Chapter 2

ESTIMATION OF GROWTH OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS OF ORDER ZERO ON THE BASIS OF SLOWLY CHANGING FUNCTIONS

CHAPTER

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2.1 Introduction, Definitions and Notations.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function and g be an entire function defined on $\mathbb{C}$. In the sequel we use the following notation:

$$\log^{[k]} x = \log(\log^{[k-1]} x) \text{ for } k = 1, 2, 3, \ldots \text{and } \log^{[0]} x = x.$$

We recall the following definitions:

Definition 2.1.1 *The order ρ_f and lower order λ_f of a meromorphic func-*

The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [**19**].

tion f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$
$$\text{and } \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}$$
$$\text{and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 2.1.2 *The hyper order $\overline{\rho}_f$ and hyper lower order $\overline{\lambda}_f$ of a meromorphic function f are defined as follows:*

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r}$$
$$\text{and } \overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r}.$$

If f is entire then

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}$$
$$\text{and } \overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition 2.1.3 *[46] Let f be meromorphic function of order zero. Then ρ^*_f, λ^*_f and $\overline{\rho}^*_f, \overline{\lambda}^*_f$ are defined as follows:*

$$\rho^*_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r},$$
$$\lambda^*_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r}$$
$$\text{and } \overline{\rho}^*_f = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r},$$
$$\overline{\lambda}^*_f = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r}$$

If f is entire, then clearly

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r},$$
$$\lambda_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}$$
$$\textit{and } \overline{\rho}_f^* = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}$$
$$\overline{\lambda}_f^* = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}.$$

Definition 2.1.4 *The type σ_f of a meromorphic function f is defined as*

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}, 0 < \rho_f < \infty.$$

If f is entire, then

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, 0 < \rho_f < \infty.$$

Somasundaram and Thamizharasi [63] introduced the notions of L-order and L-type for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. Their definitions are as follows:

Definition 2.1.5 *[63] The L-order ρ_f^L and the L-lower order λ_f^L of an entire function f are defined as follows:*

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]}$$
$$\textit{and } \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]}.$$

When f is meromorphic, then

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]}$$
$$\textit{and } \lambda_f^L = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]}.$$

Definition 2.1.6 *[63]. The L-type σ_f^L of an entire function f with L-order ρ_f^L is defined as*

$$\sigma_f^L = \limsup_{r\to\infty} \frac{\log M(r,f)}{[rL(r)]^{\rho_f^L}}, 0 < \rho_f^L < \infty.$$

For meromorphic f, the L-type σ_f^L becomes

$$\sigma_f^L = \limsup_{r\to\infty} \frac{T(r,f)}{[rL(r)]^{\rho_f^L}}, 0 < \rho_f^L < \infty.$$

Similarly one can define the L-hyper order and L-hyper lower order of entire and meromorphic f.The more generalised concept of L-order and L-type of entire and meromorphic functions are L^*-order and L^*-type. Their definitions are as follows:

Definition 2.1.7 *The L^*-order, L^*-lower order and L^*-type of a meromorphic function f are defined by*

$$\begin{aligned} \rho_f^{L^*} &= \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[re^{L(r)}]}, \\ \lambda_f^{L^*} &= \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[re^{L(r)}]} \\ \text{and } \sigma_f^{L^*} &= \limsup_{r\to\infty} \frac{T(r,f)}{[re^{L(r)}]^{\rho_f^{L^*}}}, 0 < \rho_f^{L^*} < \infty. \end{aligned}$$

When f is entire, one can easily verify that

$$\begin{aligned} \rho_f^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]}, \\ \lambda_f^{L^*} &= \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]} \\ \text{and } \sigma_f^{L^*} &= \limsup_{r\to\infty} \frac{\log M(r,f)}{[re^{L(r)}]^{\rho_f^{L^*}}}, 0 < \rho_f^{L^*} < \infty. \end{aligned}$$

In this chapter we intend to establish some results relating to the growth properties of composite entire and meromorphic functions on the basis of L-order and L^*-order improving some previous results.

