
Introduction: Hyperalgebraic and Topological Warm-up

Marty [39] first proposed the notion of hypergroup as an extension of classical group theory in the Eighth Congress of Scandinavian Mathematicians, Stockholm in 1934. For mathematicians working in different fields such as graph theory [6, 54], geometry [51, 52], cryptography [5], codes [20, 65], probability [43], lattice theory [35, 62], automata [42], artificial intelligence [38], chemical science [2, 28], physical science [21, 44], and others, this notion opened up a whole new branch of mathematics.

As in [19], a *hyperoperation* on a nonempty set \mathcal{H} is a map $*$ from the cartesian product of \mathcal{H} with itself to $\mathcal{P}^*(\mathcal{H})$; the collection of nonempty subsets of \mathcal{H} . The ordered pair $(\mathcal{H}, *)$ is said to be a *hypergroupoid*. The hyperoperation $*$ is extended to the subsets of \mathcal{H} as follows: for nonempty subsets X, Y of \mathcal{H} , $X*Y = \bigcup_{x \in X, y \in Y} x*y$ and for $h \in \mathcal{H}$, $h*X = \{h\}*X$, $X*h = X*\{h\}$. A hypergroupoid $(\mathcal{H}, *)$ is called a *semihypergroup* if the associativity property holds on \mathcal{H} , i.e., for $g, h, k \in \mathcal{H}$, $(g*h)*k = g*(h*k)$, that means $\bigcup_{p \in g*h} p*k = \bigcup_{q \in h*k} g*q$. A hypergroupoid $(\mathcal{H}, *)$ is said to be a *quasihypergroup* if for $h \in \mathcal{H}$, $h*\mathcal{H} = \mathcal{H}*h = \mathcal{H}$ (reproduction axiom). A hypergroupoid which is both semihypergroup and a quasihypergroup is said to be a *hypergroup*. In a hypergroup $(\mathcal{H}, *)$, the hyperoperations $/, \backslash : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{P}^*(\mathcal{H})$ defined by $g/h = \{z \in \mathcal{H} : g \in z*h\}$ and $h \backslash g = \{z \in \mathcal{H} : g \in h*z\}$, for all $g, h \in \mathcal{H}$ are called the *right division* and the *left division*, respectively on \mathcal{H} [3]. For $X, Y \subseteq \mathcal{H}$, $X/Y = \bigcup_{x \in X, y \in Y} x/y$ and $Y \backslash X = \bigcup_{x \in X, y \in Y} y \backslash x$. In a hypergroup

$(\mathcal{H}, *)$, the reproduction axiom is equivalent to the condition: for all $g, h \in \mathcal{H}$, $g/h \neq \phi$ and $h \setminus g \neq \phi$ (Remark 2.3 [3]). A nonempty subset \mathcal{G} of a hypergroup $(\mathcal{H}, *)$ is said to be a *subhypergroup* of \mathcal{H} if \mathcal{G} is itself a hypergroup. In other words, a nonempty subset \mathcal{G} of a hypergroup $(\mathcal{H}, *)$ is a subhypergroup of \mathcal{H} if $g*\mathcal{G} = \mathcal{G}*g = \mathcal{G}$, for all $g \in \mathcal{G}$ [19]. Consider the Example 2.2 [66] of a hypergroup, called the discrete hypergroup, defined as follows:

for a nonempty set \mathcal{H} , a hyperoperation $*$ on \mathcal{H} is defined by $h * k = \{h, k\}$, for all $h, k \in \mathcal{H}$. Then, any nonempty subset of \mathcal{H} is a subhypergroup and it is easy to find two disjoint subhypergroups of \mathcal{H} . Thus, the intersection of two subhypergroups may not be a subhypergroup in general [66].

As in [19], a subhypergroup \mathcal{G} of a hypergroup $(\mathcal{H}, *)$ is

- (i) *closed on the left (on the right)* if for $g_1, g_2 \in \mathcal{G}$ and $h \in \mathcal{H}$, $g_1 \in h * g_2$ ($g_1 \in g_2 * h$, respectively) implies $h \in \mathcal{G}$. If \mathcal{G} is closed on both left and right, then it is called a *closed* subhypergroup.
- (ii) *invertible on the left (on the right)* if for $h, k \in \mathcal{H}$, $h \in \mathcal{G} * k$ ($h \in k * \mathcal{G}$) implies $k \in \mathcal{G} * h$ ($k \in h * \mathcal{G}$, respectively). If \mathcal{G} is invertible on left as well as on right, then \mathcal{G} is called *invertible*.

A subhypergroup \mathcal{G} of the hypergroup $(\mathcal{H}, *)$ is invertible on the right $\Leftrightarrow \{h * \mathcal{G} : h \in \mathcal{H}\}$ forms a partition for \mathcal{H} . A subhypergroup \mathcal{G} of a hypergroup $(\mathcal{H}, *)$ is said to be *normal* in \mathcal{H} if for all $h \in \mathcal{H}$, $h*\mathcal{G} = \mathcal{G}*h$. An element ℓ in a hypergroup $(\mathcal{H}, *)$ is said to be a *right identity (left identity)* of \mathcal{H} if for each $h \in \mathcal{H}$, $h \in h * \ell$ ($h \in \ell * h$). If ℓ is both left and right identity of \mathcal{H} , then it is called an *identity* of \mathcal{H} , i.e., for each $h \in \mathcal{H}$, $h \in (h * \ell) \cap (\ell * h)$. A hypergroup may or may not contain an identity element, as well as multiple identities. For $h \in \mathcal{H}$, an element $h' \in \mathcal{H}$ is said to be a *right inverse (left inverse)* of h if there exists an identity ℓ in \mathcal{H} such that $\ell \in h * h'$ ($\ell \in h' * h$). h' is called an *inverse* of h , if it is both the right and left inverse of h , i.e., if $\ell \in (h * h') \cap (h' * h)$. An element in a hypergroup may or may not have an inverse, as well as more than one inverse. For $h \in \mathcal{H}$, denote by $\mathcal{I}(h)$ the set of all inverses of h .

Let $(\mathcal{H}_1, *)$, (\mathcal{H}_2, \circ) be hypergroups and $g : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a map. Then, g is

- (i) a *homomorphism* if for $h, k \in \mathcal{H}_1$, $g(h * k) \subseteq g(h) \circ g(k)$;

- (ii) a *good homomorphism* if for $h, k \in \mathcal{H}_1$, $g(h * k) = g(h) \circ g(k)$;
- (iii) a *very good homomorphism* if it is a good homomorphism and for all $h, k \in \mathcal{H}_1$, $g(h/k) = g(h)/g(k)$, $g(h \setminus k) = g(h) \setminus g(k)$;
- (iv) an *isomorphism* if g is a bijective good homomorphism [19].

Let ρ be an equivalence relation on a semihypergroup $(\mathcal{H}, *)$ and X, Y be nonempty subsets of \mathcal{H} . Then, $X \bar{\rho} Y$ means that for all $x \in X$, $\exists y \in Y$ such that $x\rho y$ and for all $y' \in Y$, $\exists x' \in X$ such that $x'\rho y'$, and $X \bar{\bar{\rho}} Y$ means that $\forall x \in X$, $\forall y \in Y$, $x\rho y$. ρ is *regular on the left (right)* if $\forall h \in \mathcal{H}$, $x\rho y \Rightarrow (h * x)\bar{\rho}(h * y)$ [$(x * h)\bar{\rho}(y * h)$, respectively] and *strongly regular on the left (right)* if $\forall h \in \mathcal{H}$, $x\rho y \Rightarrow (h * x)\bar{\bar{\rho}}(h * y)$ [$(x * h)\bar{\bar{\rho}}(y * h)$, respectively]. ρ is *regular (strongly regular)* if it is regular (strongly regular) on the left as well as on the right. If ρ is regular (strongly regular) on a semihypergroup $(\mathcal{H}, *)$, then \mathcal{H}/ρ forms a semihypergroup (semigroup) with respect to the operation \otimes defined by $\rho(h) \otimes \rho(k) = \{\rho(x) : x \in h * k\}$, for all $h, k \in \mathcal{H}$. Moreover, if \mathcal{H} is a hypergroup, then, \mathcal{H}/ρ is a hypergroup (group) when ρ is regular (strongly regular) [19]. Suppose, $(\mathcal{H}, *)$ be a semihypergroup and $n \in \mathbb{N}$. For $h, k \in \mathcal{H}$,

$$h\beta_n k \Leftrightarrow \text{there exist } h_1, h_2, \dots, h_n \in \mathcal{H} \text{ such that } \{h, k\} \subseteq h_1 * h_2 * \dots * h_n = \prod_{i=1}^n h_i.$$

