SOME BOUNDARY VALUE PROBLEMS IN LINEAR ELASTODYNAMICS

THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY (SCIENCE)
OF THE
UNIVERSITY OF NORTH BENGAL

by

Apurba Narayan Das

DECEMBER, 1993

ACKNOWLEDGMENT

I would like to express my respect and gratitude to my supervisor Prof. M. L. Ghosh, Department of Mathematics, North Bengal University for his valuable guidance and inspiration throughout the whole process of research work.

I wish to thank Dr. P.K.Chaudhui, Reader in Mathematics, North Bengal University, for his helpful discussions on numerical calculations of several research papers.

For giving me all the facilities required to complete my research work, I wish to express my sincere gratitude to the Head and other teachers and staffs of the Department of Mathematics, North Bengal University.

Further, I am thankful to all my colleagues in the department as well as in the scholars hostel. North Bengal University.

Finally, I am grateful to U.G.C., India for awarding me junior research fellowship during the tenure of this work.

Apurba Narayan Das

Department of Mathematics
North Bengal University
Darjeeling - 734 430
India.
December, 1993.

CONTENTS

CHAPTER I

SOME ELASTODYNAMIC PROBLEMS ON CRACK PROPAGATION

CHAPTER II

EXTENDING CRACK PROBLEMS IN ELASTODYNAMICS

CHAPTER III

PROBLEMS ON PROPAGATION OF ELASTIC WAVES IN THE PRESENCE OF TOPOGRAPHICAL IRREGULARITY

INTRODUCTION

The phenomenon of stress wave propagation in elastic solids offers us a rich variety of waves which was developed a century ago. Some pioneer theoretical workers in this line are Rayleigh, Love, Stokes, Kelvin and others whose contribution in the field of wave propagation in elastic solids and vibrating bodies extended the theory of elasticity. In the first three decades of this century the subject lost its interest to the research workers due to lack of sophisticated instrument, electronic devices like high-speed computer and practical methods for observing the passage of stress waves in elastic solids. But in the later stage,several theoretical and experimental worker's keen interest in this field made a large number of technical papers giving various information.

During the past two decades, seismology has made a tremendous progress, mainly because of the advent of modern computers and improvements in data acquisition systems, which are now capable of digital and analog recording of ground motion over a frequency range of five orders of magnitude. These technological developments have enabled seismologists to make measurements with far greater precision and sophistication than was previously possible. As a result, far reaching advances in our knowledge of the earth's structure and the nature of the earthquake have occurred.

We here point out the milestones of progress in elastic waves in chronological order.

1678: Robert Hooke (England) established the stress-strain relation for elastic bodies.

1760: John Michell (England) recognized that earthquakes originate within the earth and send out elastic waves through earth's interior.

1821: Louis Navier (France) derived the differential equations of the theory of elasticity.

1828: Simeo-Denis Poisson (France) predicted theoretically the existence of longitudinal and transverse elastic waves.

1849: George Gabriel Stokes (England) conceived the first mathematical model of an earthquake source.

1857: First systematic attempt to apply physical principles to earthquake effects by Robert Mallet (Ireland).

1883: Rosi-Forel scale for earthquake effects published.

1885: C.Somigliana (Italy) produced formal solutions to Navier equations for a wide class of sources and boundary conditions.

1885: Lord Rayleigh (England) predicted the existence of elastic surface waves.

1899: C.G.Knott (England) derived the general equations for the reflection and refraction of plane seismic waves at plane boundaries.

1903: A.E.H.Love (England) developed the fundamental theory of point sources in an infinite elastic space.

1904: Horace Lamb (England) solved the problem on the propagation of tremors over the surface of an elastic solid.

1907: Vito Voltera (Italy) published the theory of dislocations based on Somigliana's solution.

1909: K.Zoeppritz and L.Geiger (Germany) calculated velocities of longitudinal waves in earth's mantle.

1940: Sir Harold Jeffreys (England) and K.E.Bullen (Australia) published travel time tables for seismic waves in earth.

1949: Lapwood, E.R. first considered the distribution due to a line source in a semi- infinite elastic medium.

1959: Ari Ben-Menahem (Israel) discovered that the energy release in earthquakes takes place through a propagating rupture over the causative fault.

1967: Global seismicity patterns and earthquake generation linked to plate motions.

In recent years the problem which mostly attract the researchers both theoretical and experimental, in relation to the generation and propagation of waves in elastic medium are:

(i) diffraction of propagating waves through the medium due to an obstacle, cavity or a crack of any shape situated some where in the medium;

(ii) wave motion generated due to punch on some bounded region of the medium;

(iii) reflection and refraction of a wave at a plane surface of discontinuity;

(iv) wave motion generated in a medium when a source of disturbance is static or moving along the medium.

(v) transmission and reflection of elastic waves by topographical irregularities.

(vi) elastodynamic problems involving crack propagation, crack kinking and bifurcation.

The solution of these problems need advance level of mathematical techniques, which may roughly be grouped into the following categories:

(a) Theory of analytic function
(b) The Fredholm integral equation
(c) The singular integral equation
(d) Integral transforms and Representations
(e) Dual integral and series equations
(f) Harmonic function. Potential theory
(g) The Dirichlet and Neumann problems
(h) Green's functions
(i) The Cauchy problem
(j) Wiener- Hopf technique
(k) Riemann- Hilbert problem
(l) The method of Matched Asymptotic expansion
(m) Perturbation technique
(n) Variational method, The Ritz method
(o) The method of finite element
(p) The method of boundary element

and others.

The problem of propagation of elastic waves in the presence of topographical irregularities and also in the presence of variation of material properties in the horizontal direction have drawn the attention of many investigators of the present time, due to their applications in seismology.

The problem of transient wave propagation in a half-space composed of two elastic quarter spaces of different materials was considered by Achenbach (1969). Dutta and Mitra (1974) considered SH-wave motion in an elastic quarter space in welded contact with a uniform elastic layer of different material. The problems of SH-wave transmission across an irregularity were considered by Bose (1975),Chakroborty et. al. (1983). Wolf(1967, 1970), Sinha (1980) considered the transmission of Love waves across an topographical irregularity while scattering of Rayleigh waves by a plane barrier in a shallow ocean was considered by Mann and Deshwal (1986).

A series of problems involving the scattering of elastic waves by two dimensional and three dimensional topographical irregularities have been solved by Sanchez- Sesma (1979, 1982, 1983, 1985) by using a newly developed boundary method. The diffracted fields are constructed with linear combinations of solutions which form c- complete families for the wave equation and boundary conditions are then satisfied in a least square sense. Adopting the representation theorem due to Knopoff (1956) , Knopoff and Hudson (1964) studied the transmission of Love waves past a continental margin considering the crust to have an abrupt increase in the thickness on the continental side. Sato (1961) discussed the problem of propagation of Love waves in an elastic layer of variable thickness overlying a semi- infinite elastic medium. Approximate expressions for the transmission and reflection factors are obtained by the application of a method based on Wiener- Hopf technique.. Abubakar (1963) also studied the effect of an irregular surface with an isolated irregularity like a tough or ditch on the incident P- and SV- waves using perturbation technique.

Apart from its academic interest, the propagation of elastic waves in layered media has important applications in geophysics and seismology. Since the propagation of characteristic of earth vary with depth, the first approximation to the actual problem can be achieved by regarding earth as formed of several layers in each of which properties are constant. The problem is very cumbersome as far as the mathematics is concerned. We mention the books by Brekhovskikh (1960),Eringen and Suhubi (Vol II,1975).

Recently, the problem of propagation of waves in layered elastic medium has been solved by Zaman et.al.(1987).

The problem of fracture is the central problem of the science of resistance of materials . Fracture mechanics in the broad sense of this concept includes the part of the science of strength of its materials and structures which relates to a study of the carrying capacity of the body both with or without consideration of initial cracks and also to a study of various laws governing crack development. In general the first stage of the investigation on fracture mechanics associated with the names of Robert Hooke, C.A.de Coulomb, B. de Saint Venant, Otto Mohr is characterized by extensive studies of deformation properties of solids and by the development of various failure criteria termed strength theories.

The dynamic process of fracture is made up of two stages, crack initiation and propagation, each of these stages following its particular laws. The criterion for the initiation of crack propagation, which forms the basis of fracture mechanics, does not follow from the equations of equilibrium and motion of continuum mechanics. This is an additional boundary condition in the solution of the problem of limiting equilibrium of a cracked body. The limiting state is said to be reached if a crack-like cut can propagate. The cut then becomes a crack.

Criterion for the initiation of crack propagation can be obtained on the basis of both energy and force considerations. Historically, at first an energy fracture criterion was proposed by A.A.Griffith (1920) and G.R.Irwin (1957) formulated a force criterion.

Yoffe (1951) first investigated the propagation of a finite crack with a constant speed through a stretched isotropic elastic solid. She showed that for small crack tip velocities the maximum tensile stress acts on the line $\theta=0$. Therefore, one may reasonably anticipate that the crack extends in a straight line; but at higher crack tip velocities, starting with $0.6c_2$ the line on which the maximum tensile stress is acting begins to make an angle with the initial crack axis. The angle increases very rapidly with the crack tip velocity i.e. the crack tends to become curved at propagation velocities higher than $0.6c_2$ as shown in

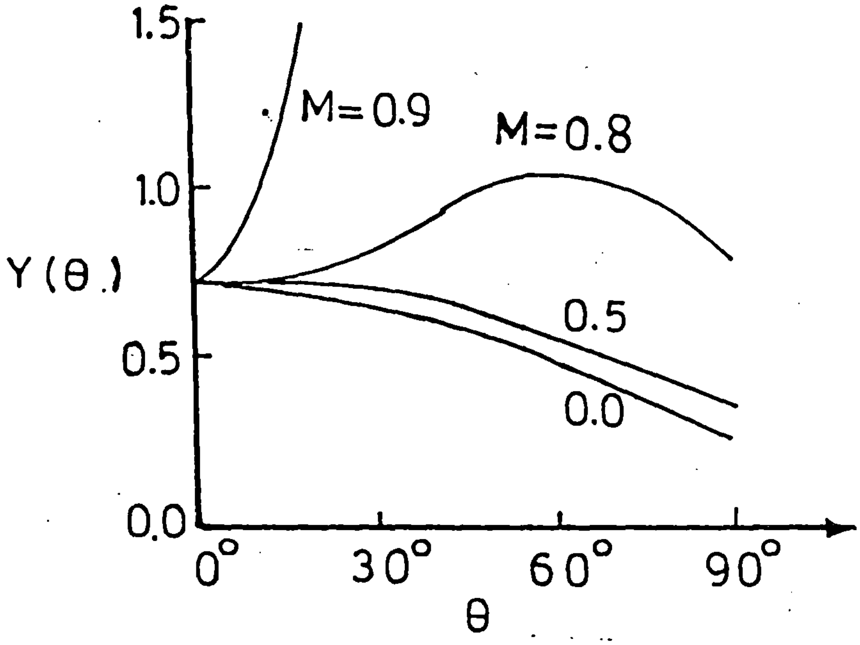


Fig. 1: Variation of Yoffe function with θ.

Fig.1. Complex variable technique to this case was latter applied by Radok (1956).

Indeed, it is now known that this variation is common to all crack tip stress distributions, as cam be verified form Irwin (1957), Williams (1957), Rice (1968). The noted variation of circumferential distribution with crack speed was proposed as an explanation for crack branching or bifurcation. While it is likely that the distribution of stress around a crack tip during acceleration may provide a mechanism by which the crack searches out alternate fracture paths, it does not explain why or how alternate paths are in fact executed.

In later stage, Yoffe's concept was utilized extensively by Singh et. al. (1981), Kassir and Tse (1983), De and Patra (1990). Three dimensional problem on moving crack was also solved by Itou (1979).

Recently, Das (1992,1993) has extended Yoffe's moving crack problems to the cases of three moving cracks.

The anti-plane problem of stress distribution around a semi-infinite crack moving with constant velocity in a strip of elastic material was solved by Sih and Chen (1972). The problem was reduced to Reimann - Hilbert problem by application of Scwartz- Christoffel transformation and the theory of complex functions.Closed form solutions were obtained for the two cases of practical importance: (i) the boundaries of the strip are clamped and displacecd in equal and opposite directions causing a tearing motion along the leading edge of the crack and (ii) the crack sheared longitudinally by a pair of concentrated forces moving with the crack while the strip boundaries are free of tractions. In both the cases, the effect of the strip width on the dynamic stress was examined.

Nilsson (1972) also solved the problem of a strip of material containing a moving semi-infinite crack using Fourier integral transform and Weiner- Hopf technique.

Here we give some techniques which are generally used in moving crack problems in elastodynamics.

1. Integral transform technique:

As the equations of motion in the theory of elasticity are partial differential equations which may be discussed with reference either to Helmholtz equation or to Laplace's equation, the method of integral transform is one of the most effective methods for solving such equations as application of this method to such equations results in the lowering of the dimension of an equation by one. There are several forms of integral transform and the choice of an integral transform depends on the structure of the equation and the geometry of the domain.

The integral transform $\overline{f}(\rho)$ of a function $f(x)$ defined on an interval (a,∞) is an expression of the form

$$\overline{f}(\rho) = \int_a^\infty f(x)\; K(x,\rho)\; dx \qquad (1)$$

where a is a real number and ρ is a complex parameter varying over some region D of the complex plane. $K(x,\rho)$ is called the kernel of the transformation. The transformation (1) becomes particularly useful if it possesses inverse mapping. In that case one can express $f(x)$ in terms of its integral transform by

$$f(x) = \frac{1}{2\pi i} \int_\Gamma \overline{f}(\rho)\; M(x,\rho)\; d\rho \qquad (2)$$

where $M(x,\rho)$ is a suitable function defined in $a<x<\infty$ and $\rho \in D$ and is called the kernel of the inverse transform, which is defined for all x in the interval (a,∞). The complex parameter ρ is in the region D while Γ is a suitable path of integration in D. After reducing the governing partial differential equation, the reduced problem can be solved for $\bar{f}(\rho)$. The solution of the original equation can be expressed in terms of the inverse integral, which may then be evaluated. The inversion from the the transformed space to the space of actual variables usually involved very complicated integrations. In many cases even the numerical integration can not be performed successfully because of the highly oscillatory character of the integrands [cf Eringen and Suhubi(1975), chap. 7; Achenbach (1975), chap. 7]. In particular, mixed boundary value problems like the dynamic response of a punch on an elastic half-space and the problem involving the presence of a crack or a strip inside an elastic medium may be reduced to Fredholm integral equation of first kind or to dual integral equations.

2. The factorization problem. The Wiener-Hopf technique:

Let a function $\phi(z)$ analytic in the interval $y_- < \mathrm{Im}\ z < y_+$ be defined in the plane of a complex variable z. It is required to express $\phi(z)$ in the form

$$\phi(z) = \phi_+(z)\phi_-(z) \tag{3}$$

where $\phi_+(z)$ and $\phi_-(z)$ are functions analytic in the half -plane $\mathrm{Im}\ z > y_-$ and the half-plane $\mathrm{Im}\ z < y_+$ respectively. The problem is called factorization problem. In a more general case, it is required to define two functions $\phi_+(z)$ and $\phi_-(z)$ which are analytic in the same half- planes respectively and which satisfy the following relation in the interval

$$A(z)\phi_+(z) + B(z)\phi_-(z) + C(z) = 0 \tag{4}$$

where A(z),B(z) and C(z) are given analytic functions in the interval. It is obvious that if C(z) = 0, we obtain the representation (3) after the corresponding changes in the notation.

Let us assume that the function $\phi(z)$ which is to be factorised does not have any zeros in the interval $y_- < \mathrm{Im}\ z < y_+$ and tends to infinity as $x \longrightarrow \infty$. In this case, neither of the functions $\phi_+(z)$ and $\phi_-(z)$ will have any zero, and we can take the logarithm of both sides of the relation (3)

$$\log \phi(z) = \log \phi_+(z) + \log \phi_-(z) \qquad (5)$$

The function $F(z) = \log \phi(z)$ satisfies the condition

$$|F(x+iy)| < c|x|^{-p}, \quad (p>0 \text{ for } x\longrightarrow \infty) \qquad (6)$$

and hence the relation (5) can always be solved with the help of the transformation

$$F(z) = F_+(z) + F_-(z) \qquad (7)$$

Finally, we get

$$\phi(z) = e^{F_+(z)}\, e^{F_-(z)} \qquad (8)$$

If the function $\phi(z)$ has zeros in the intervals we must consider a new function

$$\phi_1(z) = \frac{(z^2+b^2)^{N/2}\phi(z)}{\prod_{i=1}^{N_1} (z-z_i)^{\alpha_i}} \qquad (9)$$

where z_i and α_i are the zeros, their multiplicity in the interval $N_1 \leq N$, where N is the total number of zeros, $b>(y_+,y_-)$. The factor in the numerator of (9) ensures that the properties of auxiliary functions are conserved at infinity.

Let us now consider the relation (4) and carry out its factorisation into L_+ and $1/L_-$ for the same interval of the ratio A/B. The relation (4) can be represented in the form

$$L_+(z)\phi_+(z) + L_-(z)\phi_-(z) + L_-(z)C(z)/B(z) = 0 \qquad (10)$$

The expression $L_-(z)C(z)/B(z)$ can be represented in the following form in accordance with (7)

$$E_+(z) + E_-(z)$$

where $\phi_+(z)$ and $\phi_-(z)$ are functions analytic in the half-plane $y > y_-$ and the half-plane $y < y_+$ respectively. Taking this into account, we get

$$L_+(z)\phi_+(z) + E_+(z) = - L_-(z)\phi_-(z) - E_-(z) \qquad (11)$$

It follows from the generalized Liouville's theorem that the left as well as right hand side of (11) represents the same polynomial $P_n(z)$ of nth degree.

3. Hilbert transform technique:

If $p(y) \in L_2(a,b)$, then the equation

$$\int_a^b \frac{h(x)}{x - y}\, dx = \pi p(y), \quad y \in (a,b) \qquad (12)$$

has the solution

$$h(x) = \frac{1}{\pi}\left[\frac{x - a}{b - x}\right]^{1/2} \int_a^b \left[\frac{b - y}{y - a}\right]^{1/2} \frac{p(y)}{x - y}\, dy + \frac{C}{\sqrt{(x-a)(b-x)}} \qquad (13)$$

where C is an arbitrary constant, and the first term belongs to the class $L_2(a,b)$.

Using the above theorem, we find that the solution to the integral equation

$$\int_a^b \frac{2xh(x^2)}{x^2 - y^2}\, dx = \pi p(y), \quad y \in (a,b) \qquad (14)$$

(provided that p satisfies the conditions of the above theorem) is given by

$$h(x^2) = \frac{1}{\pi}\left[\frac{x^2 - a^2}{b^2 - x^2}\right]^{1/2} \int_a^b \left[\frac{b^2 - y^2}{y^2 - a^2}\right]^{1/2} \frac{2yp(y)}{x^2 - y^2}\, dy + \frac{C}{\sqrt{(x^2-a^2)(b^2-x^2)}}$$

where C is an arbitrary constant.

4. Schmidt method:

To solve for unknown constants $c_n(\zeta)$ occurring in

$$\sum_{n=1}^{\infty} c_n(\zeta)\, F_n(\zeta,x) = f(\zeta,x) \quad \text{for } x \in (a,b) \tag{15}$$

where $F_n(\zeta,x)$ and $f(\zeta,x)$ are known functions, we adopt Schmidt method.

Let $H_n(\zeta,x)$ be a set of orthogonal functions which satisfy

$$\int_a^b H_n(\zeta,x)H_m(\zeta,x)\, dx = N_n\delta_{nm} \tag{16}$$

where $$N_n = \int_a^b H_n^2(\zeta,x)\, dx \tag{17}$$

Then $H_n(\zeta,x)$'s can be computed from the functions $F_n(\zeta,x)$ in the following way

$$H_n(\zeta,x) = \sum_{i=1}^{\infty} \frac{c_{in}}{c_{nn}} F_i(\zeta,x) \tag{18}$$

with c_{in} as the cofactor of the e_{in} in D_n which is defined as

$$D_n = \begin{vmatrix} e_{11} & e_{12} & & e_{1n} \\ e_{21} & e_{22} & \cdots\cdots\cdots & e_{2n} \\ \cdots & \cdots\cdots & \cdots\cdots\cdots & \cdots \\ e_{n1} & e_{n2} & \cdots\cdots\cdots & e_{nn} \end{vmatrix} , \quad e_{nm} = \int_a^b F_n(\zeta,x)F_m(\zeta,x)\, dx \tag{19}$$

Now in terms of the set of orthogonal functions $H_n(\zeta,x)$, the function $f(\zeta,x)$ can be expressed as

$$f(\zeta,x) = \sum_{i=1}^{\infty} h_i\, H_i(\zeta,x) \tag{20}$$

Substituting the values of $H_n(\zeta,x)$ from (18) in (20), we obtain after some rearrangement

$$\sum_{n=1}^{\infty} c_n(\zeta)\, F_n(\zeta,x) = \sum_{n=1}^{\infty} F_n(\zeta,x) \sum_{i=n}^{\infty} \frac{c_{ni}}{c_{ii}}\, h_i \tag{21}$$

Comparing the coefficients of $F_n(\zeta,x)$ from both sides of (21) we find

$$c_n = \sum_{i=n}^{\infty} \frac{c_{ni}}{c_{ii}} h_i \tag{22}$$

where
$$h_i = \frac{1}{N_i} \int_a^b f(\zeta,x) H_i(\zeta,x)\, dx \tag{23}$$

This is in brief Schmidt method for determining the unknown coefficients c_n.

Recently extensive study on extension of crack in elastic solid has been made. Several investigations on symmetric and non-symmetric extension of crack in its own plane in an infinite elastic medium have been carried out up-till-now. Broberg (1960) first considered the problem of symmetric extension of a crack in elastic solid.

He considered the extension of a crack in a brittlelinear elastic material using Fourier transform. He assumed that the extension of crack occurs in its own plane. The plane surface is subjected

(i) to a constant pressure, acting on an infinite strip, the width of which is symmetrically increasing from zero with a constant velocity, and

(ii) to a pressure outside the strip such that the normal displacement of the surface outside the strip is zero.

The mixed boundary value problem has been treated. The stresses in the solid and the normal displacement of the surface have been solved. The result shows that the displacement of the surface is elliptic, just as the corresponding static case.

Since Broberg's investigation of the solution of a crack expanding symmetrically with constant velocity under conditions of plane stress or strain in a homogeneous elastic field of spatially and time invariant tensile stress, a number of papers have appeared analyzing different geometrical situations. Craggs (1963) later solved the same problem as that done by Broberg but he used the method of homogeneous function to obtain the solution, while Achenbach and Brock (1971) considered the corresponding anti-plane problem. Self-similarity technique, which is the most useful technique for treating extending crack problems, are used by Atkinson (1974), Brock and Achenbach (1974), and others.

Using complex variable technique Cherepanov and Afanasev(1974), Cherepanov(1979) have solved some self-similar problems of dynamic theory of elasticity. They also used the functionally invariant method of Smirnoff and Sobolev (1932). Later, this technique is used by Das (1993a) to solve the one way extension of a crack in an infinite elastic solid due to two non-parallel plane SH-waves.

Indeed, non-symmetric extension at different velocities of the crack tips is common to the fracture of geophysical settings with pre-existing rupture planes. Problems on non-symmetric extension of a small flaw into a plane crack have been studied by Brock (1975,1976), Georgiadis (1991) and Das (1993b,1993c) using self- similarity technique.

Recently, problems on extension of cracks in cruciform paths have been solved by Brock and Deng(1985), Ong and Srivastava (1985) and Georgiadis (1987).

Here we add a few lines about self-similarity technique

5. Self-similarity technique:

A self-similar solution of a physical problem can be inferred if either the data of the problem involve no characteristic length or the only characteristic length is related to a parameter to which the solution is proportional. The principal advantage of this class of solution is that the governing partial differential equations can be replaced by another set that contains one independent variable less than those in the original set.

We begin with the homogeneous solution of scalar two dimensional wave equation

$$\nabla^2\phi = c^{-2} \frac{\partial^2\phi}{\partial t^2} \tag{24}$$

which may be expressed as

$$\phi(x/t,y/t) = \phi\ (r/t,\theta)$$

where r,θ are polar coordinates in the plane. If we define a new independent variable

$$s = r/t$$

then polar form of (24)

$$\frac{\partial^2\phi}{\partial r^2} + r^{-1}\frac{\partial\phi}{\partial r} + r^{-2}\frac{\partial^2\phi}{\partial\theta^2} = c^{-2}\frac{\partial^2\phi}{\partial t^2}$$

is transformed into

$$s^2(1-s^2/c^2)\frac{\partial^2\phi}{\partial s^2} + s(1-2s^2/c^2)\frac{\partial\phi}{\partial s} + \frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{25}$$

We point out that (25) is of mixed type, i.e., it is elliptic in $s/c<1$ and hyperbolic in $s/c>1$. The domain of elliptcity and hyperbolicity of the differential equation thus correspond to the interior and exterior of the circle $r=ct$ centered at the origin of the coordinate system. Since the coefficient of $\frac{\partial^2\phi}{\partial s^2}$ vanishes at $s=c$, the circle $r=ct$ evidently represents a singular wave front across which we may expect discontinuities in the s-derivatives of the wave function. We suppose, however, that ϕ and $\partial\phi/\partial\theta$ are continuous across the wave front. equation (25) can be reduced to the canonical form for $s/c<1$ through Chaplygin's transformation

$$\beta = -\cosh^{-1}\frac{c}{s} = \log\left[\frac{c}{s} - \left(\frac{c^2}{s^2} - 1\right)^{1/2}\right] \tag{26}$$

which yields Laplace's equation in β-θ coordinates

$$\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial\beta^2} = 0. \tag{27}$$

Similarly if s/c>1, then the transformation

$$\sigma = \cos^{-1}\frac{c}{s} \tag{28}$$

reduced (25) to the equation

$$\frac{\partial^2\phi}{\partial\theta^2} - \frac{\partial^2\phi}{\partial\sigma^2} = 0. \tag{29}$$

Recently, attention are being focused to the cases of crack extension which occurs under an arbitrary angle with it's own plane (which leads that a crack may bifurcate) as shown in Fig.2 and Fig 3..Because, it is expected that once the extension of crack has started, the primary crack often bifurcates into two or more branches , each of which may propagate over a short distance, and then again split into two or more new branches.Crack bifurcation occurs in a variety of materials, and under different external conditions. The phenomenon is, however, particularly present for essentially brittle fracture, when the speed of crack propagation becomes relatively large. Experimental observations of the magnitude of the speed of crack extension at branching suggest that elastodynamic effects play a sufficient role.It has been observed that the method of self- similar solutions provide a powerful tool for the analysis of elastodynamic skew propagation and crack bifurcation.

A necessary condition for bifurcation can be determined by comparing stress prior to branching and after branching has taken place. The comparison requires expressions for the elastodynamic fields near the crack tips of the branches. For symmetric bifurcation in anti-plane strain the near tip fields were analyzed by Achenbach (1975). The propagation of a crack which emanates under an arbitrary angle from a free surface, when the surface is subjected to anti-plane mechanical disturbances was considered by Achenbach and Varatharajulu (1974). Some cases of dynamic crack propagation in elastic medium are reviewed by

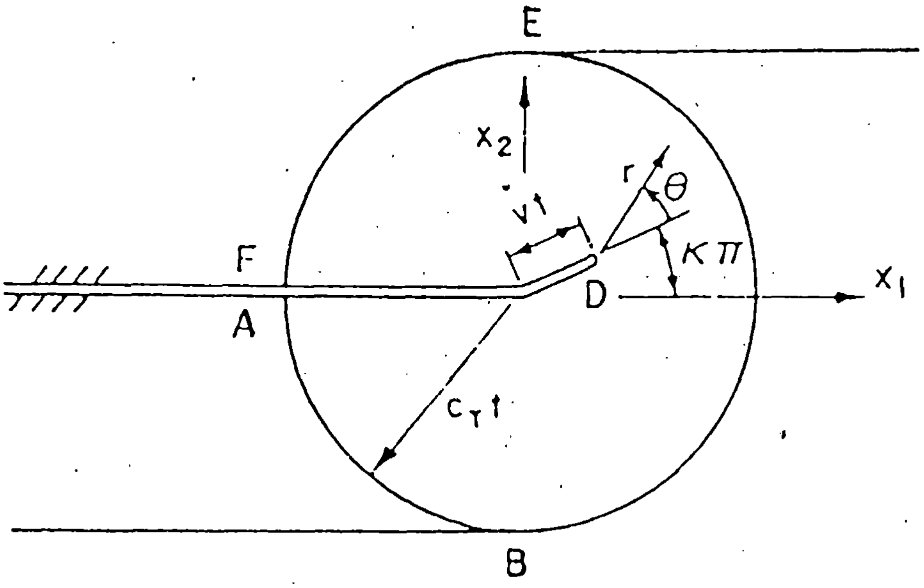


Fig. 2: Pattern of wave front and position of crack tip for skew crack propagation under the influence of a step-stress wave.

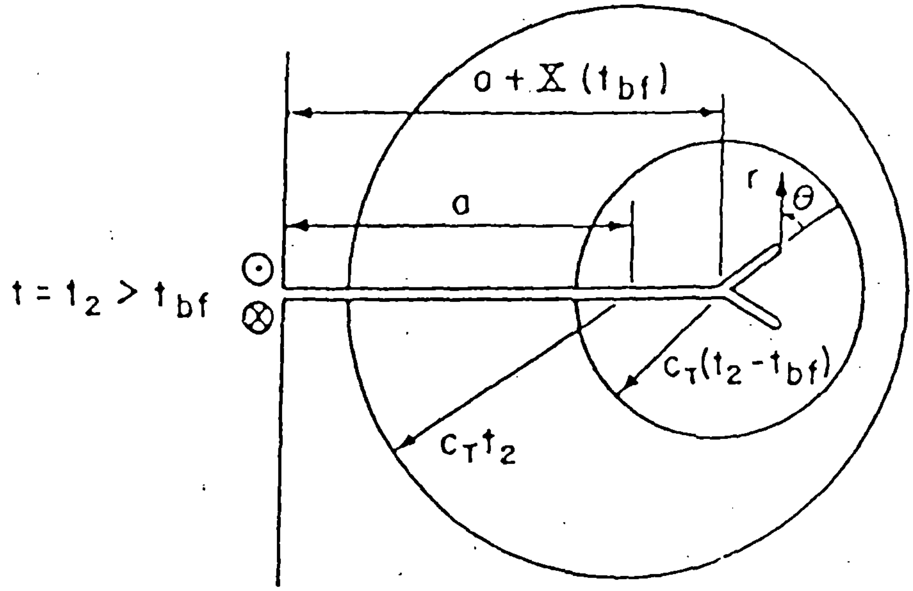


Fig. 3: Rapid propagation and bifurcation in anti-plane strain of an edge crack.

Achenbach (1972,1976) and Freund(1976).

During the last decade, problems on Mode III crack kinking and bifurcation have been studied by several investigators. Burgers and Dempsey (1982) solved the dynamic crack bifurcation in anti-plane strain for two special cases. Corrected results for Mode III kinking of a crack under an arbitrary angle was given by Dempsey et.al. (1982). A numerical approach for the study of dynamic crack propagation of a kinked or bifurcated crack in anti-plane strain has been given by Burgers (1982). Achenbach et. al. (1984) have developed a method based on superposition principle to derive approximate expression for the elastodynamic stress intensity factors of the kinked crack. The problem of rapid tearing of a half-plane was also solved by Dempsey and Smith (1985). They considered that the surface of the half-plane is subjected to sudden anti-plane mechanical disturbance, crack initiation and subsequent crack instability are examined via two idealized problems; the first is concerned with instantaneous crack bifurcation and the second with instantaneous skew crack propagation. In either problem, crack propagation occurs at a constant subsonic velocity, under an angle $k\pi$ with the normal to the surface . For various values of the angle of crack propagation, the dependence of the elastodynamic stress intensity factors on the crack propagation velocity is investigated.

Recently, transient elastodynamic non-planar self-similar Mode III crack growth in brittle materials is examined by Dempsey et. al. (1986). The dynamic similarity and Chaplygin's transformation reduced the class of problems considered to the solution of Laplace's equation in a semi-infinite strip. The Scwartz- Christoffel transformation is subsequently employed to map the semi-infinite strip on a half-plane. The theory of analytic functions are then used. Elastodynamic influences in the vicinity of a rapidly moving tip after branching are examined in a rather general fashion.

The thesis presented here consists of some problems on cracks and wave propagation. The work has been presented in three chapters.

The first chapter deals with problems on moving cracks in infinite elastic medium and in infinitely long elastic strip.

Problems on crack extension and bifurcation have been presented in the second chapter .

The third chapter deals with the problems on wave propagation in the presence of topographical irregularities.

The summary of the thesis is presented here chapter wise.

The first problem of chapter 1 has been formulated as follows:

We have considered the problem of propagation of two coplanar Griffith cracks moving steadily in infinite long finite width strip. We consider two cracks of finite width placed on X-axis from -b to -a and a to b with reference to the rectangular coordinate system (x,y,z) which referred to a fixed coordinate system (X,Y,Z), is moving with constant velocity v along X-direction within the strip of elastic material occupying the region $-h' \leq Y \leq h'$. Employing Fourier transform and finite Hilbert transform technique, closed form solutions are obtained for two cases of practical interest. Firstly, the case when the rigidly clamped edges are pulled apart in opposite directions are considered. Secondly, we have treated the case when the lateral boundaries are subjected to shearing stresses. Exact expressions for the crack opening displacement and the stress intensity factors have been derived in both the cases.

In paper 2, we have considered the problem of two coplanar Griffith cracks moving along the interface of two dissimilar elastic media. Two cases of practical importance have been considered. Firstly, the case of two coplanar Griffith cracks moving along the interface of two semi-infinite dissimilar elastic media has been treated ; secondly, the problem of propagation of two coplanar Griffith cracks along the interface of an elastic layer overlying a semi-infinite medium of different elastic properties has been considered.Employing Fourier transform we reduced these problems to solving a set of triple integral equations with cosine kernel weight functions. These equations are solved using finite Hilbert transform technique. In the second case, analytical expressions are retained up to h^{-4} , where h is the thickness of the upper layer, for deriving the dynamic stress intensity factors and crack opening displacement.

The problem of two coplanar Griffith cracks running steadily under three dimensional loading has been considered in the third paper of chapter 1. It is assumed that equal and opposite tractions which are triaxial in nature are applied to the crack surfaces. The two dimensional Fourier transforms have been used to reduce the mixed boundary value problem to the solution of triple integral equations. In order to solve the problem , the transformed surface displacement is expanded in a series of Chebyshev polynomials which is automatically zero out side the cracks and also satisfies the edge conditions. Finally, Schmidt method has been used to determine the unknown coefficients occurring in the series. The expression for stress intensity factors at the crack tips and the crack opening displacement have also been derived for different values of the parameters. An interesting feature of this paper is that there is the possibility of curving or branching of the cracks at the outer edge at very low velocities of the cracks whereas the cracks tend to become curved at the inner edge for values of crack tip velocity about $0.6c_2$.

The dynamic in-plane problem of determining the stress and displacement due to three coplanar cracks moving steadily at a subsonic speed in fixed direction in an infinite, isotropic, homogeneous medium under normal stress and the static problem of determining the stress and displacement around three coplanar Griffith cracks in an infinite isotropic elastic medium have been considered in the fourth paper of the chapter 1. In both the cases, employing Fourier integral transform, the problems have been reduced to solving a set of four integral equations. The integral equations have been solved using finite Hilbert transform technique and Cook's result to obtain the exact form of crack opening displacement and stress intensity factors which are presented in the form of graphs.

In the fifth paper of the chapter 1 we have treated the dynamic anti-plane problem of determining stress and displacement due to three coplanar cracks moving steadily at a constant speed in an infinite elastic strip.Employing the same technique as that used in solving the problem considered in paper four, the problem when the lateral boundaries of the strip are subjected to shearing stress has been solved. Numerical results for stress intensity factors have been presented in the form of graphs.

The dynamic in-plane problem of determining the stress and displacement due four coplanar Griffith cracks moving steadily at a subsonic speed in fixed direction in an infinite, isotropic, homogeneous medium under normal stress has been treated in the sixth paper of this chapter.The static problem of determining the stress and displacement around four coplanar Griffith cracks in an infinite isotropic elastic medium have also been considered in this paper. In both the cases,employing Fourier integral transform, the problems have been reduced to solving a set of five integral equations. The integral equations have been solved using finite Hilbert transform technique to obtain the exact form of crack opening displacement and stress intensity factors which are presented in the form of graphs.

In chapter 2, the first problem deals with the non-symmetric extension of a plane crack due to plane SH- waves in a pre-stressed infinite elastic medium. We considered two identical plane waves defined by

$$\sigma_{yz} = A_o W_{\pm} H(W_{\pm}), \quad \sigma_{xz} = A_o \cot\theta_o W_{\pm} H(W_{\pm})$$

referring to coordinate system (x,y,z) where

$$W_{\pm} = c_2 t \pm y\sin\theta_o + x\cos\theta_o, \quad 0 \le \theta_o \le \pi/2$$

and H() is Heaviside's unit function, to propagate through the infinite solid which is pre-stressed such that

$$\sigma^{o}_{yz} = \sigma, \quad \sigma^{o}_{xz} = 0.$$

Fracture is assumed to initiate at a point a finite time after the waves intersect there and the crack is assumed to extend non-symmetrically along the trace of wave intersection. Following Cherepanov and Afanase'v (1974) and Cherepanov (1979) the general solution has been derived in terms of analytic function of complex variable. Numerical results have been presented to illustrate the nature of the variation of stress intensity factors and the rate of energy flux into the crack edges with the speed of the crack tips and also with the time after fracture initiation.

In the second paper of the chapter 2, we investigated the problem of non-symmetric extension of an infinitesimal flaw into a plane crack at a constant rate due to the action of two non-parallel plane SH- waves of different amplitude propagating towards each other in an infinite isotropic elastic medium which is initially in a state of uniform anti-plane shear.A finite time after the crossing of the plane wave fronts, a fracture is assumed to initiate along the line where the wave fronts crossed and the crack is then assumed to travel non-symmetrically along the trace of wave intersection. Superposition considerations allow the original problem to be separated into three self-similar problems with (0,0),(0,1)and (1,0) as the indices of self-similarity. The dynamic similarity of certain field variable in each problem suggests application of the method of homogeneous functions. Expressions for the stress intensity factors and the rate of energy flux into the extending crack edges of the crack have been derived. Finally, the nature of the variation of the stress intensity factors at the crack tips and also the rate of energy flux into the edges with velocities of the crack edges and also with the time after crack initiation have been depicted by means of graphs.

The third paper of this chapter deals with the dynamic anti-plane problem of bifurcation of a semi-infinite crack due to the incidence of two linearly varying plane SH- waves with non-parallel wave fronts in an infinite elastic medium. The semi-infinite crack is assumed to bifurcate when the plane waves intersect the crack tip. The problem has been solved using self-similarity technique which is based on the observation that certain field variables show dynamic similarities. The results include the expressions for shear stress in the planes of the cracks and the stress intensity factors at the crack tips.Finally, the variations of stress intensity factors with the angle of skew for different values of the parameters have been depicted by means of graphs.

In the first paper of chapter 3, we have studied the transmission of time step SH- wave across a step like irregularity in the surface of an elastic half- space. Considering the incident wave in the form H(T-X/c) where H() is the Heaviside's step function the problem is reduced to an integral equation by using integral transform and Green's function technique and finally using Cagniard-Dehoop method of finding inverse Laplace transform, transmitted field at any distances from the step on the free surface have been determined using iterative procedure. Numerical results have been presented in the form of graphs to illustrate the nature of transmission.

Finally, we have considered the propagation of SH- wave in a medium consisting of two welded quarter spaces of different material and having a step like change in elevation at the vertical interface. The problem is reduced to an integral equation by using the Fourier transform and Green's function technique and finally, by applying the method of steepest descent, the transmitted and reflected fields at large distances form the step have been determined. To investigate the nature of the motion, we have evaluated numerically the increment in amplitude due to the presence of the step for both the transmitted and and reflected wave s which are presented in the form of graphs.

With this brief discussion we now present the thesis chapter wise. An attempt has been made to include most of the references consistent with the problems treated in this thesis, which have come to the author's knowledge.

CHAPTER I

SOME ELASTODYNAMIC PROBLEMS ON CRACK PROPAGATION

Paper 1: Two coplanar Griffith cracks moving in a strip under anti-plane shear stress.

Paper 2: Two coplanar Griffith cracks moving along the interface of two dissimilar elastic media.

paper 3: Problem of two coplanar Griffith cracks running steadily under three dimensional loading.

Paper 4: Three coplanar moving Griffith cracks in an infinite elastic medium.

Paper 5: Three coplanar moving Griffith cracks in an infinite elastic strip.

Paper 6: Four coplanar Griffith cracks in an infinite elastic medium.

TWO COPLANAR GRIFFITH CRACKS MOVING IN A STRIP UNDER ANTI-PLANE SHEAR STRESS

1.Introduction

In fracture mechanics , the problem of diffraction of elastic waves by cracks of finite dimension in a strip of elastic material has been investigated by several authors. Sih and Chen (1972) investigated the problem of propagation of a crack of finite length in a strip under plane extension. The resulting mixed boundary value problem was reduced to the solution of a Fredholm integral equation of second kind , which was solved numerically . Closed- form solutions for a finite length crack moving in a strip under anti plane shear stress was also obtained by Singh et al. (1981). As regards the dynamic crack problem , research has been restricted mainly to the case of a single crack because of the severe mathematical complexity encountered in finding solutions of two or more cracks . However , using finite Hilbert transform techniques developed by Srivastava and Lowengrub (1968), Lowengrub and Srivastava (1968) solved the statical problem of distribution of stress in an infinitely long elastic strip containing two coplanar Griffith cracks .The scattering of time harmonic normally incident plane waves by two parallel and coplanar Griffith cracks in an infinite elastic medium has been studied by Jain and Kanwal (1972) and more recently by Itou (1980).

In this paper we have considered the problem of propagation of two coplanar Yoffe (1951) cracks moving steadily in an infinitely long finite width strip . Employing Fourier transform and finite Hilbert transform technique closed-form solutions are obtained for two cases of practical interest.Firstly, the case when the rigidly clamped edges are pulled apart in opposite directions are considered. Secondly , we have treated the case when the lateral boundaries are subjected to shearing stresses. Exact expressions for the crack opening displacement and the stress intensity factors have

---

Published in "Journal Of Technical Physics" Vol 32, No.3-4, pp 479-489, 1991.

been derived in both the cases .Finally numerical results for stress intensity factors are presented graphically to show its variation with crack speed for different values of the lengths of the cracks.

2.Formulation Of The Problem

We consider two cracks of finite length to be placed on the X-axis from -b to -a and from a to b with reference to the rectangular coordinate system (x,y,z) which referred to fixed coordinate system (X,Y,Z) is moving with constant velocity v along X -direction within the strip of elastic material occupying the region $-h' \le Y \le h'$ as shown in Fig.1 .

In dynamic problem of anti plane shear, the non-vanishing component of displacement W directed in the Z-direction satisfies the equation of motion

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} = \frac{1}{c_2^2}\frac{\partial^2 W}{\partial t^2} \tag{2.1}$$

where $c_2 = (\mu/\rho)^{1/2}$ is the shear wave velocity and ρ is the density of the material.The non-vanishing components of stress are

$$\left.\begin{aligned} \sigma_{XZ} &= \mu \frac{\partial W}{\partial X} \\ \sigma_{YZ} &= \mu \frac{\partial W}{\partial Y} \end{aligned}\right] \tag{2.2}$$

Using Galilean transformation $x' = X - vt$, $y' = Y$, $z' = Z$, $t' = t$, where (x',y',z') is the translating coordinate system shown in Fig.1 and next introducing the dimensionless coordinates x,y,z such that $x' = xb$, $y' = yb$, $z' = zb$, $h' = hb$ equation (2.1) reduces to

$$s^2 \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = 0 \tag{2.3}$$

with $$s^2 = 1 - v^2/c_2^2 \tag{2.4}$$

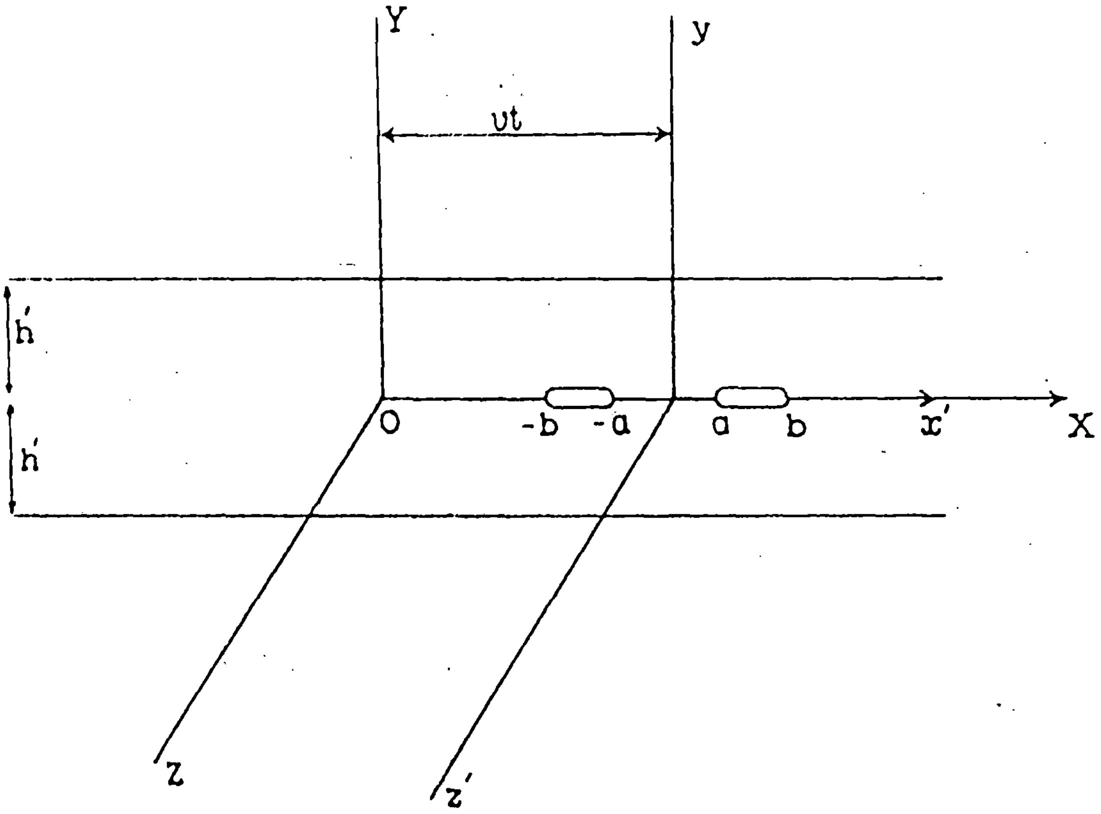


Fig 1. Moving cracks in a strip under antiplane shear.

