CHAPTER 1

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A single valued function of one complex variable which is analytic in the finite complex plane $\mathbb{C}$ is called an entire function. Let f be an entire function defined in $\mathbb{C}$. The maximum modulus function of f on the circle $|z| = r$ denoted by $M(r, f)$ or by $M(r)$ is defined as follows :

$$M(r) \equiv M(r, f) = \max_{|z|=r} |f(z)| \ .$$

Actually the maximum modulus function is a special growth scale which characterises the growth of an entire function and the distribution of its zeros.The properties of maximum modulus function are as follows :
(i) Since by Liouville's theorem a bounded entire function is constant, it follows that for nonconstant f the maximum modulus function $M(r)$ is unbounded.
(ii) It is known from Cauchy's theorem **{p.5,[43]}** that the value $M(r)$ is attained by f on $|z| = r$, which implies that for an entire function f, $M(r)$ grows monotonically as r increases. This together with the uniform continuity of f implies that $M(r)$ is a continuous function of r. Also $M(r)$ is differentiable in adjacent intervals **{p.27,[43]}**.
(iii) In view of Hadamard's theorem**{p.20,[43]}**, we can say that $\log M(r)$ is a continuous, convex and ultimately increasing function of $\log r$.

Again an entire function which has an essential singularity at the point at infinity is called a transcendental entire function. In case of a transcendental entire function f , $M(r)$ grows faster that any positive power of r. Thus in order to estimate the growth of transcendental entire functions we choose a comparison function e^{r^k}, $k > 0$ that grows more rapidly than power of r.

More precisely f is said to be a function of finite order if there exists a positive constant k such that $\log M(r) < r^k$ for all sufficiently large values of r ($r > r_0(k)$ say). The infimum of such k's is called the order of f. If no such $k(> 0)$ exists, f is said to be of infinite order.

Let $\rho\,(\geq 0)$ be the order of f . It can easily be verified that the order ρ of f has the following alternative definition

$$\rho = \limsup_{r\to\infty}\frac{\log\log M(r)}{\log r}.$$

The number

$$\lambda = \liminf_{r\to\infty}\frac{\log\log M(r)}{\log r}$$

is said to be the lower order of f. Consequently $\lambda \leq \rho$ and in particular if for a function $\lambda = \rho$,then f is said to be of regular growth. For example the function $e^z, \cos z$ are of regular growth.

With known order $\rho\,(0 < \rho < \infty)$ the growth of an entire function can be characterised more precisely by the type of the function. The number τ given by

$$\tau = \limsup_{r\to\infty}\frac{\log M(r)}{r^\rho}$$

is called the type of f.

Between two functions of same order one can be characterised to be of greater growth if its type is greater. The quantities ρ, λ and τ are extensively used to the study of growth properties of f.

The idea of order and lower order are generalised in the following way **{cf.[27],[37]}**, which are called respectively the generalised order and generalised lower order :

$$\rho_k\left(f\right) = \limsup_{r\to\infty}\frac{\log^{[k]} M(r)}{\log r} \text{ and } \lambda_k\left(f\right) = \liminf_{r\to\infty}\frac{\log^{[k]} M(r)}{\log r} \text{ where}$$

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 2, 3, 4, \ldots \text{ and } log^{[0]}x = x.$$

Let f be an entire function. Then it has an everywhere absolutely convergent Taylor series expansion

$$f\left(z\right) = \sum_{n=0}^{\infty} a_n \left(z - a\right)^n,$$

about any point 'a' in the finite complex plane.

When the Taylor series expansion is taken about the origin it takes the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots,$$

which is a natural generalization of the polynomials.

The degree of a polynomial which is equal to its number of zeros estimate the rale of growth of the polynomial as the independent variable move without bound. So the more zeros, the greater is the growth.

An analogous property that relate the set of zeros and the growth of a function can be developed for arbitrary entire functions.

Establishing relations between the distribution of the zeros of an entire function and its asymptotic behaviour as $z \to \infty$ enriched most of the classical results of the theory of entire functions. The classical investigations of Borel, Hadamard and Lindelof are of this kind.

Again the terms $|a_n|$ in the Taylor series expansion of f i.e., $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ must approach 0. So that for each a, an index $n_0 = n_0(a)$ exists for which $|a_{n_0}|$ is a maximal co efficient. B. Lepson [**23**] developed the idea of characterizing entire functions for which $n_0(a)$ is bounded, which he [**23**] called functions of bounded index.