If $\bigcup_{n=1}^{\infty} \beta_n = \beta$, then β is a reflexive and symmetric relation. Consider β^* to be the transitive closure of β . β^* is the smallest strongly regular relation on the semihypergroup \mathcal{H} [19]. β and β^* coincide on a hypergroup. For a hypergroup $(\mathcal{H}, *)$, β^* being strongly regular, \mathcal{H}/β^* is a group, called the *fundamental group*. Here, β^* is termed as *fundamental relation* on \mathcal{H} . A nonempty subset X of \mathcal{H} is said to be a *complete part* of \mathcal{H} , if for $n \in \mathbb{N}$ and $h_1, h_2, \dots, h_n \in \mathcal{H}$, $X \cap \prod_{i=1}^n h_i \neq \phi$ implies $\prod_{i=1}^n h_i \subseteq X$. If X is a complete part of \mathcal{H} , then, X^c is also a complete part of \mathcal{H} . For each $h \in \mathcal{H}$, $\beta^*(h)$ is a complete part of \mathcal{H} (by Theorem 2.5.12 [19]). In a hypergroup $(\mathcal{H}, *)$, if $\pi : \mathcal{H} \rightarrow \mathcal{H}/\beta^*$ be the canonical map, then, the *heart* of \mathcal{H} is the set $\Omega_{\mathcal{H}} = \{h \in \mathcal{H} : \pi(h) = 1\}$, where 1 is the identity of the fundamental group \mathcal{H}/β^* . $\Omega_{\mathcal{H}}$ is a complete part of \mathcal{H} [19]. For a nonempty subset X of \mathcal{H} , the *complete closure* $C(X)$ of X is the intersection of all complete parts of \mathcal{H}

containing X . For $X \subseteq \mathcal{H}$, $C(X) = \bigcup_{x \in X} C(x) = \Omega_{\mathcal{H}} * X = X * \Omega_{\mathcal{H}} = \pi^{-1}(\pi(X))$. X is a complete part of \mathcal{H} if and only if $\Omega_{\mathcal{H}} * X = X * \Omega_{\mathcal{H}} = X$. If one of the subsets X, Y of \mathcal{H} is a complete part, then, $X * Y, Y * X$ are also complete parts. $\Omega_{\mathcal{H}}$ is the smallest complete part subhypergroup of \mathcal{H} . If $CPS(\mathcal{H})$ is the class of all complete part subhypergroups of \mathcal{H} , then, $\Omega_{\mathcal{H}} = \bigcap_{G \in CPS(\mathcal{H})} G$. For a homomorphism $g : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, where $\mathcal{H}_1, \mathcal{H}_2$ are hypergroups, the *kernel* of g is defined to be the set $\ker g = \{h \in \mathcal{H} : g(h) \in \Omega_{\mathcal{H}_2}\}$. $\ker g$ is a complete part normal subhypergroup of \mathcal{H}_1 [19]. As in [17], a hypergroup $(\mathcal{H}, *)$ is said to be a *complete hypergroup* if it satisfies one of the following conditions:

- (i) for all $h, k \in \mathcal{H}$, $C(h * k) = h * k$;
- (ii) $\forall (m, n) \in \mathbb{N}^2, 2 \leq m, n, \forall (h_1, h_2, \dots, h_n) \in \mathcal{H}^n, \forall (k_1, k_2, \dots, k_m) \in \mathcal{H}^m,$
 $\prod_{i=1}^n h_i \cap \prod_{j=1}^m k_j \neq \phi \Rightarrow \prod_{i=1}^n h_i = \prod_{j=1}^m k_j$.

As in [17], a hypergroup is called *regular* if it has at least one identity and every element has at least one inverse. A regular hypergroup $(\mathcal{H}, *)$ is said to be *reversible* if for all $h, k, x \in \mathcal{H}$,

$$\begin{aligned} h \in k * x &\Rightarrow \text{there exists } x' \in \mathcal{I}(x) \text{ such that } k \in h * x' \text{ and} \\ h \in x * k &\Rightarrow \text{there exists } x'' \in \mathcal{I}(x) \text{ such that } k \in x'' * h. \end{aligned}$$

If \mathcal{H} is a complete hypergroup, then,

- (i) it is both regular and reversible;
- (ii) $\Omega_{\mathcal{H}}$ is the collection of all identities of \mathcal{H} [17].

A hypergroup \mathcal{H} is said to be *flat* if for any subhypergroup \mathcal{G} of \mathcal{H} , $\Omega_{\mathcal{G}} = \Omega_{\mathcal{H}} \cap \mathcal{G}$. Every complete hypergroup being flat, for any subhypergroup \mathcal{G} of a complete hypergroup \mathcal{H} , $\Omega_{\mathcal{G}} = \Omega_{\mathcal{H}}$. If $g : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a good homomorphism between the complete hypergroups $\mathcal{H}_1, \mathcal{H}_2$, then, $g(\Omega_{\mathcal{H}_1}) = \Omega_{\mathcal{H}_2}$ [19].

Among the different subclasses of hypergroups, the class of polygroups is the important one. A hypergroup $(P, *)$ is a *polygroup* [17] if

- (i) for all $p \in P$, there exists a unique element e , called the *scalar identity* or simply *identity*, such that $p * e = e * p = \{p\}$;

- (ii) for each $p \in P$, there exists a unique element $p^{-1} \in P$, called the *inverse* of p such that $e \in p * p^{-1} \cap p^{-1} * p$;
- (iii) for $p, q, r \in P$, if $p \in q * r$, then, there exist the inverse q^{-1} of q and r^{-1} of r , such that $r \in q^{-1} * p$ and $q \in p * r^{-1}$.

Denote a polygroup by $(P, *, e, {}^{-1})$. Consider the polygroups $(P, *, e_P, {}^{-1})$ and $(Q, \circ, e_Q, {}^{-I})$ and their cartesian product $P \times Q$. Consider a hyperoperation \otimes on $P \times Q$ in the following way:

for $(p_1, q_1), (p_2, q_2) \in P \times Q$,

$$(p_1, q_1) \otimes (p_2, q_2) = \{(p, q) | p \in p_1 * p_2, q \in q_1 \circ q_2\}.$$

Then, $P \times Q$ is a polygroup, called the *direct hyperproduct* of P and Q such that the inverse of (p_1, q_1) is the element (p_1^{-1}, q_1^{-I}) , as in [17]. Let $(P, *, e, {}^{-1})$ be a polygroup and $(\phi \neq) \mathcal{G} \subseteq P$. Then, \mathcal{G} is said to be a *subpolygroup* of P , if (i) for $g, h \in \mathcal{G}$, $g * h \subseteq \mathcal{G}$ and (ii) for $g \in \mathcal{G}$, $g^{-1} \in \mathcal{G}$. A subpolygroup \mathcal{G} is *normal* in P , if for $p \in P$, $p^{-1} * \mathcal{G} * p \subseteq \mathcal{G}$. For a normal subpolygroup \mathcal{G} of a polygroup P , the following results hold [17]:

- (i) $p * \mathcal{G} = \mathcal{G} * p$, for all $p \in P$;
- (ii) $(\mathcal{G} * p) * (\mathcal{G} * q) = \mathcal{G} * p * q$, for all $p, q \in P$;
- (iii) $\mathcal{G} * p = \mathcal{G} * q$, for all $q \in \mathcal{G} * p$.

Moreover, if \mathcal{K} is a subpolygroup of P , then,

- (iv) $\mathcal{G} \cap \mathcal{K}$ is normal in P ;
- (v) $\mathcal{G} * \mathcal{K} = \mathcal{K} * \mathcal{G}$ is a subpolygroup of P ;
- (vi) \mathcal{G} is normal in $\mathcal{G} * \mathcal{K}$.

