

C H A P T E R - I I

CRACK PROBLEMS IN ELASTODYNAMICS

Paper - 5 : Interaction of elastic waves with a
Periodic array of coplanar Griffith
cracks in an orthotropic elastic medium.

Paper - 6 : Diffraction of SH-waves by a Griffith
crack in nonhomogeneous elastic strip.

INTERACTION OF ELASTIC WAVES WITH A PERIODIC ARRAY OF COPLANAR GRIFFITH CRACKS IN AN ORTHOTROPIC ELASTIC MEDIUM

1. INTRODUCTION

In recent years, with the increased usage of macroscopically anisotropic construction materials such as fibre-reinforced composites, the study of interaction of elastic waves with cracks or inclusions in composite materials has gained much importance. Recently, Kassir and Bandyopadhyaya (1983) have studied the elastodynamic response of an infinite orthotropic solid containing a crack under the action of impact loading and the elastodynamic problem of a finite Griffith crack in an orthotropic strip under normal impact was investigated by Shindo (1986). Problem involving a moving Griffith crack in an orthotropic strip has also been studied by De and Patra (1990). But perhaps, because of mathematical complexity, elastodynamic problems involving two or more Griffith cracks in anisotropic materials have not yet received much attention. The static problem of determining the stress distribution in an infinite transversely isotropic medium containing three coplanar cracks has been considered by Dhawan and Dhaliwal (1978). Static stress distribution near periodic cracks at the interface of two bonded dissimilar orthotropic half planes has been obtained by Garg (1981). Angel and Achenbach (1985) have studied the problem of reflection and transmission of elastic waves by a periodic array of cracks in an infinite isotropic

medium. The steady state vibration of an infinite isotropic medium with a periodic system of coplanar cracks has been discussed by Parton and Morozov (1978) using the method of the finite Fourier transforms to reduce the relevant mixed boundary value problem to the solution of a pair of dual series relations.

In our problem, the interaction of normally incident time harmonic elastic waves with a periodic array of coplanar Griffith cracks in an infinite orthotropic medium has been considered. Due to geometrical symmetry the problem has been reduced to the solution of the problem of a single crack in a strip whose boundaries are shear free and constrained in a way not to permit normal displacement. Fourier transform has been used to reduce the problem to the solution of dual integral equations. By the application of Abel's integral the dual integral equations finally has been converted to a Fredholm integral equation. Stress intensity factor at the tip of the crack and crack opening displacement have been derived in closed form. To display the influence of the material orthotropy numerical values of stress intensity factor and crack opening displacement have been found out after solving the Fredholm integral equation numerically and plotted against dimensionless frequency and distance respectively for three sets of orthotropic materials.

2. FORMULATION OF THE PROBLEM

Consider an orthotropic, linearly elastic, unbounded solid weakened by a periodic array of cracks of length $2a$ as shown in

fig.1. The period of the crack-array is $2h_1$. The cracks lie in the plane $x_2=0$ and extend to infinity in the x_3 -direction which is perpendicular to the plane of the figure. Let E_i , μ_{ij} and ν_{ij} ($i, j=1, 2, 3$) denote the engineering elastic constants of the material where the subscripts 1, 2, 3 correspond to the x_1, x_2, x_3 directions which coincide with the axes of material orthotropy.

We normalize all lengths with respect to a so that $x_1/a=x$, $x_2/a=y$, $x_3/a=z$ and $h_1/a=h$. Let a time harmonic wave given by $u=0$ and $v=\exp[i(k_y y/c_{22} - \omega t)]$ where $k_y = \omega a/c_y$ and $c_y = (\mu_{12}/\rho)^{1/2}$, travelling in the direction of positive y -axis be incident normally on the crack faces so that the conditions in the plane of the cracks ($y=0$) due to the scattered field are $\tau_{xy}=0$, $\tau_{yy} = -\tau_0 e^{-i\omega t}$, where $\tau_0 = i\omega a \nu_{22}/c_y$, on the crack faces and $v=0$ at points outside the cracks in the plane $y=0$.

By simple symmetry considerations, the displacement and stress distribution due to the scattered field in the entire xy -plane can be deduced by considering only the orthotropic strip $|x| \leq h$ with a central crack $|x| \leq a$, $y=0$; the boundaries of the strip $x=\pm h$ being shear free and constrained in a way not to permit normal displacement.

Therefore, substituting $u(x, y, t) = u(x, y)e^{-i\omega t}$ and $v(x, y, t) = v(x, y)e^{-i\omega t}$ our problem reduces to the solution of the equations

$$c_{11} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (1+c_{12}) \frac{\partial^2 v}{\partial x \partial y} + \frac{a^2 \omega^2}{c_y^2} u = 0 \quad (1)$$

and

$$c_{22} \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + (1+c_{12}) \frac{\partial^2 u}{\partial x \partial y} + \frac{a^2 \omega^2}{c_y^2} v = 0 \quad (2)$$

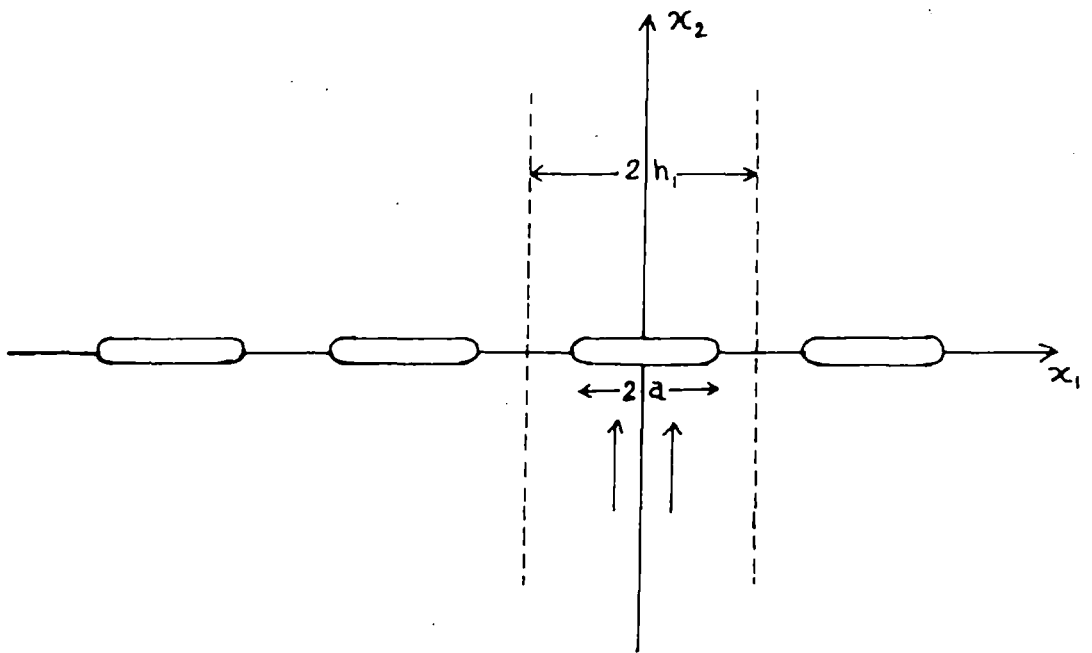


Fig. 1. Incidence of plane time-harmonic wave on a periodic array of cracks.

subject to the boundary conditions

$$\tau_{yy}(x,0) = -\tau_0 e^{-i\omega t}, \quad |x| < 1 \quad (3)$$

$$\tau_{xy}(x,0) = 0, \quad |x| \leq h \quad (4)$$

$$v(x,0) = 0, \quad 1 \leq |x| \leq h \quad (5)$$

$$\tau_{xy}(\pm h, y) = 0, \quad |y| < \infty \quad (6)$$

$$u(\pm h, y) = 0, \quad |y| < \infty \quad (7)$$

Henceforth the time factor $e^{-i\omega t}$ which is common to all field variables would be omitted in the sequel.

The nondimensional parameters c_{ij} arising in equations (1) and (2) are related to the elastic constants by the relations

$$\begin{aligned} c_{11} &= E_1 / \mu_{12} (1 - \nu_{12}^2 E_2 / E_1) \\ c_{22} &= E_2 / \mu_{12} (1 - \nu_{12}^2 E_2 / E_1) = c_{11} E_2 / E_1 \\ c_{12} &= \nu_{12} E_2 / \mu_{12} (1 - \nu_{12}^2 E_2 / E_1) = \nu_{12} c_{22} = \nu_{21} c_{11} \end{aligned} \quad (8)$$

for generalized plane stress, and by

$$\begin{aligned} c_{11} &= (E_1 / \Delta \mu_{12}) (1 - \nu_{23} \nu_{32}) \\ c_{22} &= (E_2 / \Delta \mu_{12}) (1 - \nu_{13} \nu_{31}) \\ c_{12} &= E_1 (\nu_{21} + \nu_{13} \nu_{32} E_2 / E_1) / \Delta \mu_{12} \\ &= E_2 (\nu_{12} + \nu_{23} \nu_{31} E_1 / E_2) / \Delta \mu_{12} \end{aligned} \quad (9)$$

$$\Delta = 1 - \nu_{12} \nu_{21} - \nu_{23} \nu_{32} - \nu_{31} \nu_{13} - \nu_{12} \nu_{23} \nu_{31} - \nu_{13} \nu_{21} \nu_{32}$$

for plane strain. The constants E_i and ν_{ij} satisfy the Maxwell's

relation $\nu_{ij}/E_i = \nu_{ji}/E_j$.

The stresses are related to the displacements by the equations

$$\begin{aligned}\tau_{xx}/\mu_{12} &= c_{11} \frac{\partial u}{\partial x} + c_{12} \frac{\partial v}{\partial y} \\ \tau_{yy}/\mu_{12} &= c_{12} \frac{\partial u}{\partial x} + c_{22} \frac{\partial v}{\partial y} \\ \tau_{xy}/\mu_{12} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\end{aligned}\quad (10)$$

The solutions of equations (1) and (2) are taken as

$$\begin{aligned}u(x,y) &= \frac{2}{\pi} \int_0^{\infty} \left[A_1(\xi) e^{-\gamma_1 y} + A_2(\xi) e^{-\gamma_2 y} \right] \sin(\xi x) d\xi + \\ &+ \frac{2}{\pi} \int_0^{\infty} \left[A_3(\xi) \sinh(\gamma_3 x) + A_4(\xi) \sinh(\gamma_4 x) \right] \cos(\xi y) d\xi\end{aligned}\quad (11)$$

$$\begin{aligned}\text{and } v(x,y) &= \frac{2}{\pi} \int_0^{\infty} \frac{1}{\xi} \left[\alpha_1 A_1(\xi) e^{-\gamma_1 y} + \alpha_2 A_2(\xi) e^{-\gamma_2 y} \right] \cos(\xi x) d\xi + \\ &+ \frac{2}{\pi} \int_0^{\infty} \frac{1}{\xi} \left[\alpha_3 A_3(\xi) \cosh(\gamma_3 x) + \alpha_4 A_4(\xi) \cosh(\gamma_4 x) \right] \sin(\xi y) d\xi\end{aligned}$$

where A_i ($i=1-4$) are the unknowns to be solved, γ_1^2 , γ_2^2 are the roots of the equation

$$c_{22} \gamma^4 + \left\{ (c_{12}^2 + 2c_{12} - c_{11} c_{22}) \xi^2 + (1 + c_{22}) k^2 \right\} \gamma^2 + (c_{11} \xi^2 - k^2) (\xi^2 - k^2) = 0\quad (12)$$

and γ_3^2 , γ_4^2 are the roots of the quadratic

$$c_{11}\gamma^4 + \left\{ (c_{12}^2 + 2c_{12} - c_{11}c_{22})\zeta^2 + (1+c_{11})k_0^2 \right\} \gamma^2 + (c_{22}\zeta^2 - k_0^2)(\zeta^2 - k_0^2) = 0 \quad (13)$$

α_i ($i=1-4$) occurring in (11) are given by

$$\alpha_i = \frac{c_{11}\zeta^2 - k_0^2 - \gamma_i^2}{(1+c_{12})\gamma_i} \quad (i=1,2) \quad (14)$$

and

$$\alpha_i = \frac{\zeta^2 - k_0^2 - c_{11}\gamma_i^2}{(1+c_{12})\gamma_i} \quad (i=3,4) \quad (15)$$

Therefore the stress components τ_{yy} and τ_{xy} are

$$\begin{aligned} \tau_{yy}/\mu_{12} = & \frac{2}{\pi} \int_0^\infty \left[\left(c_{12}\zeta - \frac{c_{22}\alpha_1\gamma_1}{\zeta} \right) A_1(\xi) e^{-\gamma_1 y} + \left(c_{12}\zeta - \frac{c_{22}\alpha_2\gamma_2}{\zeta} \right) A_2(\xi) e^{-\gamma_2 y} \right] \times \\ & \times \cos(\xi x) d\xi + \\ & + \frac{2}{\pi} \int_0^\infty \left[(c_{12}\gamma_3 + c_{22}\alpha_3) A_3(\zeta) \cosh(\gamma_3 x) + (c_{12}\gamma_4 + c_{22}\alpha_4) A_4(\zeta) \times \right. \\ & \left. \times \cosh(\gamma_4 x) \right] \cos(\zeta y) d\zeta \quad (16) \end{aligned}$$

$$\begin{aligned} \tau_{xy}/\mu_{12} = & - \frac{2}{\pi} \int_0^\infty \left[(\gamma_1 + \alpha_1) A_1(\xi) e^{-\gamma_1 y} + (\gamma_2 + \alpha_2) A_2(\xi) e^{-\gamma_2 y} \right] \sin(\xi x) d\xi - \\ & - \frac{2}{\pi} \int_0^\infty \left[\left(\zeta - \frac{\alpha_3\gamma_3}{\zeta} \right) A_3(\zeta) \sinh(\gamma_3 x) + \left(\zeta - \frac{\alpha_4\gamma_4}{\zeta} \right) A_4(\zeta) \times \right. \\ & \left. \times \sinh(\gamma_4 x) \right] \sin(\zeta y) d\zeta \quad (17) \end{aligned}$$

3. DERIVATION OF INTEGRAL EQUATIONS

The boundary condition (4) implies

$$A_2(\xi) = -\beta A_1(\xi), \quad \text{where} \quad \beta = \frac{\gamma_1 + \alpha_1}{\gamma_2 + \alpha_2} \quad (18)$$