3. Boundary Conditions

We consider two basic problems of practical interest with different boundary conditions

Problem I. The edges of the strip $y = \pm h$ are assumed to be rigidly clamped and displaced laterally in opposite directions by an equal amount w_o, where w_o is a constant. As a result, anti plane shear motion takes place in z-direction whereas cracks move in the x-direction and the boundary conditions are

$$W(x,\pm h) = \pm w_o , \qquad -\infty < x < \infty \qquad (3.1)$$

$$\sigma_{yz}(x,0) = 0 , \qquad d<|x|<1 \qquad (3.2)$$

$$W(x,0) = 0 , \qquad 0\leq|x|<d, |x|>1 \qquad (3.3)$$

where d=a/b .

In order to apply the integral transform technique it is necessary to solve a different but equivalent problem which can be obtained from the problem of a clamped strip (without any crack) subject to a uniform strain . The equivalent stress condition on the crack are

$$\sigma_{yz}(x,0) = -\frac{\mu w_o}{h} , \qquad d<|x|<1 \qquad (3.4)$$

and the displacement must satisfy

$$W(x,0) = 0 , \qquad 0\leq|x|<d, \quad |x|>1 \qquad (3.5)$$

$$W(x,\pm h) = 0 , \qquad -\infty < x < \infty \qquad (3.6)$$

Problem II. In this case uniform shearing stress p_o is applied to the upper and lower boundaries $y = \pm h$ of the strip. The equivalent problem in this case involves the application of the shear stress $-p_o$ to the crack faces at $y = 0$. Accordingly the boundary conditions are

$$\sigma_{yz}(x,\pm h) = 0 , \qquad -\infty < x < \infty \qquad (3.7)$$

$$\sigma_{yz}(x,0) = -p_o , \qquad d<|x|<1 \qquad (3.8)$$

$$W(x,0) = 0 , \qquad 0\leq|x|<d, |x|>1 \qquad (3.9)$$

4. Solutions Of The Problems

Due to symmetry about (x,z)- plane we need consider the region $0<y<h$ only. Employing

$$F_c[A(\xi);\xi\rightarrow x] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} A(\xi)\cos(\xi x)\,d\xi \tag{4.1}$$

and

$$F_s[A(\xi);\xi\rightarrow x] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} A(\xi)\sin(\xi x)\,d\xi \tag{4.2}$$

we obtain the solution of (2.3) as

$$W(x,y) = F_c[A_1(\xi)\exp(-\xi ys) + A_2(\xi)\exp(\xi ys);\xi\rightarrow x] \tag{4.3}$$

with

$$\sigma_{yz}(x,y)= \mu sF_c\Big[\xi\{-A_1(\xi)\exp(-\xi ys)+A_2(\xi)\exp(\xi ys)\};\xi\rightarrow x\Big] \tag{4.4}$$

Problem I. Using the expression for $W(x,y)$ given in (4.3) in (3.6) we get

$$A_1(\xi) = \frac{A(\xi)}{1 - \exp(-2\xi hs)}$$

$$A_2(\xi) = \frac{-A(\xi)\exp(-2\xi hs)}{1 - \exp(-2\xi hs)}$$

where $A(\xi)$ is to be determined.

From (3.4) and (3.5) we find that $A(\xi)$ satisfies the set of triple integral equations

$$F_c[\xi A(\xi)\,\mathrm{cth}(\xi hs)\,;\,\xi\rightarrow x] = \frac{w_0}{hs} \qquad d<x<1 \tag{4.5}$$

$$F_c[A(\xi);\xi\rightarrow x] = 0 \quad , \qquad 0\le x<d\ ,\ x>1 \tag{4.6}$$

Let us take

$$A(\xi) = \frac{1}{\xi}\sqrt{\frac{\pi}{2}}\int_d^1 g_1(\tau)\,\mathrm{Sech}^2(c\tau)\sin(\xi\tau)\,d\tau \tag{4.7}$$

It is clear that the above choice of $A(\xi)$ satisfies (4.6) if and only if

$$\int_{d}^{1} g_1(\tau)\ \mathrm{sech}^2(c\tau)\ d\tau = 0 \tag{4.8}$$

Equation (4.5) can be written as

$$\frac{d}{dx} F_s[A(\xi)\ \mathrm{cth}(\xi hs)\ ;\ \xi \longrightarrow x\ ] = \frac{w_0}{hs} \qquad d<x<1 \tag{4.9}$$

Inserting (4.7) in (4.9) and using the result [Gradshteyn and Ryzhik (1965)]

$$\int_0^{\infty} \frac{\mathrm{cth}(\xi hs)\ \sin(\xi\tau)\ \sin(\xi x)}{\xi}\, d\xi = \tfrac{1}{2}\log\left|\frac{\mathrm{th}(cx)+\mathrm{th}(c\tau)}{\mathrm{th}(cx)-\mathrm{th}(c\tau)}\right| \tag{4.10}$$

where $c = \pi/2hs$, we obtain

$$\int_{d}^{1} \frac{c g_1(\tau)\mathrm{Sech}^2(c\tau)\ \mathrm{th}(c\tau)}{\mathrm{th}^2(c\tau)\ -\ \mathrm{th}^2(cx)}\, d\tau = \frac{w_0}{hs\ \mathrm{Sech}^2(cx)}\ ,\ d<x<1 \tag{4.11}$$

Substituting $\mathrm{th}(c\tau)=T_1$, equation (4.11) is found to reduce to the form

$$\int_{D_1}^{I_1} \frac{T_1 A(T_1^2)}{T_1^2-X_1^2}\, dT_1 = \frac{w_0}{hs(1-X_1^2)} = F(X_1)\ \text{(say)},\ D_1<X_1<I_1 \tag{4.12}$$

where $D_1 = \mathrm{th}(cd)$, $I_1 = \mathrm{th}(c)$, $X_1 = \mathrm{th}(cx)$ and $A(T_1^2) = g_1(\tau)$. Using finite Hilbert transform (1968), the solutions of (4.12) is

$$A(T_1^2) = -\frac{4}{\pi^2}\sqrt{\frac{T_1^2-D_1^2}{I_1^2-T_1^2}}\int_{D_1}^{I_1}\sqrt{\frac{I_1^2-X_1^2}{X_1^2-D_1^2}}\ \frac{X_1 F(X_1)}{(X_1^2-T_1^2)}\, dX_1 + \frac{K_1}{\sqrt{(T_1^2-D_1^2)(I_1^2-T_1^2)}}$$

which can be simplified to

$$g_1(\tau) = \frac{2w_0\ \mathrm{ch}(cd)}{\pi hs(1-T_1^2)\ \mathrm{ch}(c)}\sqrt{\frac{T_1^2-D_1^2}{I_1^2-T_1^2}} + \frac{K_1}{\sqrt{(T_1^2-D_1^2)(I_1^2-T_1^2)}},\ d<\tau<1 \tag{4.13}$$

Substituting the result (4.13) in (4.8) we obtain

$$K_1\int_{D_1}^{I_1}\frac{dT}{\sqrt{(T^2-D_1^2)(I_1^2-T^2)}}=\frac{4w_o}{hs\pi^2}\int_{D_1}^{I_1}\sqrt{\frac{T^2-D_1^2}{I_1^2-T^2}}\,dT\int_{D_1}^{I_1}\sqrt{\frac{I_1^2-X_1^2}{X_1^2-D_1^2}}\;\frac{X_1\,dX_1}{(X_1^2-T^2)(1-X_1^2)}$$

which is simplified with aid of the results

$$\int_{D_1}^{I_1}\sqrt{\frac{I_1^2-X_1^2}{X_1^2-D_1^2}}\;\frac{X_1\,dX_1}{(X_1^2-T^2)}=-\frac{\pi}{2}$$

and
$$\int_{D_1}^{I_1}\sqrt{\frac{I_1^2-X_1^2}{X_1^2-D_1^2}}\;\frac{X_1\,dX_1}{(1-X_1^2)}=\frac{\pi}{2}\left[1-\sqrt{\frac{1-I_1^2}{1-D_1^2}}\right]$$

to
$$K_1=\frac{2w_o\,ch(cd)}{\pi hs\;ch(c)}\,D_1^2\left\{1-\Pi\left(\frac{\pi}{2},\;\frac{I_1^2-D_1^2}{I_1^2(1-D_1^2)},\;q\right]/F\left(\frac{\pi}{2},q\right)\right\}\qquad(4.14)$$

where $q=(I_1^2-D_1^2)^{1/2}/I_1$ and $F(\phi,k)$, $\Pi(\phi,n,k)$ are elliptic integrals of first and third kind respectively.

The expressions of displacement and shear stress on the plane of the crack are expressed as

$$W(x,o)=\frac{\pi}{2}\int_x^1 g_1(\tau)\mathrm{Sech}^2(c\tau)\,d\tau\quad,\quad d<x<1\qquad(4.15)$$

and
$$\sigma_{yz}(x,o)=\mu sc\int_d^1\frac{g_1(\tau)\mathrm{Sech}^2(c\tau)\,th(c\tau)\,\mathrm{Sech}^2(cx)}{th^2(cx)-th^2(c\tau)}\,d\tau\quad,0\le x<d,\quad x>1\qquad(4.16)$$

Now inserting (4.13)in (4.15) and (4.16) we obtain with the aid of the following results

$$\int_{X}^{I_1} \sqrt{\frac{T_1^2 - D_1^2}{I_1^2 - T_1^2}}\, \frac{dT_1}{1 - T_1^2} = - I_1^{-1} \left[F(\frac{\pi}{2}, q) + \frac{1 - D_1^2}{1 - I_1^2} \Pi\left(\lambda, \frac{I_1^2 - D_1^2}{1 - I_1^2}, q\right)\right]$$

and
$$\int_{D_1}^{I_1} \sqrt{\frac{T_1^2 - D_1^2}{I_1^2 - T_1^2}}\, \frac{T_1\, dT_1}{(X^2 - T_1^2)} = \frac{\pi}{2} \left[\sqrt{\frac{X^2 - D_1^2}{X^2 - I_1^2}} - 1 \right]$$

$$W(x,0) = - \frac{w_0\, ch(cd)}{hsc\ sh(c)} \left[F(\lambda, q)\left(1 - D_1^2 \left\{ 1 - \Pi\left(\frac{\pi}{2}, \frac{I_1^2 - D_1^2}{I_1^2 (1 - D_1^2)}, q\right) / F(\frac{\pi}{2}, q) \right\}\right) + \right.$$

$$\left. + \frac{ch^2(c)}{ch^2(cd)}\ \Pi\left(\lambda, \frac{I_1^2 - D_1^2}{1 - I_1^2}, q \right)\right], \quad d<x<1 \tag{4.17}$$

$$\text{where} \quad \sin\lambda = \sqrt{\frac{I_1^2 - X_1^2}{I_1^2 - D_1^2}}$$

$$\sigma_{yz}(x,o) = \frac{\mu w_0\, ch(cd)}{h\ ch(c)} \left[\sqrt{\frac{th^2(cx) - D_1^2}{th^2(cx) - I_1^2}} - \frac{ch(c)}{ch(cd)} + \right.$$

$$\left. + \left\{ 1 - \Pi\left(\frac{\pi}{2}, \frac{I_1^2 - D_1^2}{I_1^2 (1 - D_1^2)}, q\right) / F(\frac{\pi}{2}, q) \right\} \frac{D_1^2 Sech^2(cx)}{\sqrt{[th^2(cx) - D_1^2][th^2(cx) - I_1^2]}} \right], \quad x>1 \tag{4.18}$$

$$\sigma_{yz}(x,o) = \frac{\mu w_0\, ch(cd)}{h\ ch(c)} \left[\sqrt{\frac{D_1^2 - th^2(cx)}{I_1^2 - th^2(cx)}} - \frac{ch(c)}{ch(cd)} - \right.$$

$$\left. - \left\{ 1 - \Pi\left(\frac{\pi}{2}, \frac{I_1^2 - D_1^2}{I_1^2 (1 - D_1^2)}, q\right) / F(\frac{\pi}{2}, q) \right\} \frac{D_1^2\ Sech^2(cx)}{\sqrt{[D_1^2 - th^2(cx)][I_1^2 - th^2(cx)]}} \right], \quad o<x<d \tag{4.19}$$

where we have used the result

$$\int_d^1 \frac{1}{\sqrt{(t^2-d^2)(1-t^2)}}\,\frac{t\,dt}{t^2-x^2} = \begin{cases} \dfrac{\pi}{2\sqrt{(d^2-x^2)(1-x^2)}}, & 0<x<d \\ 0, & d<x<1 \\ \dfrac{-\pi}{2\sqrt{(x^2-d^2)(x^2-1)}}, & x>1 \end{cases} \tag{4.20}$$

The stress intensity factor at x = 1 is given by

$$S_{11} = \underset{x\to 1}{Lt} \sqrt{2(x-1)}\;\sigma_{yz}(x,o) = \frac{\mu w_0}{h\mathrm{Sech}(cd)}\left[\sqrt{\frac{I_1^2-D_1^2}{cI_1}} + \frac{D_1^2(1-I_1^2)}{\sqrt{cI_1(I_1^2-D_1^2)}}\right.$$

$$\left.\times\left\{1-\Pi\left[\frac{\pi}{2},\frac{I_1^2-D_1^2}{I_1^2(1-D_1^2)},q\right]/F(\frac{\pi}{2},q)\right\}\right] \tag{4.21}$$

and the stress intensity factor at x = d is given by

$$S_{1d} = \underset{x\to d}{Lt} \sqrt{2(d-x)}\;\sigma_{yz}(x,o) = -\frac{\mu w_0\sqrt{D_1^3(1-I_1^2)}}{h\sqrt{c(I_1^2-D_1^2)}}\times$$

$$\times\left\{1-\Pi\left[\frac{\pi}{2},\frac{I_1^2-D_1^2}{I_1^2(1-D_1^2)},q\right]/F(\frac{\pi}{2},q)\right\} \tag{4.22}$$

Letting d = a/b = 0 in the expressions for displacement , stress and stress intensity factors it can be easily shown that the results coincide with the corresponding expressions given by Singh et al. (1981).

Problem II. In this case again we take the general solution of (2.3) as

$$W(x,y) = F_c[\,C_1(\xi)\exp(-\xi ys) + C_2(\xi)\exp(\xi ys)\;;\;\xi\to x] \tag{4.23}$$

and inserting it in (3.7) we find that

$$C_1(\xi) = \frac{D(\xi)}{1 + \exp(-2\xi hs)}$$

$$C_2(\xi) = \frac{D(\xi)\exp(-2\xi hs)}{1 + \exp(-2\xi hs)}$$

From (3.8) and (3.9) it is determined that $D(\xi)$ satisfies the following set of triple integral equation

$$F_c[\xi \; D(\xi)\mathrm{th}(\xi hs) \; ;\xi \longrightarrow x \;] = \frac{p_0}{\mu s} \quad , \quad d<x<1 \tag{4.24}$$

$$F_c[D(\xi);\xi \longrightarrow x \;] = 0 \quad , \quad o\leq x<d \; ,x>1 \tag{4.25}$$

Proceeding as in problem I , we consider a trial solution

$$D(\xi) = \frac{1}{\xi}\sqrt{\frac{\pi}{2}}\int_d^1 g_2(\tau)\; \mathrm{ch}(c\tau)\sin(\xi\tau)\; d\tau \tag{4.26}$$

With this choice of $D(\xi)$, equation (4.25) will be satisfied provided the unknown function $g_2(\tau)$ in (4.26) satisfies

$$\int_d^1 g_2(\tau)\; \cosh(c\tau)\; d\tau = 0 \tag{4.27}$$

Now equation (4.24) can be written as

$$\frac{d}{dx} F_s[D(\xi)\; \mathrm{th}(\xi hs)\; ;\; \xi \longrightarrow x\;] = \frac{p_0}{\mu s} \quad d<x<1 \tag{4.28}$$

Insertion of equation (4.26) in (4.28) and use of the result [Gradshteyn and Ryzhik (1965)]

$$\int_0^\infty \frac{\mathrm{th}(\xi hs)\; \sin(\xi\tau)\; \sin(\xi x)}{\xi}\, d\xi = \frac{1}{2}\log\left|\frac{\mathrm{sh}(cx)+\mathrm{sh}(c\tau)}{\mathrm{sh}(cx)-\mathrm{sh}(c\tau)}\right| \tag{4.29}$$

where $c = \pi / 2hs$, gives

$$\int_d^1 \frac{c\; g_2(\tau)\; \mathrm{Sh}(2c\tau)}{\mathrm{sh}^2(c\tau)\; - \mathrm{sh}^2(cx)}\, d\tau = \frac{2p_0}{\mu s\; \mathrm{ch}(cx)} \quad , \; d<x<1 \tag{4.30}$$

Substituting $T_2 = \mathrm{sh}(c\tau)$, $I_2 = \mathrm{sh}(c)$, $D_2 = \mathrm{sh}(cd)$ and $X_2 = \mathrm{Sh}(cx)$ and proceeding as in problem I, we obtain the solution of (4.30) as

$$g_2(\tau) = -\frac{4p_0}{\pi^2 \mu s} \sqrt{\frac{T_2^2 - D_2^2}{I_2^2 - T_2^2}} \times \frac{1}{\sqrt{1+I_2^2}} \left[\Pi\left(\frac{\pi}{2}, \frac{I_2^2 - D_2^2}{I_2^2 - T_2^2}, q'' \right) - F\left(\frac{\pi}{2}, q'' \right) \right] +$$

$$+ \frac{K_2}{\sqrt{(T_2^2 - D_2^2)(I_2^2 - T_2^2)}} \tag{4.31}$$

where $q' = (I_2^2 - D_2^2)^{1/2}/I_2$, $q'' = q'.\mathrm{th}(c)$ and using the result (4.31) in the condition (4.27) the constant K_2 is determined with the aid of the result

$$\int_{D_2}^{I_2} \sqrt{\frac{I_2^2 - X^2}{X^2 - D_2^2}}\; \frac{X\, dX}{(X^2 - T_2^2)\sqrt{1+X^2}} = \frac{1}{\sqrt{1+I_1^2}} \left[\Pi\left(\frac{\pi}{2}, \frac{I_2^2 - D_2^2}{I_2^2 - T_2^2}, q'' \right) - F\left(\frac{\pi}{2}, q'' \right) \right]$$

as

$$K_2 = \frac{4p_0 \mathrm{th}(c)}{\pi^2 \mu s\; F\left(\frac{\pi}{2}, q' \right)} \int_{D_2}^{I_2} \sqrt{\frac{T_2^2 - D_2^2}{I_2^2 - T_2^2}} \left[\Pi\left(\frac{\pi}{2}, \frac{I_2^2 - D_2^2}{I_2^2 - T_2^2}, q'' \right) - F\left(\frac{\pi}{2}, q'' \right) \right] dT_2 \tag{4.32}$$

The relevant displacement and stress components in the plane of the cracks may be written as

$$W(x,o) = \frac{\pi}{2} \int_x^1 g_2(\tau)\; \mathrm{ch}(c\tau)\; d\tau \quad , \quad d<x<1 \tag{4.33}$$

and $$\sigma_{yz}(x,o)=\frac{\mu sc}{2}\int_{d}^{1}\frac{g_2(\tau)\ \mathrm{sh}(2c\tau)\mathrm{ch}(cx)}{\mathrm{sh}^2(cx)\ -\ \mathrm{sh}^2(c\tau)}\,d\tau\ ,\quad o\le x<d,\quad x>1 \tag{4.34}$$

Now using (4.31) in (4.33) and (4.34) we obtain

$$W(x,o)=-\ \frac{2p_o}{\pi\mu s\ \mathrm{ch}(c)}\left[\int_{x}^{1}\sqrt{\frac{\mathrm{sh}^2(c\tau)-\mathrm{sh}^2(cd)}{\mathrm{sh}^2(c)\ -\mathrm{sh}^2(c\tau)}}\times\left\{\Pi\left(\frac{\pi}{2},\frac{I_2^2-D_2^2}{I_2^2-T_2^2}\ ,q''\right)\right.\right.$$

$$\left.\left.-\ F\left(\frac{\pi}{2},q''\right)\right\}\mathrm{ch}(c\tau)\ d\tau\right]+\frac{K_2\ F\left(\lambda',\ q'\right)}{cI_2} \tag{4.35}$$

$$\text{where } \sin\lambda'\ =\ \sqrt{\frac{I_2^2-X_2^2}{I_2^2-D_2^2}}$$

$$\sigma_{yz}(x,o)=-\ \frac{2p_o\mathrm{ch}(cx)}{\pi}\sqrt{\frac{\mathrm{sh}^2(cx)-D_2^2}{\mathrm{sh}^2(cx)-I_2^2}}\int_{D_2}^{I_2}\sqrt{\frac{I_2^2-\ T_2^2}{T_2^2-\ D_2^2}}\times\frac{1}{T_2^2-\ \mathrm{sh}^2(cx)}$$

$$\times\ \frac{T_2\ dT_2}{\sqrt{1\ +\ T_2^2}}\ +\ \frac{\pi\mu s\ \mathrm{ch}(cx)K_2}{2.\sqrt{(\mathrm{sh}^2(cx)-I_2^2)(\mathrm{sh}^2(cx)-D_2^2)}}\ ,\quad \text{for } x>1 \tag{4.36}$$

$$\sigma_{yz}(x,o)=-\ \frac{2p_o\mathrm{ch}(cx)}{\pi}\sqrt{\frac{D_2^2-\mathrm{sh}^2(cx)}{I_2^2-\mathrm{sh}^2(cx)}}\int_{D_2}^{I_2}\sqrt{\frac{I_2^2-\ T_2^2}{T_2^2-\ D_2^2}}\times\frac{1}{T_2^2-\ \mathrm{sh}^2(cx)}$$

$$\times \frac{T_2 \, dT_2}{\sqrt{1 + T_2^2}} - \frac{\pi\mu s \, ch(cx) K_2}{2.\sqrt{[I_2^2 - sh^2(cx)][D_2^2 - sh^2(cx)]}} \text{, for } o<x<d \tag{4.37}$$

The stress intensity factor at x=1 is given by

$$S_{21} = \mathop{Lt}\limits_{x\to 1} \sqrt{2(x-1)} \, \sigma_{yz}(x,o) = \frac{2p_o}{\pi} \sqrt{\frac{I_2^2 - D_2^2}{cI_2 \, ch(c)}} \times F\left(\frac{\pi}{2}, q''\right) + \frac{\pi\mu s \, K_2}{2\sqrt{c.th(c)[I_2^2 - D_2^2]}} \tag{4.38}$$

and the stress intensity factor at x = d is given by

$$S_{2d} = \mathop{Lt}\limits_{x\to d} \sqrt{2(d-x)} \quad \sigma_{yz}(x,o) = \frac{-\pi\mu s \, K_2}{2\sqrt{c.th(cd)[I_2^2 - D_2^2]}} \tag{4.39}$$

Again letting d = 0 in the expressions for displacement , stress and stress intensity factors we obtain the corresponding results for a single crack as given by Singh et al. (1981).

5. Numerical results

In this section we present the variation of stress intensity factors with ratio of crack speed v to shear wave speed c_2 for both the problems. The crack length dependance of the stress intensity factors and its variations with v/c_2 have been shown in figures 2 – 5. Figures 2 – 3 depict the fact that in problem I, the stress intensity factors at both the crack tips decrease with the increase in the distance between the cracks.

But for the problem II, as seen from figures 4 – 5, it is found that the behaviour of the stress intensity factors at the crack tips is of different nature as compared to the corresponding nature of problem I. .In the problem I, the stress intensity factors at both the crack edges decrease with the increase in the value of v/c_2 and approaches to zero as $v/c_2 \rightarrow 1$. But in Problem II, the stress intensity factors at both the edges increase gradually with the increase in the value of v/c_2 and approaches infinity as $v/c_2 \rightarrow 1$. In problem II it is also found that the stress intensity factors at both the edges decrease with the increase in the values of the separating distance between the cracks. The dashed line in fig.2 and Fig.4 corresponding to the stress intensity factors at the tip of a single crack as given by Singh et al.(1981) for the case $b/h' = 1$.

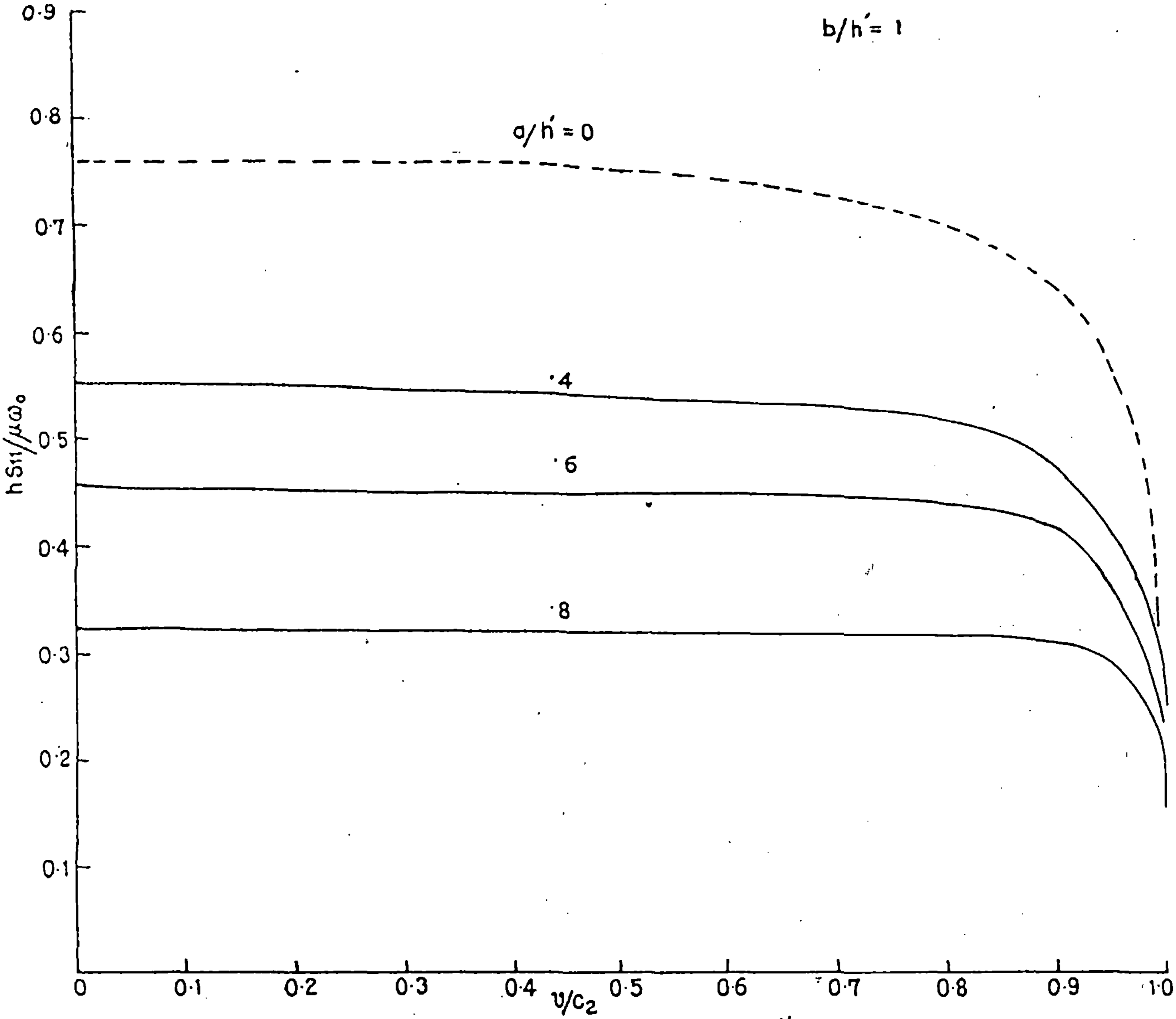


Fig 2: Stress intensity factor at the outer edge vs. v/c_2, for problem I

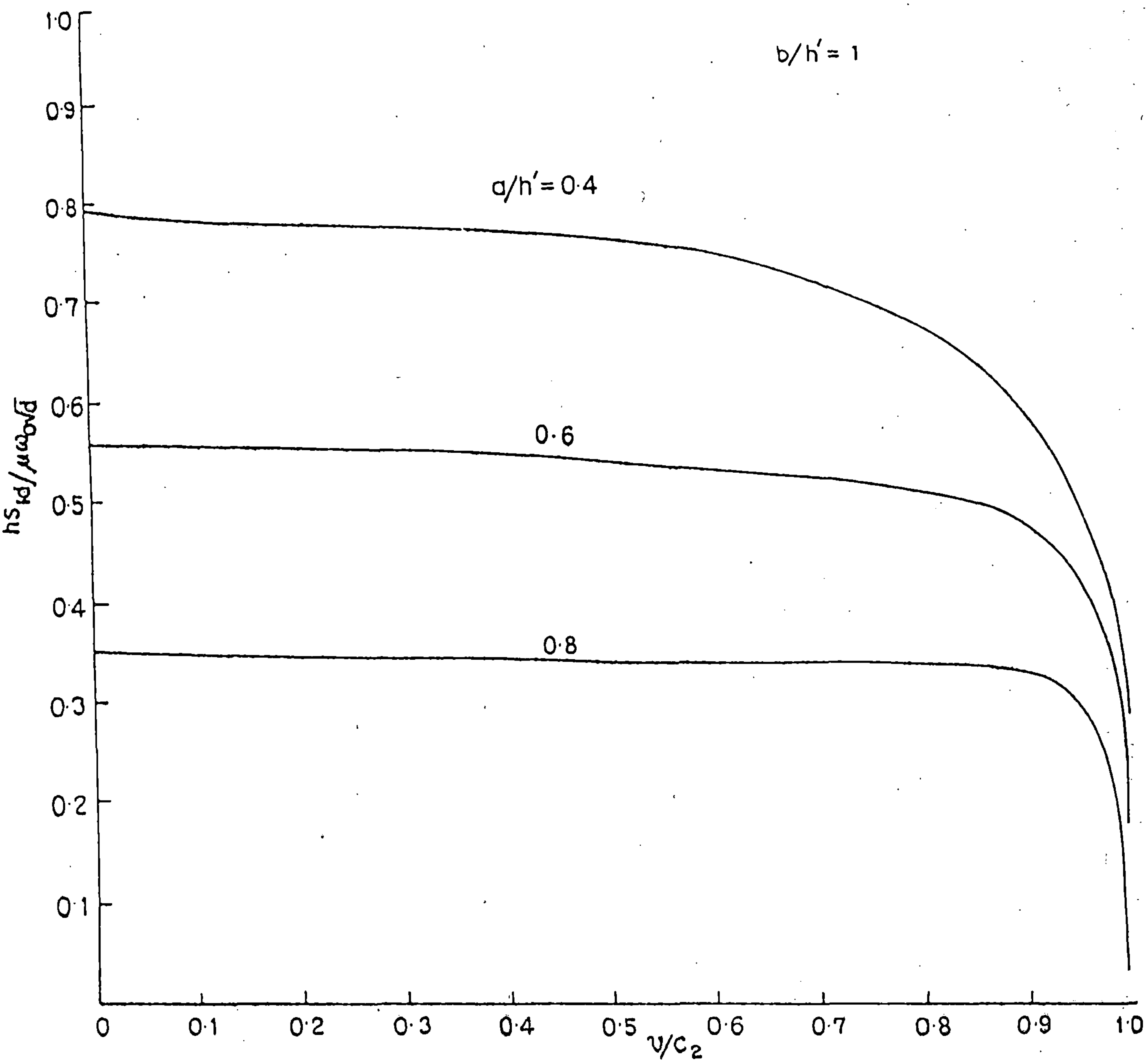


Fig 3. Stress intensity factor at the inner edge vs. v/c_2, for problem I

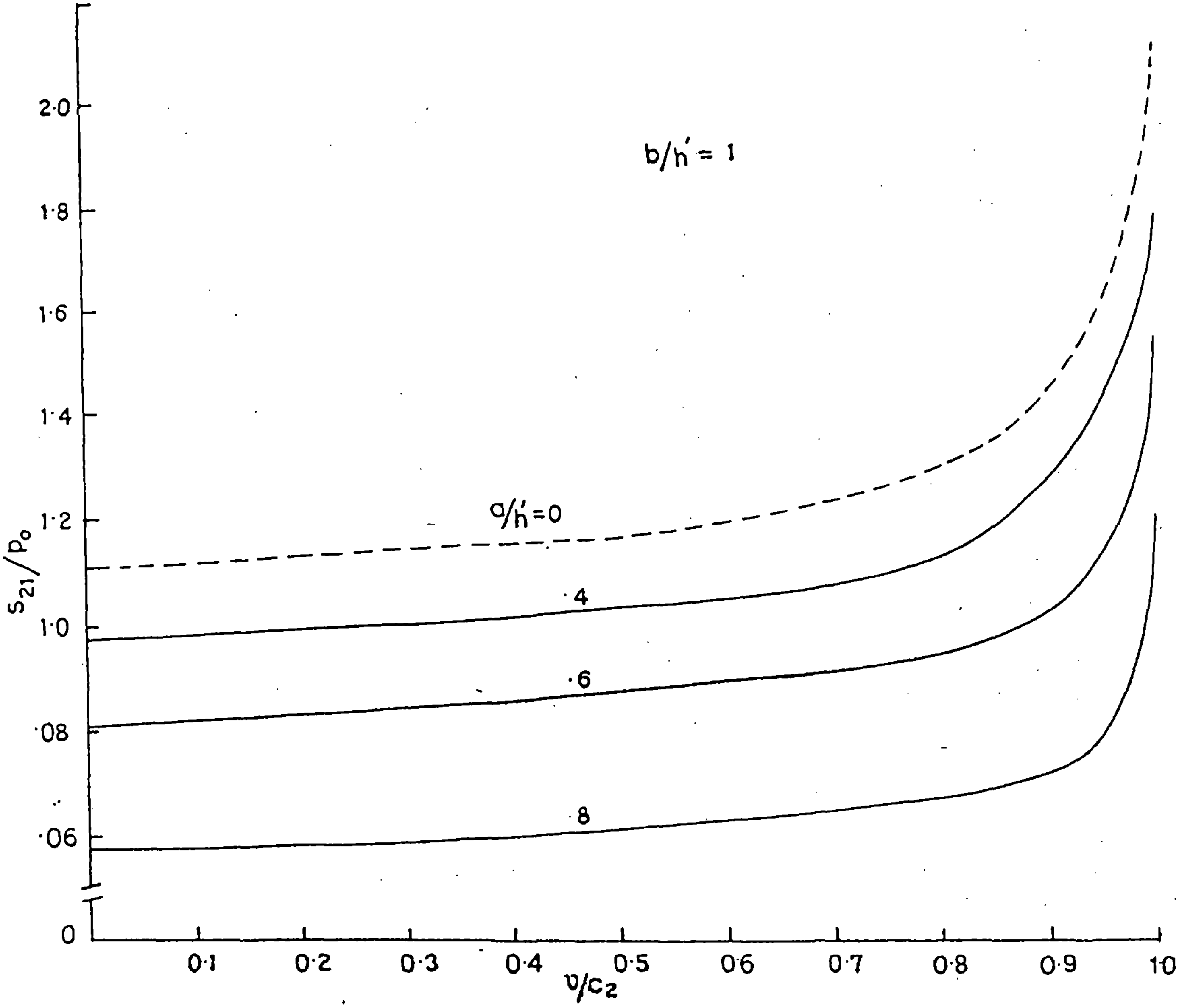


Fig 4. Stress intensity factor at the outer edge vs. v/c_2, for problem II

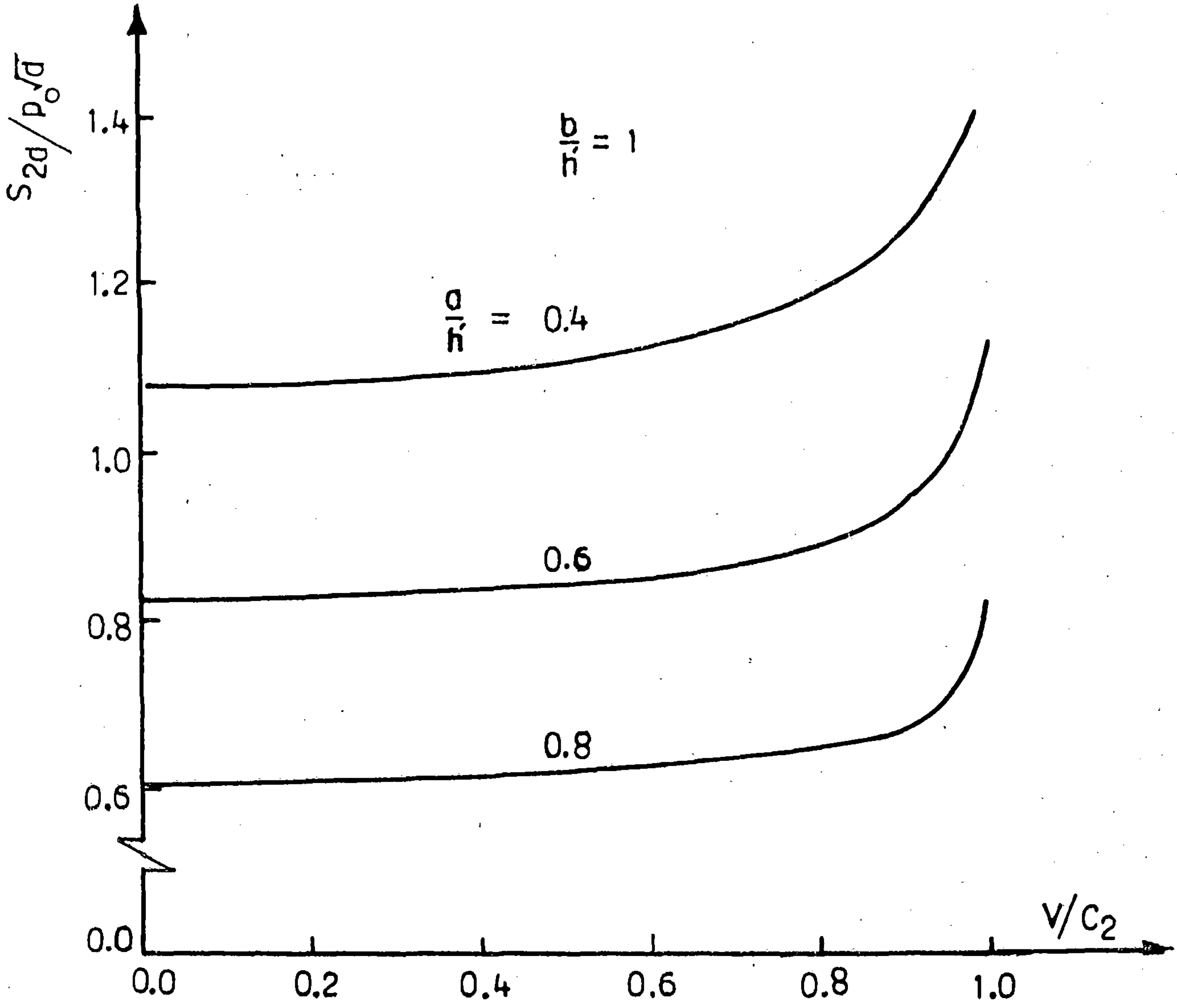


Fig. 5: Stress intensity factor at the inner edge vs v/c_2 for problem II.

TWO COPLANAR GRIFFITH CRACKS MOVING ALONG THE INTERFACE OF TWO DISSIMILAR ELASTIC MEDIA

1. Introduction

Scattering of elastic waves by cracks located in a homogeneous, isotropic medium has important applications in geophysics and seismology. If the cracks are located at the interface of layered media, the study becomes more relevant. Scattering of an elastic wave from an interface crack under anti-plane strain was solved by Bostrom (1987). Srivastava et al. (1980) solved the problem of interaction of an anti-plane shear wave by an interface crack. The problem of diffraction of Love waves by a crack of finite width in the plane interface of a layered composite has been solved by Neerhoff (1979). As regards the dynamic crack problem, research has been restricted mainly to the cases of a single crack because of the severe mathematical complexity encountered in finding solutions of problems involving two or more cracks. The diffraction of an anti-plane shear wave by two coplanar Griffith cracks in an infinite elastic medium has been treated by Itou (1980). Lowengrub and Srivastava (1968) treated the statical problem of stress distribution in the presence of two coplanar Griffith cracks in an infinite elastic strip. The scattering of time harmonic normally incident plane wave by two coplanar Griffith cracks was also solved by Jain and Kanwal (1972).

To the best knowledge of the authors, diffraction of elastic waves by two cracks moving along the interface of bonded dissimilar elastic media has not been investigated so far. In this paper, we consider the problem of determining the distribution of shear stress in the neighbourhood of the cracks, moving along the interface of the two bonded dissimilar elastic media. Two cases of practical importance have been considered here. Firstly, the case of two coplanar Griffith cracks moving along the interface of two semi-infinite dissimilar elastic media has been treated; secondly, the

---

Published in " Engineering Fracture Mechanics" Vol 41, pp 59-69, 1992

problem of the propagation of two coplanar Griffith cracks along the interface of an elastic layer overlying a semi-infinite medium of different elastic properties has been considered . Employing Fourier transforms the problem has been reduced to solving a set of triple integral equations with cosine kernel and weight functions. These equations are solved using the finite Hilbert transform technique . In the second problem, analytical expressions retain up to the order h^{-4}, where h is the thickness of the upper layer, for deriving the dynamic stress intensity factors and crack opening displacement. Numerical results have also been presented graphically.

2. Formulation Of The Problem

Two cracks of finite width are considered to be placed along the X -axis from -1 to -c and c to 1 with reference to a rectangular coordinate (x,y,z) system which, referred to fixed coordinate system (X,Y,Z), is moving with constant velocity v along X -axis, as shown in Fig.1 .

The coordinates are regarded as dimensionless, referring to the outer edge of the crack. In the dynamic problem of anti- plane shear ,there exists a single non-vanishing component of displacement in the Z -direction $W_i = W_i(X,Y,t)$, i=1,2, where W_1 and W_2 are the displacement component along the Z -direction in media Y>0 and Y<0 respectively . In the absence of body forces the equation of motion is

$$\frac{\partial^2 W_i}{\partial X^2} + \frac{\partial^2 W_i}{\partial Y^2} = \frac{1}{b_i^2}\,\frac{\partial^2 W_i}{\partial t^2} \tag{1}$$

where $b_i = (\mu_i/\rho_i)^{1/2}$, (i=1,2) are the shear wave speeds and ρ_i are the density of the materials and μ_i are the shear moduli.

