CHAPTER II

EXTENDING CRACK PROBLEMS IN ELASTODYNAMICS

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NON-SYMMETRIC EXTENSION OF A PLANE CRACK DUE TO PLANE SH-WAVES IN A PRE-STRESSED INFINITE ELASTIC MEDIUM

1. Introduction

Since Broberg's (1960) investigation of the solution of a crack expanding symmetrically with constant velocity under conditions of plane stress or strain in a homogeneous isotropic elastic medium in a field of spatially and time invariant tensile stress, a number of papers have appeared analyzing different geometrical situations. Craggs (1963) later solved the same problem as that done by Broberg but he used the method of homogeneous function to obtain the solution. Achenbach and Brock (1971) considered the wave motion generated by a uniformly extending shear crack in a body in a state of uniform anti- plane shear. The case of a crack expanding in an anisotropic medium was considered by Atkinson (1965). This work was later extended by Burridge and Willis (1963), who solved the problem of a crack with elliptical cross-section expanding symmetrically with uniform speed in an anisotropic medium. All the problems mentioned above are however self-similar ones with index (0,0) and are concerned with symmetric expansion of cracks.

Problems involving non-symmetric extension of cracks under uniform loading along the crack surface are not found much in the literature perhaps due to severe mathematical complexity encountered in solving such problems. Following the method of homogeneous functions developed by Craggs (1963)non-symmetric extension of a small flaw into a plane crack under polynomial form of loading was solved by Brock (1976). Following the same procedure, Brock (1975) also solved the problem of non-symmetric extension of a crack due to incidence of plane dilatational waves.The problem of determining the dynamic stress field due to a plane dislocation moving in an infinite elastic medium was formulaled by Ang and Williams (1959) in terms of Fourier integral equation and solved in closed form. Recently, Georgiadis (1991)has developed an integral equation approach to self-similar plane

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elastodynamic problems. He considered the elastodynamic problem of an expanding crack under homogeneous polynomial form loading and reduced it to the solution of a Cauchy integral equation.

In this paper, non-symmetric extension of an infinitesimal flaw into a plane shear crack at a constant rate due to the action of two identical non-parallel plane SH-waves propagating towards each other in an infinite isotropic elastic medium which is initially in a state of uniform anti-plane shear has been treated. A finite time after the crossing of the plane wave fronts, a fracture is assumed to initiate along the line where the wave fronts crossed and the crack edges are then assumed to travel non-symmetrically with different constants speeds. Superposition considerations allow the original problem to be separated into three self-similar problems with (0,0),(0,1) and (1,0) as the index of self-similarity. Following Cherepanov (1979), Cherepanov and Afanas'ev (1974) the mentioned self-similar problems have all been formulated as some problems of Riemann and Hilbert for half-plane, which are solved easily. Out of all the existing similarity techniques, the method of Smirnov-Sobolev (1932) which has been used extensively by Cherepanov (1979), Cherepanov and Afanas'ev (1974) being themost elegant and straight forward has been used to solve our problem.Analytical expressions for the dynamic stress intensity factors at the crack tips and also the rate of energy flux into the crack edges have been derived. Finally, the nature of the variation of the stress intensity factors and the energy flux rate at the crack tips with the velocities of the crack edges and also with the time after crack initiation have been depicted by means of graphs. The development of a crack initiating at a point being a physically realistic model from the point of view of modelling of earthquake sources, this problem also has got application in seismology.

2. Formulation Of The Problem

Let two identical plane waves defined by

$$\sigma_{yz} = A_o W_{\pm} H(W_{\pm}), \quad \sigma_{xz} = \pm A_o \cot\theta_o W_{\pm} H(W_{\pm}) \qquad (1.a,b)$$

referring to coordinate system (x,y,z) where

$$W_{\pm} = c_2 t \pm y\sin\theta_0 + x\cos\theta_0, \quad 0 \le \theta_0 \le \pi/2$$

and H() is Heaviside's unit function, propagate throughthe infinite solid which is pre-stressed such that

$$\sigma^{0}_{yz} = \sigma, \quad \sigma^{0}_{xz} = 0 \tag{1c}$$

Let us assume that at t=0 the non-parallel plane waves intersect at x=y=0. A micro crack is assumed to appear at $t=t_0$ at x=y=0 which starts to extend bilaterally along the trace of the wave intersection with uniform velocities v_1 and v_2. The expanding crack, the circular wave front associated with its motion and the plane wave front are shown in Fig.1(a).

In effect crack extension occurs by removing the stresses which would be generated in the crack plane by the combined applied static and dynamic fields if no cracks were present.

Accordingly, both the crack faces are subjected to shear tractions equal to $-\sigma - 2A_0(c_2 t + x\cos\theta_0)$.

The anti-symmetry of this loading about the crack plane implies that , it is sufficient to consider the half-plane y>0 with bounding surface y=0. The boundary conditions for this half-plane are then given by

$$y=0,\ -v_2 t' < x < v_1 t' \ : \ \sigma_{yz} = -\sigma - 2A_0 c_2 t_0 - 2A_0(c_2 t' + x\cos\theta_0)$$

$$y=0,\ x > v_1 t',\ x < -v_2 t' \ : \ W = 0 \tag{2a,b}$$

where $t' = t - t_0$.

Equation (2a) shows that invoking superposition principle the proposed problem can be divided into three separate problems of a constant shear traction, a shearing stress linearly varying with time and a shear linearly varying with distance along the crack plane.

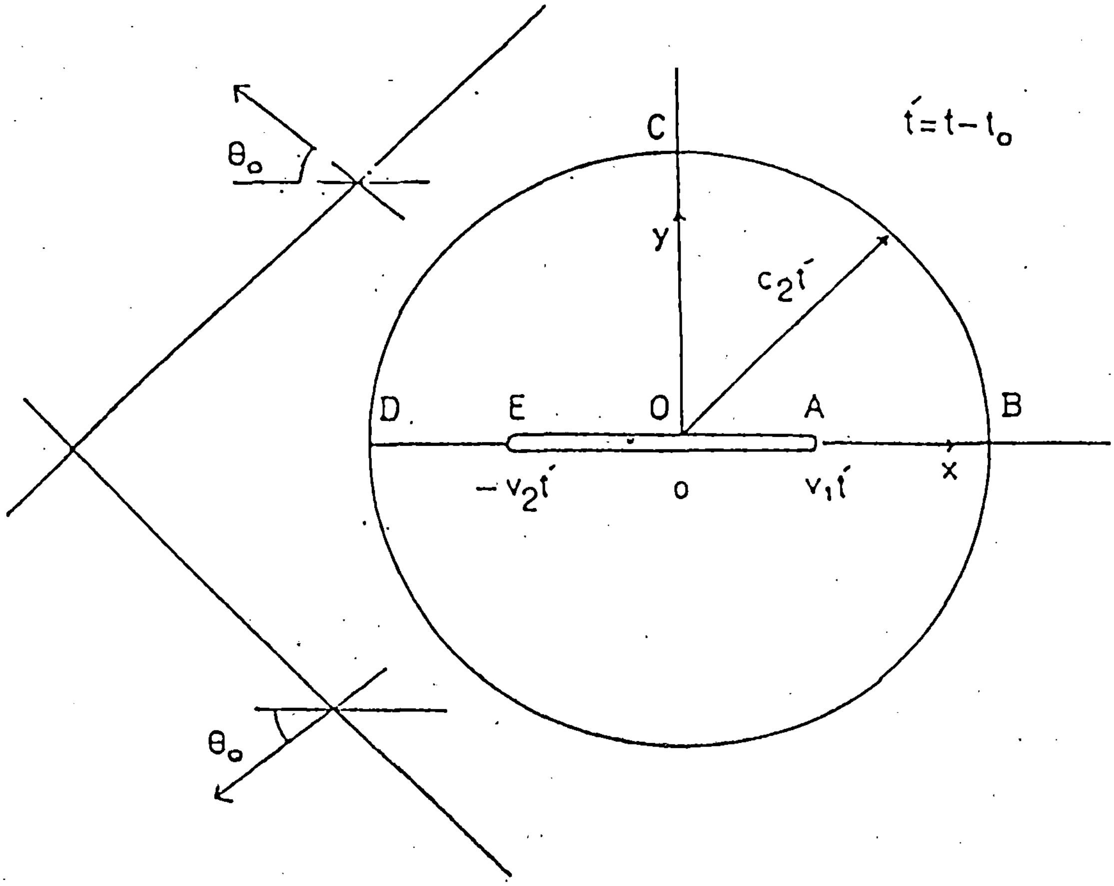

Fig. 1(a). The expanding crack and the pattern of wave front.

3. Constant Shear Traction On The Crack Faces.

The wave motion generated by constant shear tractions on the faces of the crack defined by $y=0$, $-v_2t<x<v_1t$ has been considered in this section and for simplicity t instead of t' has been used. The boundary conditions are

$$y=0,\ -v_2t<x<v_1t\ :\ \sigma_{yz} = -p_o$$

$$y=0,\ x>v_1t,\ x<-v_2t,\ W = 0 \qquad (3a,b)$$

where $p_o = \sigma+2A_oc_2t_o$

The displacement W which satisfies the wave equation

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = \frac{1}{c_2^2}\frac{\partial^2 W}{\partial t^2} \qquad (4)$$

is to be determined subject to the boundary conditions given by (3). From the boundary conditions we observe that $\frac{\partial W}{\partial t}$ shows dynamic similarity and is a homogeneous function of degree zero in x/t and y/t. Therefore, by the functionally invariant method of Smirnoff and Sobolev (1932) we can write

$$\frac{\partial W}{\partial t} = \operatorname{Re}\ \phi_o(z) \qquad (5)$$

where $$t - xz + y\sqrt{c_2^{-2}-z^2} = 0 \qquad (6)$$

The sign of the radical is to be fixed by the condition that

$$\text{as } z \to \infty,\ \sqrt{c_2^{-2}-z^2} = iz + O(z^{-1}) \qquad (7)$$

Equation (6) maps the semi-circular region of the cylindrical waves defined by OABCDE to the lower half of the complex cut z-plane given by

$$z = \frac{xt - iy\sqrt{t^2-c_2^{-2}(x^2+y^2)}}{x^2+y^2} \qquad (8)$$

as shown in Fig.1(b).

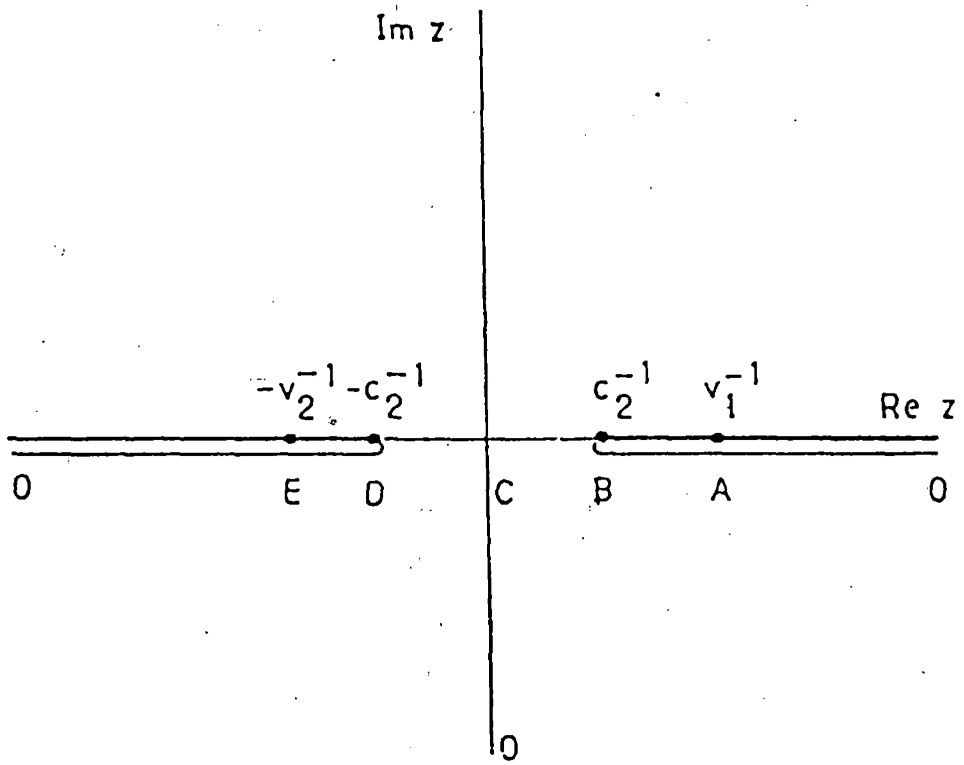

Fig. 1(b). Mapping of the interior of the semi-circle OABCDE in x–y plane on the lower half of the complex z-plane.

In view of the equations (5) and (6) we find

$$\frac{\partial \sigma_{yz}}{\partial t} = \mu \mathrm{Re}\left[\phi_o'(z)\ \frac{\partial z}{\partial y}\right]$$

so that

$$\frac{\partial \sigma_{yz}(x,0,t)}{\partial t} = \frac{1}{t}\ \mathrm{Re}\left[\ -\mu z \phi_o'(z)\sqrt{c_2^{-2}-z^2}\ \right] \tag{9}$$

Therefore the boundary conditions (3) are converted to the following conditions in z-plane

$$\mathrm{Im} z=0, \quad -v_2^{-1}<\mathrm{Re} z<v_1^{-1}\ , \ \mathrm{Re}\ \phi_o(z) = 0 \tag{10}$$

$$\mathrm{Im} z=0, \quad \mathrm{Re} z<-v_2^{-1}, \mathrm{Re} z>v_1^{-1}, \ \mathrm{Im}\ \phi_o'(z)=0 \tag{11}$$

In order to determine the analytic function $\phi_o(z)$ subject to the conditions (10),(11) it is necessary to know the behavior of the function $\phi_o(z)$ when $z \longrightarrow v_1^{-1}, -v_2^{-1}$ and $z \longrightarrow \infty$ .The infinite point of the z-plane corresponds to the origin of the coördinate of the physical plane where the displacement W is limited. Hence taking the representation (5) into account, we obtain

$$\mathrm{Re}\ \phi_o(z) = O(1) \quad \text{as } z \longrightarrow \infty \tag{12}$$

Further the condition (11) after integration with respect to z may be put in the form

$$\mathrm{Im} z=0, \quad \mathrm{Re} z<-v_2^{-1}, \mathrm{Re} z>v_1^{-1}, \ \mathrm{Im}\ \phi_o(z)=0 \tag{13}$$

Moreover, the displacement derivative $\partial W/\partial t$ near the crack tips $x = v_1 t, -v_2 t$ should show square root singularities so that at $z \longrightarrow v_1^{-1}, -v_2^{-1}$

$$\phi_o(z) = O(\ 1/\sqrt{z-v_1^{-1}}\ ),\ O(\ 1/\sqrt{z+v_2^{-1}}\ ) \tag{14}$$

respectively.

The above boundary conditions given by (10) and (13) together with the consideration (12) and (14) suggest that

$$\phi_0(z) = \frac{Az + B}{\sqrt{(z-v_1^{-1})(z+v_2^{-1})}} \tag{15}$$

where A and B are unknown constants to be determined.

