infty$ ,  $\rho_f < \infty$  and  $\sum_{a \neq \infty} \Theta(a; g) = 2$ . Then*

$$\lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{\log T(\exp r, P_0[g])} = 0 \text{ if } 0 < 1 + \alpha < \frac{1}{\rho_g}.$$

**Proof.** For all large values of  $r$  it follows from (2.18) that

$$\log T(\exp r, P_0[g]) \geq (\lambda_g - \epsilon)r. \tag{2.30}$$

From (2.16) and (2.30) it follows for all sufficiently large values of  $r$ ,

$$\frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{\log T(\exp r, P_0[g])} \leq \frac{r^{(\rho_g + \epsilon)(1+\alpha)} \left\{ (\rho_f + \epsilon) + \frac{(\rho_g + \epsilon) \log r + o(1)}{r^{\rho_g + \epsilon}} \right\}^{1+\alpha}}{(\lambda_g - \epsilon)r}. \tag{2.31}$$

Thus the theorem follows from (2.25) and (2.31). ■

**Remark 2.3.14** *The condition  $0 < 1 + \alpha < \frac{1}{\rho_g}$  is essential in Theorem 2.3.16 as we see in the following example.*

**Example 2.3.8** *Let  $f = g = \exp z$  and  $\alpha = 0$  .*

*Then*

$$\lambda_f = \rho_f = \lambda_g = \rho_g = 1, \quad \sum_{a \neq \infty} \Theta(a; g) = 2 \text{ and } f \circ g = \exp^{[2]} z.$$

Also let  $s = 1$ ,  $A_1 = 1$  and

$$\begin{aligned} n_{i1} &= 1 \text{ for } i = 1 \\ &= 0 \text{ for } i \neq 1. \end{aligned}$$

Then  $P_0[g] = \exp z$ .

Now

$$\begin{aligned} \log T(r, f \circ g) &= \log T(r, \exp^{[2]} z) \\ &\sim \log \left\{ \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \right\} \quad (r \rightarrow \infty) \\ &= r - \frac{1}{2} \log r + O(1) \quad (r \rightarrow \infty) \end{aligned}$$

and

$$\log M(r, g) = \log M(r, \exp z) = r.$$

Therefore,

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{\log T(\exp r, P_0[g])} &= \lim_{r \rightarrow \infty} \frac{\log\{T(r, f \circ g) \log M(r, g)\}}{\log T(\exp r, \exp z)} \\ &= \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ g) + \log^{[2]} M(r, g)}{\log T(\exp r, \exp z)} \\ &= \lim_{r \rightarrow \infty} \frac{r - \frac{1}{2} \log r + \log r + O(1)}{\log \left\{ \frac{\exp r}{\pi} \right\}} \\ &= \lim_{r \rightarrow \infty} \frac{r + \frac{1}{2} \log r + O(1)}{r + O(1)} \\ &= 1, \end{aligned}$$

which is contrary to Theorem 2.3.13 .

**Remark 2.3.15** Theorem 2.3.16 is still valid under the condition  $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$  or  $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$  instead of  $\sum_{a \neq \infty} \Theta(a; g) = 2$ .

**Theorem 2.3.17** Let  $f$  be meromorphic and  $g$  be transcendental entire such that  $0 < \lambda_g \leq \rho_g < \infty$ ,  $\rho_f < \infty$  and  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$ . Then

$$\lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{\log T(\exp r, M[g])} = 0 \text{ if } 0 < 1 + \alpha < \frac{1}{\rho_g}.$$

**Proof.** For all large values of  $r$  we get from (2.20) that

$$\log T(\exp r, M[g]) \geq (\lambda_g - \epsilon)r. \quad (2.32)$$

From (2.16) and (2.32) it follows for all sufficiently large values of  $r$ ,

$$\frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{\log T(\exp r, M[g])} \leq \frac{r^{(\rho_g + \epsilon)(1+\alpha)} \left\{ (\rho_f + \epsilon) + \frac{(\rho_g + \epsilon) \log r + o(1)}{r^{\rho_g + \epsilon}} \right\}^{1+\alpha}}{(\lambda_g - \epsilon)r}. \quad (2.33)$$

Thus the theorem follows from (2.25) and (2.33). ■

**Theorem 2.3.18** *Let  $f$  be meromorphic and  $g$  be transcendental entire with maximum deficiency sum such that  $0 < \lambda_g \leq \rho_g < \infty$ ,  $\rho_f < \infty$ . Then*

$$\lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{\log T(\exp r, L[g])} = 0 \text{ if } 0 < 1 + \alpha < \frac{1}{\rho_g}.$$

**Proof.** For all large values of  $r$  we get from (2.22) that

$$\log T(\exp r, L[g]) \geq (\lambda_g - \epsilon)r. \quad (2.34)$$

From (2.16) and (2.34) it follows for all sufficiently large values of  $r$ ,

$$\frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{\log T(\exp r, L[g])} \leq \frac{r^{(\rho_g + \epsilon)(1+\alpha)} \left\{ (\rho_f + \epsilon) + \frac{(\rho_g + \epsilon) \log r + o(1)}{r^{\rho_g + \epsilon}} \right\}^{1+\alpha}}{(\lambda_g - \epsilon)r}. \quad (2.35)$$

