CHAPTER - II

ELASTIC STABILITY OF A TRIANGULAR PLATE PLACED ON ELASTIC FOUNDATION.*

PAPER - I

Introduction :

Stability of thin elastic triangular plates can be investigated with the help of finite difference method. In this approximate method the differential equation is replaced by its finite difference approximation, rather than satisfying the differential equation and the boundary conditions at every point. The difference equation may be satisfied at the node points of a superposed triangular grid, as has been done by Bradley (1963). Salvadori (1951), Weingarten (1957), Timoshenko and Gere (1961) solved these types of problems by other numerical methods. To obtain accurate results by these methods, high order matrices are involved for which the solution is time consuming. Laura and Shahady (1969) have shown that complex variable method may be used for solution of such problems with less labour.

In this paper the complex variable theory has been applied to investigate the stability of a thin elastic equilateral triangular plate placed on an elastic foundation with clamped edges, and under the action of uniform compression parallel to one of the edges. The given domain is

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conformally transformed onto the unit circle and solution has been obtained with the help of error function. The foundation is assumed to be of the Winkler type.

Theory :

The differential equation, assuming small deflections, for the middle plane of a thin plate with no lateral loading but with force resultants in the middle plane is (cf. Timoshenko and Krieger, 1959)

$$\nabla^4 w = \frac{1}{D}\left(N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y}\right) \qquad \ldots (2.1)$$

in which w denotes the deflection of the middle plane of the plate ; N_x, N_y are the normal forces per unit length in the middle plane of the plate, N_{xy} is the shear force per unit length in the middle plane of the plate ; D, the plate constant, is defined by

$$D = \frac{E h^3}{12(1-\nu^2)} \qquad \ldots (2.2)$$

with E denoting the modulus of elasticity, h referring to the thickness, and ν representing Poisson's ratio ; and the biharmonic operator ∇^4 is given by

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

For a plate, shown in Fig. 2.1, with a uniform compression parallel to the base, eq. (2.1) becomes,

$$\nabla^4 w + \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2} = 0 \qquad \ldots (2.3)$$

In the case of a clamped edge boundary

$$W = 0 = \frac{\partial W}{\partial \eta}, \qquad \text{on the boundary}$$ [cf. Timoshenko and Krieger (1959), p. 87]

If $z = x + iy$, $\bar{z} = x - iy$, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \quad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

so that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and therefore

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

Thus eq.(2.3) changes into

$$16\frac{\partial^4 W}{\partial z^2 \partial \bar{z}^2} + \frac{N_x}{D}\left(\frac{\partial^2 W}{\partial z^2} + 2\frac{\partial^2 W}{\partial z \partial \bar{z}} + \frac{\partial^2 W}{\partial \bar{z}^2}\right) = 0 \qquad \ldots (2.4)$$

Let $\quad z = f(\xi) \qquad \ldots (2.4a)$

be the analytic function which maps the given shape in the z - plane onto a unit circle in the ξ - plane. Let the plate be placed on an elastic foundation having the foundation reaction k_1 per unit area per unit deflection. Substitution of eq. (2.4a) into eq. (2.4) yields

$$\frac{\partial^4 W}{\partial \xi^2 \partial \bar{\xi}^2}\cdot\frac{dz}{d\xi}\cdot\frac{d\bar{z}}{d\bar{\xi}} - \frac{\partial^3 W}{\partial \bar{\xi}^2 \partial \xi}\cdot\frac{d^2 z}{d\xi^2}\cdot\frac{d\bar{z}}{d\bar{\xi}} - \frac{\partial^3 W}{\partial \xi^2 \partial \bar{\xi}}\cdot\frac{d^2 \bar{z}}{d\bar{\xi}}\cdot\frac{dz}{d\xi} +$$

$$+ \frac{\partial^2 W}{\partial \xi \partial \bar{\xi}}\cdot\frac{d^2 z}{d\xi^2}\cdot\frac{d^2 \bar{z}}{d\bar{\xi}^2} + \frac{N_x}{D}\left[-\frac{\partial W}{\partial \xi}\cdot\frac{d^2 z}{d\xi^2}\cdot\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3 +\right.$$

$$+ \frac{\partial^2 W}{\partial \xi^2}\cdot\frac{dz}{d\xi}\cdot\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3 - \frac{\partial W}{\partial \bar{\xi}}\cdot\frac{d^2 \bar{z}}{d\bar{\xi}^2}\cdot\left(\frac{dz}{d\xi}\right)^3 +$$

$$+ \frac{\partial^2 W}{\partial \bar{\xi}^2}\left(\frac{dz}{d\xi}\right)^3 \frac{d\bar{z}}{d\bar{\xi}} + 2\frac{\partial^2 W}{\partial \xi \partial \bar{\xi}}\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2 \Bigg] +$$

$$+ \frac{K_1 W}{16D}\left(\frac{dz}{d\xi}\right)^3\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3 = 0 \qquad \text{... (2.5)}$$

in which

$\xi = re^{i\theta}$, $\bar{\xi} = re^{-i\theta}$, r being the radius of the circle.

The transformed boundary conditions in the ξ - plane are

$$W(\xi,\bar{\xi})\Big|_{r=1} = \frac{\partial W(\xi,\bar{\xi})}{\partial r}\Big|_{r=1} = 0 \qquad \text{... (2.6)}$$

Method of solution :

Since an exact solution of eq. (2.5) is, at best, very difficult, it is convenient to use an approximate method to solve it. Galerkin's method is used in this study. The procedure is as follows : $W(\xi,\bar{\xi})$ is approximated by a linear combination of independent coordinate functions which identically satisfy the boundary conditions, i.e.

$$W(\xi,\bar{\xi}) \simeq \sum_{n=1}^{K} B_n \chi_n(\xi,\bar{\xi}) \qquad \text{... (2.7a)}$$

Substituting eq. (2.7a) into eq. (2.5) results in an expression which does not vanish in general since eq.(2.7a) is not an exact solution of the partial differential equation. This nonvanishing expression is usually defined as the "error" or "residual function" and will be denoted by $\epsilon_n(\xi,\bar{\xi})$. Galerkin's method requires that the "error function" be orthogonal with respect to each co-ordinate function over the domain under consideration, i.e.,

$$\int_c \epsilon_n(\xi,\bar{\xi})\, X_n(\xi,\bar{\xi})\, dc = 0 \qquad (\eta = 1,2,3,\ldots k) \qquad \ldots (2.7b)$$

Eq. (2.7b) generates a (K x K) determinantal equation. The lowest root of this equation is the critical buckling coefficient. In the case of a clamped plate it is convenient to take

$$W(\xi,\bar{\xi}) \simeq \sum_{\eta=1}^{K} B_n \left[1-(\xi\bar{\xi})^{\eta}\right]^2 \qquad \ldots (2.7c)$$

This form of w clearly satisfies the clamped edge boundary conditions in eq. (2.6).