Integrating (9) with respect to t it can be easily shown that for $x>0$

$$\sigma_{yz}(x,0,t) = -\mu\ \mathrm{Re}\left\{\left[\phi_0(z)\sqrt{c_2^{-2}-z^2}\ \right]_{c_2^{-1}}^{t/x} + \int_{c_2^{-1}}^{t/x}\frac{z\phi_0(z)dz}{\sqrt{c_2^{-2}-z^2}}\right\}$$

$$\sigma_{yz}(-x,0,t) = -\mu\ \mathrm{Re}\left\{\left[\phi_0(z)\sqrt{c_2^{-2}-z^2}\ \right]_{-c_2^{-1}}^{-t/x} + \int_{-c_2^{-1}}^{-t/x}\frac{z\phi_0(z)dz}{\sqrt{c_2^{-2}-z^2}}\right\} \tag{16a,b}$$

Next using the boundary conditions that

$$\sigma_{yz}(x,0,t) = -p_0, \qquad 0\le x<v_1t$$

$$\sigma_{yz}(-x,0,t) = -p_0, \qquad -v_2t<-x\le 0$$

in equation (16a,b) respectively we obtain two linear equations in A and B viz;

$$AI_2(v_1^{-1},v_2^{-1}) + BI_1(v_1^{-1},v_2^{-1}) = \frac{p_0}{\mu}$$

$$AI_2(v_2^{-1},v_1^{-1}) - BI_1(v_2^{-1},v_1^{-1}) = \frac{p_0}{\mu} \tag{17a,b}$$

where

$$I_p(u,v) = \int_{c_2^{-1}}^{u}\frac{z^p dz}{\sqrt{(z^2-c_2^{-2})(u-z)(v+z)}}, \qquad (p=1,2)$$

The stress intensity factors at the crack tips $|x|=v_1t, y=0$ and $|x|=v_2t, y=0$ defined by

$$N_{01} = \mathop{\mathrm{lt}}_{x\to v_1t+}\sqrt{x-v_1t}\ \sigma_{yz}(x,0,t)$$

$$N_{02} = \mathop{\mathrm{lt}}_{x\to v_2t+}\sqrt{x-v_2t}\ \sigma_{yz}(-x,0,t)$$

respectively are obtained with the help of the equations (15) and (16) as

$$N_{01} = \frac{\mu}{c_2} \sqrt{\frac{v_2 t}{v_1}} \sqrt{\frac{c_2^2 - v_1^2}{v_1 + v_2}} \, (A+Bv_1)$$

$$N_{02} = \frac{\mu}{c_2} \sqrt{\frac{v_1 t}{v_2}} \sqrt{\frac{c_2^2 - v_2^2}{v_1 + v_2}} \, (A-Bv_2) \qquad (18a,b)$$

The rate of energy flux into the extending crack edges defined by $\frac{dE}{dt}$ is given by Achenbah and Brock (1971)

$$\frac{1}{2} \frac{dE}{dt} = \int_{-\infty}^{\infty} \sigma_{yz} \frac{\partial W}{\partial t} \, dx \qquad (19)$$

which is obtained with the aid of (5),(15) and (16) for this case as

$$\frac{dE_1}{dt} = -\frac{\mu \pi t}{c_2 (v_1 + v_2)} \left[v_2 \sqrt{c_2^2 - v_1^2} \, (A+Bv_1)^2 + v_1 \sqrt{c_2^2 - v_2^2} \, (A-Bv_2)^2 \right] \qquad (20)$$

where while carrying on the integration (19) the following result (1972)

$$\frac{H(v)}{\sqrt{v}} \frac{H(-v)}{\sqrt{-v}} = \frac{\pi}{2} \delta(v) \qquad (21)$$

has been used.

4. Problem Of Linearly Increasing Shear Traction With Time On The Crack Faces

For the case of shear tractions on the faces of the crack increasing linearly with time, the boundary conditions are

$$y=0, \quad -v_2 t < x < v_1 t \; : \; \sigma_{yz} = -p_1 t \qquad (22)$$

$$y=0, \quad x > v_1 t, \; x < -v_2 t, \; W = 0 \qquad (23)$$

where $p_1 = 2A_0 c_2$.

The second order derivative $\frac{\partial^2 W}{\partial t^2}$ now shows dynamic similarity which can be taken as the real part of the analytic function $\phi_1(z)$ so that

$$\frac{\partial^2 W}{\partial t^2} = \text{Re}\ \phi_1(z) \tag{24}$$

which implies
$$\frac{\partial^2 \sigma_{yz}(x,0,t)}{\partial t^2} = \frac{1}{t}\ \text{Re}\left[\ -\mu z \phi_1'(z)\sqrt{c_2^{-2}-z^2}\ \right] \tag{25}$$

where z is given by (8) and $\phi_1(z)$ satisfies the conditions

$$\text{Im}z=0,\quad -v_2^{-1}<\text{Re}z<v_1^{-1}\ ,\quad \text{Re}\ \phi_1(z) = 0 \tag{26}$$

$$\text{Im}z=0,\quad \text{Re}z<-v_2^{-1}, \text{Re}z>v_1^{-1},\quad \text{Im}\ \phi_1'(z) = 0 \tag{27}$$

Integrating (24), we obtain

$$W = \frac{x^2}{2}\text{Re}\int_{v_1^{-1}}^{z} (z-\tau)^2\ \phi_1'(\tau)\ d\tau \tag{28}$$

$$= \frac{x^2}{2}\ \text{Re}\int_{v_1^{-1}}^{z} 2(z-\tau)\ \phi_1(\tau)\ d\tau \tag{29}$$

so that
$$\frac{d^2}{dz^2}\left\{\frac{W}{x^2}\right\} = \text{Re}\ \phi_1(z) \tag{30}$$

Taking into consideration the facts that near the crack tips $x=v_1t,-v_2t$; $y=0$ the displacement W varies in direct proportion to the factors $\sqrt{v_1t-x}$, $\sqrt{v_2t+x}$ respectively and that as $z\longrightarrow\infty$,

$$\text{Re}\ \phi_1(z) = O(1)$$

we have in view of the conditions (26), (27) and also the equation (30), the result that

$$\phi_1(z) = \frac{d^2}{dz^2}\left[(Cz+D)\sqrt{(z-v_1^{-1})(z+v_2^{-1})}\ \right] \tag{31}$$

where the constants C,D are to be determined from the condition that on the crack surface stress $\sigma_{yz} = -p_1t$.

From (25) after integration, we derive for $x>0$

$$\sigma_{yz}(x,0,t) = -\mu x \ \mathrm{Re} \int_{c_2^{-1}}^{t/x} \left\{ \sqrt{c_2^{-2} - \tau^2} + \frac{\tau(t/x-\tau)}{\sqrt{c_2^{-2} - \tau^2}} \right\} \phi_1(\tau)\,d\tau$$

$$\sigma_{yz}(-x,0,t) = \mu x \ \mathrm{Re} \int_{-c_2^{-1}}^{-t/x} \left\{ \sqrt{c_2^{-2} - \tau^2} - \frac{\tau(t/x+\tau)}{\sqrt{c_2^{-2} - \tau^2}} \right\} \phi_1(\tau)\,d\tau \qquad (32a,b)$$

Therefore, using the boundary conditions that

$$\sigma_{yz}(x,0,t) = -p_1 t, \qquad 0 \le x < v_1 t$$

$$\sigma_{yz}(-x,0,t) = -p_1 t, \qquad -v_2 t < -x \le 0$$

we obtain by the help of (32a,b) after simplification

$$CJ_1(v_1^{-1},v_2^{-1}) + DJ_2(v_1^{-1},v_2^{-1}) = \frac{p_1}{\mu}$$

$$CJ_1(v_2^{-1},v_1^{-1}) - DJ_2(v_2^{-1},v_1^{-1}) = \frac{p_1}{\mu} \qquad (33a,b)$$

where

$$J_1(v_1^{-1},v_2^{-1}) = \int_{c_2^{-1}}^{v_1^{-1}} \Big[\left\{8\tau+3(v_2^{-1}-v_1^{-1})\right\} M(\tau,v_1^{-1},v_2^{-1}) + N(\tau,v_1^{-1},v_2^{-1}) . \left\{4\tau^2+3\tau(v_2^{-1}-v_1^{-1})-2(v_1 v_2)^{-1}\right\} \Big]\,d\tau$$

$$J_2(v_1^{-1},v_2^{-1}) = \int_{c_2^{-1}}^{v_1^{-1}} \Big[2M(\tau,v_1^{-1},v_2^{-1}) + N(\tau,v_1^{-1},v_2^{-1}) \left\{2\tau+(v_2^{-1}-v_1^{-1})\right\} \Big] . d\tau$$

with $$M(\tau,v_1^{-1},v_2^{-1}) = \frac{v_1 \tau \sqrt{v_1^{-1}-\tau}}{2\sqrt{(\tau+v_2^{-1})(\tau^2-c_2^{-2})}}$$

$$N(\tau, v_1^{-1}, v_2^{-1}) = \frac{\tau v_1}{4\sqrt{\tau^2 - c_2^{-2}}}\left[\frac{3}{\sqrt{(\tau+v_2^{-1})(v_1^{-1}-\tau)}} - \frac{\sqrt{v_1^{-1}-\tau}}{(\tau+v_2^{-1})^{3/2}}\right]$$

The stress intensity factors at the crack tips defined by

$$N_{11} = \lim_{x\rightarrow v_1 t+} \sqrt{x-v_1 t}\,\sigma_{yz}(x,0,t)$$

$$N_{12} = \lim_{x\rightarrow v_2 t+} \sqrt{x-v_2 t}\,\sigma_{yz}(-x,0,t)$$

are found to be

$$N_{11} = \frac{\mu t}{2c_2}\sqrt{\frac{t}{v_1 v_2}}\sqrt{(c_2^2 - v_1^2)(v_1 + v_2)}\;(C+Dv_1)$$

$$N_{12} = \frac{\mu t}{2c_2}\sqrt{\frac{t}{v_1 v_2}}\sqrt{(c_2^2 - v_2^2)(v_1 + v_2)}\;(C-Dv_2) \qquad (34a,b)$$

and in this case the rate of energy flux $\frac{dE_2}{dt}$ into the crack edges defined by (19) is obtained as

$$\frac{dE_2}{dt} = -\frac{\pi\mu t^3(v_1+v_2)}{4c_2}\left[v_2^{-1}\sqrt{c_2^2 - v_1^2}\,(C+Dv_1)^2 + v_1^{-1}\sqrt{c_2^2 - v_2^2}\,(C-Dv_2)^2\right] \qquad (35)$$

where while carrying on the integration (19) the use of the result (21) has again been made.

5. Problem Of Linearly Varying Shear Traction With Distance Along The Crack Plane.

Consider the initially undisturbed half-space $y\geq0$ subjected to the shear traction $-p_2 x$ over $y=0, -v_2 t<x<v_1 t$. The boundary conditions are

$$y=0,\ -v_2 t<x<v_1 t : \sigma_{yz} = -p_2 x$$

$$y=0,\ x>v_1 t,\ x<-v_2 t,\ W = 0 \qquad (36a,b)$$

where $p_2 = 2A_0 \cos\theta_0$.

In this case, $\frac{\partial^2 W}{\partial x \partial t}$ shows dynamic similarity. So we take keeping (8) in mind,

$$\frac{\partial^2 W}{\partial x \partial t} = \mathrm{Re}\ \phi_2(z)$$

with
$$\frac{\partial^2 \sigma_{yz}}{\partial x \partial t} = -\frac{\mu}{t}\ \mathrm{Re}\left[z\phi_2'(z)\sqrt{c_2^{-2}-z^2} \right] \qquad (37a,b)$$

where $\phi_2(z)$ satisfies the conditions

$$\mathrm{Im}z=0,\quad -v_2^{-1}<\mathrm{Re}z<v_1^{-1},\ \mathrm{Re}\ \phi_2(z) = 0$$

$$\mathrm{Im}z=0,\quad \mathrm{Re}z<-v_2^{-1}, \mathrm{Re}z>v_1^{-1},\ \mathrm{Im}\ \phi_2'(z)=0 \qquad (38a,b)$$

From equation (37a) after integration it is found that

$$W = -x^2 \mathrm{Re} \int_{v_1^{-1}}^{z} \tau^{-1}(z-\tau)\ \phi_2(\tau)\ d\tau$$

so that
$$-z^2 \frac{d}{dz}\left\{ \frac{1}{t}\frac{\partial W}{\partial t} \right\} = \mathrm{Re}\ \phi_2(z)$$

Since $\frac{\partial W}{\partial t}$ near the crack tips should show square root singularity and also since $\mathrm{Re}\ \phi_2(z) = O(1)$ as $z \longrightarrow \infty$, we have in view of the conditions (38)

$$\phi_2(z) = z^2 \frac{d}{dz}\left[\frac{Rz^{-1} + L}{\sqrt{(z-v_1^{-1})(z+v_2^{-1})}} \right] \qquad (39)$$

where the constants R, L are to be determined.

Equation (37b) can be integrated to obtain for x>0

$$\sigma_{yz}(x,0,t) = \mu x \ \mathrm{Re} \int_{c_2^{-1}}^{t/x} \left\{ \frac{t}{x\tau^2} \sqrt{c_2^{-2} - \tau^2} + \frac{t - \tau x}{x\sqrt{c_2^{-2} - \tau^2}} \right\} \phi_2(\tau)\, d\tau$$

$$\sigma_{yz}(-x,0,t) = \mu x \ \mathrm{Re} \int_{-c_2^{-1}}^{-t/x} \left\{ \frac{t}{x\tau^2} \sqrt{c_2^{-2} - \tau^2} + \frac{t + \tau x}{x\sqrt{c_2^{-2} - \tau^2}} \right\} \phi_2(\tau)\, d\tau \qquad (40a,b)$$

So using the boundary conditions that

$$\sigma_{yz}(x,0,t) = -p_2 x, \qquad 0 \le x < v_1 t$$

$$\sigma_{yz}(-x,0,t) = p_2 x, \qquad -v_2 t < -x \le 0$$

it is found by the help of equations (39),(40)

$$-RK_1(v_1^{-1}, v_2^{-1}) + LK_2(v_1^{-1}, v_2^{-1}) = \frac{p_2}{\mu}$$

$$RK_1(v_2^{-1}, v_1^{-1}) + LK_2(v_2^{-1}, v_1^{-1}) = \frac{p_2}{\mu}, \qquad (41)$$

where

$$K_1(v_1^{-1}, v_2^{-1}) = \int_{c_2^{-1}}^{v_1^{-1}} \left[P(\tau, v_1^{-1}, v_2^{-1}) - \tau^{-1} Q(\tau, v_1^{-1}, v_2^{-1}) \right] d\tau$$

$$K_2(v_1^{-1}, v_2^{-1}) = \int_{c_2^{-1}}^{v_1^{-1}} Q(\tau, v_1^{-1}, v_2^{-1})\, d\tau$$

$$P(\tau, v_1^{-1}, v_2^{-1}) = -\frac{\sqrt{v_1^{-1} - \tau}}{\sqrt{(\tau + v_2^{-1})(\tau^2 - c_2^{-2})}}$$

and $$Q(\tau, v_1^{-1}, v_2^{-1}) = \frac{\tau^2}{\sqrt{\tau^2 - c_2^{-2}}} \left[\frac{\sqrt{v_1^{-1} - \tau}}{2(\tau + v_2^{-1})^{3/2}} - \frac{(2v_1^{-1} + \tau)}{2\tau\sqrt{(\tau + v_2^{-1})(v_1^{-1} - \tau)}} \right]$$

In this case, the stress intensity factors are obtained as

$$N_{21} = \operatorname*{lt}_{x \to v_1 t+} \sqrt{x - v_1 t}\,\sigma_{yz}(x, 0, t) = -\frac{\mu t}{c_2} \sqrt{\frac{v_2 t}{v_1}} \sqrt{\frac{c_2^2 - v_1^2}{v_1 + v_2}}\,(Rv_1^2 + Lv_1)$$

$$N_{22} = \operatorname*{lt}_{x \to v_2 t+} \sqrt{x - v_2 t}\,\sigma_{yz}(-x, 0, t) = -\frac{\mu t}{c_2} \sqrt{\frac{v_1 t}{v_2}} \sqrt{\frac{c_2^2 - v_2^2}{v_1 + v_2}}\,(Rv_2^2 - Lv_2) \qquad (42a,b)$$

The rate of energy flux $\frac{dE_3}{dt}$ into the extending crack edges is found to be

$$\frac{dE_3}{dt} = 2\int_{-\infty}^{\infty} \sigma_{yz} \frac{\partial W}{\partial t}\,dx = -\frac{\mu\pi t^3}{c_2(v_1 + v_2)} \left[v_2\sqrt{c_2^2 - v_1^2}\,(Rv_1^2 + Lv_1)^2 + v_1\sqrt{c_2^2 - v_2^2}\,.\,(Rv_2^2 - Lv_2)^2 \right] \qquad (43)$$

where the result (21) has been used.

6. Particular Case: $v_1 = v_2$

If we set $v_1 = v_2 = v$ in all the cases solved above, the following results are obtained

(i) For the case of constant traction $\sigma_{yz} = -p_o$ on the crack faces, we find from (17) that

$B = 0$, $A = \frac{vp_o}{\mu E(q)}$, where $E(q)$ is the complete Elliptic integral of second kind and $q = \sqrt{1 - v^2/c_2^2}$. Equations (18) yield the stress intensity factor at the crack tips as

$$N_0 = N_{01} = N_{02} = \frac{A\mu\sqrt{t}}{c_2}\sqrt{\frac{c_2^2 - v^2}{2v}}$$

Also from (20) we obtain

$$\frac{dE_1}{dt} = -\frac{\mu\pi t}{c_2}\sqrt{c_2^2 - v^2}\quad A^2$$

(ii) For the case of shear traction $\sigma_{yz} = -p_1 t$ on the crack faces increasing linearly with time, it is found from equation (39) that

$$D = 0,\ C = \frac{p_1 v}{\mu\ I}$$

where

$$I = 2E(q) - F(q) + \frac{2c_2^2}{(v+c_2)(v^2-c_2^2)}\left\{2v\Pi(r^2, r) + (v+c_2)F(r)\right\}$$

$F(r)$, $\Pi(r^2, r)$ are complete Elliptic integrals of first and third kind respectively and $r = (c_2 - v)/(c_2 + v)$.