Using Galilean transformation x = X - vt, y = Y, z = Z, t'= t, where (x,y,z) represents the translating coordinates system shown in Fig.1 eqn.(1) becomes independent of t and reduces to

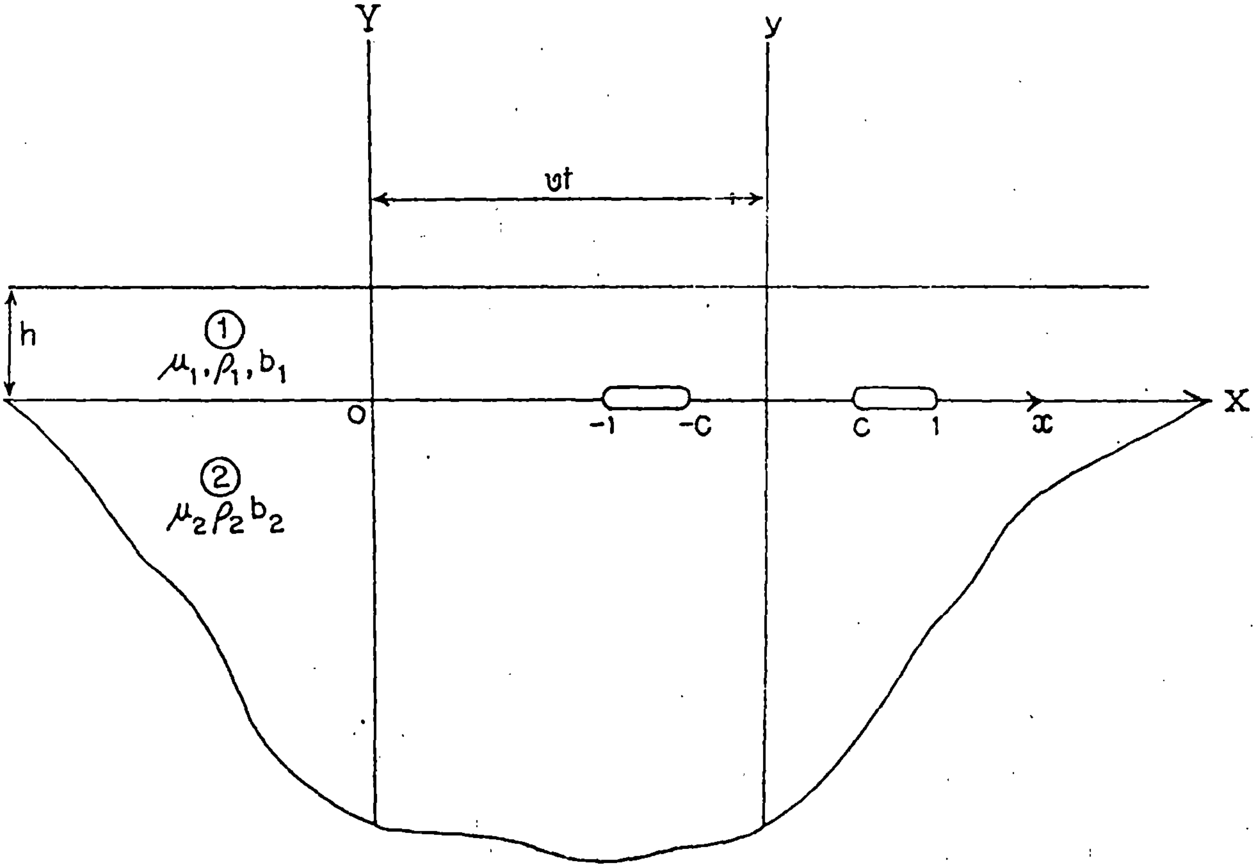


Fig 1. Geometry and coordinate system.

$$s_i^2 \frac{\partial^2 W_i}{\partial x^2} + \frac{\partial^2 W_i}{\partial y^2} = 0 \qquad (2)$$

with
$$s_i^2 = 1 - v^2/b_i^2 \qquad (3)$$

3.Boundary Conditions

Problem I:

In this case the cracks are placed along the interface of two joined dissimilar elastic half- spaces and are moving along the interface of these media. The cracks are excited by a normally incident anti-plane shear wave. The boundary conditions are

$$\begin{aligned} &[\tau_{yz}(x,0)]_1 = [\tau_{yz}(x,0)]_2 = -p, && c<|x|<1 \\ &[\tau_{yz}(x,0)]_1 = [\tau_{yz}(x,0)]_2, && 0\le|x|<c, |x|>1 \\ &W_1(x,0) = W_2(x,0), && 0\le|x|<c, |x|>1 \end{aligned} \qquad (4)$$

Problem I now consists of solving equation (2) together with the conditions (4).

Problem II:

In this case two coplanar Griffith cracks of finite width are assumed to be moving with uniform velocity under anti-plane shear stress along the interface of an elastic layer kept in welded contact with a semi-infinite medium of different elastic properties. The boundary conditions of this dynamic anti-plane problem are

$$\begin{aligned} &[\tau_{yz}(x,0)]_1 = [\tau_{yz}(x,0)]_2 = -p, && c<|x|<1 \\ &[\tau_{yz}(x,0)]_1 = [\tau_{yz}(x,0)]_2, && 0\le|x|<c, |x|>1 \\ &W_1(x,0) = W_2(x,0), && 0\le|x|<c, |x|>1 \\ &[\tau_{yz}(x,h)]_1 = 0 && -\infty < x < \infty \end{aligned} \qquad (5)$$

Problem II now consists of solving equation (2) together with the conditions (5).

4.Solution Of The Problem I

Employing Fourier cosine transforms viz.

$$f_c(\xi, y) = \int_0^\infty f(x,y)\cos(\xi x)\,dx, \quad \text{and}$$

$$f(x,y) = \frac{2}{\pi}\int_0^\infty f_c(\xi,y)\cos(\xi x)\,d\xi$$

we obtain the solution of equation (2) as

$$W_1(x,y) = \frac{2}{\pi}\int_0^\infty A_1(\xi)\exp(-s_1\xi y)\cos(\xi x)\,d\xi, \quad \text{for } y>0 \tag{6}$$

$$W_2(x,y) = \frac{2}{\pi}\int_0^\infty A_2(\xi)\exp(s_2\xi y)\cos(\xi x)\,d\xi, \quad \text{for } y<0 \tag{7}$$

where s_i is the positive root of (3) and $A_i(\xi)$ are unknown functions to be determined.

From (6) and (7) we obtain

$$[\tau_{yz}(x,y)]_1 = -\frac{2\mu_1 s_1}{\pi}\int_0^\infty \xi A_1(\xi)\exp(-s_1\xi y)\cos(\xi x)\,d\xi, \quad \text{for } y>0 \tag{8}$$

$$[\tau_{yz}(x,y)]_2 = \frac{2\mu_2 s_2}{\pi}\int_0^\infty \xi A_2(\xi)\exp(s_2\xi y)\cos(\xi x)\,d\xi, \quad \text{for } y<0 \tag{9}$$

Using (4a) and (4b) we get

$$A_2(\xi) = -\frac{\mu_1 s_1}{\mu_2 s_2} A_1(\xi) \tag{10}$$

The crack opening displacement $\Delta w(x)$ is defined as

$$\begin{aligned}\Delta w(x) &= W_1(x,0+) - W_2(x,0-) \\ &= \frac{2L}{\pi}\int_0^\infty A_1(\xi)\cos(\xi x)\,d\xi\,, && c<x<1 \qquad (11)\\ &= 0\,, && 0\le x<c\,,\ x>1\end{aligned}$$

where

$$L = \frac{\mu_1 s_1 + \mu_2 s_2}{\mu_2 s_2} \qquad (12)$$

From (8) and (4a)

$$\int_0^\infty \xi A_1(\xi)\cos(\xi x)\,d\xi = \frac{p\pi}{2\mu_1 s_1}\,,\quad c<x<1 \qquad (13)$$

Let us take $A_1(\xi) = \frac{1}{\xi}\int_c^1 h(t^2)\sin(\xi t)\,dt\,,$ (14)

Substituting (14) in (11) we see that this choice of $A_1(\xi)$ leads to

$$\int_c^1 h(t^2)\,dt = 0 \qquad (15)$$

Inserting (14) in (13) we obtain

$$\int_c^1 \frac{t h(t^2)dt}{t^2 - x^2} = \frac{p\pi}{2\mu_1 s_1}\,,\quad c<x<1 \qquad (16)$$

Using finite Hilbert transform technique (1968), the solution of (16) is

$$h(t^2) = \frac{2p}{\pi\mu_1 s_1}\sqrt{\frac{t^2-c^2}{1-t^2}}\int_c^1 \sqrt{\frac{1-x^2}{x^2-c^2}}\,\frac{x\,dx}{t^2-x^2} + \frac{K'}{\sqrt{(t^2-c^2)(1-t^2)}} \qquad (17)$$

where the unknown constant K′, determined from (15), is

$$K' = p(c^2 - E/F)/\mu_1 s_1 \qquad (18)$$

where $F = F(\Pi/2,q)$ and $E = E(\Pi/2,q)$ are complete elliptic integrals of the first kind and second kind respectively and $q = \sqrt{1-c^2}$

The relevant expressions for the crack opening displacement and stress component at the interface are

$$\Delta w(x) = L\int_x^1 h(t^2)\,dt\ , \qquad c\leq x\leq 1 \tag{19}$$

$$[\tau_{yz}(x,0)]_1 = -\frac{2\mu_1 s_1}{\pi}\int_c^1 \frac{t h(t^2)dt}{t^2-x^2}\ ,\ 0\leq x<c,\ x>1 \tag{20}$$

Substituting the value of $h(t^2)$ from (17) in (19) and (20) we obtain

$$\Delta w(x) = \frac{Lp}{\mu_1 s_1}\left[\ E(\lambda,q) - \frac{E}{F}F(\lambda,q)\right] \tag{21}$$

where

$$\sin\lambda = \sqrt{(1-x^2)/(1-c^2)} \tag{22}$$

and

$$[\tau_{yz}(x,0)]_1 = p\left[\sqrt{\frac{x^2-c^2}{x^2-1}} - 1 - \frac{E/F - c^2}{\sqrt{(x^2-c^2)(x^2-1)}}\right],\ \text{for } x>1 \tag{23}$$

$$= p\left[\sqrt{\frac{c^2-x^2}{1-x^2}} - 1 + \frac{E/F - c^2}{\sqrt{(c^2-x^2)(1-x^2)}}\right],\ \text{for } x<c \tag{24}$$

where we have used

$$\int_c^1 \frac{t\,dt}{(t^2-x^2)\sqrt{(t^2-c^2)(1-t^2)}} = \begin{cases} \dfrac{\pi}{2\sqrt{(c^2-x^2)(1-x^2)}}, & \text{for } 0<x<c \\ 0, & \text{for } c<x<1 \\ \dfrac{-\pi}{2\sqrt{(x^2-c^2)(x^2-1)}}, & \text{for } x>1 \end{cases} \tag{25}$$

The stress intensity factors at the tips of the cracks $x=1$ and $x=c$

respectively are given by

$$K_1 = \underset{x \to 1}{Lt} \sqrt{2(x-1)}\,[\tau_{yz}(x,0)]_1 = \frac{p(1-E/F)}{\sqrt{1-c^2}} \qquad (26)$$

$$K_c = \underset{x \to c}{Lt} \sqrt{2(c-x)}\,[\tau_{yz}(x,0)]_1 = \frac{p(E/F-c^2)}{\sqrt{c(1-c^2)}} \qquad (27)$$

5.Solution Of The Problem II

Employing Fourier cosine transform the solutions of is are sought in the form

$$W_1(x,y) = \frac{2}{\pi}\int_0^\infty [A_1(\xi)\exp(-s_1\xi y) + A_2(\xi)\exp(s_1\xi y)]\cos(\xi x)\,d\xi \ , \quad \text{for } 0 \le y \le h$$

$$W_2(x,y) = \frac{2}{\pi}\int_0^\infty A_3(\xi)\,\exp(s_2\xi y)\,\cos(\xi x)\,d\xi \ , \qquad \text{for } \ y<0 \qquad (28)$$

Using (28) we obtain the stress components as

$$[\tau_{yz}(x,y)]_1 = \frac{2\mu_1 s_1}{\pi}\int_0^\infty \xi[-A_1(\xi)\exp(-s_1\xi y) + A_2(\xi)\exp(s_1\xi y)]\cos(\xi x)\,d\xi ,$$

for $0 \le y \le h$

$$[\tau_{yz}(x,y)]_2 = \frac{2\mu_2 s_2}{\pi}\int_0^\infty \xi A_3(\xi)\,\exp(s_2\xi y)\,\cos(\xi x)\,d\xi , \quad \text{for } y<0 \qquad (29)$$

Applying (5a), (5b) and (5c)

$$A_3(\xi) = \frac{\mu_1 s_1}{\mu_2 s_2}[A_2(\xi) - A_1(\xi)] \qquad (30)$$

and $A_2(\xi) = A_1(\xi)\,\exp(-2\xi h s_1)$ (31)

The crack opening displacement $\Delta w(x)$ is defined as

$$\Delta w(x) = W_1(x,0+) - W_2(x,0-)$$

$$= \frac{2L}{\pi} \int_0^\infty f(\xi)\cos(\xi x)\, d\xi \qquad , \qquad c<x<1$$

$$= 0 \qquad , \qquad 0\le x<c \;,\; x>1 \qquad (32)$$

where

$$f(\xi) = A_1(\xi)\left[1 + \frac{\mu_2 s_2 - \mu_1 s_1}{\mu_2 s_2 + \mu_1 s_1} \exp(-2\xi h s_1) \right] \qquad (33)$$

Therefore, by (5c) and (5a), $f(\xi)$ is found to be the solution of the following triple integral equations

$$\int_0^\infty f(\xi)\cos(\xi x)\, d\xi = 0 \qquad 0\le x<c,\, x>1 \qquad (34)$$

$$\int_0^\infty \xi f(\xi)[1+M(\xi h)]\cos(\xi x)\, d\xi = \frac{p\pi}{2\mu_1 s_1} \;,\; c<x<1 \qquad (35)$$

with

$$M(\xi h) = -\frac{1 - \tanh(\xi h s_1)}{\left[1 + \frac{\mu_1 s_1}{\mu_2 s_2}\tanh(\xi h s_1)\right]} \qquad (36)$$

Assuming

$$f(\xi) = \frac{1}{\xi}\int_c^1 h(t^2)\sin(\xi t)\, dt \quad , \qquad (37)$$

it is found from (35) and (36)

$$\int_c^1 \frac{t h(t^2)dt}{t^2 - x^2} = \frac{p\pi}{2\mu_1 s_1} - Q(x)$$

where

$$Q(y) = \int_c^1 h(t^2)\, K_1(y,t)\, dt$$

and

$$K_1(y,t) = \int_0^\infty M(\xi h)\,\cos(\xi y)\,\sin(\xi t)\,d\xi . \qquad (38)$$

Now using Hilbert transform technique (1968) we find that $h(x^2)$ is the solution of the following Fredholm integral equation

$$h(x^2) + \int_c^1 h(t^2)\,K(x^2,t)\,dt = F(x^2)\,, \qquad c<x<1 \qquad (39)$$

satisfying the condition

$$\int_c^1 h(x^2)\quad dx = 0 \qquad (40)$$

where

$$K(x^2,t) = -\frac{4}{\pi^2}\sqrt{\frac{x^2-c^2}{1-x^2}}\int_c^1 \sqrt{\frac{1-y^2}{y^2-c^2}} \times \frac{y\;K_1(y,t)}{y^2-x^2}\,dy \qquad (41)$$

and

$$F(x^2) = -\frac{2p}{\pi\mu_1 s_1}\sqrt{\frac{x^2-c^2}{1-x^2}}\int_c^1 \sqrt{\frac{1-y^2}{y^2-c^2}}\,\frac{y\,dy}{y^2-x^2} + \frac{K''}{\sqrt{(x^2-c^2)(1-x^2)}}\,, \qquad (42)$$

K'' being an arbitrary constant determined by condition (40). If $h \gg 1$ is taken, then by substituting $\eta = \xi h$ and expanding $\cos(\eta y/h)$, $\sin(\eta y/h)$, it is possible to write (38) in the form

$$K_1(y,t) = \frac{I_0 t}{h^2} + \frac{I_1 t}{h^4}(t^2+3y^2) + O(h^{-6}) \qquad (43)$$

where

$$I_j = \frac{(-1)^j}{(2j+1)!}\int_0^\infty \eta^{2j+1}\,M(\eta)\,d\eta \quad , \ (j=0,1) \qquad (44)$$

and hence

$$K(x^2,t) = \frac{2}{\pi}\sqrt{\frac{x^2-c^2}{1-x^2}}\left[\frac{I_o t}{h^2} + \frac{I_1 t}{h^4}\left(t^2+3x^2-\frac{3}{2}k^2\right)\right] + O(h^{-\sigma}) \tag{45}$$

where $k^2 = 1 - c^2$.

Integrating both sides of (39) with respect to x from c to 1 and using (40)

$$K'' = \frac{p}{\mu_1 s_1}\left[c^2 - \frac{E}{F}\right] + \frac{1}{F}\int_c^1 h(t^2)\, K(t)dt \tag{46}$$

with

$$K(t) = \frac{2}{\pi}\left[\frac{I_o t}{h^2}(E - c^2F) + \frac{I_1 t}{h^4}\left\{\left(t^2-\frac{3}{2}k^2\right)(E-c^2F) - c^2(E+F)+2E\right\}\right] + O(h^{-\sigma})$$

where E and F defined by $E = E(\pi/2, q)$ and $F = F(\pi/2, q)$ with $q = k$ are known as elliptic integrals of first and second kind respectively. Using the results (46) and (42) in the equation (38) we see that $h(x^2)$ must satisfy the integral equation

$$h(x^2) + \int_c^1 h(t^2)\, M(x^2,t)\, dt = S(x^2) \tag{47}$$

where

$$M(x^2,t) = K(x^2,t) - \frac{K(t)}{F\sqrt{(x^2-c^2)(1-x^2)}} = \frac{2t}{\pi\sqrt{(x^2-c^2)(1-x^2)}}\left[\frac{I_o}{h^2}\left(x^2-\frac{E}{F}\right) + \frac{I_1}{h^4}\right.$$

$$\left.\times\left\{\left(t^2+\frac{3}{2}k^2\right)\left(x^2-\frac{E}{F}\right) + 3x^2(x^2-1) + \frac{E}{F} + c^2 - \frac{2c^2E}{F}\right\}\right] + O(h^{-\sigma}) \tag{48}$$

and

$$S(x^2) = \frac{p\left[x^2-\frac{E}{F}\right]}{\mu_1 s_1\sqrt{(x^2-c^2)(1-x^2)}} \tag{49}$$

Since $h \gg 1$, and $|M(x^2,t)| < 1$, the solution of (47) may be written in the form

$$h(x^2) = h_0(x^2) + \frac{1}{h^2}\, h_1(x^2) + \frac{1}{h^4}\, h_2(x^2) + O(h^{-6}) \qquad (50)$$

where

$$h_0(x^2) = \frac{p\left[\, x^2 - \frac{E}{F} \,\right]}{\mu_1 s_1 \sqrt{(x^2-c^2)(1-x^2)}} \qquad (51)$$

$$h_1(x^2) = \frac{-\, I_0 C_0 p\left[\, x^2 - \frac{E}{F} \,\right]}{2\mu_1 s_1 \sqrt{(x^2-c^2)(1-x^2)}} \qquad (52)$$

$$h_2(x^2) = \frac{p\, C_0}{4\mu_1 s_1 \sqrt{(x^2-c^2)(1-x^2)}} \left[I_0^2 C_0 \left\{ x^2 - \frac{E}{F} \right\} - 2I_1 (3x^4 + C_1 x^2 + C_2) \right] \qquad (53)$$

with

$$C_0 = 1 + c^2 - 2\,\frac{E}{F}$$

$$C_1 = k^4/4C_0 - (1 + c^2)$$

$$C_2 = c^2 + \frac{E}{F}\left\{ C_1 - \frac{k^4}{2C_0} \right\}$$

The relevant crack opening displacement and stress component at the interface are

$$\Delta w(x) = L \int_x^1 h(t^2)\, dt\,, \qquad c \le x \le 1 \qquad (54)$$

$$[\tau_{yz}(x,0)]_1 = -\,\frac{2\mu_1 s_1}{\pi} \left[\int_c^1 \frac{t h(t^2) dt}{t^2 - x^2} + \int_c^1 h(t^2) K_1(x,t) dt \right], 0 \le x < c,\ x > 1 \qquad (55)$$

where $K_1(x,t)$ is given in (38).

Using (43) and equations (50)– (53)

$$\int_c^1 h(t^2) K_1(x,t) dt = \frac{p\pi}{8\mu_1 s_1} \left[\frac{2I_0 C_0}{h^2} - \frac{I_0^2 C_0^2}{h^4} + \frac{2I_1 C_0}{h^4} \left\{ 3x^2 + C_1 + \frac{3}{2}(1+c^2) \right\} \right] + O(h^{-6})$$

(56)

we get for $0<x<c$

$$\int_c^1 \frac{th(t^2)dt}{t^2-x^2} = \frac{p\pi}{2\mu_1 s_1}\left[\left\{1-\frac{I_0C_0}{2h^2}+\frac{I_0^2C_0^2}{4h^4}\right\}\times\left\{\frac{x^2-\frac{E}{F}}{X_1}+1\right\}-\frac{I_1C_0}{2h^4}\times\right.$$

$$\left.\times\left\{\frac{3x^4+C_1x^2+C_2}{X_1}+3\left(\frac{1+c^2}{2}+x^2\right)+C_1\right\}\right]+O(h^{-6}) \quad , \tag{57}$$

and for $x>1$

$$\int_c^1 \frac{th(t^2)dt}{t^2-x^2} = \frac{p\pi}{2\mu_1 s_1}\left[\left\{1-\frac{I_0C_0}{2h^2}+\frac{I_0^2C_0^2}{4h^4}\right\}\times\left\{\frac{\frac{E}{F}-x^2}{X_2}+1\right\}+\frac{I_1C_0}{2h^4}\times\right.$$

$$\left.\times\left\{\frac{3x^4+C_1x^2+C_2}{X_2}-3\left(\frac{1+c^2}{2}+x^2\right)-C_1\right\}\right]+O(h^{-6}) \quad , \tag{58}$$

where

$$X_1 = \sqrt{(c^2-x^2)(1-x^2)}$$

$$X_2 = \sqrt{(x^2-c^2)(x^2-1)}$$

Using equations (50) to (53) the crack opening displacement is obtained from (54) after integration as

$$\Delta w(x) = \frac{pL}{\mu_1 s_1}\left[\left\{1-\frac{I_0C_0}{2h^2}+\frac{I_0^2C_0^2+2I_1C_0(C_1-k^4/2C_0)}{4h^4}\right\}\times\right.$$

$$\left.\times\left\{E(\lambda,q)-\frac{E}{F}F(\lambda,q)\right\}-\frac{2I_1C_0}{4h^4}\times\sqrt{(1-x^2)(x^2-c^2)}\right]+O(h^{-6}) \tag{59}$$

where λ is given by (22).

Substituting the results obtained in (56),(57) and (58) on the right hand side of (55) the stress in the plane of the crack can be derived and from it stress intensiy factors at the crack tips can be determined easily.

The stress intensity factor at x = 1 is given by

$$N_1 = \underset{x \to 1}{Lt} \sqrt{2(x-1)}\,[\tau_{yz}(x,0)]_1 = \frac{-p}{\sqrt{1-c^2}}\left[(E/F-1)\left\{1-\frac{I_0C_0}{2h^2}+\frac{I_0^2C_0^2}{4h^4}\right\} + \frac{I_1C_0}{2h^4}(3+C_1+C_2)\right] + O(h^{-6}) \tag{60}$$

and the stress intensity factor at x = c is found to be

$$N_c = \underset{x \to c}{Lt} \sqrt{2(c-x)}\,[\tau_{yz}(x,0)]_1 = \frac{-p}{\sqrt{c(1-c^2)}}\left[(c^2-E/F)\left\{1-\frac{I_0C_0}{2h^2}+\frac{I_0^2C_0^2}{4h^4}\right\} - \frac{I_1C_0}{2h^4}(3c^4+C_1c^2+C_2)\right] + O(h^{-6}) \tag{61}$$

5.NUMERICAL RESULTS

In this section numerical results are presented for the stress intensity factors at the crack tips and also the crack opening displacement for different values of the layer thickness and the crack speed and for $b_1/b_2 = 0.6$ .The crack opening displacement is found to increase gradually with increase in the value of the crack speed. Further, for a fixed crack speed, the crack opening displacement increases with decrease in the value of the separating distance between the cracks.

Variation of the stress intensity factors at both the crack tips with the crack speed is depicted in Figures 4 – 7. It is interesting to note that the stress intensity factors at both the crack tips increase very slowly at the onset with increase in the value of v/b_1 but change rapidly and goes to infinity as v/b_1 approaches unity.This fact becomes prominent as the layer thickness becomes large.

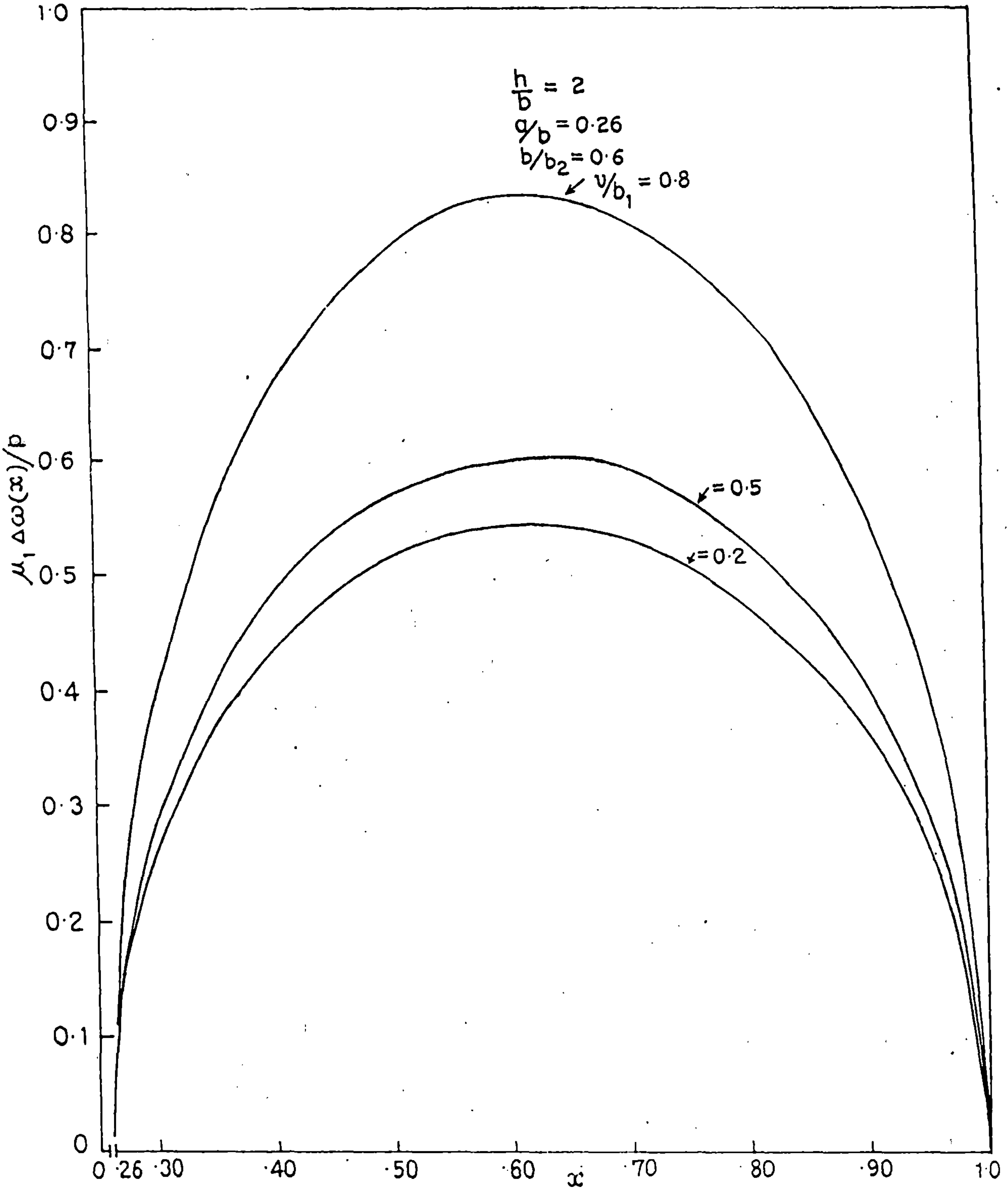


Fig. 2. Variation of crack opening displacement with x for problem II

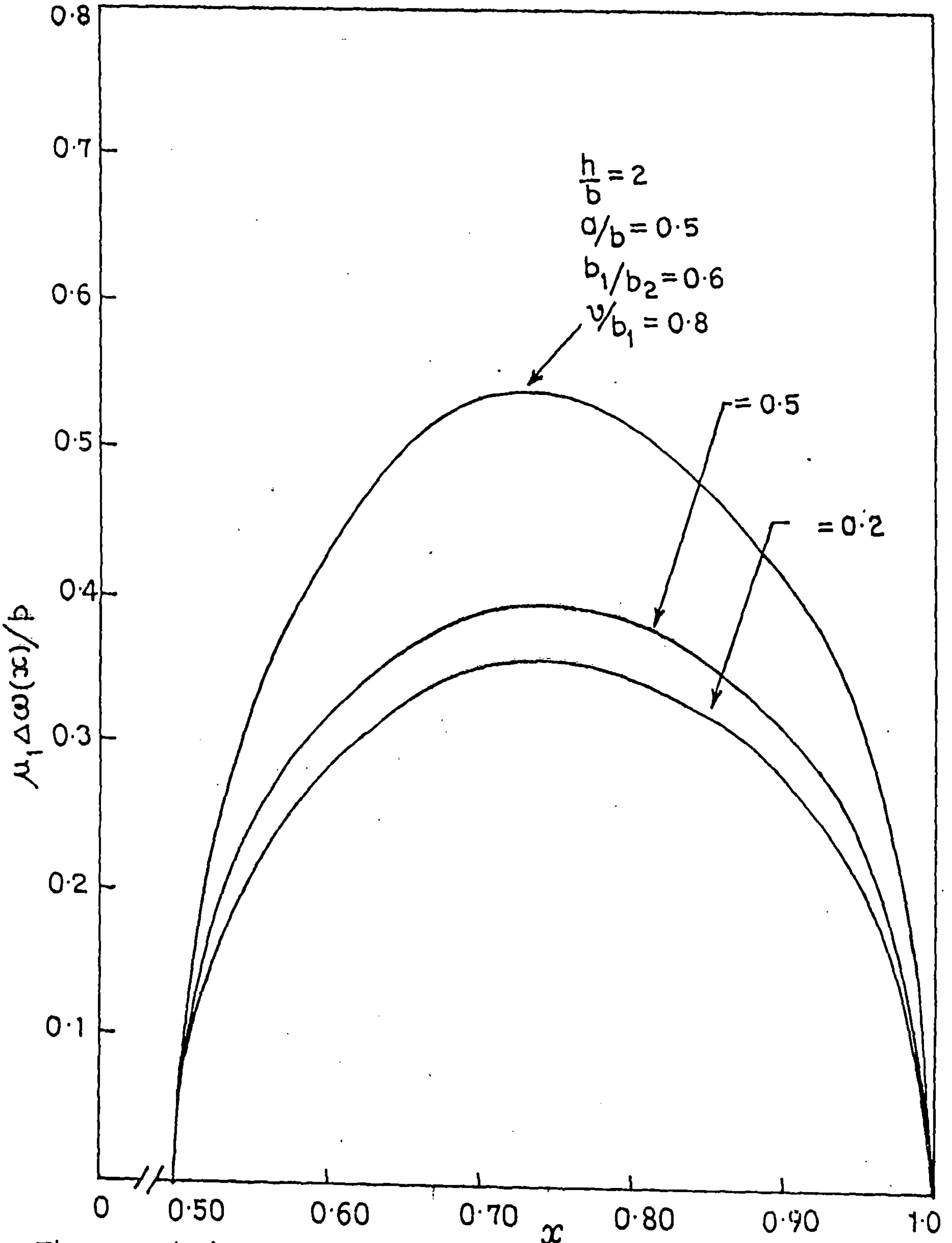


Fig 3. Variation of crack opening displacement with x for problem II.

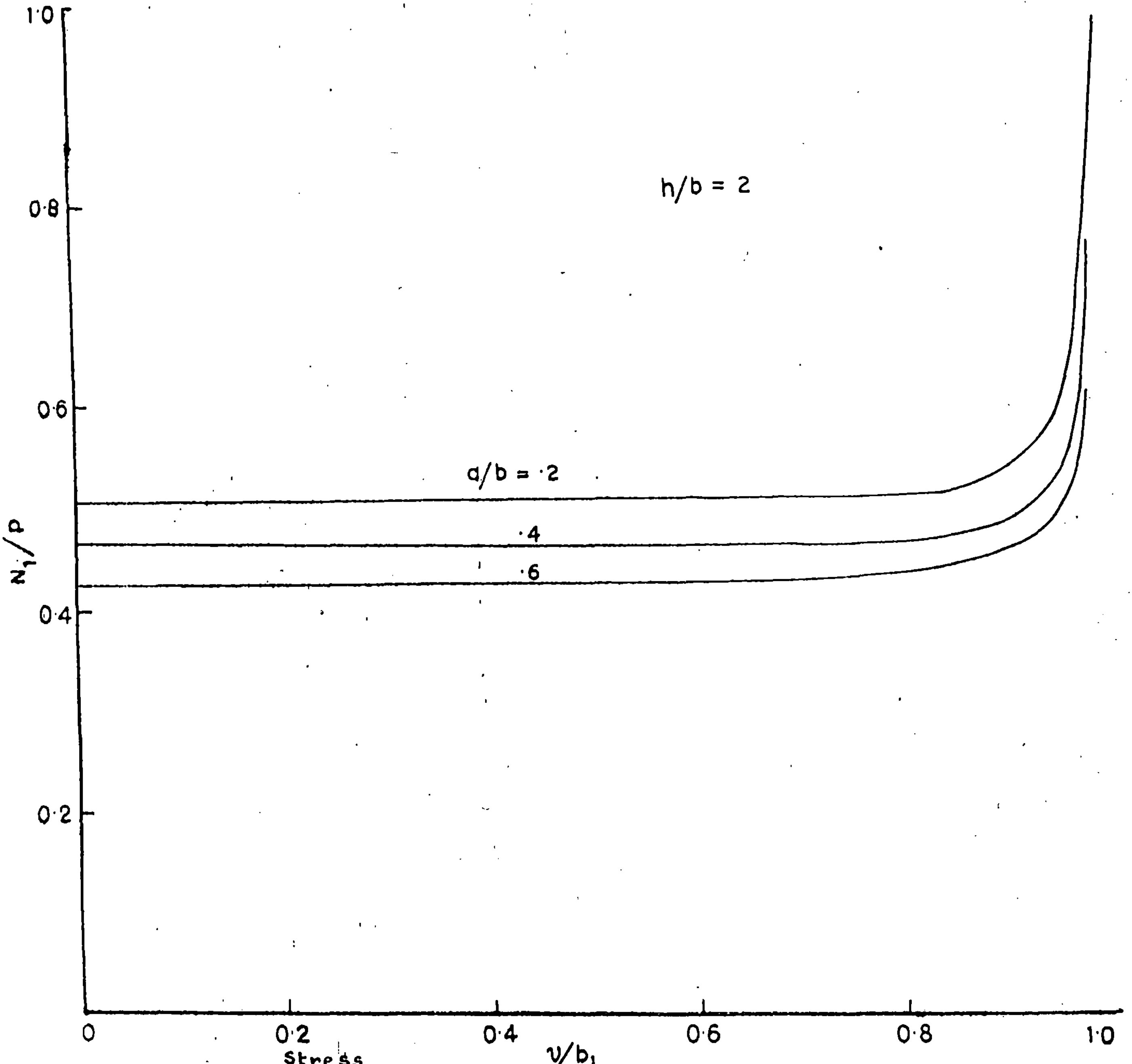


Fig 4. Variation of ~~~~ Stress intensity factor at the outer edge with r/b_1 for problem II

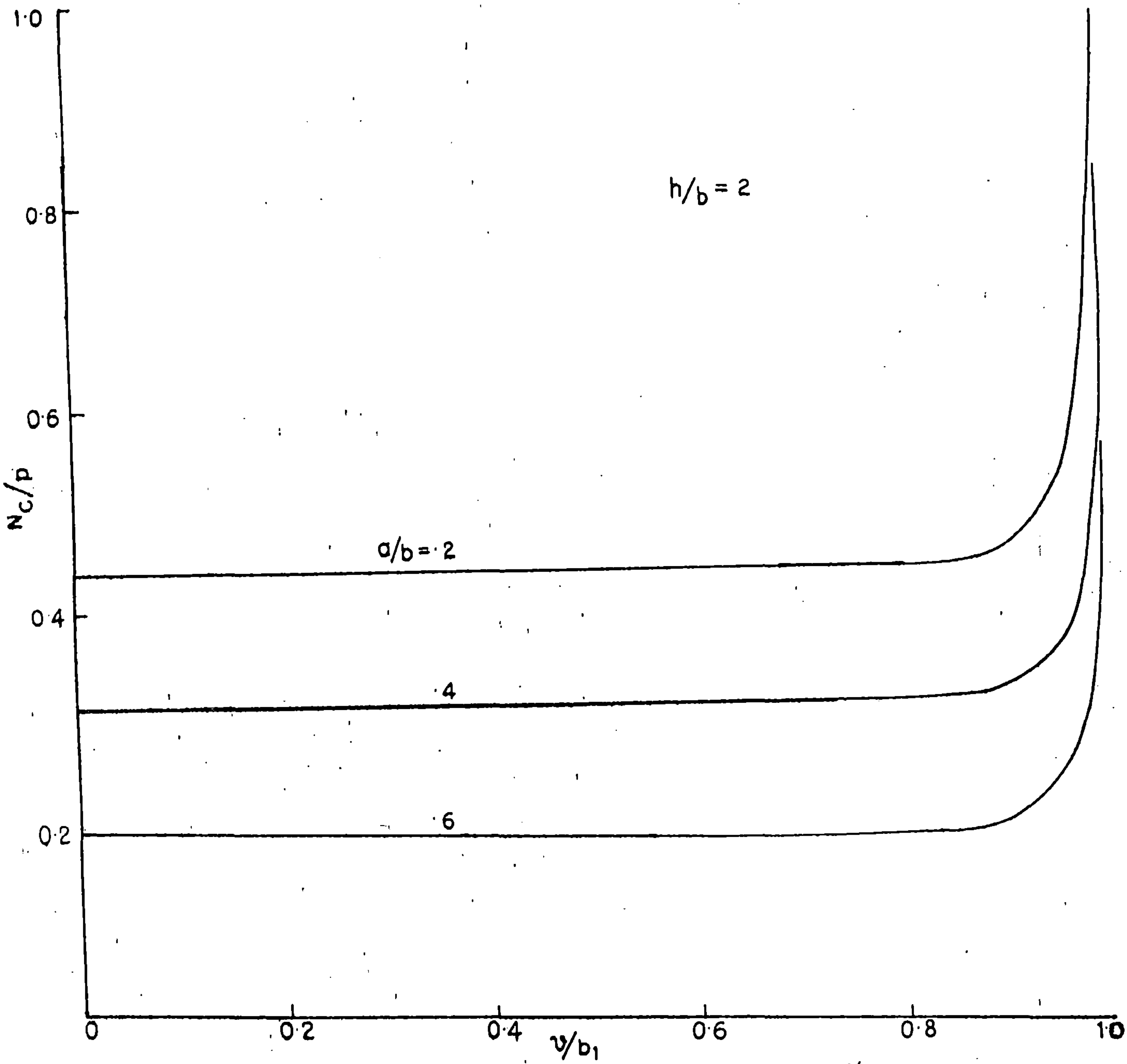


Fig 5. Variation of stress intensity factor at the inner edge with υ/b_1 for problem II.

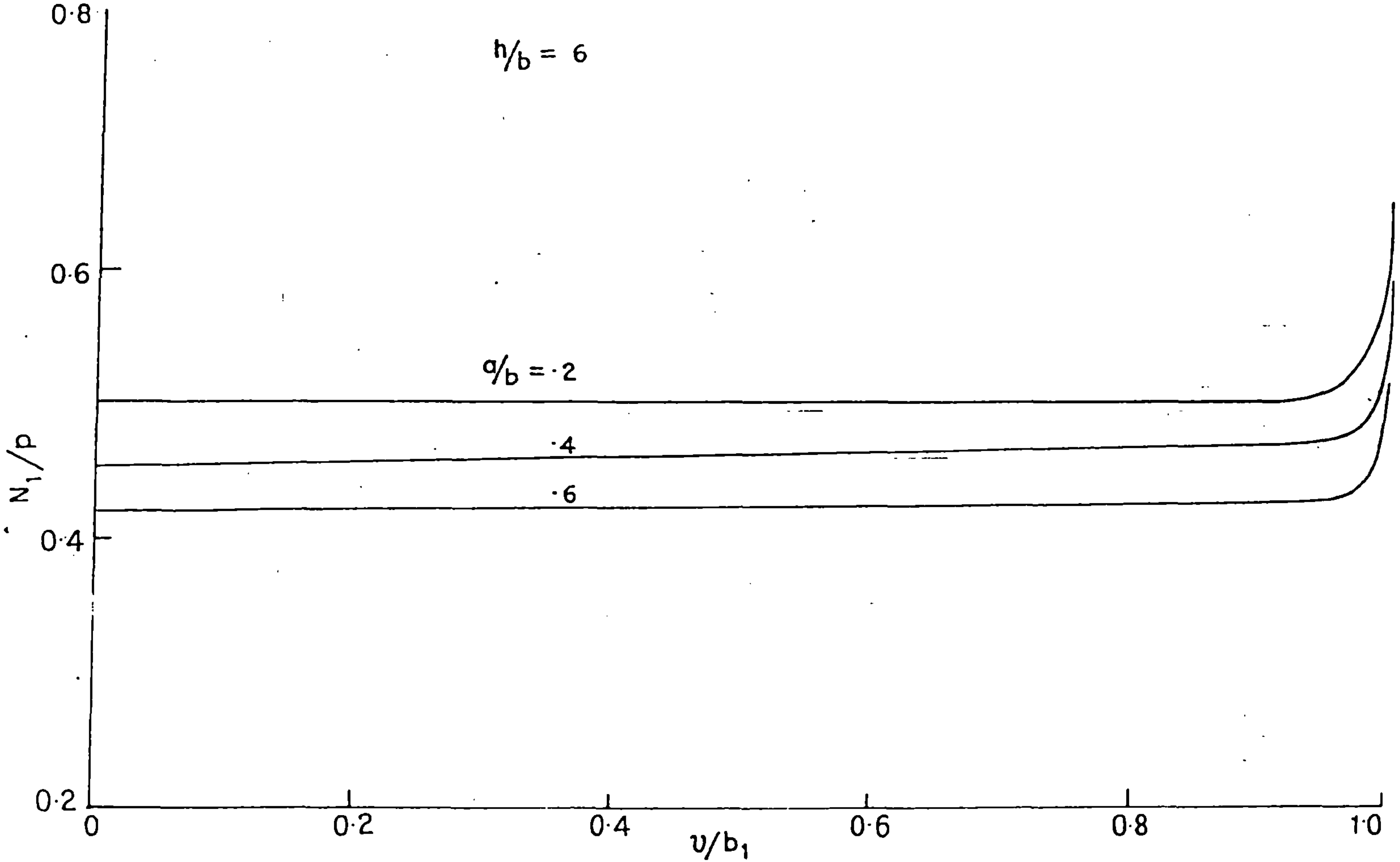


Fig 6. Variation of stress intensity factor at the outer edge with υ/b_1 for problem II .

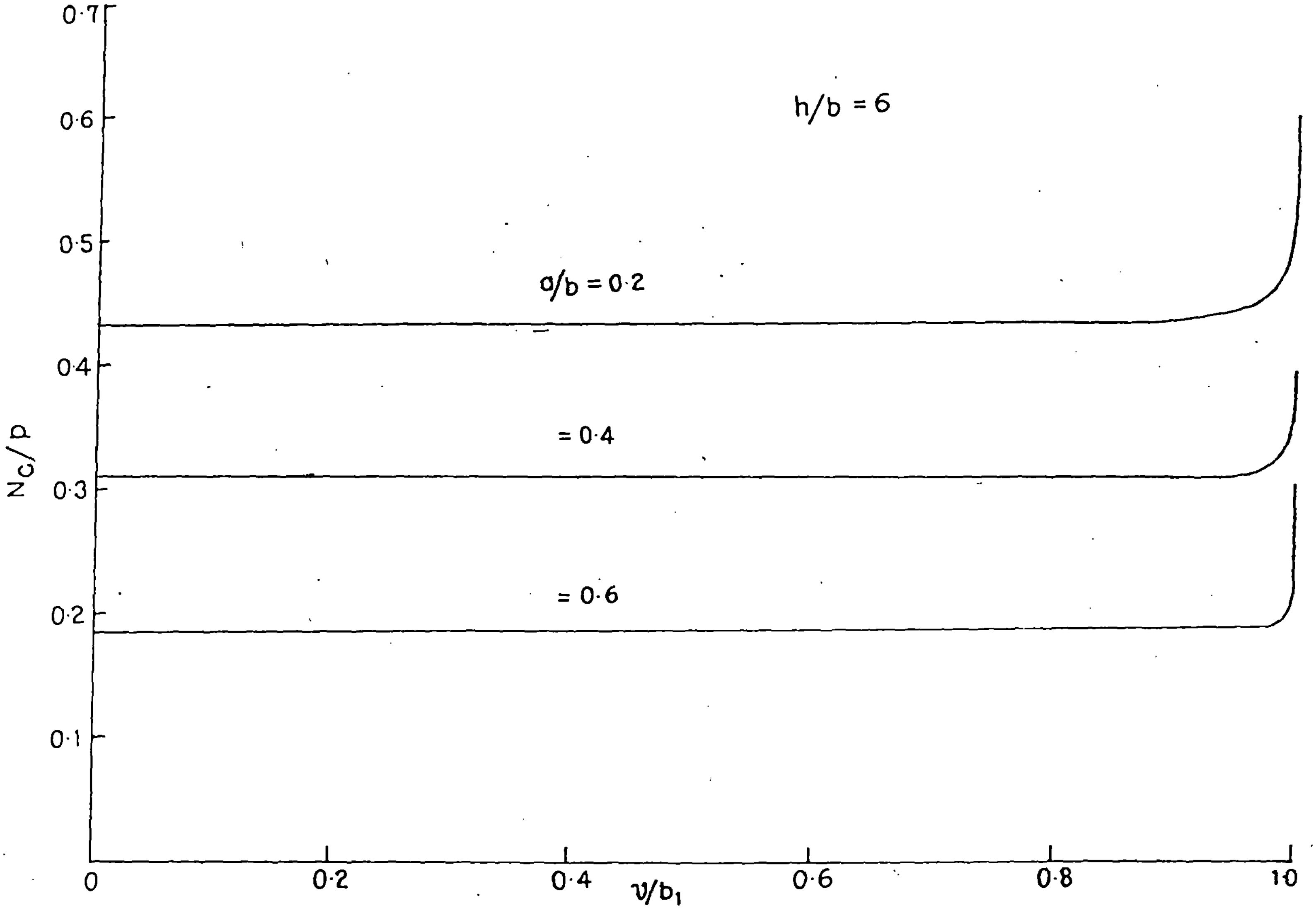


Fig 7. Variation of stress intensity factor at the inner edge with ν/b_1 for problem II.