Thus the theorem follows from (2.25) and (2.35). ■

**Remark 2.3.16** *The condition  $0 < 1 + \alpha < \frac{1}{\rho_g}$  is essential in Theorem 2.3.15 as we see in the following example.*

**Example 2.3.9** *Let  $f = g = \exp z$  and  $\alpha = 0$ .*

*Then*

$$\lambda_f = \rho_f = \lambda_g = \rho_g = 1, \quad \sum_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2 \text{ and } f \circ g = \exp^{[2]} z.$$

*Taking  $a_1 = 1$ ,  $a_2 = \dots = a_k = 0$  and  $k = 1$  in Definition 2.1.12 we get that*

$$L(g) = \begin{vmatrix} 1 & \exp z \\ 0 & \exp z \end{vmatrix} = \exp z.$$

Now

$$\begin{aligned}\log T(r, f \circ g) &= \log T(r, \exp^{[2]} z) \\ &\sim \log \left\{ \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \right\} \quad (r \rightarrow \infty) \\ &= r - \frac{1}{2} \log r + O(1) \quad (r \rightarrow \infty)\end{aligned}$$

and

$$\log M(r, g) = \log M(r, \exp z) = r.$$

Therefore,

$$\begin{aligned}\lim_{r \rightarrow \infty} \frac{[\log\{T(r, f \circ g) \log M(r, g)\}]^{1+\alpha}}{\log T(\exp r, P_0[g])} &= \lim_{r \rightarrow \infty} \frac{\log\{T(r, f \circ g) \log M(r, g)\}}{\log T(\exp r, \exp z)} \\ &= \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ g) + \log^{[2]} M(r, g)}{\log T(\exp r, \exp z)} \\ &= \lim_{r \rightarrow \infty} \frac{r - \frac{1}{2} \log r + \log r + O(1)}{\log \left\{ \frac{\exp r}{\pi} \right\}} \\ &= \lim_{r \rightarrow \infty} \frac{r + \frac{1}{2} \log r + O(1)}{r + O(1)} \\ &= 1,\end{aligned}$$

which is contrary to Theorem 2.3.18.

**Theorem 2.3.19** *Let  $f$  be rational and  $g$  be a meromorphic function such that  $0 < \bar{\lambda}_{f \circ g} \leq \bar{\rho}_{f \circ g} < \infty$  and  $0 < \bar{\rho}_g < \infty$ . Also let  $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$ . Then for any positive number  $A$ ,*

$$\liminf_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r^A, P_0[g])} \leq \frac{\bar{\rho}_{f \circ g}}{A \bar{\rho}_g} \leq \limsup_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r^A, P_0[g])}.$$

**Proof.** From the definition of hyper order we get for all sufficiently large values of  $r$ ,

$$\log^{[2]} T(r, f \circ g) \leq (\bar{\rho}_{f \circ g} + \epsilon) \log r. \quad (2.36)$$

Again in view of Lemma 2.2.10 and Remark 2.2.3 we have for a sequence of values of  $r$  tending to infinity,

$$\begin{aligned} \log^{[2]} T(r^A, P_0[g]) &\geq (\bar{\rho}_{P_0[g]} - \epsilon) \log r^A \\ \text{i.e., } \log^{[2]} T(r^A, P_0[g]) &\geq A(\bar{\rho}_g - \epsilon) \log r. \end{aligned} \quad (2.37)$$

Now combining (2.36) and (2.37) it follows for a sequence of values of  $r$  tending to infinity,

$$\frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r^A, P_0[g])} \leq \frac{(\bar{\rho}_{f \circ g} + \epsilon) \log r}{A(\bar{\rho}_g - \epsilon) \log r}.$$

Since  $\epsilon (> 0)$  is arbitrary, it follows from above that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r^A, P_0[g])} \leq \frac{\bar{\rho}_{f \circ g}}{A\bar{\rho}_g}. \quad (2.38)$$

Also for arbitrary positive  $\epsilon$  and for all sufficiently large values of  $r$ ,

$$\begin{aligned} \log^{[2]} T(r^A, P_0[g]) &\leq (\bar{\rho}_{P_0[g]} + \epsilon) \log r^A \\ \text{i.e., } \log^{[2]} T(r^A, P_0[g]) &\leq A(\bar{\rho}_g + \epsilon) \log r. \end{aligned} \quad (2.39)$$

Again for a sequence of values of  $r$  tending to infinity,

$$\log^{[2]} T(r, f \circ g) \geq (\bar{\rho}_{f \circ g} - \epsilon) \log r. \quad (2.40)$$

Now from (2.39) and (2.40) we obtain for a sequence of values of  $r$  tending to infinity,

$$\frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r^A, P_0[g])} \geq \frac{(\bar{\rho}_{f \circ g} - \epsilon) \log r}{A(\bar{\rho}_g + \epsilon) \log r}.$$

As  $\epsilon (> 0)$  is arbitrary, we have from above that

$$\limsup_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r^A, P_0[g])} \geq \frac{\bar{\rho}_{f \circ g}}{A\bar{\rho}_g}. \quad (2.41)$$

Thus the theorem follows from (2.38) and (2.41). ■

**Remark 2.3.17** *The conclusion of Theorem 2.3.19 can also be drawn under the hypothesis  $\sum_{a \neq \infty} \Theta(a; g) = 2$  or  $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$  instead of  $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$ .*

The following theorem can be proved in the line of Theorem 2.3.19 and in view of Lemma 2.2.13.