2.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.2.1 *[14] If f and g are two entire functions, then for all sufficiently large values of r*

$$M(\frac{1}{8}M(\frac{r}{2},g)-|g(0)|,f)\leq M(r,fog)\leq M(M(r,g),f).$$

Lemma 2.2.2 *[62] Let f be entire and g be a transcendental entire function of finite lower order. Then for any $\delta>0$,*

$$M(r^{1+\delta},fog)\geq M(M(r,g),f)(r\geq r_0).$$

Lemma 2.2.3 *[29] Let f be a meromorphic function and g be transcendental entire. If $\lambda_{fog}<\infty$ then $\lambda_f=0$.In the line of Lemma 2.2.3 we may state the following lemma without proof.*

Lemma 2.2.4 *Let f be a meromorphic function and g be transcendental entire. If $\lambda^L_{fog}<\infty$ then $\lambda^L_f=0$.*

Lemma 2.2.5 *[31] Let f be meromorphic and g be transcendental entire. If $\rho_{fog}<\infty$ then $\rho_f=0$.*

Lemma 2.2.6 *Let f be meromorphic and g be transcendental entire. If $\rho^L_{fog}<\infty$ then $\rho^L_f=0$.*

Lemma 2.2.7 *Let f be a meromorphic function and g be transcendental entire. If $\lambda^{L^*}_{fog}<\infty$ then $\lambda^{L^*}_f=0$.*

Lemma 2.2.8 *Let f be meromorphic and g be transcendental entire. If $\rho^{L^*}_{fog}<\infty$ then $\rho^{L^*}_f=0$.*

Lemma 2.2.9 *Let f be entire and g be transcendental entire with $\lambda^L_g<\infty$.Also let $\rho^L_{fog}=0$.Then ${}^*\rho_f\ {}^*\lambda^L_g\leq{}^*\ \rho^L_{fog}\leq{}^*\rho_f\ {}^*\rho^L_g$.*

Proof. By Lemma 2.2.2,

$$\begin{aligned}
{}^*\rho^L_{fog} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r^{1+\delta}, fog)}{\log^{[2]}[rL(r)]} \\
&\geq \limsup_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \liminf_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log^{[2]}[rL(r)]} \\
&= \rho^*_f \, {}^*\lambda^L_g .
\end{aligned}$$

Again by Lemma 2.2.1,

$$\begin{aligned}
{}^*\rho^L_{fog} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]}[rL(r)]} \\
&\leq \limsup_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log^{[2]}[rL(r)]} \\
&= \rho^*_f \, {}^*\rho^L_g .
\end{aligned}$$

From the above two inequalities we get that $\rho^*_f \, {}^*\lambda^L_g \leq {}^*\rho^L_{fog} \leq \rho^*_f \, {}^*\rho^L_g$. ■

Remark 2.2.1 *The second part of Lemma 2.2.9 is also valid under the same conditions for meromorphic f and entire g.*

Lemma 2.2.10 *Let f and g be two entire functions such that $\rho^L_f = 0$ and $\lambda^L_g < \infty$. Also let g be transcendental entire. Then*

$$\lambda^*_f \rho^L_g \leq \rho^L_{fog} \leq \rho^*_f \rho^L_g .$$

Proof. In view of Lemma 2.2.1, we get that

$$\begin{aligned}
\rho^L_{fog} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log[rL(r)]} \\
&\leq \limsup_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log[rL(r)]} \\
&= \rho^*_f \rho^L_g .
\end{aligned}$$

Also from Lemma 2.2.2 it follows that

$$\begin{aligned}\rho^L_{fog} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r^{1+\delta}, fog)}{\log[rL(r)]^{1+\delta}} \\ &\geq \liminf_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log[rL(r)]^{1+\delta}} \\ &= \lambda^*_f \rho^L_g .\end{aligned}$$

Now combining the above two inequalities we obtain that

$$\lambda^*_f \rho^L_g \leq \rho^L_{fog} \leq \rho^*_f \rho^L_g .$$

■

Remark 2.2.2 *Under the conditions of Lemma 2.2.10,*

$$\rho^*_f \lambda^L_g \leq \rho^L_{fog} \leq \rho^*_f \rho^L_g .$$

Lemma 2.2.11 *If f be an entire function and g be transcendental entire with*

$$\lambda^L_{fog} = 0, \lambda^L_g < \infty.$$

$$\textit{Then } {}^*\, \lambda^L_{fog} \geq {}^*\, \lambda^{L*}_f \lambda^L_g .$$

Proof. By Lemma 2.2.2,

$$\begin{aligned}{}^*\, \lambda^L_{fog} &= \liminf_{r\to\infty} \frac{\log^{[2]} M(r^{1+\delta}, fog)}{\log^{[2]}[rL(r)]^{1+\delta}} \\ &\geq \liminf_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \liminf_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log^{[2]}[rL(r)]} \\ &= \lambda^*_f\, {}^*\lambda^L_g .\end{aligned}$$

This proves the lemma. ■

Lemma 2.2.12 *Let f be entire and g be transcendental entire with $\lambda^{L^*}_g < \infty$. Also let $\rho^{L^*}_{fog} = 0$. Then*

$$^*\rho^*_f \lambda^{L^*}_g \leq^* \rho^{L^*}_{fog} \leq^* \rho^*_f \rho^{L^*}_g .$$