In this case, the stress intensity factors and the rate of energy flux into the extending crack tips given by (34) and (35) can be simplified to

$$N_1 = N_{11} = N_{12} = \frac{C\mu t}{c_2}\sqrt{\frac{t}{2v}}\sqrt{c_2^2 - v^2}$$

and
$$\frac{dE_2}{dt} = -\frac{\mu\pi t^3}{c_2}\sqrt{c_2^2 - v^2}\quad C^2$$

(iii) For the case of shear traction $\sigma_{yz} = -p_2 x$ on the crack faces, it is obvious from equations (41) that

$$R = 0,\ L = \frac{p_2 v}{\mu\ J}$$

where

$$J = \frac{2c_2^2}{(v+c_2)(v^2-c_2^2)}\left\{2v\Pi(r^2, r) + (v+c_2)F(r)\right\} - E(q) - F(q)$$

and it is found from equations (42),(43) that stress intensity factors at the crack tips and the rate of energy flux into the extending crack edges in this case are given by

$$N_2 = N_{21} = -N_{22} = \frac{-\mu t L}{c_2}\sqrt{\frac{vt}{2}}\sqrt{c_2^2 - v^2}$$

and
$$\frac{dE_3}{dt} = -\frac{\mu\pi t^3}{c_2}\sqrt{c_2^2 - v^2}\quad L^2 v^2$$

7. Numerical Results And Discussions.

The solution of the original crack problem is obtained by taking $p_o = \sigma + 2A_o c_2 t_o$, $p_1 = 2A_o c_2$ and $p_2 = 2A_o \cos\theta_o$ and superposing the results obtained in sections 3-5 with the stress fields given by (1). Taking together the results obtained in the sections 3-5 it is possible to write the stress intensity factors at the edges of the crack and the rate of energy flux into the extending crack edges as

$$S_1 = \frac{N_{o1} + N_{11} + N_{21}}{\sigma\sqrt{v_1 t_o}} = \sqrt{\frac{u_2 \tau}{u_1 + u_2}}\quad \mu H_+(u_1, u_2, \tau)$$

$$S_2 = \frac{N_{o2} + N_{12} + N_{22}}{\sigma\sqrt{v_1 t_o}} = \sqrt{\frac{u_2 \tau}{u_1 + u_2}}\quad \mu H_-(u_2, u_1, \tau) \qquad (44a,b)$$

and

$$En = \frac{\mu}{t_o c_2^2 \sigma^2}\frac{d}{dt}\left(E_1 + E_2 + E_3\right) = -\frac{\pi u_2 \mu^2}{u_1 + u_2}\left[G_+(u_1, u_2, \tau) + \frac{u_1}{u_2} G_-(u_2, u_1, \tau)\right] \qquad (45)$$

where

$$H_\pm(u_1, u_2, \tau) = \sqrt{1 - u_1^2}\left[\frac{1+\Delta}{p_o}\left(\frac{A}{c_2 u_1} \pm B\right) + \Delta\tau\left\{\frac{u_1 + u_2}{2p_1 u_2}\left(\frac{C}{c_2 u_1} \pm D\right) - \right.\right.$$
$$\left.\left. - \frac{u_1 \cos\theta_o}{p_2}\left(\pm\frac{L}{c_2 u_1} + R\right)\right\}\right]$$

$$G_{\pm}(u_1,u_2,\tau)=\tau\sqrt{1-u_1^2}\left[\left(\frac{1+\Delta}{p_0}\right)^2\left(\frac{A}{c_2}\pm Bu_1\right)^2+(\Delta\tau)^2\left\{\frac{(u_1+u_2)^2}{4p_1^2u_2^2}\left(\frac{C}{c_2}\pm Du_1\right)+\right.\right.$$

$$\left.\left.+\frac{u_1^2\cos^2\theta_0}{p_2^2}\left(\frac{L}{c_2}\pm Ru_1\right)^2\right\}\right]$$

and the parameter $\tau = \frac{t}{t_0}-1$ is the non-dimensionalized time after crack initiation and $\Delta = \frac{2A_0c_2t_0}{\sigma}$ is the ratio at x=y=0 at initiation of the crack plane stress due to the plane waves and the pre-stress.

Also u_1, u_2 are the non-dimensional crack tip velocities given by $u_1 = v_1/c_2$ and $u_2 = v_2/c_2$.

The variations of stress intensity factors and energy flux rate given by (44) and (45) respectively with

(i) v_1/c_2 for different values of v_2/c_2 and

(ii) τ for different values of v_1/c_2 and Δ

have been presented in Figs. 2-4. It has been shown in Fig.2 that Stress intensity factors at the edge $x=v_1t'$, $y=0$ decreases with the increase in the values of v_1/c_2 but increases with the increase in the values of v_2/c_2 and for $v_1/c_2<0.45$, the stress intensity factor at the edge $x=v_2t'$, $y=0$ increases as v_2/c_2 increases but for $v_1/c_2>0.45$, the variation of stress intensity factor at that edge shows an opposite character. It has also been depicted in Fig.2 that the value of energy flux rate $|En|$ increases with the increase in the value of v_1/c_2, shows maximum at $v_1/c_2 = 0.8$ after which it decreases with the increase in the value of v_1/c_2.

In Fig.3. the variations of S_1, S_2 and $|En|$ with τ for various values of $v_1/c_2 \le v_2/c_2$ have been depicted. It may be observed from this figure that $S_1, S_2, |En|$ all increase rapidly with the increase in the value of τ. It may be noted further that for fixed value of v_2/c_2, values of stress intensity factors at the crack tips decrease with the increase in the value of v_1/c_2 whereas energy flux rate $|En|$ increases with the gradual increase in the value of v_1/c_2.

In Fig.4, S_1, S_2 and $|En|$ are again plotted vs τ but in this case, crack tip velocities are kept fixed whereas Δ is assumed to vary. It may be seen that increase in the values of Δ produce marked increase in the value of S_1, S_2 and $|En|$ for any fixed value of τ.

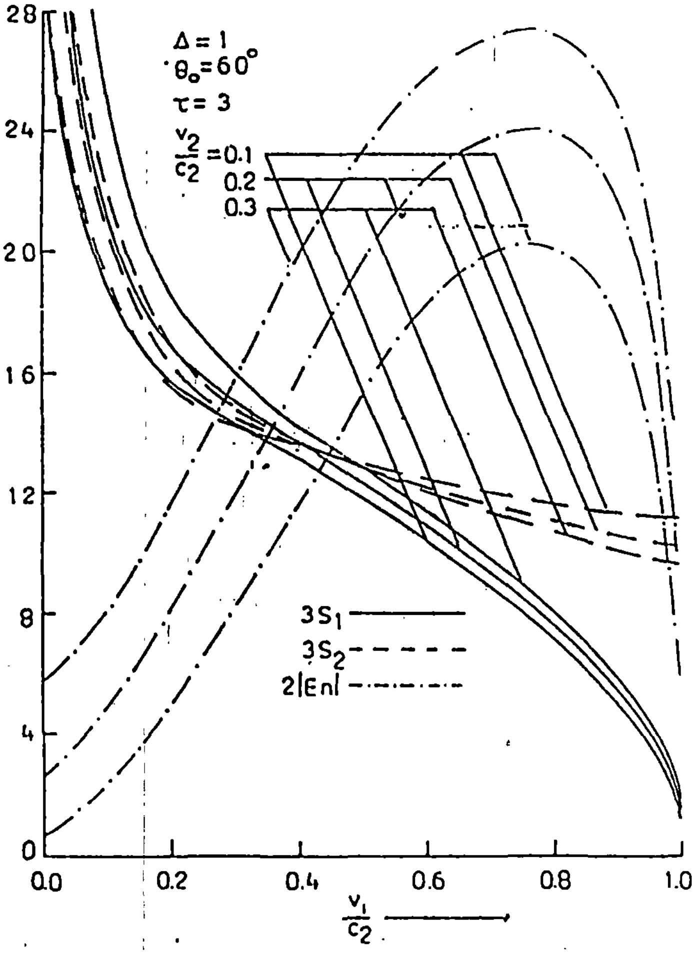

Fig. 2. Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional speed v_1/c_2.

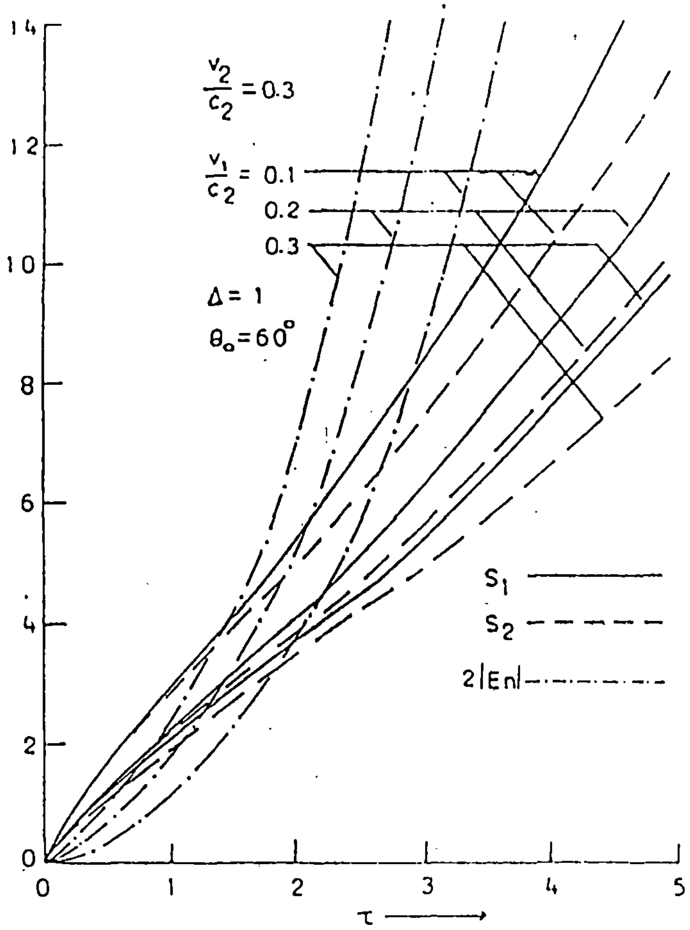

Fig. 3. Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional time after fracture initiation τ.

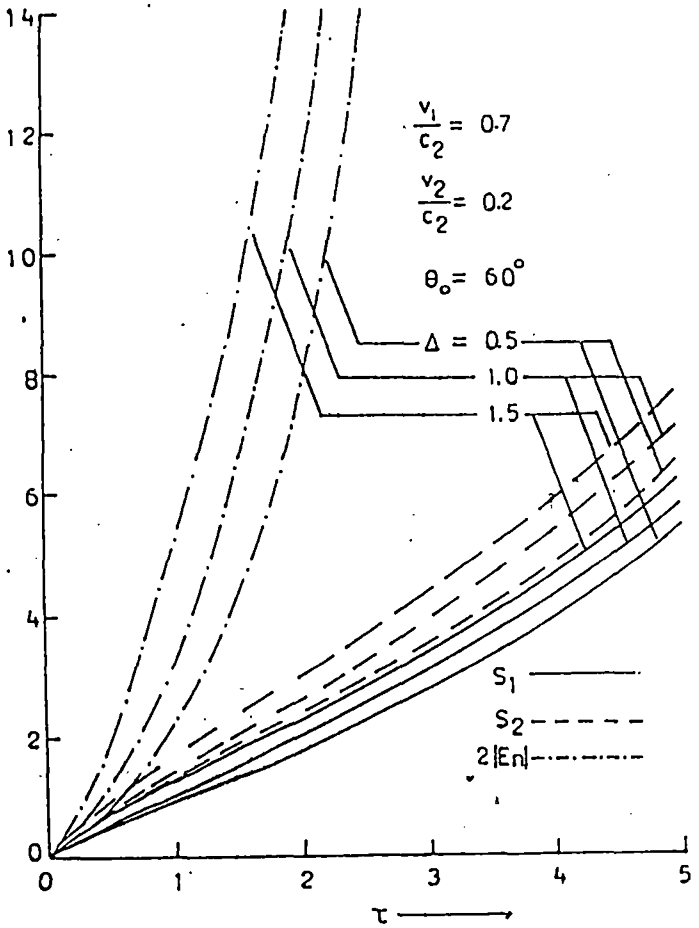

Fig. 4. Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional time after fracture initiation τ.

EXTENSION OF A CRACK DUE TO PLANE SH WAVE IN A PRE-STRESSED INFINITE ELASTIC MEDIUM

1. Introduction

Since Broberg's (1960) investigation of the solution of crack expanding symmetrically with constant velocity under conditions of plane stress or strain in a homogeneous isotropic elastic medium in a field of spatially and time invariant tensile stress, a number of papers have appeared analyzing different geometrical conditions. The problem of Broberg (1960) was also solved by Craggs (1963)using the method of homogeneous function and the corresponding anti-plane problem was examined by Achenbach and Brock (1971). All the problems mentioned above are however self similar ones with index (0,0) and are concerned with symmetric extension of a crack.

Problems involving non-symmetric extension of cracks under uniform loading along the crack surface are not found much in the literature perhaps due to severe mathematical complexity encountered in solving such problems. Following the method of homogeneous function Brock (1976,1975) solved the problems of non-symmetric extension of a small flaw into a plane crack. Recently, the elastodynamic problem of non-uniform expansion of a crack under homogeneous polynomial form loading has been solved by Georgiadis (1991) by means of complex variable method.

The problem of non-symmetric extension of an infinitesimal flaw into a plane crack at a constant rate due to the action of two non-parallel plane SH-waves having different amplitudes propagating towards each other in an infinite isotropic elastic medium which is initially in a state of uniform anti-plane shear has been analyzed in this paper. A finite time after the crossing of the plane wave fronts, a fracture is assumed to initiate along the line where the wave fronts crossed and the crack is then assumed to extend

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non-symmetrically along the trace of the wave intersection. Superposition considerations allow the original problem to be separated into three self-similar problems with (0,0) (0,1) and (1,0) as the index of similarity. The dynamic similarity of certain field variable in each problem suggests application of the method of homogeneous functions. Expressions for the stress intensity factors and the rate of energy flux into the extending crack tips have been derived. Finally, the nature of the variation of the stress intensity factors at the crack tips and also of the rate of energy flux into the crack edges with velocities of the crack edges and also with the time after crack initiation have been depicted by means of graphs.

2. Formulation Of The Problem

Let two identical plane waves defined by

$$\sigma_{yz} = S_{\pm}W_{\pm}H(W_{\pm}), \quad \sigma_{xz} = S_{\pm}\cot\theta_{0}W_{\pm}H(W_{\pm}) \qquad (1a,b)$$

referred to coordinate system (x,y,z) where

$$W_{\pm} = t \pm y\sin\theta_{0}/c_{2} + x\cos\theta_{0}/c_{2}$$

and H() is Heaviside's unit function, propagate through the infinite solid which is pre-stressed such that

$$\sigma^{0}_{yz} = \sigma, \quad \sigma^{0}_{xz} = 0 \qquad (2a,b)$$

Let us assume that at t=0 the non-parallel plane waves intersect along the line y=0. A micro crack is assumed to appear at $t=t_{0}$ at x=y=0 which starts to extend bilaterally along the line y=0 with uniform velocities V_{R} and V_{L}. The expanding crack, the circular wave front associated with its motion and the plane wave fronts are shown in Fig.1.

In effect crack extension occurs by removing the stresses which would be generated in the crack plane by the combined applied static and dynamic fields if no cracks were present. So, both the crack faces are subjected to shear tractions equal to $-\sigma - (S_{+}+S_{-})t - (S_{+}+S_{-})x\cos\theta_{0}/c_{2}$.