More precisely F. Gross [**20**] defined a function of bounded index as follows.

Definition 1.0.1 *An entire function f is said to be of bounded index if and only if there exists an integer N, such that for all z in in the finite complex plane*

$$\max\left(|f|, \left|f^{(1)}\right|, \frac{\left|f^{(2)}\right|}{2!}, \ldots, \frac{\left|f^{(N)}\right|}{N!}\right) \geq \frac{\left|f^{(j)}\right|}{j!}$$

for $j = 0, 1, 2, 3, \ldots$, where $f^{(\nu)}$ denotes the ν th- derivative of f and $f^{(0)} = f$. The smallest integer for which the above inequality holds is called the index of f. For example the function e^{2z} is of bounded index, the index being 1. An entire function which is not of bounded index is said to be of unbounded index.

Let $L \equiv L(r)$ be a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a.

Lakshminarasimhan [24] defined a function of L- bounded index as follows:

Definition 1.0.2 *f is said to be of L- bounded index if there exists a positive integer M such that*

$$\max_{0\leq i\leq M}\left(\frac{L(i+2)}{i!}\left|f^{(i)}(z)\right|\right)\geq\frac{L(j+2)}{j!}\left|f^{(j)}(z)\right|$$

for all z belonging to the open complex plane $\mathbb{C}$ and $j=0,1,2,3,\ldots$

The least value N_L of the integer M for which the above inequality holds is called the $L-$index of f. Functions of the form $f(z)=\exp(\alpha z+\beta),\alpha,\beta$ being constants, are of L- bounded index.

Weakening the idea of a function of bounded index F. Gross [20] defined a function of non-uniform bounded index as follows:

Definition 1.0.3 *An entire function f is called a function of non-uniform bounded index if there exist integers N_j and an integer N such that*

$$\sum_{i=0}^{N}\frac{\left|f^{(i)}(z)\right|}{i!}\geq\frac{d\left|f^{(j)}(z)\right|}{j!}$$

for $|z|>N_j\ (j=0,1,2,3,\ldots)$ and d is any fixed constant.

With this definition he [20] investigates the relation between a function of bounded index and that of non-uniform bounded index.

Let f be a meromorphic function in the finite complex plane i.e., a single valued function whose only singularities in the finite plane are poles.

During the present century the most important occurrence in function theory is the theory of meromorphic functions developed by Rolf Nevanlinna. The revolutionary invention of his first and second Fundamental theorem gave a new dimension to the value distribution theory.

The first Fundamental theorem provides an upper bound to the number of roots of the equation $f=a$, while the second Fundamental theorem solve the more difficult question of lower bounds for the number of roots of the equation $f=a$. The main beauty of the second fundamental theorem remains in the significant extension of Picard's famous theorem that f must assume all values in the complex plane with at most two exceptions. It turns out that the number of deficient values is always countable.

From mathematical view point his first fundamental theorem is essentially a rewriting of the Poisson -Jensen formula.

Now we state Poisson -Jensen formula { p.1,[**21**]} as the following theorem.

Theorem 1.0.1 *Let f be meromorphic in $|z| \leq R\,(0 < R < \infty)$ and a_μ $(\mu = 1, 2, ..., M)$ be the zeros, b_ν $(\nu = 1, 2, ..., N)$ be the poles of f in $|z| < R$. Then if $z = re^{i\theta}$ $(0 < r < R)$ and if $f(z) \neq 0, \infty$ we have*

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(\mathrm{Re}^{i\phi}\right)\right| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi$$
$$+ \sum_{\mu=1}^{M} \log\left|\frac{R(z - a_\mu)}{R^2 - \bar{a}_\mu z}\right| - \sum_{\nu=1}^{N} \log\left|\frac{R(z - b_\nu)}{R^2 - \bar{b}_\nu z}\right|.$$

The theorem holds good also when f has zeros and poles on $|z| = R$. For $z = 0$ this reduces to Jensen's formula

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(\mathrm{Re}^{i\phi}\right)\right| d\phi + \sum_{\mu=1}^{M} \log\frac{|a_\mu|}{R} - \sum_{\nu=1}^{N} \log\frac{|b_\nu|}{R},$$

provided that $f(0) \neq 0, \infty$.