**Theorem 2.3.20** *Let  $f$  be rational and  $g$  be a transcendental meromorphic function such that  $0 < \bar{\lambda}_{f \circ g} \leq \bar{\rho}_{f \circ g} < \infty$  and  $0 < \bar{\rho}_g < \infty$ . Then for any positive number  $A$ ,*

$$\liminf_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r^A, L[g])} \leq \frac{\bar{\rho}_{f \circ g}}{A \bar{\rho}_g} \leq \limsup_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r^A, L[g])}.$$

**Theorem 2.3.21** *Let  $f$  be rational and  $g$  be a transcendental meromorphic function having maximum deficiency sum such that  $0 < \bar{\lambda}_{f \circ g} \leq \bar{\rho}_{f \circ g} < \infty$  and  $0 < \bar{\rho}_g < \infty$ . Also let  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$ . Then for any positive number  $A$ ,*

$$\liminf_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r^A, M[g])} \leq \frac{\bar{\rho}_{f \circ g}}{A \bar{\rho}_g} \leq \limsup_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r^A, M[g])}.$$

We omit the proof of the theorem because it can be carried out in the line of Theorem 2.3.19 with the help of Lemma 2.2.16.

**Theorem 2.3.22** *Let  $f$  be rational and  $g$  be transcendental meromorphic satisfying the following conditions:*

(i)  $0 < \rho_g < \infty$ , (ii)  $\sigma_g < \infty$ , (iii)  $\rho_{f \circ g} = \rho_g$ , (iv)  $0 < \sigma_{f \circ g} < \infty$  and (v)  $\sum_{a \neq \infty} \Theta(a; g) = 2$ . Then

$$\liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\sigma_{f \circ g}}{\Gamma_{P_0} \sigma_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}.$$

**Proof.** By the definition of type, we have for arbitrary positive  $\epsilon$  and for all sufficiently large values of  $r$ ,

$$T(r, f \circ g) \leq (\sigma_{f \circ g} + \epsilon) r^{\rho_{f \circ g}}. \quad (2.42)$$

In view of Lemma 2.2.6 we get for a sequence of values of  $r$  tending to infinity,

$$\begin{aligned} T(r, P_0[g]) &\geq (\sigma_{P_0[g]} - \epsilon)r^{\rho_{P_0[g]}} \\ \text{i.e., } T(r, P_0[g]) &\geq (\Gamma_{P_0} \cdot \sigma_g - \epsilon)r^{\rho_g} . \end{aligned} \quad (2.43)$$

Since  $\rho_{f \circ g} = \rho_g$ , from (2.42) and (2.43) it follows for a sequence of values of  $r$  tending to infinity,

$$\frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{(\sigma_{f \circ g} + \epsilon)}{(\Gamma_{P_0} \cdot \sigma_g - \epsilon)} . \quad (2.44)$$

As  $\epsilon (> 0)$  is arbitrary, it follows from (2.44) that

$$\liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\sigma_{f \circ g}}{\Gamma_{P_0} \cdot \sigma_g} . \quad (2.45)$$

Again for a sequence of values of  $r$  tending to infinity,

$$T(r, f \circ g) \geq (\sigma_{f \circ g} - \epsilon)r^{\rho_{f \circ g}} . \quad (2.46)$$

Also for all sufficiently large values of  $r$  we get by Lemma 2.2.6 that

$$\begin{aligned} T(r, P_0[g]) &\leq (\sigma_{P_0[g]} + \epsilon)r^{\rho_{P_0[g]}} \\ \text{i.e., } T(r, P_0[g]) &\leq (\Gamma_{P_0} \cdot \sigma_g + \epsilon)r^{\rho_g} . \end{aligned} \quad (2.47)$$

Now in view of condition (iii) we get from (2.46) and (2.47), for a sequence of values of  $r$  tending to infinity,

$$\frac{T(r, f \circ g)}{T(r, g)} \geq \frac{(\sigma_{f \circ g} - \epsilon)}{\Gamma_{P_0} \cdot \sigma_g + \epsilon} .$$

Since  $\epsilon (> 0)$  is arbitrary, we obtain from above that

$$\limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \geq \frac{\sigma_{f \circ g}}{\Gamma_{P_0} \cdot \sigma_g} . \quad (2.48)$$

Thus the theorem follows from (2.45) and (2.48). ■

**Remark 2.3.18** *Theorem 2.3.22 remains true with  $\gamma_{P_0}$  instead of  $\Gamma_{P_0}$  if we replace the condition  $\sum_{a \neq \infty} \Theta(a; g) = 2$  by  $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$  or  $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$  and the other conditions are same.*

**Remark 2.3.19** *Considering*

$$\begin{aligned} f &= z, \quad g = \exp z, \\ s &= 1, \quad A_1 = 1 \text{ and} \\ n_{i1} &= 1, \text{ for } i = 1 \\ &= 0, \text{ for } i \neq 1 \end{aligned}$$

one can easily verify that the sign ' $\leq$ ' can not be replaced by ' $<$ ' only in Theorem 2.3.22 .