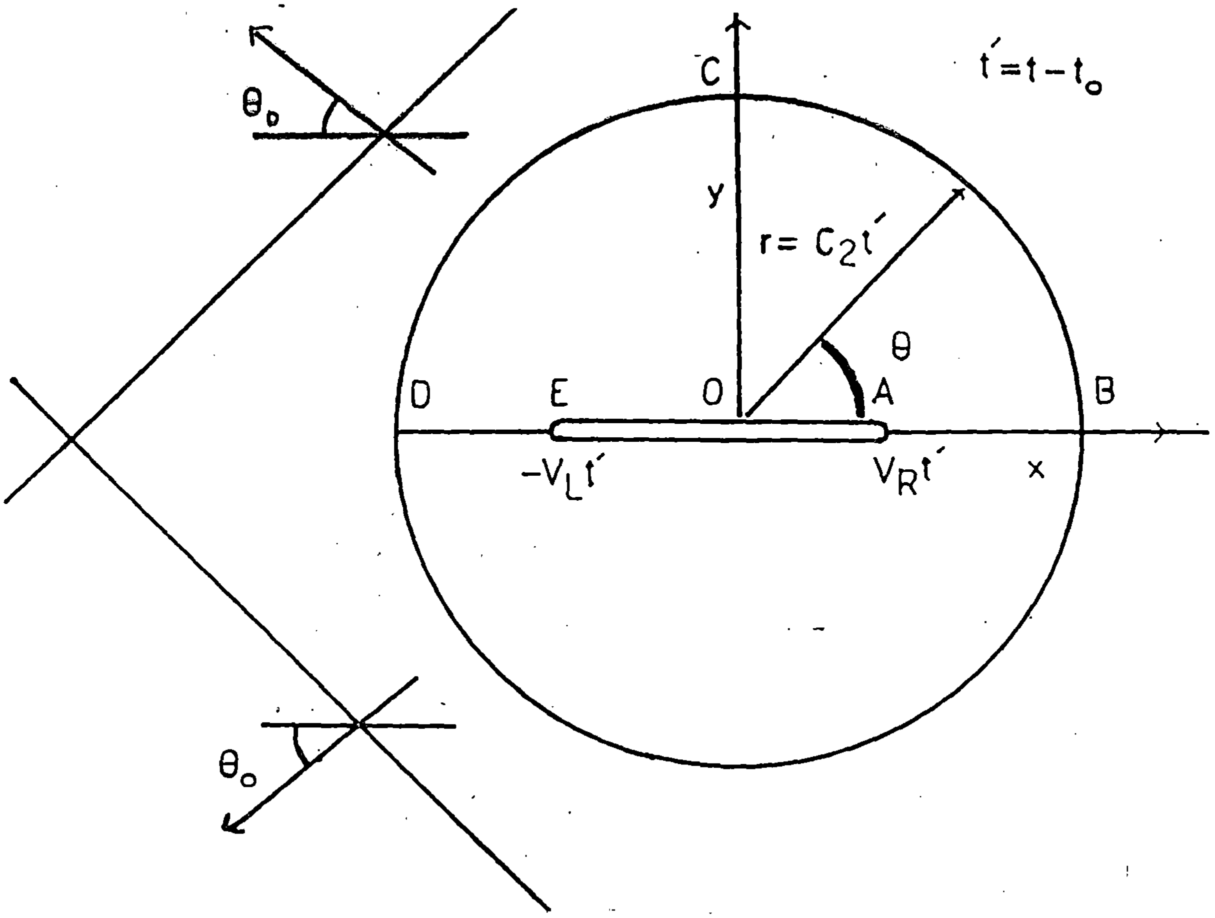

Fig.1: The x-y plane.

The anti-symmetry of this loading about the crack plane implies that, it is sufficient to consider the half-plane $y>0$ with bounding surface $y=0$. The boundary conditions for this half-plane are then given by

$$y=0,\quad -V_L t' < x < V_R t' : \sigma_{yz} = -\sigma-(S_+ + S_-)t_o-(S_+ + S_-)t' - (S_+ + S_-)x\cos\theta_o/c_2$$

$$y=0,\quad x > V_R t',\ x < -V_L t' : W = 0 \qquad (3a,b)$$

where $t'=t-t_o$

Equation (3a) shows that invoking superposition principle the proposed problem can be divided into three separate problems of a constant shear traction and a shearing stress linearly varying with time and a shear linearly varying with distance along the crack plane.

The two dimensional wave equation in polar coordinates r, θ and t where $r = (x^2+y^2)^{1/2}$ and $\theta = \tan^{-1}(y/x)$, for a field variable $\phi(r,\theta,t)$ is

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial\phi}{\partial r}\right\} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{1}{c_2^2}\frac{\partial^2\phi}{\partial t^2} \qquad (4)$$

where $c_2 = (\mu/\rho)^{1/2}$, μ is the shear modulus and ρ is the material density.

The absence of the characteristic length in the formulation of the problem suggests that the solution of (4) will be dynamically similar i.e. depends on r/t and θ rather on r,θ,t separately. Introducing the variable

$$s = r/t$$

we see that $\phi(s,\theta)$ satisfies the equation

$$s^2\left[1-\frac{s^2}{c^2}\right]\frac{\partial^2\phi}{\partial s^2} + s\left[1-2\frac{s^2}{c^2}\right]\frac{\partial\phi}{\partial s} + \frac{\partial^2\phi}{\partial\theta^2} = 0 \qquad (5)$$

For $s<c$, the Chaplygin's transformation

$$\beta = \cosh^{-1}(c/s) \tag{6}$$

reduces (5) to Laplace equation

$$\frac{\partial^2\phi}{\partial\beta^2} + \frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{7}$$

and maps the interior of semi circular region in the upper half of the physical plane into a semi-infinite strip in θ-β plane.

A convenient method to solve the equation (7) is to express $\phi(s,\theta)$ as the real part of an analytic function and to construct an appropriate analytic function of the complex variable $\beta+i\theta$. Superposition in (3a) is invoked to consider the problems separately in next three sections.

3. Constant Shear Traction In The Crack Faces.

The wave motion generated by constant shear tractions on the faces of the crack defined by $y=0$, $-V_L t<x<V_R t$ has been considered in this section and for simplicity t instead of t′ has been used. The boundary conditions are

$$y = 0,\ -V_L t<x<V_R t\ :\ \sigma_{yz} = -p_o$$
$$y = 0,\ x<-V_L t,\ x>V_R t\ :\ W = 0 \tag{8a,b}$$

where $p_o = \sigma + (S_- + S_+)t_o$

From the conditions (8) we observe that $\frac{\partial W}{\partial x}$, $\frac{\partial W}{\partial y}$ and $\frac{\partial W}{\partial t}$ show dynamic similarity. We can choose $\frac{\partial W}{\partial t}$ to take place of ϕ in (4) — (7).

Considering $y\geq0$, the boundary conditions (8) are converted to the following conditions in β-θ plane

$$\theta = 0,\ V_R<s<c_2\ :\ \Omega(s,\theta) = 0$$
$$\theta = \pi,\ -c_2<s<-V_L\ :\ \Omega(s,\theta) = 0$$
$$\theta = 0,\ 0<s<V_R\ :\ \frac{\partial\Omega(s,\theta)}{\partial\theta} = 0$$
$$\theta = \pi,\ -V_L<s<0\ :\ \frac{\partial\Omega(s,\theta)}{\partial\theta} = 0 \tag{9a-d}$$

where $\frac{\partial W}{\partial t} = \Omega(s,\theta)$ and s is related to β by (6). (10)

Further, $\frac{\partial W}{\partial t}$ vanishes at the wave front, $r = c_2 t$, which yields

$$s = c,\ 0 \le \theta \le \pi,\ \Omega(s,\theta) = 0 \tag{11}$$

The derivative $\frac{\partial W}{\partial t}$ may be written as real part of an analytic function which can be obtained by mapping the interior of the strip in θ-β plane, see Fig.2, on the lower half-plane of the ζ-plane by means of the conformal transformation

$$\zeta = \xi + i\eta = \mathrm{Sech}(\beta + i\theta) \tag{12}$$

The mappings of the various points are indicated in Figs 1-3. In the ζ-plane we take

$$\frac{\partial W}{\partial t} = \mathrm{Re}\ \phi_o(\zeta) \tag{13}$$

In view of (12) and (13) we find

$$\frac{\partial \sigma_{yz}(x,0,t)}{\partial t} = \frac{\mu}{x}\ \mathrm{Im}\left[\zeta\sqrt{1-\zeta^2}\ \phi_o'(\zeta)\right] \tag{14}$$

Therefore, the boundary conditions given by (9) and (11) are converted into the following conditions in complex ζ-plane

$$\eta = 0,\ -\infty < \xi < -V_L/c_2,\ V_R/c_2 < \xi < \infty\ :\ \mathrm{Re}\phi_o(\zeta) = 0$$

$$\eta = 0, -V_L/c_2 < \xi < V_R/c_2\ \ :\ \mathrm{Im}\ \phi_o'(\zeta) = 0 \tag{15a,b}$$

In order to determine the analytic function $\phi_o(\zeta)$ subject to the conditions (15) it is necessary to know the behaviour of the function $\phi_o(\zeta)$ when $\zeta \to -V_L/c_2, V_R/c_2$ and $\zeta \to \infty$. The infinite point in the ζ-plane corresponds to the point $x=0$, $y=c_2 t$ in the physical plane where $\partial W/\partial t$ is limited. Hence taking (13) into account, we obtain

$$\mathrm{Re}\ \phi_o(\zeta) = O(1) \text{ as } \zeta \to \infty \tag{16}$$

Further, the condition (15b) after integration with respect to ζ may

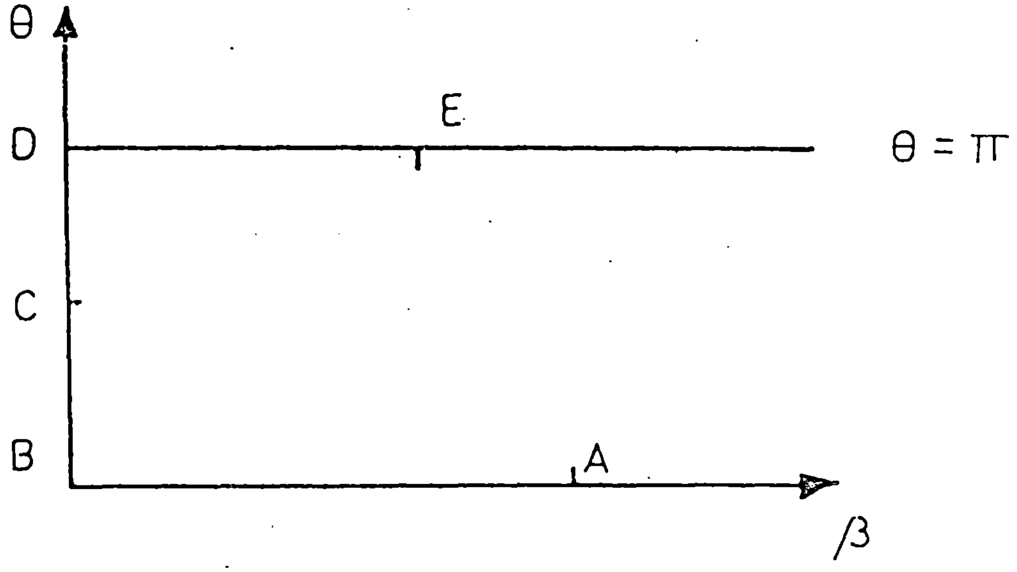

Fig.2: The θ-β plane.

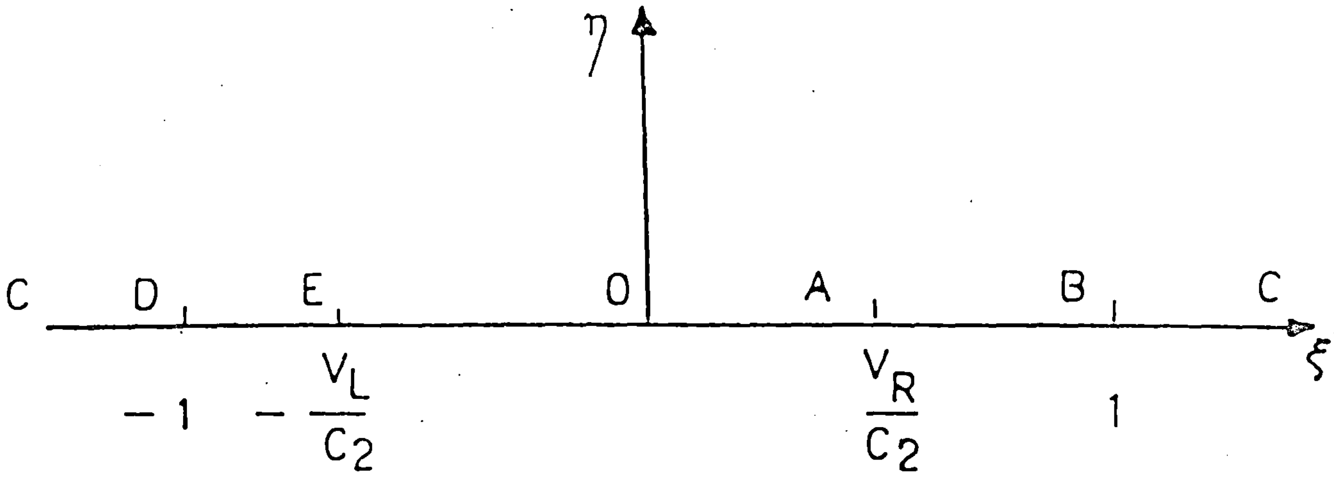

Fig. 3: The ξ-η plane.

be put in the form

$$\eta = 0, -V_L/c_2 < \xi < V_R/c_2 \quad : \operatorname{Im} \phi_o(\zeta) = 0 \qquad (17)$$

Now from (13) we obtain

$$\operatorname{Re} \phi_o(\zeta) = \zeta^2 \frac{d}{d\zeta} \left[- \frac{c_2}{x} W \right] \qquad (18)$$

Taking into the consideration the fact that near the crack tips $x = -V_L t, V_R t$, $y = 0$ the displacement W varies in direct proportion to the factors $\sqrt{V_R t - x}$ and $\sqrt{V_L t + x}$ respectively we have in view of the equation (18), the result that

$$\phi_o(\zeta) = \frac{A\zeta + B}{\sqrt{(V_R/c_2 - \zeta)(V_L/c_2 + \zeta)}} \qquad (19)$$

where the constants A and B are to be determined

Integrating (14) with respect to t it can be easily shown that for $x > 0$

$$\sigma_{yz}(x,0,t) = \frac{\mu}{c_2} \operatorname{Im} \left[\frac{\sqrt{1-\zeta^2}}{\zeta} \phi_o(\zeta) \Big|_{x/c_2 t}^{1} + \int_{x/c_2 t}^{1} \frac{\phi_o(\zeta)}{\zeta^2 \sqrt{1-\zeta^2}} d\zeta \right],$$

$$\sigma_{yz}(-x,0,t) = \frac{\mu}{c_2} \operatorname{Im} \left[\frac{\sqrt{1-\zeta^2}}{\zeta} \phi_o(\zeta) \Big|_{-x/c_2 t}^{-1} + \int_{-x/c_2 t}^{-1} \frac{\phi_o(\zeta)}{\zeta^2 \sqrt{1-\zeta^2}} d\zeta \right]$$

(20a,b)

Next using the boundary conditions

$$\sigma_{yz}(x,0,t) = -p_o, \qquad 0 \le x < V_R t$$

$$\sigma_{yz}(-x,0,t) = -p_o, \qquad -V_L t < -x \le 0$$

in equations (20a,b) respectively we obtain two linear equation in A and B viz;

$$A I_1\left(\frac{V_R}{c_2},\frac{V_L}{c_2}\right) + B I_2\left(\frac{V_R}{c_2},\frac{V_L}{c_2}\right) = \frac{c_2 p_o}{\mu}$$

$$-A I_1\left(\frac{V_L}{c_2},\frac{V_R}{c_2}\right) + B I_2\left(\frac{V_L}{c_2},\frac{V_R}{c_2}\right) = \frac{c_2 p_o}{\mu} \qquad (21a,b)$$

where

$$I_p(u,v) = \int_u^1 \frac{z^{-p}\,dz}{\sqrt{(1-z^2)(z-u)(z+v)}}, \quad (p=1,2)$$

The stress intensity factors at the crack tips $|x| = V_R t$, $y = 0$ and $|x| = V_L t$, $y = 0$ defined by

$$N_{01} = \underset{x \to V_R t+}{\mathrm{Lt}} \sqrt{x - V_R t}\;\sigma_{yz}(x,0,t)$$

$$N_{02} = \underset{x \to V_L t+}{\mathrm{Lt}} \sqrt{x - V_L t}\;\sigma_{yz}(-x,0,t)$$

respectively are obtained with the help of equations (20a,b) as

$$N_{01} = \frac{\mu}{c_2}\sqrt{\frac{(c_2^2 - V_R^2)t}{V_R + V_L}}\left[A + B\frac{c_2}{V_R}\right]$$

$$N_{02} = \frac{\mu}{c_2}\sqrt{\frac{(c_2^2 - V_L^2)t}{V_R + V_L}}\left[-A + B\frac{c_2}{V_L}\right] \qquad (21c,d)$$

The rate of energy flux into the extending crack edges defined by dE/dt is given by Achenbach and Brock (1971)

$$\frac{1}{2}\frac{dE}{dt} = -\int_{-\infty}^{\infty} \sigma_{yz}\frac{\partial w}{\partial t}\,dx \qquad (22)$$

which is obtained with the aid of (13),(19) and (20) for this case as

$$\frac{1}{2}\frac{dE_1}{dt} = -\frac{\mu\pi}{2c_2}\frac{t}{V_R+V_L}\left[(AV_R+Bc_2)^2\sqrt{\frac{c_2^2}{V_R^2}-1} + (AV_L-Bc_2)^2\sqrt{\frac{c_2^2}{V_L^2}-1}\right] \tag{23}$$

where while carrying on the integration (22) the following result (1972)

$$\frac{H(v)}{\sqrt{v}}\,\frac{H(-v)}{\sqrt{-v}} = \frac{\pi}{2}\,\delta(v) \tag{24}$$

has been used.