From boundary conditions (3) and (5), we obtain the following dual integral equations

$$\begin{aligned} & \int_0^\infty \left[\left(c_{12} \xi - \frac{c_{22} \alpha_1 \gamma_1}{\xi} \right) - \beta \left(c_{12} \xi - \frac{c_{22} \alpha_2 \gamma_2}{\xi} \right) \right] A_1(\xi) \cos(\xi x) d\xi + \\ & + \int_0^\infty \left[(c_{12} \gamma_3 + c_{22} \alpha_3) A_3(\xi) \cosh(\gamma_3 x) + (c_{12} \gamma_4 + c_{22} \alpha_4) A_4(\xi) \cosh(\gamma_4 x) \right] d\xi \\ & = -\frac{\pi \tau_0}{2\mu_{12}}, \quad |x| < 1 \end{aligned} \quad (19)$$

$$\text{and} \quad \int_0^\infty \frac{1}{\xi} (\alpha_1 - \beta \alpha_2) A_1(\xi) \cos(\xi x) d\xi = 0, \quad 1 \leq |x| \leq h \quad (20)$$

$$\text{Assuming} \quad A(\xi) = \frac{\alpha_1 - \beta \alpha_2}{\xi} A_1(\xi) = \frac{\beta \alpha_2 - \alpha_1}{\beta \xi} A_2(\xi) \quad (21)$$

equations (19) and (20) can be rewritten as

$$\begin{aligned} & \int_0^\infty \left[\frac{c_{12} \xi^2 - c_{22} \alpha_1 \gamma_1 - \beta (c_{12} \xi^2 - c_{22} \alpha_2 \gamma_2)}{(\alpha_1 - \beta \alpha_2)} \right] A(\xi) \cos(\xi x) d\xi + \\ & + \int_0^\infty \left[(c_{12} \gamma_3 + c_{22} \alpha_3) A_3(\xi) \cosh(\gamma_3 x) + (c_{12} \gamma_4 + c_{22} \alpha_4) A_4(\xi) \cosh(\gamma_4 x) \right] d\xi \\ & = -\frac{\pi \tau_0}{2\mu_{12}}, \quad |x| < 1 \end{aligned} \quad (22)$$

$$\text{and } \int_0^{\infty} A(\xi) \cos(\xi x) d\xi = 0, \quad 1 \leq |x| \leq h, \quad (23)$$

For large ξ the equation (12) takes the form

$$c_{22} \gamma^4 + (c_{12}^2 + 2c_{12} - c_{11} c_{22}) \xi^2 \gamma^2 + c_{11} \xi^4 = 0 \quad (24)$$

Let the roots of the equation (23) be $\xi^2 N_1^2$ and $\xi^2 N_2^2$, where

$$N_1^2 = \frac{1}{2c_{22}} \left\{ c_{11} c_{22} - c_{12}^2 - 2c_{12} + [(c_{12}^2 + 2c_{12} - c_{11} c_{22})^2 - 4c_{11} c_{22}]^{1/2} \right\} \quad (25)$$

$$N_2^2 = \frac{1}{2c_{22}} \left\{ c_{11} c_{22} - c_{12}^2 - 2c_{12} - [(c_{12}^2 + 2c_{12} - c_{11} c_{22})^2 - 4c_{11} c_{22}]^{1/2} \right\} \quad (26)$$

Also for large ξ , α_i ($i=1,2$) and β given by (14) and (18) respectively become

$$\alpha_i = \xi \left[\frac{c_{11} - N_i^2}{(1 + c_{12}) N_i} \right], \quad (i=1,2) \quad (27)$$

$$\text{and } \beta = \frac{(c_{11} + c_{12} N_1^2) N_2}{(c_{11} + c_{12} N_2^2) N_1} \quad (28)$$

Now, for large ξ

$$\begin{aligned} & \frac{c_{12} \xi^2 - c_{22} \alpha_1 \gamma_1 - \beta (c_{12} \xi^2 - c_{22} \alpha_2 \gamma_2)}{(\alpha_1 - \beta \alpha_2) \xi} \\ &= \frac{(c_{12}^2 + c_{12} - c_{11} c_{22}) (c_{12} N_1 N_2 - c_{11}) - c_{22} [c_{12} N_1^2 N_2^2 + c_{11} (N_1^2 + N_1 N_2 + N_2^2)]}{c_{11} (1 + c_{12}) (N_1 + N_2)} \\ &= \theta \text{ (say)} \end{aligned} \quad (29)$$

Therefore we find that

$$H(\xi) = \frac{c_{12}\xi^2 - c_{22}\alpha_1\gamma_1 - \beta(c_{12}\xi^2 - c_{22}\alpha_2\gamma_2)}{(\alpha_1 - \beta\alpha_2)\xi\theta} - 1 \quad (30)$$

tends to zero as ξ tends to ∞ .

Using (30) in equation (22) we finally obtain from (22) and (23) the following dual integral equations for the determination of the unknown function $A(\xi)$:

$$\int_0^{\infty} \xi [1+H(\xi)] A(\xi) \cos(\xi x) d\xi = p(x) \quad , \quad |x| < 1 \quad (31)$$

$$\int_0^{\infty} A(\xi) \cos(\xi x) d\xi = 0 \quad , \quad 1 \leq |x| \leq h \quad (32)$$

where

$$p(x) = -\frac{\pi\tau_0}{2\mu_{12}\theta} - \frac{1}{\theta} \int_0^{\infty} [(c_{12}\gamma_3 + c_{22}\alpha_3) A_3(\zeta) \cosh(\gamma_3 x) + (c_{12}\gamma_4 + c_{22}\alpha_4) A_4(\zeta) \cosh(\gamma_4 x)] d\zeta \quad (33)$$

The boundary conditions (6) and (7) yields

$$\int_0^{\infty} \left[\left(\zeta - \frac{\alpha_3\gamma_3}{\zeta} \right) A_3(\zeta) \sinh(\gamma_3 h) + \left(\zeta - \frac{\alpha_4\gamma_4}{\zeta} \right) A_4(\zeta) \sinh(\gamma_4 h) \right] \sin(\zeta y) d\zeta$$

$$= \int_0^{\infty} \left[\frac{(\gamma_1 + \alpha_1)\xi}{(\beta\alpha_2 - \alpha_1)} e^{-\gamma_1 y} - \frac{(\gamma_2 + \alpha_2)\xi\beta}{(\beta\alpha_2 - \alpha_1)} e^{-\gamma_2 y} \right] A(\xi) \sin(\xi h) d\xi \quad (34)$$

and

$$\int_0^{\infty} [A_3(\zeta) \sinh(\gamma_3 h) + A_4(\zeta) \sinh(\gamma_4 h)] \cos(\zeta y) d\zeta$$

$$= - \int_0^{\infty} \frac{\xi}{(\alpha_1 - \beta\alpha_2)} \left[e^{-\gamma_1 y} - \beta e^{-\gamma_2 y} \right] A(\xi) \sin(\xi h) d\xi \quad (35)$$

Applying Fourier sine and cosine inverse transforms in equations (34) and (35) respectively, we get

$$\left(\zeta - \frac{\alpha_3 \gamma_3}{\zeta}\right) A_3(\zeta) \sinh(\gamma_3 h) + \left(\zeta - \frac{\alpha_4 \gamma_4}{\zeta}\right) A_4(\zeta) \sinh(\gamma_4 h) = \frac{2}{\pi} \int_0^{\infty} g_1(\xi, \zeta) A(\xi) d\xi \quad (36)$$

$$\text{and } A_3(\zeta) \sinh(\gamma_3 h) + A_4(\zeta) \sinh(\gamma_4 h) = \frac{2}{\pi} \int_0^{\infty} g_2(\xi, \zeta) A(\xi) d\xi \quad (37)$$

$$\text{where } g_1(\xi, \zeta) = \frac{\xi \zeta (\gamma_1 + \alpha_1)}{\beta \alpha_2 - \alpha_1} \left[\frac{1}{\zeta^2 + \gamma_1^2} - \frac{1}{\zeta^2 + \gamma_2^2} \right] \sin(\xi h) \quad (38)$$

$$g_2(\xi, \zeta) = \frac{\xi}{\beta \alpha_2 - \alpha_1} \left[\frac{\gamma_1}{\zeta^2 + \gamma_1^2} - \frac{\beta \gamma_2}{\zeta^2 + \gamma_2^2} \right] \sin(\xi h) \quad (39)$$

Solving the equations (36) and (37), the unknown functions $A_3(\zeta)$ and $A_4(\zeta)$ can be found to be related to $A(\xi)$ as :

$$A_3(\zeta) = f_1(\zeta) \int_0^{\infty} g_1(\xi, \zeta) A(\xi) d\xi + f_2(\zeta) \int_0^{\infty} g_2(\xi, \zeta) A(\xi) d\xi \quad (40)$$

$$A_4(\zeta) = f_3(\zeta) \int_0^{\infty} g_1(\xi, \zeta) A(\xi) d\xi + f_4(\zeta) \int_0^{\infty} g_2(\xi, \zeta) A(\xi) d\xi$$

$$\text{where } f_1(\zeta) = \frac{2}{\pi D(\zeta) \sinh(\gamma_3 h)}$$

$$f_2(\zeta) = \frac{2}{\pi D(\zeta) \sinh(\gamma_3 h)} \left[\frac{\alpha_4 \gamma_4}{\zeta} - \zeta \right]$$

$$f_3(\zeta) = \frac{-2}{\pi D(\zeta) \sinh(\gamma_4 h)}$$

$$f_4(\zeta) = \frac{-2}{\pi D(\zeta) \sinh(\gamma_4 h)} \left[\frac{\alpha_3 \gamma_3}{\zeta} - \zeta \right]$$

$$D(\zeta) = \frac{\alpha_4 \gamma_4 - \alpha_3 \gamma_3}{\zeta}$$

(41)

Now, substituting the values of α_i ($i=1,2$) and β given by (14) and (18) and using the relations

$$\gamma_1^2 + \gamma_2^2 = -\frac{1}{c_{22}} \left\{ (c_{12}^2 + 2c_{12} - c_{11} c_{22}) \xi^2 + (1 + c_{22}) k^2 \right\}$$

$$\gamma_1^2 \gamma_2^2 = \frac{1}{c_{22}} (\xi^2 - k^2) (c_{11} \xi^2 - k^2) \quad (42)$$

$$(\xi^2 + \gamma_1^2) (\xi^2 + \gamma_2^2) = \frac{c_{11}}{c_{22}} (\xi^2 + \gamma_3^2) (\xi^2 + \gamma_4^2),$$

$g_1(\xi, \zeta)$ and $g_2(\xi, \zeta)$ from (38) and (39) finally can be obtained as:

$$g_1(\xi, \zeta) = \xi \zeta \left[\frac{\gamma_3^2 q_1 - q_2}{\xi^2 + \gamma_3^2} - \frac{\gamma_4^2 q_1 - q_2}{\xi^2 + \gamma_4^2} \right] \frac{\sin(\xi h)}{\gamma_4^2 - \gamma_3^2}$$

(43)

$$g_2(\xi, \zeta) = \xi \left[-\frac{\gamma_3^2 q_3 - q_4}{\xi^2 + \gamma_3^2} + \frac{\gamma_4^2 q_3 - q_4}{\xi^2 + \gamma_4^2} \right] \frac{\sin(\xi h)}{\gamma_4^2 - \gamma_3^2}$$

where

$$q_1 = \frac{c_{11} c_{22} - c_{12}^2}{c_{11}}, \quad q_2 = -\frac{k^2 (c_{12} + c_{22})}{c_{11}}$$

(44)

$$q_3 = -\frac{c_{12}}{c_{11}}, \quad q_4 = \frac{c_{22} \xi^2 + c_{12} k^2}{c_{11}}$$

4. METHOD OF SOLUTION

In order to reduce the dual integral equations (31) and (32) to a single Fredholm integral equation, we assume

$$A(\xi) = -\frac{\pi\tau_0}{2\mu_{12}\theta} \int_0^\infty \eta\phi(\eta)J_0(\xi\eta)d\eta \quad (45)$$

so that equation (32) is automatically satisfied.

Next substituting the value of $A(\xi)$ from (45) in $A_3(\zeta)$ and $A_4(\zeta)$ given by (40) and using the result (Gradshteyn and Ryzhik, 1965)

$$\int_0^\infty \frac{\xi \sin(\xi h) J_0(\xi t)}{\xi^2 + \gamma^2} d\xi = \frac{\pi}{2} e^{-h\gamma} I_0(t\gamma)$$

$A_3(\zeta)$ and $A_4(\zeta)$ can be written in terms of $\phi(t)$ as

$$\begin{aligned} A_3(\zeta) = & -\frac{\pi^2\tau_0}{4\mu_{12}\theta(\gamma_4^2 - \gamma_3^2)} \int_0^1 \left[\zeta f_1(\zeta) \left\{ (\gamma_3^2 q_1 - q_2) I_0(\gamma_3 t) e^{-\gamma_3 h} - (\gamma_4^2 q_1 - q_2) \right. \right. \\ & \times I_0(\gamma_4 t) e^{-\gamma_4 h} \left. \left. \right\} + f_2(\zeta) \left\{ -(\gamma_3^2 q_3 - q_4) I_0(\gamma_3 t) e^{-\gamma_3 h} + (\gamma_4^2 q_3 - q_4) \right. \right. \\ & \left. \left. \times I_0(\gamma_4 t) e^{-\gamma_4 h} \right\} \right] t\phi(t) dt \quad (46) \end{aligned}$$

and

$$\begin{aligned} A_4(\zeta) = & -\frac{\pi^2\tau_0}{4\mu_{12}\theta(\gamma_4^2 - \gamma_3^2)} \int_0^1 \left[\zeta f_3(\zeta) \left\{ (\gamma_3^2 q_1 - q_2) I_0(\gamma_3 t) e^{-\gamma_3 h} - (\gamma_4^2 q_1 - q_2) \right. \right. \\ & \times I_0(\gamma_4 t) e^{-\gamma_4 h} \left. \left. \right\} + f_4(\zeta) \left\{ -(\gamma_3^2 q_3 - q_4) I_0(\gamma_3 t) e^{-\gamma_3 h} + (\gamma_4^2 q_3 - q_4) \right. \right. \\ & \left. \left. \times I_0(\gamma_4 t) e^{-\gamma_4 h} \right\} \right] t\phi(t) dt \quad (47) \end{aligned}$$

Substitution of the values of $A(\xi)$, $A_3(\zeta)$ and $A_4(\zeta)$ in terms of $\phi(t)$ from equations (45), (46) and (47) respectively in (31) yields the following Fredholm integral equation of second kind for the determination of $\phi(\eta)$:

$$\phi(\eta) + \int_0^1 t \{K_1(\eta, t) + K_2(\eta, t)\} \phi(t) dt = 1 \quad (48)$$