PROBLEM OF TWO COPLANAR GRIFFITH CRACKS RUNNING STEADILY UNDER THREE- DIMENSIONAL LOADING

1. Introduction

Yoffe (1951) considered the problem of propagation of a crack of fixed length at a constant speed through a stretched isotropic elastic solid of infinite extent. In recent years, Yoffe's investigation was extended to include different types of materials and different material geometries. Sih and Chen (1972) considered the problem of uniformly propagating finite crack in a strip of isotropic elastic material. Recently, Kassir and Tse (1983) solved the plane stress problem of a moving crack in an infinite orthotropic stressed medium by using integral transform technique and the same technique has been employed by De and Patra (1990) to solve Yoffe's problem in an stressed orthotropic strip of finite thickness.

However all the problems mentioned above have been solved using dynamic equation of elasticity in two dimension. But in most instance, cracks are subjected to a state of stress that is triaxial in nature. Cracks problems involving three dimensional loading have generally not been attempted so far.

Recently, Angel and Achenbach (1985) derived the elastodynamic stress intensity factor for three dimensional loading of a cracked half- space. Freund (1971) also solved the three dimensional problem of the oblique reflection of a Rayleigh wave from the edge of a semi- infinite crack employing a Wiener- Hopf technique. The problem of a uniformly propagating finite crack in an elastic medium has been solved by Itou (1979) using dynamic equations of elasticity in three dimension.

Regarding the dynamic crack problem, research has

Published in " International Journal Of Fracture" Vol 56,
pp 279- 298, 1992

been restricted mainly to a single crack because of severe mathematical complexity encountered in finding the solution of two or more cracks. Recently Jain and Kanwal (1972) presented the low frequency solution of diffraction of normally incident longitudinal waves by two coplanar Griffith cracks in an isotropic elastic medium. They used the finite Hilbert transform technique developed by Srivastava and Lowengrub (1968) to solve the mixed boundary value problem. Using a completely different technique Itou (1980) solved the diffraction problem of elastic waves by two coplanar Griffith cracks in an infinite elastic medium.

In this paper, we have considered the problem of propagation of two coplanar Griffith cracks propagating steadily with uniform velocity under three dimensional loading. The application of two dimensional Fourier transform reduced this problem to that of solving triple integral equations in which the double Fourier transform of the crack opening displacement appear as the unknown. In an attempt to solve the problem the transformed surface displacement has been expanded in a series of a function which is automatically zero outside the cracks. Finally, Schmidt method (1978) has been employed to solve the integral equations . The dynamic stress intensity factors and the crack opening displacement have been evaluated numerically for various values of crack speed and distance between the cracks.

2. Formulation of the problem.

Let (X,Y,Z) be a fixed rectangular coordinate system. Two coplanar Griffith cracks of infinite length but of finite width located in the XZ- plane, the Z- axis being in the direction of the length of the cracks, are, assumed to be moving steadily with velocity U in the direction of X- axis. It is convenient to introduce Galilean transformation $x = X-UT$, $y = Y$, $z = Z$, $t = T$ where (x,y,z) represents the translating coordinate system shown in Fig.1. Referred to this moving system of the coordinate the cracks are assumed to occupy the positions $b<|x|<a, y=0, |z|<\infty$.

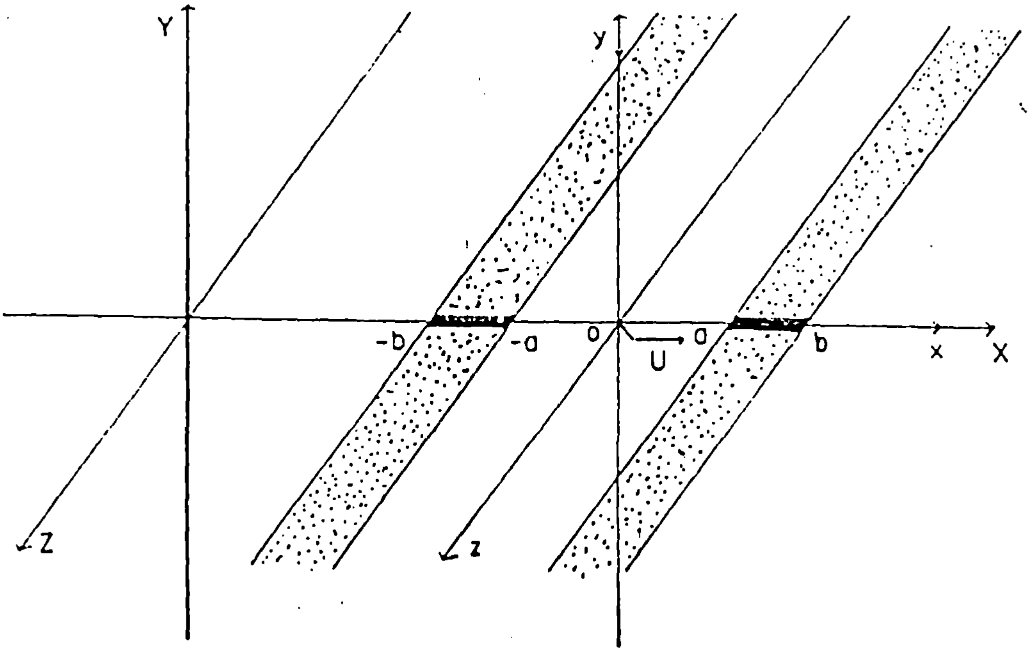


Fig.1: Geometry and coordinate system.

The equations on motion in the absence of body force are

$$(\lambda+\mu)\frac{\partial}{\partial X}\left[\frac{\partial u^*}{\partial X}+\frac{\partial v^*}{\partial Y}+\frac{\partial w^*}{\partial Z}\right]+\mu\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+\frac{\partial^2}{\partial Z^2}\right)u^*=\rho\frac{\partial^2 u^*}{\partial T^2}$$

$$(\lambda+\mu)\frac{\partial}{\partial Y}\left[\frac{\partial u^*}{\partial X}+\frac{\partial v^*}{\partial Y}+\frac{\partial w^*}{\partial Z}\right]+\mu\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+\frac{\partial^2}{\partial Z^2}\right)v^*=\rho\frac{\partial^2 v^*}{\partial T^2}$$

$$(\lambda+\mu)\frac{\partial}{\partial Z}\left[\frac{\partial u^*}{\partial X}+\frac{\partial v^*}{\partial Y}+\frac{\partial w^*}{\partial Z}\right]+\mu\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+\frac{\partial^2}{\partial Z^2}\right)w^*=\rho\frac{\partial^2 w^*}{\partial T^2} \qquad (2.1)$$

where u^*, v^*, w^* are the displacement components, λ and μ are Lame's constants and ρ is the material density. Using Galilean transformation

$$x = X - UT, \quad y = Y, \quad z = Z, \quad t = T$$

(2.1) reduces to

$$(\lambda+\mu)\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]+\mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u=\rho U^2\frac{\partial^2 u}{\partial x^2}$$

$$(\lambda+\mu)\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]+\mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)v=\rho U^2\frac{\partial^2 v}{\partial x^2}$$

$$(\lambda+\mu)\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]+\mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)w=\rho U^2\frac{\partial^2 w}{\partial x^2} \qquad (2.2)$$

where u,v,w are the displacement components in the moving coordinate system so that

$$u^*(X,Y,Z,T) = u(x,y,z)$$

$$v^*(X,Y,Z,T) = v(x,y,z)$$

$$w^*(X,Y,Z,T) = w(x,y,z)$$

The stress components for the three dimensional problem are

$$\sigma_x = (\lambda+2\mu)\frac{\partial u}{\partial x} + \lambda\left[\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right],\quad \tau_{xy} = \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]$$

$$\sigma_y = (\lambda+2\mu)\frac{\partial v}{\partial y} + \lambda\left[\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right],\quad \tau_{yz} = \left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right]$$

$$\sigma_z = (\lambda+2\mu)\frac{\partial w}{\partial z} + \lambda\left[\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right],\quad \tau_{xz} = \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \tag{2.3}$$

The boundary conditions are

$$\sigma_y/2\mu = -p(x,z),\quad \text{for } y=0,\ a\le|x|\le b,\ |z|<\infty,$$

$$v=0,\quad \text{for } y=0,\ |x|>b, |x|<a, |z|<\infty,$$

$$\tau_{xy} = 0 = \tau_{yz},\quad \text{for } y=0,\ |x|<\infty, |z|<\infty. \tag{2.4}$$

3. Solution of the problem

Using Fourier transformations viz.

$$\bar{g}(\xi,y,\zeta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y,z)\, e^{i(\xi x+\zeta z)}\, dx\, dz,$$

and

$$g(x,y,z) = (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{g}(\xi,y,\zeta)\, e^{-i(\xi x+\zeta z)}\, d\xi\, d\zeta, \tag{3.1}$$

(2.2) reduces to

$$\left\{\frac{d^2}{dy^2} - (\alpha^2-M^2)\xi^2 - \zeta^2\right\}\bar{\bar{u}} - i(\alpha^2-1)\xi\frac{d\bar{\bar{v}}}{dy} - (\alpha^2-1)\xi\zeta\bar{\bar{w}} = 0,$$

$$-i(\alpha^2-1)\xi\frac{d\bar{\bar{u}}}{dy} + \left\{\alpha^2\frac{d^2}{dy^2} - (1-M^2)\xi^2 - \zeta^2\right\}\bar{\bar{v}} - i(\alpha^2-1)\xi\frac{d\bar{\bar{w}}}{dy} = 0,$$

$$-(\alpha^2-1)\xi\zeta\bar{\bar{u}} - i(\alpha^2-1)\zeta\frac{d\bar{\bar{v}}}{dy} + \left\{\frac{d^2}{dy^2} - (1-M^2)\xi^2 - \zeta^2\alpha^2\right\}\bar{\bar{w}} = 0, \tag{3.2}$$

with $\alpha^2 = (\lambda+2\mu)/\mu$, $\beta^2=\rho/\mu$ and $M^2 = \beta^2U^2$.

Due to symmetry given in (2.4), we need to consider the region $y\geq0$ only. The solutions of (3.2) in the region $y\geq0$ can easily be found to be of the form

$$\bar{\bar{u}} = A_1\, e^{-s_1 y} + B_1\, e^{-s_2 y}$$

$$\bar{\bar{v}} = A_2\, e^{-s_1 y} + B_2\, e^{-s_2 y}$$

$$\bar{\bar{w}} = A_3\, e^{-s_1 y} + B_3\, e^{-s_2 y} \qquad (3.3)$$

where

$$s_1 = \sqrt{(1-M^2/\alpha^2)\xi^2+\zeta^2}$$

$$s_2 = \sqrt{(1-M^2)\xi^2+\zeta^2} \qquad (3.4)$$

and

$$A_1= i\xi A_2/s_1, \quad A_3= i\zeta A_2/s_1, \quad B_2= -i(\xi B_1+\zeta B_3)/s_2 \qquad (3.5)$$

The transformed stress components $\bar{\bar{\sigma}}_x$, $\bar{\bar{\sigma}}_y$, $\bar{\bar{\tau}}_{xy}$, $\bar{\bar{\tau}}_{yz}$ obtained from (3.3), (3.5) and (2.3) are

$$\bar{\bar{\sigma}}_x /2\mu = \left[\, \xi^2M^2(1-2/\alpha^2) + 2\xi^2 \,\right] A_2e^{-s_1 y}/2s_1 - i\xi B_1e^{-s_2 y},$$

$$\bar{\bar{\sigma}}_y /2\mu = \left[\, \xi^2M^2- 2(\xi^2+\zeta^2)\right] A_2e^{-s_1 y}/2s_1 + i(\xi B_1+\zeta B_3)e^{-s_2 y},$$

$$\bar{\bar{\tau}}_{xy} /2\mu = -i\xi A_2\, e^{-s_1 y} - \left[\, \xi\zeta B_3 +(s_2^2+\xi^2)B_1\right] e^{-s_2 y}/2s_2$$

$$\bar{\bar{\tau}}_{yz} /2\mu = -i\zeta A_2\, e^{-s_1 y} - \left[\, \xi\zeta B_1 +(s_2^2+\zeta^2)B_3\right] e^{-s_2 y}/2s_2 \qquad (3.6)$$

Using the conditions (2.4.3) B_1 and B_3 can be expressed in terms of

A_2 as

$$B_1 = \frac{-2i\xi s_2 A_2}{(2-M^2)\xi^2+2\zeta^2}$$

$$B_3 = \frac{-2i\zeta s_2 A_2}{(2-M^2)\xi^2+2\zeta^2} \tag{3.7}$$

Hence we find that all the components of stress and displacement can be expressed in terms of the unknown function $A_2(\xi,\zeta)$. Now insertion of (3.5) and (3.7) in $\bar{\bar{v}}$ given in (3.3) yields

$$A_2 = -\frac{(2-M^2)\xi^2+2\zeta^2}{M^2\xi^2}\,\bar{\bar{v}}_0 \tag{3.8}$$

where $\bar{\bar{v}}_0$ is the transformed displacement on y=0.

Using (3.7) and (3.8) we obtain from (3.6)

$$\bar{\bar{\sigma}}_x/2\mu = -\bar{\bar{v}}_0\left[\left\{(2-M^2)\xi^2+2\zeta^2\right\}\left\{2+M^2(1-2/\alpha^2)\right\}\frac{e^{-s_1y}}{2M^2s_1} - 2s_2\frac{e^{-s_2y}}{M^2}\right],$$

$$\bar{\bar{\sigma}}_y/2\mu = \bar{\bar{v}}_0\left[\left\{(2-M^2)\xi^2+2\zeta^2\right\}^2\frac{e^{-s_1y}}{2M^2s_1\xi^2} - 2(\xi^2+\zeta^2)s_2\frac{e^{-s_2y}}{M^2\xi^2}\right],$$

$$\bar{\bar{\tau}}_{xy}/2\mu = i\xi\bar{\bar{v}}_0\left\{(2-M^2)\xi^2+2\zeta^2\right\}\frac{e^{-s_1y}-e^{-s_2y}}{M^2\xi^2},$$

$$\bar{\bar{\tau}}_{yz}/2\mu = i\zeta\bar{\bar{v}}_0\left\{(2-M^2)\xi^2+2\zeta^2\right\}\frac{e^{-s_1y}-e^{-s_2y}}{M^2\xi^2}. \tag{3.9}$$

Using the conditions (2.4.1) and (2.4.2) we obtain the following triple integral equations

$$\bar{\sigma}_y/2\mu = (2\pi)^{-1}\int_{-\infty}^{\infty}\bar{\bar{v}}_0\,G(\xi,\zeta)\,e^{-i\xi x}\,d\xi = -\bar{p}(x,\zeta),\ \text{for } a<|x|<b$$

and

$$\bar{v}_o = (2\pi)^{-1} \int_{-\infty}^{\infty} \bar{\bar{v}}_o \; e^{-i\xi x} \, d\xi = 0, \text{ for } |x|>b, \ |x|<a. \tag{3.10}$$

with

$$G(\xi,\zeta) = \frac{1}{2M^2\xi^2 s_1} \left[((2-M^2)\xi^2+2\zeta^2)^2 - 4(\xi^2+\zeta^2)s_1 s_2 \right] \tag{3.11}$$

Taking p(x,z) as the even function of x, the solution may be assumed as

$$\bar{v}_o(x,\zeta) = \sum_{n=1}^{\infty} c_n(\zeta) \frac{(-1)^{n+1}}{n} \sin\left[n \cos^{-1}\left\{ \frac{a+b-2|x|}{b-a} \right\}\right], \text{ for } a \leq |x| \leq b$$

$$= 0, \text{ for } 0 \leq |x| < a, \ |x|>b, \tag{3.12}$$

where $c_n(\zeta)$ are the unknown functions to be determined.

Applying Fourier tranformation on (3.12) and using the result

$$\int_a^b \sin\left[n \cos^{-1}\left\{\frac{a+b-2x}{b-a}\right\}\right] \cos(\xi x)dx = (-1)^{n+1} \frac{n\pi}{\xi} \sin\left[\frac{a+b}{2}\xi - \frac{n\pi}{2}\right] J_n\left(\frac{b-a}{2}\xi\right)$$

we obtain

$$\bar{\bar{v}}_o(\xi,\zeta) = 2\pi\xi^{-1} \sum_{n=1}^{\infty} c_n(\zeta) \sin\left[\frac{a+b}{2}\xi - \frac{n\pi}{2}\right] J_n\left(\frac{b-a}{2}\xi\right), \tag{3.13}$$

where $J_n()$ are Bessels functions.

Insertion of the expression (3.13) in the first equation of (3.10) yields

$$2 \sum_{n=1}^{\infty} c_n(\zeta) \int_0^{\infty} \frac{G(\xi,\zeta)}{\xi} \sin\left[\frac{a+b}{2}\xi - \frac{n\pi}{2}\right] J_n\left(\frac{b-a}{2}\xi\right) \cos(\xi x) d\xi = -\bar{p}(x,\zeta),$$

$$\text{for } a<x<b. \tag{3.14}$$

Using the following results [Gradshteyn and Ryzhik (1965)]

$$\int_0^{\infty} \cos(a_1\xi)\, J_n(a_2\xi)\,d\xi = \frac{\cos(n\varepsilon)}{\sqrt{a_2^2 - a_1^2}}, \qquad \text{for } a_2 > a_1 > 0$$

$$= \frac{a_2^n \sin(n\pi/2)}{\sqrt{a_1^2 - a_2^2}\,\left[a_1 + \sqrt{a_1^2 - a_2^2}\,\right]^n}, \quad \text{for } a_1 > a_2 > 0,$$

and

$$\int_0^{\infty} \sin(a_1\xi)\, J_n(a_2\xi)\,d\xi = \frac{\sin(n\varepsilon)}{\sqrt{a_2^2 - a_1^2}}, \qquad \text{for } a_2 > a_1 > 0$$

$$= \frac{a_2^n \cos(n\pi/2)}{\sqrt{a_1^2 - a_2^2}\,\left[a_1 + \sqrt{a_1^2 - a_2^2}\,\right]^n}, \quad \text{for } a_1 > a_2 > 0,$$

where $\varepsilon = \sin^{-1}(a_1/a_2)$

in (3.14) we obtain,

$$\sum_{n=1}^{\infty} \bar{c}_n(\zeta) \Bigg[\Bigg[\int_0^{\infty} \left\{ \frac{G(\xi,\zeta)}{\xi} - \frac{G(\delta,\zeta)}{\delta} \right\} \Bigg[\cos(n\pi/2)\left\{\sin\left(\frac{a+b+2x}{2}\,\xi\right) + \sin\left(\frac{a+b-2x}{2}\,\xi\right)\right\}$$

$$- \sin(n\pi/2)\left\{\cos\left(\frac{a+b+2x}{2}\,\xi\right) + \cos\left(\frac{a+b-2x}{2}\,\xi\right)\right\}\Bigg] J_n\left(\frac{b-a}{2}\,\xi\right) d\xi + \frac{G(\delta,\zeta)}{\delta} \times$$

$$\times \left[\left(\frac{b-a}{2}\right)^n \Big/ \left[\sqrt{\left(\frac{a+b+2x}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2}\, \left\{\frac{a+b+2x}{2} + \sqrt{\left(\frac{a+b+2x}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2}\, \right\}^n\right] + \right.$$

$$\left. + \sin\left\{n \sin^{-1}\left\{\frac{a+b-2x}{b-a}\right\} - \frac{n\pi}{2}\right\} \Big/ \sqrt{\left(\frac{b-a}{2}\right)^2 - \left(\frac{a+b-2x}{2}\right)^2}\,\right]\Bigg]\Bigg] = -\bar{p}(x,\zeta),$$

(3.16)

where

$$\mathop{\mathrm{Lt}}_{\xi \to \infty} \frac{G(\xi,\zeta)}{\xi} = \frac{G(\delta,\zeta)}{\delta} = \left[(2-M^2)^2-4\sqrt{1-M^2}\,\sqrt{1-M^2/\alpha^2}\right]/2M^2\sqrt{1-M^2/\alpha^2} \qquad (3.17)$$

since the function $\frac{G(\xi,\zeta)}{\xi} - \frac{G(\delta,\zeta)}{\delta}$ behaves as ξ^{-2} for large ξ, the semi-infinite integral on the left hand side of (3.16) can easily be evaluated by Filon's method.

To solve (3.16) for unknown coefficients $c_n(\zeta)$ we adopt the Schmidt method (1958) and write (3.16) as

$$\sum_{n=1}^{\infty} c_n(\zeta)\, F_n(\zeta,x) = -f(\zeta,x), \qquad \text{for } a<x<b, \qquad (3.18)$$

where $F_n(\zeta,x)$ and $f(\zeta,x) = \bar{p}(\zeta,x)$ are known functions. Let $H_n(\zeta,x)$'s be a set of orthogonal functions which satisfy

$$\int_a^b H_n(\zeta,x)H_m(\zeta,x)\,dx = N_n\delta_{nm},$$

where $$N_n = \int_a^b H_n^2(\zeta,x)\,dx \qquad (3.19)$$

Then $H_n(\zeta,x)$'s can be constructed from the functions $F_n(\zeta,x)$ in the following way

$$H_n(\zeta,x) = \sum_{i=1}^{\infty} \frac{C_{in}}{C_{nn}}\, F_i(\zeta,x) \qquad (3.20)$$

with C_{in} as the cofactor of the element e_{in} of D_n which is defined as

$$D_n = \begin{vmatrix} e_{11}, e_{12} \cdots\cdots\cdots\cdots e_{1n} \\ e_{21}, e_{22} \cdots\cdots\cdots\cdots e_{2n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ e_{n1}, e_{n2} \cdots\cdots\cdots\cdots e_{nn} \end{vmatrix}, \quad e_{in} = \int_a^b F_n(\zeta, x) F_i(\zeta, x)\, dx. \qquad (3.21)$$

Now in terms of the set of orthogonal functions $H_n(\zeta, x)$, the function $-f(\zeta, x)$ can be expressed as

$$-f(\zeta, x) = \sum_{i=1}^{\infty} h_i H_i(\zeta, x) \qquad (3.22)$$

Substituting values of $H_n(\zeta, x)$ from (3.20) into (3.22), we obtain from (3.18) after some rearrangement

$$\sum_{n=1}^{\infty} c_n(\zeta)\ F_n(\zeta, x) = \sum_{n=1}^{\infty} F_n(\zeta, x) \sum_{i=n}^{\infty} h_i \frac{C_{ni}}{C_{ii}} \qquad (3.23)$$

Comparing the coefficients of $F_n(\zeta, x)$ from both sides of (3.23) we find

$$c_n = \sum_{i=n}^{\infty} h_i \frac{C_{ni}}{C_{ii}} \qquad (3.24)$$

where $$h_i = -\frac{1}{N_i} \int_a^b f(\zeta, x)\ H_i(\zeta, x)\ dx \qquad (3.25)$$

4. Stress intensity factors and crack opening displacement

To evaluate the stress intensity factors at the vicinity of the crack ends we put $x=b+r\cos\theta$. $y=r\sin\theta$ for the stress intensity factor at the outer edge and $x=a-r\cos\theta$, $y=r\sin\theta$ for the stress intensity factor at the inner edge.

The required stress σ_θ given by

$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta \cos\theta \qquad (4.1)$$

is to be evaluated for small values of r.

Using asymptotic values of $J_n\left(\frac{b-a}{2}\xi\right)$ for large value of ξ, we obtain

$$\sin\left[\frac{a+b}{2}\xi - \frac{n\pi}{2}\right] J_n\left(\frac{b-a}{2}\xi\right)\cos(\xi x) = \frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4\pi\xi\,(b-a)}}\left[\cos\frac{n\pi}{2}\left\{\sin(b-x)\xi - \sin(x-a)\xi\right.\right.$$

$$\left.+\left[\cos(x-a)\xi - \cos(b-x)\xi\right]\tan\left(\frac{2n+1}{\pi}\right)\right\} - \sin\frac{n\pi}{2}\left\{\cos(b-x)\xi + \cos(x-a)\xi +\right.$$

$$\left.\left.+\left[\sin(x-a)\xi + \sin(b-x)\xi\right]\tan\left(\frac{2n+1}{\pi}\right)\right\}\right].$$

Further using the following results [Gradhsteyn and Ryzhik (1965)]

$$\int_0^\infty x^{\mu-1} e^{-\beta x} \sin(\delta x)\, dx = \frac{\Gamma(\mu)}{(\beta^2+\delta^2)^{\mu/2}} \sin\left[\mu \tan^{-1}\left(\frac{\delta}{\beta}\right)\right], \quad \mu>-1,\ \beta>0$$

$$\int_0^\infty x^{\mu-1} e^{-\beta x} \cos(\delta x)\, dx = \frac{\Gamma(\mu)}{(\beta^2+\delta^2)^{\mu/2}} \cos\left[\mu \tan^{-1}\left(\frac{\delta}{\beta}\right)\right], \quad \mu>0\ ,\ \beta>0$$

it is found that for small values of r

$$\int_0^\infty e^{-\sqrt{1-q^2}\,\xi y} \sin\left[\frac{a+b}{2}\xi - \frac{n\pi}{2}\right] J_n\left(\frac{b-a}{2}\xi\right)\cos(\xi x)\,d\xi = -\frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4r(b-a)}} \times$$

$$\left[\cos(n\pi/2)\sqrt{\frac{(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}} + \sin(n\pi/2)\sqrt{\frac{-(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right]$$

$$+ O(r^0), \quad \text{for } x>b \qquad (4.2)$$

$$= \frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4r(b-a)}} \left[\cos(n\pi/2)\sqrt{\frac{(-1)^n\cos\theta + \sqrt{1-q^2\sin^2\theta}}{1\ -q^2\sin^2\theta}}\ -\right.$$

$$\left. -\ \sin(n\pi/2)\sqrt{\frac{-(-1)^n\cos\theta + \sqrt{1-q^2\sin^2\theta}}{1\ -q^2\sin^2\theta}}\right] + O(r^0),\ \text{for}\ x<a \qquad (4.3)$$

and

$$\int_0^\infty e^{-\sqrt{1-q^2}\,\xi y} \sin\left[\frac{a+b}{2}\xi - \frac{n\pi}{2}\right] J_n\left(\frac{b-a}{2}\xi\right)\sin(\xi x)\,d\xi = -\frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4r(b-a)}} \times$$

$$\left[\cos(n\pi/2)\sqrt{\frac{-(-1)^n\cos\theta + \sqrt{1-q^2\sin^2\theta}}{1\ -q^2\sin^2\theta}} - \sin(n\pi/2)\sqrt{\frac{(-1)^n\cos\theta + \sqrt{1-q^2\sin^2\theta}}{1\ -q^2\sin^2\theta}}\right]$$

$$+\ O(r^0),\ \text{for}\ x>b \qquad (4.4)$$

$$= \frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4r(b-a)}} \left[\cos(n\pi/2)\sqrt{\frac{-(-1)^n\cos\theta + \sqrt{1-q^2\sin^2\theta}}{1\ -q^2\sin^2\theta}}\ +\right.$$

$$\left. +\ \sin(n\pi/2)\sqrt{\frac{(-1)^n\cos\theta + \sqrt{1-q^2\sin^2\theta}}{1\ -q^2\sin^2\theta}}\right] + O(r^0),\ \text{for}\ x<a \qquad (4.5)$$

Inserting (3.13) into (3.9) and taking inverse Fourier transform of (3.9) we obtain the stress intensity factor at $x=b$ with the aid of (4.2)-(4.5) as

$$K_b = \frac{\sigma_\theta}{2\mu}\sqrt{r}\ \Big|_{r\to 0} = \sum_{n=1}^{\infty}\frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{b\ -\ a}}\left[\ \frac{2\ -\ M^2}{2M^2\sqrt{1-M^2/\alpha^2}}\ Q_1^+\left\{(2+M^2(1-2/\alpha^2))\sin^2\theta-\right.\right.$$

$$- (2 - M^2)\cos^2\theta\Big\} + \frac{2\sqrt{1-M^2}\,\cos 2\theta}{M^2}\, Q_2^+ - \frac{2-M^2}{M^2}\,(P_1^- - P_2^-)\,\sin 2\theta\Big]\times$$

$$\times\,\frac{1}{2\pi}\int_{-\infty}^{\infty} c_n(\zeta)\,e^{-i\zeta z}\,d\zeta \tag{4.6}$$

and also the stress intensity factor at x=a is found to be

$$K_a = \frac{\sigma_\theta}{2\mu}\sqrt{r}\,\Big|_{r\to 0} = \sum_{n=1}^{\infty}\frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{b-a}}\Big[\,\frac{-(2-M^2)}{2M^2\sqrt{1-M^2/\alpha^2}}\,Q_1^-\Big\{(2+M^2(1-2/\alpha^2))\sin^2\theta -$$

$$- (2 - M^2)\cos^2\theta\Big\} - \frac{2\sqrt{1-M^2}\,\cos 2\theta}{M^2}\, Q_2^- - \frac{2-M^2}{M^2}\,(P_1^+ - P_2^+)\,\sin 2\theta\Big]\times$$

$$\times\,\frac{1}{2\pi}\int_{-\infty}^{\infty} c_n(\zeta)\,e^{-i\zeta z}\,d\zeta \tag{4.7}$$

where

$$\left.\begin{aligned} Q_i^{\pm} &= \Big[\cos(n\pi/2)\sqrt{q_i+(-1)^n\cos\theta} \pm \sin(n\pi/2)\sqrt{q_i-(-1)^n\cos\theta}\,\Big]/q_i \\ P_i^{\pm} &= \Big[\cos(n\pi/2)\sqrt{q_i-(-1)^n\cos\theta} \pm \sin(n\pi/2)\sqrt{q_i+(-1)^n\cos\theta}\,\Big]/q_i \end{aligned}\right\}\quad i=1,2$$

and

$$q_1 = \sqrt{1-M^2\alpha^{-2}\sin^2\theta}$$

$$q_2 = \sqrt{1-M^2\sin^2\theta}$$

It is to be noted that in (4.6) $\theta = \tan^{-1} y/(x-b)$ whereas in (4.7) it is given by $\theta = \tan^{-1} y/(a-x)$.

Taking Fourier inversion of (3.12) we obtain the crack surface displacement as

$$v_0(x,z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left[n \cos^{-1}\left\{\frac{a+b-2|x|}{b-a}\right\}\right] \frac{1}{2\pi} \int_{-\infty}^{\infty} c_n(\zeta) e^{-i\zeta z} d\zeta$$

$$\text{for } a \le |x| \le b \qquad (4.8)$$

5. Numerical discussions

In order to evaluate the stress intensity factors and crack surface displacement we take the function $p(x,z)$ as

$$p(x,z) = \frac{P}{1+d^2 z^2},$$

where d governs the distribution of the applied force and P is a constant. Numerical calculations have been done taking $\lambda=\mu$ and $d=1$. The semi infinite integral in (3.16) is evaluated by Filon's method as the integral converges rapidly because of the rapid decay of the function

$$\left\{\frac{G(\xi,\zeta)}{\xi} - \frac{G(\delta,\zeta)}{\delta}\right\}$$

with the increase in ξ. Adopting the first seven terms of the infinite series given in the left hand side of (3.18) we used the Schmidt method to determine the coefficients $c_n(\zeta)$. For the check of accuracy the value of $\sum_{n=1}^{7} c_n(\zeta) F_n(\zeta,x)/Pb$ and $-f(\zeta,x)/Pb$ are given in Table 1 for ζb = 0.0, 0.2, M=0.4 and for a/b = 0.3, 0.4.

Table 1.

ζb	a/b	x/b	$\sum_{n=1}^{7} c_n(\zeta)F_n(\zeta,x)/Pb$	$-f(\zeta,x)/Pb$
0.0	0.3	0.3	-3.140993	-3.140994
		0.4	-3.140995	
		0.5	-3.140993	
		0.6	-3.140996	
		0.7	-3.140991	
		0.8	-3.140994	
		0.9	-3.140993	
		1.0	-3.140992	
	0.4	0.4	-3.140995	
		0.5	-3.140994	
		0.6	-3.140994	
		0.7	-3.140994	
		0.8	-3.140994	
		0.9	-3.140995	
		1.0	-3.140994	
0.2	0.3	0.3	-2.572111	-2.572113
		0.4	-2.572113	
		0.5	-2.572111	
		0.6	-2.572116	
		0.7	-2.572110	
		0.8	-2.572113	
		0.9	-2.572108	
		1.0	-2.572106	
	0.4	0.4	-2.572114	
		0.5	-2.572114	
		0.6	-2.572114	
		0.7	-2.572113	
		0.8	-2.572113	
		0.9	-2.572113	
		1.0	-2.572113	

Table 2.

ζb	$c_1(\zeta)$	$c_2(\zeta)$	....	$c_7(\zeta)$
0.0	-0.165871×10^{1}	-0.923569×10^{-4}	....	-0.759039×10^{-8}
0.2	-0.135194×10^{1}	-0.734980×10^{-4}	....	0.105638×10^{-6}
0.4	-0.109342×10^{1}	-0.556495×10^{-4}	....	0.357814×10^{-6}
....	....	....	....	....
....	....	....	....	....
3.0	-0.578184×10^{-3}	-0.601254×10^{-7}	....	0.114694×10^{-5}
4.0	-0.182994×10^{-3}	0.883491×10^{-7}	....	0.659423×10^{-6}
5.0	-0.573139×10^{-4}	0.489839×10^{-7}	....	0.342023×10^{-6}
....	....	....	....	....
....	....	....	....	....
9.6	0.366305×10^{-5}	-0.816894×10^{-8}		-0.907244×10^{-7}
9.8	0.362848×10^{-5}	-0.829789×10^{-8}		-0.938769×10^{-7}
10.0	0.358409×10^{-5}	-0.843117×10^{-8}		-0.967438×10^{-7}

From Table 1 it is clear that the Schmidt method is carried out satisfactorily. The values of $c_n(\zeta)$ are given in Table 2 for M=0.4, a/b =0.4.

The variation of stress intensity factor at the outer edge and at the inner edge with M is shown in Fig.2. and Fig.3. respectively for $\theta = 0°, 18°, 36°$ and a/b = 0.2, 0.3, 0.4. Fig.2 depicts the fact that the value of stress intensity factor at the outer edge decreases with the increase in the values of a/b, whereas from Fig.3 it is evident that the stress intensity factor at the inner edge is of an opposite character. It increases with the increases in the values of a/b.

The variations of stress intensity factors both at the inner edge and outer edge with z have been presented in Figs. 4-7 for different values of a/b, M and θ. The values of the stress intensity factor in all the cases are found to decrease gradually with the increase in the values of z, which is expected from physical stand point.

The variation of stress intensity factor corresponding to the circumferential stress σ_θ given by (4.1) with θ at both the crack tips has been shown in in Figs. 8-12 for different values of a/b and M.

It is known that there are several factors which contribute to crack curving and branching. One factor, of course, is based upon the criterion that a crack may propagate in a direction normal to the maximum tensile stress and it is interesting to note from Fig.8 and Fig.10, there is the possibility of curving and branching of the cracks at the outer edge at very low velocities of the cracks whereas from Fig.9, Fig.11 and Fig.12 it is clear that for a/b = 0.3, the crack tends to become curved at the inner edge for values of M about 0.65.

Finally the crack opening displacement in the plane z=0 has been shown by means of graphs in Figs. 15-16 for different values of a/b and M. The variation of crack opening displacement with z for some fixed values of M and a/b has been depicted in Figs.13-14.

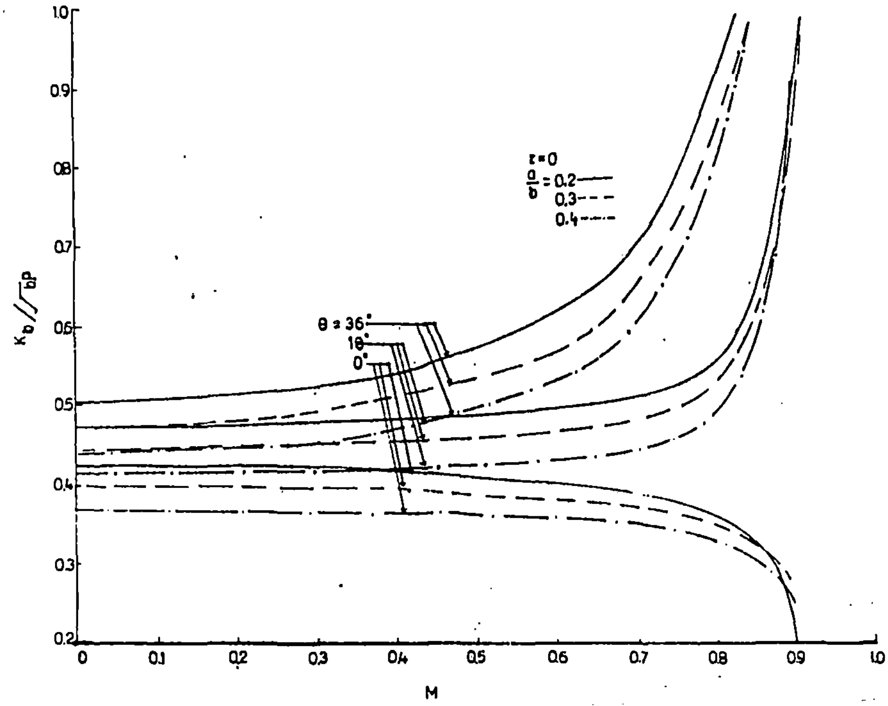


Fig. 2: Variation of stress intensity factor at the outer edge with M.

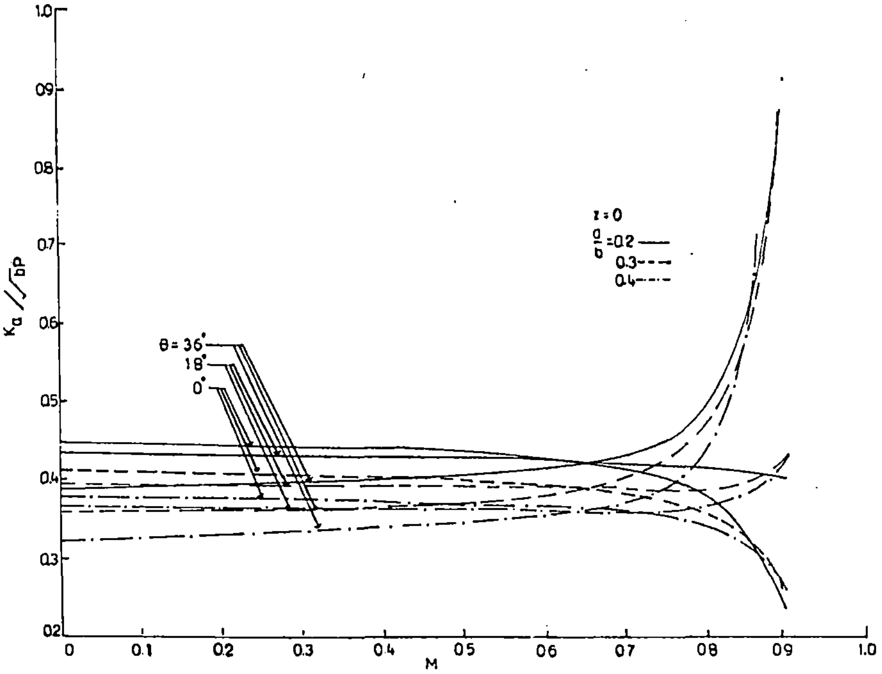


Fig.3: Variation of stress intensity factor at the inner edge with M.

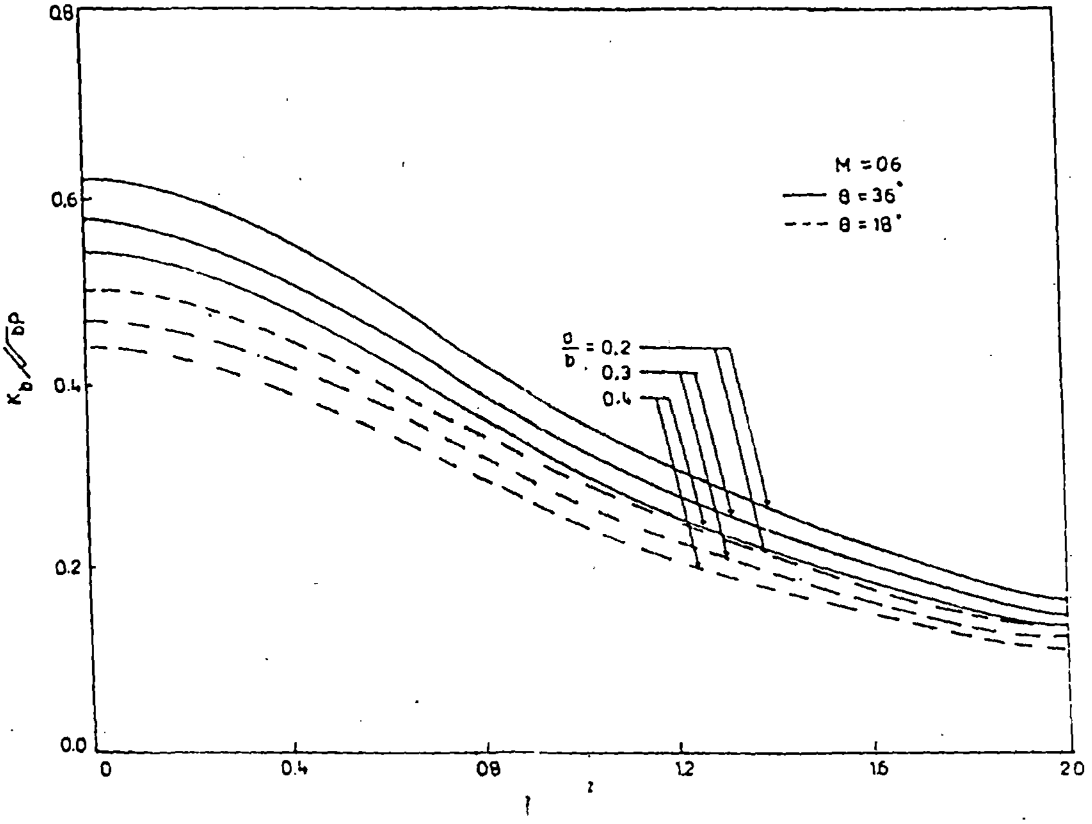


Fig. 4: Stress intensity factor at the outer edge vs z.