**Theorem 2.3.23** *Let  $f$  be rational and  $g$  be transcendental meromorphic satisfying the following conditions :*

- (i)  $0 < \rho_g < \infty$ , (ii)  $\sigma_g < \infty$ , (iii)  $\rho_{f \circ g} = \rho_g$ , (iv)  $0 < \sigma_{f \circ g} < \infty$  and (v)  $\sum_{a \in \text{CU}\{\infty\}} \delta_1(a; g) = 4$ . Then

$$\liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, M[g])} \leq \frac{\sigma_{f \circ g}}{\{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; g)\}\sigma_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, M[g])} .$$

The proof of the theorem can be established in the line of Theorem 2.3.22 and with the help of Lemma 2.2.12 and therefore is omitted.

The following theorem can also be proved in the line of Theorem 2.3.22 and in view of Lemma 2.2.16.

**Theorem 2.3.24** *Let  $f$  be rational and  $g$  be transcendental meromorphic having maximum deficiency sum satisfying the following conditions :*

- (i)  $0 < \rho_g < \infty$ , (ii)  $\sigma_g < \infty$ . (iii)  $\rho_{f \circ g} = \rho_g$ , and (iv)  $0 < \sigma_{f \circ g} < \infty$ . Then

$$\liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, L[g])} \leq \frac{\sigma_{f \circ g}}{\{1 + k - k\delta(\infty; f)\}\sigma_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, L[g])} .$$

The proof is omitted.

**Remark 2.3.20** *Let us consider*

$$f = z, \quad g = \exp z .$$

Taking  $a_1 = 1$ ,  $a_2 = \dots = a_k = 0$  and  $k = 1$  in Definition 2.1.12 we get tha

$$L(g) = \begin{vmatrix} 1 & \exp z \\ 0 & \exp z \end{vmatrix} = \exp z .$$

Now one can easily verify that the sign ' $\leq$ ' can not be replaced by ' $<$ ' only in Theorem 2.3.24.

**Theorem 2.3.25** *Let  $f$  be rational and  $g$  be transcendental meromorphic satisfying the following conditions:*

(i)  $0 < \rho_g < \infty$ , (ii)  $\eta_g \leq \sigma_g < \infty$ , (iii)  $\rho_{f \circ g} = \rho_g$ , (iv)  $0 < \eta_{f \circ g} \leq \sigma_{f \circ g} < \infty$  and (v)  $\sum_{a \neq \infty} \Theta(a; g) = 2$ . Then

$$\frac{\eta_{f \circ g}}{\Gamma_{P_0} \cdot \sigma_g} \leq \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\eta_{f \circ g}}{\Gamma_{P_0} \cdot \eta_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\sigma_{f \circ g}}{\Gamma_{P_0} \cdot \eta_g}.$$

**Proof.** By the definition of lower type, we have for arbitrary positive  $\epsilon$  and for all sufficiently large values of  $r$

$$T(r, f \circ g) \geq (\eta_{f \circ g} - \epsilon)r^{\rho_{f \circ g}}. \quad (2.49)$$

Since  $\rho_{f \circ g} = \rho_g$ , from (2.47) and (2.49) it follows for all sufficiently large values of  $r$  that

$$\frac{T(r, f \circ g)}{T(r, P_0[g])} \geq \frac{(\eta_{f \circ g} - \epsilon)}{(\Gamma_{P_0} \cdot \sigma_g - \epsilon)}.$$

As  $\epsilon (> 0)$  is arbitrary, it follows from above that

$$\liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \geq \frac{\eta_{f \circ g}}{\Gamma_{P_0} \cdot \sigma_g}. \quad (2.50)$$

Again for a sequence of values of  $r$  tending to infinity,

$$T(r, f \circ g) \leq (\eta_{f \circ g} + \epsilon)r^{\rho_{f \circ g}}. \quad (2.51)$$

Also for all sufficiently large values of  $r$  we get by Lemma 2.2.6 that

$$\begin{aligned} T(r, P_0[g]) &\geq (\eta_{P_0[g]} - \epsilon)r^{\rho_{P_0[g]}} \\ \text{i.e., } T(r, P_0[g]) &\geq (\Gamma_{P_0} \cdot \eta_g - \epsilon)r^{\rho_g}. \end{aligned} \quad (2.52)$$

Now in view of condition (iii) we get from (2.51) and (2.52), for a sequence of values of  $r$  tending to infinity,

$$\frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{(\eta_{f \circ g} + \epsilon)}{\Gamma_{P_0} \cdot \eta_g + \epsilon}.$$

Since  $\epsilon (> 0)$  is arbitrary, we obtain from above that

$$\liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\eta_{f \circ g}}{\Gamma_{P_0} \cdot \eta_g}. \quad (2.53)$$

Also for a sequence of values of  $r$  tending to infinity we get by Lemma 2.2.6 that

$$\begin{aligned} T(r, P_0[g]) &\leq (\eta_{P_0[g]} + \epsilon)r^{\rho_{P_0[g]}} \\ \text{i.e.. } T(r, P_0[g]) &\leq (\Gamma_{P_0} \cdot \eta_g + \epsilon)r^{\rho_g}. \end{aligned} \quad (2.54)$$