The kernels $K_1(\eta, t)$ and $K_2(\eta, t)$ are given by

$$K_1(\eta, t) = \int_0^\infty \xi H(\xi) J_0(\xi t) J_0(\xi \eta) d\xi \quad (49)$$

$$\text{and } K_2(\eta, t) = \int_0^\infty \left\{ S_1(\zeta) I_0(\gamma_3 \eta) I_0(\gamma_3 t) + S_2(\zeta) I_0(\gamma_3 \eta) I_0(\gamma_4 t) + \right. \\ \left. + S_3(\zeta) I_0(\gamma_4 \eta) I_0(\gamma_3 t) + S_4(\zeta) I_0(\gamma_4 \eta) I_0(\gamma_4 t) \right\} d\zeta \quad (50)$$

where

$$S_1(\zeta) = \frac{\pi(c_{12}\gamma_3 + c_{22}\alpha_3)}{2(\gamma_4^2 - \gamma_3^2)\theta} \left\{ \zeta f_1(\zeta)(\gamma_3^2 q_1 - q_2) - f_2(\zeta)(\gamma_3^2 q_3 - q_4) \right\} e^{-\gamma_3 h}$$

$$S_2(\zeta) = - \frac{\pi(c_{12}\gamma_3 + c_{22}\alpha_3)}{2(\gamma_4^2 - \gamma_3^2)\theta} \left\{ \zeta f_1(\zeta)(\gamma_4^2 q_1 - q_2) - f_2(\zeta)(\gamma_4^2 q_3 - q_4) \right\} e^{-\gamma_4 h}$$

$$S_3(\zeta) = \frac{\pi(c_{12}\gamma_4 + c_{22}\alpha_4)}{2(\gamma_4^2 - \gamma_3^2)\theta} \left\{ \zeta f_3(\zeta)(\gamma_3^2 q_1 - q_2) - f_4(\zeta)(\gamma_3^2 q_3 - q_4) \right\} e^{-\gamma_3 h}$$

$$S_4(\zeta) = - \frac{\pi(c_{12}\gamma_4 + c_{22}\alpha_4)}{2(\gamma_4^2 - \gamma_3^2)\theta} \left\{ \zeta f_3(\zeta)(\gamma_4^2 q_1 - q_2) - f_4(\zeta)(\gamma_4^2 q_3 - q_4) \right\} e^{-\gamma_4 h}$$

Next the crack opening displacement $\Delta v(x,0) = v(x,0+) - v(x,0-)$ on the surface of the crack has been obtained with the help of the equation (45) in the form

$$\begin{aligned} \Delta v(x,0) &= \frac{4}{\pi} \int_0^{\infty} A(\xi) \cos(\xi x) d\xi \\ &= - \frac{2\tau_0}{\mu_{12} \theta} \int_x^1 \frac{\eta \phi(\eta)}{(\eta^2 - x^2)^{1/2}} d\eta, \quad |x| < 1 \end{aligned} \quad (54)$$

6. NUMERICAL CALCULATIONS AND DISCUSSIONS

The method of Fox and Goodwin (1953) has been used to solve the integral equation (48) numerically for different values of dimensionless frequency k_0 and h , the separating distance of the cracks. The integral in (48) has been represented by a quadrature formula involving values of the desired function ϕ at pivotal points in the range of integration which leads to a set of algebraic linear simultaneous equations. The solution of the set of linear algebraic equations gives a first approximation to the required pivotal values of ϕ which has been improved by the use of difference-correction technique. The kernel $K_1(\eta, t)$ has been transformed into two finite integrals and have been evaluated by using Gauss-quadrature integration formula. The second kernel $K_2(\eta, t)$ has been evaluated by Simpson's method. After solving the integral equation (48) for different values of engineering elastic constants of several orthotropic materials listed in table

1, the stress intensity factor K given by (53) has been plotted against k_{\square} for different values of h (Fig.2-Fig.4) and the crack opening displacement $\Delta v(x,0)\mu_{12}/\tau_0$ given in (54) has been plotted against dimensionless distance x ($0 \leq x \leq 1$) (Fig.5-Fig.7).

The nature of crack opening displacements which have been plotted for three different orthotropic materials for different values of the separating distance h is found to show two maxima within the range ($0 \leq x \leq 1$) one near $x=0$ and the other near $x=1$, the maximum value near $x=0$ being the greatest.

From fig.2-fig.4 it may be noted that the stress intensity factor increases with the increase in the values of frequency k_{\square} , attains maximum value and then decreases for all orthotropic materials for $0 < k_{\square} \leq 1$. The stress intensity factor is found to decrease with the increase in the values h , the separating distance of the cracks and also decreases sharply after the maximum value is attained.

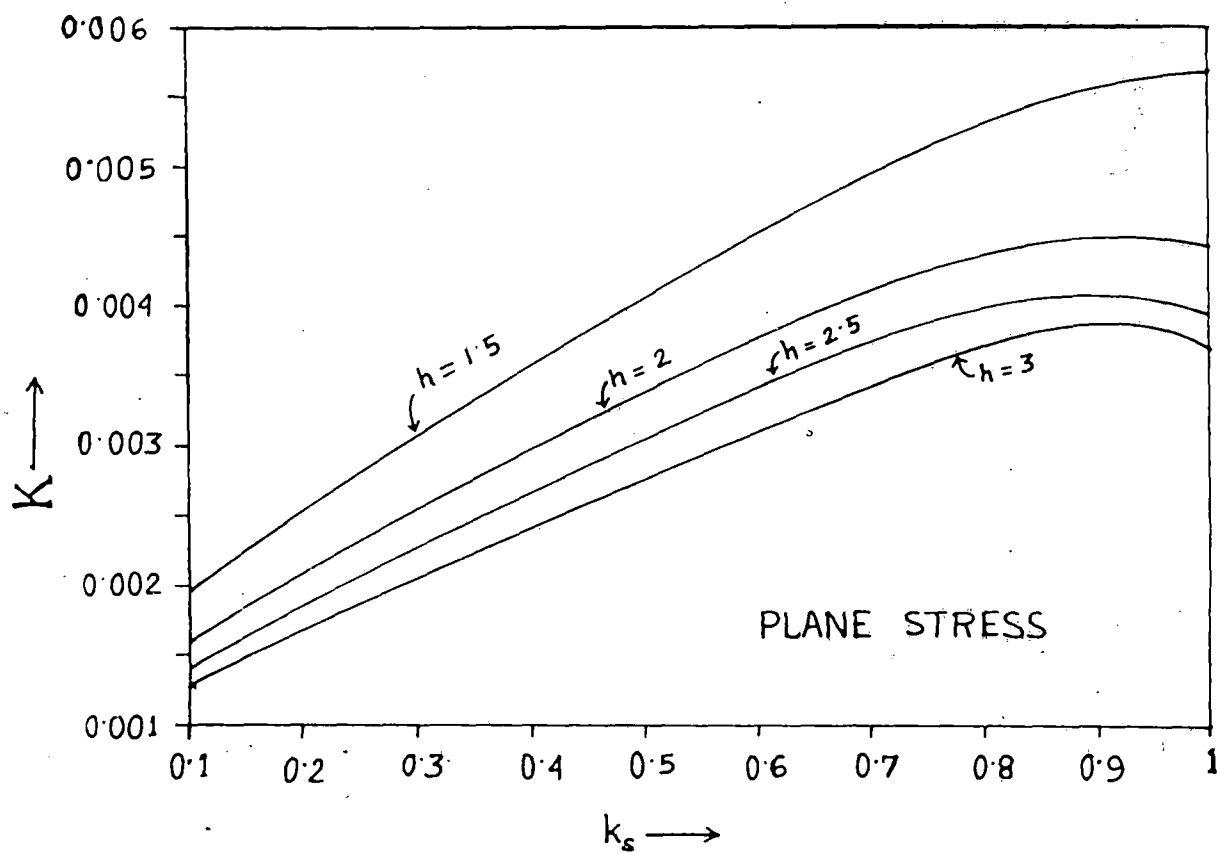


Fig. 2. Dynamic stress intensity factor K vs. dimensionless frequency k_s for Boron-Expoxi composite.

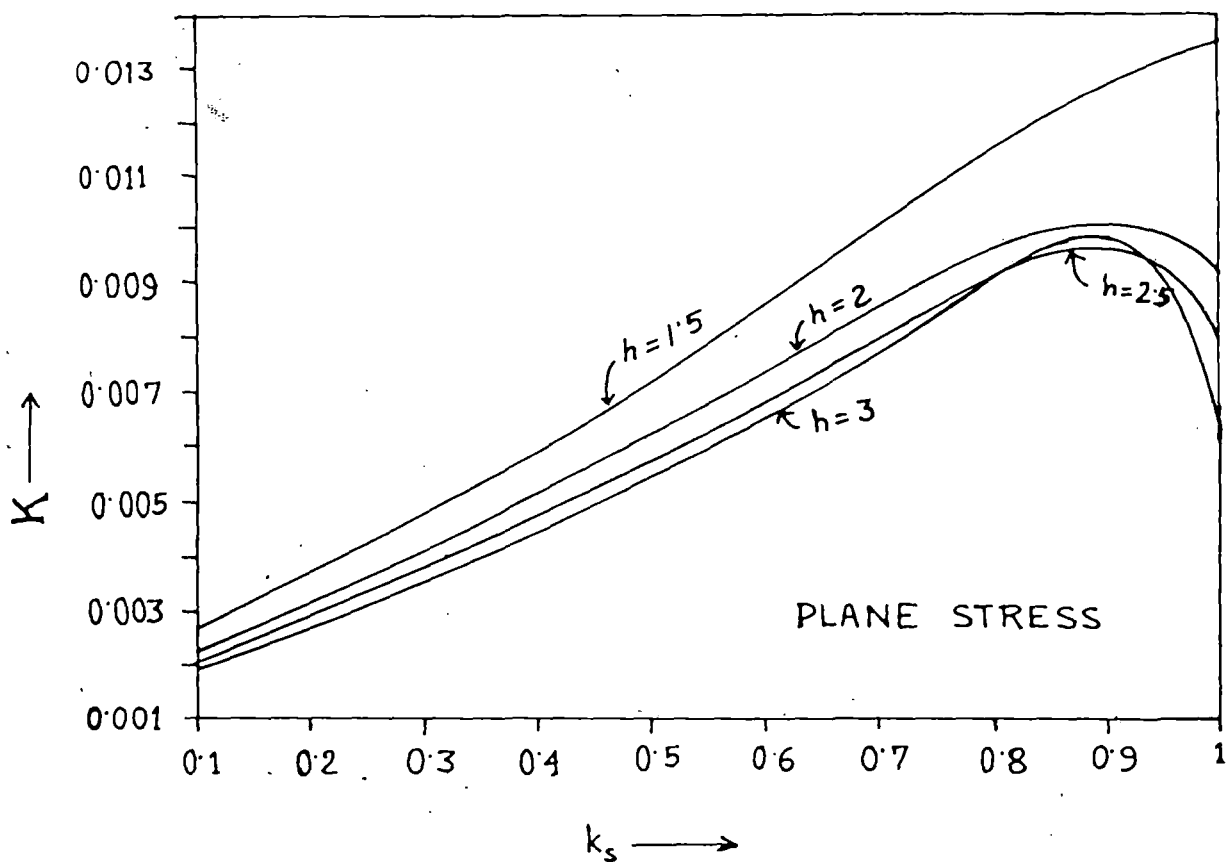


Fig. 3. Dynamic stress intensity factor K vs. dimensionless frequency k_s for Steel-Mylar composite.

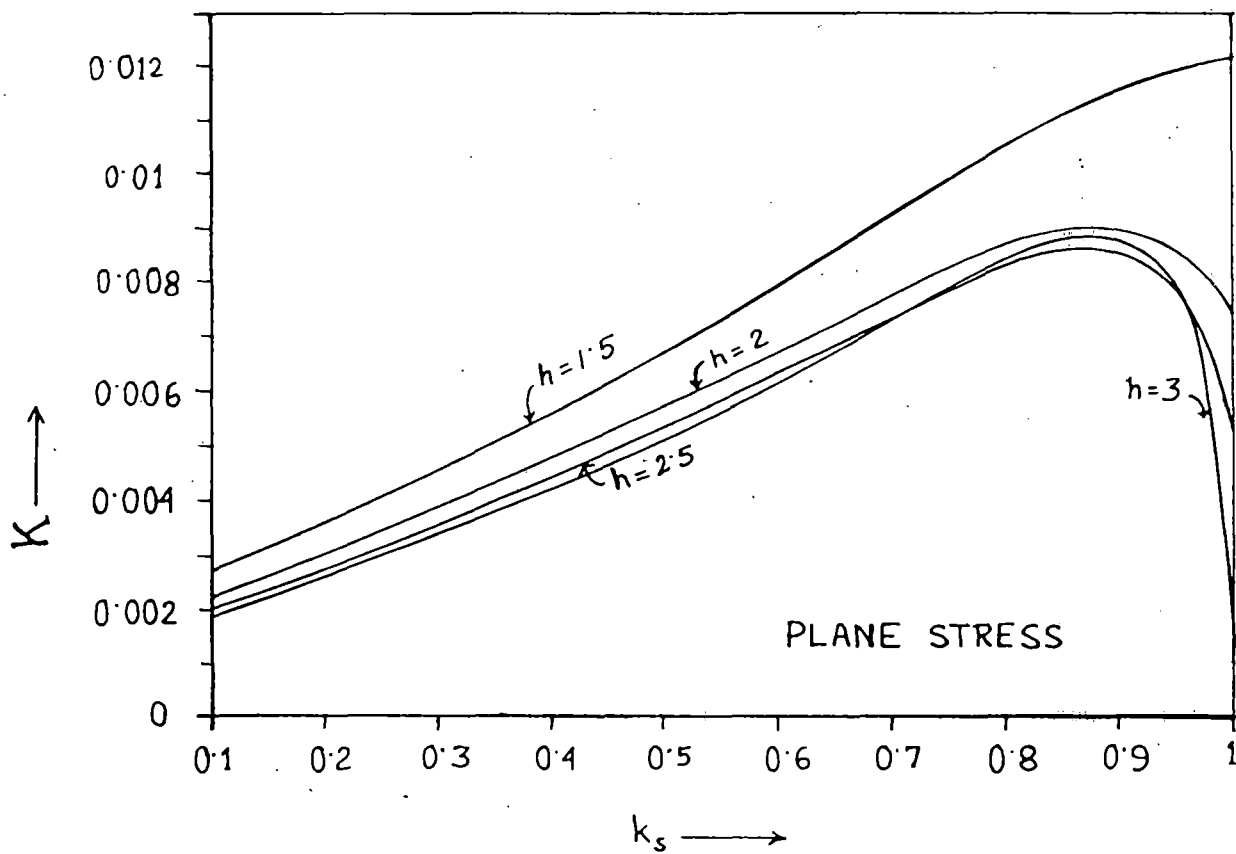


Fig. 4. Dynamic stress intensity factor K vs. dimensionless frequency k_s for Graphite-Epoxy composite.