If at the point $z = 0$, f has a zero of multiplicity λ or a pole of multiplicity $-\lambda$ such that $f(z) = C_\lambda z^\lambda + ...$ then we get from Jensen's formula

$$\log|C_\lambda| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(\mathrm{Re}^{i\phi}\right)\right| d\phi + \sum_{\mu=1}^{M} \log\frac{|a_\mu|}{R} - \sum_{\nu=1}^{N} \log\frac{|b_\nu|}{R} - \lambda\log R.$$

This modification becomes a bit tiresome to be recognized explicitly. Hence in general we shall assume that our function behave in such a way that no term of Jensen's formula become infinite knowing that these exceptional cases, if they occur, can be treated.

To rewrite Poisson-Jensen formula we need the following notions due to Nevanlinna [**21**].

For a complex number 'a', finite or infinite let $n\left(r, \frac{1}{f-a}\right) \equiv n(r,a)$ denote the number of $a-$points of f in $|z| \leq r$, counted with proper multiplicities.

We define the function $N(r,a)$ as follows:

$$N(r,a) = \int_0^r \frac{n(t,a) - n(0,a)}{t} dt + n(0,a) \log r$$

and $N(r,\infty) \equiv N(r,f)$.

Obviously if f has no $a-$points at $z = 0$ then

$$N(r,a) \equiv \int_0^r \frac{n(t,a)}{t} dt.$$

Then from Riemann-Stieltjes integral it follows that

$$\sum_{\nu=1}^{N} \log \left| \frac{R}{b_\nu} \right| = \int_0^R \frac{n(t,f)}{t} dt = N(R,f)$$

and

$$\sum_{\mu=1}^{M} \log \left| \frac{R}{a_\mu} \right| = \int_0^R \frac{n\left(t, \frac{1}{f}\right)}{t} dt = N(R, \frac{1}{f}).$$

Next we define

$$\begin{aligned} log^+ x &= \log x \text{ if } x \geq 1 \\ &= 0 \text{ if } 0 \leq x < 1. \end{aligned}$$

Clearly the function satisfies the following properties :
(i) $\log^+ x \geq 0$ if $x \geq 0$
(ii) $\log^+ x \geq \log x$ if $x > 0$
(iii) $\log^+ x \geq \log^+ y$ if $x > y$
(iv) $\log x = \log^+ x - \log^+ \frac{1}{x}$ if $x > 0$.
The function $m(r,f)$ defined as

$$m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| d\theta$$

is called the proximity function of f.

Clearly $m(r, f)$ is a sort of averaged magnitude of $\log|f(z)|$ on arcs of $|z| = r$ where $|f(z)|$ is large.

The sum function $T(r, f) = m(r, f) + N(r, f)$ is called Nevanlinna's Characteristic function of f {p.4,[**21**]} and it plays an important role in the theory of meromorphic functions as the function $M(r, f)$ plays in the theory of entire functions.

Now with the above notations Jensen's formula becomes

$$\log|f(0)| = m(R, f) - m\left(R, \frac{1}{f}\right) + N(R, f) - N\left(R, \frac{1}{f}\right)$$

$$\text{i.e., } T(R, f) = T\left(R, \frac{1}{f}\right) + \log|f(0)|.$$

In case of dealing with only one function f at a time it is usual to write $T(R)$ for $T(R, f)$ and for finite 'a', $M(R, a)$, $N(R, a)$, $n(R, a)$ instead of $M(R, \frac{1}{f-a})$, $N(R, \frac{1}{f-a})$, $n(R, \frac{1}{f-a})$ and $M(R, \infty)$, $N(R, \infty)$, $n(R, \infty)$ instead of $M(R, f)$, $N(R, f)$, $n(R, f)$. On the other hand to distinguish between several functions under consideration at the same tome, we will write $T(r, f)$, $N(r, a; f)$, $m(r, a; f)$ etc.