Since  $\rho_{f \circ g} = \rho_g$ , from (2.49) and (2.54) it follows for all sufficiently large values of  $r$  that

$$\frac{T(r, f \circ g)}{T(r, P_0[g])} \geq \frac{(\eta_{f \circ g} - \epsilon)}{(\Gamma_{P_0} \cdot \eta_g + \epsilon)}.$$

Choosing  $\epsilon \rightarrow 0$  we get that

$$\limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \geq \frac{\eta_{f \circ g}}{\Gamma_{P_0} \cdot \eta_g}. \quad (2.55)$$

Again from (2.42) and (2.52) and in view of condition (iii) it follows for all sufficiently large values of  $r$  that

$$\limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \geq \frac{(\sigma_{f \circ g} - \epsilon)}{(\Gamma_{P_0} \cdot \eta_g + \epsilon)}.$$

As  $\epsilon (> 0)$  is arbitrary, we obtain from above that

$$\limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \geq \frac{\sigma_{f \circ g}}{\Gamma_{P_0} \cdot \eta_g}. \quad (2.56)$$

Thus the theorem follows from (2.50), (2.53), (2.55) and (2.56). ■

**Remark 2.3.21** *Theorem 2.3.25 remains true with  $\gamma_{P_0}$  instead of  $\Gamma_{P_0}$  if we replace the condition  $\sum_{a \neq \infty} \Theta(a; g) = 2$  by  $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$  or  $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$  and the other conditions are same.*

The following two theorems can be proved in the line of Theorem 2.3.25 with the help of Lemma 2.2.12 and Lemma 2.2.15 respectively and so their proofs are omitted.

**Theorem 2.3.26** Let  $f$  be rational and  $g$  be transcendental meromorphic satisfying the following conditions:

(i)  $0 < \rho_g < \infty$ , (ii)  $\eta_g \leq \sigma_g < \infty$ , (iii)  $\rho_{f \circ g} = \rho_g$ , (iv)  $0 < \eta_{f \circ g} \leq \sigma_{f \circ g} < \infty$  and (v)  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$ . Then

$$\begin{aligned} & \frac{\eta_{f \circ g}}{\{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; g)\} \cdot \sigma_g} \\ & \leq \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\eta_{f \circ g}}{\{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; g)\} \cdot \eta_g} \\ & \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\sigma_{f \circ g}}{\{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; g)\} \cdot \eta_g}. \end{aligned}$$

**Theorem 2.3.27** Let  $f$  be rational and  $g$  be transcendental meromorphic having maximum deficiency sum satisfying the following conditions:

(i)  $0 < \rho_g < \infty$ , (ii)  $\eta_g \leq \sigma_g < \infty$ , (iii)  $\rho_{f \circ g} = \rho_g$ , and (iv)  $0 < \eta_{f \circ g} \leq \sigma_{f \circ g} < \infty$ . Then

$$\begin{aligned} & \frac{\eta_{f \circ g}}{\{1 + k - k\delta(\infty; f)\} \cdot \sigma_g} \leq \\ & \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\eta_{f \circ g}}{\{1 + k - k\delta(\infty; f)\} \cdot \eta_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \\ & \frac{\sigma_{f \circ g}}{\{1 + k - k\delta(\infty; f)\} \cdot \eta_g}. \end{aligned}$$

The following theorem is a natural consequence of Theorem 2.3.22 and Theorem 2.3.23.

**Theorem 2.3.28** Let  $f$  be rational and  $g$  be transcendental meromorphic satisfying the following conditions:

(i)  $0 < \rho_g < \infty$ , (ii)  $\eta_g \leq \sigma_g < \infty$ , (iii)  $\rho_{f \circ g} = \rho_g$ , (iv)  $0 < \eta_{f \circ g} \leq \sigma_{f \circ g} < \infty$  and (v)  $\sum_{a \neq \infty} \Theta(a; g) = 2$ . Then

$$\begin{aligned} & \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \min \left\{ \frac{\eta_{f \circ g}}{\Gamma_{P_0} \cdot \eta_g}, \frac{\sigma_{f \circ g}}{\Gamma_{P_0} \cdot \sigma_g} \right\} \leq \\ & \max \left\{ \frac{\eta_{f \circ g}}{\Gamma_{P_0} \cdot \eta_g}, \frac{\sigma_{f \circ g}}{\Gamma_{P_0} \cdot \sigma_g} \right\} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}. \end{aligned}$$

**Remark 2.3.22** Theorem 2.3.28 remains true with  $\gamma_{P_0}$  instead of  $\Gamma_{P_0}$  if we replace the condition  $\sum_{a \neq \infty} \Theta(a; g) = 2$  by  $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$  or  $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$  and the other conditions are same.

Analogously one may state the following two theorems without proof.