Application :

In order to illustrate the procedure for the determination of the critical buckling condition, an equilateral triangular plate with clamped edges, shown in Fig. 2.1, with a uniform compression parallel to one of the sides is considered.

The mapping function is given by

$$z = f(\xi) = 1.1352\,a\left[\xi + \frac{1}{6}\xi^4 + \frac{5}{63}\xi^7 + \frac{4}{81}\xi^{10}\right] \qquad \ldots (2.8)$$

For first approximation, let

$K=1$ i.e., $W = B_1(1-\xi\bar{\xi})^2$ ... (2.9)

With the mapping function in eq.(2.8), putting eq.(2.9) in eq.(2.5) one gets the following error function

$$
\begin{aligned}
\epsilon_1(\zeta,\bar{\zeta}) = {} & 4B_1(1.1352\,a)^2(\phi\psi - \zeta\phi_1\psi - \bar{\zeta}\psi_1\phi) + \\
& + 2B_1(1.1352\,a)^2(2\zeta\bar{\zeta}-1)\phi_1\psi_1 + \\
& + \frac{B_1 N_x}{16D}\Big[2(1.1352a)^4\{(\bar{\zeta}-\zeta\bar{\zeta}^2)\phi_1\psi^3 + \\
& + (\zeta-\zeta^2\bar{\zeta})\psi_1\phi^3 + \bar{\zeta}^2\phi\psi^3 + \zeta^2\phi^3\psi + \\
& + 2(2\zeta\bar{\zeta}-1)\phi^2\psi^2\}\Big] + \\
& + \frac{B_1(1.1352a)^6 K_1}{16D}(1-\zeta\bar{\zeta})^2\phi^3\psi^3
\end{aligned}
\qquad \ldots (2.10)
$$

where

$$\phi = 1 + \frac{2}{3}\zeta^3 + \frac{5}{9}\zeta^6 + \frac{40}{81}\zeta^9$$

$$\psi = 1 + \frac{2}{3}\bar{\zeta}^3 + \frac{5}{9}\bar{\zeta}^6 + \frac{40}{81}\bar{\zeta}^9$$

$$\phi_1 = 2\zeta^2 + \frac{10}{3}\zeta^5 + \frac{40}{9}\zeta^8$$

$$\psi_1 = 2\bar{\zeta}^2 + \frac{10}{3}\bar{\zeta}^5 + \frac{40}{9}\bar{\zeta}^8$$

Let us now follow Galerkin's procedure given in eq.(2.7b).

Multiplying eq.(2.10) by $(1-\zeta\bar{\zeta})^2 r\,d\theta\,dr$ and integrating between the limits 0 to 2π and 0 to 1, the critical buckling condition is obtained as

$$\frac{N_{cr}a^2}{D} = 20.96 + 1.79\frac{K_1 a^4}{D} \qquad \ldots (2.11)$$

In terms of the altitude of the plate, eq.(2.11) gives the critical buckling condition as

$$\frac{N_{cr} a^2}{D} = 188{\cdot}64 + 16{\cdot}11 \frac{k_1 a^4}{D} \qquad \ldots (2.12)$$

If the plate is considered without resting on the elastic foundation,

$$N_{cr} = 188{\cdot}64 \frac{D}{a^2} \qquad \ldots (2.13)$$

Eq.(2.13) is in excellent agreement with the corresponding result obtained by Bradley [cf. Bradley (1963) P. 55] as

$$N_{cr} = 189{\cdot}0 \frac{D}{a^2} \qquad \ldots (2.14)$$

C O N C L U S I O N S

The analytical procedure presented is straightforward. The same procedure may be followed for a unified treatment of elastic instability problems involving complicated boundary shape, such as regular polygonal shape, circular boundary with flat sides, epitrochoidal boundary etc., since the co-ordinate function which satisfies the given boundary conditions will be the same for any shape. The co-ordinate function used in this study satisfying the boundary conditions is simple. Only the first term approximation is found to yield fairly accurate result and thus minimises labour. No other form of w can reduce the error so greatly with such minimum labour.

It must also be noted that the accuracy of the results obtained by different numerical techniques applied to boundary value problems involving irregular shapes can be verified by the conformal mapping technique exhibited in this study.