4. Linearly Increasing Shear Traction With Time On The Crack Faces

For the case of shear traction on the crack faces increasing linearly with time, the boundary conditions are

$$y = 0,\ -V_L t<x<V_R t\ :\ \sigma_{yz} = -p_1 t$$

$$y = 0,\ x<-V_L t,\ x>V_R t\ :\ W = 0 \tag{25a,b}$$

where $p_1 = (S_- + S_+)$.

The second order derivatives $\frac{\partial^2 W}{\partial x^2}$, $\frac{\partial^2 W}{\partial y^2}$, $\frac{\partial^2 W}{\partial x\partial y}$ and $\frac{\partial^2 W}{\partial t^2}$ now show dynamic similarity. We select $\frac{\partial^2 W}{\partial t^2}$ to take the place of ϕ in (4)-(7). Accordingly we assume

$$\frac{\partial^2 W}{\partial t^2} = \mathrm{Re}\ \phi_1(\zeta) \tag{26}$$

so that

$$\frac{\partial^2 \sigma_{yz}(x,0,t)}{\partial t^2} = \frac{\mu}{x}\ \mathrm{Im}\left[\zeta\sqrt{1-\zeta^2}\ \phi_1'(\zeta)\right] \tag{27}$$

where $\phi_1(\zeta)$ satisfies

$$\eta = 0,\ -\infty<\xi<-V_L/c_2,\ V_R/c_2<\xi<\infty\ :\ \mathrm{Re}\ \phi_1(\zeta) = 0$$

$$\eta = 0, \; -V_L/c_2 < \xi < V_R/c_2 \quad : \; \mathrm{Im}\, \phi_1'(\zeta) = 0 \qquad (28a,b)$$

From (26) we find that

$$\frac{\partial W}{\partial t} = -\frac{x}{c_2}\,\mathrm{Re}\int_{\frac{V_R}{c_2}}^{\zeta} \frac{\phi_1(\tau)}{\tau^2}\, d\tau$$

so that

$$\mathrm{Re}\,\phi_1(\zeta) = \zeta^2 \frac{d}{d\zeta}\left[-\frac{c_2}{x}\frac{\partial W}{\partial t}\right] \qquad (29)$$

Since $\partial W/\partial t$ near the crack tips should show square root singularity and also since $\mathrm{Re}\,\phi_1(\zeta) \longrightarrow O(1)$ as $\zeta \longrightarrow \infty$ we have in view of the conditions (28) and the equation (29)

$$\phi_1(\zeta) = \zeta^2 \frac{d}{d\zeta}\left[\frac{C + D/\zeta}{\sqrt{(V_R/c_2 - \zeta)(V_L/c_2 + \zeta)}}\right] \qquad (30)$$

where the constants C and D are to be determined from the condition that on the crack surface $\sigma_{yz} = -p_1 t$.

Integrating (27), we derive for x>0

$$\sigma_{yz}(x,0,t) = -\frac{\mu x}{c_2^2}\,\mathrm{Im}\int_{x/c_2 t}^{1}\left[\frac{(2\zeta - \tau)\sqrt{1-\tau^2}}{\zeta\tau^3} - \frac{(\tau - \zeta)}{\zeta\tau\sqrt{1-\tau^2}}\right]\phi_1(\tau)\, d\tau$$

$$\sigma_{yz}(-x,0,t) = \frac{\mu x}{c_2^2}\,\mathrm{Im}\int_{-x/c_2 t}^{-1}\left[\frac{(2\zeta - \tau)\sqrt{1-\tau^2}}{\zeta\tau^3} - \frac{(\tau - \zeta)}{\zeta\tau\sqrt{1-\tau^2}}\right]\phi_1(\tau)\, d\tau$$

(31a,b)

Therefore, using the boundary condition

$$\sigma_{yz}(x,0,t) = -p_1 t, \qquad 0 \le x < V_R t$$

$$\sigma_{yz}(-x,0,t) = -p_1 t, \qquad -V_L t < -x \leq 0$$

in equations (31a,b) respectively we obtain after simplification

$$CJ_1\left(\frac{V_R}{c_2},\frac{V_L}{c_2}\right) + DJ_2\left(\frac{V_R}{c_2},\frac{V_L}{c_2}\right) = -\frac{c_2 p_1}{\mu}$$

$$-CJ_1\left(\frac{V_L}{c_2},\frac{V_R}{c_2}\right) + DJ_2\left(\frac{V_L}{c_2},\frac{V_R}{c_2}\right) = -\frac{c_2 p_1}{\mu} \qquad (32a,b)$$

where

$$J_1\left(\frac{V_R}{c_2},\frac{V_L}{c_2}\right) = \int_{V_R/c_2}^{1} M\left(\tau,\frac{V_R}{c_2},\frac{V_L}{c_2}\right)d\tau$$

$$J_2\left(\frac{V_R}{c_2},\frac{V_L}{c_2}\right) = \int_{V_R/c_2}^{1} \tau^{-1}\left[M\left(\tau,\frac{V_R}{c_2},\frac{V_L}{c_2}\right) + N\left(\tau,\frac{V_R}{c_2},\frac{V_L}{c_2}\right) \right]d\tau$$

with
$$M\left(\tau,\frac{V_R}{c_2},\frac{V_L}{c_2}\right) = \left(\frac{2V_R}{c_2\tau^2} - \frac{\tau}{2}\right)\frac{1}{\sqrt{(1-\tau^2)(\tau-V_R/c_2)(\tau+V_L/c_2)}} + \frac{\tau\sqrt{\tau-V_R/c_2}}{2\sqrt{(1-\tau^2)(\tau+V_L/c_2)^3}}$$

$$N\left(\tau,\frac{V_R}{c_2},\frac{V_L}{c_2}\right) \Bigg] = \frac{\sqrt{\tau - V_R/c_2}}{\sqrt{(1-\tau^2)(\tau+V_L/c_2)}}$$

The stress intensity factors at the crack tips defined by

$$N_{11} = \underset{x \to V_R t+}{\mathrm{Lt}} \sqrt{x - V_R t}\; \sigma_{yz}(x,0,t)$$

$$N_{12} = \operatorname*{Lt}_{x \to V_L t+} \sqrt{x - V_L t}\, \sigma_{yz}(-x,0,t)$$

are found to be

$$N_{11} = -\frac{\mu t^{3/2}}{V_R c_2}\sqrt{\frac{c_2^2 - V_R^2}{V_R + V_L}}\,(CV_R + Dc_2)$$

$$N_{12} = \frac{\mu t^{3/2}}{V_L c_2}\sqrt{\frac{c_2^2 - V_L^2}{V_R + V_L}}\,(CV_L - Dc_2) \qquad (33a,b)$$

and in this case the rate of energy flux $\frac{1}{2}\frac{dE_2}{dt}$ into the crack edges defined by (22) is obtained as

$$\frac{1}{2}\frac{dE_2}{dt} = -\frac{\mu\pi}{2c_2}\frac{t^3}{V_R + V_L}\left[(CV_R + Dc_2)^2\sqrt{\frac{c_2^2}{V_R^2} - 1} + (CV_L - Dc_2)^2\sqrt{\frac{c_2^2}{V_L^2} - 1}\right] \qquad (34)$$

where while carrying on the integration (22) the use of the result (24) has again been made.

5. Problem Of Linearly varying shear Traction With Distance Along The Crack Plane

Consider the initially undisturbed half-space $y \geq 0$ subjected to the shear traction $-\,p_2 x$ over $y=0$, $-\,V_L t < x < V_R t$. The boundary conditions are

$$y = 0,\ -V_L t < x < V_R t \ :\ \sigma_{yz} = -p_2 x$$

$$y = 0,\ x < -V_L t,\ x > V_R t \ :\ W = 0 \qquad (35a,b)$$

where $p_2 = (S_+ + S_-)\cos\theta_o / c_2$

In this case we select $\frac{\partial^2 W}{\partial x \partial t}$ to take place of ϕ in (4)-(7). So we take

$$\frac{\partial^2 W}{\partial x \partial t} = \mathrm{Re}\ \phi_2(\zeta)$$

with

$$\frac{\partial^2 \sigma_{yz}}{\partial x \partial t} = \frac{\mu}{x}\ \mathrm{Im}\left[\zeta\sqrt{1-\zeta^2}\ \phi_2'(\zeta)\right] \qquad (36a,b)$$

where $\phi_2(\zeta)$ satisfies

$$\eta = 0,\ -\infty<\xi<-V_L/c_2,\ V_R/c_2<\xi<\infty\ :\ \mathrm{Re}\ \phi_2(\zeta) = 0$$

$$\eta = 0,\ -V_L/c_2<\xi<V_R/c_2\ :\ \mathrm{Im}\ \phi_2'(\zeta) = 0 \qquad (37a,b)$$

From (36a) we have

$$\frac{\partial W}{\partial t} = c_2 t\ \mathrm{Re}\int_{\frac{V_R}{c_2}}^{\zeta} \phi_2(\tau)\ d\tau$$

so that

$$\mathrm{Re}\ \phi_2(\zeta) = \frac{d}{d\zeta}\left(\frac{1}{c_2 t}\frac{\partial W}{\partial t}\right). \qquad (38)$$

Taking into consideration the fact that near the crack tips $x = V_R t$, $-V_L t$, $y=0$ the displacement derivative $\partial W/\partial t$ varies in inverse proportion to the factors $\sqrt{V_R t - x}$, $\sqrt{V_L t + x}$ respectively and as $\zeta \to \infty$ $\mathrm{Re}\ \phi_2(\zeta) = O(1)$. We can take, keeping (37)and (38) in mind,

$$\phi_2(\zeta) = \frac{d}{d\zeta}\left(\frac{(R\zeta+L)\ \zeta}{\sqrt{(V_R/c_2-\zeta)(V_L/c_2+\zeta)}}\right) \qquad (39)$$

where the constants R and L are to be determined.

Equation (36b) can be integrated to derive for x>0

$$\sigma_{yz}(x,0,t) = \mu t \;\mathrm{Im}\int_{x/c_2 t}^{1} \left[\frac{\zeta\sqrt{1-\tau^2}}{\tau^2} - \frac{(\tau-\zeta)}{\sqrt{1-\tau^2}} \right] \phi_2(\tau)\; d\tau$$

$$\sigma_{yz}(-x,0,t) = \mu t \;\mathrm{Im}\int_{-x/c_2 t}^{-1} \left[\frac{\zeta\sqrt{1-\tau^2}}{\tau^2} - \frac{(\tau-\zeta)}{\sqrt{1-\tau^2}} \right] \phi_2(\tau)\; d\tau \qquad (40a,b)$$

Therefore, using the boundary condition

$$\sigma_{yz}(x,0,t) = -p_2 x, \qquad 0 \le x < V_R t$$

$$\sigma_{yz}(-x,0,t) = p_2 x, \qquad -V_L t < -x \le 0$$

in equations (40a,b) respectively we obtain after simplification

$$RK_1\left(\frac{V_R}{c_2}, \frac{V_L}{c_2}\right) + LK_2\left(\frac{V_R}{c_2}, \frac{V_L}{c_2}\right) = \frac{V_R p_2}{\mu}$$

$$-\,RK_1\left(\frac{V_L}{c_2}, \frac{V_R}{c_2}\right) + LK_2\left(\frac{V_L}{c_2}, \frac{V_R}{c_2}\right) = \frac{V_L p_2}{\mu} \qquad (41a,b)$$

where

$$K_1\left(\frac{V_R}{c_2}, \frac{V_L}{c_2}\right) = \int_{V_R/c_2}^{1} \tau \left[P\left(\tau, \frac{V_R}{c_2}, \frac{V_L}{c_2}\right) - 2Q\left(\tau, \frac{V_R}{c_2}, \frac{V_L}{c_2}\right) \right] d\tau$$

$$K_2\left(\frac{V_R}{c_2}, \frac{V_L}{c_2}\right) = \int_{V_R/c_2}^{1} \left[P\left(\tau, \frac{V_R}{c_2}, \frac{V_L}{c_2}\right) - Q\left(\tau, \frac{V_R}{c_2}, \frac{V_L}{c_2}\right) \right] d\tau$$

with $$P\left(\tau, \frac{V_R}{c_2}, \frac{V_L}{c_2}\right) = \left(\frac{2V_R}{c_2 \tau^2} + \frac{\tau}{2} - \frac{V_R}{c_2} \right) \frac{1}{\sqrt{(1-\tau^2)(\tau - V_R/c_2)(\tau + V_L/c_2)}} +$$

$$+\frac{\tau\sqrt{\tau-V_R/c_2}}{2\sqrt{(1-\tau^2)(\tau+V_L/c_2)^3}}$$

$$Q\left[\tau,\frac{V_R}{c_2},\frac{V_L}{c_2}\right] = \frac{\sqrt{\tau-V_R/c_2}}{\sqrt{(1-\tau^2)(\tau+V_L/c_2)}}$$

In this case, the stress intensity factors at the crack tips are obtained as

$$N_{21} = \underset{x\to V_R t+}{\mathrm{Lt}} \sqrt{x-V_R t}\,\sigma_{yz}(x,0,t) = \frac{\mu t^{3/2}}{c_2}\sqrt{\frac{c_2^2-V_R^2}{V_R+V_L}}\,(RV_R+Lc_2)$$

$$N_{22} = \underset{x\to V_L t+}{\mathrm{Lt}} \sqrt{x-V_L t}\,\sigma_{yz}(-x,0,t) = \frac{\mu t^{3/2}}{c_2}\sqrt{\frac{c_2^2-V_L^2}{V_R+V_L}}\,(RV_L-Lc_2) \qquad (42a,b)$$

and in this case the rate of energy flux $\frac{1}{2}\frac{dE_2}{dt}$ into the crack edges defined by (22) is obtained as

$$\frac{dE_3}{dt} = \frac{\mu\pi}{c_2}\frac{t^3}{V_R+V_L}\left[V_R^2(RV_R+Lc_2)^2\sqrt{\frac{c_2^2}{V_R^2}-1} + V_L^2(RV_L-Lc_2)^2\sqrt{\frac{c_2^2}{V_L^2}-1}\right] \qquad (43)$$

where while carrying on the integration (22) the use of the result (24) has again been made.

6. Particular case - I : $V_R=V_L$

If we set $V_R=V_L=V$ in all the cases solved above , the following results are obtained

(i) For the case of constant shear traction $\sigma_{yz} = -p_o$ on the crack faces, we find from (21a,b) that

$A=0, B = \frac{Vmp_o}{\mu E(q)}$, where E(q) is the complete Elliptic integral of second kind and $m = V/c_2$, $q = \sqrt{1-m^2}$. Equations

(21c,d) yield the stress intensity factors at the crack tips as

$$N_0 = N_{01} = N_{02} = \mu B\sqrt{t}\,\sqrt{\frac{c_2^2 - V^2}{2v^3}}$$

Also from (23) we obtain

$$\frac{dE_1}{dt} = -\,\pi\mu t q B^2/m^2$$

(ii) For the case of shear traction $\sigma_{yz} = -p_1 t$ on the crack surface increasing linearly with time, it is found from equation (32) that

$$C = 0,\ D = -\,p_1 c_2/\mu I$$

where

$$I = m^{-2}\left[\frac{\pi}{2} + m^2 F(q) - \frac{2m^3}{(1-m^2)(m+1)}\left\{2\Pi(r^4, r^2) - (m+1)F(r^2)\right\}\right]$$

$F(r^2)$, $\Pi(r^4, r^2)$ are the complete Elliptic integral of first and third kind respectively and $r = \sqrt{(1-m)/(1+m)}$.