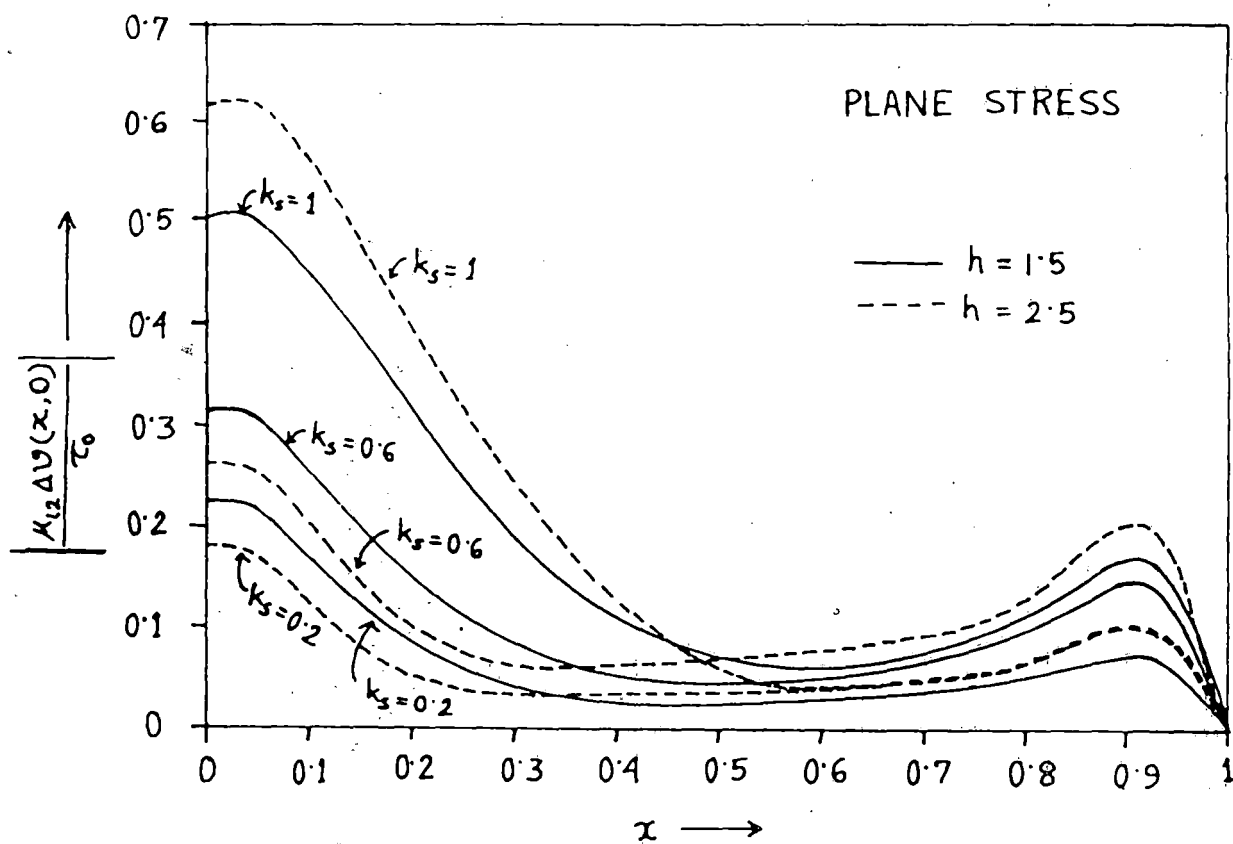


Fig. 5. Crack opening displacement vs. dimensionless distance for Graphite-Epoxy composite.

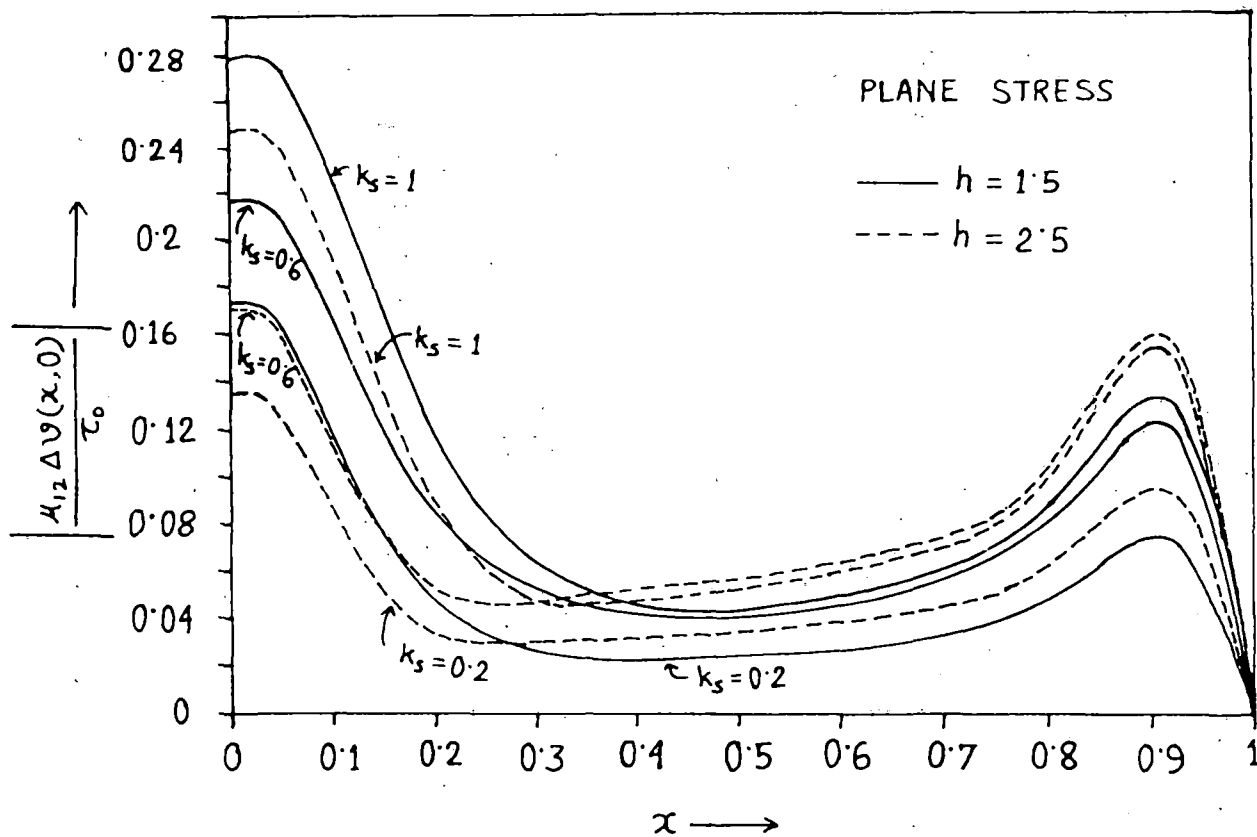


Fig. 6. Crack opening displacement vs. dimensionless distance for Boron-Epoxy composite.

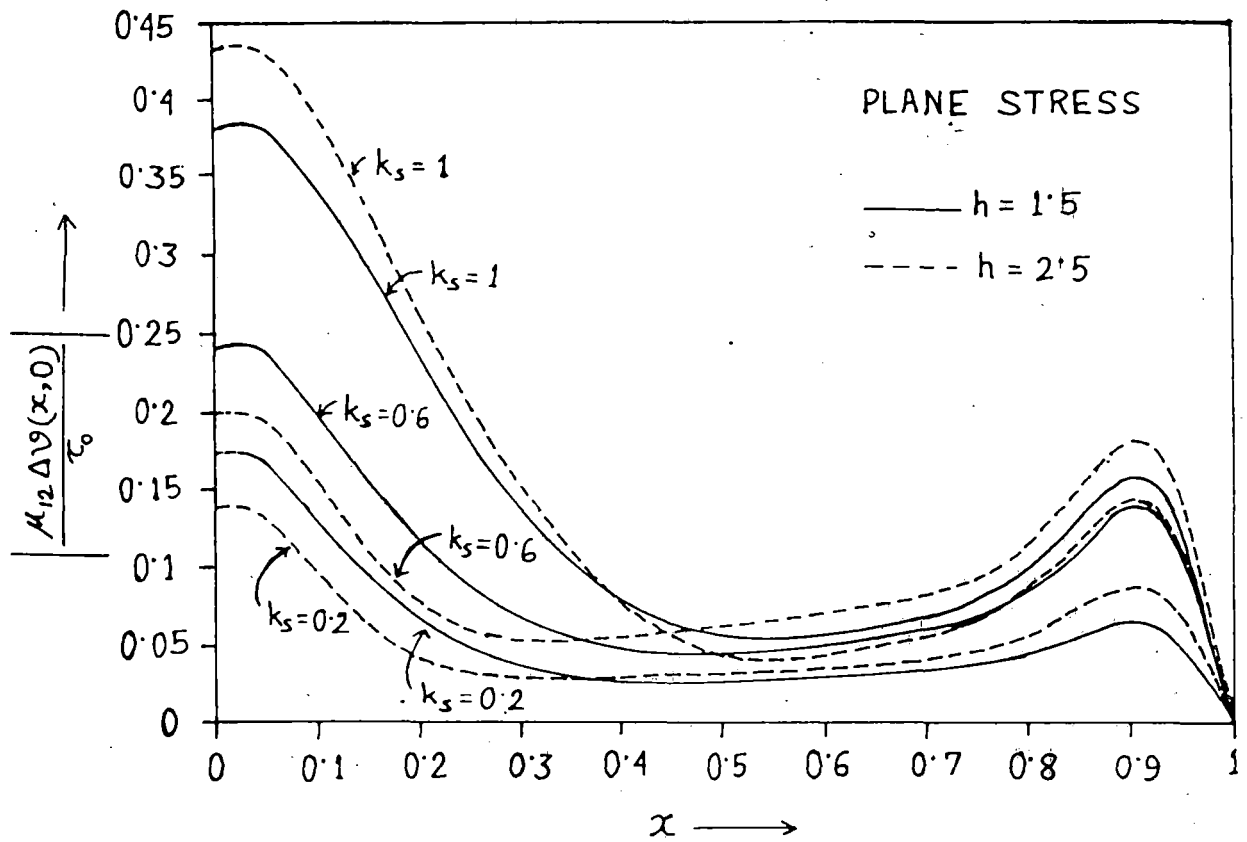


Fig. 7. Crack opening displacement vs. dimensionless distance for Steel-Mylar composite.

APPENDIX

EVALUATION OF THE KERNAL $K_1(\eta, t)$:

The kernal $K_1(\eta, t)$ given by equation (49) is

$$K_1(\eta, t) = \int_0^{\infty} \xi H(\xi) J_0(\xi t) J_0(\xi \eta) d\xi \quad (A1)$$

where $H(\xi)$ is given in equation (30)

To evaluate the integral (A1) we consider two contour integrals :

$$I_1 = \int_{C_1} M(\xi, \gamma_1, \gamma_2) J_0(\xi t) H_0^{(1)}(\eta \xi) d\xi, \quad \eta > t \quad (A2)$$

$$I_2 = \int_{C_2} M(\xi, \gamma_1, \gamma_2) J_0(\xi t) H_0^{(2)}(\eta \xi) d\xi, \quad \eta > t$$

where

$$M(\xi, \gamma_1, \gamma_2) = \frac{c_{11} \xi^2 - \alpha \gamma_1 c_{22} - \beta (c_{12} \xi^2 - \alpha \gamma_2 c_{22})}{(\alpha_1 - \beta \alpha_2) \theta} - \xi$$

$$\gamma_1 = \left[\frac{1}{2} \left\{ -B_1 + (B_1^2 - 4B_2)^{1/2} \right\} \right]^{1/2}$$

$$\gamma_2 = \left[\frac{1}{2} \left\{ -B_1 - (B_1^2 - 4B_2)^{1/2} \right\} \right]^{1/2}$$

$$B_1 = \frac{1}{c_{22}} \left\{ (c_{12}^2 + 2c_{12} - c_{11} c_{22}) \xi^2 + (1 + c_{22}) k_s^2 \right\}$$

$$B_2 = \frac{1}{c_{22}} (\xi^2 - k_s^2) (c_{11} \xi^2 - k_s^2)$$

and C_1, C_2 are the closed contours defined in Fig.8.

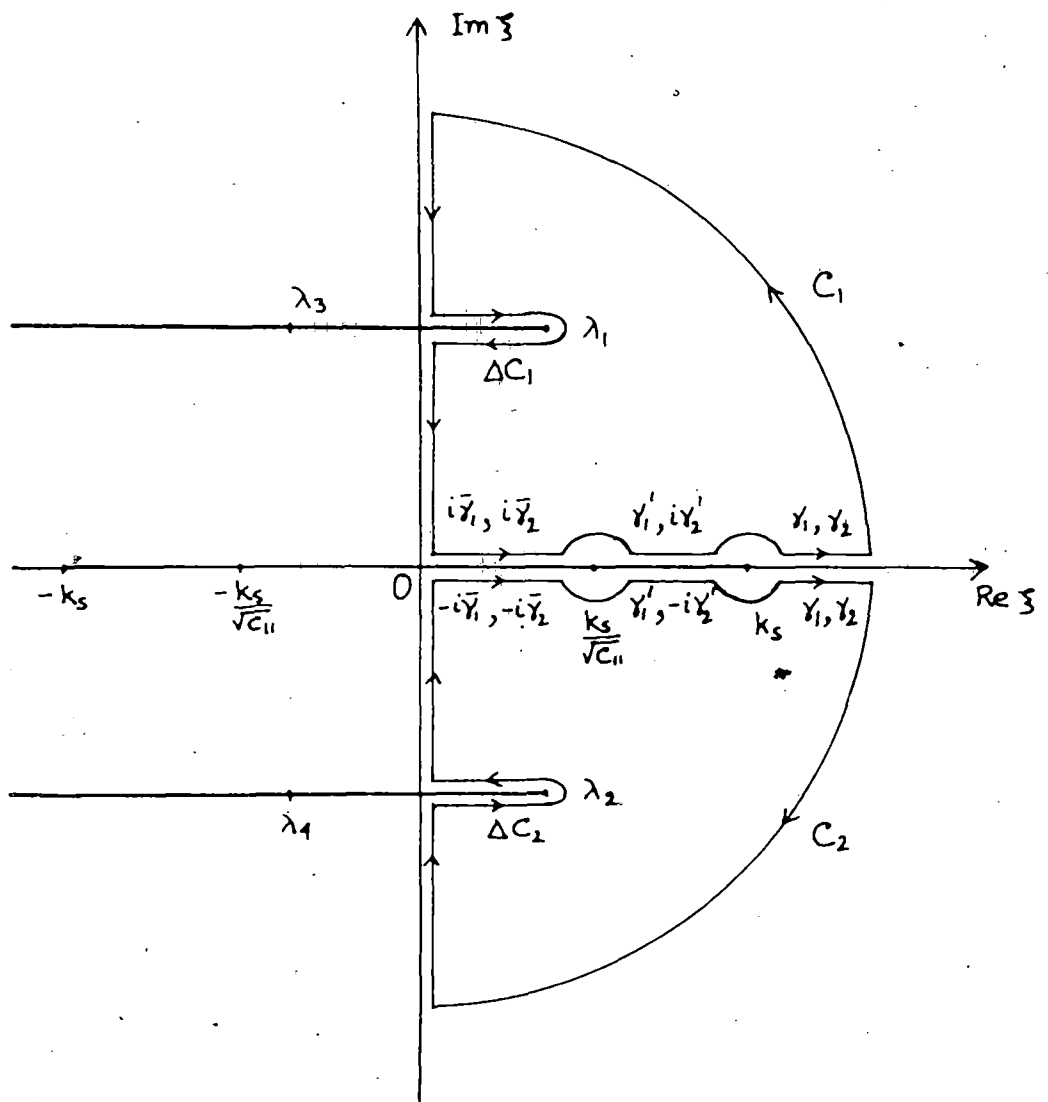


Fig. 8. Contours of integration for integral in equation (A1).