Proof. By Lemma 2.2.2,

$$\begin{aligned}
{}^*\rho_{fog}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r^{1+\delta}, fog)}{\log^{[2]}[re^{L(r)}]} \\
&\geq \limsup_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \liminf_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log^{[2]}[re^{L(r)}]} \\
&= \rho_f^* \, {}^*\lambda_g^{L^*}.
\end{aligned}$$

Again by Lemma 2.2.1,

$$\begin{aligned}
{}^*\rho_{fog}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]}[re^{L(r)}]} \\
&\leq \limsup_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log^{[2]}[re^{L(r)}]} \\
&= \rho_f^* \, {}^*\rho_g^{L^*}.
\end{aligned}$$

From the above two inequalities we get that

$$\rho_f^* \, {}^*\lambda_g^{L^*} \leq^* \rho_{fog}^{L^*} \leq \rho_f^* \, {}^*\rho_g^{L^*}.$$

This proves the lemma. ■

Remark 2.2.3 *The second part of Lemma 2.2.12 is also valid under the same conditions for meromorphic f and entire g.*

Lemma 2.2.13 *Let f and g be two entire functions such that $\rho_f^{L^*} = 0$ and $\lambda_g^{L^*} < \infty$. Also let g be transcendental entire. Then*

$$\lambda_f^* \rho_g^{L^*} \leq \rho_{fog}^{L^*} \leq \rho_f^* \rho_g^{L^*}$$

Proof. In view of Lemma 2.2.1, we get that

$$\begin{aligned}
\rho_{fog}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log[re^{L(r)}]} \\
&\leq \limsup_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log[re^{L(r)}]} \\
&= \rho_f^* \rho_g^{L^*}.
\end{aligned}$$

Again from Lemma 2.2.2 it follows that

$$\begin{aligned}\rho^{L^*}_{fog} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r^{1+\delta}, fog)}{\log[re^{L(r)}]^{1+\delta}} \\ &\geq \liminf_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log[re^{L(r)}]^{1+\delta}} \\ &= \lambda^*_f \rho^{L^*}_g .\end{aligned}$$

Now combining the above two inequalities we obtain that

$$\lambda^*_f \rho^{L^*}_g \leq \rho^{L^*}_{fog} \leq \rho^*_f \rho^{L^*}_g .$$

Thus the lemma is established. ■

Remark 2.2.4 *Under the conditions of Lemma 2.2.13,*

$$\rho^*_f \lambda^{L^*}_g \leq \rho^{L^*}_{fog} \leq \rho^*_f \rho^{L^*}_g .$$

Lemma 2.2.14 *If f is an entire function and g be transcendental entire with*

$$\lambda^{L^*}_{fog} = 0, \lambda^{L^*}_g < \infty.$$

Then

$${}^*\lambda^{L^*}_{fog} \geq {}^*\lambda^{L^*}_f \; {}^*\lambda^{L^*}_g .$$

Proof. By Lemma 2.2.2,

$$\begin{aligned}{}^*\lambda^{L^*}_{fog} &= \liminf_{r\to\infty} \frac{\log^{[2]} M(r^{1+\delta}, fog)}{\log^{[2]}[re^{L(r)}]^{1+\delta}} \\ &\geq \liminf_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \liminf_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log^{[2]}[re^{L(r)}]} \\ &= \lambda^*_f \; {}^*\lambda^{L^*}_g .\end{aligned}$$

This proves the lemma. ■

2.3 Theorems.

In this section we present the main results of this chapter.

Theorem 2.3.1 *Let f be meromorphic and g be transcendental entire such that $\lambda_{fog}^{L} > 0$. Then for every positive number A,*

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^{A}, f^{(k)})} \leq \frac{{}^{*}\lambda_{fog}^{L}}{A^{*}\lambda_{f}^{L}}, \text{ where } k = 0, 1, 2, 3, \ldots$$

Proof. Case (a). If $\lambda_{fog}^{L} = \infty$, the theorem is obvious.