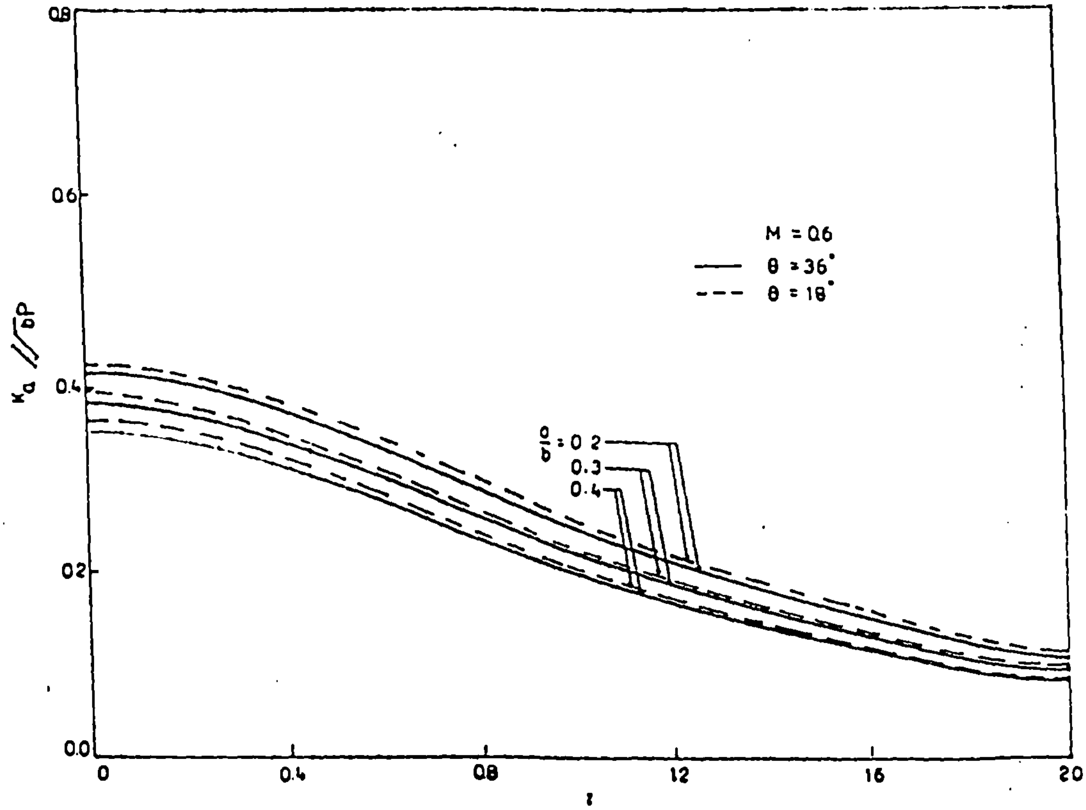


Fig. 5: Stress intensity factor at the inner edge vs z.

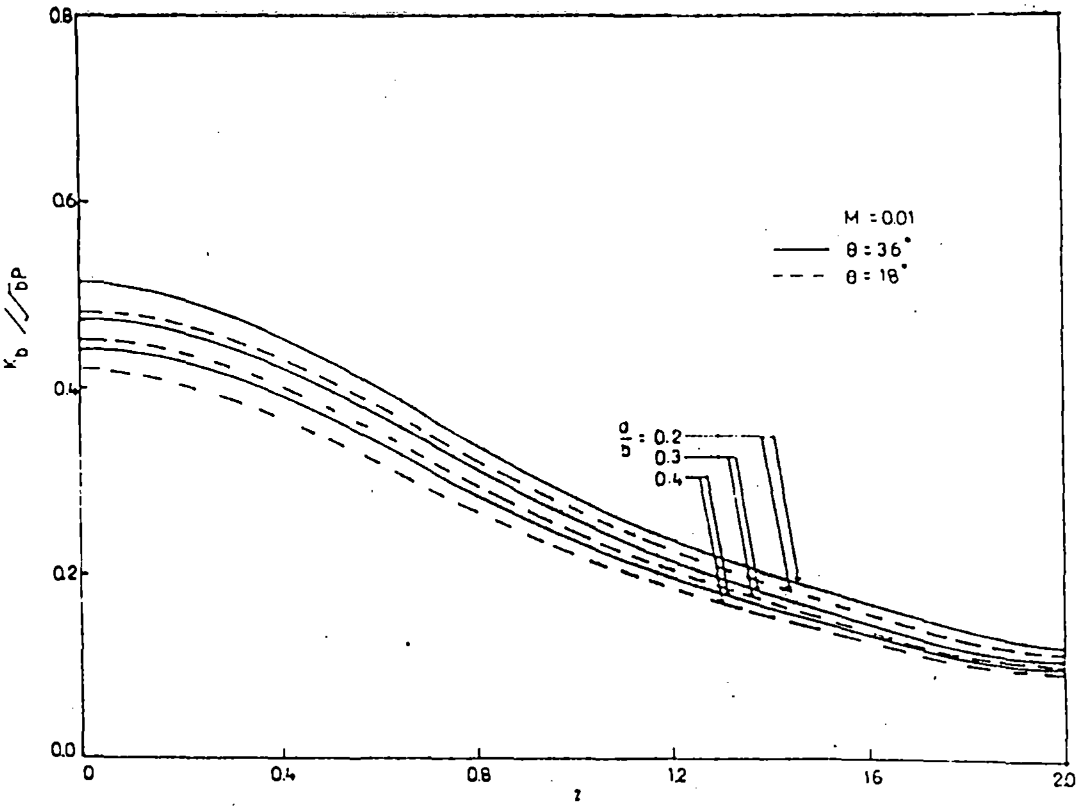


Fig. 6: Stress intensity factor at the outer edge vs z.

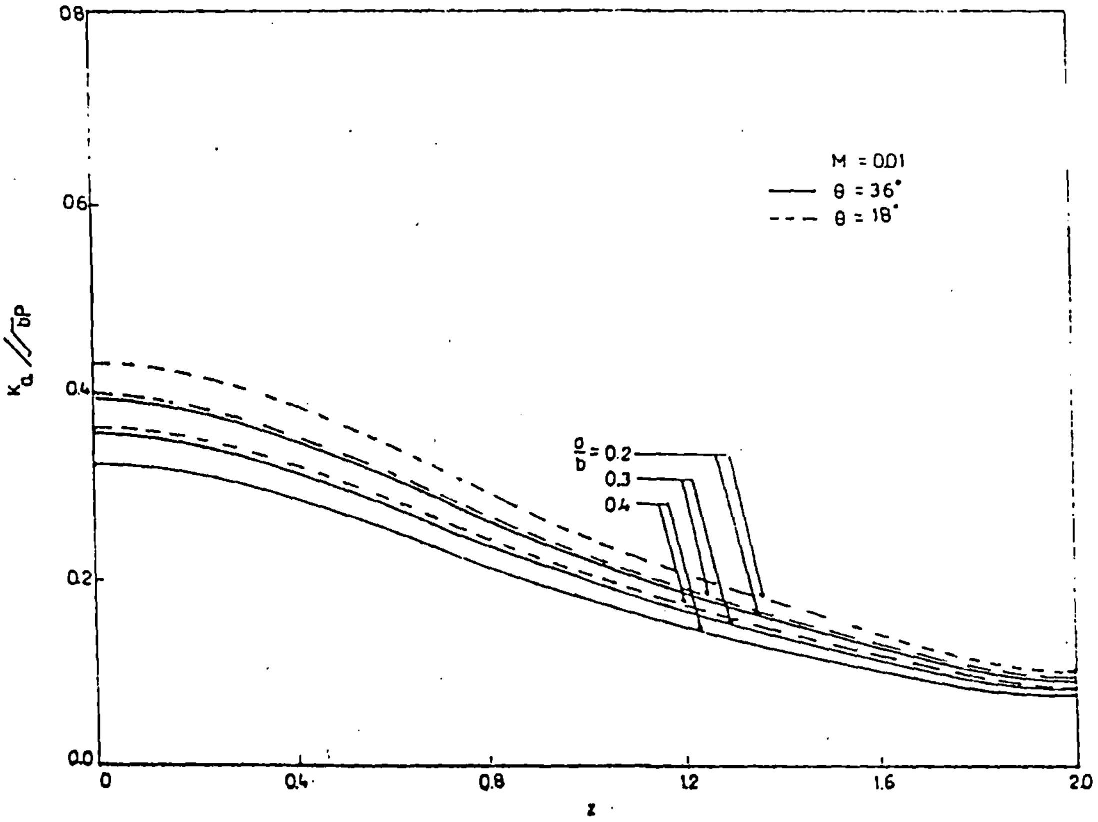


Fig.7: Stress intensity factor at the inner edge vs z.

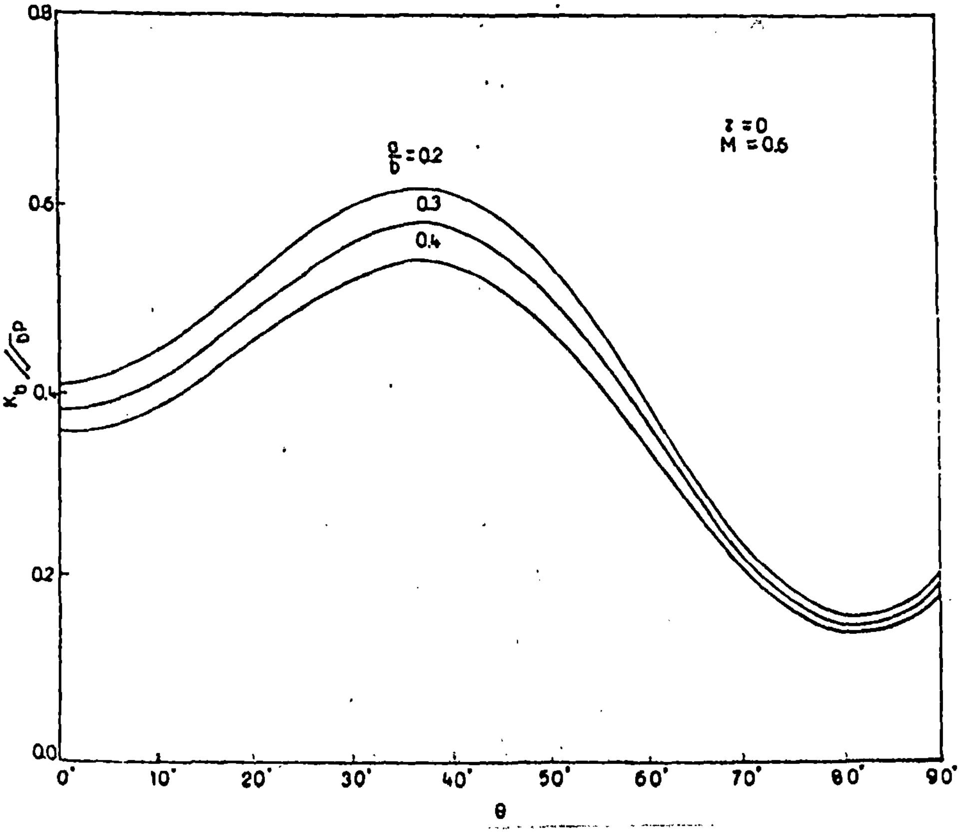


Fig. 8: Variation of stress intensity factor at the outer edge with θ.

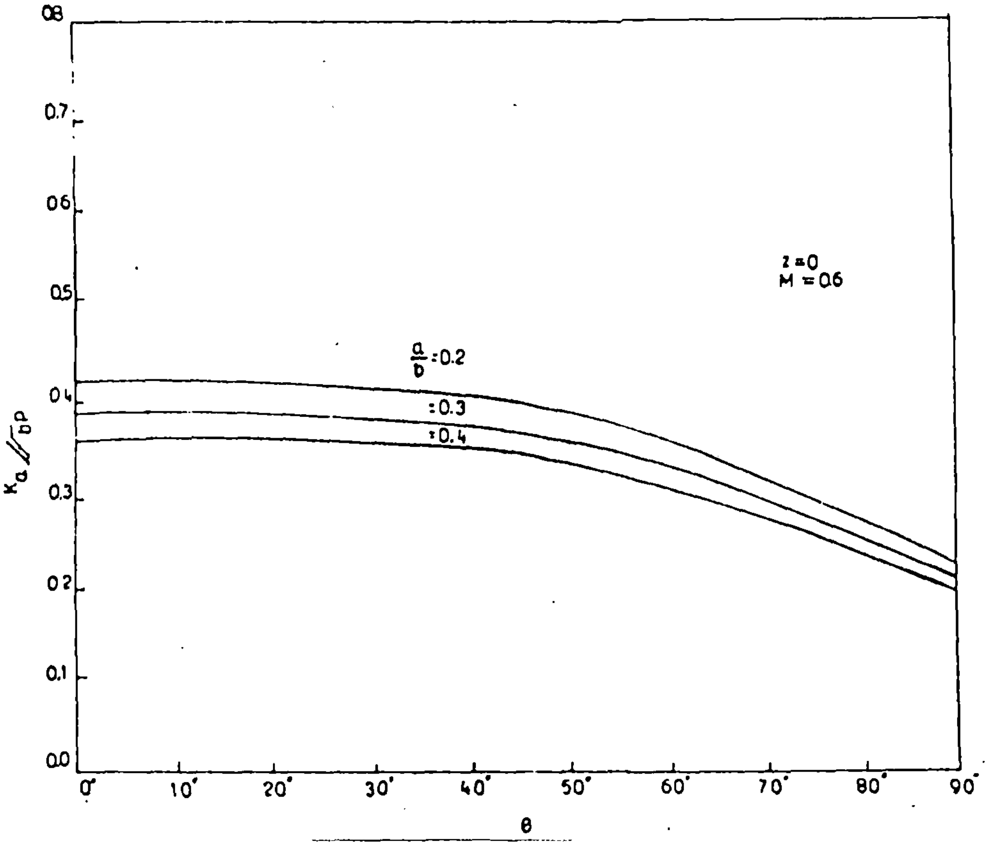


Fig. 9: Variation of stress intensity factor at the outer edge with θ.

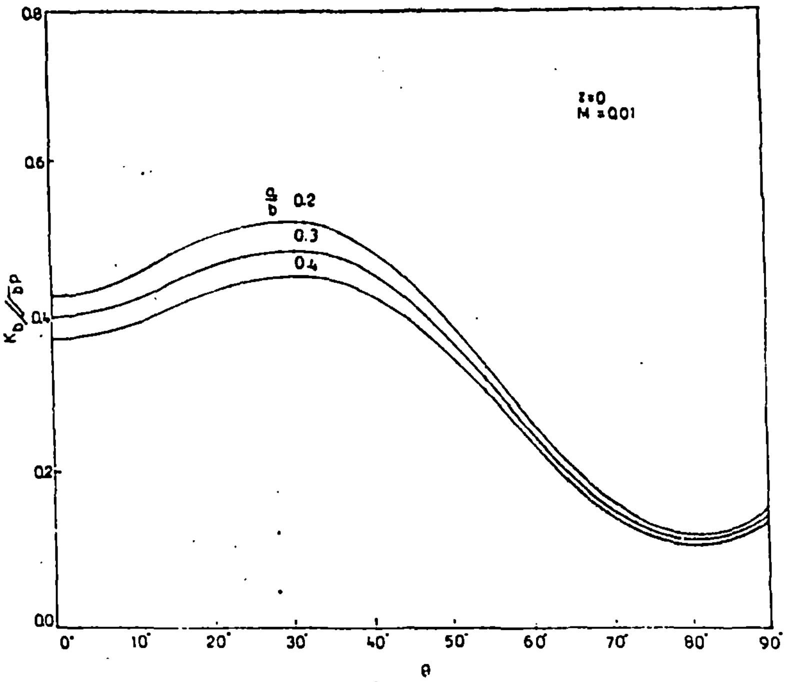


Fig.10: Variation of stress intensity factor at the outer edge with θ.

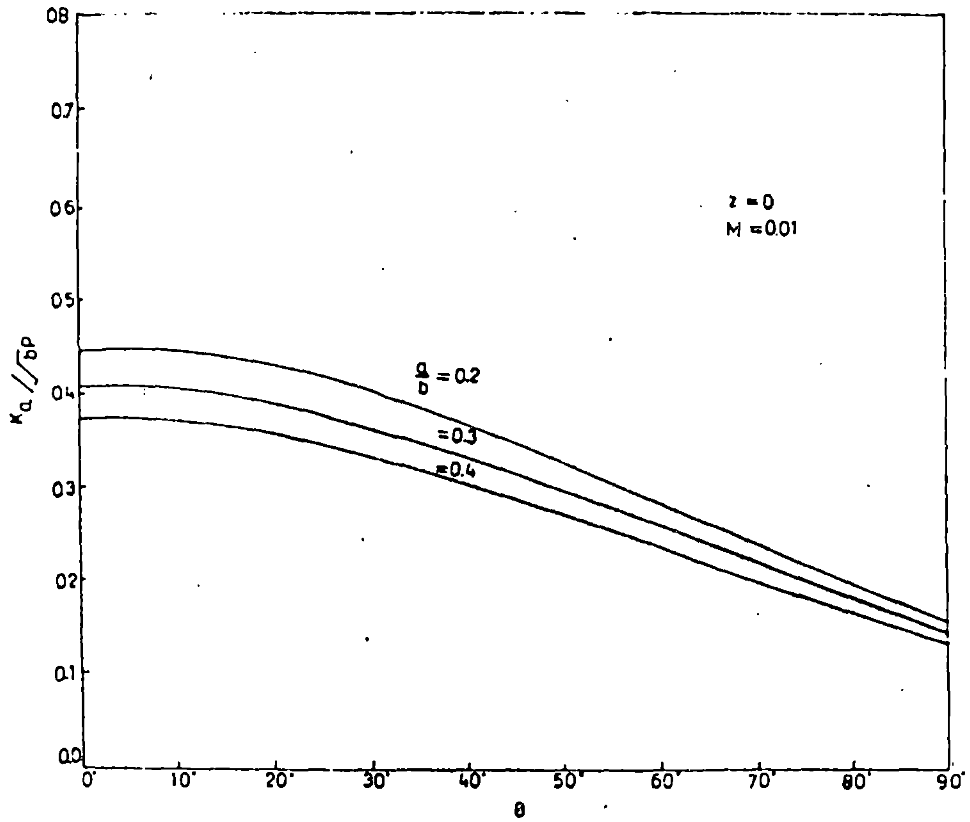


Fig.11: Variation of stress intensity factor at the outer edge with θ.

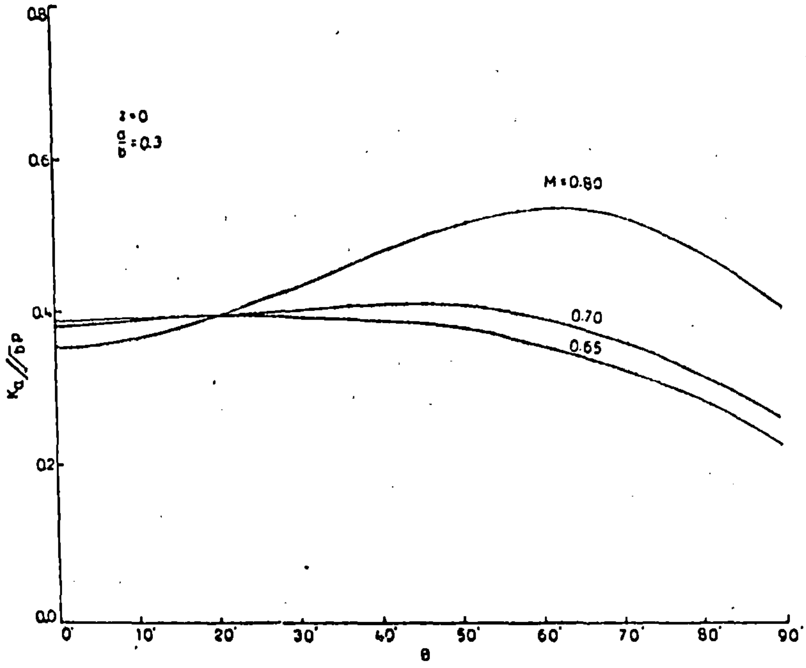


Fig.12: Variation of stress intensity factor at the outer edge with θ.

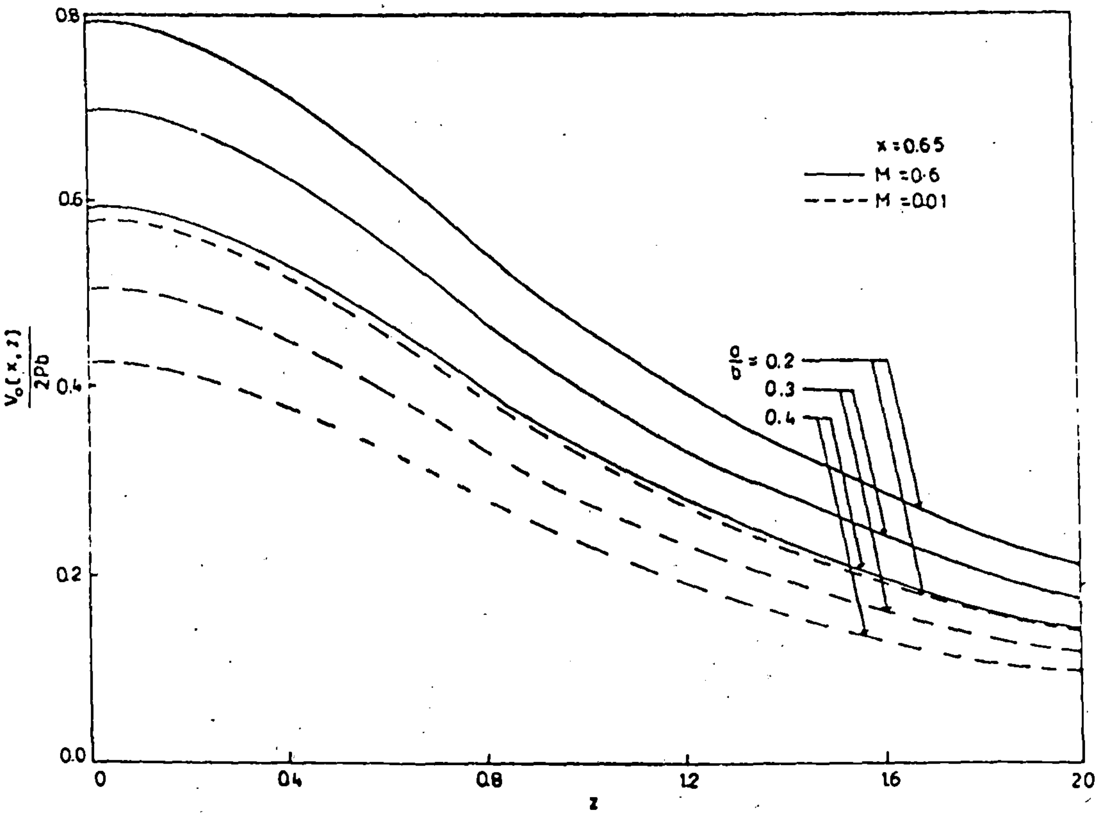


Fig. 13: Crack opening displacement vs z.

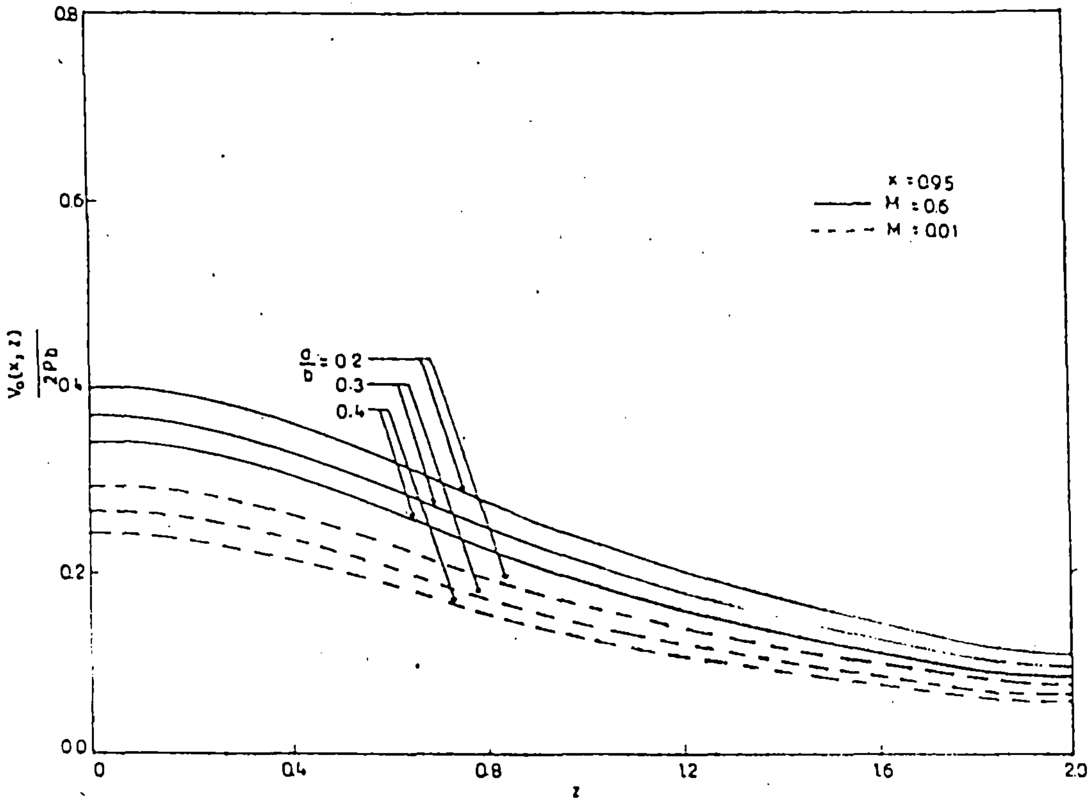


Fig.14: Crack opening displacement vs z.

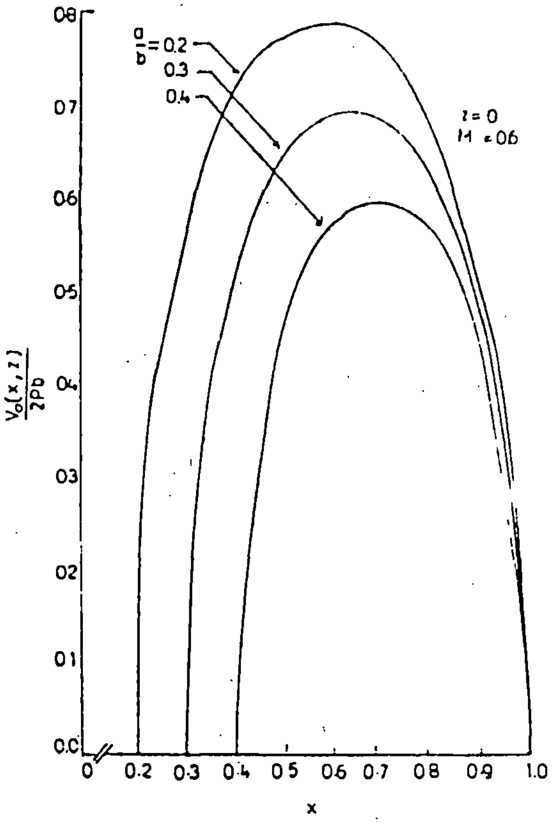


Fig. 15: Variation of crack opening displacement with x.

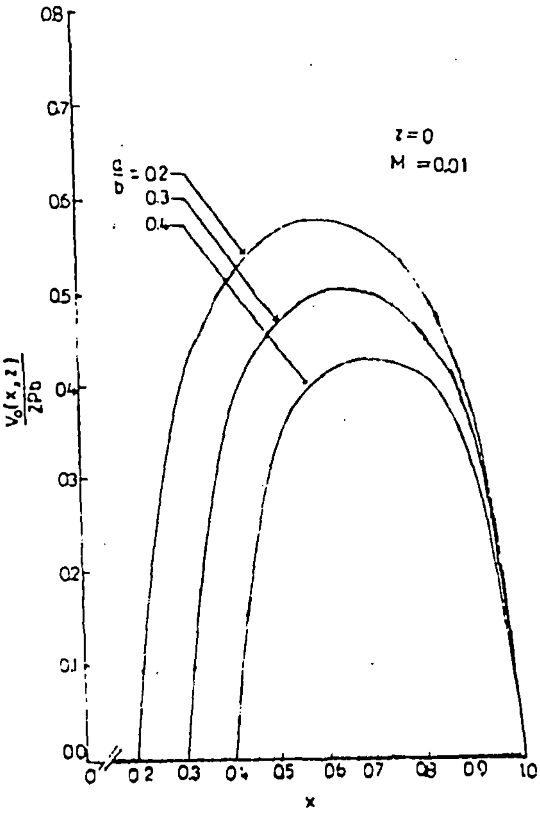


Fig. 16: Variation of crack opening displacement with x.

THREE CO-PLANAR MOVING GRIFFITH CRACKS IN AN INFINITE ELASTIC MEDIUM

1. Introduction

In fracture mechanics, scattering of elastic waves by cracks of finite dimension in an infinite elastic medium has been investigated by several investigators.The problem of scattering of elastic wave from an interface crack was solved by Bostrom (1987). Srivastava et. al. (1980) solved the problem of interaction of anti-plane shear wave by an interface crack. The problem of diffraction of Love waves by a crack of finite width in the plane interface of a layered composite has been solved by Neerhoff (1979). Itou (1980) solved problem of diffraction of anti-plane shear wave by two co-planar Griffith cracks in an infinite elastic medium. The scattering of time harmonic normally incident plane wave by two co-planar Griffith cracks was solved by Jain and Kanwal (1972). Itou (1978) also solved the problem of stress concentration around two co-planar Griffith cracks in an infinite elastic medium. Yoffe (1951) considered the problem of propagation of a crack of fixed length at a constant speed through a stretched isotropic elastic solid of infinite extent. The problem of diffraction of horizontal shear waves by a moving interface crack has been solved by Nishida et. al. (1984). Recently Kassir and Tse (1983) have solved the plane stress problem of a moving Griffith crack in an infinite orthotropic stressed medium by using integral transform technique and the same technique has been employed by De and Patra (1990) to solve Yoffe's problem in a stressed orthotropic strip of finite thickness.

As regards the crack problem, research has been restricted mainly to the case of single crack or a pair of cracks because of severe mathematical complexity encountered in solving the problems of three or more cracks. Recently, Dhawan and Dhaliwal (1978)solved the statical problem of determining the stress distribution in an infinite transversely isotropic medium containing three co-planar cracks

---

Published in " International Journal Of Fracture" Vol 59, pp 291-306, 1992

To the best knowledge of the author, the problem of stress distribution around three co-planar moving Griffith cracks in an infinite isotropic elastic medium has not been investigated so far. In this paper, two cases regarding stress distribution around three co-planar Griffith cracks in an infinite homogeneous, isotropic medium have been investigated. In the first case, cracks are assumed to be moving steadily along a fixed direction with constant velocity V. In the second case, the statical problem of determining the stress and displacement in an infinite homogeneous, isotropic medium weakened by three co-planar Griffith cracks has been considered. Using Fourier integral transform both the problems have been reduced to solving a set of four integral equations. Employing finite Hilbert transform technique (1968) and Cook's result (1970) the integral equations have been solved to derive crack opening displacement and stress intensity factors which are presented in the form of graphs.

2. Statement Of Problem I And Its Formulation

Consider an infinite homogeneous isotropic material weakened by three co-planar Griffith cracks, moving steadily at a constant velocity V in the X- direction referred to a fixed coordinate system (X,Y,Z) as shown in the Fig 1. In absence of body force equations of motion in terms of displacement are

$$(\lambda+2\mu)\left[u_{,XX} + v_{,XY} \right] + \mu\left[u_{,YY} - v_{,XY} \right] = \rho\, u_{,TT} \tag{2.1}$$

$$(\lambda+2\mu)\left[u_{,XY} + v_{,YY} \right] + \mu\left[v_{,XX} - u_{,XY} \right] = \rho\, v_{,TT} \tag{2.2}$$

where u,v denote the displacement components in X and Y directions and λ, μ are Lame's constants and $u_{,X}$ represents partial derivatives of u with respect to X.

For cracks moving with constant velocity V in the X- direction it is convenient to introduce the Galilean transformation

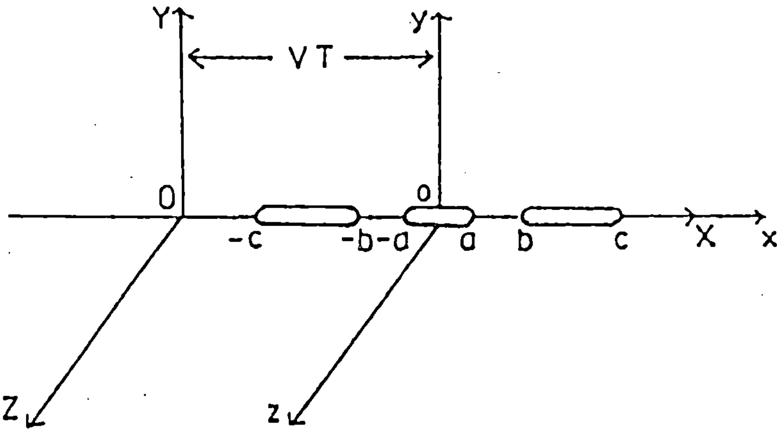


Fig. 1. Geometry and coordinate system.

$$x = X - VT, \quad y = Y, \quad z = Z, \quad t = T \tag{2.3}$$

where (x,y,z) represents the translating coordinate system as shown in Fig 1.

Let the positions of the co-planar Griffith cracks referred to translating coordinate (x,y,z) be $-a<x<a$, $-c<x<-b$, $b<x<c$ on $y=0$.

In the moving coordinates, The equations of motion (2.1) and (2.2)become independent of time and take the form

$$(\lambda+2\mu-\rho V^2)u_{,xx} + (\lambda+\mu)v_{,xy} + \mu u_{,yy} = 0$$

$$(\lambda+2\mu)v_{,yy} + (\mu-\rho V^2)v_{,xx} + (\lambda+\mu)u_{,xy} = 0 \tag{2.4}$$

The cracks are assumed to be moving steadily in an infinite medium subjected to a homogeneous stress such that the state of stress at infinity is given by $\sigma^{\infty}_{yy} = p$, $\sigma^{\infty}_{xx} = \sigma^{\infty}_{xy} = 0$.

For symmetry about the x- axis, only a half plane need be considered.

The state conditions at $y=\infty$ can all be made zero by superposing the simple static problem $\sigma^{\infty}_{yy} = -p$, $\sigma^{\infty}_{xx} = \sigma^{\infty}_{xy} = 0$.

The boundary conditions of the resulting dynamic problem are in terms of moving coordinates.

$$\begin{aligned} v &= 0, & & y=0,\ a\le|x|\le b\ ,\ |x|\ge c \\ \sigma_{xy} &= 0, & & |x|<\infty \\ \sigma_{yy} &= -p, & & |x|<a,\ b<|x|<c \end{aligned} \tag{2.5}$$

In view of the symmetry of the proposed problem with respect to y-axis, we introduce

$$\bar{u}_s(\xi,y) = \int_0^{\infty} u(x,y)\sin(\xi x)\,dx$$

$$\overline{v}_c(\xi,y) = \int_0^\infty v(x,y)\cos(\xi x)\,dx$$

and
$$u(x,y) = \frac{2}{\pi}\int_0^\infty \overline{u}_s(\xi,y)\sin(\xi x)\,d\xi$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \overline{v}_c(\xi,y)\cos(\xi x)\,d\xi$$

in equation (2.4) so that equations given by (2.4) reduce to

$$\mu\overline{u}_{s,yy} - \xi(\lambda+\mu)\overline{v}_{c,y} - \xi^2(\lambda+2\mu-\rho V^2)\overline{u}_s = 0$$

$$(\lambda+2\mu)\overline{v}_{c,yy} + \xi(\lambda+\mu)\overline{u}_{s,y} - \xi^2(\mu-\rho V^2)\overline{v}_c = 0 \qquad (2.6)$$

Elimination of $\overline{u}_s$ from (2.6) yields the following ordinary differential equation

$$\left[\left\{\frac{d^2}{dy^2} - (1-M^2k^2)\xi^2\right\}\left\{\frac{d^2}{dy^2} - (1-M^2)\xi^2\right\}\right]\overline{v}_c = 0 \qquad (2.7)$$

where $M = V/c_2$, $k = c_2/c_1$.

The solution of the differential equation given by (2.7), for $y \geq 0$, is

$$\overline{v}_c(\xi,y) = A(\xi)\,e^{-\xi y\sqrt{1-M^2k^2}} + B(\xi)\,e^{-\xi y\sqrt{1-M^2}} \qquad (2.8)$$

where the unknown functions $A(\xi)$ and $B(\xi)$ are to be determined using the boundary conditions of the proposed problem.

Employing (2.8) in equations (2.6) it can be shown that

$$\overline{u}_s(\xi,y) = \frac{A(\xi)}{\sqrt{1-M^2k^2}}\,e^{-\xi y\sqrt{1-M^2k^2}} + \sqrt{1-M^2}\,B(\xi)\,e^{-\xi y\sqrt{1-M^2}},\quad y\geq 0 \qquad (2.9)$$

Therefore, the stress components given by

$$\sigma_{yy} = \lambda(u_{,x} + v_{,y}) + 2\mu v_{,y}$$

$$\sigma_{xy} = \mu(u_{,y} + v_{,x}) \qquad (2.10)$$

become

$$\sigma_{yy}(x,y) = -\frac{2\mu}{\pi}\int_0^\infty \xi \left[\frac{2-M^2}{\sqrt{1-M^2k^2}}\, A(\xi)\, e^{-\xi y\sqrt{1-M^2k^2}} + 2\sqrt{1-M^2}\, B(\xi)\, e^{-\xi y\sqrt{1-M^2}}\right].$$
$$.\cos(\xi x)\, d\xi$$

$$\sigma_{xy}(x,y) = -\frac{2\mu}{\pi}\int_0^\infty \xi \left[2A(\xi)\, e^{-\xi y\sqrt{1-M^2k^2}} + (2-M^2)B(\xi)\, e^{-\xi y\sqrt{1-M^2}}\right]\sin(\xi x)\, d\xi$$

$$(2.11)$$

with

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[\frac{A(\xi)}{\sqrt{1-M^2k^2}}\, e^{-\xi y\sqrt{1-M^2k^2}} + \sqrt{1-M^2}\, B(\xi)\, e^{-\xi y\sqrt{1-M^2}}\right]\sin(\xi x)\, d\xi$$

and

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \left[A(\xi)\, e^{-\xi y\sqrt{1-M^2k^2}} + B(\xi)\, e^{-\xi y\sqrt{1-M^2}}\right]\cos(\xi x)\, d\xi \qquad (2.12)$$

On account of symmetry with respect to y-axis the boundary conditions (2.5) can be rewritten as

$$v(x,0) = 0\ , \qquad x \in I_2, I_4 \qquad (2.13)$$

$$\sigma_{xy}(x,0) = 0, \qquad 0<x<\infty \qquad (2.14)$$

$$\sigma_{yy}(x,0) = -p, \qquad x \in I_1, I_3 \qquad (2.15)$$

where $I_1 = (0,a)$, $I_2 = (a,b)$, $I_3 = (b,c)$, $I_4 = (c,\infty)$

Using the condition (2.14) in (2.11.2) it is found that $A(\xi), B(\xi)$ are related by

$$B(\xi) = -\frac{2}{2-M^2}\, A(\xi) \qquad (2.16)$$

With the help of the boundary condition (2.13), equation (2.12.2) reduces to

$$\int_0^\infty A(\xi)\cos(\xi x)\ d\xi = 0, \quad x \in I_2, I_4 \tag{2.17}$$

Substitution of (2.11.1) in (2.15) yields with the aid of (2.16)

$$\int_0^\infty \xi\ A(\xi)\cos(\xi x)\ d\xi = \frac{P\pi}{2\mu}, \quad x \in I_1, I_3 \tag{2.18}$$

where $$P = \frac{p}{K}, \quad K = \frac{(2-M^2)^2 - 4\sqrt{(1-M^2k^2)(1-M^2)}}{(2-M^2)\sqrt{1-M^2k^2}}$$

3.Method Of Solution

In order to solve the set of four integral equations given in equations (2.17) and (2.18) let us take

$$A(\xi) = \frac{1}{\xi}\int_0^a h(s)\ \sin(\xi s)\ ds + \frac{1}{\xi}\int_b^c g(t^2)\ \sin(\xi t)\ dt \tag{3.1}$$

where h(s) and $g(t^2)$ are unknown functions to be determined from the boundary conditions.

Inserting the value of $A(\xi)$ from equation (3.1) in equation (2.17) and using the following result [Gradshteyn and Ryzhik (1965)]

$$\int_0^\infty \frac{\sin(\xi x)\cos(\xi y)}{\xi}\ d\xi = \begin{cases} \pi/2, & x>y>0 \\ \pi/4, & x=y>0 \\ 0, & y>x>0 \end{cases}$$

it is found that this choice of $A(\xi)$ leads to the equation

$$\int_b^c g(t^2)\ dt = 0 \tag{3.2}$$

Further substitution of $A(\xi)$ from equation (3.1) in (2.18.1) and use of the result [Gradshteyn and Ryzhik (1965)]

$$\int_0^{\infty} \frac{\sin(\xi x)\sin(\xi u)}{\xi}\, d\xi = \frac{1}{2} \log\left|\frac{u+x}{u-x}\right|$$

yields

$$\frac{d}{dx} \int_0^a h(s)\log\left|\frac{s+x}{s-x}\right| ds + \frac{d}{dx} \int_b^c g(t^2)\log\left|\frac{t+x}{t-x}\right| dt = \frac{\pi P}{\mu}, \qquad x \in I_1$$

Rewriting this equation as

$$\int_0^a h(s)\log\left|\frac{s+x}{s-x}\right| ds = \pi\, F(x), \qquad x \in I_1$$

where $$F(x) = \int_0^x \left[\frac{P}{\mu} - \frac{2}{\pi} \int_c^d \frac{t g(t^2)}{t^2 - x'^2} dt \right] dx'$$

and using Cook's result (1970) it is found that

$$h(s) = \frac{P}{\mu} \frac{s}{\sqrt{a^2 - s^2}} - \frac{2}{\pi} \frac{s}{\sqrt{a^2 - s^2}} \int_b^c \frac{\sqrt{t^2 - a^2}\; g(t^2)}{t^2 - s^2} dt \tag{3.3}$$

where the result

$$\int_0^a \sqrt{a^2 - x^2} \frac{dx}{(s^2 - x^2)(t^2 - x^2)} = \frac{\pi}{2} \frac{\sqrt{t^2 - a^2}}{t} - \frac{1}{t^2 - s^2}$$

has been used.

Substituting the value of h(s) from (3.3) in (3.1) and using the resulting value of $A(\xi)$ in the boundary condition (2.18.2) and using the results

$$\int_0^a \frac{1}{\sqrt{a^2 - s^2}} \frac{s^2 ds}{(s^2 - x^2)(t^2 - s^2)} = \frac{\pi}{2}\left[\frac{t}{\sqrt{t^2 - a^2}} - \frac{x}{\sqrt{x^2 - a^2}}\right] \frac{1}{t^2 - x^2},$$

and $$\int_0^a \frac{1}{\sqrt{a^2 - s^2}} \frac{s^2\, ds}{(s^2 - x^2)} = \frac{\pi}{2}\left[1 - \frac{x}{\sqrt{x^2 - a^2}}\right], \qquad \text{for } x \in I_3$$

it can be shown that $g(t^2)$ is solution of the singular integral equation

$$\int_b^c \sqrt{\frac{t^2 - a^2}{t^2 - x^2}}\, g(t^2)\, dt = \frac{\pi P}{2\mu}\ , \quad x \in I_3$$

Using finite Hilbert transform technique (1968) the solution of this integral equation is obtained with the aid of the result

$$\int_b^c \sqrt{\frac{c^2 - x^2}{x^2 - b^2}}\ \frac{x\, dx}{(x^2 - v^2)} = -\frac{\pi}{2}\ , \qquad \text{for } x \in I_3$$

as

$$g(t^2) = \frac{P}{\mu}\sqrt{\frac{t^2(t^2 - b^2)}{(t^2 - a^2)(c^2 - t^2)}} + \frac{t\, C_1}{\sqrt{(t^2 - a^2)(t^2 - b^2)(c^2 - t^2)}} \qquad (3.4)$$

the constant C_1 is to be determined using the condition given by equation (3.2).