TABLE - 1

ENGINEERING ELASTIC CONSTANTS

Materials	E_1 (psi)	E_2 (psi)	μ_{12} (psi)	ν_{12}
Boron-Epoxi composite	32.5×10^6	1.84×10^6	0.642×10^6	0.256
Graphite-Epoxi composite	2.22×10^6	22.93×10^6	0.8×10^6	0.033
Steel-Mylar composite	26.28×10^6	4.1×10^6	0.9×10^6	0.44

Assuming the relation

$$\left\{ \frac{(c_{12}^2 + 2c_{12}c_{11}c_{22})(1+c_{22})}{c_{22}^2} + \frac{2(1+c_{11})}{c_{22}} \right\}^2 - \left\{ \frac{(c_{12}^2 + 2c_{12}c_{11}c_{22})^2}{c_{22}^2} - \frac{4c_{11}}{c_{22}} \right\} \times \left\{ \frac{(1+c_{22})^2}{c_{22}^2} - \frac{4}{c_{22}} \right\} < 0 \quad (A3)$$

it is noted that the branch points $\xi = \lambda_i$ ($i=1-4$) corresponding to the roots of the equation $B_1^2 - 4B_2 = 0$ are always complex.

Now the branch points corresponding to the roots of the equations

$$-B_1 + (B_1^2 - 4B_2)^{1/2} = 0 \quad \text{and} \quad -B_1 - (B_1^2 - 4B_2)^{1/2} = 0$$

are $\xi = \pm k_0$ and $\xi = \pm k_0 / \sqrt{c_{22}}$ respectively, where it is assumed that

$$c_{11}c_{22} - c_{12}^2 - 2c_{12} > 1 + c_{22} \quad (A4)$$

and
$$c_{12}^2 + 2c_{12} + c_{11} > 0$$

Most of the orthotropic materials satisfy the relations (A3) and (A4). Therefore under the above condition, $\xi = \pm k_0 / \sqrt{c_{22}}$ and $\xi = \pm k_0$ are the branch points of γ_1 and γ_2 respectively.

The integrals in equation (A2) are found to be zero on the contours ΔC_1 and ΔC_2 (fig.8) around the branch cuts from λ_1 and λ_2 . Thus integrating the integrals I_1 and I_2 along the contours C_1 and C_2 (fig.8) we obtain

for contour C_1 :

$$\begin{aligned}
 & \int_0^{k_0/\sqrt{c_{11}}} M(\xi, i\bar{\gamma}_1, i\bar{\gamma}_2) J_0(\xi t) H_0^{(1)}(\eta\xi) d\xi + \\
 & + \int_{k_0/\sqrt{c_{11}}}^{k_0} M(\xi, \gamma_1', i\gamma_2') J_0(\xi t) H_0^{(1)}(\eta\xi) d\xi + \\
 & + \int_{k_0}^{\infty} M(\xi, \gamma_1, \gamma_2) J_0(\xi t) H_0^{(1)}(\eta\xi) d\xi + \\
 & + i \int_0^{\infty} M(iy, i\bar{\gamma}_1', i\bar{\gamma}_2') J_0(iyt) H_0^{(1)}(iy\eta) dy = 0, \quad \eta > t \quad (A5)
 \end{aligned}$$

and for the contour C_2 :

$$\begin{aligned}
 & \int_0^{k_0/\sqrt{c_{11}}} M(\xi, -i\bar{\gamma}_1, -i\bar{\gamma}_2) J_0(\xi t) H_0^{(2)}(\eta\xi) d\xi + \\
 & + \int_{k_0/\sqrt{c_{11}}}^{k_0} M(\xi, \gamma_1', -i\gamma_2') J_0(\xi t) H_0^{(2)}(\eta\xi) d\xi + \\
 & + \int_{k_0}^{\infty} M(\xi, \gamma_1, \gamma_2) J_0(\xi t) H_0^{(2)}(\eta\xi) d\xi - \\
 & - i \int_0^{\infty} M(-iy, -i\bar{\gamma}_1', -i\bar{\gamma}_2') J_0(-iyt) H_0^{(2)}(-iy\eta) dy = 0, \quad \eta > t \quad (A6)
 \end{aligned}$$

where

$$\bar{\gamma}_1 = \left[\frac{1}{2} \left\{ B_1 - (B_1^2 - 4\bar{B}_2)^{1/2} \right\} \right]^{1/2}$$

$$\bar{\gamma}_2 = \left[\frac{1}{2} \left\{ B_1 + (B_1^2 - 4\bar{B}_2)^{1/2} \right\} \right]^{1/2}$$

$$\bar{\gamma}'_1 = \left[\frac{1}{2} \left\{ \bar{B}'_1 - (\bar{B}'_1{}^2 - 4\bar{B}'_2)^{1/2} \right\} \right]^{1/2}$$

$$\bar{\gamma}'_2 = \left[\frac{1}{2} \left\{ \bar{B}'_1 + (\bar{B}'_1{}^2 - 4\bar{B}'_2)^{1/2} \right\} \right]^{1/2}$$

$$\gamma'_1 = \left[\frac{1}{2} \left\{ -B_1 + (B_1^2 + 4B_2)^{1/2} \right\} \right]^{1/2}$$

$$\gamma'_2 = \left[\frac{1}{2} \left\{ B_1 + (B_1^2 + 4B_2)^{1/2} \right\} \right]^{1/2}$$

$$\bar{B}_2 = \frac{c_{11}}{c_{22}} \left(k_s^2 - \xi^2 \right) \left(\frac{k_s^2}{c_{11}} - \xi^2 \right)$$

$$\bar{B}'_1 = \frac{1}{c_{22}} \left\{ -(c_{12}^2 + 2c_{12} - c_{11}c_{22})y^2 + (1 + c_{22})k_s^2 \right\}$$

$$\bar{B}'_2 = \frac{c_{11}}{c_{22}} \left(y^2 + k_s^2 \right) \left(y^2 + \frac{k_s^2}{c_{11}} \right)$$

$$B'_2 = \frac{c_{11}}{c_{22}} \left(k_s^2 - \xi^2 \right) \left(\xi^2 - \frac{k_s^2}{c_{11}} \right)$$

Using the relations

$$J_0(-iyt) = J_0(iyt) \quad \text{and} \quad H_0^{(2)}(-iy\eta) = -H_0^{(1)}(iy\eta)$$

(A6) can be written as

$$\begin{aligned} & 2 \int_0^\infty M(\xi, \gamma_1, \gamma_2) J_0(\xi t) J_0(\xi \eta) d\xi - \\ & - \int_0^{k_0/\sqrt{c_{11}}} M(\xi, -i\bar{\gamma}'_1, -i\bar{\gamma}'_2) J_0(\xi t) H_0^{(1)}(\eta\xi) d\xi - \\ & - \int_{k_0/\sqrt{c_{11}}}^k M(\xi, \gamma'_1, -i\gamma'_2) J_0(\xi t) H_0^{(1)}(\eta\xi) d\xi - \\ & - \int_{k_0}^\infty M(\xi, \gamma_1, \gamma_2) J_0(\xi t) H_0^{(1)}(\eta\xi) d\xi + \\ & + i \int_0^\infty M(-iy, -i\bar{\gamma}'_1, -i\bar{\gamma}'_2) J_0(iyt) H_0^{(1)}(iy\eta) dy = 0, \quad \eta > t \end{aligned} \quad (A7)$$

Adding (A5) and (A7) and using

$$M(-iy, -i\bar{\gamma}'_1, -i\bar{\gamma}'_2) = -M(iy, i\bar{\gamma}'_1, i\bar{\gamma}'_2)$$

the kernel $K_1(\eta, t)$ for $\eta > t$ can be finally written as

$$K_1(\eta, t) = -i \left[\int_0^{k/\sqrt{c_{11}}} \left(\bar{X}_1 \bar{\gamma}_1 - \bar{X}_2 \bar{\gamma}_2 \right) J_0(\xi t) H_0^{(1)}(\xi \eta) d\xi + \int_{k/\sqrt{c_{11}}}^k \left(X'_2 \gamma'_2 \right) J_0(\xi t) H_0^{(1)}(\xi \eta) d\xi \right], \quad \eta > t \quad (A8)$$

where

$$\bar{X}_1 = \frac{[c_{12}(1+c_{12})\xi^2 - c_{22}(c_{11}\xi^2 - k^2 + \bar{\gamma}_1^2)](c_{11}\xi^2 - k^2 - c_{12}\bar{\gamma}_2^2)}{(c_{11}\xi^2 - k^2)(1+c_{12})(\bar{\gamma}_1^2 - \bar{\gamma}_2^2)\theta} \quad (A9)$$

$$\bar{X}_2 = \frac{[c_{12}(1+c_{12})\xi^2 - c_{22}(c_{11}\xi^2 - k^2 + \bar{\gamma}_2^2)](c_{11}\xi^2 - k^2 - c_{12}\bar{\gamma}_1^2)}{(c_{11}\xi^2 - k^2)(1+c_{12})(\bar{\gamma}_1^2 - \bar{\gamma}_2^2)\theta} \quad (A10)$$

$$X'_2 = \frac{[c_{12}(1+c_{12})\xi^2 - c_{22}(c_{11}\xi^2 - k^2 + \gamma_2'^2)](c_{11}\xi^2 - k^2 + c_{12}\gamma_1'^2)}{(c_{11}\xi^2 - k^2)(1+c_{12})(\gamma_1'^2 + \gamma_2'^2)\theta} \quad (A11)$$

The corresponding expression of $K_1(\eta, t)$ for $\eta < t$ follows from (A8) by interchanging η and t .

DIFFRACTION OF SH-WAVES BY A GRIFFITH CRACK IN NONHOMOGENEOUS ELASTIC STRIP

1. INTRODUCTION

The natural or artificial materials are usually inhomogeneous; so in recent years a great attention has been given in the study of diffraction of elastic waves by cracks or obstacles in inhomogeneous medium in view of their application in fracture mechanics. Many problems have been solved involving one or more cracks in an infinite homogeneous elastic medium. Loeber and Sih (1968) and Mal (1970) have studied the problem of diffraction of elastic waves by a Griffith crack in an infinite medium. The problem of finite crack at the interface of two elastic half spaces has been discussed by Srivastava et al (1980) and Bostrom (1987). Singh et al (1977, 1980) considered the problem of scattering of SH-wave by cracks or strips in nonhomogeneous infinite elastic medium. Papers involving crack or strip in infinitely long elastic strip are very few. The problem of an infinite elastic strip containing an arbitrary number of unequal size Griffith cracks, located parallel to its surfaces and opened by an arbitrary internal pressure has been treated by Adams (1980). Finite crack perpendicular to the surface of the infinitely long elastic strip has been studied by Chen (1978) for impact load and by Srivastava et al (1981) for normally incident waves. Recently Shindo et al (1986) considered the problem of

impact response of a finite crack in an orthotropic strip.

In our paper, the diffraction of normally incident SH-waves by a Griffith crack situated in an infinitely long inhomogeneous elastic strip has been discussed. The shear modulus (μ) and the density (ρ) of the material have been assumed to vary both in horizontal and vertical directions. Applying Fourier transform the mixed boundary value problem has been converted to the solution of dual integral equations. The dual integral equations finally has been reduced to a Fredholm integral equation of second kind by applying Abel transform. Expressions for stress intensity factor and crack opening displacement have been derived. The numerical values of stress intensity factor and crack opening displacement have been depicted by means of graphs to show the effect of material inhomogeneity.

2. FORMULATION OF THE PROBLEM

Consider the problem of diffraction of SH-waves by a Griffith crack in an inhomogeneous elastic strip of width $2h_1$. The crack is located in the region $-d \leq x_1 \leq d$, $-\infty < y_1 < \infty$, $z_1 = 0$ (fig.1). Normalizing all the lengths with respect to d and putting $x_1/d = x$, $y_1/d = y$, $z_1/d = z$, $h_1/d = h$ it is found that the location of the crack is $-1 \leq x \leq 1$, $-\infty < y < \infty$, $z = 0$ referred to a cartesian co-ordinate system (x, y, z) . Let a plane harmonic SH-wave originating at $z = -\infty$ impinge on the crack normally to the x -axis. The variation of the shear modulus μ and the density ρ is taken in both the vertical (z) and

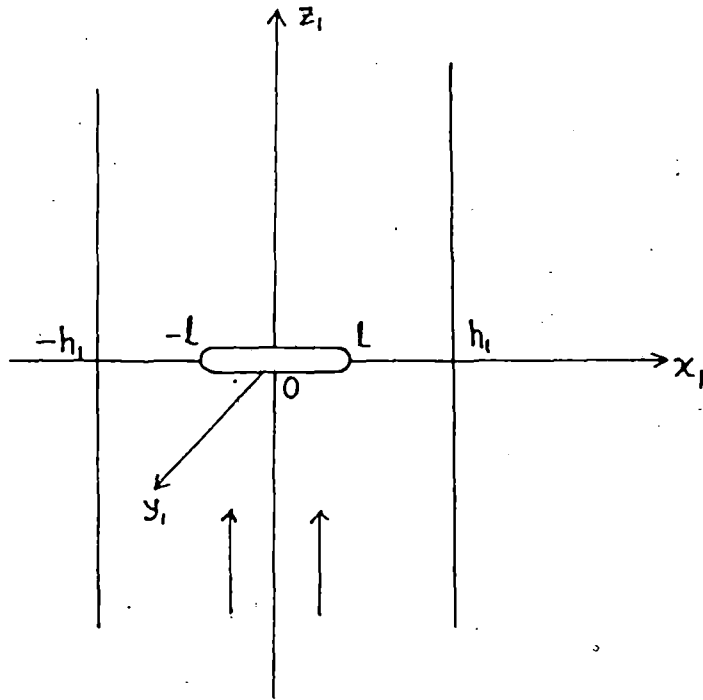


Fig.1 crack in the inhomogeneous strip .

horizontal(x) directions in such a manner that the shear velocity $(\mu/\rho)^{1/2}$ is constant.

The only non-vanishing y-component of the displacement which is independent of y is $v=v(x,z,t)$.

The equation of motion is given by

$$\frac{\partial}{\partial x} \left[\mu \frac{\partial v}{\partial x} \right] + \frac{\partial}{\partial z} \left[\mu \frac{\partial v}{\partial z} \right] = \rho \frac{\partial^2 v}{\partial t^2} \quad (1)$$

If we consider $v(x,z,t)$ in the form

$$v(x,z,t) = \frac{V(x,y,t)}{\sqrt{\mu(x,z)}} \quad (2)$$

equation (1) takes the form

$$\mu \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial z^2} \right) + \frac{1}{2} \left[\frac{1}{2\mu} \left\{ \left(\frac{\partial \mu}{\partial x} \right)^2 + \left(\frac{\partial \mu}{\partial z} \right)^2 \right\} - \frac{\partial^2 \mu}{\partial z^2} \right] V = \rho \frac{\partial^2 V}{\partial t^2} \quad (3)$$

Putting $V(x,z,t) = F(x)G(z)e^{-i\omega t}$ and $\mu(x,z) = \mu_0 f(x)g(z)$, $\rho(x,z) = \rho_0 f(x)g(z)$ in equation (3) where μ_0, ρ_0 are constants, such that $(\mu_0/\rho_0)^{1/2} = c_2$ is the shear wave velocity, it is found that $F(x)$ and $G(z)$ satisfy the following equations

$$\frac{\partial^2 F}{\partial x^2} + n^2 F = 0 \quad (4)$$

$$\frac{\partial^2 G}{\partial z^2} + \left[\frac{d^2 \omega^2}{c_2^2} - a^2 - b^2 - n^2 \right] G = 0 \quad (5)$$

provided $f(x)$ and $g(z)$ are of the form

$$-\frac{1}{4}\left(\frac{\partial f}{\partial x} / f\right)^2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} / f\right) = a^2 \quad (6)$$

$$-\frac{1}{4}\left(\frac{\partial g}{\partial z} / g\right)^2 + \frac{1}{2}\left(\frac{\partial^2 g}{\partial z^2} / g\right) = b^2 \quad (7)$$

where n , a and b are constants.