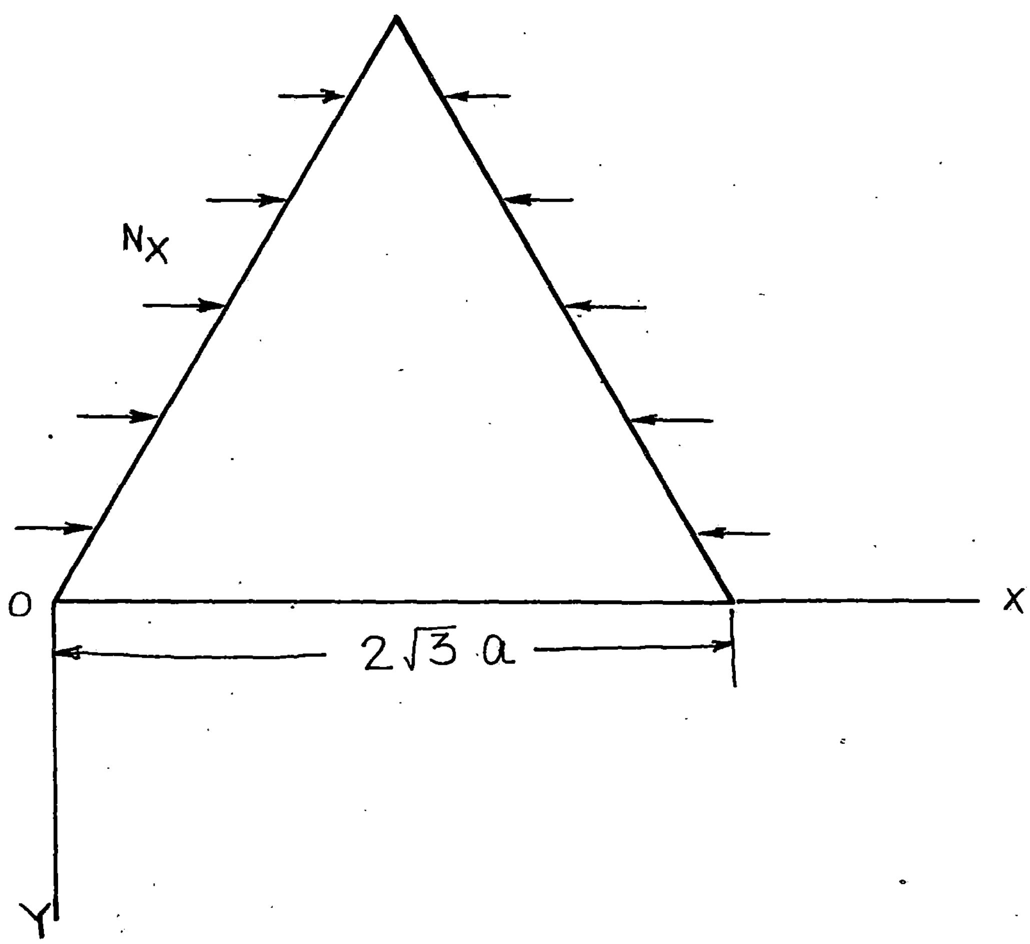

FIG.2·1 EQUILATERAL TRIANGULAR PLATE.

BUCKLING OF A NON-HOMOGENEOUS RECTANGULAR PLATE PLACED ON ELASTIC FOUNDATION*

PAPER - II

Introduction :

Critical buckling conditions of homogeneous thin rectangular plates subjected to combined bending and compression were investigated by Timoshenko and Gere (1961), Johnson and Noel (1953) and many other investigators. The object of this paper is to use error function to obtain the approximate solutions in the case of buckling of a non-homogeneous thin rectangular plate under the action of combined bending and compression in the middle plane of the plate. The plate is placed on an elastic foundation and is simply supported. Bradley (1963) used finite difference approximations to the governing differential equations to investigate stability of equilateral triangular plates. There are other numerical methods for the solutions of these types of buckling problems. But these methods are time consuming.

Since the governing differential equation obtained in this paper cannot be exactly solved, error function and Galerkin's method have been utilised to obtain an approximate solution of the differential equation. It is observed that the results obtained from this

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method for the homogeneous plate not resting on foundation are in good agreement with the known results obtained by strain energy method. Flexural rigidity of the plate is assumed to vary exponentially and the foundation is taken of the Winkler type. Results obtained have been presented in the form of graphs.

Analysis :

Consider a simply supported rectangular plate of varying flexural rigidity and along whose sides $x = 0$ and $x = a$ (Fig.2.2) distributed forces, acting in the middle plane of the plate, are applied, their intensity being given by the equation

$$N_x = N_0\left(1-\alpha\frac{y}{b}\right) \qquad \ldots (2.15)$$

where N_0 is the intensity of compressive force at edge $y = 0$ and α is a numerical factor. The plate is placed on an elastic foundation having the reaction, K_1 per unit area per unit deflection and is subjected to a uniform transverse load, q.

The governing differential equation of equilibrium of an element of the plate not resting on foundation is [Timoshenko and Krieger (1959) P. 379]

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\left(q + N_x\frac{\partial^2 w}{\partial x^2}\right) \qquad \ldots (2.16)$$

where

$$\left.\begin{aligned} M_x &= -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\\ M_y &= -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)\\ M_{xy} &= D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}\end{aligned}\right\} \qquad \ldots (2.17)$$

Substituting Eqs. (2.15) and (2.17) in Eq.(2.16) and observing that the flexural rigidity is a function of the co-ordinates x and y, one gets the differential equation of equilibrium for a plate resting on elastic foundation in the following form

$$\nabla^2(D\nabla^2 W) - (1-\nu)\left\{\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 D}{\partial x^2} - 2\frac{\partial^2 D}{\partial x\partial y}\cdot\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 D}{\partial y^2}\cdot\frac{\partial^2 w}{\partial x^2}\right\} +$$

$$+ K_1 W = q + N_0\left(1-\alpha\frac{y}{b}\right)\frac{\partial^2 w}{\partial x^2} \qquad \text{... (2.18)}$$

For simply supported edges the deflection can be represented by the double series

$$W = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \text{... (2.19)}$$

As the flexural rigidity is variable, let

$$D = D_0 e^{-2\alpha_1\frac{x}{a}} \qquad \text{... (2.20)}$$

where D_0 and α_1 are constants.

Eq. (2.19) is an approximate solution of Eq. (2.18) and therefore substitution of Eq.(2.19) into Eq.(2.18) results the following error function,

$$E_{(x,y)} = C_{mn} D_0\left[\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}^2 - \frac{4\alpha_1^2}{a^2}\left\{\left(\frac{m\pi}{a}\right)^2 + \nu\left(\frac{n\pi}{b}\right)^2\right\}\right] \times$$

$$\times e^{-2\alpha_1\frac{x}{a}} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} +$$

$$+ C_{mn} K_1 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} - q -$$

$$- C_{mn} N_0\left(\frac{m\pi}{a}\right)^2\left(1-\alpha\frac{y}{b}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \text{... (2.21)}$$

According to Galerkin's principle, the following conditions are imposed on the error function, $E(x,y)$

$$\int_0^a \int_0^b E_{(x,y)} W_{(x,y)} \, dx \, dy = 0 \qquad \dots (2.22)$$

Substituting Eq.(2.19) into Eq.(2.22) and observing that

$$\int_0^b y \sin\frac{i\pi y}{b} \sin\frac{j\pi y}{b} \, dy = \frac{b^2}{4} \quad \text{for } i = j$$

$= 0$ for $i \neq j$ and $i \pm j$ an even number

$= -\frac{4b^2}{\pi^2} \frac{ij}{(i^2 - j^2)^2}$ for $i \neq j$ and $i \pm j$ an odd number