Since for any positive integer p and complex number a_ν,

$$\log^+\left|\prod_{\nu=1}^{p} a_\nu\right| \le \sum_{\nu=1}^{p} \log^+|a_\nu| \text{ and}$$

$$\log^+\left|\sum_{\nu=1}^{p} a_\nu\right| \le \log^+\left(p. \max_{\nu=1,2,\cdots p} |a_\nu|\right) \le \sum_{\nu=1}^{p} \log^+|a_\nu| + \log p\ ,$$

it is easy to show that {p.5, [**21**]} for p meromorphic functions $f_1(z), f_2(z), \cdots f_p(z)$

$$m\left(r, \sum_{\nu=1}^{p} f_\nu(z)\right) \le \sum_{\nu=1}^{p} m(r, f_\nu(z)) + \log p\ ,$$

$$N\left(r, \sum_{\nu=1}^{p} f_\nu(z)\right) \le \sum_{\nu=1}^{p} N(r, f_\nu(z))$$

and so

$$T\left(r,\sum_{\nu=1}^{p}f_{\nu}(z)\right)\leq\sum_{\nu=1}^{p}T\left(r,f_{\nu}(z)\right)+\log p\ .$$

Now we express Nevanlinna's First Fundamental theorem in the following form:

Theorem 1.0.2 *{p.6, [21]}If f is a meromorphic function in $|z|<\infty$ and a is any complex number, finite or infinite, then*

$$m(r,a)+N(r,a)=T(r,f)+O(1)\ .$$

This result indicates the symmetrical behaviour exhibited by a meromorphic function relative to different complex number a, finite or infinite. for different values of a, the sum $m(r,a)+N(r,a)$ maintains a total, given by the quantity $T(r,f)$ which is invariant apart from a bounded additive term .

The first term $m(r,a)$ of this invariant sum treated as the mean of $\log^{+}\left|\frac{1}{f-a}\right|$ on the circle $|z|=r$, contribute remarkably for those arce on the circle where the functional values are very closer to the given value a. The magnitude of the proximity function can thus be considered as a measure for the mean deviation on the circle $|z|=r$ of the functional value f from the value a.

The term $N(r,a)$ measures the density of the average distribution of the roots of the equation $f=a$ in the disc $|z|<r$. This counting function for $a-points$ grows faster with r if the number of $a-points$ is large.

If the $a-points$ are relatively scarce for a certain 'a', relatively slow growth of the function $N(r,a)$ as $r\to\infty$ are found and in the extreme case where $f\neq a$ in $|z|<\infty$ (i.e., 'a' is a Picard's exceptional value of the function), $N(r,a)$ is identically zero. But in this circumstance the function deviates in the mean slightly from the value 'a' in question; the corresponding proximity function $m(r,a)$ then be relatively large, keeping the sum $m(r,a)+N(r,a)$ invariant, reaching the magnitude $T(r,f)$.

If f is an entire function, $N(r,f)\equiv 0$ and $T(r,f)=m(r,f)$. For an entire function f the study of the comparative growth properties of $T(r,f)$ and $\log M(r,f)$ is a popular problem for the researchers.

Here we note that for a meromorphic fiunction f, $T(r,f)$ is an increasing convex function of $\log r$ {p.9, **[21]**}. Also by the Hadamard's there circles theorem for an entire function f, $\log M(r)$ is an increasing convex function of $\log r$ and this is one of many respects in which for entire functions $T(r,f)$ and $\log M(r)$ have similar behaviour, even though they are different.

Relating $T(r,f)$ and $\log M(r,f)$ the following inequality is the f undamental one.

Theorem 1.0.3 *$\{p.18,[21]\}$. If f is regular for $|z| \leq R$ then*

$$T(r,f) \leq \log^{+} M(r,f) \leq \frac{R+r}{R-r} T(R,f), \ 0 \leq r < R.$$

Now we state the following definition.

Definition 1.0.4 *$\{p.16,[21]\}$. Let S be a non-negative real valued function increasing for $r_0 \leq r < \infty$, $r_0 > 0$. The order k and lower order λ of the function $S(r)$ are defined as*

$$k = \limsup_{r\to\infty} \frac{\log S(r)}{\log r} \quad \text{and } \lambda = \liminf_{r\to\infty} \frac{\log S(r)}{\log r}$$

respectively.

Let k be finite and positive. If $c = \limsup\limits_{r\to\infty} \frac{\log S(r)}{r^k}$ then we have the following possibilities:

(a) $S(r)$ has minimal type if $c = 0$;
(b) $S(r)$ has mean type if $0 < c < +\infty$;
(c) $S(r)$ has maximal type if $c = +\infty$;
(d) $S(r)$ has convergence class if $\int\limits_{r_0}^{\infty} \frac{S(t)}{t^{k+1}} dt$ converges.