If \mathcal{G} is normal in P , then, the collection of distinct left or right cosets of \mathcal{G} in P , i.e., $P/\mathcal{G} = \{\mathcal{G} * p : p \in P\}$ forms a polygroup, called the *quotient polygroup* with respect to the hyperoperation \otimes defined as, for $p, q \in P$, $\mathcal{G} * p \otimes \mathcal{G} * q = \{\mathcal{G} * r | r \in p * q\}$, where the inverse of $\mathcal{G} * p$ is the element $\mathcal{G} * p^{-1}$. Let $(P, *, e_P, {}^{-1})$ and $(Q, \circ, e_Q, {}^{-I})$ be polygroups and $g : P \rightarrow Q$ be a map satisfying $g(e_P) = e_Q$. Then, g is said to be (as in [17])

- (i) a *homomorphism* if $g(p * q) \subseteq g(p) \circ g(q)$, for all $p, q \in P$;
- (ii) a *good homomorphism* or *strong homomorphism* if $g(p * q) = g(p) \circ g(q)$, for all $p, q \in P$;
- (iii) an *isomorphism* if g is a bijective good homomorphism.

If there exists such an isomorphism between P and Q , then the polygroups are called *isomorphic* and written as $P \cong Q$. If $g : P \rightarrow Q$ is a homomorphism, then, for each $p \in P$, $g(p^{-1}) = [g(p)]^{-1}$. Moreover, if g is a good homomorphism, then, the kernel of g , $\ker g = \{p \in P : g(p) = e_Q\}$ is a subpolygroup of P , but it is not normal in P in general [17]. If $g : P \rightarrow Q$ is a good homomorphism, then g is injective $\Leftrightarrow \ker g = \{e_P\}$. If in a polygroup $(P, *)$, for each $\alpha \in I$, A_α is a complete part of P , then

- (i) $p * (\bigcap_{\alpha \in I} A_\alpha) = \bigcap_{\alpha \in I} (p * A_\alpha)$ and $(\bigcap_{\alpha \in I} A_\alpha) * p = \bigcap_{\alpha \in I} (A_\alpha * p)$, for all $p \in P$;
- (ii) $X * (\bigcap_{\alpha \in I} A_\alpha) * Y = \bigcap_{\alpha \in I} (X * A_\alpha * Y)$, for nonempty subsets X, Y of P [56].

More information about hypergroup and its sub-classes can be found in [19, 17, 18, 41, 25, 39, 1, 48, 24, 32, 36, 66, 8, 12, 13, 45, 9, 10, 11, 46, 47, 31, 69].

A hyperring in the general sense the largest class of multi-valued systems that satisfies the ring-like axioms. Let \mathfrak{R} be a nonempty set. Then, the triplet $(\mathfrak{R}, +, \cdot)$ is a *hyperring* if $(\mathfrak{R}, +)$ is a hypergroup, (\mathfrak{R}, \cdot) is a semihypergroup, and \cdot is distributive over $+$, i.e., for $p, q, r \in \mathfrak{R}$, $p \cdot (q + r) = p \cdot q + p \cdot r$ and $(p + q) \cdot r = p \cdot r + q \cdot r$. In literature, there are different notions of hyperrings. If the hyperoperation \cdot is a binary operation in the above hyperring $(\mathfrak{R}, +, \cdot)$, then, it is called an *additive hyperring*. A special case of this type of hyperring is the hyperring defined by Krasner [37]. A *Krasner hyperring* is an algebraic structure $(\mathfrak{R}, +, \cdot)$ with the following axioms:

- (i) $(\mathfrak{R}, +)$ is a canonical hypergroup (commutative polygroup), i.e., $+$: $\mathfrak{R} \times \mathfrak{R} \rightarrow \mathcal{P}^*(\mathfrak{R})$ is a map satisfying the following axioms:
 - (a) for all $p, q, r \in \mathfrak{R}$, $p + (q + r) = (p + q) + r$;
 - (b) for all $p, q \in \mathfrak{R}$, $p + q = q + p$;

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- (c) for all $p \in \mathfrak{A}$, there exists $0 \in \mathfrak{A}$, called the *zero* of \mathfrak{A} such that $0+p = \{p\}$;
 - (d) for each $p \in \mathfrak{A}$, there exists a unique element $-p \in \mathfrak{A}$ such that $0 \in p + (-p)$;
 - (e) if $p, q, r \in \mathfrak{A}$ such that $r \in p + q$, then, $q \in -p + r$ and $p \in r - q$.
- (ii) (\mathfrak{A}, \cdot) forms a semigroup, where zero is a bilaterally absorbing element, i.e., for all $p \in \mathfrak{A}$, $0 \cdot p = p \cdot 0 = 0$.
 - (iii) The binary operation ‘ \cdot ’ is distributive over the hyperoperation $+$.

Throughout the thesis, the term ‘hyperring’ will be used to denote a Krasner hyperring. In a hyperring \mathfrak{A} , for $p, q \in \mathfrak{A}$, $-(-p) = p$ and $-(p + q) = -p - q$, where $-A = \{-a : a \in A\}$, for $(\phi \neq)A \subseteq \mathfrak{A}$. Also, for $p, q, r, s \in \mathfrak{A}$, $(p + q) \cdot (r + s) \subseteq p \cdot r + q \cdot r + p \cdot s + q \cdot s$. A Krasner hyperring $(\mathfrak{A}, +, \cdot)$ is *commutative* if the semigroup (\mathfrak{A}, \cdot) is commutative. $(\phi \neq)\mathcal{I} \subseteq \mathfrak{A}$ is said to be a *subhyperring* of \mathfrak{A} if \mathcal{I} is itself a hyperring. The subhyperring \mathcal{I} is *normal* in \mathfrak{A} if and only if $p + \mathcal{I} - p \subseteq \mathcal{I}$, for all $p \in \mathfrak{A}$. In a hyperring $(\mathfrak{A}, +, \cdot)$, a subhyperring \mathcal{I} is said to be a *left (right) hyperideal* of \mathfrak{A} if $r \cdot i \in \mathcal{I}$ ($i \cdot r \in \mathcal{I}$), for all $r \in \mathfrak{A}, i \in \mathcal{I}$, i.e., $\mathfrak{A} \cdot \mathcal{I} \subseteq \mathcal{I}$ ($\mathcal{I} \cdot \mathfrak{A} \subseteq \mathcal{I}$). \mathcal{I} is said to be a *hyperideal* of \mathfrak{A} if it is both left and right hyperideal of \mathfrak{A} . $(\phi \neq)\mathcal{I}$ is a hyperideal of a hyperring $(\mathfrak{A}, +, \cdot)$ if and only if (i) for all $x, y \in \mathcal{I}$, $x - y \subseteq \mathcal{I}$, (ii) for all $r \in \mathfrak{A}, x \in \mathcal{I}$, $r \cdot x \in \mathcal{I}$ and $x \cdot r \in \mathcal{I}$. For a normal hyperideal \mathcal{I} of a hyperring \mathfrak{A} , the following results [19] hold:

- (i) for all $p \in \mathfrak{A}$, $p + \mathcal{I} = \mathcal{I} + p$;
- (ii) for all $p, q \in \mathfrak{A}$, $(p + \mathcal{I}) + (q + \mathcal{I}) = p + q + \mathcal{I}$;
- (iii) if $p, q \in \mathfrak{A}$ and $r \in p + q$, then, $p + q + \mathcal{I} = r + \mathcal{I}$;
- (iv) if $p, q \in \mathfrak{A}$ such that $q \in p + \mathcal{I}$, then, $p + \mathcal{I} = q + \mathcal{I}$.

Let $\mathcal{I}_1, \mathcal{I}_2$ be two hyperideals of a hyperring \mathfrak{A} such that \mathcal{I}_2 is normal in \mathfrak{A} . Then,

- (i) $\mathcal{I}_1 \cap \mathcal{I}_2$ is a normal hyperideal of \mathcal{I}_1 ;
- (ii) \mathcal{I}_2 is a normal hyperideal of $\mathcal{I}_1 + \mathcal{I}_2$.