One gets the following

$$\frac{\pi^6}{2\alpha_1}\left(1-e^{-2\alpha_1}\right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \left[\pi^2\left(m^2+n^2\frac{a^2}{b^2}\right)^2 - 4\alpha_1^2\left(m^2+\nu n^2\frac{a^2}{b^2}\right)\right] \times$$

$$\times \frac{m^2}{(\pi^2 m^2+\alpha_1^2)} + \pi^2 K_F \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} - \frac{16 q a^4}{D_0} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} -$$

$$- \frac{N_0}{D_0} \pi^4 a^2 \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 C_{mn} - \frac{\alpha}{2} \sum_{m=1}^{\infty} m^2 \left\{\sum_{n=1}^{\infty} C_{mn} - \frac{16}{\pi^2} \sum_{n=1}^{\infty} \sum_{i}^{\infty} \frac{C_{mi} n i}{(n^2-i^2)^2}\right\}\right]$$

$$= 0 \qquad \dots (2.23)$$

Where the nondimensional foundation modulus,

$$K_F = \frac{K_1 a^4}{D_0} \quad \text{and} \quad n \pm i \quad \text{is always odd.}$$

Taking $n = i$, the deflection, W is obtained from Eq.(2.23)

$$W = \frac{16 q a^4}{D_0} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \times$$

$$\times 1 \Bigg/ \left[\frac{\pi^6(1-e^{-2\alpha_1})}{2\alpha_1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\pi^2\left(m^2+n^2\frac{a^2}{b^2}\right)^2 - 4\alpha_1^2\left(m^2+\nu n^2\frac{a^2}{b^2}\right)\right] \times \right.$$

$$\left. \times \frac{m^2}{(\pi^2 m^2+\alpha_1^2)} + \pi^2 K_F - \frac{N_0 \pi^4 a^2}{D_0} \left\{ \sum_{m=1}^{\infty} m^2\left(1-\frac{\alpha}{2}\right) \right\} \right]$$

... (2.24)

From Eq. (2.23) the critical buckling condition is obtained when

$$\frac{N_0 a^2}{D_0} \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 C_{mn} - \frac{\alpha}{2} \sum_{m=1}^{\infty} m^2 \left\{ \sum_{n=1}^{\infty} C_{mn} - \frac{16}{\pi^2} \sum_{n=1}^{\infty} \sum_{i}^{\infty} \frac{C_{mi}\, ni}{(n^2-i^2)^2} \right\} \right]$$

$$= \frac{\pi^2(1-e^{-2\alpha_1})}{2\alpha_1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \left[\pi^2\left(m^2+n^2\frac{a^2}{b^2}\right)^2 - 4\alpha_1^2\left(m^2+\nu n^2\frac{a^2}{b^2}\right)\right] \times$$

$$\times \frac{m^2}{(\pi^2 m^2+\alpha_1^2)} + \frac{1}{\pi^2} K_F \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}$$

... (2.25)

The plate may buckle in such a way that there can be several half-waves in the direction of compression but only one half-wave in the perpendicular direction. For one half wave buckling m = 1, for two half-waves buckling m = 2 and so on.

If the plate buckles in one half-wave, one gets from Eq.(2.25) by taking m = 1 a system of equations of the following kind :

$$c_{1n}\left[\frac{(1-e^{-2\alpha_1})}{2\alpha_1(\pi^2+\alpha_1^2)}\left\{\pi^2\left(1+n^2\frac{a^2}{b^2}\right)^2-4\alpha_1^2\left(1+\nu\frac{a^2n^2}{b^2}\right)\right\}+\frac{K_F}{\pi^4}-\right.$$

$$\left.-\sigma_{cr}\frac{a^2h}{\pi^2D_0}\left(1-\frac{\alpha}{2}\right)\right]-8\alpha\sigma_{cr}\frac{a^2h}{\pi^4D_0}\sum_{i}^{\infty}\frac{c_{1i}\,ni}{(n^2-i^2)^2}=0 \qquad \ldots (2.26)$$

where $\sigma_{cr}=\dfrac{(N_0)_{cr}}{h}$, h being the plate thickness.

The lowest root of the determinantal equation thus formed will determine the critical buckling load. From the first approximate lowest root one gets by taking n = 1

$$\sigma_{cr}=K\frac{\pi^2D_0}{b^2h} \qquad \ldots (2.27)$$

where

$$K=\frac{1}{1-\frac{\alpha}{2}}\left[\frac{1-e^{-2\alpha_1}}{2\alpha_1(\pi^2+\alpha_1^2)}\left\{\pi^2\left(\frac{b}{a}+\frac{a}{b}\right)^2-4\alpha_1^2\left(\frac{b^2}{a^2}+\nu\right)\right\}+\frac{K_F}{\pi^4}\cdot\frac{b^2}{a^2}\right]$$

Thus the buckling load is a function of $\frac{a}{b}$ and the foundation modulus, K_F.

For $\alpha = 0$, the critical buckling load σ_{cr} is obtained from Eq.(2.25) by taking n = 1

$$\sigma_{cr}=K\frac{\pi^2D_0}{b^2h} \qquad \ldots (2.28)$$

where

$$K = \frac{1-e^{-2\alpha_1}}{2\alpha_1(\pi^2 m^2+\alpha_1^2)}\left\{\pi^2\left(m^2\frac{b}{a}+\frac{a}{b}\right)^2 - 4\alpha_1^2\left(m^2\frac{b^2}{a^2}+\nu\right)\right\} +$$

$$+ \frac{K_F}{\pi^4}\cdot\frac{1}{m^2}\cdot\frac{b^2}{a^2} \qquad (m = 1, 2, 3, \ldots\ldots) \qquad \ldots (2.29)$$

For homogeneous material, $D_0 \to D$ when $\alpha_1 \to 0$. Setting $\alpha_1 \to 0$ in Eq.(2.27) one gets the critical buckling load for a homogeneous plate on elastic foundation for one half-wave buckling

$$\sigma_{cr} = K\frac{\pi^2 D}{b^2 h} \qquad \ldots (2.30)$$

where

$$K = \frac{1}{1-\alpha/2}\left[\left(\frac{b}{a}+\frac{a}{b}\right)^2 + \frac{K_F}{\pi^4}\cdot\frac{b^2}{a^2}\right]$$

For $K_F = 0$, Eq.(2.30) is the result obtained by Timoshenko and Gere. For m half-waves buckling, K in Eq.(2.30) can be expressed as

$$K = \frac{1}{m^2}\left(m^2\frac{b}{a}+\frac{a}{b}\right)^2 + \frac{K_F}{\pi^4}\cdot\frac{1}{m^2}\cdot\frac{b^2}{a^2} \qquad \ldots (2.31)$$