Equations (6) and (7) are of similar type. We rewrite the equation (6) as

$$\frac{d}{df} \left[\left(\frac{\partial f}{\partial x} \right)^2 / f \right] = 4a^2 \quad (8)$$

Integrating the above equation we obtain

$$\frac{df}{dx} = \pm 2(a^2 f^2 + a_1 f)^{1/2} \quad (9)$$

where a_1 is a constant.

Again integrating (9) it can be shown

$$x + a_2 = \pm \frac{1}{2} \int \frac{df}{(a^2 f^2 + a_1 f)^{1/2}} \quad (10)$$

where a_2 is another constant.

Now, we consider the following three possible cases :

Case - 1 : $a_1 = 0$.

In this case equation (10) takes the form

$$x + a_2 = \pm \frac{1}{2} \int \frac{df}{af}$$

which on integration yields

$$f(x) = C_1 e^{\pm 2ax} \quad (11)$$

where C_1 is a constant.

Case - 2 : $a = 0$.

Putting $a=0$ in equation (10) we get

$$x + a_2 = \pm \frac{1}{2} \int \frac{df}{\sqrt{a_1 f}}$$

from which we get

$$f(x) = C_2 \left(1 + \frac{x}{a_2} \right)^2 \quad (12)$$

where $C_2 = a_1 a_2^2$ is a constant.

Case - 3 : $a \neq 0$, $a_1 \neq 0$.

In this case (10) yields

$$f(x) = C_3 \cosh^2(ax + a_3) \quad (13)$$

where $C_3 = -a_1/a^2$, $a_3 = aa_2$.

Equation (13) gives the general solution of equation (8) when both the constants a and a_1 are non zero.

In view of the above results, we assume the forms of $f(x)$ and $g(z)$ as

$$f(x) = \cosh^2(ax) \quad \text{and} \quad g(z) = \cosh^2(bz) \quad (14)$$

so that equations (6) and (7) are automatically satisfied.

Now the shear modulus $\mu(x,z)$ and density of the medium $\rho(x,z)$ are

$$\mu = \mu_0 \cosh^2(ax) \cosh^2(bz) \quad , \quad \rho = \rho_0 \cosh^2(ax) \cosh^2(bz) \quad (15)$$

The displacement component $v^{(i)}(x, z, t)$ and stress $\tau^{(i)}(x, z, t)$ due to incident waves are given by

$$v^{(i)}(x, z, t) = \frac{A_0 e^{i(k_2 z - \omega t)}}{\sqrt{\mu_0} \cosh(ax) \cosh(bz)} \quad (16)$$

$$\text{and } \tau_{yz}^{(i)}(x, z, t) = A_0 \sqrt{\mu_0} \cosh(ax) [ik_2 \cosh(bz) - b \sinh(bz)] e^{i(k_2 z - \omega t)} \quad (17)$$

where A_0 is a constant and $k_2 = \omega d / c_2$.

Henceforth the time factor $e^{-i\omega t}$ will be suppressed in the sequel.

Using (8) and (2), equation (1) takes the form

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial z^2} + k^2 V = 0 \quad , \quad k^2 = (k_2^2 - a^2 - b^2) \quad (18)$$

whose solution is

$$v(x, z) = \int_0^\infty B_1(\xi) e^{-\beta z} \cos(\xi x) d\xi + \int_0^\infty C_1(\zeta) \cosh(\alpha x) \sin(\zeta z) d\zeta \quad (19)$$

$$\text{where } \alpha = (\zeta^2 - k^2)^{1/2} \quad , \quad \zeta > k \quad , \quad \beta = (\xi^2 - k^2)^{1/2} \quad , \quad \xi > k \\ = -i(k^2 - \zeta^2)^{1/2} \quad , \quad \zeta < k \quad , \quad = -i(k^2 - \xi^2)^{1/2} \quad , \quad \xi < k$$

Now displacement $v(x, z)$ and stresses $\tau_{yz}(x, z)$, $\tau_{xy}(x, z)$ due to

scattered field are

$$v(x, z) = \frac{1}{\cosh(ax)\cosh(bz)} \left[\int_0^{\infty} B(\xi) e^{-\beta z} \cos \xi x d\xi + \int_0^{\infty} C(\zeta) \cosh(ax) \sin \zeta z d\zeta \right] \quad (20)$$

$$\begin{aligned} \tau_{yz}(x, z) = & -\mu_0 b \cosh(ax) \sinh(bz) \left[\int_0^{\infty} B(\xi) e^{-\beta z} \cos \xi x d\xi + \right. \\ & \left. + \int_0^{\infty} C(\zeta) \cosh(ax) \sin \zeta z d\zeta \right] + \mu_0 \cosh(ax) \cosh(bz) \times \\ & \times \left[-\int_0^{\infty} \beta B(\xi) e^{-\beta z} \cos \xi x d\xi + \int_0^{\infty} \zeta C(\zeta) \cosh(ax) \cos \zeta z d\zeta \right] \quad (21) \end{aligned}$$

$$\begin{aligned} \tau_{xy}(x, z) = & -\mu_0 a \sinh(ax) \cosh(bz) \left[\int_0^{\infty} B(\xi) e^{-\beta z} \cos \xi x d\xi + \right. \\ & \left. + \int_0^{\infty} C(\zeta) \cosh(ax) \sin \zeta z d\zeta \right] + \mu_0 \cosh(ax) \cosh(bz) \times \\ & \times \left[-\int_0^{\infty} \xi B(\xi) e^{-\beta z} \sin \xi x d\xi + \int_0^{\infty} \alpha C(\zeta) \sinh(ax) \sin \zeta z d\zeta \right] \quad (22) \end{aligned}$$

$$\text{where } B(\xi) = \frac{1}{\sqrt{\mu_0}} B_1(\xi) \quad , \quad C(\zeta) = \frac{1}{\sqrt{\mu_0}} C_1(\zeta) \quad .$$

The boundary conditions are

$$\tau_{yz}(x, 0) = -\tau_0 \cosh(ax) \quad , \quad |x| \leq 1 \quad (23)$$

$$v(x,0) = 0, \quad 1 \leq |x| \leq h \quad (24)$$

$$\text{and} \quad \tau_{xy}(\pm h, z) = 0, \quad |z| < \infty \quad (25)$$

where $\tau_0 = ik_2 \sqrt{\mu_0}$.

From the boundary condition (25) $C(\zeta)$ is found to be expressible in terms of $B(\xi)$ as follows:

$$C(\zeta) = \frac{2\zeta}{\pi F_2(\alpha, h, a)} \int_0^\infty \frac{F_1(\xi, h, a) B(\xi)}{\beta^2 + \zeta^2} d\xi \quad (26)$$

$$\text{where } F_1(\xi, h, a) = a \sinh(a h) \cos(\xi h) + \xi \cosh(a h) \sin(\xi h) \quad (27)$$

$$\text{and } F_2(\alpha, h, a) = \alpha \cosh(a h) \sinh(\alpha h) - a \sinh(a h) \cosh(\alpha h) \quad (28)$$

Next, the use of (26) in the boundary condition (23) and (24) yields the following dual integral equations from which the unknown function $B(\xi)$ is to be determined:

$$\int_0^\infty \xi [1 + M(\xi)] B(\xi) \cos(\xi x) d\xi = p(x), \quad |x| \leq 1 \quad (29)$$

$$\text{and} \quad \int_0^\infty B(\xi) \cos(\xi x) d\xi = 0, \quad 1 \leq |x| \leq h \quad (30)$$

$$\text{where } M(\xi) = \left[\frac{\beta}{\xi} - 1 \right] \quad (31)$$

$$p(x) = \frac{\tau_0}{\mu_0} + \frac{2}{\pi} \int_0^\infty \frac{\zeta^2 \cosh(\alpha x)}{F_2(\alpha, h, a)} d\zeta \int_0^\infty \frac{F_1(\xi, h, a) B(\xi)}{\beta^2 + \zeta^2} d\xi \quad (32)$$

3. METHOD OF SOLUTION

In order to solve the dual integral equations (29) and (30), $B(\xi)$ is taken in the form

$$B(\xi) = \frac{\tau_0}{\mu_0} \int_0^1 t \phi(t) J_0(\xi t) dt \quad (33)$$

so that equation (30) is automatically satisfied.

Substitution of the value of $B(\xi)$ from (33) in (29), yields a Fredholm integral equation of second kind

$$\phi(t) + \int_0^1 u [L_1(u, t) + L_2(u, t)] \phi(u) du = 1 \quad (34)$$

where

$$L_1(u, t) = \int_0^\infty \xi M(\xi) J_0(\xi u) J_0(\xi t) d\xi \quad (35)$$

$$L_2(u, t) = -\frac{2}{\pi} \int_0^\infty \frac{I_0(\alpha t) \zeta^2 K(\zeta, u)}{F_2(\alpha, h, a)} d\zeta \quad (36)$$

$$K(\zeta, u) = \int_0^\infty \frac{F_1(\xi, h, a) J_0(\xi u)}{\beta^2 + \zeta^2} d\xi \quad (37)$$

Using the results (Gradshteyn et al, 1965)

$$\int_0^\infty \frac{\xi \sin(\xi h) J_0(\xi u)}{\xi^2 + \alpha^2} d\xi = \frac{\pi}{2} e^{-\alpha h} I_0(\alpha u)$$

$$\int_0^\infty \frac{\cos(\xi h) J_0(\xi u)}{\xi^2 + \alpha^2} d\xi = \frac{\pi}{2\alpha} e^{-\alpha h} I_0(\alpha u)$$

$$\int_0^{\infty} \frac{\xi \sin(\xi h) J_0(\xi u)}{\xi^2 - \alpha_1^2} d\xi = \frac{\pi}{2} \cos(\alpha_1 h) J_0(u \alpha_1)$$

and

$$\int_0^{\infty} \frac{\cos(\xi h) J_0(\xi u)}{\xi^2 - \alpha_1^2} d\xi = -\frac{\pi}{2\alpha_1} \sin(\alpha_1 h) J_0(\alpha_1 u)$$

$K(\zeta, u)$ takes the form

$$\begin{aligned} K(\zeta, u) &= \frac{\pi}{2} \left[\frac{a \sinh(ah)}{\alpha} + \cosh(ah) \right] e^{-\alpha h} I_0(u\alpha) \quad , \quad \zeta > k \\ &= \frac{\pi}{2} \left[\cosh(ah) \cos(\alpha_1 h) - \frac{a \sinh(ah) \sin(\alpha_1 h)}{\alpha_1} \right] J_0(u\alpha_1) \quad , \quad \zeta < k \end{aligned} \quad (38)$$

where $\alpha_1 = (k^2 - \zeta^2)^{1/2}$.

Using contour integration technique (Srivastava et al, 1980), the infinite integral arising in the kernel $L_1(u, t)$ can be converted to finite integral and is given by

$$\begin{aligned} L_1(u, t) &= -ik^2 \int_0^1 (1-\eta^2)^{1/2} J_0(k\eta t) H_0^{(1)}(k\eta u) d\eta \quad , \quad u > t \\ &= -ik^2 \int_0^1 (1-\eta^2)^{1/2} J_0(k\eta u) H_0^{(1)}(k\eta t) d\eta \quad , \quad u < t \end{aligned} \quad (39)$$

Now

$$L_2(u, t) = -\frac{2}{\pi} \left[\int_0^k + \int_k^{\infty} \right] \frac{I_0(\alpha t) \zeta^2 K(\zeta, u)}{F_2(\alpha, h, a)} d\zeta \quad (40)$$

where

$$F_2(\alpha, h, a) = \alpha \cosh(\alpha h) \sinh(\alpha h) - a \sinh(\alpha h) \cosh(\alpha h) \quad , \quad \zeta > k$$

$$= -[\alpha_1 \cosh(\alpha h) \sin(\alpha_1 h) + a \sinh(\alpha h) \cos(\alpha_1 h)] \quad , \quad \zeta < k$$

Using the value of $K(\zeta, u)$ from (38) and putting $\zeta^2 = k^2(1-y^2)$, $\zeta^2 = k^2(1+y^2)$ in first and second integrals of (40) respectively, it is found that

$$L_2(u, t) = k^2 \left[\int_0^1 \frac{(1-y^2)^{1/2} [k y \cosh(\alpha h) \cos(kyh) - a \sinh(\alpha h) \sin(kyh)]}{k y \cosh(\alpha h) \sin(kyh) + a \sinh(\alpha h) \cos(kyh)} \times \right. \\ \left. \times J_0(kyu) J_0(kyt) dy - \int_0^\infty \frac{(1+y^2)^{1/2} [a \sinh(\alpha h) + k y \cosh(\alpha h)] e^{-kyh}}{k y \cosh(\alpha h) \sinh(kyh) - a \sinh(\alpha h) \cosh(kyh)} \times \right. \\ \left. \times I_0(kyu) I_0(kyt) dy \right] \quad (41)$$

4. STRESS INTENSITY FACTOR AND CRACK OPENING DISPLACEMENT

From (15) the stress τ_{yz} on the plane $z=0$ can be written as

$$\tau_{yz}(x, 0) = \mu_0 \cosh(ax) \left[- \int_0^\infty \beta B(\xi) \cos \xi x d\xi + \int_0^\infty \zeta C(\zeta) \cosh(\alpha x) d\zeta \right]$$

Substituting the value of $C(\zeta)$ and $B(\xi)$ from (20) and (27) the

expression for the stress finally can be deduced as

$$\tau_{yz}(x,0) = \frac{\tau_0 x \cosh(ax)}{(x^2-1)^{1/2}} \phi(1) + o(1) \quad , \quad |x| > 1.$$

Defining the stress intensity factor T by

$$T = \lim_{x \rightarrow 1^+} \left| \frac{(x-1)^{1/2} \tau_{yz}(x,0)}{\tau_0} \right|$$

we obtain,

$$T = \frac{1}{\sqrt{2}} \cosh(a) |\phi(1)| \quad (42)$$

Now the crack opening displacement $\Delta v(x,0) = v(x,0+) - v(x,0-)$ can be obtained from (20) as

$$\Delta v(x,0) = \frac{2}{\cosh(ax)} \int_0^\infty B(\xi) \cos(\xi x) d\xi \quad , \quad |x| \leq 1$$

which on substitution of the value of B(ξ) from (33) takes the form

$$\Delta v(x,0) = \frac{2\tau_0}{\mu_0 \cosh(ax)} \left[(1-x^2)^{1/2} \phi(1) - \int_x^1 (t^2-x^2)^{1/2} \phi'(t) dt \right] \quad , \quad |x| \leq 1 \quad (43)$$

5. NUMERICAL RESULTS AND DISCUSSION

Using the method of Fox and Goodwin (1953) the Fredholm integral equation given by equation (34) has been solved numerically for different values of material inhomogeneity parameters. In this

method the integral in (34) at first has been represented by a quadrature formula involving values of the desired function $\phi(t)$ at pivotal points inside the specified range of integration and then converted to a set of linear algebraic simultaneous equations, solving which the first approximation to the required pivotal values of $\phi(t)$ has been obtained. Applying difference-correction technique the first approximations has been improved. The kernel $L_1(u,t)$ given by (39) and the first integral of the kernel $L_2(u,t)$ given by (41) have been evaluated numerically by using Gauss quadrature integration formula. The second infinite integral of the kernel $L_2(u,t)$ has been evaluated by Simpson's method. After solving the integral equation (34) numerically, the stress intensity factor T and the crack opening displacement $2\mu_0 v(x,0)/\tau_0$ have been calculated numerically and plotted separately against dimensionless frequency k_2 ($0 < k_2 \leq 1.5$) and dimensionless distance x ($0 \leq x \leq 1$) respectively for different values of material inhomogeneity parameters a , b and strip width $2h$.