**Theorem 2.3.29** Let  $f$  be rational and  $g$  be transcendental meromorphic satisfying the following conditions:

(i)  $0 < \rho_g < \infty$ , (ii)  $\eta_g \leq \sigma_g < \infty$ , (iii)  $\rho_{f \circ g} = \rho_g$ , (iv)  $0 < \eta_{f \circ g} \leq \sigma_{f \circ g} < \infty$  and (v)  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$ . Then

$$\begin{aligned} & \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \\ & \leq \min \left\{ \frac{\eta_{f \circ g}}{\{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; g)\} \cdot \eta_g}, \frac{\sigma_{f \circ g}}{\{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; g)\} \cdot \sigma_g} \right\} \\ & \leq \max \left\{ \frac{\eta_{f \circ g}}{\{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; g)\} \cdot \eta_g}, \frac{\sigma_{f \circ g}}{\{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; g)\} \cdot \sigma_g} \right\} \\ & \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}. \end{aligned}$$

**Theorem 2.3.30** Let  $f$  be rational and  $g$  be transcendental meromorphic having maximum deficiency sum satisfying the following conditions:

(i)  $0 < \rho_g < \infty$ , (ii)  $\eta_g \leq \sigma_g < \infty$ , (iii)  $\rho_{f \circ g} = \rho_g$  and (iv)  $0 < \eta_{f \circ g} \leq \sigma_{f \circ g} < \infty$ . Then

$$\begin{aligned} & \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \\ & \leq \min \left\{ \frac{\eta_{f \circ g}}{\{1 + k - k\delta(\infty; f)\} \cdot \eta_g}, \frac{\sigma_{f \circ g}}{\{1 + k - k\delta(\infty; f)\} \cdot \sigma_g} \right\} \\ & \leq \max \left\{ \frac{\eta_{f \circ g}}{\{1 + k - k\delta(\infty; f)\} \cdot \eta_g}, \frac{\sigma_{f \circ g}}{\{1 + k - k\delta(\infty; f)\} \cdot \sigma_g} \right\} \\ & \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}. \end{aligned}$$

**Theorem 2.3.31** *Let  $f$  be rational and  $g$  be transcendental meromorphic satisfying the following conditions:*

(i)  $0 < \lambda_g < \infty$ , (ii)  $\tau_g < \infty$ , (iii)  $\lambda_{f \circ g} = \lambda_g$ , (iv)  $0 < \tau_{f \circ g} < \infty$  and (v)  $\sum_{a \neq \infty} \Theta(a; g) = 2$ . Then

$$\liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\tau_{f \circ g}}{\Gamma_{P_0} \cdot \tau_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}.$$

**Proof.** By the definition of weak type, we have for arbitrary positive  $\epsilon$  and for all sufficiently large values of  $r$ ,

$$T(r, f \circ g) \geq (\tau_{f \circ g} - \epsilon) r^{\lambda_{f \circ g}}. \quad (2.57)$$

In view of Lemma 2.2.8 we get for a sequence of values of  $r$  tending to infinity,

$$\begin{aligned} T(r, P_0[g]) &\leq (\tau_{P_0[g]} + \epsilon) r^{\lambda_{P_0[g]}} \\ \text{i.e., } T(r, P_0[g]) &\leq (\Gamma_{P_0} \cdot \tau_g + \epsilon) r^{\lambda_g}. \end{aligned} \quad (2.58)$$

Since  $\lambda_{f \circ g} = \lambda_g$ , from (2.57) and (2.58) it follows for a sequence of values of  $r$  tending to infinity,

$$\frac{T(r, f \circ g)}{T(r, P_0[g])} \geq \frac{(\tau_{f \circ g} - \epsilon)}{(\Gamma_{P_0} \cdot \tau_g + \epsilon)}.$$

As  $\epsilon (> 0)$  is arbitrary, it follows from above that

$$\limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \geq \frac{\tau_{f \circ g}}{\Gamma_{P_0} \cdot \tau_g}. \quad (2.59)$$

Again for a sequence of values of  $r$  tending to infinity,

$$T(r, f \circ g) \leq (\tau_{f \circ g} + \epsilon) r^{\lambda_{f \circ g}}. \quad (2.60)$$

Also for all sufficiently large values of  $r$  we get by Lemma 2.2.6 that

$$\begin{aligned} T(r, P_0[g]) &\geq (\tau_{P_0[g]} - \epsilon) r^{\lambda_{P_0[g]}} \\ \text{i.e., } T(r, P_0[g]) &\geq (\Gamma_{P_0} \cdot \tau_g - \epsilon) r^{\lambda_g}. \end{aligned} \quad (2.61)$$

Now in view of condition (iii) we get from (2.60) and (2.61), for a sequence of values of  $r$  tending to infinity,

$$\frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{(\tau_{f \circ g} + \epsilon)}{\Gamma_{P_0} \cdot \tau_g - \epsilon}.$$

Since  $\epsilon (> 0)$  is arbitrary, we obtain from above that

$$\limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\tau_{f \circ g}}{\Gamma_{P_0} \cdot \tau_g}. \quad (2.62)$$

Thus the theorem follows from (2.59) and (2.62). ■

**Remark 2.3.23** *The conclusion of Theorem 2.3.31 remains valid with  $\gamma_{P_0}$  instead of  $\Gamma_{P_0}$  if we replace the condition  $\sum_{a \neq \infty} \Theta(a; g) = 2$  by  $\Theta(\infty; g) =$*

*$\sum_{a \neq \infty} \delta_p(a; g) = 1$  or  $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$  and the other conditions are same .*

**Theorem 2.3.32** *Let  $f$  be rational and  $g$  be transcendental meromorphic satisfying the following conditions:*

*(i)  $0 < \lambda_g < \infty$ , (ii)  $\tau_g < \infty$ , (iii)  $\lambda_{f \circ g} = \lambda_g$ , (iv)  $0 < \tau_{f \circ g} < \infty$  and (v)  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$ . Then*

$$\liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\tau_{f \circ g}}{\{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; g)\} \cdot \tau_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}.$$

The proof of the theorem can be established in the line of Theorem 2.3.31 and with the help of Lemma 2.2.12 and therefore is omitted.