The ratio $\frac{a}{b}$ for which σ_{cr} becomes a minimum for uniform compression is obtained from Eq.(2.29) and denoting this ratio by $\left(\frac{a}{b}\right)_{cr}$, one gets for a homogeneous plate

$$\left(\frac{a}{b}\right)_{cr} = \frac{1}{\pi}\left(K_F + \pi^4 m^4\right)^{1/4} \qquad \ldots (2.32)$$

and for a nonhomogeneous plate

$$\left(\frac{a}{b}\right)_{cr} = \left[m^4 + \frac{K_F}{\pi^6 m^2}\cdot\frac{2\alpha_1(\pi^2 m^2+\alpha_1^2)}{(1-e^{-2\alpha_1})} - \frac{4\alpha_1^2 m^2}{\pi^2}\right]^{1/4} \qquad \ldots (2.33)$$

The ratio $\frac{a}{b}$ at which the transition from m to m+1 half-waves buckling occur can also be computed from Eq. (2.29). For homogeneous plate under uniform compression, transition from one to two half-waves occurs when

$$\frac{a}{b} = \left(4 - \frac{K_F}{\pi^4}\right)^{1/4}$$

and transition from two to three half-waves occurs when

$$\frac{a}{b} = \left(36 - \frac{K_F}{\pi^4}\right)^{1/4}$$

Eq.(2.27) gives satisfactory results for small values of α. An improved result is obtained by taking two equations of the system Eq.(2.26) with coefficients C_{11} and C_{12} and setting the determinant equal to zero. Thus for one half-wave buckling

$$\sigma_{cr} = K\frac{\pi^2 D_0}{b^2 h} \qquad \ldots (2.34)$$

where

$$K = \left[\frac{1}{\{\cdot 0065\alpha^2 - \cdot 2(1-\alpha/2)^2\}} \times \left\{\cdot 01(1-\alpha/2)^2\left(B + \frac{2k_F}{\pi^4}\right)^2 + \right.\right.$$
$$\left. + \left[\cdot 0013\alpha^2 - \cdot 041(1-\alpha/2)^2\right] \times \left[A + B\frac{k_F}{\pi^4} + \frac{k_F^2}{\pi^8}\right]\right\}^{1/2} -$$
$$\left. - \frac{(1-\alpha/2)\left(B + \frac{2k_F}{\pi^4}\right)}{\cdot 064\alpha^2 - 2(1-\alpha/2)^2}\right] \times \frac{b^2}{a^2}$$

and

$$A = \left\{\frac{1-e^{-2\alpha_1}}{2\alpha_1(\pi^2+\alpha_1^2)}\right\}^2 \left\{\pi^2\left(1+\frac{a^2}{b^2}\right)^2 - 4\alpha_1^2\left(1+\nu\frac{a^2}{b^2}\right)\right\}\left\{\pi^2\left(1+\frac{4a^2}{b^2}\right)^2 - \right.$$
$$\left. - 4\alpha_1^2\left(1+4\nu\frac{a^2}{b^2}\right)\right\}$$

$$B = \frac{1-e^{-2\alpha_1}}{2\alpha_1(\pi^2+\alpha_1^2)}\left\{\pi^2\left(2+10\frac{a^2}{b^2}+17\frac{a^4}{b^4}\right) - 4\alpha_1^2\left(2+5\nu\frac{a^2}{b^2}\right)\right\}$$

For pure bending when $\alpha = 2$ Eq.(2.34) reduces to

$$\sigma_{cr} = \frac{\pi^2 D_0}{b^2 h}\left[2.77\frac{b^2}{a^2}\left(A + B\frac{K_F}{\pi^4} + \frac{K_F^2}{\pi^8}\right)^{1/2}\right] \quad \ldots (2.35)$$

Setting $\alpha_1 \to 0$ in Eq.(2.35) one gets the buckling load under pure bending for a homogeneous plate for one half-wave buckling

$$\sigma_{cr} = K\frac{\pi^2 D}{b^2 h} \quad \ldots (2.36)$$

where

$$K = 2.77\frac{b^2}{a^2}\left[\left(1+\frac{a^2}{b^2}\right)^2\left(1+\frac{4a^2}{b^2}\right)^2 + \left(2 + 10\frac{a^2}{b^2} + 17\frac{a^4}{b^4}\right) \times \right.$$
$$\left. \times \frac{K_F}{\pi^4} + \frac{K_F^2}{\pi^8}\right]^{1/2}$$

For m half-wave buckling K in Eq.(2.36) can be written as

$$K = \frac{1}{m^2} \times 2.77\frac{b^2}{a^2}\left[\left(m^2 + \frac{a^2}{b^2}\right)^2\left(m^2 + \frac{4a^2}{b^2}\right)^2 + \right.$$

$$\left. + \left(2m^4 + 10m^2\frac{a^2}{b^2} + 17\frac{a^4}{b^4}\right)\frac{K_F}{\pi^4} + \frac{K_F^2}{\pi^8}\right]^{1/2} \quad \ldots (2.37)$$

The presence of the foundation modulus, K_F in Eq.(2.24) reduces the deflection, w and hence the bending stress. Therefore Eqs.(2.34, 2.35 and 2.36) resulting from two term approximation of Eq.(2.26) may be taken as fairly accurate.

RESULTS AND CONCLUSIONS

Numerical results are obtained for two cases : a) when the plate is under uniform compression, and b) when it is under pure bending. For uniform compression of a homogeneous plate, the values of K are calculated for different values of $\frac{a}{b}$ with the help of Eq.(2.30) for one half-wave buckling, taking $K_F = \pi^4$, and $\alpha = 0$. For two half-waves and three half-waves buckling Eq.(2.31) is used with the same value of K_F. These results are presented in the form of graphs in Fig. 2.3. For a non-homogeneous plate under uniform compression, the values of K are calculated for one half-wave, two and three half-waves buckling by taking the same value of K_F and $\alpha_1 = 0.1$ in Eq.(2.29). These results are presented in Fig.2.4. The values of K when the plates are not on the foundation are also presented in Fig. 2.3 and 2.4 for comparison.

For pure bending of a homogeneous plate Eq. (2.37) is used for calculation of K for different values of $\frac{a}{b}$ taking $K_F = \pi^4$ and the results are presented in Fig. 2.5. In the same figure the corresponding results for K without foundation are also presented for comparison.