Next substituting the value of $g(t^2)$ from (3.4) in equation (3.3) and finally using the following results

$$\int_b^c \sqrt{\frac{t^2 - b^2}{c^2 - t^2}}\ \frac{t\, dt}{(t^2 - s^2)} = \frac{\pi}{2}\left[\, 1 - \sqrt{\frac{b^2 - s^2}{c^2 - s^2}}\,\right]$$

$$\int_b^c \frac{t\, dt}{(t^2 - s^2)\sqrt{(t^2 - b^2)(c^2 - t^2)}} = \frac{\pi}{2\sqrt{(c^2 - s^2)(b^2 - s^2)}} \quad \text{for } s \in I_1$$

h(s) is derived in the form

$$h(s) = \frac{P}{\mu}\sqrt{\frac{s^2(b^2 - s^2)}{(a^2 - s^2)(c^2 - s^2)}} - \frac{s\, C_1}{\sqrt{(a^2 - s^2)(b^2 - s^2)(c^2 - s^2)}} \qquad (3.5)$$

Now insertion of (3.4) in condition (3.2) yields

$$C_1 = -\frac{P}{\mu}\left[(c^2-a^2)\frac{E(\pi/2,l)}{F(\pi/2,l)} - (b^2-a^2)\right] \tag{3.6}$$

where $F(\phi,l)$ and $E(\phi,l)$ are elliptic integrals of first kind and second kind respectively and $l = \sqrt{\dfrac{c^2-b^2}{c^2-a^2}}$

The relevant displacement and stress components in the plane of crack can now be shown to be given by

$$v(x,0) = \int_x^a h(s)\,ds, \qquad 0\le x\le a$$

$$= \int_x^c g(t^2)\,dt, \qquad b\le x\le c \tag{3.7}$$

and

$$[\sigma_{yy}(x,o)]_{a<x<b} = \frac{2\mu K}{\pi}\left[\int_0^a \frac{sh(s\ )}{x^2-s^2}\,ds - \int_b^c \frac{tg(t^2)}{t^2-x^2}\,dt\right]$$

$$[\sigma_{yy}(x,o)]_{x>c} = \frac{2\mu K}{\pi}\left[\int_0^a \frac{sh(s\ )}{x^2-s^2}\,ds + \int_b^c \frac{tg(t^2)}{x^2-t^2}\,dt\right] \tag{3.8}$$

Insertion of the values of $h(s)$ and $g(t^2)$ as given by the equations (3.5) and (3.4) in the expressions (3.8) yields after some algebraic manipulation,

$$[\sigma_{yy}(x,o)]_{a<x<b} = \frac{2\mu K}{\pi}\left[F_1(x)-F_2(x)+F_3(x)-F_5(x)-F_6(x)\right]$$

$$[\sigma_{yy}(x,o)]_{x>c} = \frac{2\mu K}{\pi}\left[F_1(x)-F_2(x)+F_4(x)-F_5(x)+F_6(x)\right] \tag{3.9}$$

where

$$F_1(x) = \left[\frac{P}{\mu} (b^2-a^2) - C_1 \right] \left[\sqrt{\frac{x^2}{x^2-a^2}} - 1 \right] \frac{\pi}{2\sqrt{(c^2-a^2)(b^2-a^2)}}$$

$$F_2(x) = \int_0^a \left[\frac{P}{\mu} (c^2-b^2) - C_1 \frac{2u^2-b^2-c^2}{b^2-u^2} \right] \frac{g_1(u,x)}{c^2-u^2} \, du$$

$$F_{3,4}(x) = \left\{ \frac{P}{\mu} \left[\sqrt{\frac{b^2-x^2}{c^2-x^2}} - 1 \right] \mp \frac{C_1}{\sqrt{(c^2-x^2)(b^2-x^2)}} \right\} \frac{\pi c}{2\sqrt{c^2-a^2}}$$

$$F_5(x) = \frac{P}{\mu} a^2 \int_b^c \left[\tan^{-1} \sqrt{\frac{v^2-b^2}{c^2-v^2}} - \sqrt{\frac{b^2-x^2}{c^2-x^2}} \tan^{-1} \sqrt{\frac{(c^2-x^2)(v^2-b^2)}{(b^2-x^2)(c^2-v^2)}} \right] \frac{dv}{\sqrt{(v^2-a^2)^3}}$$

$$F_6(x) = \frac{a^2 C_1}{\sqrt{(c^2-x^2)(b^2-x^2)}} \int_b^c \frac{\tan^{-1} \sqrt{\frac{(u^2-b^2)(x^2-c^2)}{(c^2-u^2)(x^2-b^2)}}}{\sqrt{(u^2-a^2)^3}} \, du$$

$$g_1(u,x) = \frac{u}{\sqrt{(b^2-u^2)(c^2-u^2)}} \left[\sin^{-1}\left(\frac{u}{a}\right) - \frac{x}{\sqrt{x^2-a^2}} \tan^{-1} \sqrt{\frac{(x^2-a^2)u^2}{(a^2-u^2)x^2}} \right] \tag{3.10}$$

The dynamic stress intensity factors are given by

$$N_a = \operatorname{Lt}_{x \to a+} \sqrt{2(x-a)} \left[\sigma_{yy}(x,0) \right]_{a<x<b}$$

$$N_b = \operatorname{Lt}_{x \to b-} \sqrt{2(b-x)} \left[\sigma_{yy}(x,0) \right]_{a<x<b}$$

$$N_c = \operatorname{Lt}_{x \to c+} \sqrt{2(x-c)} \left[\sigma_{yy}(x,0) \right]_{x>c} \tag{3.11}$$

Employing (3.9) in (3.11) it can be shown that

$$N_a = p\sqrt{a}\sqrt{\frac{c^2 - a^2}{b^2 - a^2}}\,\frac{E(\pi/2, l)}{F(\pi/2, l)}$$

$$N_b = \frac{p\sqrt{b}}{\sqrt{(c^2 - b^2)(b^2 - a^2)}}\left[(c^2 - a^2)\frac{E(\pi/2, l)}{F(\pi/2, l)} - (b^2 - a^2)\right]$$

$$N_c = p\sqrt{c}\sqrt{\frac{c^2 - a^2}{c^2 - b^2}}\left[1 - \frac{E(\pi/2, l)}{F(\pi/2, l)}\right]$$

Now using the values of h(s) and g(t^2) from (3.5) and (3.4) in the expressions given by equations (3.7) displacement on the cracks are obtained as

$$[v(x,0)]_{0\le x\le a} = \frac{P}{\mu}\sqrt{c^2 - a^2}\,F(\beta, l)\left[\frac{E(\pi/2, l)}{F(\pi/2, l)} - \frac{E(\beta, l)}{F(\beta, l)}\right] + \frac{P}{\mu}\frac{\sqrt{(c^2 - x^2)(a^2 - x^2)}}{\sqrt{b^2 - x^2}}$$

$$[v(x,0)]_{b\le x\le c} = \frac{P}{\mu}\sqrt{c^2 - a^2}\,F(\lambda, l)\left[\frac{E(\lambda, l)}{F(\lambda, l)} - \frac{E(\pi/2, l)}{F(\pi/2, l)}\right]$$

$$\text{where} \quad \sin\lambda = \sqrt{\frac{c^2 - x^2}{c^2 - b^2}} \quad \text{and} \quad \sin\beta = \sqrt{\frac{a^2 - x^2}{b^2 - x^2}}\,.$$

It is interesting to note that the crack opening displacements depend on the crack velocity V but in the plane of the cracks the stresses and stress intensity factors are independent of the velocity of the moving cracks in an infinite elastic medium.

4. Statement Of Problem II And Its Formulation

In this case, consider an infinite homogeneous isotropic material with three coplanar Griffith cracks, located at $Y=0, -a\le X\le a$, $b\le|X|\le c$ and subjected to uniform internal pressure q. In absence of body force equation of equilibrium in terms of displacement are

$$(\lambda+2\mu)\left[u_{,XX} + v_{,XY}\right] + \mu\left[u_{,YY} - v_{,XY}\right] = 0$$

and
$$(\lambda+2\mu)\left[u_{,XY} + v_{,YY}\right] + \mu\left[v_{,XX} - u_{,XY}\right] = 0 \tag{4.1}$$

Since the problem exhibits a state of symmetry about $Y = 0$, attention can be given to a single half-space occupying the region $Y\ge 0$.

The equations (4.1) are to be solved subject to the boundary conditions

$$v(X,0) = 0, \qquad a\le|X|\le b,\ |X|\ge c \tag{4.2}$$

$$\sigma_{XY}(X,0) = 0, \qquad -\infty < X < \infty \tag{4.3}$$

$$\sigma_{YY}(X,0) = -q, \quad |X|\le a,\ b\le|X|\le c \tag{4.4}$$

In view of the boundary conditions, appropriate integral solutions of equation (4.1) are

$$u(X,Y) = \frac{2}{\pi}\int_0^\infty\left[C(\xi) + D(\xi)\left\{Y - \frac{1}{\xi}\,\frac{\lambda+3\mu}{\lambda+\mu}\right\}\right]e^{-\xi Y}\sin(\xi X)\,d\xi$$

and
$$v(X,Y) = \frac{2}{\pi}\int_0^\infty\left[C(\xi) + Y\,D(\xi)\right]e^{-\xi Y}\cos(\xi X)\,d\xi \tag{4.5}$$

Therefore,

$$\sigma_{YY}(X,Y) = -\frac{4\mu}{\pi}\int_0^\infty\left[\xi C(\xi) + \left\{Y\xi - \frac{\mu}{\lambda+\mu}\right\}D(\xi)\right]e^{-\xi Y}\cos(\xi X)\,d\xi$$

$$\sigma_{XY}(X,Y) = -\frac{4\mu}{\pi}\int_0^\infty\left[\xi C(\xi) + \left\{Y\xi - \frac{\lambda+2\mu}{\lambda+\mu}\right\}D(\xi)\right]e^{-\xi Y}\sin(\xi X)\, d\xi \qquad (4.6)$$

It may be noted that the displacement and stress components given by (4.5) and (4.6) can not be derived from the corresponding expressions of the dynamic problem given in (2.11) and (2.12) on setting $M = 0$.

The functions $C(\xi)$and $D(\xi)$ are to be determined from the boundary conditions (4.2)-(4.4), which yield

$$C(\xi) = \frac{1}{\xi}\frac{\lambda+2\mu}{\lambda+\mu} D(\xi) \qquad (4.7)$$

and the following set of four integral equations

$$\int_0^\infty C(\xi)\cos(\xi X)\, d\xi = 0, \quad X \in I_2, I_4 \qquad (4.8)$$

$$\int_0^\infty \xi\, C(\xi)\cos(\xi X)\, d\xi = \frac{Q\pi}{2\mu}, \quad X \in I_1, I_3 \qquad (4.9)$$

where

$Q = \frac{(\lambda+2\mu)}{2(\lambda+\mu)} q$ and I_j $(j=1,2,3,4)$ are the intervals defined earlier in problem I.

5.Method Of Solution And Quantities Of Physical Interest

Integral equations given by (4.8) and (4.9) are found to be the same as given by equations (2.17) and (2.18) with the exception that P is replaced by Q. Therefore, the same technique as that used in problem I can be employed to obtain

$$\left[v(X,0)\right]_{0\le X\le a} = \frac{Q}{\mu}\sqrt{c^2 - a^2}\; F(\beta', 1)\left[\frac{E(\pi/2, 1)}{F(\pi/2, 1)} - \frac{E(\beta', 1)}{F(\beta', 1)}\right]$$

$$+ \frac{Q}{\mu}\frac{\sqrt{(c^2 - X^2)(a^2 - X^2)}}{\sqrt{b^2 - X^2}}$$

$$[v(X,0)]_{b\le X\le c} = \frac{Q}{\mu}\sqrt{c^2-a^2}\;F(\lambda',1)\left[\frac{E(\lambda',1)}{F(\lambda',1)}-\frac{E(\pi/2,1)}{F(\pi/2,1)}\right] \qquad (5.1)$$

where $\sin\lambda' = \sqrt{\dfrac{c^2-X^2}{c^2-b^2}}$ and $\sin\beta' = \sqrt{\dfrac{a^2-X^2}{b^2-X^2}}$.

Stresses in the regions $a<X<b$, $X>c$ are found to be the same as that given in (3.9), the only change being that P is to replaced by Q.

6.Numerical Results and Discussions

Numerical results for the stress intensity factors and crack opening displacement, defined as $\Delta v(x,0) = v(x,0^+) - v(x,0^-)$, for different values of the parameters and $\lambda = \mu$ are presented in this section. Numerical calculations have been carried out for both the dynamic and static problems. As the crack velocity is less than Rayleigh wave velocity, it is reasonable to take the value of M less than 0.9194 .

Problem I: Variations of crack opening displacement for different values of crack speed, crack lengths and the separating distance between the cracks have been plotted in Figures 2-4. It is interesting to note from the Fig.2 that crack opening displacement on both the cracks decreases with the increase in the value of M at the onset and takes its minimum value at M=0.7415 ,after which it increases with the increase in the value of M. It has also been depicted in figures 3-4 that on each of the cracks,crack opening displacement decreases as the crack length decreases.

It has been mentioned earlier that the stress intensity factors at the crack tips are independent of crack speed and are found to depend on the crack lengths and the separating distance between the cracks. Variation of stress intensity factors with a/b for different values of c/b, and that with b/a for different values c/a are plotted in Fig.5 and Fig.6 respectively.

It has been found form these graphs that when the separating distance between the inner crack and outer pair of cracks decreases the variations of stress intensity factors at the tips x=a and x=b become more prominent than that at the edge x=c. Fig.7 shows that the stress intensity factors at the edges of the inner crack and outer pair of cracks increases as the length of the outer pair of cracks increases keeping the separating distance between the inner crack and outer pair of cracks fixed.

Problem II: Fig.8 shows the variations of crack opening displacement for different values of the parameters a/b, c/b, They exhibit that crack opening displacement on a crack of fixed length increases with the increase in the length of the other crack as expected from physical stand point.

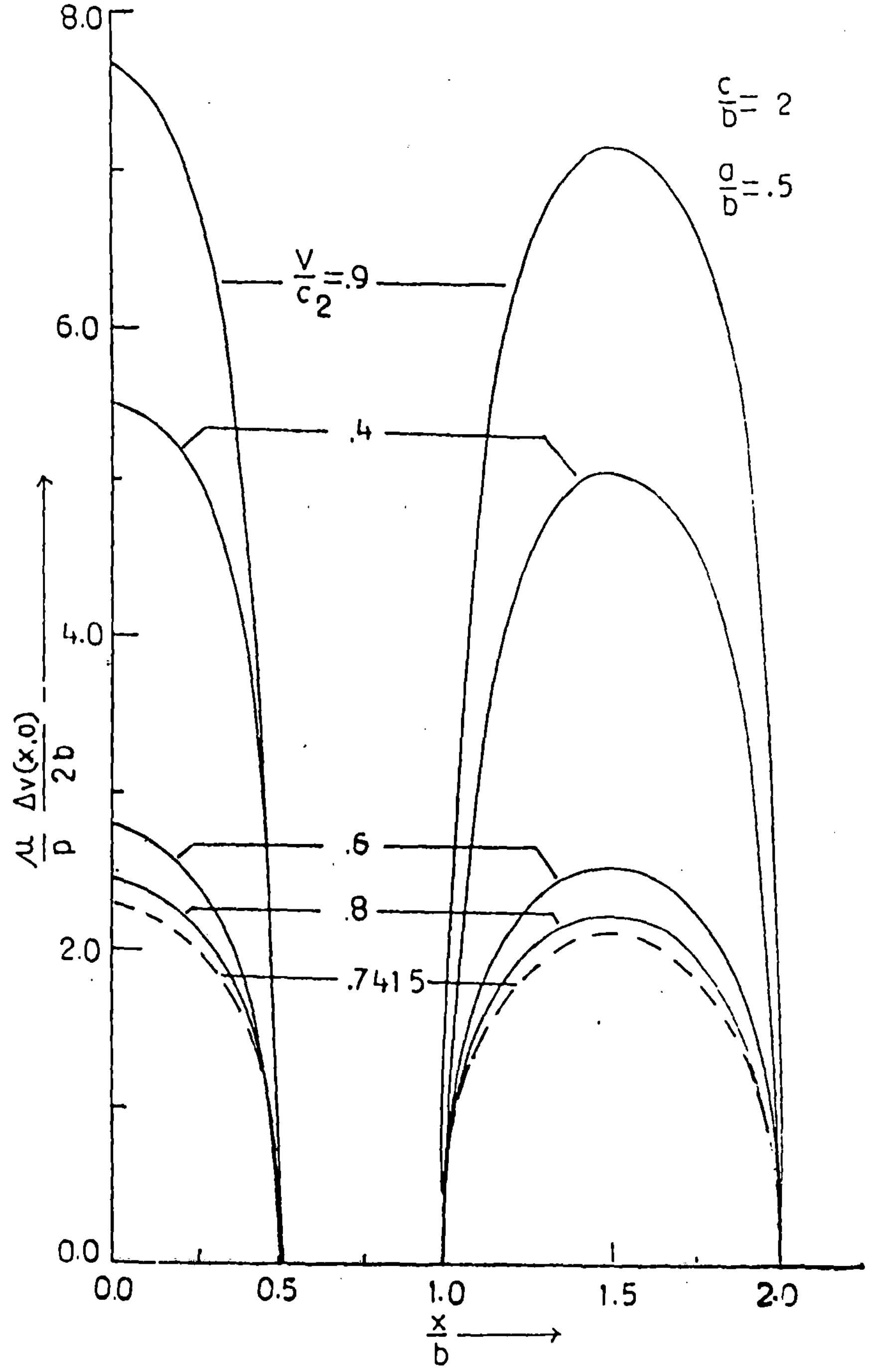


Fig. 2. Variation of crack opening displacement with x/b on both the cracks for the problem I.

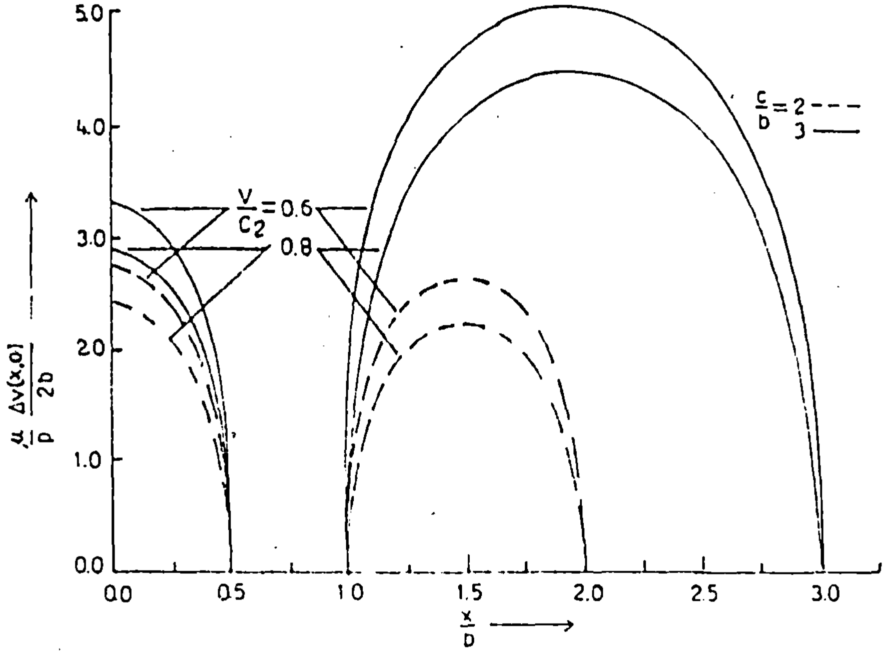


Fig. 3. Variation of crack opening displacement with x/b on both the cracks for the problem I.

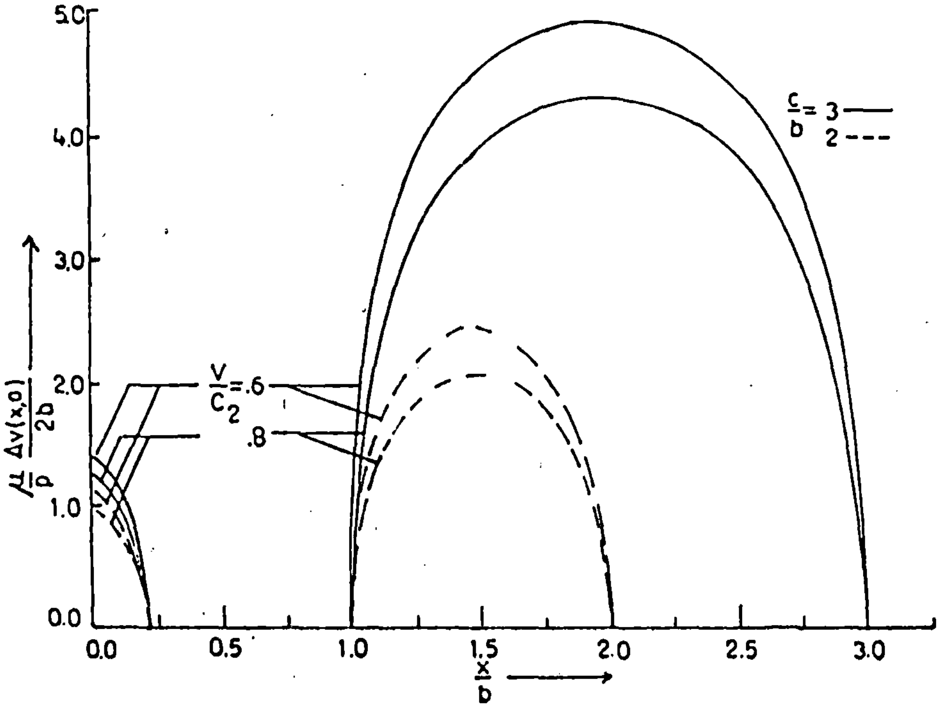


Fig. 4. Variation of crack opening displacement with x/b on both the cracks for the problem I.

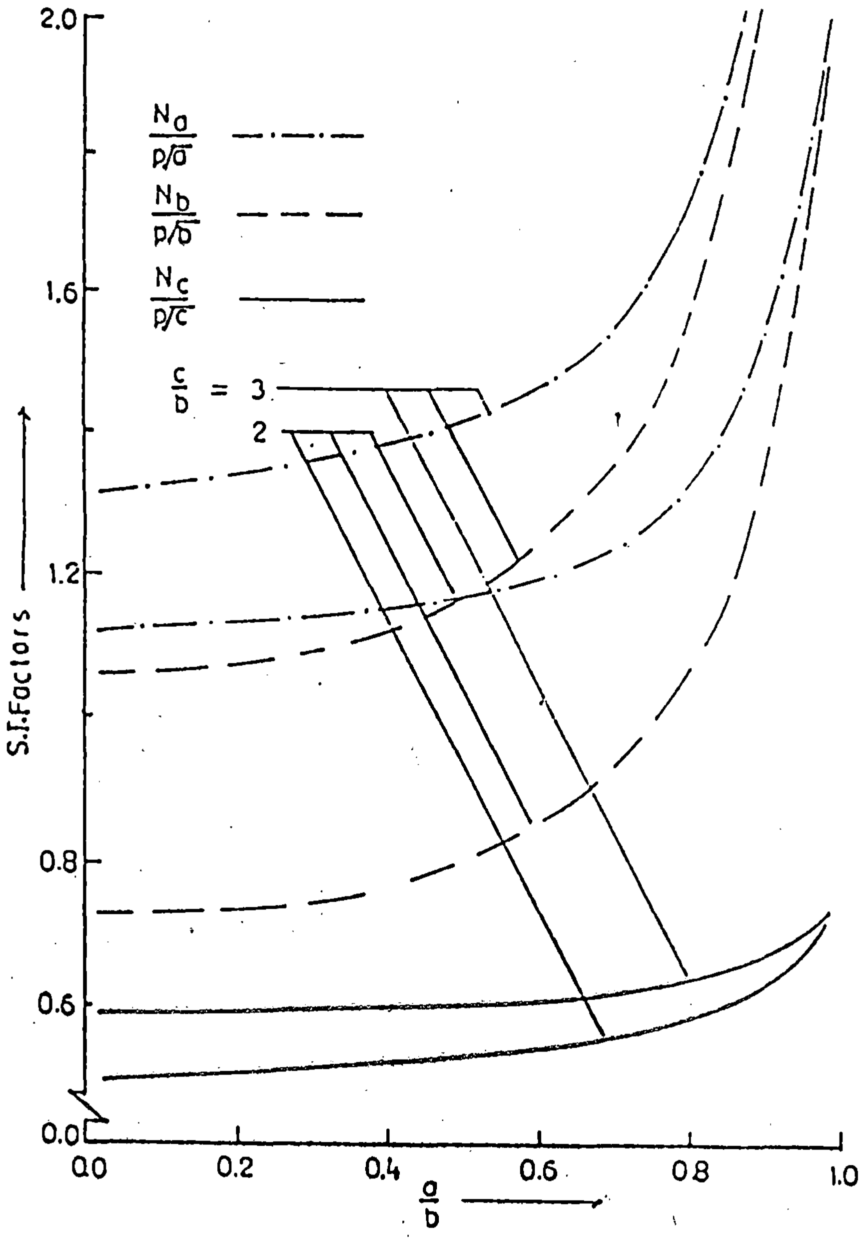


Fig. 5. Stress intensity factors Vs. a/b.

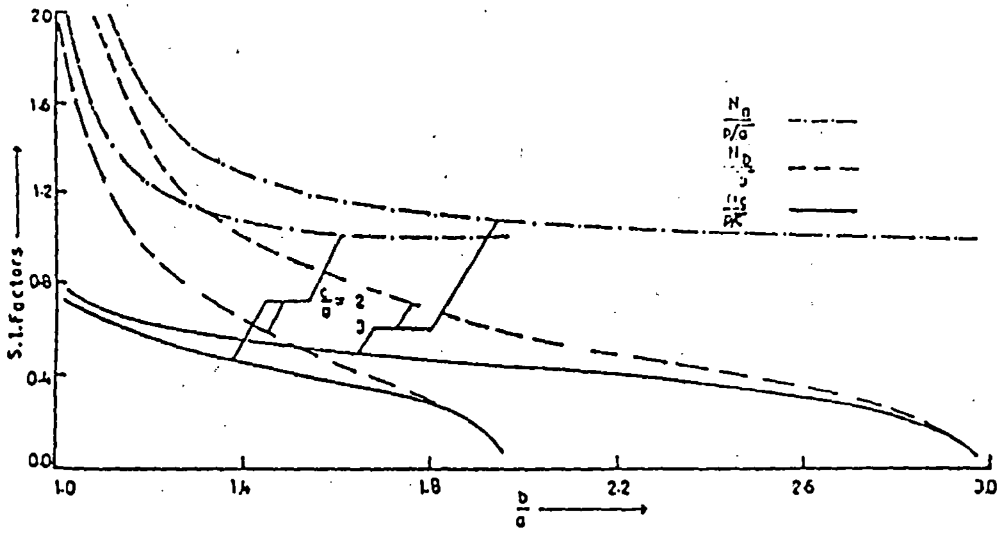


Fig. 6. Stress intensity factors Vs. *b*/*a*.

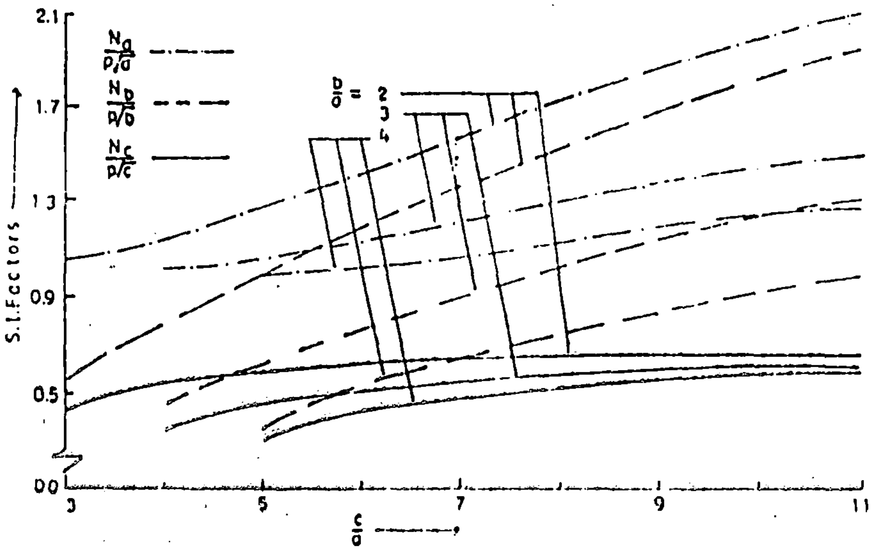


Fig. 7. Stress intensity factors Vs. c/a.

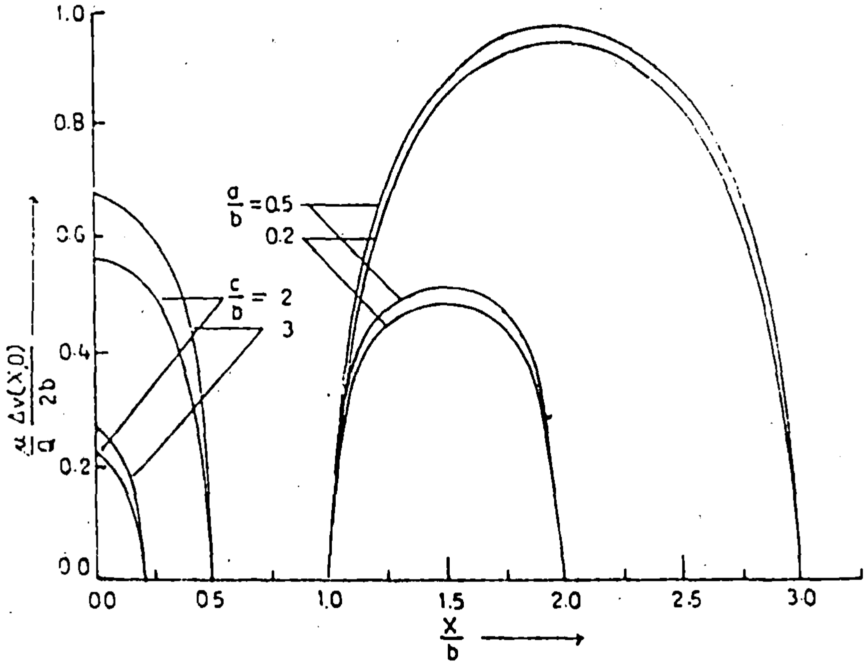


Fig. 8 Variation of crack opening displacement with X/b on both the cracks for the problem II.

THREE CO-PLANAR MOVING GRIFFITH CRACKS IN AN INFINITE ELASTIC STRIP

1. Introduction

In fracture mechanics, the problem of diffraction of elastic waves by cracks of finite dimension in a strip of elastic material has been investigated by several investigators. Sih and Chen (1972) investigated the problem of propagation of a crack of finite length in a strip under plane extension. Closed-from solutions for a finite length crack moving in a strip under anti-plane shear stress was obtained by Singh et. al (1981). Using finite Hilbert transform technique developed by Srivastava and Lowengrub (1968), Lowengrub and Srivastava (1968) solved the statical problem of distribution of stress and displacement in an infinitely long elastic strip containing two co-planar Griffith cracks.

As regards the crack problem, research has been restricted mainly to the case of a single crack or a pair of cracks because of severe mathematical complexity encountered in solving the problems of three or more cracks. Recently, Dhawan and Dhaliwal (1978) solved the statical problem of determining the stress distribution in an infinite transversely isotropic medium containing three co-planar Griffith cracks.

To the best knowledge of the author, the problem of stress distribution around three co-planar moving Griffith cracks in an infinite elastic strip has not been investigated so far. In this paper, the problem of propagation of three co-planar Griffith cracks in a fixed direction with constant velocity V in an infinitely long but of finite width elastic strip has been considered. Employing Fourier integral transform the problem, when the lateral boundaries are subjected to shearing stress, has been reduced to solving a set of four integral equations which are solved using finite Hilbert transform technique and Cook's result (1970) to derive the exact form of stress intensity factors and crack opening displacement. Numerical results for stress intensity factors are

---

Published in " Archives Of Mechanics" Vol 44, pp 465- 476, 1992.

presented graphically to show its variations with crack speed, crack lengths and the separating distance between the cracks.

2. Statement Of The Problem

Consider an infinitely long elastic strip occupying the region $-h \le Y \le h$, weakened by three co-planar Griffith cracks moving steadily at a constant velocity V in the X-direction referred to a fixed coordinate system (X,Y,Z) as shown in the Fig.1.

In dynamic problem of anti-plane shear, the non-vanishing component of displacement W directed in the Z-direction satisfies the equation of motion

$$W_{,XX} + W_{,YY} = \frac{1}{C_2^2} W_{,TT} \tag{2.1}$$

where $C_2 = (\mu/\rho)^{1/2}$ is the shear wave velocity, ρ is the material density and $W_{,X}$ represents partial derivatives of W with respect to X.

For cracks moving with constant velocity V in the X-direction it is convenient to introduce the Galilean transformation

$$x = X - VT,\ y = Y,\ z = Z,\ t = T \tag{2.2}$$

where (x,y,z) represents the translating coordinate system as shown in the Fig.1.

Let the positions of the co-planar Griffith cracks referred to coordinate (x,y,z) be $-c<x<-b$, $-a<x<a$ and $b<x<c$ on $y=0$, and let the uniform shearing stress p be apllied to the lataral boundaries $y = \pm h$ of the strip. The equivalent problems involves the application of shear stress $-p$ to the crack faces at $y=0$. Accordingly, the boundary conditions of the proposed problem are

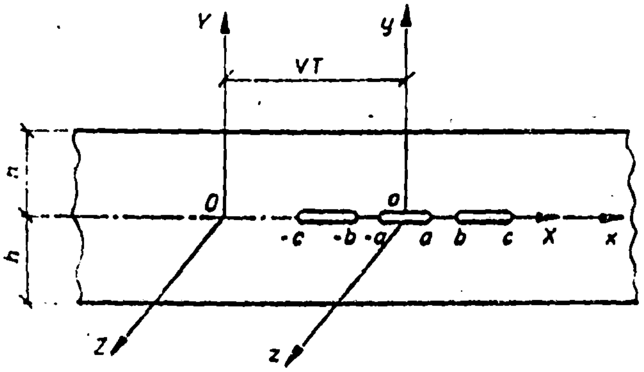


FIG. 1. Geometry and coordinate system.

$$\sigma_{yz}(x,0) = -p, \quad |x|<a, \; b<|x|<c \tag{2.3}$$

$$\sigma_{yz}(x,\pm h) = 0, \quad -\infty<x<\infty \tag{2.4}$$

$$W(x,0) = 0 , \quad a<|x|<b, \; |x|>c \tag{2.5}$$

In the moving coordinate system , the equation of motion becomes independent of time and takes the form

$$s^2 W_{,xx} + W_{,yy} = 0 \tag{2.6}$$

with

$$s = \sqrt{1 - V^2/C_2^2} \tag{2.7}$$

Due to the symmetry about x-z plane we need to consider the region $0<y\leq h$ only. Introducing the Fourier transform

$$\overline{W}_c(\xi,y)=\int_0^\infty W(x,y)\cos(\xi x)dx$$

$$W(x,y)= \frac{2}{\pi}\int_0^\infty \overline{W}_c(\xi,y)\cos(\xi x)d\xi \tag{2.8}$$

in equation (2.6), the solution of equation (2.6) is obtained as

$$W(x,y)= \frac{2}{\pi}\int_0^\infty \left[C_1(\xi)e^{-\xi ys}+C_3(\xi)e^{\xi ys}\right]\cos(\xi x)d\xi \tag{2.9}$$

with

$$\sigma_{yz}(x,y) = -\frac{2\mu s}{\pi}\int_0^\infty \xi\left[C_1(\xi)e^{-\xi ys}-C_3(\xi)e^{\xi ys}\right]\cos(\xi x)d\xi \tag{2.10}$$

Using the expression for $\sigma_{yz}(x,y)$ given in (2.10) in equation (2.4) it has been found that

$$C_1(\xi) = \frac{C(\xi)}{1 + e^{-2\xi hs}}$$

$$C_3(\xi) = \frac{C(\xi)e^{-2\xi hs}}{1 + e^{-2\xi hs}}$$

where the unknown function $C(\xi)$ is to be determined. From conditions (2.3) and (2.5) it is determined that $C(\xi)$ satisfies the following quadruple integral equations

$$\int_0^\infty \xi C(\xi)\,\mathrm{th}(\xi hs)\cos(\xi x)\,d\xi = \frac{\pi p}{2\mu s}, \quad x \in I_1, I_3 \tag{2.11}$$

and

$$\int_0^\infty C(\xi)\cos(\xi x)\,d\xi = 0, \quad x \in I_2, I_4 \tag{2.12}$$

where $I_1=(0,a)$, $I_2=(a,b)$, $I_3=(b,c)$, $I_4=(c,\infty)$

3. Method Of Solution

In order to solve the quadruple integral equations given by equations (2.11) and (2.12), let us take

$$C(\xi) = \frac{1}{\xi}\int_0^a h(u)\sin(\xi u)\,du + \frac{1}{\xi}\int_b^c g(v^2)\,\mathrm{ch}(ev)\sin(\xi v)\,dv \tag{3.1}$$

where $h(u)$ and $g(v^2)$ are the unknown functions to be determined from the boundary conditions of the proposed problem.

Substituting the value of $C(\xi)$ given by (3.1) in (2.12) and using the well known result

$$\int_0^\infty \frac{\sin(\xi x)\cos(\xi y)}{\xi}\,d\xi = \left.\begin{matrix} \pi/2, & x>y>0 \\ \pi/4, & x=y>0 \\ 0, & y>x>0 \end{matrix}\right\}$$

it is found that this choice of $C(\xi)$ leads to the condition

$$\int_b^c g(v^2)\,ch(ev)\,dv = 0 \tag{3.2}$$

Rewriting equation (2.11.1) as

$$\frac{d}{dx}\int_0^\infty C(\xi)\,th(\xi hs)\sin(\xi x)\,d\xi = \frac{\pi p}{2\mu s}\ , \quad x \in I_1 \tag{3.3}$$

and inserting the value of $C(\xi)$ from equation (3.1) in (3.3) it is found that $h(u)$ is the solution of the following singular integral equation

$$\int_0^a h(u)\log\left|\frac{sh(ex)+sh(eu)}{sh(ex)-sh(eu)}\right|du = \pi f(x), \qquad x \in I_1 \tag{3.4}$$

with $\quad f(x) = \int_0^x \left[\frac{p}{\mu s} - \frac{1}{\pi}\int_b^c \frac{e g(v^2)\,ch(ex')\,sh(2ev)}{sh^2(ev) - sh^2(ex')}\,dv\right]dx'$

where the following result [Gradshteyn, I.S. and Ryzhik, I.M. (1965)] has been used

$$\int_0^\infty th(\xi hs)\frac{\sin(\xi x)\sin(\xi u)}{\xi}\,d\xi = \frac{1}{2}\log\left|\frac{sh(ex)+sh(eu)}{sh(ex)-sh(eu)}\right|, \qquad e = \frac{\pi}{2hs} \tag{3.5}$$

Now using the Cook's result (1970), the solution of (3.4) has been obtained with the aid of following result

$$\int_0^a \frac{\sqrt{sh^2(ea)-sh^2(ex)}\ e\ ch(ex)\ dx}{[sh^2(ex)-sh^2(eu)][sh^2(ev)-sh^2(ex)]} = -\frac{\pi}{2sh(ev)}\ \frac{\sqrt{sh^2(ev)-sh^2(ea)}}{sh^2(ev)-sh^2(eu)}$$

for $u \in I_1$ and $v \in I_3$,

$$h(u)=\frac{-e\ sh(2eu)}{\pi\sqrt{sh^2(ea)-sh^2(eu)}}\left[\frac{p}{\mu s}\int_0^a\frac{\sqrt{sh^2(ea)-sh^2(ex)}}{sh^2(ex)-sh^2(eu)}dx+\int_b^c\frac{\sqrt{sh^2(ev)-sh^2(ea)}}{sh^2(ev)-sh^2(eu)}\times\right.$$

$$\left.\times g(v^2)ch(ev)dv\right] \qquad (3.6)$$

Substituting the resulting value of $C(\xi)$, obtained using equation (3.6) in equation (3.1), in condition (2.11.2) and making use of the following results

$$\int_0^a\frac{e\ sh^2(eu)ch(eu)\ du}{[sh^2(eu)-sh^2(ex)][sh^2(ev)-sh^2(eu)]\sqrt{sh^2(ea)-sh^2(eu)}}=$$

$$=\frac{\pi}{2[sh^2(ev)-sh^2(ex)]}\left[\frac{sh(ev)}{\sqrt{sh^2(ev)-sh^2(ea)}}-\frac{sh(ex)}{\sqrt{sh^2(ex)-sh^2(ea)}}\right],$$

$$\int_0^a\frac{e\ sh^2(eu)ch(eu)\ du}{[sh^2(eu)-sh^2(ex)][sh^2(ey')-sh^2(eu)]\sqrt{sh^2(ea)-sh^2(eu)}}=$$

$$=\frac{\pi}{2[sh^2(ex)-sh^2(ey')]}\ \frac{sh(ex)}{\sqrt{sh^2(ex)-sh^2(ea)}},\ \text{for } x,v\in l_3 \text{ and } y'\in l_1$$

it can be shown that $g(v^2)$ is the solution of the following singular integral equation

$$\int_b^c\frac{\sqrt{sh^2(ev)-sh^2(ea)}}{sh^2(ev)-sh^2(ex)}\ eg(v^2)ch(ev)\ dv=\frac{\pi p}{\mu s}\left[\frac{\sqrt{sh^2(ex)-sh^2(ea)}}{sh(2ex)}+\right.$$

$$\left.+\frac{1}{\pi}\int_0^a\frac{\sqrt{sh^2(ea)-sh^2(ey')}}{sh^2(ex)-sh^2(ey')}\ dy'\right],\ \text{for } x\in l_3. \qquad (3.7)$$

Using finite Hilbert transform technique (1968) and the following result

$$\int_b^c \sqrt{\frac{sh^2(ec)-sh^2(ex)}{sh^2(ex)-sh^2(eb)}}\ \frac{sh(2ex)\ \ dx}{[sh^2(ex)-sh^2(ey')][sh^2(ex)-sh^2(ev)]} =$$

$$= -\ \frac{\pi}{e[sh^2(ev)-sh^2(ey')]}\sqrt{\frac{sh^2(ec)-sh^2(ey')}{sh^2(eb)-sh^2(ey')}}\ ,$$

the solution of equation (3.7) is found as

$$g(v^2) = -\ \frac{2ep}{\mu\pi s}\ \frac{sh(ev)\sqrt{sh^2(ev)-sh^2(eb)}}{\sqrt{[sh^2(ev)-sh^2(ea)][sh^2(ec)-sh^2(ev)]}}\Bigg[\int_b^c \sqrt{\frac{sh^2(ec)-sh^2(ex)}{sh^2(ex)-sh^2(eb)}}\times$$

$$\times\sqrt{\frac{sh^2(ex)-sh^2(ea)}{sh^2(ex)-sh^2(ev)}}\ dx\ -\int_0^a \sqrt{\frac{sh^2(ec)-sh^2(ey')}{sh^2(eb)-sh^2(ey')}}\ \sqrt{\frac{sh^2(ea)-sh^2(ey')}{sh^2(ev)-sh^2(ey')}}\ dy'\Bigg]+$$

$$+\ \frac{C_1\ \ sh(ev)}{\sqrt{[sh^2(ev)-sh^2(ea)][sh^2(ev)-sh^2(eb)][sh^2(ec)-sh^2(ev)]}} \qquad (3.8)$$

Next substituting the value of $g(v^2)$ from equation (3.8) in equation (3.6) and finally using the following result

$$\int_b^c \sqrt{\frac{sh^2(ev)-sh^2(eb)}{sh^2(ec)-sh^2(ev)}}\ \frac{sh(2ev)\ \ dv}{[sh^2(ev)-sh^2(eu)][sh^2(ex')-sh^2(ev)]} =$$

$$= \frac{\pi}{e[sh^2(eu)-sh^2(ex')]}\left[\sqrt{\frac{sh^2(eb)-sh^2(eu)}{sh^2(ec)-sh^2(eu)}} - \sqrt{\frac{sh^2(eb)-sh^2(ex')}{sh^2(ec)-sh^2(ex')}}\right],$$

for $u,\ x' \in l_1$

h(u) is derived in the form

$$h(u) = -\frac{2ep}{\mu\pi s}\,\frac{ch(eu)sh(eu)\sqrt{sh^2(eb)-sh^2(eu)}}{\sqrt{[sh^2(ea)-sh^2(eu)][sh^2(ec)-sh^2(eu)]}}\Bigg[\int_0^a\sqrt{\frac{sh^2(ea)-sh^2(ey')}{sh^2(eb)-sh^2(ey')}}\times$$

$$\times\sqrt{\frac{sh^2(ec)-sh^2(ey')}{sh^2(ey')-sh^2(eu)}}\,dy' - \int_b^c\sqrt{\frac{sh^2(ec)-sh^2(ex)}{sh^2(ex)-sh^2(eb)}}\,\sqrt{\frac{sh^2(ex)-sh^2(ea)}{sh^2(ex)-sh^2(eu)}}\,dx\Bigg] -$$

$$-\frac{C_1\ sh(eu)ch(eu)}{\sqrt{[sh^2(ea)-sh^2(eu)][sh^2(eb)-sh^2(eu)][sh^2(ec)-sh^2(eu)]}} \qquad (3.9)$$

Substitution of the value of $g(v^2)$ from equation (3.8) in the condition (3.2) yields

$$C_1 = -\frac{2ep}{\pi\mu s}\Bigg[\int_b^c\sqrt{\frac{sh^2(ec)-sh^2(ex)}{sh^2(ex)-sh^2(eb)}}\,\sqrt{sh^2(ex)-sh^2(ea)}\Bigg\{\frac{sh^2(ex)-sh^2(eb)}{sh^2(ec)-sh^2(ex)}\times$$

$$\Pi\Big\{\frac{\pi}{2},\frac{sh^2(ec)-sh^2(eb)}{sh^2(ec)-sh^2(ex)};q\Big\}/F(\frac{\pi}{2},q)+1\Bigg\}dx+\int_0^a\sqrt{\frac{sh^2(ec)-sh^2(es)}{sh^2(eb)-sh^2(es)}}\,\sqrt{sh^2(ea)-sh^2(es)}$$

$$\Bigg\{1-\frac{sh^2(eb)-sh^2(es)}{sh^2(ec)-sh^2(es)}\,\Pi\Big\{\frac{\pi}{2},\ \frac{sh^2(ec)-sh^2(eb)}{sh^2(ec)-sh^2(es)},q\Big\}/F(\frac{\pi}{2},q)\Bigg\}\ ds \qquad (3.10)$$

where $F(\phi,q)$ and $\Pi(\phi,n,q)$ are elliptic integrals of first and third kind respectively and $q = \sqrt{\frac{sh^2(ec)-sh^2(eb)}{sh^2(ec)-sh^2(ea)}}$