Case (b). If $\lambda_{fog}^{L} < \infty$, then by Lemma 2.2.4, $\lambda_{f}^{L} = 0$ and the theorem follows. ■

Theorem 2.3.2 *Let f be meromorphic and g be transcendental entire such that $\rho_{fog}^{L} = 0$. Also, let $0 < {}^{*}\lambda_{fog}^{L} \leq {}^{*}\rho_{fog}^{L} < \infty$ and $0 < {}^{*}\lambda_{f}^{L} \leq {}^{*}\rho_{f}^{L} < \infty$. Then for any positive number A,*

$$\frac{{}^{*}\lambda_{fog}^{L}}{A^{*}\rho_{f}^{L}} \leq \liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^{A}, f)} \leq \frac{{}^{*}\lambda_{fog}^{L}}{A^{*}\lambda_{f}^{L}}$$

$$\leq \limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^{A}, f)} \leq \frac{{}^{*}\rho_{fog}^{L}}{A^{*}\lambda_{f}^{L}}.$$

Proof. Since $\rho_{fog}^{L} = 0 < \infty$ by Lemma 2.2.6, $\rho_{f}^{L} = 0$. Now from the definition of ${}^{*}\rho_{f}^{L}$ and ${}^{*}\lambda_{f}^{L}$ we have for arbitrary positive ε and for all large values of r,

$$\log T(r, fog) \geq ({}^{*}\lambda_{fog}^{L} - \varepsilon) \log^{[2]}[rL(r)] \tag{2.3.1}$$

$$\text{and } \log T(r^{A}, f) \leq A({}^{*}\rho_{f}^{L} + \varepsilon) \log^{[2]}[rL(r)]. \tag{2.3.2}$$

Now from (2.3.1) and (2.3.2) it follows for all large values of r,

$$\frac{\log T(r, fog)}{\log T(r^{A}, f)} \geq \frac{{}^{*}\lambda_{fog}^{L} - \varepsilon}{A({}^{*}\rho_{f}^{L} + \varepsilon)}.$$

As $\varepsilon (> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^{A}, f)} \geq \frac{{}^{*}\lambda_{fog}^{L}}{A^{*}\rho_{f}^{L}}. \tag{2.3.3}$$

Again for a sequence of values of r tending to infinity,

$$\log T(r, fog) \leq ({}^{*}\lambda_{fog}^{L} + \varepsilon) \log^{[2]}[rL(r)] \tag{2.3.4}$$

and for all large values of r,

$$\log T(r^A, f) \geq A({}^*\lambda_f^L - \varepsilon) \log^{[2]}[rL(r)]. \tag{2.3.5}$$

Combining (2.3.4) and (2.3.5) we get for a sequence of values of r tending to infinity,

$$\frac{\log T(r, fog)}{\log T(r^A, f)} \leq \frac{{}^*\lambda_{fog}^L + \varepsilon}{A({}^*\lambda_f^L - \varepsilon)}.$$

Since $\varepsilon(>0)$ is arbitrary it follows that

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)} \leq \frac{{}^*\lambda_{fog}^L}{A^*\lambda_f^L}. \tag{2.3.6}$$

Also, for a sequence of values of r tending to infinity,

$$\log T(r^A, f) \leq A({}^*\lambda_f^L + \varepsilon) \log^{[2]}[rL(r)]. \tag{2.3.7}$$

Now from (2.3.1) and (2.3.7) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log T(r, fog)}{\log T(r^A, f)} \geq \frac{{}^*\lambda_{fog}^L - \varepsilon}{A({}^*\lambda_f^L + \varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary, we get that

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)} \geq \frac{{}^*\lambda_{fog}^L}{A^*\lambda_f^L}. \tag{2.3.8}$$

Also for all large values of r,

$$\log T(r, fog) \leq ({}^*\rho_{fog}^L + \varepsilon) \log^{[2]}[rL(r)]. \tag{2.3.9}$$

From (2.3.5) and (2.3.9) it follows for all large values of r,

$$\frac{\log T(r, fog)}{\log T(r^A, f)} \leq \frac{{}^*\rho_{fog}^L + \varepsilon}{A({}^*\lambda_f^L - \varepsilon)}.$$

Since $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)} \leq \frac{{}^*\rho_{fog}^L}{A^*\lambda_f^L}. \tag{2.3.10}$$

Thus the theorem follows from (2.3.3), (2.3.6), (2.3.8) and (2.3.10). ■

Theorem 2.3.3 *Let f be entire and g be transcendental entire satisfying the following conditions*

$$(i)\rho^L_{fog} = 0 \text{ and } \lambda^L_g < \infty$$
$$(ii) 0 <^* \lambda^L_{fog} \leq^* \rho^L_{fog} < \infty$$
$$\text{and } (iii) 0 <^* \lambda^L_f \leq^* \rho^L_f < \infty.$$

Then

$$\frac{{}^*\lambda^{L*}_f \lambda^L_g}{A^* \rho^L_f} \leq \frac{{}^*\lambda^L_{fog}}{A^* \rho^L_f} \leq \liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, f)} \leq \frac{{}^*\lambda^L_{fog}}{A^* \lambda^L_f}$$
$$\leq \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, f)} \leq \frac{\rho^L_{fog}}{A^* \lambda^L_f} \leq \frac{{}^*\rho^{L*}_f \rho^L_g}{A^* \lambda^L_f}.$$