In fig.2, the effect of the width of the strip on the stress intensity factor when the material is assumed to be homogeneous has been shown where as the effect of inhomogeneity of the material on the stress intensity factor for different width of the strip has been depicted in fig.3-fig.5.

It is found that the effect of the strip width increases prominently with the increase of the frequency where as the inhomogeneity parameters a and b have no remarkable effect on the stress intensity factor for a fixed value of the strip width $2h$.

In fig.6-fig.9 the crack opening displacement against dimensionless distance x for different values of material inhomogeneity parameters a , b and the strip width $2h$ have been illustrated by means of graphs. In each case the maximum value of the crack opening displacement is found to occur at $x=0.08$ and then after a few oscillations it gradually decreases to zero at $x=1$. It is also to be noted that the magnitude of the crack opening displacements increases with the increase in the value of the dimensionless frequency k_2 ; the increase being more prominent in the case of the presence of lateral inhomogeneity.

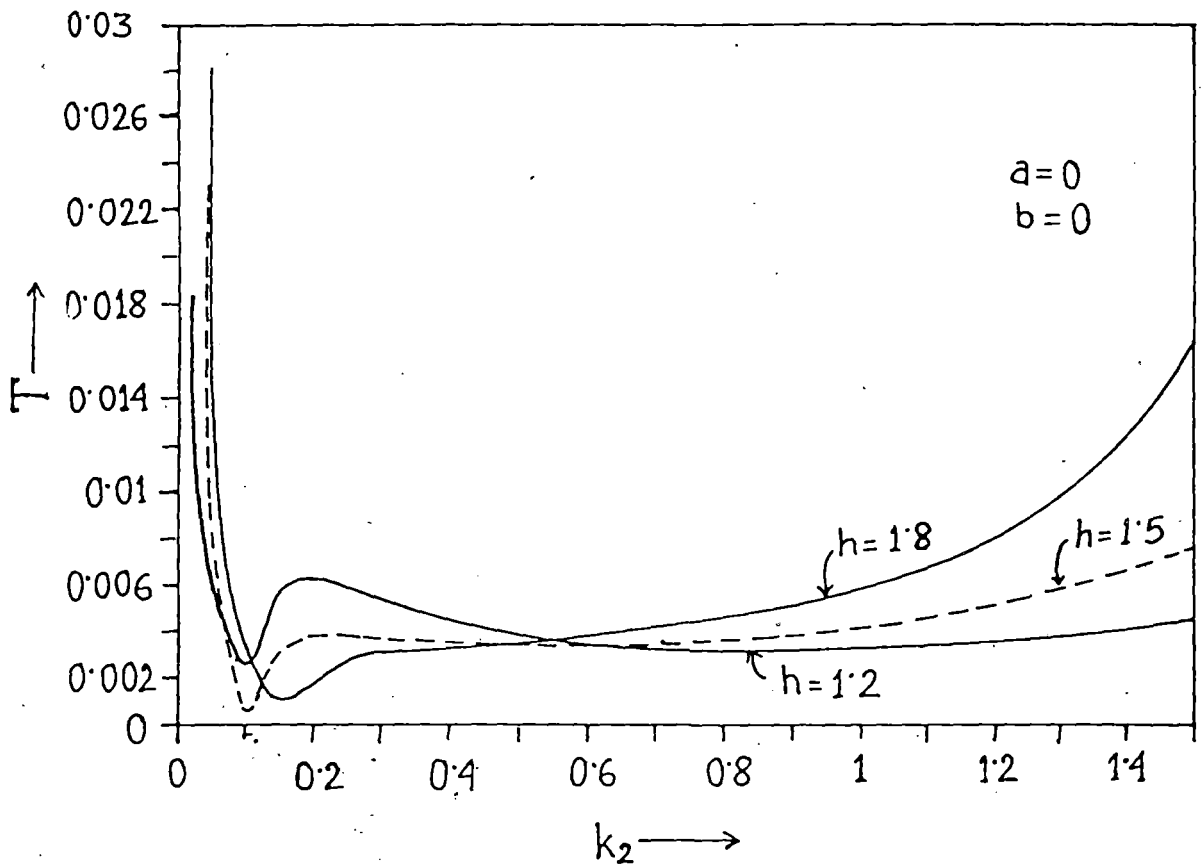


Fig. 2 Stress intensity factor T vs. dimensionless frequency k_2 for homogeneous medium ($a=0$, $b=0$).

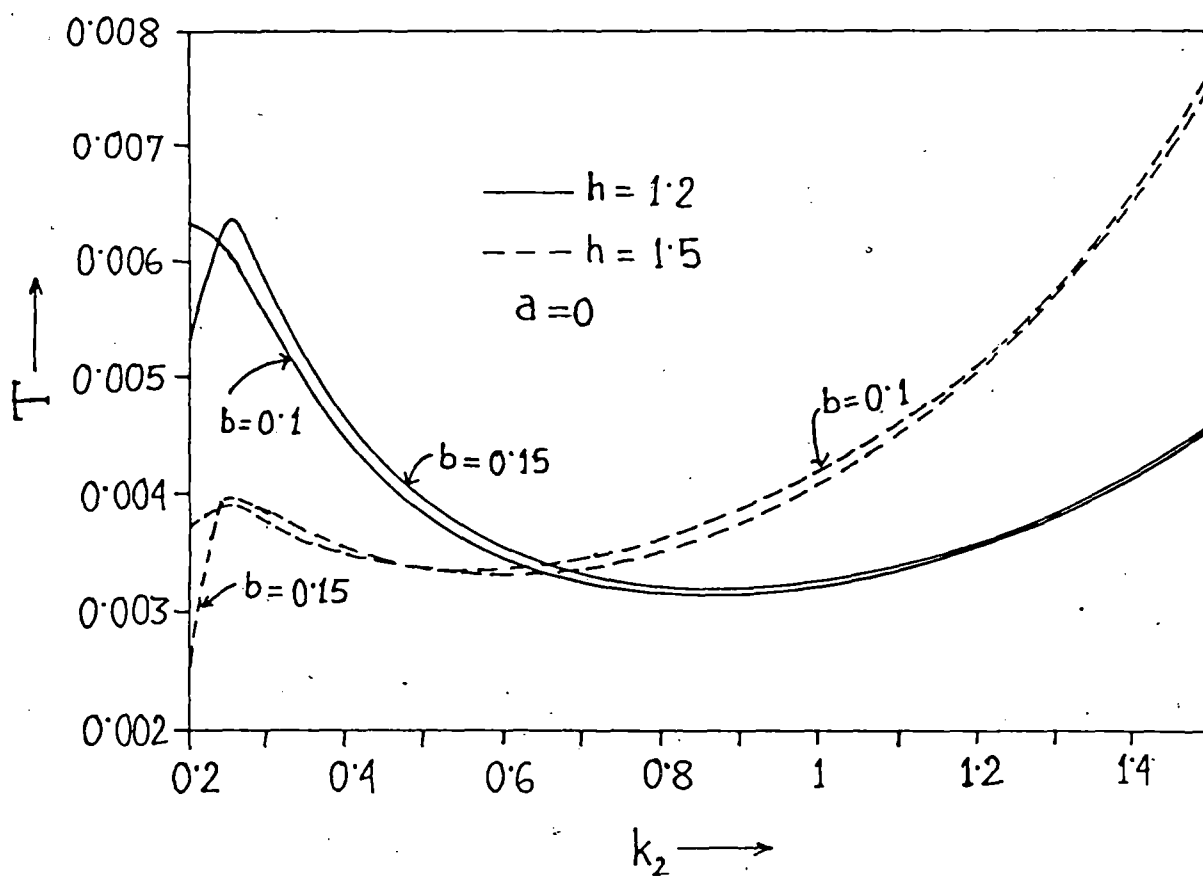


Fig. 3 Stress intensity factor T vs. dimensionless frequency k_2 .

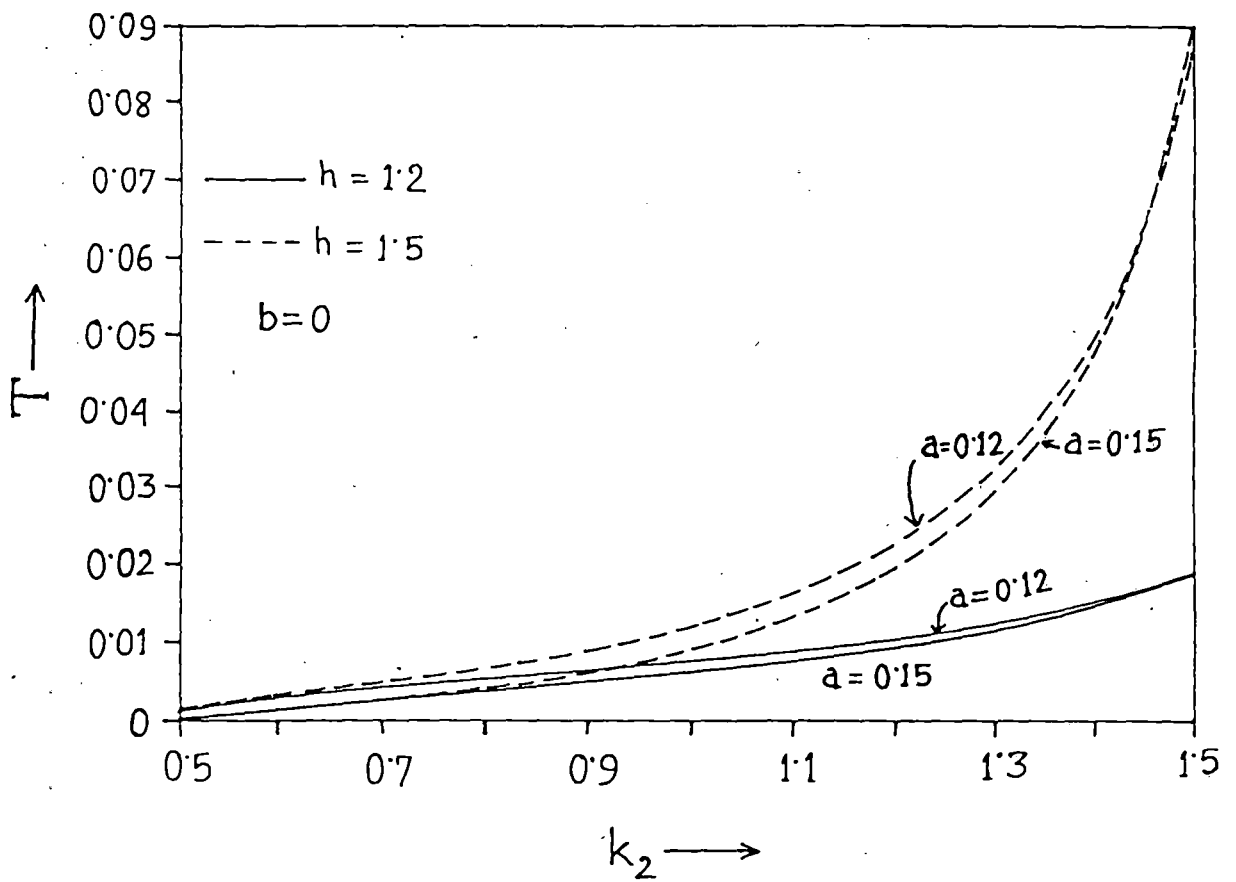


Fig. 4 Stress intensity factor T vs. dimensionless frequency k_2 .

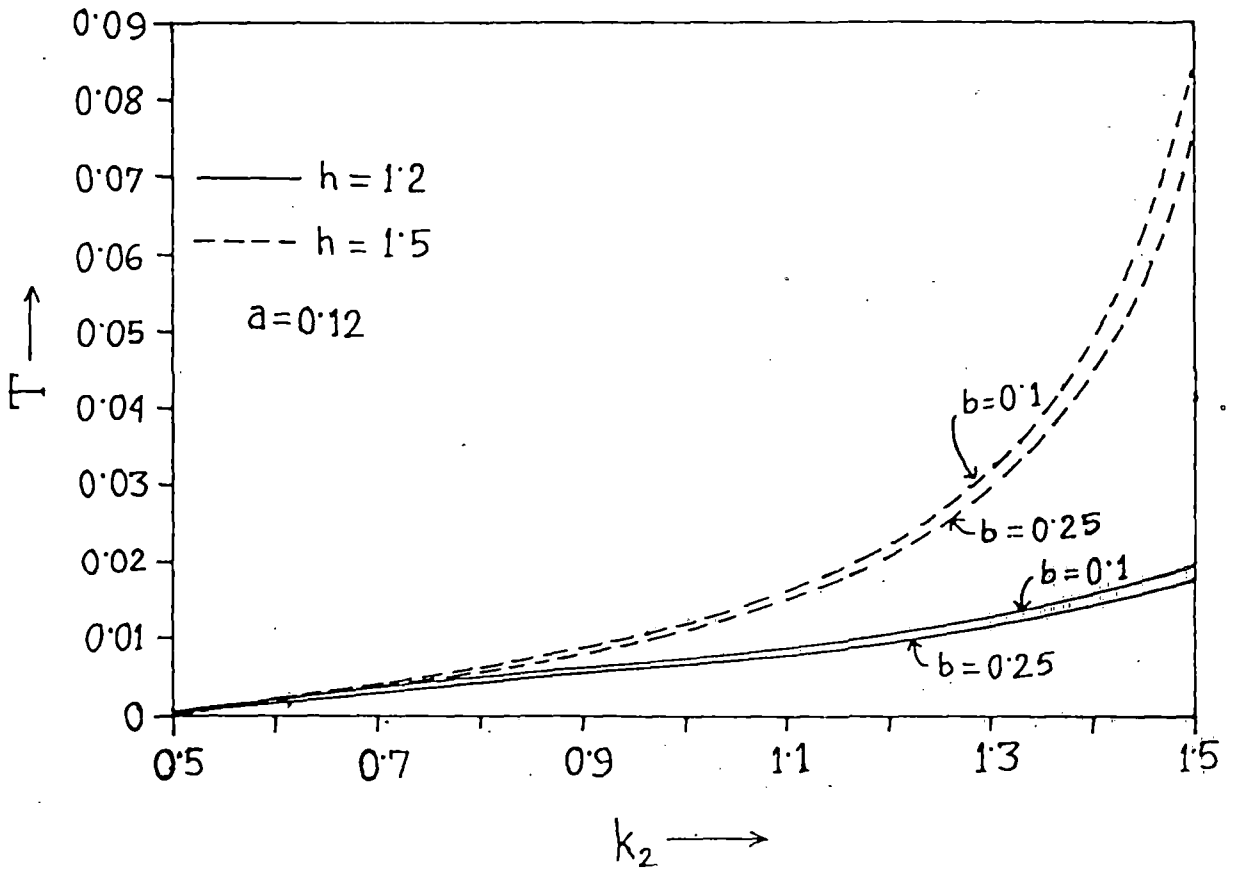


Fig. 5. Stress intensity factor T vs. dimensionless frequency k_2 .