The following theorem is trivial.

Theorem 1.0.4 *$\{p.18,[21]\}$. If f is an entire function then the order k of the function $S_1(r) = \log^{+} M(r,f)$ and $S_2(r) = T(r,f)$ is the same. Further if $0 < k < \infty$, $S_1(r)$ and $S_2(r)$ belong to the same classes (a), (b), (c) or (d).*

Here we note that $S_1(r)$ and $S_2(r)$ have the same lower order.

A function f meromorphic in the plane is said to have order ρ , lower order λ and minimal, mean, maximal type or convergence class if the function $T(r,f)$ has this property. For entire functions these coincide by the above theorem with the corresponding definition in terms of $M(r,f)$ which is classical. For a meromorphic function of finite positive order ρ, the term τ defined by

$$\tau = \limsup_{r\to\infty}\frac{\log T(r,f)}{r^{\rho}}$$

is called the type of f.

Here we also note that like an entire function , a meromorphic function f and its derivative are of same order.

After the knowledge of the remarkable symmetry exhibited by a meromorphic function f through the invariance of the sum $m(r,a)+N(r,a)$, it is natural to study more carefully the relative strength of two terms in the sum namely the proximity component $m(r,a)$ and the counting component $N(r,a)$ .Some individual results in this direction are the following{p.234,[**21**]} :

1. Picard's theorem shows that the counting function for a non-constant meromorphic function in the finite complex plane can vanish for atmost two values of a.

2.For a meromorphic function of finite non-integral order there is atmost one Picard's exceptional value.

3. That the counting function $N(r,a)$ is in general i.e., for the great majority of the values of 'a', large in comparison with the proximity function.

We now state Nevanlinna's Second Fundamental theorem.

Theorem 1.0.5 *{p.31,[**21**]}. Suppose that f is a non-constant meromorphic function in $|z| \le r$.Let $f(0) \ne 0$, ∞ and $f'(0) \ne 0$.Let $a_1, a_2, \ldots a_q$ where $q \ge 2$, be distinct finite complex numbers, $\delta > 0$ and suppose that*

$$|a_\mu - a_\nu| \ge \delta\,(0 < \delta < 1) \;\; for\; 1 \le \mu < \nu \le q.$$

Then

$$m(r,\infty) + \sum_{\nu=1}^{q} m(r,a_\nu) \le 2T(r,f) - N_1(r) + S(r),$$

where $N_1(r)$ is positive quantity given by

$$N_1(r) = N\left(r,\frac{1}{f'}\right) + 2N(r,f) - N\left(r,f'\right)$$

$$\text{and } S(r) = m\left(r, \frac{f'}{f}\right) + m\left\{r, \sum_{\nu=1}^{q} \frac{f'}{(f-a_\nu)}\right\} + q\log^+ \frac{3q}{\delta} + \log 2 + \log \frac{1}{|f'(0)|}.$$

The cases $f(0) = 0$ or ∞ , $f'(0) = 0$ can be dealt with suitable modifications.

The quantity $S(r)$ will in general play the role of an unimportant error term. The combination of this fact with the above theorem yields the second fundamental theorem.

The following theorem gives an estimation of $S(r)$.

Theorem 1.0.6 *{p.34,[21]}. Let f be a non constant meromorphic function in $|z| < R_0 \leq \infty$ and that $S(r) \equiv S(r, f)$ is as in the above theorem. Then*

(i) If $R_0 = +\infty$, $S(r,f) = O\{\log T(r,f)\} + O(\log r)$, as $r \to \infty$ through all values if f has finite order and as $r \to \infty$ outside a set E of finite linear measure otherwise

(ii) If $0 < R_0 < +\infty$, $S(r,f) = O\left\{\log^+ T(r,f) + \log \frac{1}{R_0 - r}\right\}$ as $r \to R_0$ outside a set E such that $\int_E \frac{dr}{R_0 - r} < \infty$.

Further there is a point r outside E for which $\rho < r < \rho'$ provided that $0 < R - \rho' < e^{-2}(R - \rho)$.