In this case the stress intensity factors and the rate of energy flux into the extending crack edges given by (33) and (34) can be simplified to

$$N_1 = N_{11} = N_{12} = -\,\mu D t^{3/2}\sqrt{\frac{c_2^2 - V^2}{2v^3}}$$

and

$$\frac{dE_2}{dt} = -\,\pi\mu t^3 q D^2/m^2$$

(iii) For the case of shear traction $\sigma_{yz} = -p_2 x$ on the crack faces, it is obvious form equation (41) that

$$R = 0,\ L = p_2 V/\mu J$$

where

$$J = \frac{2}{m}E(q) - m\,F(q) + \frac{2m^2}{(1-m^2)(m+1)}\left\{2\Pi(r^4, r^2) - (m+1)F(r^2)\right\}$$

and it is found from equations (42),(43) that the stress intensity factors and the rate of energy flux into the extending crack edges in this case are given by

$$N_2 = N_{21} = -N_{22} = \mu L t^{3/2} \sqrt{\frac{c_2^2 - V^2}{2v}}$$

and

$$\frac{dE_2}{dt} = \pi \mu t^3 q L^2 c_2^2$$

7. Particular Case - II: $V_L = 0$, $V_R = V$.

If we set $V_R = V$ and $V_L = 0$ in all the cases solved above , the following results are obtained

(i) For the case of constant shear traction $\sigma_{yz} = - p_o$ on the crack faces, we find from (21d) that

$$B=0, A = \frac{c_2 p_o}{\mu I_1} ,$$

where

$$I_1 = \frac{2}{\sqrt{m+1}} \left[2\Pi(r^2, r) - F(r) \right]$$

(21c,d) yield the stress intensity factors at the crack tips as

$$N_{01} = \mu A q \sqrt{t/v}$$

$$N_{02} = - \mu A \sqrt{t/v}$$

Also from (23) we obtain

$$\frac{dE_1}{dt} = - \pi \mu t q A^2$$

(ii) For the case of shear traction $\sigma_{yz} = -p_1 t$ increasing linearly with time on the crack surface , it is found from equation (33b) that

$$D = 0, \; C = - p_1 c_2 / \mu J_1$$

where

$$J_1 = \frac{8}{3}\frac{\sqrt{m+1}}{m}E(r) + \frac{m-8}{3\sqrt{m+1}}F(r)$$

In this case the stress intensity factors and the rate of energy flux into the extending crack edges given by (33) and (34) can be simplified to

$$N_{11} = -\mu C t^{3/2} q/\sqrt{v}$$

$$N_{12} = \mu C t^{3/2}/\sqrt{v}$$

and

$$\frac{dE_2}{dt} = -\pi\mu t^3 q C^2$$

(iii) For the case of shear traction $\sigma_{yz} = -p_2 x$ on the crack faces, it is obvious form equation (41) that

$$L = 0, \quad R = p_2 V/\mu K_1$$

where

$$K_1 = \frac{m}{\sqrt{m+1}}\left[8\Pi(r^2, r) + 2\Pi(-r^2, r) - 5F(r)\right] - G(m)$$

with

$$G(m) = \int_m^1 \frac{x^{3/2}}{\sqrt{(1-x^2)(x-m)}}\,dx$$

and it is found from equations (42),(43) that the stress intensity factors and the rate of energy flux into the extending crack edges in this case are given by

$$N_{21} = \mu R t^{3/2}\, q/\sqrt{v}$$

$$N_{22} = 0$$

and

$$\frac{dE_2}{dt} = \pi\mu t^3 q R^2 v^2$$

8. Numerical Results and Discussions

The solution of the original problem is obtained by taking $p_0 = \sigma + (S_+ + S_-)t_0$, $p_1 = (S_+ + S_-)$ $p_2 = (S_+ + S_-)\cos\theta_0/c_2$ and superposing the results obtained in sections 3-5 with the stress fields given by (1) and (2). Taking together the results obtained in the sections 3-5 it is possible to write the stress intensity factors at the crack edges and the rate of energy flux into the extending crack edges as

$$S_1 = \frac{N_{01} + N_{11} + N_{21}}{\sigma\sqrt{c_2 t_0}} = H_+(v_1, v_2, \tau)$$

$$S_2 = \frac{N_{02} + N_{12} + N_{22}}{\sigma\sqrt{c_2 t_0}} = H_-(v_2, v_1, \tau) \qquad (44a,b)$$

and

$$En = \frac{\mu}{t_0 c_2^2 \sigma^2}\left[\frac{dE_1}{dt} + \frac{dE_2}{dt} + \frac{dE_3}{dt}\right] = -\pi\tau\left(\frac{\mu}{c_2}\right)^2\left[G_+(v_1, v_2, \tau) + G_-(v_2, v_1, \tau)\right] \qquad (45)$$

where

$$H_\pm(v_1, v_2, \tau) = \frac{\mu}{c_2}\sqrt{\frac{\tau\,(1-v_1^2)}{(v_1+v_2)}}\left[\frac{1+\Delta(1+S)}{p_0}\left(\pm A + B/v_1\right) - \Delta\tau(1+S)\left\{\frac{1}{p_1}\left(\pm C + D/v_1\right) - \frac{\cos\theta_0}{p_2}\left(Rv_1 \pm L\right)\right\}\right]$$

$$G_\pm(v_1, v_2, \tau) = \frac{\sqrt{(1-v_1^2)}}{v_1+v_2}v_1\left[\left(\frac{1+\Delta(1+S)}{p_0}\right)^2\left(\pm A + B/v_1\right)^2 + \left(\Delta\tau(1+S)\right)^2\left\{\frac{1}{p_1^2}\left(\pm C + D/v_1\right)^2 - \frac{\cos^2\theta_0}{p_2^2}\left(Rv_1 \pm L\right)^2\right\}\right]$$

and the parameter $\tau = t/t_0 - 1$ is the non-dimensionalized time after crack initiation and $\Delta = S_+ t_0/\sigma$ is the ratio at x=y=0 at initiation of the crack plane stress due to one of the plane waves and the pre-stress and $S = S_-/S_+$ is the ratio of the stresses due to plane waves.

Also v_1, v_2 are the non-dimensional crack tip velocities given by $v_1 = V_R/c_2$, $v_2 = V_L/c_2$.

The variations of stress intensity factors and energy flux rate given by (44) and (45) respectively with (i) V_R/c_2 for different values of V_L/c_2, S and with (ii) V_L/c_2 for different values of V_R/c_2 have been presented in Figs.4-6. It has been shown in Fig.4 and Fig.5 that the stress intensity factor at the edge $x = V_R t$, $y = 0$ increases slowly with the increase in the values of V_R/c_2, shows maximum at 0.4 after which it decreases gradually with the increase in the values of V_R/c_2 and also increases with the increase in the values of V_L/c_2 while the stress intensity factor at the other edge increases with the increase in the values of V_R/c_2, V_L/c_2. It has also been depicted in Fig.4 & 6 that $|En|$ increases with the increase in the values of V_R/c_2, showing its maximum value at $V_R/c_2 = 0.8$ after which it decreases with the increase in the value of V_R/c_2. The variations shown in Fig.4 & 5 are expected from physical stand point. Fig.6 shows that S_1, S_2 and $|En|$ also increase with the increase in the values of S.

In Fig.7, the variations of S_1, S_2 and $|En|$ with τ for various values of Δ have been depicted. It may be observed from this figure that $S_1, S_2, |En|$ all increase rapidly with the increase in the value of τ. It may be noted further that for fixed value of τ values S_1, S_2 and $|En|$ increase with the increase in the values Δ.

In Fig.8, S_1, S_2 and $|En|$ are again plotted Vs τ but in this case, Δ is kept fixed whereas S is assumed to vary. It may be seen that increase in the values of S produces marked increase in the value of S_1, S_2 and $|En|$ for any fixed value of τ.

9. Conclusions

Up-to-now the Chaplygin's technique is the most simple and descriptive formulation among different similarity techniques and has been employed in several Elastodynamic problems on crack extension. However, this technique presents some disadvantages, especially in the plane-stress strain cases, during the final steps of the analysis where the appropriate form of the complex functions are sought to accomplish the solution.

But the method of determining the complex function presented in this paper is the correct one and ,therefore, there is no possibility of losing features of the solution.

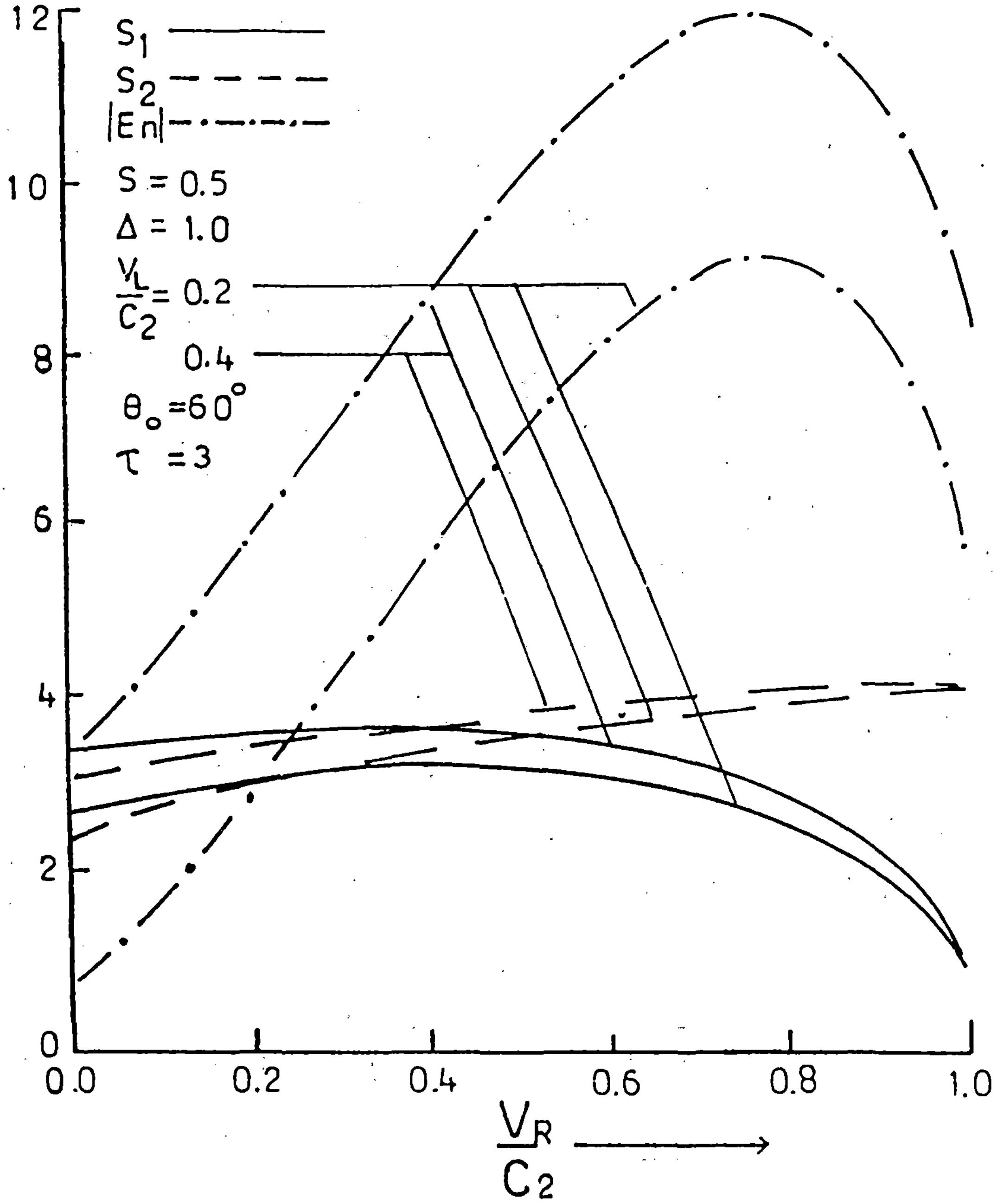

Fig. 4: Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate |En| with non-dimensional speed v_R/c_2.

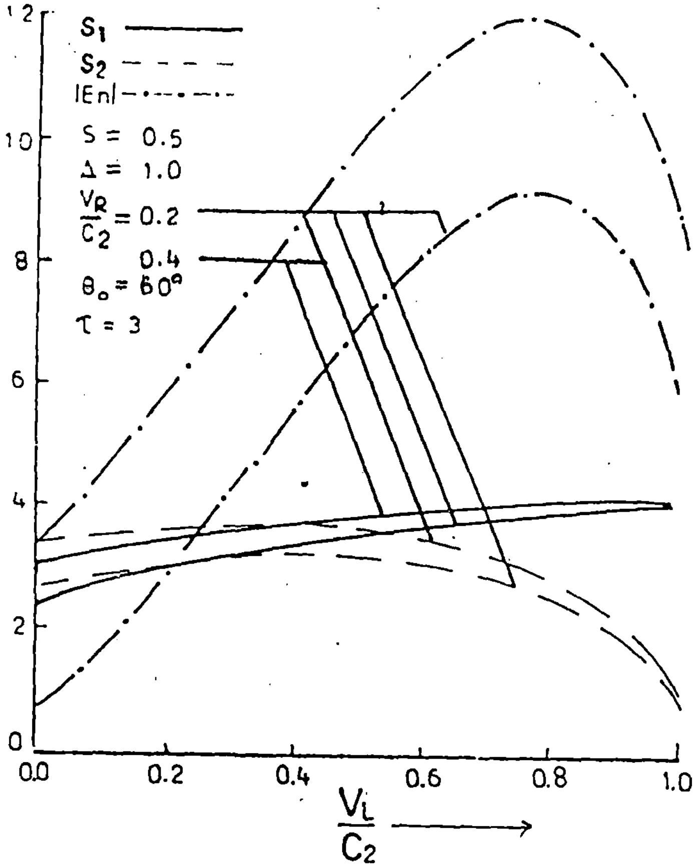

Fig. 5: Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional speed v_L/c_2.

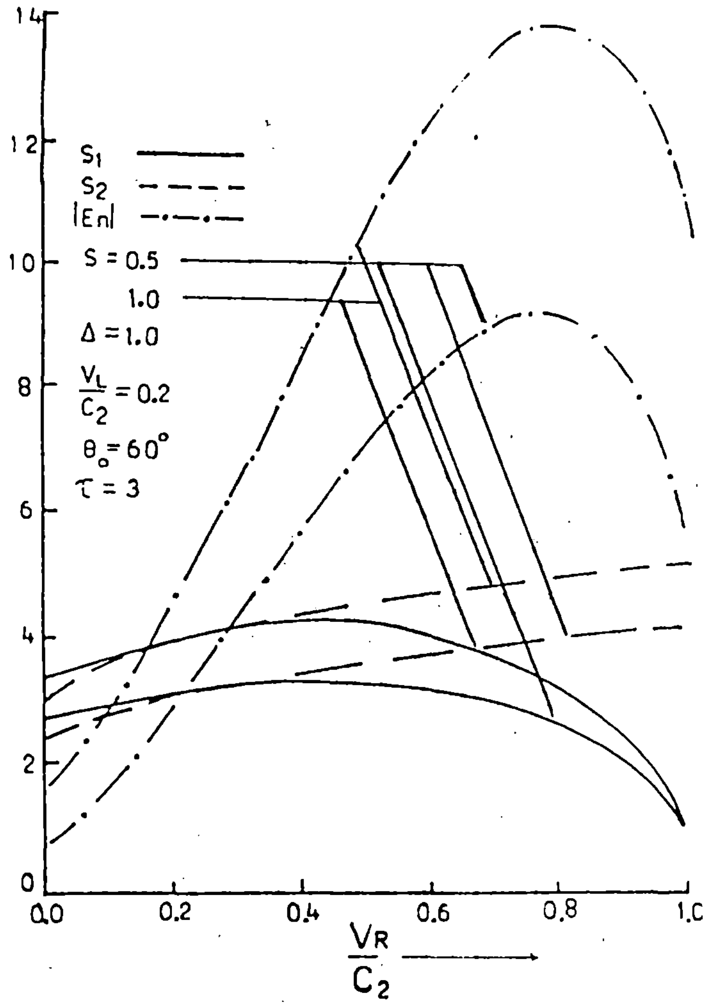

Fig.6: Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional speed v_R/c_2.