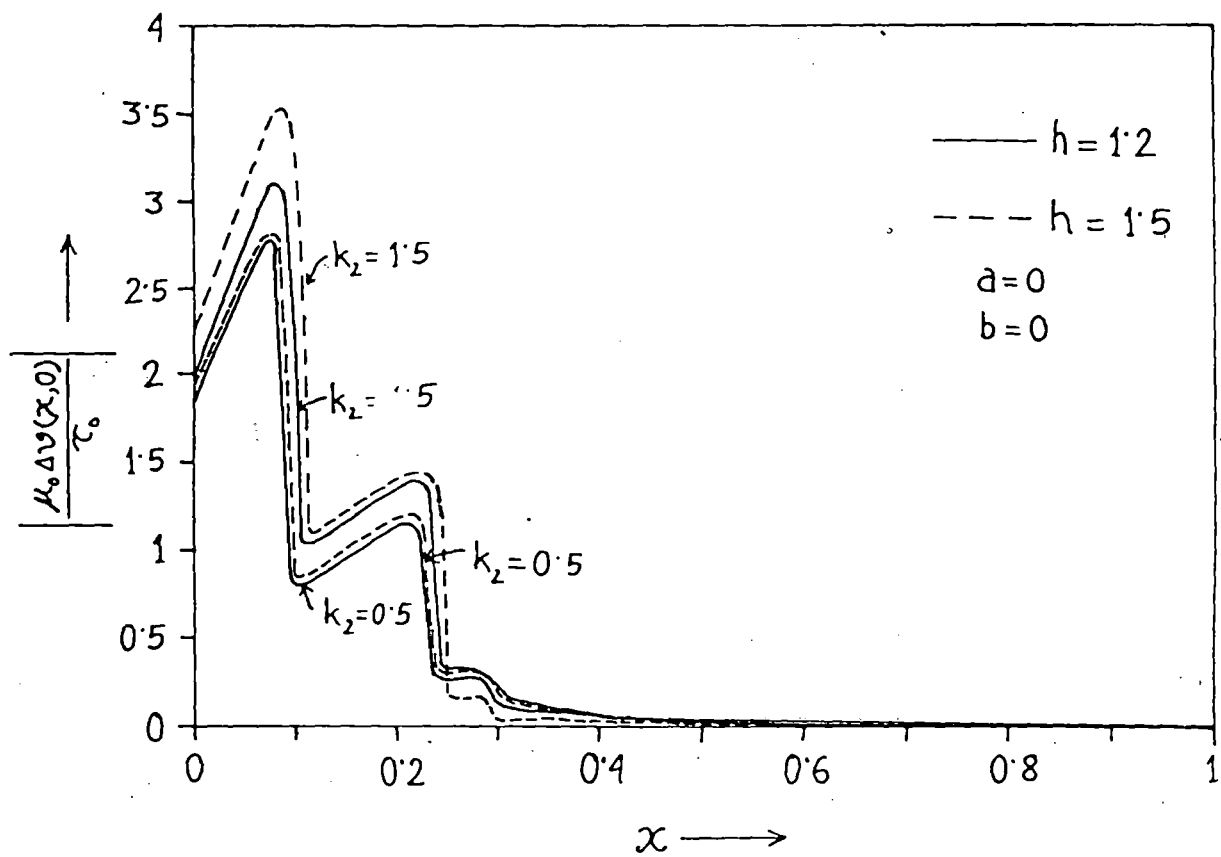


Fig. 6. Crack opening displacement vs. dimensionless distance x ($a=0, b=0$).

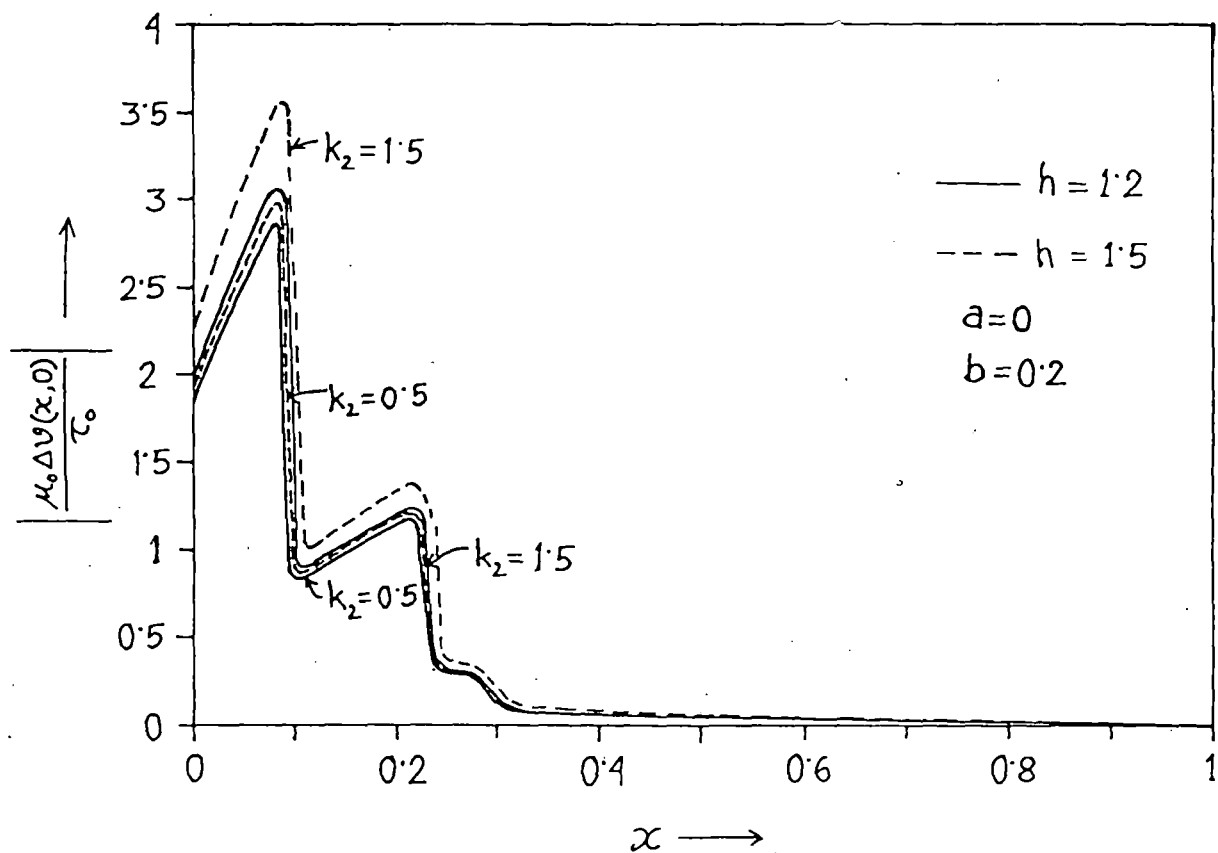


Fig. 7 Crack opening displacement vs. dimensionless distance x
($a = 0, b = 0.2$).

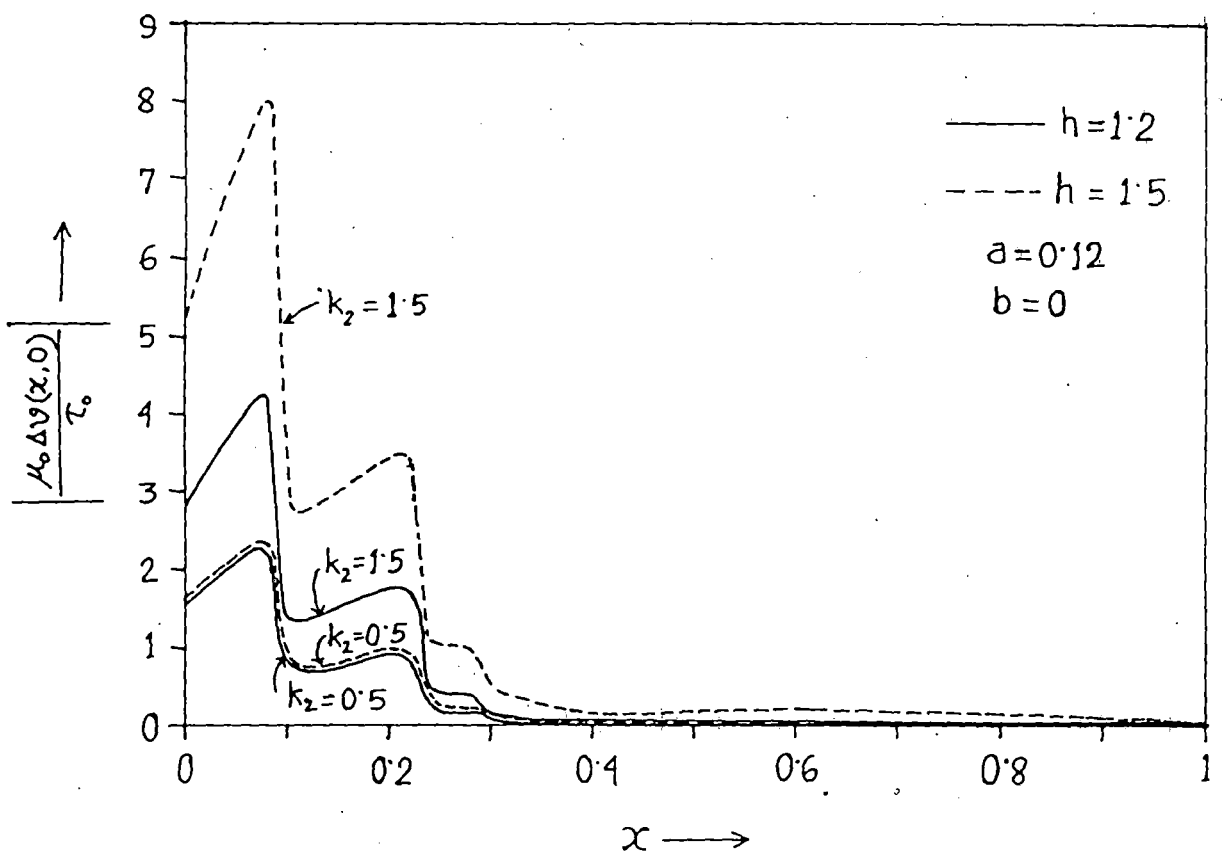


Fig. 8 Crack opening displacement vs. dimensionless distance x
($a=0.12, b=0$)

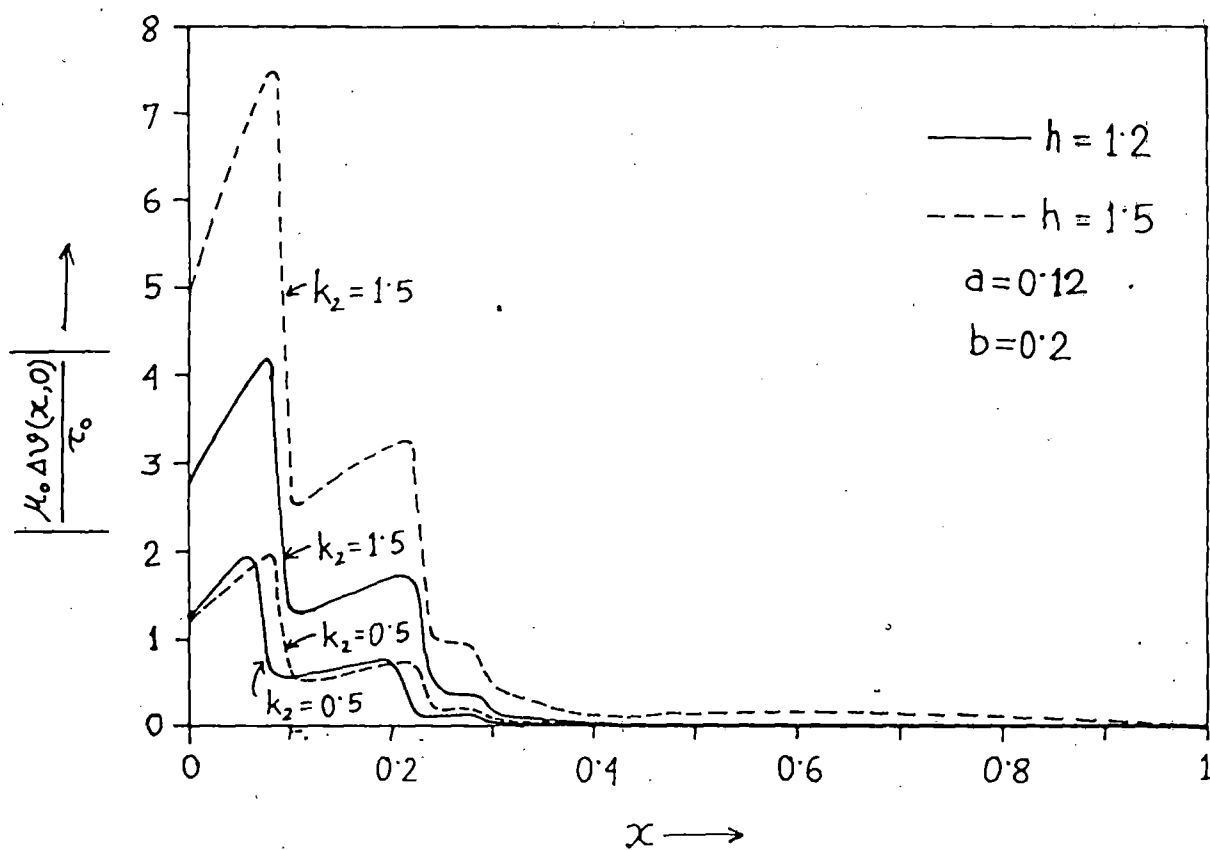


Fig. 9. Crack opening displacement vs. dimensionless distance x ($a=0.12, b=0.2$).