From the foregoing analysis and from Figs. 2.3, 2.4 and 2.5 the following conclusions may be drawn :

i) Foundation increases the buckling load

ii) Resistance offered by the foundation is more for one half-wave buckling compared to multiple half-waves buckling. When buckling is in more than one half-wave, the foundation resistance remains practically constant.

iii) Foundation increases the $\left(\frac{a}{b}\right)_{cr}$ ratio and reduces the $\frac{a}{b}$ ratio at which transition takes place as compared to a plate not resting on foundation.

iv) A non-homogeneous plate will have lower buckling load as compared to a homogeneous plate.

v) Use of error function reduces labour considerably compared to any other numerical method.

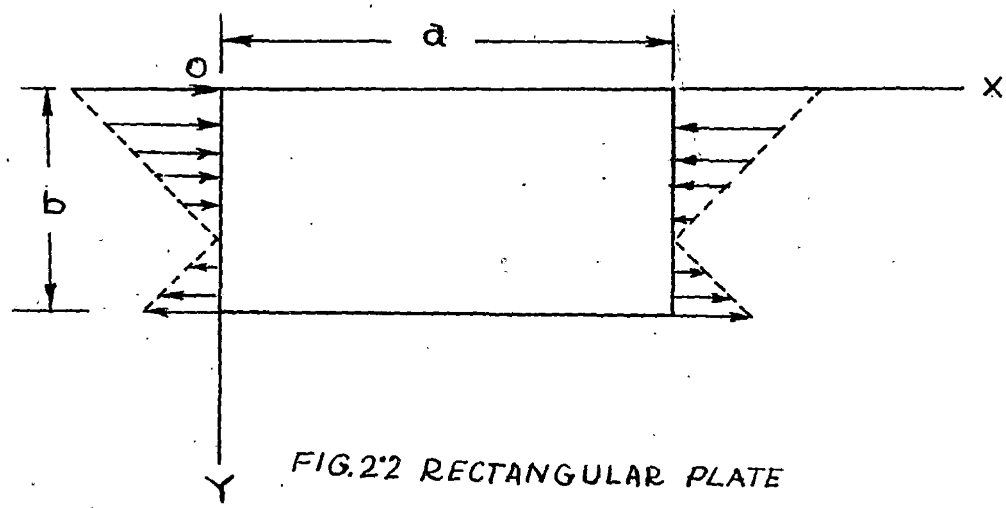

FIG. 2·2 RECTANGULAR PLATE

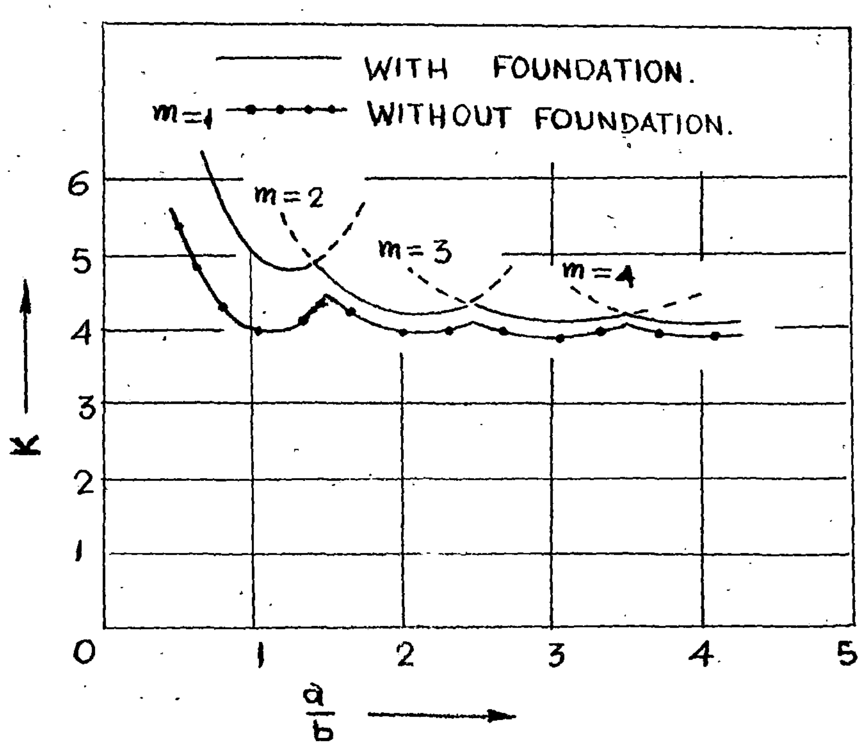

FIG. 2·3 HOMOGENEOUS PLATE UNDER UNIFORM COMPRESSION.

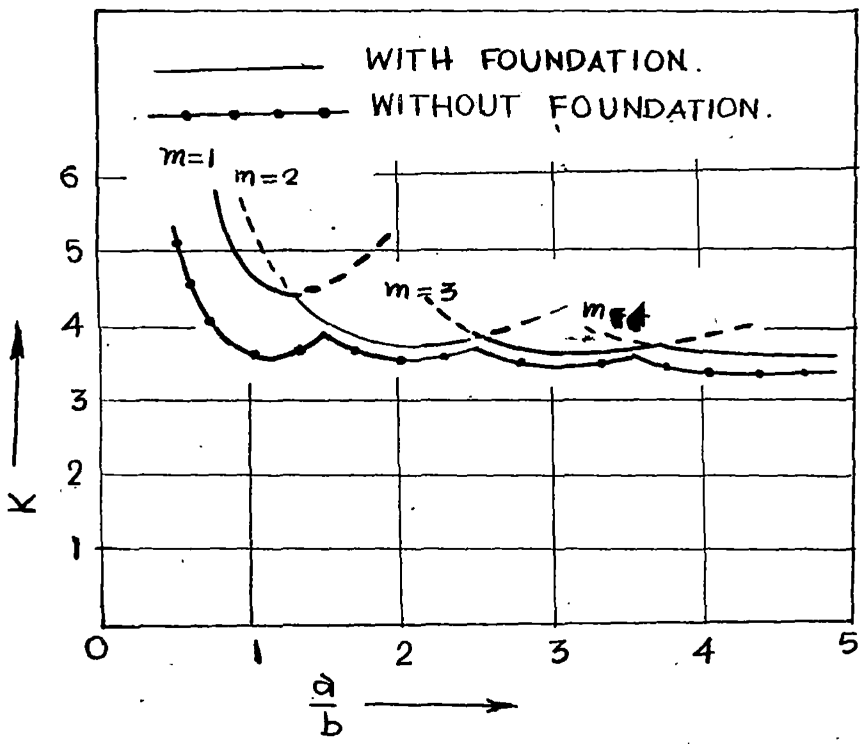

FIG. 2.4 NON HOMOGENEOUS PLATE UNDER UNIFORM COMPRESSION.