The relevant displacement and stress components in the plane of the crack can now be shown to be given by

$$W(x,0) = \int_x^a h(u)du \ , \qquad 0 \leq x \leq a$$

$$= \int_x^c g(v^2)ch(ev)dv \ , \qquad b \leq x \leq c \qquad (3.11)$$

and

$$\left[\sigma_{yz}(x,0)\right]_{a<x<b} = \frac{2\mu s}{\pi}\left[\int_0^a \frac{eh(u)sh(eu)\ du}{sh^2(ex)-sh^2(eu)} - \int_b^c \frac{eg(v^2)sh(ev)ch(ev)}{sh^2(ev) - sh^2(ex)}dv\right]ch(ex)$$

$$\left[\sigma_{yz}(x,0)\right]_{x>c} = \frac{2\mu s}{\pi}\left[\int_0^a \frac{eh(u)sh(eu)\ du}{sh^2(ex)-sh^2(eu)} + \int_b^c \frac{eg(v^2)sh(ev)ch(ev)}{sh^2(ex) - sh^2(ev)}dv\right]ch(ex)$$

$$(3.12)$$

Now insertion of the values of $h(u)$ and $g(v^2)$ as given by equations (3.9) and (3.8) , in the expressions (3.12) yields after some algebraic manipulations

$$\left[\sigma_{yz}(x,0)\right]_{a<x<b} = \frac{2pe}{\pi}\left[-\sqrt{\frac{sh^2(eb)-sh^2(ea)}{sh^2(ec)-sh^2(ea)}}\ \frac{sh(ex)}{\sqrt{sh^2(ex)-sh^2(ea)}}\left\{\int_0^a F_2(u,x)du\ +\right.\right.$$

$$\left. + \int_b^c F_2(v,x)dv \right\} - \frac{2e[sh^2(ec)-sh^2(eb)]}{\pi}\left\{ \int_0^a F_2(u',x)du' \int_0^a F_4(c,u) \times \right.$$

$$\left. \times\ F_3(0,x,u)\ du + \int_b^c F_2(v,x)dv\int_0^a F_4(c,u)F_3(v,x,u)\ du \right\} + \frac{\mu s}{ep}\ C_1\left\{ \frac{\pi}{2} \times \right.$$

$$\left. \times \frac{1 - sh(ex)/\sqrt{sh^2(ex)-sh^2(ea)}}{\sqrt{[sh^2(eb)-sh^2(ea)][sh^2(ec)-sh^2(ea)]}} + e\int_0^a F_4(c,u)F_5(u,x)\ du \right\} +$$

$$+ \frac{e[sh^2(eb)-sh^2(ea)]}{\pi}\left\{\int_b^c F_2(v',x)dv' \int_b^c F_4(a,v)F_6(v',x,v)dv + \int_0^a F_2(u,x)du \times\right.$$

$$\left.\int_b^c F_4(a,v)F_6(u,x,v)dv - \frac{sh^2(ec)-sh^2(eb)}{sh^2(eb)-sh^2(ea)}\int_0^a F_1(u,x)du\int_0^a F_4(c,u')F_9(u,u')du'\right\}$$

$$\left.- \frac{\mu s}{pe}\frac{C_1}{X_1}\left\{\frac{\pi}{2}\frac{sh(ec)}{\sqrt{sh^2(ec)-sh^2(ea)}} + e\ sh^2(ea)\int_b^c F_7(x,v)\ dv\right\}\right]ch(ex)$$

and

$$\left[\sigma_{yz}(x,0)\right]_{x>c} = \frac{2pe}{\pi}\left[-\sqrt{\frac{sh^2(eb)-sh^2(ea)}{sh^2(ec)-sh^2(ea)}}\ \frac{sh(ex)}{\sqrt{sh^2(ex)-sh^2(ea)}}\left\{\int_0^a F_2(u,x)du +\right.\right.$$

$$\left.+ \int_b^c F_2(v,x)dv\right\} - \frac{2e[sh^2(ec)-sh^2(eb)]}{\pi}\left\{\int_0^a F_2(u',x)du' \int_0^a F_4(c,u)\times\right.$$

$$\left.\times F_8(0,x,u)\ du + \int_b^c F_2(v,x)dv \int_0^a F_4(c,u)F_3(v,x,u)\ du\right\} + \frac{\mu s}{ep}\ C_1\left\{\frac{\pi}{2}\times\right.$$

$$\left.\times \frac{1 - sh(ex)/\sqrt{sh^2(ex)-sh^2(ea)}}{\sqrt{[sh^2(ec)-sh^2(ea)][sh^2(eb)-sh^2(ea)]}} + e\int_0^a F_4(c,u)F_5(u,x)du\right\} -$$

$$- \frac{e[sh^2(eb)-sh^2(ea)]}{\pi}\left\{\int_b^c F_2(v',x)dv' \int_b^c F_4(a,v)F_8(v',v,x)dv + \int_0^a F_2(u,x)du \times\right.$$

$$\left.\int_b^c F_4(a,v)F_8(u,v,x)dv + \frac{sh^2(ec)-sh^2(eb)}{sh^2(eb)-sh^2(ea)}\int_0^a F_1(u,x)du\int_0^a F_4(c,u')F_9(u,u')du'\right\}$$

$$+\frac{\mu s C_1}{p e X_1}\left\{\frac{\pi}{2}\,\frac{\mathrm{sh}(ec)}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}+e\;\mathrm{sh}^2(ea)\int_b^c F_7(x,v)\;dv\right\}-\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}\times$$

$$\times\frac{\mathrm{sh}(ex)}{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ec)}}\left\{\int_0^a F_2(u,x)du+\int_b^c F_2(v,x)dv\right\}\Bigg]\mathrm{ch}(ex) \qquad (3.13)$$

where

$$F_1(u,x)=\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}}\;\frac{\mathrm{sh}(eu)}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eu)}$$

$$F_2(v,x)=\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}}\;\frac{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ex)}$$

$$F_3(v,x,u)=\frac{\mathrm{sh}(ex)}{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}\;\tan^{-1}\left\{\frac{\mathrm{sh}(eu)}{\mathrm{sh}(ex)}\sqrt{\frac{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}\right\}-$$

$$-\frac{\mathrm{sh}(ev)}{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}}\;\tan^{-1}\left\{\frac{\mathrm{sh}(eu)}{\mathrm{sh}(ev)}\sqrt{\frac{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}\right\}$$

$$F_4(\omega,u)=\frac{\mathrm{ch}(eu)\;\mathrm{sh}(eu)}{\sqrt{[\mathrm{sh}^2(e\omega)-\mathrm{sh}^2(eu)]^3[\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)]}}$$

$$F_5(u,x)=[\;2\mathrm{sh}^2(eu)-\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)]\left\{\sin^{-1}\left(\frac{\mathrm{sh}(eu)}{\mathrm{sh}(ea)}\right)-F_3(0,x,u)\right\}$$

$$F_6(u,x,v) = \frac{sh(ex)}{\sqrt{sh^2(ec)-sh^2(ex)}} \times$$

$$\times \log\left|\frac{sh(ex)\sqrt{sh^2(ec)-sh^2(ev)}+sh(ev)\sqrt{sh^2(ec)-sh^2(ex)}}{sh(ex)\sqrt{sh^2(ec)-sh^2(ev)}-sh(ev)\sqrt{sh^2(ec)-sh^2(ex)}}\right| - \frac{sh(eu)}{\sqrt{sh^2(ec)-sh^2(eu)}}$$

$$\times\ \log\left|\frac{sh(eu)\sqrt{sh^2(ec)-sh^2(ev)}+sh(ev)\sqrt{sh^2(ec)-sh^2(eu)}}{sh(eu)\sqrt{sh^2(ec)-sh^2(ev)}-sh(ev)\sqrt{sh^2(ec)-sh^2(eu)}}\right|$$

$$F_7(x,v) = \tan^{-1}\left[\frac{\sqrt{sh^2(ec)-sh^2(ex)}}{\sqrt{sh^2(ec)-sh^2(ev)}}\ \frac{\sqrt{sh^2(ev)-sh^2(eb)}}{\sqrt{sh^2(eb)-sh^2(ex)}}\right]\frac{ch(eu)}{\sqrt{[sh^2(ev)-sh^2(ea)]^3}}$$

$$F_8(u,v,x) = -\frac{2sh(ex)}{\sqrt{sh^2(ex)-sh^2(ec)}}\ \tan^{-1}\left\{\frac{sh(ev)}{sh(ex)}\sqrt{\frac{sh^2(ex)-sh^2(ec)}{sh^2(ec)-sh^2(ev)}}\right\} +$$

$$+\ \frac{sh(eu)}{\sqrt{sh^2(ec)-sh^2(eu)}}\ \log\left|\frac{sh(eu)\sqrt{sh^2(ec)-sh^2(ev)}+sh(ev)\sqrt{sh^2(ec)-sh^2(eu)}}{sh(eu)\sqrt{sh^2(ec)-sh^2(ev)}-sh(ev)\sqrt{sh^2(ec)-sh^2(eu)}}\right|$$

$$F_9(u,u') = \log\left|\frac{sh(eu)\sqrt{sh^2(ea)-sh^2(eu')}+sh(eu')\sqrt{sh^2(ea)-sh^2(eu)}}{sh(eu)\sqrt{sh^2(ea)-sh^2(eu')}-sh(eu')\sqrt{sh^2(ea)-sh^2(eu)}}\right|$$

and $$X_1 = \sqrt{[sh^2(eb)-sh^2(ex)][sh^2(ec)-sh^2(ex)]} \qquad (3.14)$$

The dynamic stress intensity factors are defined by

$$N_a = \operatorname*{Lt}_{x \to a+} \sqrt{2(x-a)} \left[\sigma_{yz}(x,0) \right]_{a<x<b}$$

$$N_b = \operatorname*{Lt}_{x \to b-} \sqrt{2(b-x)} \left[\sigma_{yz}(x,0) \right]_{a<x<b}$$

$$N_c = \operatorname*{Lt}_{x \to c+} \sqrt{2(x-c)} \left[\sigma_{yz}(x,0) \right]_{x>c} \qquad (3.15)$$

Substitution of the results given by equations (3.13) in expressions (3.15) yields

$$N_a = \sqrt{\frac{sh(2ea)}{e}} \left[- \sqrt{\frac{sh^2(eb)-sh^2(ea)}{sh^2(ec)-sh^2(ea)}} \, \frac{2pe}{\pi} \left\{ \int_0^a F_2(u,a)du + \int_b^c F_2(v,a)dv \right\} - \right.$$

$$\left. - \frac{\mu s \, C_1}{\sqrt{[sh^2(eb)-sh^2(ea)][sh^2(ec)-sh^2(ea)]}} \right]$$

$$N_b = - \frac{\mu s \, C_1}{\sqrt{[sh^2(eb)-sh^2(ea)][sh^2(ec)-sh^2(eb)]}} \sqrt{\frac{sh(2eb)}{e}}$$

$$N_c = \sqrt{\frac{sh(2ec)}{e}} \left[- \sqrt{\frac{sh^2(ec)-sh^2(eb)}{sh^2(ec)-sh^2(ea)}} \, \frac{2pe}{\pi} \left\{ \int_0^a F_2(u,c)du + \int_b^c F_2(v,c)dv \right\} + \right.$$

$$\left. + \frac{\mu s \, C_1}{\sqrt{[sh^2(ec)-sh^2(ea)][sh^2(ec)-sh^2(eb)]}} \right] \qquad (3.16)$$

Again insertion of the values of $h(u)$ and $g(v^2)$, given by equations (3.8) and (3.9), in the expressions for displacements given by equations (3.11) yields

$$\Big[W(x,0)\Big]_{0\le x\le a} = -\frac{p}{\mu\pi s}\Bigg[\frac{2[\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)]}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}\Bigg\{\int_b^c \Pi\Big\{\lambda,\frac{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)},q\Big\}\times$$

$$\times\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}}\;\frac{dv}{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}} - \int_0^a \Pi\Big\{\lambda,\frac{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)},q\Big\}\times$$

$$\times\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}}\;\frac{du}{\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}\Bigg\}\Bigg] - \frac{C_1 F(\lambda,q)}{e\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}$$

and

$$\Big[W(x,0)\Big]_{b\le x\le c} = \Bigg[\frac{2p}{\mu\pi s}\Bigg(\int_b^c\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}}\;\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}\Bigg\{F(\lambda',q)\;+$$

$$+\frac{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}\,\Pi\Big\{\lambda',\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)},q\Big\}\Bigg\}\,dv\;+\int_0^a\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}}\times$$

$$\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}\Bigg\{F(\lambda',q)-\frac{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}\Pi\Big\{\lambda',\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)},q\Big\}\Bigg\}du\Bigg]\;+$$

$$+\;\frac{C_1}{e}F(\lambda',q)\Bigg]\frac{1}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}} \qquad (3.17)$$

where $\sin\lambda = \sqrt{\dfrac{sh^2(ea)-sh^2(ex)}{sh^2(eb)-sh^2(ex)}}$, $\sin\lambda' = \sqrt{\dfrac{sh^2(ec)-sh^2(ex)}{sh^2(ec)-sh^2(eb)}}$ and $F(\phi,q)$, $\Pi(\phi,n,q)$ and q have been defined earlier.

On putting b=c and simplifying, it may be noted that, the results (3.16.1) and (3.17.1) become those given by equation (4.18) and (4.19) of Singh et. al (1981) and for a=0 the results given by (3.16.2),(3.16.3) and (3.17.2) coincide with those given by equation (4.38), (4.39) and (4.35) of Das and Ghosh (1991).

4. Numerical Results and Discussions.

Numerical results for stress intensity factors at the tips of the cracks for different values of crack speed, crack lengths and the separating distance between the cracks have been presented in this section. The crack length dependence of the stress intensity factors and its variations with V/C_2 have been shown in Figures 2-5. It has been depicted in Figures 2-3 that stress intensity factors at the edges of the cracks have a prominent variation when $V/C_2 \rightarrow 1$ and variations of stress intensity factors at the edge x=a become more prominent than that at the tips x=b and x=c when the length of the inner crack increases.

Variations of stress intensity factors at the edges of the cracks with a/b for different values of c/b and that with b/a for different values of c/a are plotted in Figures 4-5 respectively. It has been found that when the separating distance between the inner crack and outer pair of cracks decreases the stress intensity factors at the tips x=a and x=b become greater than that at the edge x=c.

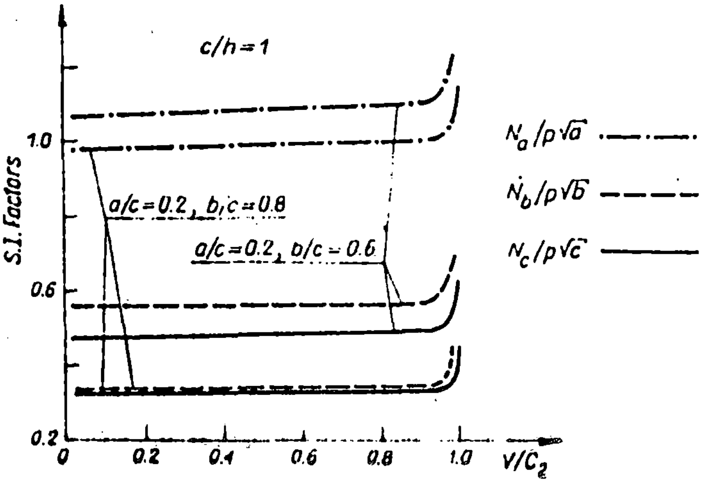


FIG. 2. Variations of stress intensity factors with V/C_2.

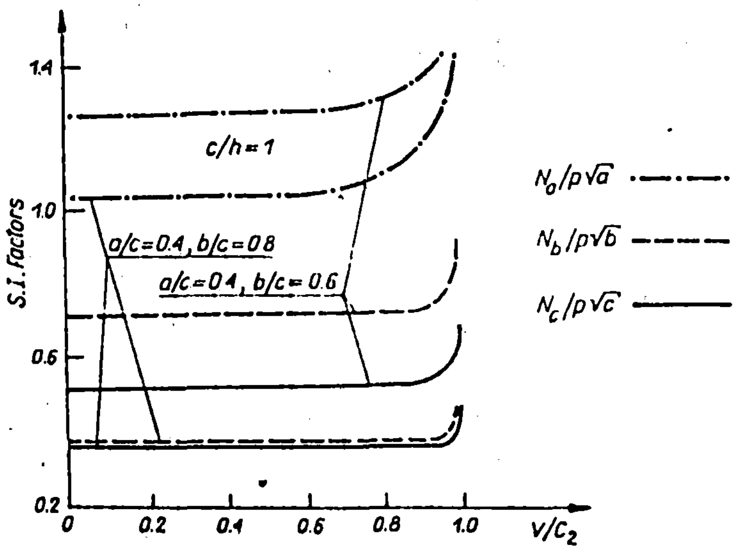


FIG. 3. Variations of stress intensity factors with V/C_2.

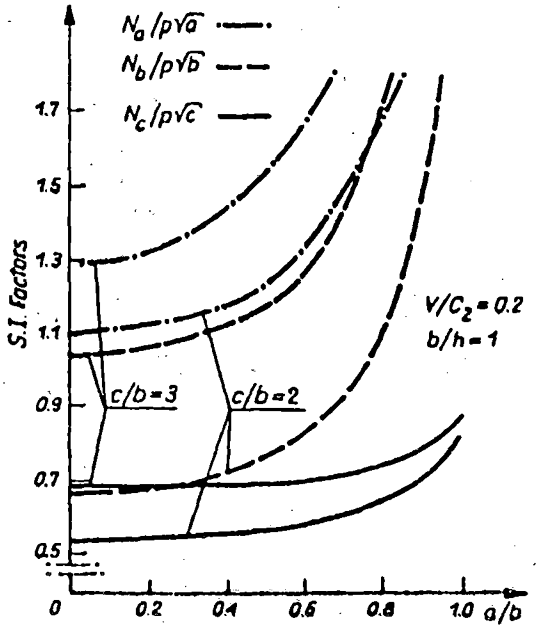


FIG. 4. Stress intensity factors Vs. a/b.

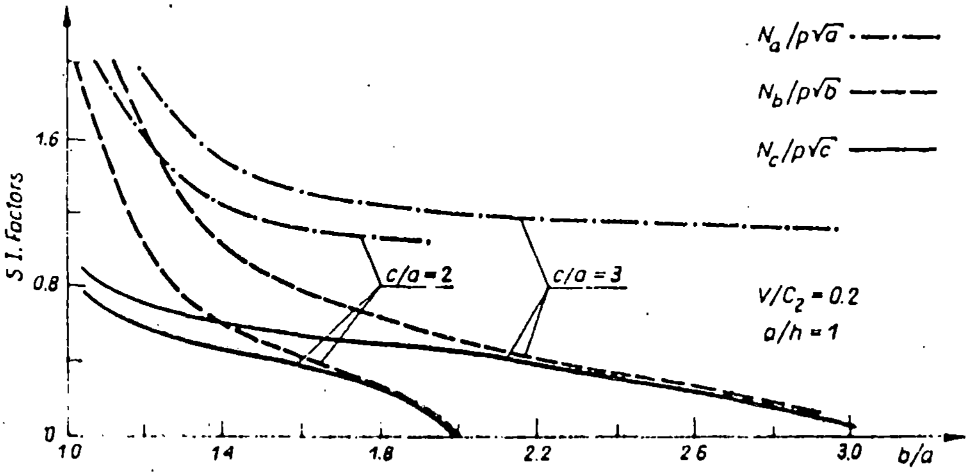


FIG. 5. Stress intensity factors Vs. b/a.

FOUR CO-PLANAR GRIFFITH CRACKS IN AN INFINITE ELASTIC MEDIUM

1. Introduction

In fracture mechanics, scattering of elastic waves by cracks of finite dimension in an infinite elastic medium has been investigated by several investigators. The problem of scattering of elastic wave from an interface crack was solved by Bostrom (1987). Srivastava et. al. (1980) solved the problem of interaction of anti-plane shear wave by an interface crack. The problem of diffraction of Love waves by a crack of finite width in the plane interface of a layered composite has been solved by Neerhoff (1979). Itou (1980) solved problem of diffraction of anti-plane shear wave by two co-planar Griffith cracks in an infinite elastic medium. The scattering of time harmonic normally incident plane wave by two co-planar Griffith cracks was solved by Jain and Kanwal (1972). Itou (1978) also solved the problem of stress concentration around two co-planar Griffith cracks in an infinite elastic medium.

As regards the crack problem, research has been restricted mainly to the case of single crack or a pair of cracks because of severe mathematical complexity encountered in solving the problems of three or more cracks. Recently, Dhawan and Dhaliwal (1978) solved the statical problem of determining the stress distribution in an infinite transversely isotropic medium containing three co-planar cracks.

To the best knowledge of the authors, the problem of stress distribution around four co-planar Griffith cracks has not been investigated so far. In this paper, we consider two cases regarding stress distribution around four co-planar Griffith cracks in an infinite homogeneous, isotropic medium. In the first case, cracks are assumed to be moving steadily along a fixed direction with constant velocity V. In the second case, the statical problem of determining the stress and displacement in an infinite homogeneous, isotropic medium weakened by four co-planar Griffith

---

Published in " Engineering Fracture Mechanics " Vol 43, pp 941-955, 1992.

cracks has been considered. Using Fourier integral transform both the problems have been reduced to solving a set of five integral equations. Employing finite Hilbert transform technique (1968) the integral equations have been solved to derive crack opening displacement and stress intensity factors which are presented in the form of graphs.

2.Statement Of Problem I And Its Formulation

Consider an infinite homogeneous isotropic material weakened by four co-planar Griffith cracks, moving steadily at a constant velocity V in the X- direction referred to a fixed coordinate system (X,Y,Z) as shown in the Fig 1. In absence of body force equations of motion in terms of displacement are

$$(\lambda+2\mu)\left[u_{,XX} + v_{,XY} \right] + \mu\left[u_{,YY} - v_{,XY} \right] = \rho\, u_{,TT}$$

and $$(\lambda+2\mu)\left[u_{,XY} + v_{,YY} \right] + \mu\left[v_{,XX} - u_{,XY} \right] = \rho\, v_{,TT} \qquad (1a,b)$$

where u,v denote the displacement components in X and Y directions and λ,μ are Lame's constants and $u_{,X}$ represents partial derivatives of u with respect to X.

For cracks moving with constant velocity V in the X- direction it is convenient to introduce the Galilean transformation

$$x = X - VT,\ y = Y,\ z = Z,\ t = T \qquad (2)$$

where (x,y,z) represents the translating coordinate system as shown in Fig 1.

In the moving coordinates, The equations of motion (1) become independent of time and take the form

$$(\lambda+2\mu-\rho V^2)u_{,xx} + (\lambda+\mu)v_{,xy} + \mu u_{,yy} = 0$$

$$(\lambda+2\mu)v_{,yy} + (\mu-\rho V^2)v_{,xx} + (\lambda+\mu)u_{,xy} = 0 \qquad (3a,b)$$

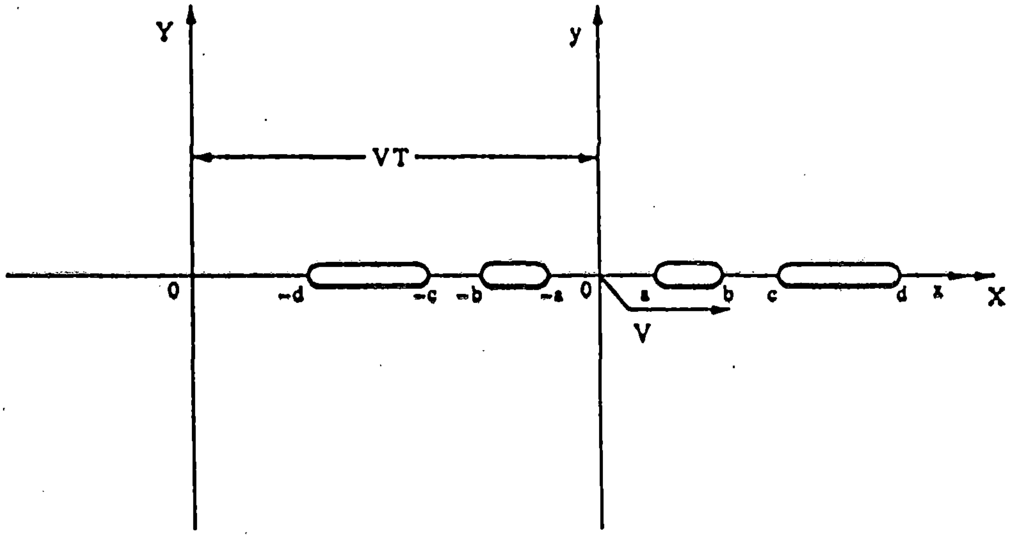


Fig. 1. Geometry and coordinate system.

Introducing $\bar{u}_s(\xi,y) = \int_0^\infty u(x,y)\,\sin(\xi x)\,dx$

$$\bar{v}_c(\xi,y) = \int_0^\infty v(x,y)\,\cos(\xi x)\,dx \qquad (4a,b)$$

and $u(x,y) = \frac{2}{\pi}\int_0^\infty \bar{u}_s(\xi,y)\,\sin(\xi x)\,d\xi$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \bar{v}_c(\xi,y)\,\cos(\xi x)\,d\xi \qquad (5a,b)$$

in equation (3) we obtain

$$\mu\bar{u}_{s,yy} - \xi(\lambda+\mu)\bar{v}_{c,y} - \xi^2(\lambda+2\mu - \rho V^2)\bar{u}_s = 0$$

$$(\lambda+2\mu)\bar{v}_{c,yy} + \xi(\lambda+\mu)\bar{u}_{s,y} - \xi^2(\mu - \rho V^2)\bar{v}_c = 0 \qquad (6a,b)$$

Elimination of $\bar{u}_s$ from (6a,b) yields the following ordinary differential equation

$$\left[\left\{\frac{d^2}{dy^2} - (1-M^2k^2)\xi^2\right\}\left\{\frac{d^2}{dy^2} - (1-M^2)\xi^2\right\}\right]\bar{v}_c = 0 \qquad (7)$$

where $M = V/c_2$, $k = c_2/c_1$.

The solution of the differential equation given by (7), for $y \geq 0$, is

$$\bar{v}_c(\xi,y) = A(\xi)\,e^{-\xi y\sqrt{1-M^2k^2}} + B(\xi)\,e^{-\xi y\sqrt{1-M^2}} \qquad (8)$$

where the unknown functions $A(\xi)$ and $B(\xi)$ are to be determined using the boundary conditions of the proposed problem.

Employing (8) in equations (6a,b), we obtain

$$\bar{u}_s(\xi,y) = \frac{A(\xi)}{\sqrt{1-M^2k^2}}\, e^{-\xi y\sqrt{1-M^2k^2}} + \sqrt{1-M^2}\, B(\xi)e^{-\xi y\sqrt{1-M^2}}, \quad y\geq 0 \qquad (9)$$

Therefore, the stress components given by

$$\sigma_{yy} = \lambda(u_{,x} + v_{,y}) + 2\mu v_{,y}$$

$$\sigma_{xy} = \mu(u_{,y} + v_{,x}) \qquad (10a,b)$$

become

$$\sigma_{yy}(x,y) = -\frac{2\mu}{\pi}\int_0^\infty \xi \left[\frac{2-M^2}{\sqrt{1-M^2k^2}}\, A(\xi)e^{-\xi y\sqrt{1-M^2k^2}} + 2\sqrt{1-M^2}\, B(\xi)e^{-\xi y\sqrt{1-M^2}}\right]. \cos(\xi x)\, d\xi$$

$$\sigma_{xy}(x,y) = -\frac{2\mu}{\pi}\int_0^\infty \xi \left[2A(\xi)e^{-\xi y\sqrt{1-M^2k^2}} + (2-M^2)B(\xi)e^{-\xi y\sqrt{1-M^2}}\right]\sin(\xi x)\, d\xi$$

$$(11a,b)$$

with

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[\frac{A(\xi)}{\sqrt{1-M^2k^2}}\, e^{-\xi y\sqrt{1-M^2k^2}} + \sqrt{1-M^2}\, B(\xi)e^{-\xi y\sqrt{1-M^2}}\right]\sin(\xi x)\, d\xi$$

and

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \left[A(\xi)\, e^{-\xi y\sqrt{1-M^2k^2}} + B(\xi)\, e^{-\xi y\sqrt{1-M^2}}\right]\cos(\xi x)\, d\xi$$

$$(12a,b)$$

Let four co-planar Griffith cracks of finite length located along X-axis be moving steadily with velocity V in the direction of X axis so that their position referred to translating coordinate (x,y,z) are $a\leq|x|\leq b$, $c\leq|x|\leq d$ on $y=0$.

The boundary conditions of the proposed problem on account of symmetry with respect to y-axis are

$$v(x,0)=0\,,\qquad x\in I_1,I_3,I_5 \tag{13a-c}$$

$$\sigma_{xy}(x,0)=0,\qquad 0<x<\infty \tag{14}$$

$$\sigma_{yy}(x,0)=-p,\qquad x\in I_2,I_4 \tag{15a,b}$$

where $I_1=(0,a)$, $I_2=(a,b)$, $I_3=(b,c)$, $I_4=(c,d)$, $I_5=(d,\infty)$

Using the condition (14) in (11b) we find that $A(\xi)$, $B(\xi)$ are related by

$$B(\xi)=-\frac{2}{2-M^2}A(\xi) \tag{16}$$

With the help of the boundary condition (13), we obtain from (12b)

$$\int_0^\infty A(\xi)\cos(\xi x)\,d\xi=0,\qquad x\in I_1,I_3,I_5 \tag{17a-c}$$

Substitution of (11a) in (15) yields with the aid of (16)

$$\int_0^\infty \xi\,A(\xi)\cos(\xi x)\,d\xi=\frac{P\pi}{2\mu},\qquad x\in I_2,I_4 \tag{18a,b}$$

where $\qquad P=\dfrac{p}{K},\quad K=\dfrac{(2-M^2)^2-4\sqrt{(1-M^2k^2)(1-M^2)}}{(2-M^2)\sqrt{1-M^2k^2}}$

3.Method Of Solution

In order to solve the set of five integral equations given in equations (17) and (18) we assume

$$A(\xi)=\frac{1}{\xi}\int_a^b h(s^2)\sin(\xi s)\,ds+\frac{1}{\xi}\int_c^d g(t^2)\sin(\xi t)\,dt \tag{19}$$

where $h(s^2)$ and $g(t^2)$ are unknown functions to be determined from the boundary conditions.

Inserting the value of $A(\xi)$ from equation (19) in equation (17) it is found that this choice of $A(\xi)$ leads to the equations

$$\int_a^b h(s^2)\,ds = 0 \text{ and } \int_c^d g(t^2)\,dt = 0 \qquad (20a,b)$$

Further substituting $A(\xi)$ from equation (19) in (18a), we obtain

$$\int_a^b \frac{sh(s^2)}{s^2-x^2}\,ds + \int_c^d \frac{tg(t^2)}{t^2-x^2}\,dt = \frac{\pi P}{2\mu}, \quad x \in I_2$$

Rewriting this equation as

$$\int_a^b \frac{sh(s^2)}{s^2-x^2}\,ds = \frac{\pi}{2}F(x), \quad x \in I_2$$

where $$F(x) = \frac{P}{\mu} - \frac{2}{\pi}\int_c^d \frac{tg(t^2)}{t^2-x^2}\,dt$$

and using finite Hilbert transform technique (1968), we obtain

$$h(s^2) = \frac{P}{\mu}\sqrt{\frac{s^2-a^2}{b^2-s^2}} - \frac{2}{\pi}\sqrt{\frac{s^2-a^2}{b^2-s^2}}\int_c^d \sqrt{\frac{t^2-b^2}{t^2-a^2}}\,\frac{tg(t^2)}{t^2-s^2}\,dt + \frac{C_1}{\sqrt{(s^2-a^2)(b^2-s^2)}}$$

where we have used (21)

$$\int_a^b \sqrt{\frac{b^2-x^2}{x^2-a^2}}\,\frac{x\,dx}{(s^2-x^2)(t^2-x^2)} = \frac{\pi}{2}\sqrt{\frac{t^2-b^2}{t^2-a^2}}\,\frac{1}{t^2-s^2}$$

The constant C_1 is to be determined from equation (20).

Substituting the value of $h(s^2)$ from (21) in (19) and using the resulting value of $A(\xi)$ in the boundary condition (18b) we obtain, using the results

$$\int_a^b \sqrt{\frac{s^2-a^2}{b^2-s^2}}\,\frac{s\,ds}{(s^2-x^2)(t^2-s^2)} = \frac{\pi}{2}\left[\sqrt{\frac{t^2-a^2}{t^2-b^2}} - \sqrt{\frac{x^2-a^2}{x^2-b^2}}\right]\frac{1}{t^2-x^2}$$

and $\displaystyle\int_a^b \frac{s\,ds}{(s^2-x^2)\sqrt{(s^2-a^2)(b^2-s^2)}} = -\frac{\pi}{2\sqrt{(x^2-a^2)(x^2-b^2)}}$ for $x \in I_4$

the singular integral equation

$$\int_c^d \sqrt{\frac{t^2-b^2}{t^2-a^2}}\,\frac{tg(t^2)}{t^2-x^2}\,dt = \frac{\pi}{2}\left[\frac{P}{\mu} + \frac{C_1}{x^2-a^2}\right],\quad x \in I_4$$

Again using finite Hilbert transform technique [9], we obtain

$$g(t^2) = \frac{P}{\mu}\sqrt{\frac{(t^2-a^2)(t^2-c^2)}{(t^2-b^2)(d^2-t^2)}} + \sqrt{\frac{d^2-a^2}{c^2-a^2}}\,\frac{C_1\sqrt{t^2-c^2}}{\sqrt{(t^2-a^2)(t^2-b^2)(d^2-t^2)}} + \frac{C_2\sqrt{t^2-a^2}}{\sqrt{(t^2-b^2)(t^2-c^2)(d^2-t^2)}} \tag{22}$$

where we have used

$$\int_c^d \sqrt{\frac{d^2-x^2}{x^2-c^2}}\,\frac{x\,dx}{(x^2-a^2)(x^2-t^2)} = -\frac{\pi}{2}\sqrt{\frac{d^2-a^2}{c^2-a^2}}\,\frac{1}{t^2-a^2}$$

and the constant C_2 is to be determined using the condition given by equation (20).

Next substituting the value of $g(t^2)$ from (22) in equation (21) and finally using the following results

$$\int_c^d \sqrt{\frac{t^2-c^2}{d^2-t^2}}\,\frac{t\,dt}{(t^2-a^2)(t^2-s^2)} = \frac{\pi}{2}\left[\sqrt{\frac{c^2-a^2}{d^2-a^2}} - \sqrt{\frac{c^2-s^2}{d^2-s^2}}\right]\frac{1}{s^2-a^2}$$

$$\int_c^d \frac{t\,dt}{(t^2-s^2)\sqrt{(t^2-c^2)(d^2-t^2)}} = \frac{\pi}{2\sqrt{(c^2-s^2)(d^2-s^2)}} \quad \text{for } s \in I_2$$

$h(s^2)$ is derived in the form

$$h(s^2) = \frac{P}{\mu}\sqrt{\frac{(s^2-a^2)(c^2-s^2)}{(b^2-s^2)(d^2-s^2)}} + \sqrt{\frac{d^2-a^2}{c^2-a^2}}\;\frac{C_1\sqrt{c^2-s^2}}{\sqrt{(s^2-a^2)(b^2-s^2)(d^2-s^2)}} - \frac{C_2\sqrt{s^2-a^2}}{\sqrt{(b^2-s^2)(c^2-s^2)(d^2-s^2)}} \tag{23}$$

To determine the values of the unknown constants C_1 and C_2 we substitute $g(t^2)$ and $h(s^2)$ given by (22) and (23) in (20) and obtain

$$C_1 = \frac{K^{c,d}_{a,b}\, I^{a,b}_{c,d} + K^{a,b}_{c,d}\, J^{c,d}_{a,b}}{I^{c,d}_{a,b}\, I^{a,b}_{c,d} + J^{c,d}_{a,b}\, J^{a,b}_{c,d}}\;\frac{P}{\mu}\sqrt{\frac{c^2-a^2}{d^2-a^2}}$$

$$C_2 = \frac{K^{a,b}_{c,d}\, I^{c,d}_{a,b} - K^{c,d}_{a,b}\, J^{a,b}_{c,d}}{I^{c,d}_{a,b}\, I^{a,b}_{c,d} + J^{c,d}_{a,b}\, J^{a,b}_{c,d}}\;\frac{P}{\mu}$$

where

$$I^{r,s}_{p,q} = \int_p^q \frac{\sqrt{x^2-r^2}\;dx}{\sqrt{(x^2-p^2)(x^2-q^2)(s^2-x^2)}}$$

$$J^{r,s}_{p,q} = \int_p^q \frac{\sqrt{x^2-p^2}\;dx}{\sqrt{(x^2-q^2)(x^2-r^2)(s^2-x^2)}}$$

$$K^{r,s}_{p,q} = -\int_p^q \frac{\sqrt{(x^2-p^2)(x^2-r^2)}}{\sqrt{(x^2-q^2)(s^2-x^2)}}\,dx$$

The relevant displacement and stress components in the plane of crack can now be shown to be given by

$$v(x,0) = \int_x^b h(s^2)\,ds\ , \qquad a\le x\le b$$

$$= \int_x^d g(t^2)\,dt\ , \qquad c\le x\le d \qquad (24a,b)$$

and

$$[\sigma_{yy}(x,o)]_{0<x<a} = -\frac{2\mu K}{\pi}\left[\int_a^b \frac{sh(s^2)}{s^2-x^2}\,ds + \int_c^d \frac{tg(t^2)}{t^2-x^2}\,dt\right]$$

$$[\sigma_{yy}(x,o)]_{b<x<c} = \frac{2\mu K}{\pi}\left[\int_a^b \frac{sh(s^2)}{x^2-s^2}\,ds - \int_c^d \frac{tg(t^2)}{t^2-x^2}\,dt\right]$$

$$[\sigma_{yy}(x,o)]_{x>d} = \frac{2\mu K}{\pi}\left[\int_a^b \frac{sh(s^2)}{x^2-s^2}\,ds + \int_c^d \frac{tg(t^2)}{x^2-t^2}\,dt\right] \qquad (25a\text{-}c)$$

Insertion of the values of $h(s^2)$ and $g(t^2)$ as given by the equations (22) and (23) in the expressions (25) yields after some algebraic manipulation,

$$[\sigma_{yy}(x,o)]_{0<x<a} = -\frac{2\mu K}{\pi}\left[F_1(x)+F_2(x)+F_3(x)+F_4(x)+F_6(x)+F_7(x)\right]$$

$$[\sigma_{yy}(x,o)]_{b<x<c} = -\frac{2\mu K}{\pi}\left[F_1(x)+F_2(x)+F_3(x)+F_4(x)-F_5(x)-F_8(x)\right]$$

$$[\sigma_{yy}(x,o)]_{x>d} = -\frac{2\mu K}{\pi}\left[F_1(x)+F_2(x)+F_3(x)+F_4(x)-F_6(x)-F_7(x)\right]$$

(26a-c)

where

$$F_1(x) = \left[\frac{P}{\mu} (c^2-a^2) - C_2 \right]\left[1- \sqrt{\frac{a^2- x^2}{b^2- x^2}} \right] \frac{\pi}{2\sqrt{(c^2- a^2)(d^2- a^2)}}$$

$$F_2(x) = \int_a^b \left[\frac{P}{\mu} (d^2-c^2) - C_2 \frac{2u^2- d^2- c^2}{c^2- u^2} \right] \frac{g_1(u,x)}{d^2- u^2} du$$

$$F_3(x) = \left[\frac{P}{\mu}(c^2-a^2) + C_1 \sqrt{\frac{d^2- a^2}{c^2- a^2}} \right]\left[1- \sqrt{\frac{c^2- x^2}{d^2- x^2}} \right] \frac{\pi}{2\sqrt{(c^2- a^2)(c^2- b^2)}}$$

$$F_4(x) = \int_c^d \left[\frac{P}{\mu} (b^2-a^2) + C_1 \sqrt{\frac{d^2- a^2}{c^2- a^2}} \frac{2u^2- a^2- b^2}{u^2- a^2} \right] \frac{g_2(u,x)}{u^2- b^2} du$$

$$F_{5,6}(x) = \frac{\pi}{2} \sqrt{\frac{d^2- a^2}{d^2- b^2}} \left[\frac{C_1}{X_1} \sqrt{\frac{c^2- b^2}{c^2- a^2}} \mp \frac{C_2}{X_2} \right]$$

$$F_{7,8}(x) = \frac{C_1}{X_1} \sqrt{\frac{d^2- a^2}{c^2- a^2}} L^{a,b}_{c,d}(x) \mp \frac{C_2}{X_2} L^{c,d}_{a,b}(x)$$

$$g_1(u,x) = \frac{u}{\sqrt{(d^2- u^2)(c^2- u^2)}} \left[\sqrt{\frac{a^2- x^2}{b^2- x^2}} \tan^{-1}\sqrt{\frac{(a^2- x^2)(b^2- u^2)}{(b^2- x^2)(u^2- a^2)}} - \right.$$

$$\left. - \tan^{-1}\sqrt{\frac{b^2- u^2}{u^2- a^2}} \right]$$

$$g_2(u,x) = \frac{u}{\sqrt{(u^2- b^2)(u^2- a^2)}} \left[\sqrt{\frac{c^2- x^2}{d^2- x^2}} \tan^{-1}\sqrt{\frac{(c^2- x^2)(d^2- u^2)}{(d^2- x^2)(u^2- c^2)}} - \right.$$

$$\left. - \tan^{-1}\sqrt{\frac{d^2- u^2}{u^2- c^2}} \right]$$

$$X_1 = \sqrt{(x^2 - a^2)(x^2 - b^2)}$$

$$X_2 = \sqrt{(x^2 - c^2)(x^2 - d^2)}$$

$$L^{p,q}_{r,s}(x) = \int_p^q \frac{(s^2 - r^2)u \tan^{-1}\sqrt{\frac{(u^2 - p^2)(x^2 - q^2)}{(q^2 - u^2)(x^2 - p^2)}}}{\sqrt{(s^2 - u^2)^3(r^2 - u^2)}}\, du \qquad (27a\text{-}k)$$

4. Stress Intensity Factors:

The dynamic stress intensity factors are given by

$$N_a = \underset{x \to a-}{\mathrm{Lt}} \sqrt{2(a-x)} \left[\sigma_{yy}(x,0) \right]_{0<x<a}$$

$$N_b = \underset{x \to b+}{\mathrm{Lt}} \sqrt{2(x-b)} \left[\sigma_{yy}(x,0) \right]_{b<x<c}$$

$$N_c = \underset{x \to c-}{\mathrm{Lt}} \sqrt{2(c-x)} \left[\sigma_{yy}(x,0) \right]_{b<x<c}$$

$$N_d = \underset{x \to d+}{\mathrm{Lt}} \sqrt{2(x-d)} \left[\sigma_{yy}(x,0) \right]_{x>d} \qquad (28a\text{-}d)$$

Employing (26) in (28) we obtain

$$N_a = -\frac{\mu K C_1}{\sqrt{a(b^2 - a^2)}}$$

$$N_b = \mu K \left[\frac{P}{\mu}\sqrt{\frac{(b^2 - a^2)(c^2 - b^2)}{b(d^2 - b^2)}} + C_1\sqrt{\frac{(d^2 - a^2)(c^2 - b^2)}{b(b^2 - a^2)(d^2 - b^2)(c^2 - a^2)}} - \right.$$

$$\left. - C_2\sqrt{\frac{(b^2 - a^2)}{b(c^2 - b^2)(d^2 - b^2)}} \right]$$

$$N_c = -\frac{\mu K C_2}{\sqrt{c(d^2-c^2)}}\sqrt{\frac{c^2-a^2}{c^2-b^2}}$$

$$N_d = \mu K\left[\frac{P}{\mu}\sqrt{\frac{(d^2-a^2)(d^2-c^2)}{d(d^2-b^2)}} + C_1\sqrt{\frac{(d^2-c^2)}{d(c^2-a^2)(d^2-b^2)}} + \right.$$

$$\left. + C_2\sqrt{\frac{(d^2-a^2)}{d(d^2-c^2)(d^2-b^2)}}\right] \tag{29}$$

It is interesting to note that the crack opening displacements depend on the crack velocity V but in the plane of the cracks the stresses and stress intensity factors are independent of the velocity of the moving cracks in an infinite elastic medium.