For a normal hyperideal \mathcal{I} of a hyperring \mathfrak{R} , let's consider a relation \mathcal{I}^* in the following way:

$$p \equiv q \pmod{\mathcal{I}} \Leftrightarrow (p - q) \cap \mathcal{I} \neq \phi.$$

Then, \mathcal{I}^* is an equivalence relation on \mathfrak{R} [19] and for $p, q \in \mathfrak{R}$,

$$(i) \quad \mathcal{I}^*(\mathcal{I}^*(p) + \mathcal{I}^*(q)) = \mathcal{I}^*(p) + \mathcal{I}^*(q);$$

$$(ii) \quad \mathcal{I}^*(\mathcal{I}^*(p \cdot q)) = \mathcal{I}^*(p \cdot q).$$

In this case, for $p \in \mathfrak{R}$, $\mathcal{I}^*(p) = p + \mathcal{I}$, and $[\mathfrak{R} : \mathcal{I}^*]$ denotes the collection of all equivalence classes, i.e., $[\mathfrak{R} : \mathcal{I}^*] = \{\mathcal{I}^*(p) : p \in \mathfrak{R}\}$, which forms a hyperring with respect to the hyperoperation \oplus and multiplication \odot defined in the following way: for $p, q \in \mathfrak{R}$,

$$\mathcal{I}^*(p) \oplus \mathcal{I}^*(q) = \{\mathcal{I}^*(r) : r \in \mathcal{I}^*(p) + \mathcal{I}^*(q)\} \text{ and } \mathcal{I}^*(p) \odot \mathcal{I}^*(q) = \mathcal{I}^*(p \cdot q).$$

As in [50], a map $g : \mathfrak{R} \rightarrow \mathfrak{R}'$, where $(\mathfrak{R}, +, \cdot)$, $(\mathfrak{R}', +', \cdot')$ are hyperrings, is said to be

$$(i) \quad \text{a } \textit{homomorphism} \text{ if } g(p + q) \subseteq g(p) +' g(q) \text{ and } g(p \cdot q) = g(p) \cdot' g(q), \text{ for all } p, q \in \mathfrak{R};$$

$$(ii) \quad \text{a } \textit{good homomorphism} \text{ if } g(p + q) = g(p) +' g(q) \text{ and } g(p \cdot q) = g(p) \cdot' g(q), \text{ for all } p, q \in \mathfrak{R};$$

$$(iii) \quad \text{an } \textit{epimorphism} \text{ if } g \text{ is an onto homomorphism};$$

$$(iv) \quad \text{an } \textit{isomorphism} \text{ if } g \text{ is a bijective good homomorphism.}$$

The hyperrings $(\mathfrak{R}, +, \cdot)$, $(\mathfrak{R}', +', \cdot')$ are said to be isomorphic, and written as $\mathfrak{R} \cong \mathfrak{R}'$ if there exists an isomorphism from \mathfrak{R} onto \mathfrak{R}' . If $g : \mathfrak{R} \rightarrow \mathfrak{R}'$ is an isomorphism, then, $g^{-1} : \mathfrak{R}' \rightarrow \mathfrak{R}$ is an isomorphism. The *kernel* of a homomorphism $g : \mathfrak{R} \rightarrow \mathfrak{R}'$ is defined to be the set $\ker g = \{p \in \mathfrak{R} : g(p) = 0_{\mathfrak{R}'}\}$. The kernel of a homomorphism $g : \mathfrak{R} \rightarrow \mathfrak{R}'$ may be an empty set (Example 1.2 [50]), but, if it is nonempty, then the following results hold:

$$(i) \quad g(0_{\mathfrak{R}}) = 0_{\mathfrak{R}'};$$

- (ii) for all $p \in \mathfrak{A}$, $g(-p) = -g(p)$;
- (iii) $\ker g$ is a hyperideal of \mathfrak{A} ;
- (iv) if g is injective, then $\ker g = \{0_{\mathfrak{A}}\}$;
- (v) g is injective, when g is a good homomorphism and $\ker g = \{0_{\mathfrak{A}}\}$;
- (vi) if g is a good homomorphism, then $g(\mathfrak{A})$ is a subhyperring of \mathfrak{A}' .

Moreover, if g is onto, then, there exists $r \in \mathfrak{A}$ such that $g(r) = 0_{\mathfrak{A}'}$, i.e., $\ker g$ is nonempty. If g is a onto homomorphism between the hyperrings \mathfrak{A} and \mathfrak{A}' and $\psi : \mathfrak{A}/\ker g \rightarrow \mathfrak{A}'$ defined by $\psi(r + \ker g) = g(r)$, for all $r \in \mathfrak{A}$, then, from [50] the following results hold:

- (i) ψ is a homomorphism from $\mathfrak{A}/\ker g$ onto \mathfrak{A}' ;
- (ii) ψ is an isomorphism and $\mathfrak{A}/\ker g \cong \mathfrak{A}'$, when g is a good homomorphism.

Let's recall the definition of complete part in a hyperring. A nonempty subset X of a hyperring \mathfrak{A} is said to be a complete part of \mathfrak{A} if for $n \in \mathbb{N}$ and for all $p_1, p_2, \dots, p_n \in \mathfrak{A}$, $X \cap \sum_{i=1}^n p_i \neq \phi \Rightarrow \sum_{i=1}^n p_i \subseteq X$. For nonempty subsets X, Y of a hyperring \mathfrak{A} with X as a complete part and $p \in \mathfrak{A}$, from [56] the following results hold:

- (i) $-p + p + X = p - p + X = X$;
- (ii) $-X$ is also a complete part of \mathfrak{A} ;
- (iii) $p + X, X + p$ are complete parts of \mathfrak{A} ;
- (iv) $Y \subseteq -p + X \Leftrightarrow p + Y \subseteq X$.

More about hyperrings can be found in [19, 16, 40, 64, 37, 63, 50].

Let's turn into some other issues as follows:

Consider a nonempty set Y and its cartesian product with itself $Y \times Y$. Then, the diagonal of $Y \times Y$ is defined to be the set $\Delta_Y = \{(y, y) : y \in Y\}$. For nonempty sets X, Y, Z , and for $H \subseteq X \times Y, K \subseteq Y \times Z, M \subseteq X$ consider the following:

1. $H \circ K = \{(x, z) \in X \times Z : \exists y \in Y \text{ such that } (x, y) \in X \times Y \text{ and } (y, z) \in Y \times Z\}$;
2. $H^{-1} = \{(y, x) \in Y \times X : (x, y) \in H\}$;
3. $H[M] = \{y \in Y : \exists m \in M \text{ such that } (m, y) \in H\}$.

For $M = \{m\}$, write $H[m]$ instead of $H[\{m\}]$.

Consider a nonempty set X and $\mathcal{U} \subseteq \mathcal{P}(X \times X)$ such that $\mathcal{U} \neq \phi$. Then, \mathcal{U} is said to be a *uniformity* (or a *uniform structure*)[53] on X if the following conditions hold:

- (i) each member of \mathcal{U} contains Δ_X ;
- (ii) if $H \in \mathcal{U}$, then, $H^{-1} \in \mathcal{U}$;
- (iii) if $H \in \mathcal{U}$ and $K \in \mathcal{P}(X \times X)$ such that $H \subseteq K$, then $K \in \mathcal{U}$;
- (iv) for all $H, K \in \mathcal{U}$, $H \cap K \in \mathcal{U}$;
- (v) for each $K \in \mathcal{U}$, there exists $H \in \mathcal{U}$ such that $H \circ H \subseteq K$.

Now, let (X, τ) be a topological space and \mathcal{U} be a uniformity on X , then, \mathcal{U} is called a *compatible uniformity* on X if the topology generated by the uniformity \mathcal{U} on X coincides with τ . The following is an another way to rephrase the notion of compatibility:

For each $U \in \mathcal{U}$ and for each $x \in X$, write $U[x] = \{y \in X : (x, y) \in U\}$. Here, the set $U[x]$ is termed as the *U-ball* centered at x . The uniformity \mathcal{U} on X is compatible with X if for all $U \in \mathcal{U}$ and for all $x \in X$, $U[x]$ is a neighborhood of x in X and the collection $\{U[x] : U \in \mathcal{U}, x \in X\}$ is a neighborhood base for the topology τ .