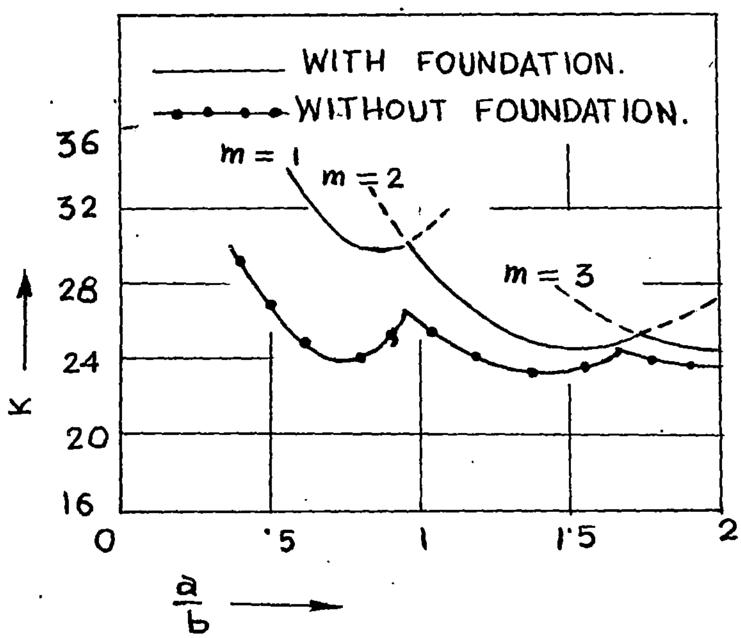

FIG. 2·5 HOMOGENEOUS PLATE UNDER PURE BENDING.

THERMAL BUCKLING OF SOME HEATED PLATES PLACED ON ELASTIC FOUNDATION*

PAPER - III

INTRODUCTION :

Thermal Buckling of thin elastic plates is of much practical importance in modern engineering. Nowacki (1962) has discussed the thermal buckling of a rectangular plate under different boundary conditions. Mansfield (1964) has investigated the buckling and curling of a heated thin circular plate of constant thickness. Klosner and Forray (1958) have studied the thermal buckling of simply supported plates under symmetrical temperature distribution.

In this paper thermal buckling of a heated equilateral triangular plate of simply-supported edges and a clamped elliptic plate placed on elastic foundation has been investigated. The foundation is assumed to be of the Winkler type. The boundary has been transformed conformally onto the unit circle and the stability criterion has been obtained with the help of Galerkin's procedure.

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THEORY :

Let us consider a plate of thickness, h, subjected to a temperature distribution T which is independent of x and y, but varies arbitarily through the thickness, i.e.,

$$T = T(z)$$

The plate is subjected to no external load and motion of all supports in the plane of the plate is prevented. It justifies then, that under the above condition there are no displacements in the plane of the plate, i.e.,

$$u = v = 0$$

On the above propositions the differential equation for the displacement [Boley and Weiner, 1967] is

$$D\nabla^4 W + \frac{N_T}{1-\nu}\nabla^2 W = 0 \qquad \ldots (2.38)$$

For a plate placed on elastic foundation having the foundation reaction, K_1 , Eq. (2.38) becomes

$$D\nabla^4 W + \frac{N_T}{1-\nu}\nabla^2 W + K_1 W = 0 \qquad \ldots (2.39)$$

Eq.(2.39) may be written as

$$\left(\nabla^2 + P_1^2\right)\left(\nabla^2 + P_2^2\right) W = 0 \qquad \ldots (2.40)$$

in which

$$p_1^2 p_2^2 = \frac{K_1}{D} \qquad \ldots (2.41)$$

$$p_1^2 + p_2^2 = \frac{N_T}{D(1-\nu)} \qquad \ldots (2.42)$$

and

$$N_T = \alpha E \int_{-h/2}^{+h/2} T\,dz \qquad \ldots (2.43)$$

α being the coefficient of thermal expansion.

If $Z = x + iy$, $\bar{Z} = x - iy$,

Eq.(2.40) changes into

$$\left(4\frac{\partial^2}{\partial z \partial \bar{z}} + p_1^2\right)\left(4\frac{\partial^2}{\partial z \partial \bar{z}} + p_2^2\right) W = 0 \qquad \ldots (2.44)$$

Let $Z = f(\xi)$ be the analytic function which maps the given shape in the ξ plane onto a unit circle.

Thus Eq.(2.44) transforms into

$$\frac{\partial^4 W}{\partial \xi^2 \partial \bar{\xi}^2} \frac{dz}{d\xi}\frac{d\bar{z}}{d\bar{\xi}} - \frac{\partial^3 W}{\partial \bar{\xi}^2 \partial \xi}\frac{d^2 z}{d\xi^2}\frac{d\bar{z}}{d\bar{\xi}} - \frac{\partial^3 W}{\partial \xi^2 \partial \bar{\xi}}\frac{d^2 \bar{z}}{d\bar{\xi}^2}\frac{dz}{d\xi} +$$

$$+ \frac{\partial^2 W}{\partial \xi \partial \bar{\xi}}\frac{d^2 z}{d\xi^2}\frac{d^2 \bar{z}}{d\bar{\xi}^2} + \frac{(p_1^2+p_2^2)}{4}\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2\frac{\partial^2 W}{\partial \xi \partial \bar{\xi}} +$$

$$+ p_1^2 p_2^2 \left(\frac{dz}{d\xi}\right)^3\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3 W = 0$$

$$\ldots (2.45)$$

Let us assume

$$W = \sum_{n=1}^{\infty} B_n \left[1-(\xi\bar{\xi})^n\right] \qquad \ldots (2.46)$$

Clearly the above form of W satisfies the edge condition W = 0 at r = 1 . Eq.(2.46) is an admissible function for the simply supported edge condition in the sense that this satisfies the kinematic boundary W = 0 at r = 1, but does not satisfy the force boundary condition $M_n = 0$. Putting Eq.(2.46) in Eq.(2.45) one gets the error function, $\epsilon_{\pi,\theta}$. Galerkin's procedure requires that the error function to be orthogonal over the domain, i.e.,

$$\int_C \epsilon_{\pi,\theta}(\xi\bar{\xi})\, w(\xi\bar{\xi})\, dc = 0 \quad (n = 1, 2, \cdots K) \qquad \ldots (2.47)$$

This generates (K x K) determinantal equation. The lowest root of this gives the critical buckling temperature.