As an immediate consequence the next theorem follows :

Theorem 1.0.7 *{p.41,[21]}. Let f be meromorphic and nonconstant in $|z| < R_0$. Then $\frac{S(r,f)}{T(r,f)} \to 0$ (A) as $r \to R_0$ with the following provisions:*

(a) (A) holds without restrictions if $R_0 = +\infty$ and f is of finite order in the plane.

(b) If f has infinite order in the plane, (A) still holds as $r \to \infty$ outside a certain exceptional set E_0 of finite length. Here E_0 depends only on f.

(c) If $R_0 < +\infty$ and $\limsup_{r \to \infty} \frac{T(r,f)}{\log\left\{\frac{1}{(R_0 - r)}\right\}} = +\infty$, then (A) holds as $r \to R_0$ through a suitable sequence r_n, which depends on f only.

This theorem points out the role of $S(r)$ as an unimportant error term.

Let f be a non constant meromorphic function in the plane. By $S(r, f)$ we shall denote any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ possibly outside a set r of finite linear measure.

Now we shall present an interesting consequence of Nevanlinna's second fundamental theorem.Precisely it is a second version of Nevanlinna's second Fundamental Theorem which is less informative but may sometimes be more easily used. To do this we need the following definitions :

Definition 1.0.5 . *For a complex number 'a' we put*

$$\delta(a) = \delta(a; f) = \liminf_{r\to\infty} \frac{m(r,a)}{T(r)} = 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r)}.$$

The quantities $\delta(a; f)$, $\Theta(a; f)$ and $\theta(a; f)$ are called respectively the deficiency, the ramification index and the index of multiplicity of the value 'a'. Evidently $\delta(a; f)$ is positive only if there are relatively few roots of the equation $f(z) = a$. Maximum value of deficiency is 1 when the roots of the equation $f(z) = a$ is very sparsely distributed, in particular when the value 'a' is a Picard's exceptional value. In general $0 \le m(r, a; f) \le T(r, f)$ so we have $0 \le \delta(a; f) \le 1$. Thus the quantity gives us a very accurate measure for relative density of the points where the function f assumes the value 'a'. We shall call a value normal if $\delta(a; f)$ vanishes. Again $\theta(a; f)$ is positive only when $f(z) = a$ has relatively many multiple roots.*and*

$$\Theta(a; f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}(r, a; f)}{T(r, f)},$$

where $\bar{N}(r, a; f)$ is the counting function for distinct a-points of f.
Also

$$\theta(a; f) = \liminf_{r\to\infty} \frac{N(r, a; f) - \bar{N}(r, a; f)}{T(r, f)}.$$

The quantities $\delta(a)$ is called the deficiency of the value 'a'. Evidently $\delta(a)$ is positive only if there are relatively few roots of the equation $f = a$. If the value 'a' is a Picard's exceptional value.i.e., a complex number which is not assumed by the function f , then $N(r, a) \equiv 0$ and so $\delta(a) = 1$.In any case since $0 \le m(r, a) \le T(r)$ we have $0 \le \delta(a) \le 1$. Thus the quantity $\delta(a)$ gives us a very accurate measure for relative density of the points where the function f assumes the value 'a' in question. The larger the deficiency is, the more rare are latter points, We shall call every value of vanishing deficiency $\delta(a)$, a normal value in contrast to the deficient values for which $\delta(a)$ is positive.

Next we have the following analogous definitions.

Definition 1.0.6 .

$$\Theta(a) = \Theta(a; f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}(r, a)}{T(r, f)},$$

where $\bar{N}(r, a; f) = \bar{N}(r, a)$ *is the counting function for distinct* a*-points.*

Definition 1.0.7 .

$$\Theta(a) = \Theta(a; f) = \liminf_{r\to\infty} \frac{N(r, a) - \bar{N}(r, a)}{T(r, f)}.$$

The quantity $\Theta(a)$ *is called the index of multiplicity of* a. *Clearly* $\Theta(a)$ *is positive if there are relatively many multiple roots.*

Let ε be any arbitrary positive quantity. Then from the above definitions we have for sufficiently large values of r

$$N(r, a) - \bar{N}(r, a) > \{\theta(a) - \varepsilon\}T(r, f),$$

$$N(r, a) < \{1 - \delta(a) + \varepsilon\}T(r, f)$$

and hence

$$\bar{N}(r, a) < \{1 - \delta(a) - \theta(a) + 2\varepsilon\}T(r, f)$$

so that

$$\Theta(a) \geq \delta(a) + \theta(a).$$

The quantity

$$\Delta(a; f) = 1 - \liminf_{r\to\infty} \frac{N(r, a)}{T(r)} = \limsup_{r\to\infty} \frac{m(r, a)}{T(r)}$$

gives another measure of deficiency and is called the Valiron deficiency, Clearly $0 \leq \delta(a; f) \leq \Delta(a; f) \leq 1$.