A set Y in a topological space (X, τ) is called *semi-open* if there exists $U \in \tau$ such that $U \subseteq Y \subseteq \overline{U}$. Let's denote the collection of all semi-open sets in X by $SO(X)$. The arbitrary union of semi-open sets is semi-open, but the intersection of two semi-open sets is not semi-open in general. While taking the intersection of a semi-open set along with an open set, the intersection becomes semi-open. Consider $x \in X$ and $N \subseteq X$. Then, N is said to be a *semi-neighborhood* of x if there exists $M \in SO(X)$

such that $x \in M \subseteq N$. N is called *semi-open* if it is a semi-neighborhood of each of its points. Consider the topological spaces (X_i, τ_i) and $M_i \in SO(X_i)$, for $i \in \{1, 2\}$. Then, $(X_1 \times X_2, \tau_1 \times \tau_2)$ is a topological space and $M_1 \times M_2 \in SO(X_1 \times X_2)$. $Y \subseteq X$ is called *semi-closed* if $Y^c \in SO(X)$; for $Y \subseteq X$, the *semi-closure* of Y is defined to be the set $sCl(Y) = \cap\{A \subseteq X : Y \subseteq A \text{ and } A^c \in SO(X)\}$ [14, 15]. $x \in sCl(Y) \Leftrightarrow$ for every semi-open neighborhood V of x , $Y \cap V \neq \emptyset$. A map $g : X \rightarrow Y$, where X, Y are topological spaces, is said to be *semi-continuous* if for every open subset U of Y , $g^{-1}(U) \in SO(X)$. Equivalently, a mapping $g : X \rightarrow Y$ is semi-continuous \Leftrightarrow for $x \in X$ and open set U of Y containing $g(x)$, there exists $W \in SO(X)$ containing x such that $g(W) \subseteq U$. Separation axioms $semi-T_i$, for $i \in \{0, 1, 2\}$ and s -regularity are analogous to the classical axioms T_i , for $i \in \{0, 1, 2\}$ and regularity, respectively, and can be obtained by replacing everywhere open sets with semi-open sets [23].

Metametric [68] generalizes the concept of metric. It has been widely used in the study of Gromov hyperbolic spaces. A *metametric* on a nonempty set M is a function $\sigma : M \times M \rightarrow [0, \infty)$ satisfying the following conditions:

(MM1) σ is symmetric, i.e., for all $m, n \in M$, $\sigma(m, n) = \sigma(n, m)$;

(MM2) for all $m, n, p \in M$, $\sigma(m, n) \leq \sigma(m, p) + \sigma(p, n)$;

(MM3) $\sigma(m, n) = 0 \Rightarrow m = n$.

In this case, the ordered pair (M, σ) is called a *metametric space*. For $m \in M$ and $\delta > 0$, consider $\mathcal{B}_\sigma(m, \delta) = \{n \in M : \sigma(m, n) < \delta\}$. Every metametric σ on M generates a Hausdorff topology τ_σ in the usual way, i.e., $U \subseteq M$ is open in M if for each $u \in U$, there exists $\delta > 0$ such that $\mathcal{B}_\sigma(u, \delta) \subseteq U$.

Let (X, τ) be a topological space and $\mathcal{P}^*(X)$ denotes the collection of all nonempty subsets of X . For $U \in \tau$, suppose $S_U = \{Y \in \mathcal{P}^*(X) : Y \subseteq U\}$ and $I_U = \{Y \in \mathcal{P}^*(X) : Y \cap U \neq \emptyset\}$. Then, the collections $\{S_U : U \in \tau\}$ is a base for some topology τ_U , called the *upper topology* [30] on $\mathcal{P}^*(X)$, and the collection $\{I_U : U \in \tau\}$ is a subbase for some topology $\tau_{\mathcal{L}}$, called the *lower topology* [30] on $\mathcal{P}^*(X)$.

In 2014, the notion of topological hypergroups [26] was introduced as a generalization of the classical topological group. For a given hypergroup $(\mathcal{H}, *)$ endowed with some topology τ , there is no straightforward way to obtain a topology on $\mathcal{P}^*(\mathcal{H})$. So, while defining a topological hypergroup, the upper topology τ_U [30]

is considered on $\mathcal{P}^*(\mathcal{H})$. A hypergroup $(\mathcal{H}, *)$ together with some topology τ is called a *topological hypergroup* [26] if the hyperoperations $* : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{P}^*(\mathcal{H})$ and $/ : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{P}^*(\mathcal{H})$ are continuous while considering the product topology on $\mathcal{H} \times \mathcal{H}$ and topology $\tau_{\mathcal{U}}$ on $\mathcal{P}^*(\mathcal{H})$. Denote the topological hypergroup by $(\mathcal{H}, *, \tau)$. As in [26], if τ is a topology on a hypergroup $(\mathcal{H}, *)$, then,

- (i) $*$ is continuous \Leftrightarrow for $g, h \in \mathcal{H}$ and $V \in \tau$ satisfying $g * h \subseteq V$, there exist $U, W \in \tau$ containing g, h , respectively such that $U * W \subseteq V$;
- (ii) $/$ is continuous \Leftrightarrow for $g, h \in \mathcal{H}$ and $V \in \tau$ satisfying $g/h \subseteq V$, there exist $U, W \in \tau$ containing g, h , respectively such that $U/W \subseteq V$.

Some results of topological groups do not hold in the new setting in general, e.g., in a topological group G , for an open subset U of G and for $a \in G$, aU is open in G . As in Example 5 [26], the set of real numbers \mathbb{R} equipped with the standard topology forms a topological hypergroup with respect to the hyperoperation \circ defined by: $x \circ y = \{x, y\}$, for all $x, y \in \mathbb{R}$. For the open set $(0, 1)$, $2 \circ (0, 1) = \{2\} \cup (0, 1)$, which is not open in \mathbb{R} .

Consider an equivalence relation ρ on a topological space X . For $x \in X$, $[x]$ is the equivalence class corresponding to x . Let $X/\rho = \{[x] : x \in X\}$ denotes the *quotient space* of X modulo ρ . Let X/ρ be equipped with the topology induced by the projection map $p : X \rightarrow X/\rho$ defined by $p(x) = [x]$, for all $x \in X$, i.e., $\mathcal{A} \subseteq X/\rho$ is open in X/ρ if and only if $p^{-1}(\mathcal{A})$ is open in X . For $Y \subseteq X$, the *saturation* of Y with respect to ρ is the set $\hat{Y} = \{x \in X | \exists y \in Y \text{ such that } x\rho y\}$. If $\hat{Y} = Y$, then, Y is called *saturated* [26]. For the fundamental relation β^* on a topological hypergroup $(\mathcal{H}, *, \tau)$,

- (i) $(\mathcal{H}/\beta^*, \bar{\tau})$ is a topological space, where $\bar{\tau}$ is the quotient topology induced by the natural map $\pi : \mathcal{H} \rightarrow \mathcal{H}/\beta^*$, i.e., $\mathcal{A} \subseteq \mathcal{H}/\beta^*$ is open in \mathcal{H}/β^* if and only if $\pi^{-1}(\mathcal{A}) \in \tau$ [26];
- (ii) every saturated subset of \mathcal{H} is a complete part [26].

If $(\mathcal{H}, *, \tau)$ is a topological hypergroup such that the members of τ are complete parts, then

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- (i) the natural map $\pi : \mathcal{H} \rightarrow \mathcal{H}/\beta^*$ is open [26];
 - (ii) $(\mathcal{H}/\beta^*, \otimes, \bar{\tau})$ is a topological group, where, for all $x, y, z \in \mathcal{H}$, $\beta^*(x) \otimes \beta^*(y) = \beta^*(z)$, when $z \in x * y$ [26];
 - (iii) each $U \in \tau$ can be written as $U = \bigcup_{u \in U} \beta^*(u)$ [27];
 - (iv) \mathcal{H} is a group, when (\mathcal{H}, τ) is a T_0 space [27].

In a topological hypergroup $(\mathcal{H}, *, \tau)$, if every saturated subset belongs to τ , then $(\mathcal{H}/\beta^*, \otimes, \bar{\tau})$ is also a topological group. If β^* is the fundamental relation on a hypergroup $(\mathcal{H}, *)$, then the collection $\{\beta^*(h) : h \in \mathcal{H}\}$ is a base for some topology τ_β on \mathcal{H} such that the members of τ are complete parts [27]. In this case, the topology τ_β is the finest topology on \mathcal{H} such that $(\mathcal{H}, *, \tau_\beta)$ is a topological hypergroup and the members of τ are complete parts [27].