The following theorem can also be proved in the line of Theorem 2.3.31 and in view of Lemma 2.2.15.

**Theorem 2.3.33** *Let  $f$  be rational and  $g$  be transcendental meromorphic having maximum deficiency sum satisfying the following conditions :*

*(i)  $0 < \lambda_g < \infty$ , (ii)  $\tau_g < \infty$ , (iii)  $\lambda_{f \circ g} = \lambda_g$ , and (iv)  $0 < \tau_{f \circ g} < \infty$  . Then*

$$\liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\tau_{f \circ g}}{\{1 + k - k\delta(\infty; f)\} \cdot \tau_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}.$$

The proof is omitted.

**Theorem 2.3.34** *Let  $f$  and  $g$  be two entire functions satisfying (i)  $0 < \lambda_g \leq \rho_g < \infty$ , (ii)  $\lambda_f > 0$ , (iii)  $\Delta(0; f) < 1$  and (iv)  $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ .*

*Then*

$$\lim_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, P_0[g])} = \infty,$$

*where  $A$  is a real constant.*

**Proof.** We suppose that  $A > 0$  because otherwise the theorem is obvious.

For given  $\epsilon$  ( $0 < \epsilon < 1 - \Delta(0; f)$ ),

$$N(r, 0; f) > (1 - \Delta(0; f) - \epsilon)T(r, f)$$

for all sufficiently large values of  $r$ . So from Lemma 2.3.17 we get for all large values of  $r$ ,

$$\begin{aligned} & T(r, f \circ g) + O(1) \\ & \geq (\log \frac{1}{\eta}) \frac{(1 - \Delta(0; f) - \epsilon)T \left\{ M((\eta r)^{\frac{1}{1+\alpha}}, g), f \right\}}{\log M((\eta r)^{\frac{1}{1+\alpha}}, g) - O(1)} - O(1). \end{aligned} \quad (2.63)$$

Since for all large values of  $r$ ,  $\log M(r, g) < r^{\rho_g + \epsilon}$ , it follows from (2.63) that for all sufficiently large values of  $r$ ,

$$\begin{aligned} \log T(r, f \circ g) + O(1) & \geq O(\log r) + \log T \left\{ M((\eta r)^{\frac{1}{1+\alpha}}, g), f \right\} \\ & \quad + \log \left[ 1 - \frac{\log M((\eta r)^{\frac{1}{1+\alpha}}, g)O(1)}{(1 - \Delta(0; f) - \epsilon)T \left\{ M((\eta r)^{\frac{1}{1+\alpha}}, g), f \right\}} \right]. \end{aligned}$$

Since  $f$  is transcendental, it follows that

$$\lim_{r \rightarrow \infty} \frac{\log M((\eta r)^{\frac{1}{1+\alpha}}, g)}{T \left\{ M((\eta r)^{\frac{1}{1+\alpha}}, g), f \right\}} = 0.$$

So from above we get for all large values of  $r$ ,

$$\log T(r, f \circ g) \geq O(\log r) + \log T \left\{ M((\eta r)^{\frac{1}{1+\alpha}}, g), f \right\} + o(1). \quad (2.64)$$

Also we see that for all large values of  $r$ ,

$$M(r, g) > \exp \left\{ (r)^{(1/2)\lambda_g} \right\}$$

$$\log T(r, f) > \frac{1}{2}\lambda_f \log r$$

and  $T(r, P_0[g]) < r^{\rho_{P_0[g]} + 1} = r^{\rho_g + 1}$ , by Lemma 2.2.7.

So from (2.64) we obtain for all sufficiently large values of  $r$ ,

$$\frac{\log T(r, f \circ g)}{\log T(r^A, P_0[g])} > \frac{O(\log r)}{A(1 + \rho_g) \log r} + \frac{\lambda_f}{2} \cdot \frac{(\eta r)^{\frac{\lambda_g}{2(1+\alpha)}}}{A(1 + \rho_g) \log r} + o(1),$$

which implies that

$$\lim_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, P_0[g])} = \infty .$$

This proves the theorem. ■

**Remark 2.3.24** *The conclusion of Theorem 2.3.34 can also be deduced under the hypothesis  $\sum_{a \neq \infty} \Theta(a; g) = 2$  or  $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$  instead of  $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ .*

**Remark 2.3.25** *If we replace the condition  $\Delta(0; f) < 1$  by  $\delta(0; f) < 1$  and the other conditions remain the same then Theorem 2.3.34 is still valid with ‘limit’ replaced by ‘limit superior’.*

In the way of Theorem 2.3.34 and in view of Lemma 2.2.12 we may state the following theorem without proof.