Proof. In view of Lemma 2.2.11 and the second part of Lemma 2.2.9, Theorem 2.3.3 follows from Theorem 2.3.2. ■

Theorem 2.3.4 *Let f be meromorphic and g be entire such that $\rho^L_{fog} = 0$. Also let $0 < {}^*\lambda^L_{fog} \leq {}^*\rho^L_{fog} < \infty$ and $0 < {}^*\rho^L_f < \infty$. Then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)} \leq \frac{{}^*\rho^L_{fog}}{A^* \rho^L_f} \leq \limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)}.$$

Proof. In view of Lemma 2.2.6, $\rho^L_{fog} = 0$ implies that $\rho^L_f = 0$. From the definition of L-order we get for a sequence of values of r tending to infinity,

$$\log T(r^A, f) \geq A({}^*\rho^L_f - \varepsilon) \log^{[2]}[rL(r)]. \tag{2.3.11}$$

Now from (2.3.9) and (2.3.11) it follows for a sequence of values of r tending to infinity,

$$\frac{\log T(r, fog)}{\log T(r^A, f)} \leq \frac{{}^*\rho^L_{fog} + \varepsilon}{A({}^*\rho^L_f - \varepsilon)}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain,

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)} \leq \frac{{}^*\rho^L_{fog}}{A^* \rho^L_f}. \tag{2.3.12}$$

Again for a sequence of values of r tending to infinity,

$$\log T(r, fog) \geq ({}^*\rho^L_{fog} - \varepsilon)\log^{[2]}[rL(r)]. \tag{2.3.13}$$

So combining (2.3.2) and (2.3.13) we get for a sequence of values of r tending to infinity,

$$\frac{\log T(r, fog)}{\log T(r^A, f)} \geq \frac{{}^*\rho^L_{fog} - \varepsilon}{A({}^*\rho^L_f + \varepsilon)}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows that

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)} \geq \frac{{}^*\rho^L_{fog}}{A^*\rho^L_f}. \tag{2.3.14}$$

Thus the theorem follows from (2.3.12) and (2.3.14). ■

Theorem 2.3.5 *If f be an entire function and g be a transcendental entire function satisfying*

$$\begin{gathered}(i)\rho^L_{fog} = 0 \text{ and } \lambda^L_g < \infty,\\ (ii)0 <^* \lambda^L_{fog} \leq^* \rho^L_{fog} < \infty\\ \text{and } (iii)0 <^* \rho^L_f < \infty, \text{ then}\end{gathered}$$

$$\begin{gathered}\limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r, f)} \geq^* \lambda^L_g\\ \text{and } \liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r, f)} \leq^* \rho^L_g.\end{gathered}$$

Proof. In view of Lemma 2.2.9 we obtain from Theorem 2.3.4 for $A = 1$,

$$\begin{gathered}\limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r, f)} \geq \frac{{}^*\rho^{L*}_f\lambda^L_g}{{}^*\rho^L_f} =^* \lambda^L_g\\ \text{and } \liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r, f)} \leq \frac{{}^*\rho^{L*}_f\rho^L_g}{{}^*\rho^L_f} =^* \rho^L_g.\end{gathered}$$

Thus the theorem follows from the above two inequalities. The following theorem is a natural consequence of Theorem 2.3.2 and Theorem 2.3.4. ■

Theorem 2.3.6 *Let* f *be meromorphic and* g *be entire such that* $\rho^{L}_{fog} = 0$.*Also let*

$$0 < {}^{*}\lambda^{L}_{fog} \leq {}^{*}\rho^{L}_{fog} < \infty$$
$$\text{and } 0 <^{*} \lambda^{L}_{f} \leq^{*} \rho^{L}_{f} < \infty.$$

Then for any positive number A,

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^{A}, f)} &\leq \min\{\frac{{}^{*}\lambda^{L}_{fog}}{A^{*}\lambda^{L}_{f}}, \frac{{}^{*}\rho^{L}_{fog}}{A^{*}\rho^{L}_{f}}\} \\ &\leq \max\{\frac{{}^{*}\lambda^{L}_{fog}}{A^{*}\lambda^{L}_{f}}, \frac{{}^{*}\rho^{L}_{fog}}{A^{*}\rho^{L}_{f}}\} \\ &\leq \limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^{A}, f)}.\end{aligned}$$

The proof is omitted.

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