Now we introduce Milloux's theorem which is important in studying the properties of derivatives of meromorphic functions.

Theorem 1.0.8 .{*p.55, [21]*} *Let l be a positive integer and $\psi = \sum_{\nu=0}^{l} a_\nu f^{(\nu)}$, where $T(r, a_\nu) = S(r, f)$, $\nu = 0, 1, ..., l$. Then*

$$m\left(r, \frac{\psi}{f}\right) = S(r, f)$$
$$\text{and } T(r, \psi) \leq (l+1)T(r, f) + S(r, f).$$

Milloux showed that in the second fundamental theorem we can replace the counting functions for certain roots of $f = a$ by roots of the equation $\psi = b$ where ψ is gives as in the above theorem . In this connection we state the following theorem.

Theorem 1.0.9 .{*p.57, [21]*} *Let f be meromorphic and non-constant in the plane and $\psi = \sum_{\nu=0}^{l} a_\nu f^{(\nu)}$ where l is a positive integer, be non-constant . Then*

$$T(r, f) \leq \bar{N}(r, f) + N(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{\psi - 1})$$
$$- N_0(r, \frac{1}{\psi'}) + S(r, f),$$

where in $N_0(r, \frac{1}{\psi'})$ only zeros of ψ' not corresponding to the repeated roots of $\psi = 1$ are to be considered.

Here we note that this results reduces to the second fundamental theorem if $\psi = f$ and $q = 3$.

Apart from **Chapter 1** the thesis contains six chapters which are structured as follows:

- In **Chapter 2** we study the comparative growth of composite entire or meromorphic functions and differential monomials, differential polynomials generated by one of the factors which improves some earlier results. Since the natural extension of a derivative is a differential polynomial, in this chapter we also prove our results for special type of linear differential polynomials viz., the wronskians.

- In **Chapter 3** we establish some theorems on the generalised growth estimates of the composition of entire and meromorphic functions. Some examples are also provided to show that the conditions in the theorems are essential. The results of this chapter have been published in **News Bulletin of Calcutta Mathematical Society,** see [**13**].

- In **Chapter 4** we study some growth properties of composite functions analytic in the unit disc. Some results related to the relative order (relative lower order) , relative Nevanlinna order (relative Nevanlinna lower order) and relative Nevanlinna hyper order (relative Nevanlinna hyper lower order) of an analytic function with respect to an entire function are also established in this paper. The results of this chapter have been published in **Wesleyan Journal of Research, see** [**19**] and in **International Journal of Contemporary Mathematical Sciences,** see [**14**].

- In **Chapter 5** we extend a few results of Datta and Jha [**9**]. Also we generalise a result of Datta and Jha [**9**]. The results of this chapter have been published in **International Mathematical Forum,** see [**16**] and **International Journal of Mathematical Analysis,** see [**17**].

- In **Chapter 6** we consider several meromorphic functions having common roots and find some relations involving their relative defects.The results of this chapter have been published in **International Journal of Mathematical Analysis,** see [**18**].

- In **Chapter 7** we prove several results on the deficiencies of differential polynomials considered by Bhoosnurmath and Prasad [**4**].The results of this chapter have been published in **International Mathematical Forum,** see [**15**].

From **Chapter 2** onwards when we write **Theorem** $a.b.c$ (or **Corollary** $a.b.c$ etc.) where a, b and c are positive integers, we mean the c-**th theorem** (or c-**th corollary** etc.) of the b-**th section** in the a-**th chapter**. Also by **equation number** $(a.b)$ we mean the b-**th equation** in the a-**th chapter** for positive integers a and b. Individual chapters have been presented in such a manner that they are almost independent of the other chapters. The references to books and journals have been classified as bibliography and are given at the end of the thesis.

The author of the thesis is thankful to the authors of various papers and books which have been consulted during the preparation of the entire thesis.

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