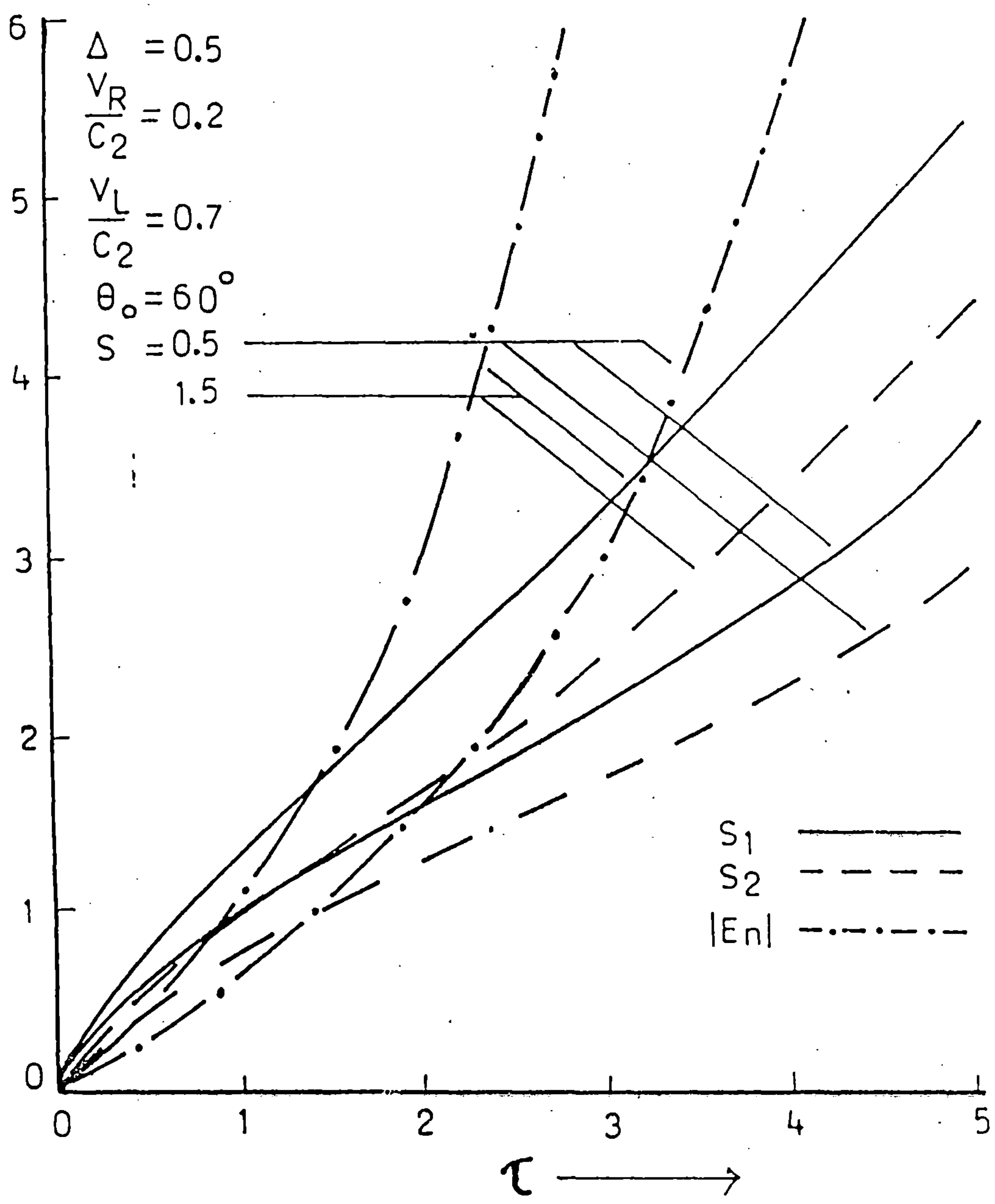

Fig. 7: Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional time after fracture initiation, τ.

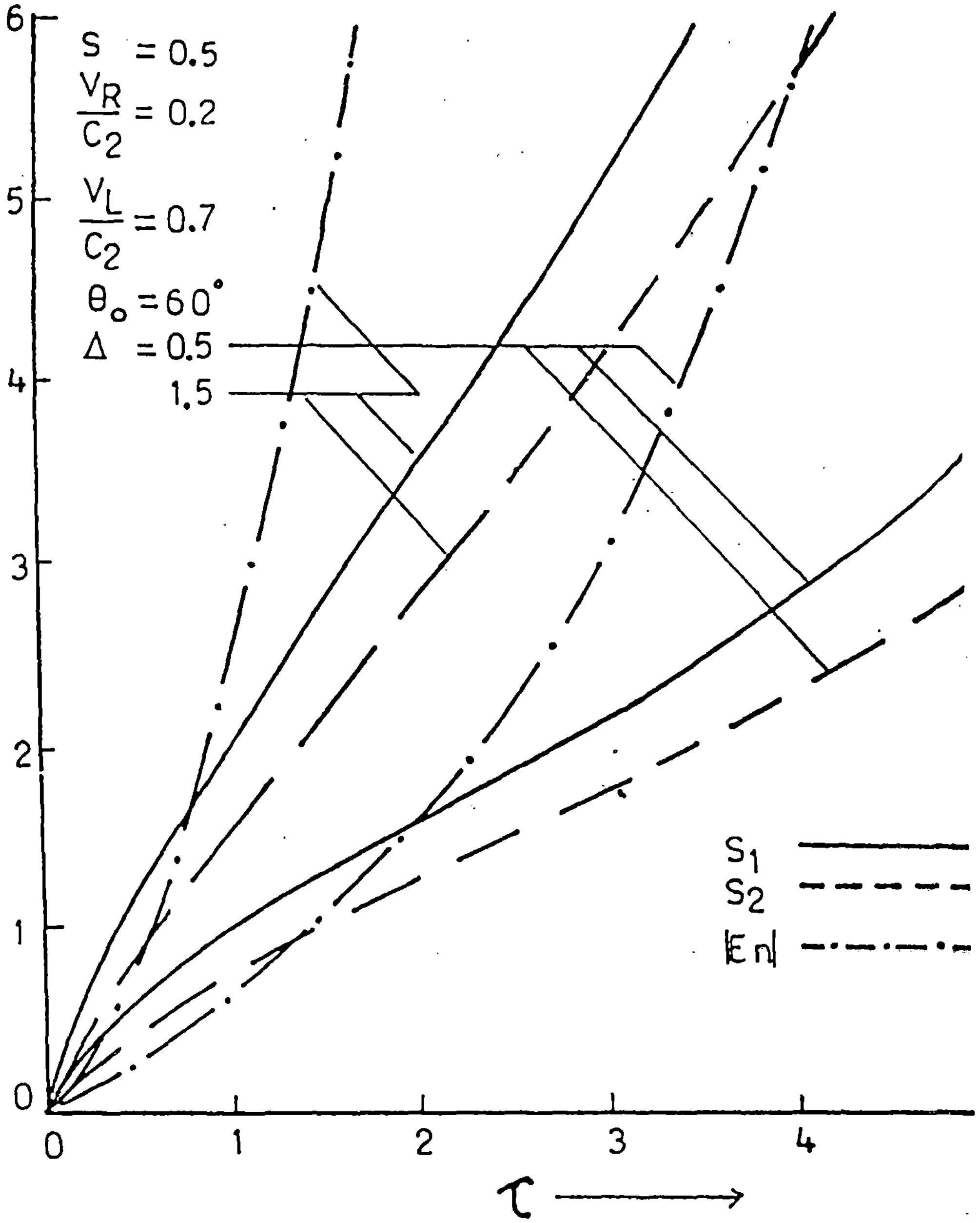

Fig. 8: Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional time after fracture initiation, τ.

BIFURCATION OF A CRACK DUE TO PLANE SH-WAVES IN AN INFINITE ELASTIC MEDIUM

1. Introduction

Several investigations on symmetric or non-symmetric extension of crack in its own plane in an infinite elastic medium have been carried out up-till-now. But when the extension of the crack occurs under an arbitrary angle with its own plane (which leads that a primary crack may bifurcate) the study becomes more relevant. Solutions for dynamic crack bifurcation in anti-plane strain for two special cases were solved by Burgers and Dempsey (1982). Corrected results for mode III kinking of crack under an arbitrary angle was given by Dempsey et. al.(1982). A numerical approach for the study of dynamic propagation of a kinked or bifurcated crack in anti-plane strain and also the dynamic kinking of a crack in plane strain have been given by Burgers(1982,1983). Recently, Achenbach et. al. (1984) have developed a method based on superposition principle to derive approximate expressions for the elastodynamic stress intensity factors of the kinked crack.

In this paper,the dynamic anti-plane problem of bifurcation of a semi-infinite crack due to the incidence of two linearly varying plane SH-waves with non-parallel wave fronts in an infinite elastic medium has been considered. The semi-infinite crack is assumed to bifurcate when the plane waves intersect the crack tip. For constant crack tip velocities the shear stress and particle velocity are self-similar which allow Chaplygin's transformation to reduce the problem to the solution of Laplace equation in semi-infinite strip containing a slit. The Schwarz-Christoffel transformation is employed to map the semi-infinite strip on a half-space.Expressions for shear stress in the planes of the cracks and stress intensity factors in the vicinity of the crack tips have been derived. Finally, numerical results for stress intensity factors have been presented graphically to show its variations with angle of skew for different values of the angle of incidence and the crack tip velocity.

2. Statement Of The Problem

Let two identical plane waves defined by

$$w^{\pm}_{inc} = \mp \frac{c\sigma}{\mu} \tau_{\pm} H(\tau_{\pm}) \tag{1}$$

referred to the coordinate system (r,θ,z) where

$$\tau_{\pm} = t + r\cos(\theta \mp \theta_o)/c, \quad 0 \le \theta_o \le \pi/2$$

and $H()$ is the Heaviside step function, strike the tip of a stationary semi-infinite crack at $t=0$ and causes the crack to bifurcate symmetrically from the tip under an angle $k\pi$ with the plane of the crack and each of the branches starts to extend with velocity $v(<c)$. Thus, at time $t>0$, crack tips are defined by $r=vt$, $\theta=\pm k\pi$. The expanding crack, the circular wave front associated with its motion and the plane wave fronts are shown in Fig.1. The shear stress component $\sigma^o_{\theta z}$ corresponding to incident waves is

$$\sigma^o_{\theta z} = \sigma\left[\sin(\theta-\theta_o)H(\tau_+) - \sin(\theta+\theta_o)H(\tau_-) \right] \tag{2}$$

Superposing the fields due to incident waves and scattered wave we see that the conditions on the crack faces due to the scattered wave are

$$\theta=\pm\pi;\ r>0 : \sigma_{\theta z} = -2\sigma \sin\theta_o H(t - r\cos\theta_o/c)$$

$$\theta=\pm k\pi;\ 0\le r<vt : \sigma_{\theta z} = 2\sigma \sin\theta_o \cos k\pi \tag{3a,b}$$

The shear traction given by (3a) generate the plane waves with constant particle velocity, i.e., of magnitude $\mp 2c\sigma/\mu$. Since stresses and velocities are continuous across the cylindrical wave front, on the cylindrical wave front the conditions in particle velocity are

$$\pi-\theta_o<\theta<\pi,\ r=ct : \dot{w} = -2c\sigma/\mu$$

$$-\pi<\theta<-\pi+\theta_o,\ r=ct : \dot{w} = 2c\sigma/\mu$$

and $$-\pi+\theta_o<\theta<\pi-\theta_o,\ r=ct : \dot{w} = 0 \tag{4a-c}$$

The problem is obviously anti-symmetric about x- axis with respect to particle velocity, so only a half-plane need be considered.

In polar coordinate (r,θ); the two dimensional anti-plane wave motions are governed by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \qquad (5)$$

where $w(r,\theta,t)$ is the displacement in the z- direction and $c=\sqrt{\mu/\rho}$ is the velocity of transverse waves.

Absence of any characteristic length in the geometrical configuration of the problem and the boundary conditions (3),(4) suggest that the particle velocity $\dot{w}$ is self-similar, implying thereby that depends on $r/t,\theta$ rather than on r,θ,t separately.

Introducing the variable

$$s=r/t$$

it is found that $\dot{w}(s,\theta)$ satisfies the equation

$$s^2(1-s^2/c^2)\frac{\partial^2 \dot{w}}{\partial s^2} + s(1-2s^2/c^2)\frac{\partial \dot{w}}{\partial s} + \frac{\partial^2 \dot{w}}{\partial \theta^2} = 0 \qquad (6)$$

Within the half-circular region ABEMDCA, see Fig.1, the boundary conditions on $\dot{w}(s,\theta)$ are

$$\theta=\pi,\ s\le c\ :\ \frac{\partial \dot{w}}{\partial \theta} = 0$$

$$\pi-\theta_o<\theta<\pi,\ s=c\ :\ \dot{w} = -\frac{2c\sigma}{\mu}$$

$$0<\theta<\pi-\theta_o,\ s=c\ :\ \dot{w} = 0$$

$$\theta=0,\ 0\le s\le c:\quad \dot{w} = 0$$

$$\theta=k\pi\pm\varepsilon,\ 0\le s<v:\ \frac{\partial \dot{w}}{\partial \theta} = 0 \qquad (7a-e)$$

For $s<c$, the Chaplygin's transformation

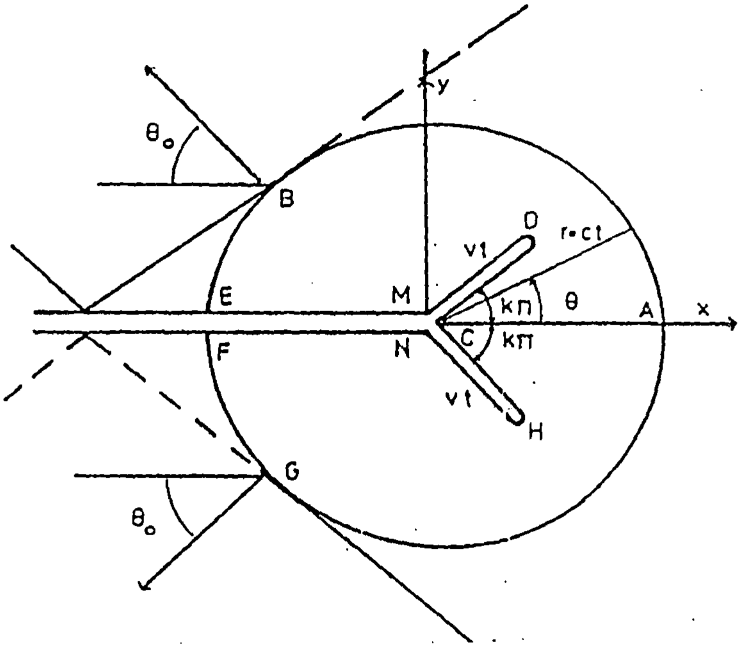

Fig.1: Pattern of incident, reflected and diffracted waves.

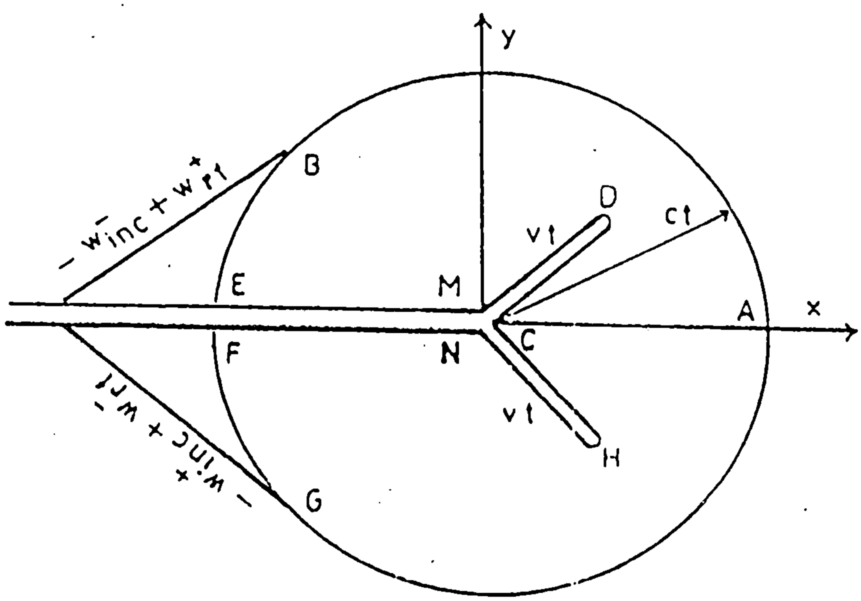

Fig.2: Pattern of waves for the superposition problem.

$$\beta = \cosh^{-1}(c/s) \tag{8}$$

reduces equation (6) to Laplace equation

$$\frac{\partial^2 \dot{w}}{\partial \beta^2} + \frac{\partial^2 \dot{w}}{\partial \theta^2} = 0 \tag{9}$$

and maps the interior of the half-circular region $[0 \le \theta \le \pi, s \le c]$ of the physical plane into a semi-infinite strip $[0 \le \theta \le \pi, 0 \le \beta < \infty]$ in θ-β plane as shown in Fig 3.