A map between two topological groups is called a *topological isomorphism* [26] if it is a group isomorphism and a topological homeomorphism. For an arbitrary group (G, \cdot) , let $\{A_g : g \in G\}$ denotes the collection of nonempty disjoint sets. Take $H = \bigcup_{g \in G} A_g$ and define for $h, k \in H$, $h * k = A_{g_h g_k}$, when $h \in A_{g_h}$ and $k \in A_{g_k}$, then, $(H, *)$ forms a hypergroup, called the (G, H) -hypergroup [22]. Now, if (G, \cdot, τ) is a topological group, then $(H, *, \tau_H)$ is a topological hypergroup with respect to the topology $\tau_H = \{\bigcup_{v \in V} A_v : V \in \tau\} \cup \{\emptyset\}$. Moreover, the fundamental group of (G, H) -hypergroup H and G are topologically isomorphic [26]. Let H be a non-normal subgroup of a topological group (G, \cdot, τ) . Then, $G/H = \{xH : x \in G\}$ is a topological space with respect to the quotient topology induced by the canonical map $\pi : G \rightarrow G/H$ and every open subset of G/H is of the form $\{uH : u \in U\}$ for $U \in \tau$ [29]. Consider the hyperoperation \circ on G/H defined by $xH \circ yH = \{zH : z \in xHy\}$, for all $xH, yH \in G/H$, then, $(G/H, \circ)$ is a hypergroup [19]. As in [26], if β^* is the fundamental relation of the hypergroup $(G/H, \circ)$, then

- (i) $(G/H)/\beta^*$ is a topological space with respect to the quotient topology induced by the natural map $\pi : G/H \rightarrow (G/H)/\beta^*$, where $\pi(xH) = \beta^*(xH)$, i.e., $A \subseteq (G/H)/\beta^*$ is open in $(G/H)/\beta^*$ if and only if $\pi^{-1}(A) = \{uH : u \in U\}$ for $U \in \tau$;

- (ii) π is an open map;
- (iii) $((G/H)/\beta^*, \otimes)$ is a topological group;
- (iv) there exists a normal subgroup K of G such that the topological groups $(G/H)/\beta^*$ and G/K are topologically isomorphic.

The literature contains another notion of topological hypergroup, called the β -topological hypergroup [34]. Before describing the notion, let's deliver some prerequisites here. Let X be a subset of a hypergroup $(\mathcal{H}, *)$. Then, the β -inverse of X is defined to be the set $X_\beta^{-1} = \{h \in \mathcal{H} : h * x \subseteq \Omega_{\mathcal{H}} \text{ for some } x \in X\}$. For $x \in X$, (i) $\{x_\beta^{-1}\}$ is denoted by x_β^{-1} , (ii) $x * h \subseteq \Omega_{\mathcal{H}} \Rightarrow h * x \subseteq \Omega_{\mathcal{H}}$, when $h \in \mathcal{H}$. For nonempty subsets X, Y of $(\mathcal{H}, *)$, the following results [34] hold:

- (i) $\pi(X_\beta^{-1}) = (\pi(X))^{-1}$. Thus, $X_\beta^{-1} = \pi^{-1}((\pi(X))^{-1})$.
- (ii) $X \subseteq Y \Rightarrow X_\beta^{-1} \subseteq Y_\beta^{-1}$.
- (iii) X_β^{-1} is a complete part of \mathcal{H} .
- (iv) $X \subseteq (X_\beta^{-1})_\beta^{-1}$ and $(X_\beta^{-1})_\beta^{-1} = C(X)$.
- (v) $X_\beta^{-1} = (C(X))_\beta^{-1}$
- (vi) If $X * Y \subseteq \Omega_{\mathcal{H}}$, then $Y * X \subseteq \Omega_{\mathcal{H}}$. Moreover, $X \subseteq Y_\beta^{-1}$ and $Y \subseteq X_\beta^{-1}$.
- (vii) If $\Omega_{\mathcal{H}} \subseteq X * Y$, then $X \cap Y_\beta^{-1} \neq \phi$ and $Y \cap X_\beta^{-1} \neq \phi$.
- (viii) $(X * Y)_\beta^{-1} = Y_\beta^{-1} * X_\beta^{-1}$.

If $(\mathcal{H}, *)$ is a hypergroup and $h \in \mathcal{H}$, then $h * h_\beta^{-1} = h_\beta^{-1} * h = \Omega_{\mathcal{H}}$. A subhypergroup \mathcal{G} of the hypergroup $(\mathcal{H}, *)$ is a complete part \Leftrightarrow for $g \in \mathcal{G}$, $g_\beta^{-1} \subseteq \mathcal{G}$ [34]. A hypergroup $(\mathcal{H}, *)$ endowed with some topology τ is a β -topological hypergroup of type-I [34] if the following conditions hold:

- (i) if $h * k \subseteq U$ for some $U \in \tau$ and $h, k \in \mathcal{H}$, then there exist neighborhoods V_h, V_k of h, k , respectively such that $V_h * V_k \subseteq U$;

- (ii) for each $h \in \mathcal{H}$ and $V \in \tau$ if $h_\beta^{-1} \subseteq V$, then there exists an open subset as well as a complete part W containing h such that $W_\beta^{-1} \subseteq V$.

Let $(\mathcal{H}, *, \tau)$ be a β -topological hypergroup of type-I and $V \in \tau$ be a complete part. Then,

- (i) for each $h \in \mathcal{H}$, $h * V, V * h$ are open complete parts;
- (ii) V_β^{-1} is an open complete part [34].

In a β -topological hypergroup of type-I $(\mathcal{H}, *, \tau)$, for each $V \in \tau$ containing $\Omega_{\mathcal{H}}$, there exists an open complete part W such that $\Omega_{\mathcal{H}} \subseteq W \subseteq V$ [34]. For a β -topological hypergroup of type-I $(\mathcal{H}, *, \tau)$, the fundamental group $(\mathcal{H}/\beta, \otimes)$ is a quasitopological group with respect to the topology $\bar{\tau}$. Moreover, if $(\mathcal{H}/\beta, \bar{\tau})$ is a locally compact Hausdorff space, then $(\mathcal{H}/\beta, \otimes, \bar{\tau})$ is a topological group [34]. A hypergroup $(\mathcal{H}, *)$ endowed with some topology τ is a β -topological hypergroup of type-II [34] if the following conditions hold:

- (i) if $h * k \subseteq U$ for some $U \in \tau$ and $h, k \in \mathcal{H}$, then there exist neighborhoods V_h, V_k of h, k , respectively such that $V_h * V_k \subseteq U$;
- (ii) for each $U \in \tau$, if $\Omega_{\mathcal{H}} \subseteq U$, then there exists $W \in \tau$ such that W is a complete part and $W * W_\beta^{-1} \subseteq U$.

Let $(\mathcal{H}, *, \tau)$ be a β -topological hypergroup of type-II and $V \in \tau$ be a complete part. Then,

- (i) for each $h \in \mathcal{H}$, $h * V, V * h$ are open complete parts;
- (ii) V_β^{-1} is an open complete part [34].

If $(\mathcal{H}, *, \tau)$ is a β -topological hypergroup of type-II, then $(\mathcal{H}/\beta, \otimes, \bar{\tau})$ is a topological group [34]. In a β -topological hypergroup of type-II $(\mathcal{H}, *, \tau)$, if $h, k \in \mathcal{H}$ and $U \in \tau$ such that $h * k \subseteq U$, then there exist open complete parts V_h, V_k containing h, k respectively such that $V_h * V_k \subseteq U$. Every β -topological hypergroup of type-II is a β -topological hypergroup of type-I [34]. Let $(\mathcal{H}, *)$ be a hypergroup endowed with some topology τ . Then,

- (i) \mathcal{H} is a *quasi β -topological hypergroup*, if it is a β -topological hypergroup of type-I;
- (ii) \mathcal{H} is a *β -topological hypergroup*, if it is a β -topological hypergroup of type-II [34].