**Theorem 2.3.35** *Let  $f$  and  $g$  be two transcendental entire functions satisfying (i)  $0 < \lambda_g \leq \rho_g < \infty$ , (ii)  $\lambda_f > 0$ , (iii)  $\Delta(0; f) < 1$  and (iv)  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) =$*

4. Then

$$\lim_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, M[g])} = \infty ,$$

where  $A$  is a real constant.

**Remark 2.3.26** *If we change the condition  $\Delta(0; f) < 1$  by  $\delta(0; f) < 1$  and the other conditions remain the same then Theorem 2.3.35 is still valid with ‘limit’ replaced by ‘limit superior’.*

Analogously one may state the following theorem without proof because it can be carried out in the line of Theorem 2.3.34 and with the help of Lemma 2.2.15.

**Theorem 2.3.36** *Let  $f$  and  $g$  be two transcendental entire functions where  $g$  has the maximum deficiency sum satisfying (i)  $0 < \lambda_g \leq \rho_g < \infty$ , (ii)  $\lambda_f > 0$  and (iii)  $\Delta(0; f) < 1$ . Then*

$$\lim_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, L[g])} = \infty ,$$

where  $A$  is a real constant.

**Remark 2.3.27** If we change the condition  $\Delta(0; f) < 1$  by  $\delta(0; f) < 1$  and the other conditions remain the same then Theorem 2.3.36 is still valid with 'limit' replaced by 'limit superior'.

**Remark 2.3.28** If we consider  $\rho_f > 0$  instead of  $\lambda_f > 0$  then Theorem 2.3.34, Theorem 2.3.35 and Theorem 2.3.36 remain true for 'limit' replaced by 'limit superior'.

Now we state the following theorems without proof.

**Theorem 2.3.37** Let  $f$  be rational and  $g$  be transcendental meromorphic satisfying the following conditions : (i)  $0 < \lambda_g < \infty$ , (ii)  $\tau_g < \infty$  (iii)  $\lambda_{f \circ g} = \lambda_g$  (iv)  $0 < \tau_{f \circ g} < \infty$  and (v)  $\sum_{a \neq \infty} \Theta(a; g) = 2$ . If  $f \circ g$  and  $g$  be both of regular growth then

$$(i) \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\sigma_{f \circ g}}{\Gamma_{P_0} \cdot \sigma_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}$$

$$(ii) \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\tau_{f \circ g}}{\Gamma_{P_0} \cdot \tau_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}$$

$$(iii) \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\bar{\sigma}_{f \circ g}}{\Gamma_{P_0} \cdot \bar{\sigma}_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}$$

$$\text{and (iv) } \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\bar{\tau}_{f \circ g}}{\Gamma_{P_0} \cdot \bar{\tau}_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}.$$

**Remark 2.3.29** The conclusion of Theorem 2.3.37 can also be drawn with  $\gamma_{P_0}$  instead of  $\Gamma_{P_0}$  under the hypothesis  $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$  or  $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$  and the other conditions remain the same.

**Theorem 2.3.38** Let  $f$  be rational and  $g$  be transcendental meromorphic satisfying the following conditions : (i)  $0 < \lambda_g < \infty$ , (ii)  $\tau_g < \infty$  (iii)  $\lambda_{f \circ g} = \lambda_g$  (iv)  $0 < \tau_{f \circ g} < \infty$  and (v)  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$ . If  $f \circ g$  and  $g$  be both of regular growth then

$$(i) \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\sigma_{f \circ g}}{\{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; g)\} \cdot \sigma_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}$$

$$(ii) \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\tau_{f \circ g}}{\{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; g)\} \cdot \tau_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}$$

$$(iii) \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\bar{\sigma}_{f \circ g}}{\{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; g)\} \cdot \bar{\sigma}_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}$$

$$\text{and } (iv) \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\bar{\tau}_{f \circ g}}{\{\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty; g)\} \cdot \bar{\tau}_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}$$

**Theorem 2.3.39** *Let  $f$  be rational and  $g$  be transcendental meromorphic satisfying the following conditions : (i)  $0 < \lambda_g < \infty$ , (ii)  $\tau_g < \infty$  (iii)  $\lambda_{f \circ g} = \lambda_g$  (iv)  $0 < \tau_{f \circ g} < \infty$  and (v)  $g$  have maximum deficiency sum. If  $f \circ g$  and  $g$  be both of regular growth then*

$$(i) \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\sigma_{f \circ g}}{\{1 + k - k\delta(\infty; g)\} \cdot \sigma_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}$$

$$(ii) \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\tau_{f \circ g}}{\{1 + k - k\delta(\infty; g)\} \cdot \tau_g} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}$$

$$(iii) \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\bar{\sigma}_{f \circ g}}{\{1 + k - k\delta(\infty; g)\} \cdot \bar{\sigma}_g} \\ \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])}$$

$$\text{and (iv) } \liminf_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{\bar{\tau}_{f \circ g}}{\{1 + k - k\delta(\infty; g)\} \cdot \bar{\tau}_g} \\ \leq \limsup_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} .$$

-----X-----