A convenient method to solve the equation (9) is to express $\dot{w}(s,\theta)$ as the real part of an analytic function and to construct an appropriate analytic function of the complex variable $\beta + i\theta$.

The domain in the γ-plane can be related to the upper half-plane of the ζ-plane by means of Schwarz-Christoffel transformation

$$\gamma = \omega(\zeta), \quad \zeta = \xi + i\eta$$

An appropriate transformation is

$$\gamma = C_o \int_1^{\zeta} \frac{u \, du}{(u+\xi_C)(u-\xi_M)\sqrt{1-u^2}} + i\pi \tag{10}$$

where C_o is an arbitrary complex constant. The ζ-plane is shown in Fig.4. The transformation given by (10) implies that the points E, A and D are mapped into $\zeta = 1$, $\zeta = -1$ and $\zeta = 0$ respectively.

Equation (10) may be integrated to yield

$$\gamma = -\frac{\xi_C}{\xi_M + \xi_C} \frac{C_o}{\sqrt{1-\xi_C^2}} \left[\ln\left\{ \sqrt{(1-\xi_C^2)(1-\zeta^2)} + \zeta\xi_C + 1 \right\} - \ln(\zeta + \xi_C) \right] -$$

$$- \frac{\xi_M}{\xi_M + \xi_C} \frac{C_o}{\sqrt{1-\xi_M^2}} \left[\ln\left\{ \sqrt{(1-\xi_M^2)(1-\zeta^2)} - \zeta\xi_M + 1 \right\} - \ln(\zeta - \xi_M) \right] + i\pi \tag{11}$$

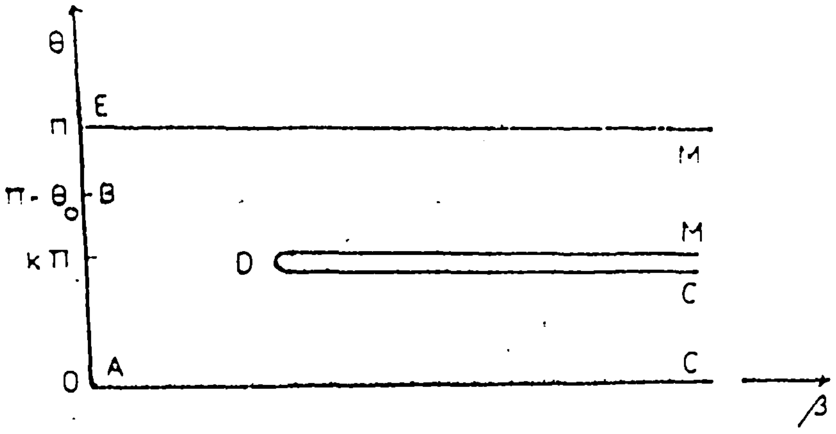

Fig. 3: The θ-β plane.

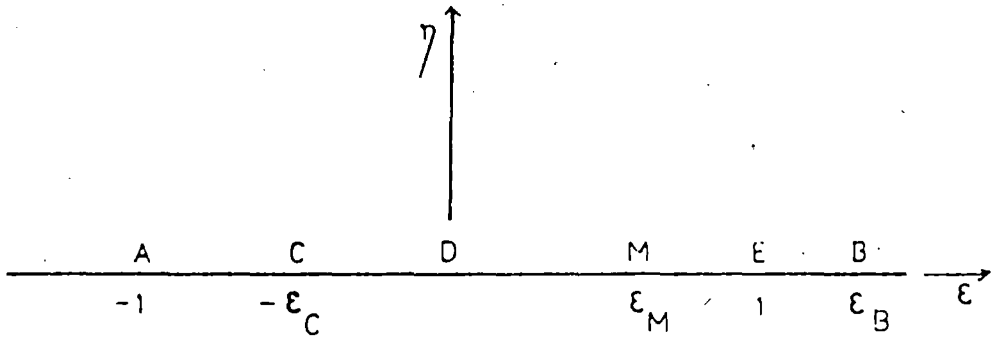

Fig. 4: The ξ-η plane.

Considering the change in imaginary parts in M and C, we obtain

$$-\frac{\xi_M}{\xi_M+\xi_C}\,\frac{C_o}{\sqrt{1-\xi_M^2}} = 1-k \tag{12}$$

and

$$-\frac{\xi_C}{\xi_M+\xi_C}\,\frac{C_o}{\sqrt{1-\xi_C^2}} = k \tag{13}$$

respectively. Thus the result (11) becomes with the aid of (12) and (13)

$$\gamma = k\left[\ln\left\{\sqrt{(1-\xi_C^2)(1-\zeta^2)} + \zeta\xi_C+1\right\} - \ln(\zeta+\xi_C)\right] + (1-k)\left[\ln\left\{\sqrt{(1-\xi_M^2)(1-\zeta^2)}\right.\right.$$

$$\left.\left.-\zeta\xi_M+1\right\} - \ln(\zeta-\xi_M)\right] + i\pi \tag{14}$$

Comparing the coordinate of the point D in the γ-plane and ζ-plane we obtain

$$\gamma_D = \cosh^{-1}\frac{c}{v} + ik\pi = k\ \ln\frac{1+\sqrt{1-\xi_C^2}}{\xi_C} + (1-k)\ln\frac{1+\sqrt{1-\xi_M^2}}{\xi_M} + ik\pi \tag{15}$$

Comparison of the coordinate of the point B in γ- plane and ζ- plane results in the relation

$$k\ \sin^{-1}\frac{1+\xi_C\xi_B}{\xi_B+\xi_C} - (1-k)\ \sin^{-1}\frac{\xi_B\xi_M-1}{\xi_B-\xi_M} = \pi/2 - \theta_o \tag{16}$$

Equations (12), (13) (15) and (16) can be used to solve for C_o, ξ_M, ξ_C, ξ_B.

The boundary conditions given by (7a-e) turn into the following conditions in the ζ-plane

$$\eta=0,\quad -\infty<\xi\leq-\xi_C \;:\; \dot{w} = 0$$

$$\eta=0,\quad -\xi_C<\xi<1 \;:\; \frac{\partial\dot{w}}{\partial\eta} = 0$$

$$\eta=0,\quad 1\leq\xi\leq\xi_B \;:\; \dot{w} = -\frac{2c\sigma}{\mu}$$

$$\eta=0,\quad \xi_B<\xi<\infty \;:\; \dot{w} = 0 \qquad (17a\text{-}d)$$

Before we proceed to construct an analytic function which satisfies the conditions (17a-d), we will investigate the relations between small distances from the point D in the physical plane, in the γ-plane and in the ζ- plane. In the physical plane we consider the distance r-vt in the crack plane $\theta=k\pi$. For $(r-vt)/r<<1$ we easily establish that

$$\frac{r-vt}{r} = \frac{v}{c}\left[\mathrm{Cosh}\beta_D - \mathrm{Cosh}\beta\right] \approx \left[1-v^2/c^2\right]^{1/2}(\beta_D - \beta)$$

In the ζ- plane we find for $|\zeta|<<1$

$$\gamma-\gamma_D = \frac{1}{2}\,\omega_o\zeta^2$$

i.e.,

$$\zeta^2 = -\frac{2}{\omega_o}\left[1-v^2/c^2\right]^{-1/2}(r-vt)/vt \qquad (18)$$

where

$$\omega_o = k\,\frac{\sqrt{1-\xi_C^2}}{\xi_C^2} + (1-k)\,\frac{\sqrt{1-\xi_M^2}}{\xi_M^2} \qquad (19)$$

Equation (18) has been derived by expanding (14) and maintaining terms of $O(\zeta^2)$. The terms $O(\zeta)$ in the expansion is found to vanish with the aid of the results (12), (13).Further the equation (18) suggests that $\zeta=i\eta$

Mhere

$$\eta = \left[\frac{2}{\omega_o}\left[1-v^2/c^2\right]^{-1/2}\frac{c}{v}\,\frac{r-vt}{ct}\right]^{1/2} \qquad (20)$$

Next, if we take $\dot{w}$ =Re $F(\zeta)$ then in view of the conditions given by (17) it is found convenient to work with $F'(\zeta)$. Accordingly, we consider

$$F'(\zeta) = F_1'(\zeta) + F_2'(\zeta) \tag{21}$$

where

$$F_1'(\zeta) = \frac{A}{(\zeta-\xi_B)\sqrt{(1-\zeta)(\zeta+\xi_C)}} \tag{22}$$

and

$$F_2'(\zeta) = \frac{1}{\sqrt{(1-\zeta)(\zeta+\xi_C)}}\left[\frac{B}{\zeta} + \frac{C}{\zeta^2}\right] \tag{23}$$

Integrating (21) with respect to ζ and using the condition that $\dot{w}$ possesses a jump discontinuity at $\xi=\xi_B$ as seen from (17c) and (17d) we find that

$$A = 2\frac{c\sigma}{\pi\mu}\sqrt{(\xi_B+\xi_C)(\xi_B-1)} \tag{24}$$

Integrating (23) we obtain

$$F_2(\zeta) = \xi_C^{-1}\left[-C\zeta^{-1}\sqrt{(\zeta+\xi_C)(1-\zeta)} + \xi_C^{-1/2}\left\{(1-\xi_C)C-2B\xi_C\right\}\ln\left\{\frac{\sqrt{\xi_C(1-\zeta)}+\sqrt{\zeta+\xi_C}}{\sqrt{\zeta\xi_C}}\right\}\right] \tag{25}$$

Since the term involving logarithm gives rise to a logarithmic singularity at $\zeta=0$ which is not acceptable, we require

$$B = (1-\xi_C)C/2\xi_C \tag{26}$$

The shear stress at $r>vt$, $\theta=k\pi$ can be obtained using the relation

$$\frac{\partial\dot{w}}{\partial\theta} = -\operatorname{Im}\left[F'(\zeta)\frac{d\zeta}{d\gamma}\right]$$

as

$$\tau_{\theta z} = -\frac{\mu}{r}\operatorname{Im}\int_{r/c}^{t} F'(\zeta)\frac{d\zeta}{d\gamma}\,dt \tag{27}$$

As for all values of k, ζ can not be expressed in terms of t explicitly. Hence for $k \neq 0$ the integration (27) is to be carried out numerically.

In order to extract the singular term we change the integration of $F'_2(\zeta)$ over t in (27) to an integration over ζ as follows

$$I_2 = -\frac{\mu}{c}\,\mathrm{Im}\int_{\Gamma} \sinh\beta(\zeta)\,F'_2(\zeta)\,d\zeta \tag{28}$$

where Γ is the corresponding contour in the ζ-plane.

Integrating (28) by parts and then changing the variable in the integration over $F'_1(\zeta)$ and $F_2(\zeta)$ to the variable s we obtain from (27)

$$\tau_{\theta z} = -\frac{\mu}{s}\left(1-\frac{s^2}{c^2}\right)^{1/2}\mathrm{Im}\,F_2(\zeta) - \mu I \tag{29}$$

where

$$I = I_3 + C I_4 \tag{30}$$

with

$$I_3 = 2\frac{c\sigma}{\pi\mu}\sqrt{(\xi_B+\xi_C)(\xi_B-1)}\;\mathrm{Im}\int_s^c s^{-2}\,\frac{d\zeta}{d\gamma}\,\frac{ds}{(\zeta-\xi_B)\sqrt{(1-\zeta)(\zeta+\xi_C)}} \tag{31}$$

and

$$I_4 = \xi_C^{-1}\;\mathrm{Im}\int_s^c \zeta^{-1}\sqrt{(\zeta+\xi_C)(1-\zeta)}\;\frac{ds}{s^2\sqrt{1-s^2/c^2}} \tag{32}$$

The stress intensity factor N, at the crack tip defined by r=vt, $\theta=k\pi$ is obtained using (20),(25) and (29) as

$$N = \underset{r\longrightarrow vt}{\mathrm{Lt}}\sqrt{2\pi(r-vt)}\;\tau_{\theta z} = -\left[\frac{\pi\omega_o}{vc}\left(1-\frac{v^2}{c^2}\right)^{3/2}\right]^{1/2}\mu C\sqrt{ct/\xi_C} \tag{33}$$

From the asymptotic analysis of the deformation field about a dynamically extending crack tip, we see that if $\tau_{\theta z}\longrightarrow\tau$ as $r\longrightarrow vt-0$ on

$\theta=k\pi$, then the regular term in $\tau_{\theta z}$ as $r\rightarrow vt+0, \theta=k\pi$ should also be equal to τ. So, we require

$$I_3\big|_{s=v}+CI_4\big|_{s=v} = -\frac{2\sigma}{\mu}\sin\theta_0\cos k\pi \qquad (34)$$

where I_3 and I_4 are given by (31) and (32) respectively. Equation (34) gives the value of C.

3. Numerical Results and Discussions

In this section numerical results for the dimensionless stress intensity factor S where $S = N/\sigma\sqrt{ct}$ and N is defined by (33) have been plotted in Fig.5 versus the parameter k which defines the angle of skew, for different values of v/c and θ_0. It has been shown in Fig.5 that for fixed value of v/c the values of S decreases with the increase in the value of k and as the value of v/c increases the values of S is found to decrease which is expected from physical stand point. Again it is to be noted from Fig.4 that the values of S increases with the increase in the value of the angle of incidence of the plane waves.

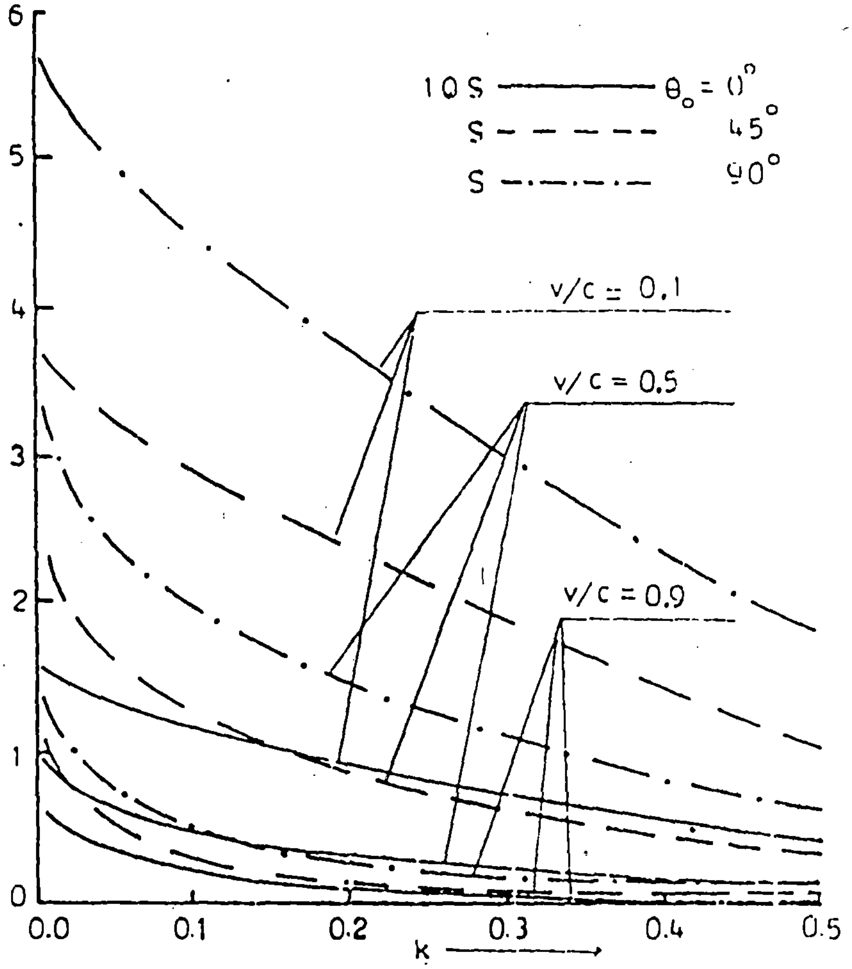

Fig. 5: Variation of stress intensity factor with k.