As in [34], a quasi β -topological hypergroup $(\mathcal{H}, *, \tau)$ is a β -topological hypergroup if and only if for each $U \in \tau$ containing $\Omega_{\mathcal{H}}$ there exist $V, W \in \tau$ containing $\Omega_{\mathcal{H}}$ such that $V * W \subseteq W$. Every quasi β -topological hypergroup with compact heart is a β -topological hypergroup. Let's recall the definition of a scalar element in a hypergroup. An element a in a hypergroup $(\mathcal{H}, *)$ is called *scalar*, if for all $h \in \mathcal{H}$, $a * h$ and $h * a$ are singletons. Denote the collection of all scalars in \mathcal{H} by $S(\mathcal{H})$ and $a * h, h * a$ by $\{a \odot h\}, \{h \odot a\}$ respectively, for $a \in S(\mathcal{H})$ and $h \in \mathcal{H}$. If \mathcal{H} has a scalar, then, $(S(\mathcal{H}), \odot)$ is a group [12]. For a hypergroup $(\mathcal{H}, *)$, the collection $\tau_c = \{A \subseteq \mathcal{H} : A \text{ is a complete part}\} \cup \{\phi\}$ is a topology on \mathcal{H} [66]. For a β -topological hypergroup $(\mathcal{H}, *, \tau)$, $S(\mathcal{H}, \odot)$ is a topological group with respect to the subspace topology induced by the topology $\tau \cap \tau_c$ on \mathcal{H} [34]. Let $(\mathcal{H}, *, \tau)$ be a β -topological hypergroup. Then, $(\mathcal{H}/\beta, \bar{\tau})$ is a Hausdorff space if and only if for all $h \notin \Omega_{\mathcal{H}}$, there exists an open complete part U containing h such that $U \cap \Omega_{\mathcal{H}} = \phi$. Moreover, if \mathcal{H}/β is a Hausdorff space, then $\Omega_{\mathcal{H}}$ is a closed subset of \mathcal{H} . For a β -topological hypergroup $(\mathcal{H}, *, \tau)$, the followings are equivalent:

- (i) π is an open map;
- (ii) if $V \in \tau$, then $C(V) \in \tau$;
- (iii) the inverse of each open set is open;
- (iv) for each $U \in \tau$ and $h \in \mathcal{H}$, if $h_{\beta}^{-1} \cap U \neq \phi$, then there exists an open complete part V containing h such that $V_{\beta}^{-1} \subseteq C(U)$.

Let \mathcal{G} be a subhypergroup of a β -topological hypergroup \mathcal{H} and $\pi : \mathcal{H} \rightarrow \mathcal{H}/\beta$ be an open map. Then, $\bar{\mathcal{G}}$ is a subhypergroup of \mathcal{H} , if \mathcal{G} is a complete part [34]. Let $(\mathcal{H}, *)$ be a hypergroup and (H, τ) be a topological space. Then, \mathcal{H} is a quasi β -topological hypergroup if and only if \mathcal{H} is a topological hypergroup (as defined

in [26]) and the map $i : \mathcal{H} \rightarrow \mathcal{P}^*(\mathcal{H})$ defined by $i(x) = x_\beta^{-1}$, for all $x \in \mathcal{H}$ is continuous, while considering the upper topology $\tau_{\mathcal{U}}$ [30] on $\mathcal{P}^*(\mathcal{H})$ [34].

Let $(P, *, e_P, {}^{-1})$ be a polygroup equipped with topology τ . Then, P is said to be a *topological polygroup* [27], if the maps $*$: $P \times P \rightarrow \mathcal{P}^*(P)$ and ${}^{-1}$: $P \rightarrow P$ are continuous while considering the product topology on $P \times P$ and the upper topology $\tau_{\mathcal{U}}$ [30] on $\mathcal{P}^*(P)$. Denote the topological polygroup by $(P, *, e_P, {}^{-1}, \tau)$. Let P be a topological polygroup with respect to the topology τ , then $*$ is continuous \Leftrightarrow for $p, q \in P$ and $U \in \tau$ satisfying $p * q \subseteq U$, there exist $V, W \in \tau$ containing p, q , respectively such that $V * W \subseteq U$ [27]. In a topological polygroup $(P, *)$, for $p \in P$, the maps $x \mapsto p*x, x \mapsto x*p$ and $x \mapsto p*x*p^{-1}$ from P into $\mathcal{P}^*(P)$ are continuous. A subpolygroup of a topological polygroup inherits the subspace topology and becomes a topological polygroup, such subpolygroup is called *topological subpolygroup* [27]. If in a topological polygroup $(P, *, e, {}^{-1}, \tau)$, the members of τ are complete parts, then,

- (i) for $p \in P, X \subseteq P$, and $V \in \tau$,
 - (a) $X \subseteq p^{-1} * V \Leftrightarrow p * X \subseteq V$;
 - (b) $p * V, V * p, X * V$, and $V * X \in \tau$ [27].
- (ii) for $p \in P$ and $V \in \mathcal{V}(e)$, there exists $W \in \mathcal{V}(e)$ such that $p * W * p^{-1} \subseteq V$ [55].
- (iii) for a basis \mathcal{U} at e , the families $\{x*U : x \in P, U \in \mathcal{U}\}, \{U*x : x \in P, U \in \mathcal{U}\}$ are also basis for P [27].
- (iv) for every $W \in \mathcal{V}(e)$, there exists $U \in \mathcal{V}(e)$ satisfying $\overline{U} \subseteq W$ [27].
- (v) for a given neighborhood U of e , and a compact subset C of P , there exists a neighborhood W of e such that $x * W * x^{-1} \subseteq U$, for all $x \in C$ [27].
- (vi) for a given neighborhood U of e , and a compact subset C of P satisfying $C \subseteq U$, there exists a neighborhood W of e such that $(C * W) \cup (W * C) \subseteq U$ [27].

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- (vii) for a compact subset G and a closed subset C of P satisfying $G \cap C = \phi$, there exists $U \in \mathcal{V}(e)$ such that $(G * U) \cap C = \phi$ and $(U * G) \cap C = \phi$ [55].
 - (viii) for a compact subset G of P and $p \in P$, $p * G$ and $G * p$ are compact. Moreover, if X is a closed subset of P , then $X * G$ and $G * X$ are closed subsets of P [55].
 - (ix) for subsets X, Y of P ,
 - (a) $\overline{X * Y} \subseteq \overline{X * Y}$;
 - (b) $(\overline{X})^{-1} = \overline{(X^{-1})}$ [27].
 - (x) for a nonempty subset A of P ,
 - (a) $\overline{A} = \bigcap_{V \in \mathcal{V}(e)} A * V = \bigcap_{V \in \mathcal{V}(e)} V * A = \bigcap_{U, V \in \mathcal{V}(e)} V * A * U$ [55].
 - (b) $\overline{p * A} = p * \overline{A}$, for all $p \in P$ [56].
 - (xi) if K is
 - (a) a subpolygroup of P , then, \overline{K} is also a subpolygroup of P [27].
 - (b) normal in P , then, \overline{K} is also normal in P [55].
 - (xii) a subpolygroup K of P is open if and only if $K^\circ \neq \phi$ [27].
 - (xiii) open subpolygroups of P are closed [27].
 - (xiv) for a symmetric neighborhood V of e , the set $M = \bigcup_{n=1}^{\infty} V^n$ is an open as well as a closed subpolygroup of P , where $V^2 = V * V$ and $V^n = V^{n-1} * V$, for $n \in \mathbb{N}$ [27].
 - (xv) a subpolygroup K of P is closed \Leftrightarrow there exists $V \in \tau$ such that $V \cap K = V \cap \overline{K} \neq \phi$ [27].
 - (xvi) for a non-closed subpolygroup K of P , $\overline{K} \cap K^c$ is dense in \overline{K} [27].

Consider a topological polygroup $(P, *, e, {}^{-1}, \tau)$ and its normal subpolygroup \mathcal{G} . Then, the quotient polygroup P/\mathcal{G} can be topologized using the canonical map $\pi : P \rightarrow P/\mathcal{G}$ in the following way:

for a subset Y of P , $Y/\mathcal{G} = \{\mathcal{G} * y \mid y \in Y\}$ is open in P/\mathcal{G} if and only if $\pi^{-1}(Y/\mathcal{G})$ is open in P . In this case, the topology is called the *quotient topology* on P/\mathcal{G} and denoted by $\bar{\tau}$. Then, for $X \subseteq P$,

- (i) $\pi^{-1}(\{\mathcal{G} * x : x \in X\}) = \mathcal{G} * X$ and $\pi^{-1}(\pi(X)) = \mathcal{G} * X$;
- (ii) $\{\mathcal{G} * x : x \in X\} = \{\mathcal{G} * p : p \in \mathcal{G} * X\}$.

Now, if the members of τ are complete parts, then,

- (i) π is an open map [27];
- (ii) $(P/\mathcal{G}, \otimes, \mathcal{G}, {}^{-I}, \bar{\tau})$ is a topological polygroup, where $\mathcal{G} * p \otimes \mathcal{G} * q = \{\mathcal{G} * r : r \in p * q\}$ and $(\mathcal{G} * p)^{-I} = \mathcal{G} * p^{-1}$ [27].
- (iii) for $p \in P$, the family $\{\pi(V * p) \mid V \in \mathcal{V}(e)\}$ forms a local base for P/\mathcal{G} centered at $\mathcal{G} * p$ [56].
- (iv) for $V, W \in \mathcal{V}(e)$ satisfying $W * W^{-1} \subseteq V$, $\overline{\pi(W)} \subseteq \pi(V)$ holds [56].
- (v) for $\mathcal{V} \in \mathcal{V}(\mathcal{G})$, there exists $\mathcal{W} \in \mathcal{V}(\mathcal{G})$ such that $\overline{\mathcal{W}} \subseteq \mathcal{V}$ [56].
- (vi) \mathcal{G} is compact $\Rightarrow \pi$ is a perfect map [56].
- (vii) P is compact $\Rightarrow P/\mathcal{G}$ is compact [56].
- (viii) \mathcal{G} and P/\mathcal{G} are compact $\Rightarrow P$ is compact [56].