APPLICATIONS :

A. Let us consider a simply-supported equilateral triangular plate of side $2\sqrt{3}\,a$ placed on an elastic foundation.

To solve the differential equation for W let us put

$$W = W_1 + W_2 \qquad \ldots (2.48)$$

From Eq.(2.40) one gets

$$(\nabla^2 + P_1^2)W_1 = 0 \qquad \ldots (2.49)$$

$$(\nabla^2 + P_2^2)W_2 = 0 \qquad \ldots (2.50)$$

For the edge condition W = 0 along the boundary, let

$$W_2 = \sum_{n=1}^{K} B_n\left[1-(\xi\bar{\xi})^n\right] = \sum_{n=1}^{K} B_n\left(1-r^{2n}\right) \qquad \ldots (2.51)$$

For 1st term approximation it is sufficient to solve either Eq.(2.49) or Eq.(2.50). Changing either of the two Eqs.(2.49) and (2.50) in the complex coordinates one gets

$$\nabla^2 W_1(\xi\bar{\xi}) + P_2^2\left(\frac{dz}{d\xi}\right)^2 W_1(\xi\bar{\xi}) = 0 \qquad \ldots (2.52)$$

The mapping function [Laura and Shahady, 1969]

$$Z = 1{\cdot}13520\left[\xi + \frac{1}{6}\xi^4 + \frac{5}{63}\xi^7 + \frac{4}{81}\xi^{10}\right] \qquad \ldots (2.52a)$$

maps an equilateral triangular plate into a unit circle in the ξ - plane. With this mapping function, putting Eq.(2.51) in Eq.(2.52) and remembering $\xi = re^{i\theta}$ one gets the required error function. After evaluating the integral given by Eq.(2.47) and taking $K = 1$ one gets

$$\lambda_2^2 = 6 \qquad \ldots (2.53)$$

where $\lambda_2^2 = P_2^2(1{\cdot}1352a)^2$

From Eq.(2.53) one gets the following critical buckling temperature

$$(N_T)_{cr} = D(1-\nu)\left[\frac{6}{(1{\cdot}1352a)^2} + \frac{K_1(1{\cdot}1352a)^2}{6D}\right] \qquad \ldots (2.54)$$

For $K_1 = 0$, Eq.(2.54) reduces to

$$(N_T)_{cr} = \frac{4{\cdot}6}{a^2} D(1-\nu) \qquad \ldots (2.54a)$$

Expressing this value in terms of the altitude h of the triangle one gets

$$(N_T)_{cr} = \frac{41{\cdot}4}{h^2} D(1-\nu) \qquad \ldots (2.54b)$$

which agrees well with the result obtained by Banerjee (1975) who used tri-linear co-ordinates to obtain the results for the corresponding problem without elastic foundation.

B. Let us consider an elliptic plate having centre at the origin. Let h be the thickness of the plate.

For clamped edge boundary condition let us take w in the following form

$$W = \sum_{n=1}^{K} B_n \left[1-(\zeta\bar{\zeta})^n\right]^2 \qquad \ldots (2.55)$$

Clearly the above form of w satisfies the clamped edge conditions

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{at} \quad r = 1$$

Now for the ellipse

$$\frac{x^2}{\frac{4}{3}} + \frac{y^2}{\frac{4}{5}} = 1$$

mapping function is [Laura and Shahady, 1969]

$$Z = 0{\cdot}99b\left[\zeta + 0{\cdot}12\zeta^3 + 0{\cdot}03\zeta^5 + 0{\cdot}01\zeta^7\right] \quad \ldots (2.56)$$

which maps the above ellipse a unit circle in the ζ - plane. With this mapping function, putting Eq.(2.55) in Eq.(2.45) and remembering $\zeta = re^{i\theta}$ one gets the required error function. After evaluating the integral given by Eq.(2.47) and taking K = 2, the following determinant is obtained.

$$\begin{vmatrix} \frac{32}{3} - \frac{2}{3}(\lambda_1^2+\lambda_2^2) + \frac{\lambda_1^2\lambda_2^2}{10} & \frac{256}{15} - \frac{4}{5}(\lambda_1^2+\lambda_2^2) + \frac{33}{70}\lambda_1^2\lambda_2^2 \\ \frac{256}{15} - \frac{4}{5}(\lambda_1^2+\lambda_2^2) + \frac{29}{576}\lambda_1^2\lambda_2^2 & \frac{5632}{105} - \frac{4}{3}(\lambda_1^2+\lambda_2^2) + \frac{127}{315}\lambda_1^2\lambda_2^2 \end{vmatrix} = 0 \quad \ldots (2.57)$$

where

$$\lambda_1^2 = p_1^2(0{\cdot}99b)^2$$
$$\lambda_2^2 = p_2^2(0{\cdot}99b)^2$$

Solving Eq.(2.57) for the lowest root, the critical buckling temperature is obtained as

$$(N_T)_{cr} = D(1-\nu)\left[\frac{44{\cdot}6}{b^2} + 0{\cdot}094\, b^2\frac{K_1}{D} - \sqrt{\left\{\frac{29{\cdot}2}{b^2} + 0{\cdot}094\frac{K_1}{D}\right\}^2 + 0{\cdot}106\, b^4\left(\frac{K_1}{D}\right)^2}\;\right] \quad \ldots (2.58)$$

CONCLUDING REMARKS

Solutions obtained in this study are only approximate, because only the first term of the mapping function is considered and K is taken to be 2. More accurate results are obtained by considering the remaining terms of the mapping function and taking K more than 2. Solution of the eigenvalue problem governing the stability of the thin elastic plates having various configurations, such as regular polygonal shape, circular boundary with flat sides, epitrochoidal boundary etc, is easily accomplished with the help of the complex variable theory applied in this study.