Consider the topological polygroups $(P_1, *_1, e_1, {}^{-1}, \tau_1)$ and $(P_2, *_2, e_2, {}^{-1}, \tau_2)$. Then, as in [27], a map $f : P_1 \rightarrow P_2$ is a *good topological homomorphism* if for $p, q \in P_1$,

1. $f(e_1) = e_2$;
2. $f(p *_1 q) = f(p) *_2 f(q)$;
3. f is continuous;
4. f is open.

A good topological homomorphism is a *topological isomorphism* if f is bijective. Let's write $P_1 \cong P_2$, when P_1, P_2 are topologically isomorphic.

Let $(P_1, *_1, e_1, ^{-1}, \tau_1), (P_2, *_2, e_2, ^{-1}, \tau_2)$ be topological polygroups such that the members of τ_1 are complete parts and $f : P_1 \rightarrow P_2$ be a homomorphism. Then, the continuity of f on P_1 is equivalent to the continuity of f at e_1 [27]. The isomorphism theorems for topological polygroups (as in [27]) are stated below:

- Let $(P_1, *_1, e_1, ^{-1}, \tau_1), (P_2, *_2, e_2, ^{-1}, \tau_2)$ be topological polygroups such that the members of τ_1 are complete parts and $f : P_1 \rightarrow P_2$ be an open, continuous, good topological homomorphism such that $\ker f$ is a normal subpolygroup of P_1 . Then, $P_1/\ker f \cong P_2$.
- Let K be an open and N be a normal subpolygroup of a topological polygroup $(P, *, e, ^{-1}, \tau)$ such that the members of τ are complete parts. Then, $K/(N \cap K) \cong (N * K)/N$.
- Let K, N be normal subpolygroups of a topological polygroup $(P, *, e, ^{-1}, \tau)$ such that the members of τ are complete parts and $N \subseteq K$. Then, $(P/N)/(K/N) \cong P/K$.

If the open subsets of a topological polygroup are complete parts, then, the open subsets of its any subpolygroup equipped with the subspace topology are also complete parts [56]. Consider a topological space X and $x \in X$. Then, the *character* of X at x is the least cardinality of a local base at x , denoted by $\chi(x, X)$. Let P be a topological polygroup such that the open subsets are complete parts. Then,

- (i) for a dense subspace K of P , $\chi(k, K) = \chi(k, P)$, for all $k \in K$ [56].
- (ii) Q is a first-countable subpolygroup $\Rightarrow \overline{Q}$ is also first-countable subpolygroup [56].

As in [56], a nonempty subset X of a topological polygroup P is said to be *cp-resolvable* if there exist two disjoint dense subsets of X such that at least one of them is a complete part of P . Let $(P, *, \tau)$ is a topological polygroup such that the members of τ are complete parts. Then, the following results [56] hold:

- (i) If K is a cp-resolvable subpolygroup of P and $p \in P$, then $p*K$ is cp-resolvable.

- (ii) If $\{P_\alpha : \alpha \in I\}$ is a pairwise disjoint family of nonempty subsets of P such that each P_α is cp-resolvable and $P = \bigcup_{\alpha \in I} P_\alpha$, then, P is cp-resolvable.
- (iii) If K is a proper dense subpolygroup of P , which is also a complete part, then, P is cp-resolvable.
- (iv) A normal subpolygroup of P is cp-resolvable $\Rightarrow P$ is cp-resolvable.
- (v) If P contains a non-closed normal subpolygroup which is also a complete part, then, P is cp-resolvable.

A nonempty subset X of a hypergroupoid $(H, *)$ is called *left (right) big* if there exists a finite subset F of H such that $H = F * X(X * F)$. If $H = F * X = X * F$, then X is called *big*. In a polygroup $(P, *, e, {}^{-1})$, the following conditions hold:

- (i) $X \subseteq P$ is left big $\Rightarrow X^{-1}$ is right big;
- (ii) If Y is left big and $Y \subseteq X \subseteq P$, then X is left big;
- (iii) If X is a left big subset as well as a complete part of P and K is a subpolygroup of P , then $(X^{-1} * X) \cap K$ is a left big subset of K [55].

Let P_1, P_2 be polygroups and $f : P_1 \rightarrow P_2$ be a surjective good homomorphism. Then,

- (i) for a left big subset Y of P_2 , $f^{-1}(Y)$ is a left big subset of P_1 ;
- (ii) for a left big subset X of P_1 , $f(X)$ is a left big subset of P_2 .

A topological polygroup P is *totally bounded* if every nonempty open subset of P is left (right) big [55]. Let $(P, *, e, {}^{-1}, \tau)$ be a totally bounded topological polygroup such that the members of τ are complete parts, then for $V \in \mathcal{V}(e)$, there exists $W \in \mathcal{V}(e)$ such that for every $p \in P$, $p * W * p^{-1} \subseteq V$ [55]. If \mathcal{U} is a base of a topological polygroup $(P, *, e, {}^{-1}, \tau)$, where the members of τ are complete parts, then P is totally bounded \Leftrightarrow every element of \mathcal{U} is left big [55]. If P is a topological polygroup such that the open subsets are complete parts, then,

- (i) P is totally bounded \Rightarrow every subpolygroup of P is also totally bounded;

- (ii) the existence of a dense and totally bounded subpolygroup of P implies P is totally bounded [55].

A topological polygroup P is not totally bounded \Leftrightarrow there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of P and a nonempty open set V such that $(x_n * V) \cap (V * x_m) = \phi$, for every $m \neq n$ [55]. Let $(P_1, *_1, e_1, ^{-1}, \tau_1)$, $(P_2, *_2, e_2, ^{-1}, \tau_2)$ be topological polygroups and $f : P_1 \rightarrow P_2$ be a continuous surjective good homomorphism. Then, the following results hold:

- (i) P_1 is totally bounded $\Rightarrow P_2$ is totally bounded;
- (ii) If $\mathcal{B} = \{f^{-1}(U) : U \in \tau_2\}$ is a base for the topological space (P_1, τ_1) and P_2 is totally bounded, then P_1 is totally bounded [55].

Let $(P, *, e, ^{-1})$ be a polygroup endowed with some topology τ . Then, the system $(P, *, e, ^{-1}, \tau)$ is called a *paratopological polygroup* [33] if the hyperoperation $* : P \times P \rightarrow \mathcal{P}^*(P)$ is continuous, while considering the upper topology $\tau_{\mathcal{U}}$ [30] on $\mathcal{P}^*(P)$. If the unitary operation $^{-1} : P \rightarrow P$ is continuous, then P is a topological polygroup. Consider a hypergroupoid $(H, *)$ equipped with a topology τ , then the hyperoperation $*$ is τ -closed [33] if for all $p, q \in H$, $p * q$ is a closed subset of H . Let $(P, *, e, ^{-1}, \tau)$ be a paratopological polygroup such that (P, τ) is a regular space and $*$ is τ -closed. Then,

- (i) the graph of the map $^{-1} : P \rightarrow P$ is a closed subset of $P \times P$;
- (ii) $^{-1} : P \rightarrow P$ is continuous, if P is compact;
- (iii) $^{-1} : P \rightarrow P$ is continuous, if P is sequential countably compact [33].

Let $(P, *, e, ^{-1}, \tau)$ be a paratopological polygroup such that the members of τ are complete parts. Then, the following results [33] hold:

- (i) The inversion is continuous if and only if it is continuous at e .
- (ii) If $V, W \in \mathcal{V}(e)$ satisfying $W^2 \subseteq V$, then $(\overline{W^{-1}})^{-1} \subseteq V$.
- (iii) If P is countably compact, then P is a topological polygroup.

If (P, τ) is a countably compact topological polygroup such that the members of τ are complete parts, then, P is a regular space [33]. More literature about topological hypergroup and its different subclasses can be found in [3, 26, 27, 33, 34, 55, 56].