5.Statement Of Problem II And Its Formulation

In this case, we consider an infinite homogeneous isotropic material with four coplanar Griffith cracks located at $Y = 0$, $a \le |X| \le b$, $c \le |X| \le d$ and subjected to uniform internal pressure q. In absence of body force, the equations of equilibrium in terms of displacement are

$$(\lambda+2\mu)\left[u_{,XX} + v_{,XY}\right] + \mu\left[u_{,YY} - v_{,XY}\right] = 0$$

and $$(\lambda+2\mu)\left[u_{,XY} + v_{,YY}\right] + \mu\left[v_{,XX} - u_{,XY}\right] = 0 \tag{30a,b}$$

Since the problem exhibits a state of symmetry about $Y = 0$, we can restrict our attention to a single half-space occupying the region $Y \ge 0$.

The equations (30) are to be solved subject to the boundary conditions

$$v(X,0) = 0, \quad |X|\leq a, \quad b\leq|X|\leq c, \quad |X|\geq d \tag{31a-c}$$

$$\sigma_{XY}(X,0) = 0, \quad -\infty < X < \infty \tag{32}$$

$$\sigma_{YY}(X,0) = -\,q\ , \quad a\leq|X|\leq b, \quad c\leq|X|\leq d \tag{33a,b}$$

In view of the boundary conditions, appropriate integral solutions of equation (30) are

$$u(X,Y) = \frac{2}{\pi}\int_0^\infty\left[C(\xi) + D(\xi)\left\{Y - \frac{1}{\xi}\,\frac{\lambda+3\mu}{\lambda+\mu}\right\}\right]e^{-\xi Y}\sin(\xi X)\,d\xi$$

and
$$v(X,Y) = \frac{2}{\pi}\int_0^\infty\left[C(\xi) + Y\,D(\xi)\right]e^{-\xi Y}\cos(\xi X)\,d\xi \tag{34a,b}$$

Therefore,

$$\sigma_{YY}(X,Y) = -\frac{4\mu}{\pi}\int_0^\infty\left[\xi C(\xi) + \left\{Y\xi - \frac{\mu}{\lambda+\mu}\right\}D(\xi)\right]e^{-\xi Y}\cos(\xi X)\,d\xi$$

$$\sigma_{XY}(X,Y) = -\frac{4\mu}{\pi}\int_0^\infty\left[\xi C(\xi) + \left\{Y\xi - \frac{\lambda+2\mu}{\lambda+\mu}\right\}D(\xi)\right]e^{-\xi Y}\sin(\xi X)\,d\xi \tag{35a,b}$$

It may be noted that the displacement and stress components given by (34) and (35) can not be derived from the corresponding expressions of the dynamic problem given in (11) and (12) on setting M = 0.

The functions $C(\xi)$and $D(\xi)$ are to be determined from the boundary conditions (31)-(33), which yield

$$C(\xi) = \frac{1}{\xi}\,\frac{\lambda+2\mu}{\lambda+\mu}\,D(\xi) \tag{36}$$

and the following set of five integral equations

$$\int_0^\infty C(\xi)\cos(\xi X)\,d\xi = 0, \quad X \in I_1, I_3, I_5 \tag{37a-c}$$

$$\int_0^\infty \xi\ C(\xi)\cos(\xi X)\ d\xi = \frac{Q\pi}{2\mu}\ , \quad X \in I_2, I_4 \qquad (38a,b)$$

where

$Q = \frac{(\lambda+2\mu)}{2(\lambda+\mu)}\ q$ and I_j $(j=1,2,..,5)$ are the intervals defined earlier in problem I.

6.Method Of Solution And Quantities Of Physical Interest

Integral equations given by (37) and (38) are found to be the same as given by equations (17) and (18) with the exception that P is replaced by Q. Therefore, the same technique as that used in problem I can be employed to obtain

$$v(X,0)=\int_X^b\Bigg[\ \frac{Q}{\mu}\sqrt{\frac{(s^2-a^2)(c^2-s^2)}{(b^2-s^2)(d^2-s^2)}}+\sqrt{\frac{d^2-a^2}{c^2-a^2}}\ \frac{C_1\sqrt{c^2-s^2}}{\sqrt{(s^2-a^2)(b^2-s^2)(d^2-s^2)}}-$$

$$-\frac{C_2\sqrt{s^2-a^2}}{\sqrt{(b^2-s^2)(c^2-s^2)(d^2-s^2)}}\Bigg]ds, \qquad a\le X\le b$$

$$=\int_X^d\Bigg[\ \frac{Q}{\mu}\sqrt{\frac{(t^2-a^2)(t^2-c^2)}{(t^2-b^2)(d^2-t^2)}}+\sqrt{\frac{d^2-a^2}{c^2-a^2}}\ \frac{C_1\sqrt{t^2-c^2}}{\sqrt{(t^2-a^2)(t^2-b^2)(d^2-t^2)}}+$$

$$+\frac{C_2\sqrt{t^2-a^2}}{\sqrt{(t^2-b^2)(t^2-c^2)(d^2-t^2)}}\Bigg]dt, \qquad c\le X\le d$$

(39a,b)

Stresses in the regions $0<X<a$, $b<X<c$, $X>d$ are found to be the same as that given in (26), the only change being that P is to replaced by Q.

Amount of energy in opening the cracks $a \le |X| \le b$, $c \le |X| \le d$ are given by $E = 2E_1 + 2E_2$, where

$$E_1 = 2\left| \int_a^b \left[\sigma_{YY}(X,0)\ v(X,0) \right] dX \right|$$

$$E_2 = 2\left| \int_c^d \left[\sigma_{YY}(X,0)\ v(X,0) \right] dX \right| \qquad (40a,b)$$

Equations (40) can be simplified, with the aid of (33) and (39), to

$$E_1 = -2q\left[\frac{Q}{\mu} M^{c,d}_{a,b} + (c^2-b^2)L_1\, \Pi\left\{\frac{\pi}{2},\ \frac{b^2-a^2}{c^2-a^2},\ r\right\} + \frac{(c^2-a^2)C_2 - c^4\frac{Q}{\mu}}{\sqrt{(d^2-b^2)(c^2-a^2)}} F\left(\frac{\pi}{2}, r\right) \right]$$

$$E_2 = 2q\left[\frac{Q}{\mu} M^{a,b}_{c,d} - (d^2-a^2)L_2\, \Pi\left\{\frac{\pi}{2}, \frac{c^2-d^2}{c^2-a^2}, r\right\} - \frac{\sqrt{(c^2-a^2)(d^2-a^2)}\ C_1 + a^4\frac{Q}{\mu}}{\sqrt{(d^2-c^2)(c^2-a^2)}} F\left(\frac{\pi}{2}, r\right) \right]$$

where

$$L_{1,2} = \frac{\left[(a^2+c^2)\frac{Q}{\mu} \mp C_1\sqrt{\frac{d^2-a^2}{c^2-a^2}} \mp C_2 \right]}{\sqrt{(d^2-b^2)(c^2-a^2)}},$$

$$r = \sqrt{\frac{(d^2-c^2)(b^2-a^2)}{(d^2-b^2)(c^2-a^2)}}, \quad 2M^{r,s}_{p,q} = \int_{p^2}^{q^2} \frac{z^2\,dz}{\sqrt{(z-p^2)(z-q^2)(z-r^2)(s^2-z)}}$$

and $F(\phi, r)$, $\Pi(\phi, n, r)$ are the elliptic integrals of first and third kinds respectively.

7. Numerical Results and Discussions

Numerical results for the stress intensity factors and crack opening displacement, defined as $\Delta v(x,0) = v(x,0^+) - v(x,0^-)$, for different values of the parameters are presented in this section. Numerical calculations have been carried out for both the dynamic and static problems. As the crack velocity is less than Rayleigh wave velocity, it is reasonable to take the value of M less than 0.9194.

Problem I: Variations of crack opening displacement for different values of crack speed, crack lengths and the separating distance between the cracks have been plotted in figures 2-4. It is interesting to note from these graphs that crack opening displacement on both the cracks decreases with the increase in the value of M at the onset and takes its minimum value at M=0.7415 ,after which it increases with the increase in the value of M. It has also been depicted in figures 3-4 that on each of the cracks,crack opening displacement decreases as the crack length decreases.

It has been mentioned earlier that the stress intensity factors at the crack tips are independent of crack speed and are found to depend on the crack lengths and the separating distance between the cracks. Variation of stress intensity factors with a/b for different values of c/b, d/b and that with c/b for different values a/b, d/b are plotted in figures 5-8 and figures 9-12 respectively.

It has been found that the effect of variation of the length of either the inner or the outer pair of cracks is more prominent on the stress intensity factors at the edges of the cracks whose lengths are varying compared to its effect on the stress intensity factors at the tips of the cracks whose lengths are kept fixed.

Problem II: Figures 13-15 show the variations of crack opening displacement for different values of the parameters a/b, c/b, d/b. They exhibit that crack opening displacement on a crack of fixed length increases with the increase in the length of the other crack as expected from physical stand point.

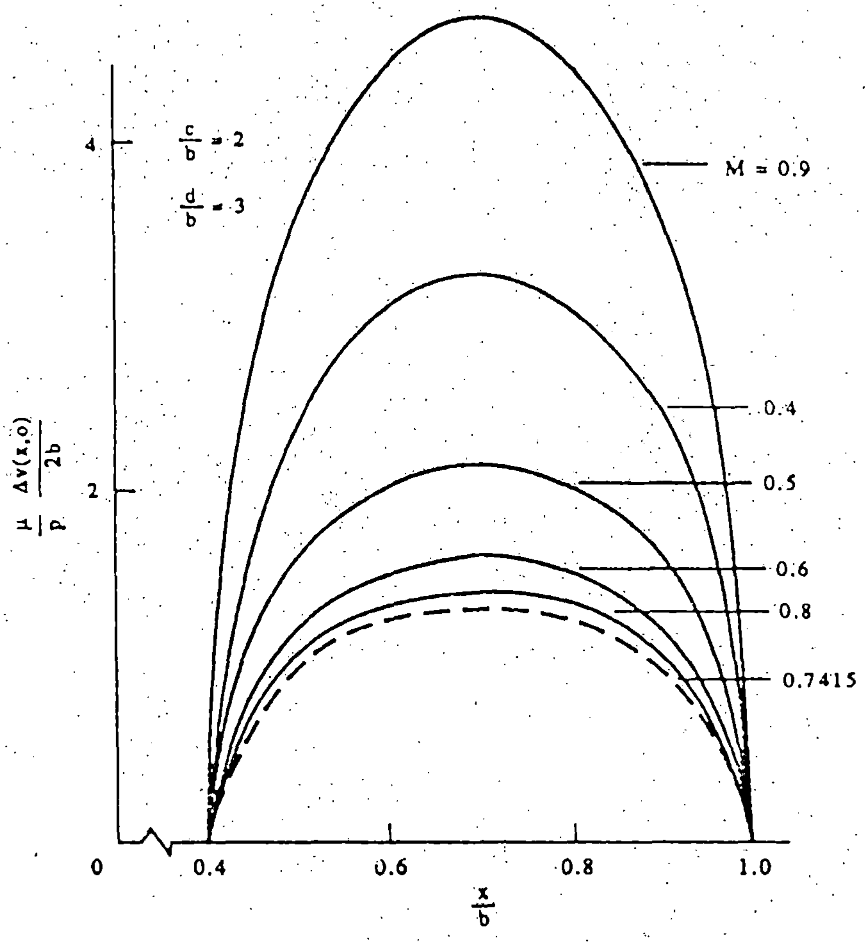


Fig. 2. Variation of crack opening displacement with x/b on the crack of the outer pair for problem I.

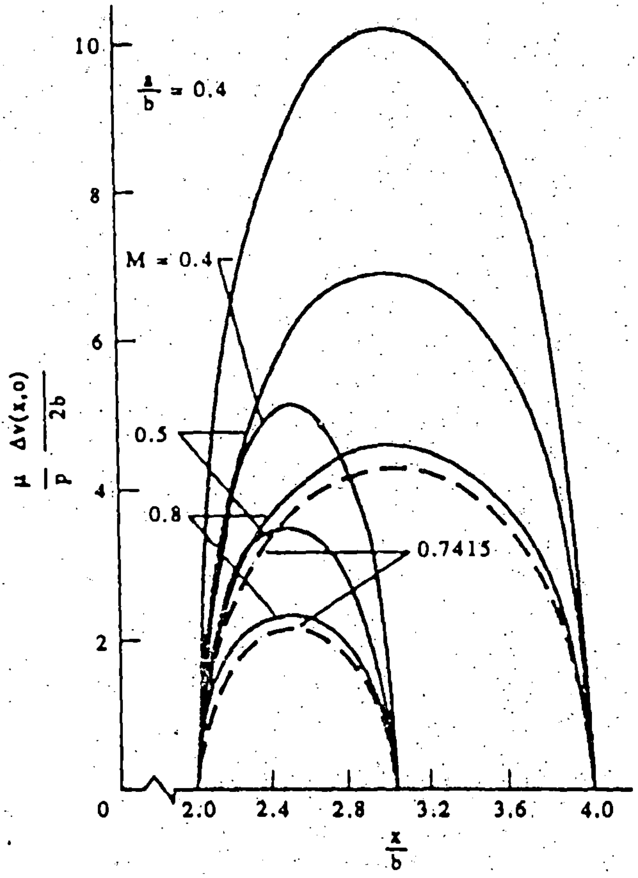


Fig. 3. Variation of crack opening displacement with x/b on the crack of the inner pair for problem I.

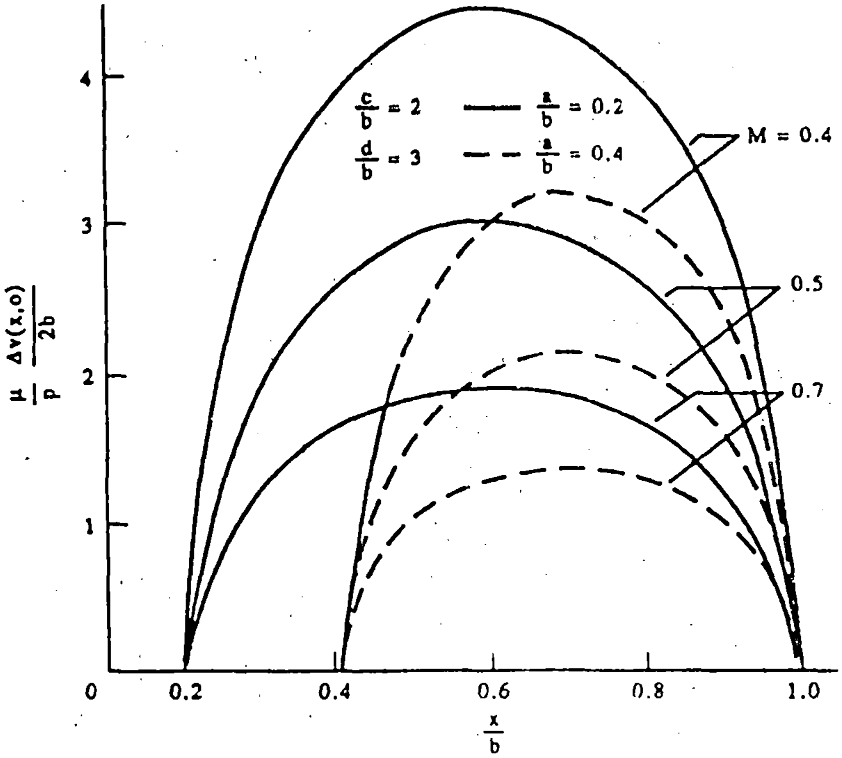


Fig. 4. Variation of crack opening displacement with x/b on the crack of the inner pair for problem I.

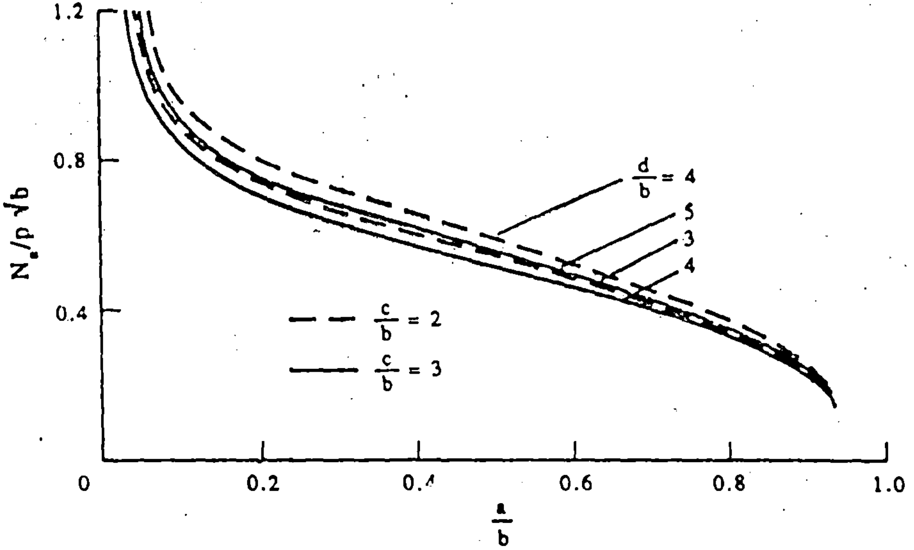


Fig. 5. Stress intensity factor vs a/b at the edge $x = a$.

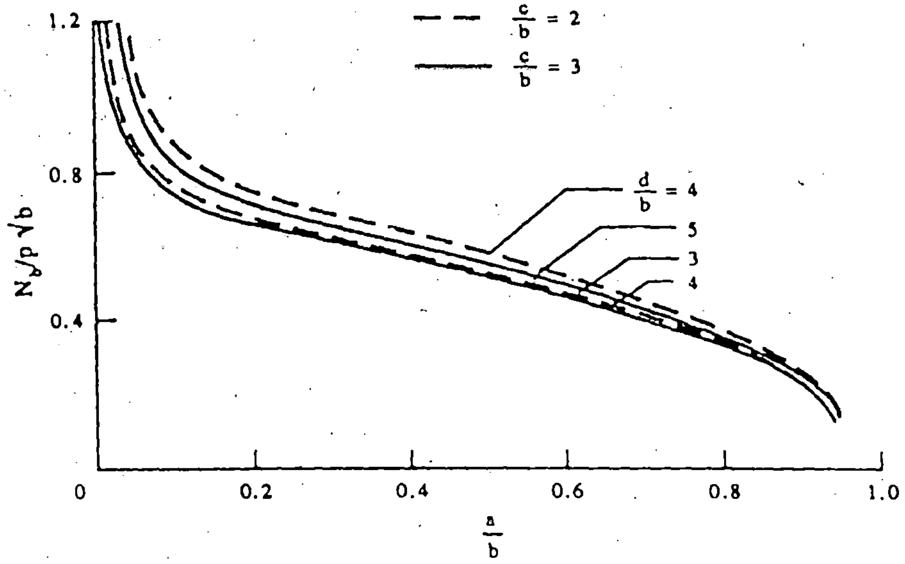


Fig. 6. Stress intensity factor vs a/b at the edge $x = b$.

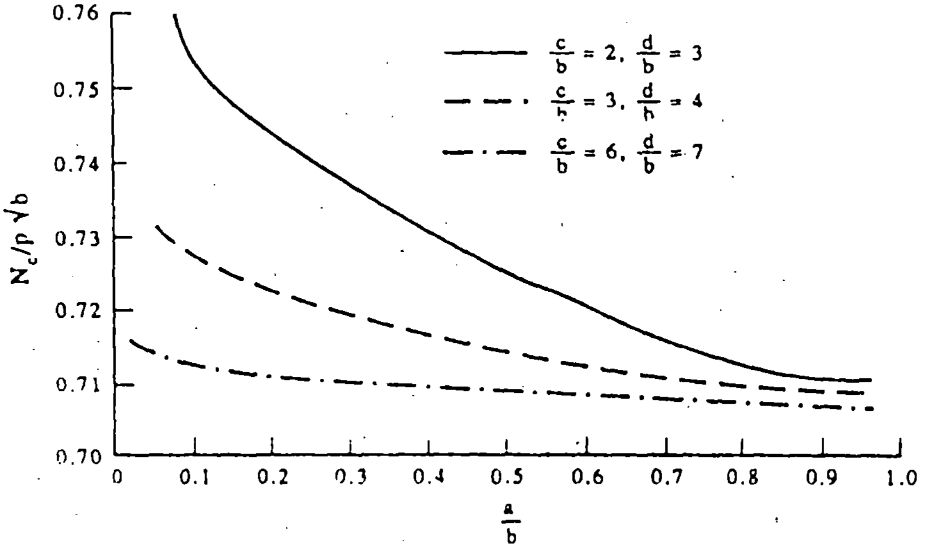


Fig. 7. Stress intensity factor vs a/b at the edge $x = c$

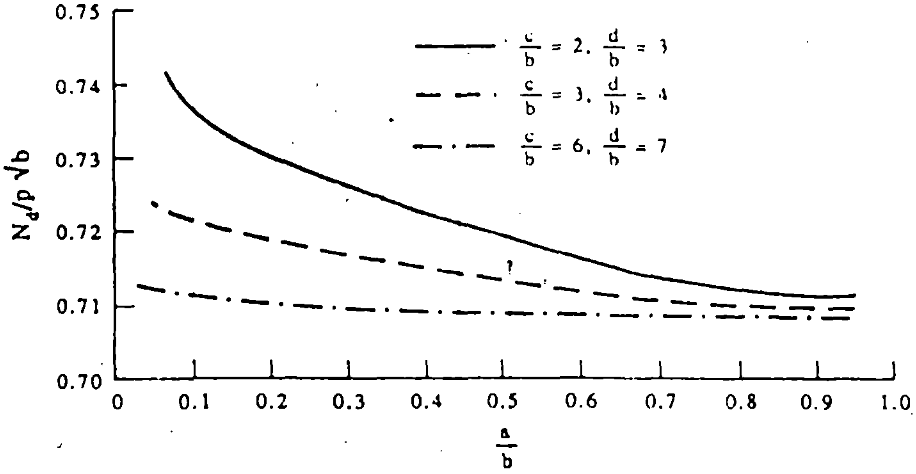


Fig. 8. Stress intensity factor vs a/b at the edge $x = d$.

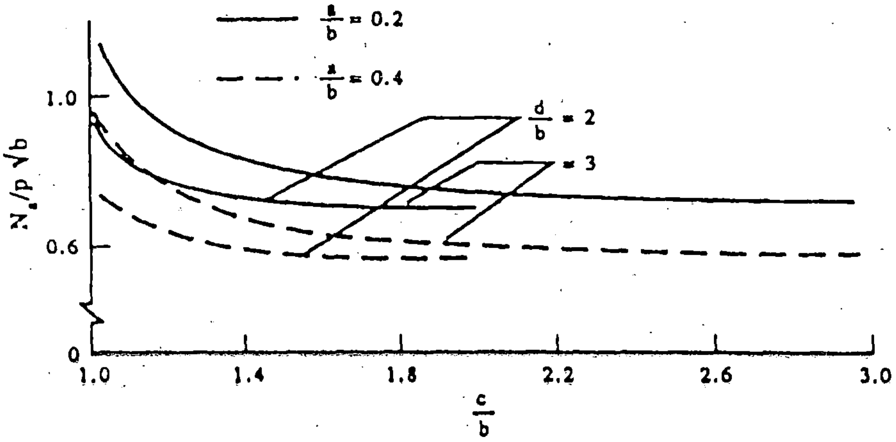


Fig. 9. Stress intensity factor vs c/b at the edge $x = a$.

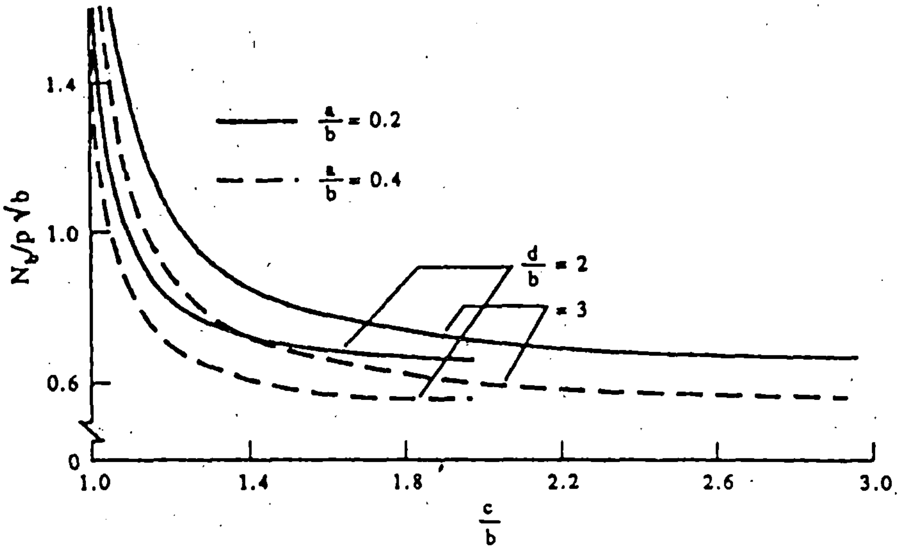


Fig. 10. Stress intensity factor vs c/b at the edge $x = b$.

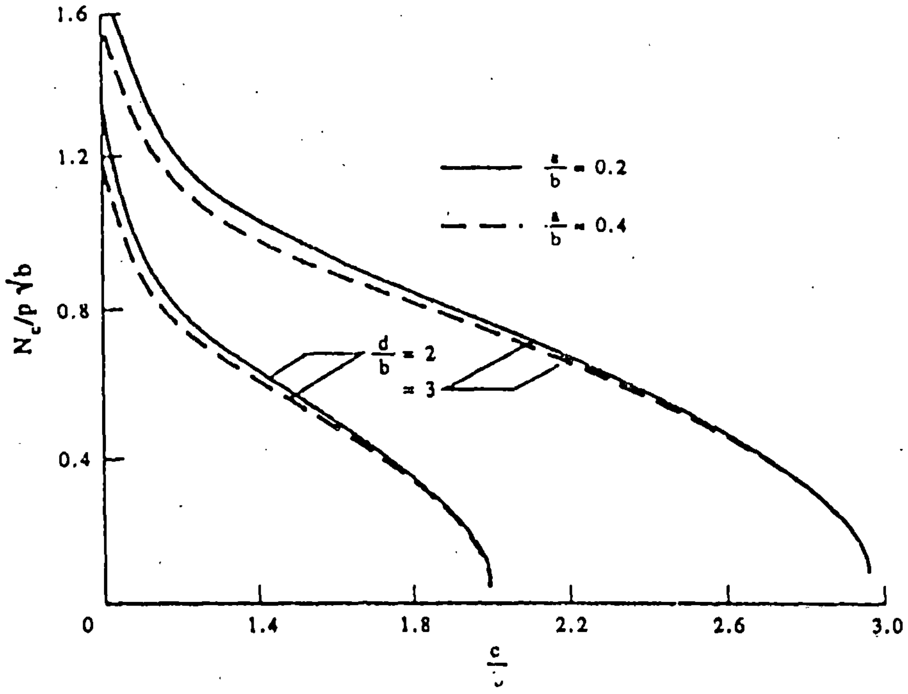


Fig. 11. Stress intensity factor vs c/b at the edge $x = c$.

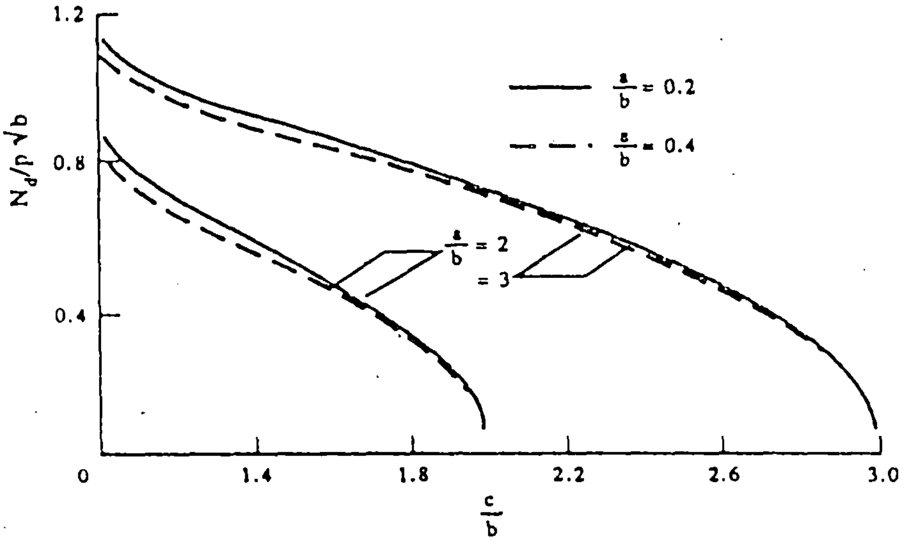


Fig. 12. Stress intensity factor vs c/b at the edge $x = d$.

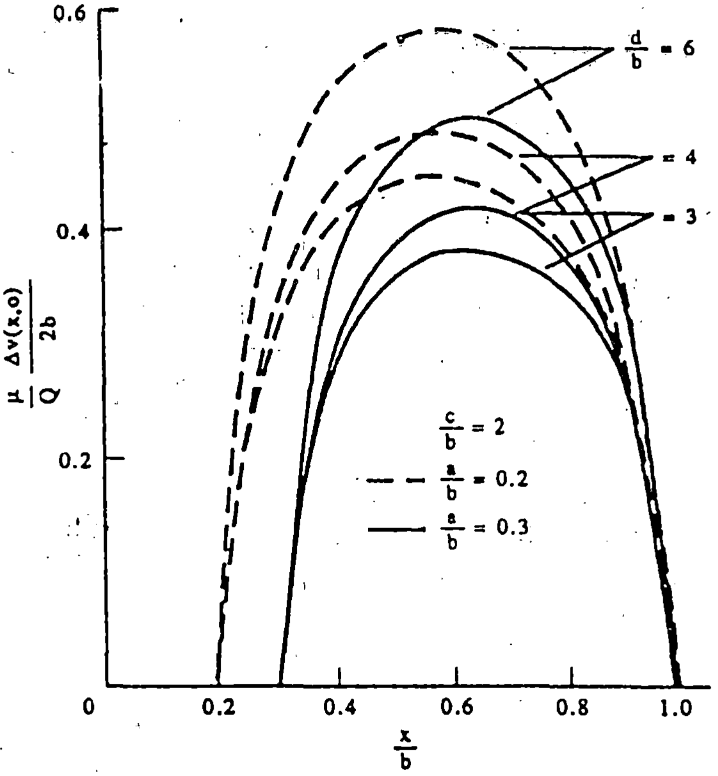


Fig. 13. Variation of crack opening displacement with X/b on the crack of the inner pair for problem II.

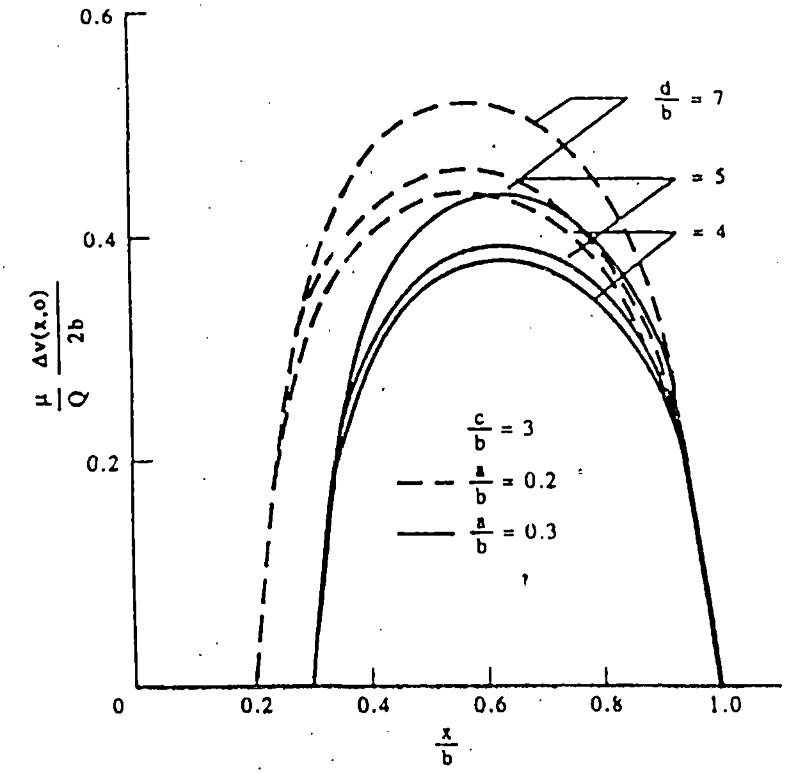


Fig. 14. Variation of crack opening displacement with X/b on the crack of the inner pair for problem II.

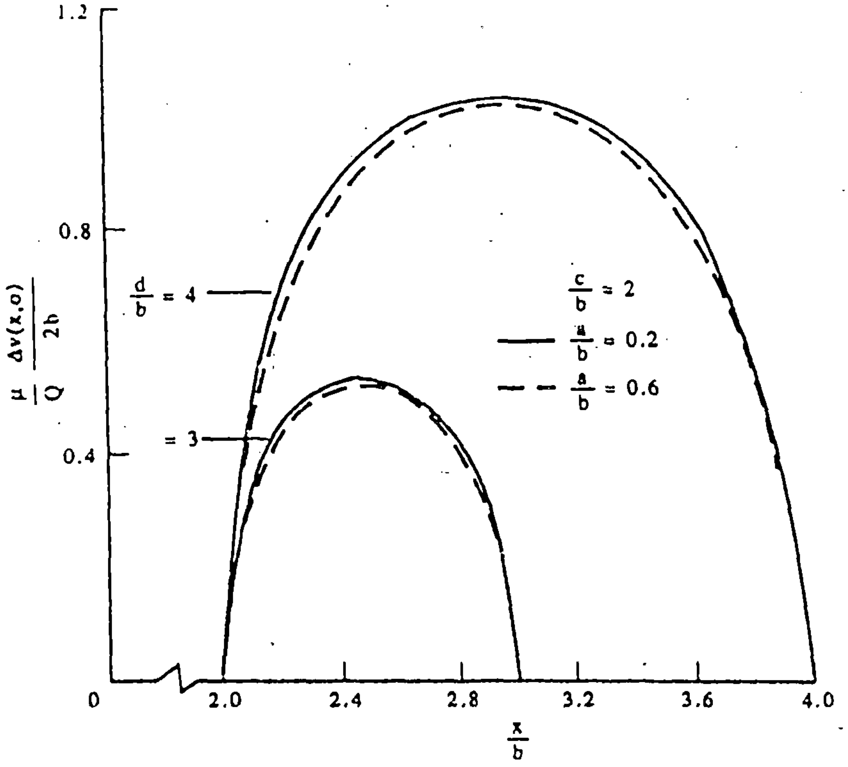


Fig. 15. Variation of crack opening displacement with X/b on the crack of the outer pair for problem II.

CHAPTER II

EXTENDING CRACK PROBLEMS IN ELASTODYNAMICS

Paper 7: Non- symmetric extension of a plane crack due to plane SH-waves in a pre- stressed infinite elastic medium.

Paper 8: Extension of a crack due to plane SH- wave in a pre-stressed infinite elastic medium.

Paper 9: Bifurcation of a crack due to plane SH- waves in a pre-stressed infinite elastic medium.

NON-SYMMETRIC EXTENSION OF A PLANE CRACK DUE TO PLANE SH-WAVES IN A PRE-STRESSED INFINITE ELASTIC MEDIUM

1. Introduction

Since Broberg's (1960) investigation of the solution of a crack expanding symmetrically with constant velocity under conditions of plane stress or strain in a homogeneous isotropic elastic medium in a field of spatially and time invariant tensile stress, a number of papers have appeared analyzing different geometrical situations. Craggs (1963) later solved the same problem as that done by Broberg but he used the method of homogeneous function to obtain the solution. Achenbach and Brock (1971) considered the wave motion generated by a uniformly extending shear crack in a body in a state of uniform anti- plane shear. The case of a crack expanding in an anisotropic medium was considered by Atkinson (1965). This work was later extended by Burridge and Willis (1963), who solved the problem of a crack with elliptical cross-section expanding symmetrically with uniform speed in an anisotropic medium. All the problems mentioned above are however self-similar ones with index (0,0) and are concerned with symmetric expansion of cracks.

Problems involving non-symmetric extension of cracks under uniform loading along the crack surface are not found much in the literature perhaps due to severe mathematical complexity encountered in solving such problems. Following the method of homogeneous functions developed by Craggs (1963)non-symmetric extension of a small flaw into a plane crack under polynomial form of loading was solved by Brock (1976). Following the same procedure, Brock (1975) also solved the problem of non-symmetric extension of a crack due to incidence of plane dilatational waves.The problem of determining the dynamic stress field due to a plane dislocation moving in an infinite elastic medium was formulaled by Ang and Williams (1959) in terms of Fourier integral equation and solved in closed form. Recently, Georgiadis (1991)has developed an integral equation approach to self-similar plane

---

Published in " International Journal Of Fracture" Vol 61, pp 361-377
1993

elastodynamic problems. He considered the elastodynamic problem of an expanding crack under homogeneous polynomial form loading and reduced it to the solution of a Cauchy integral equation.

In this paper, non-symmetric extension of an infinitesimal flaw into a plane shear crack at a constant rate due to the action of two identical non-parallel plane SH-waves propagating towards each other in an infinite isotropic elastic medium which is initially in a state of uniform anti-plane shear has been treated. A finite time after the crossing of the plane wave fronts, a fracture is assumed to initiate along the line where the wave fronts crossed and the crack edges are then assumed to travel non-symmetrically with different constants speeds. Superposition considerations allow the original problem to be separated into three self-similar problems with (0,0),(0,1) and (1,0) as the index of self-similarity. Following Cherepanov (1979), Cherepanov and Afanas'ev (1974) the mentioned self-similar problems have all been formulated as some problems of Riemann and Hilbert for half-plane, which are solved easily. Out of all the existing similarity techniques, the method of Smirnov-Sobolev (1932) which has been used extensively by Cherepanov (1979), Cherepanov and Afanas'ev (1974) being themost elegant and straight forward has been used to solve our problem.Analytical expressions for the dynamic stress intensity factors at the crack tips and also the rate of energy flux into the crack edges have been derived. Finally, the nature of the variation of the stress intensity factors and the energy flux rate at the crack tips with the velocities of the crack edges and also with the time after crack initiation have been depicted by means of graphs. The development of a crack initiating at a point being a physically realistic model from the point of view of modelling of earthquake sources, this problem also has got application in seismology.

2. Formulation Of The Problem

Let two identical plane waves defined by

$$\sigma_{yz} = A_o W_{\pm} H(W_{\pm}), \quad \sigma_{xz} = \pm A_o \cot\theta_o W_{\pm} H(W_{\pm}) \qquad (1.a,b)$$

referring to coordinate system (x,y,z) where

$$W_{\pm} = c_2 t \pm y\sin\theta_0 + x\cos\theta_0, \quad 0 \le \theta_0 \le \pi/2$$

and H() is Heaviside's unit function, propagate throughthe infinite solid which is pre-stressed such that

$$\sigma^0_{yz} = \sigma \, , \; \sigma^0_{xz} = 0 \tag{1c}$$

Let us assume that at t=0 the non-parallel plane waves intersect at x=y=0. A micro crack is assumed to appear at $t=t_0$ at x=y=0 which starts to extend bilaterally along the trace of the wave intersection with uniform velocities v_1 and v_2. The expanding crack, the circular wave front associated with its motion and the plane wave front are shown in Fig.1(a).

In effect crack extension occurs by removing the stresses which would be generated in the crack plane by the combined applied static and dynamic fields if no cracks were present.

Accordingly, both the crack faces are subjected to shear tractions equal to $-\sigma - 2A_0(c_2 t + x\cos\theta_0)$.

The anti-symmetry of this loading about the crack plane implies that , it is sufficient to consider the half-plane y>0 with bounding surface y=0. The boundary conditions for this half-plane are then given by

$$y=0, \; -v_2 t' < x < v_1 t' \; : \; \sigma_{yz} = -\sigma - 2A_0 c_2 t_0 - 2A_0(c_2 t' + x\cos\theta_0)$$

$$y=0, \; x > v_1 t', \; x < -v_2 t' \; : \; W = 0 \tag{2a,b}$$

where $t' = t - t_0$.

Equation (2a) shows that invoking superposition principle the proposed problem can be divided into three separate problems of a constant shear traction, a shearing stress linearly varying with time and a shear linearly varying with distance along the crack plane.

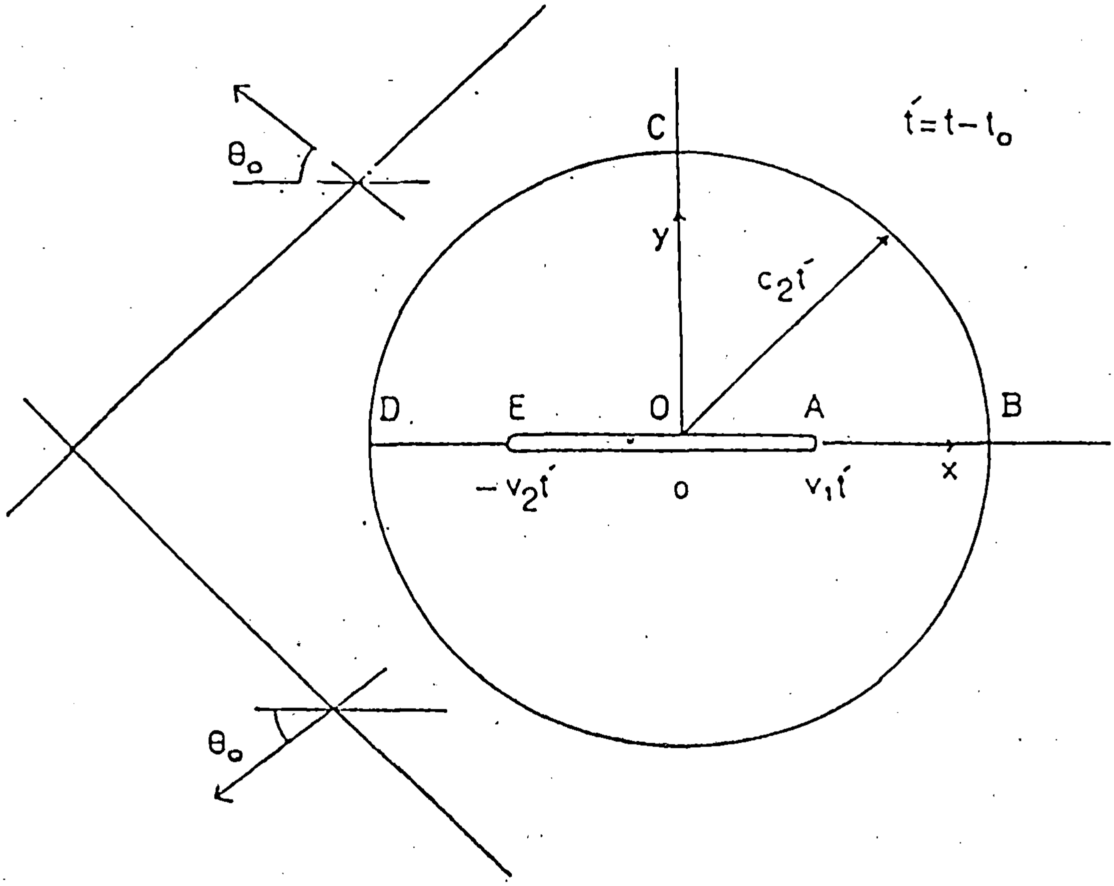


Fig. 1(a). The expanding crack and the pattern of wave front.

3. Constant Shear Traction On The Crack Faces.

The wave motion generated by constant shear tractions on the faces of the crack defined by $y=0$, $-v_2t<x<v_1t$ has been considered in this section and for simplicity t instead of t' has been used. The boundary conditions are

$$y=0,\ -v_2t<x<v_1t\ :\ \sigma_{yz}\ =\ -p_0$$

$$y=0,\ x>v_1t,\ x<-v_2t,\ W\ =\ 0 \qquad (3a,b)$$

where $p_0=\sigma+2A_0c_2t_0$

The displacement W which satisfies the wave equation

$$\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}=\frac{1}{c_2^2}\frac{\partial^2 W}{\partial t^2} \qquad (4)$$

is to be determined subject to the boundary conditions given by (3). From the boundary conditions we observe that $\frac{\partial W}{\partial t}$ shows dynamic similarity and is a homogeneous function of degree zero in x/t and y/t. Therefore, by the functionally invariant method of Smirnoff and Sobolev (1932) we can write

$$\frac{\partial W}{\partial t}=\text{Re}\ \phi_0(z) \qquad (5)$$

where $$t\ -\ xz\ +\ y\sqrt{c_2^{-2}-z^2}\ =\ 0 \qquad (6)$$

The sign of the radical is to be fixed by the condition that

$$\text{as } z\rightarrow\infty,\ \sqrt{c_2^{-2}-z^2}\ =\ iz\ +O(z^{-1}) \qquad (7)$$

Equation (6) maps the semi-circular region of the cylindrical waves defined by OABCDE to the lower half of the complex cut z-plane given by

$$z\ =\ \frac{xt\ -\ iy\sqrt{t^2-c_2^{-2}(x^2+\ y^2)}}{x^2+\ y^2} \qquad (8)$$

as shown in Fig.1(b).

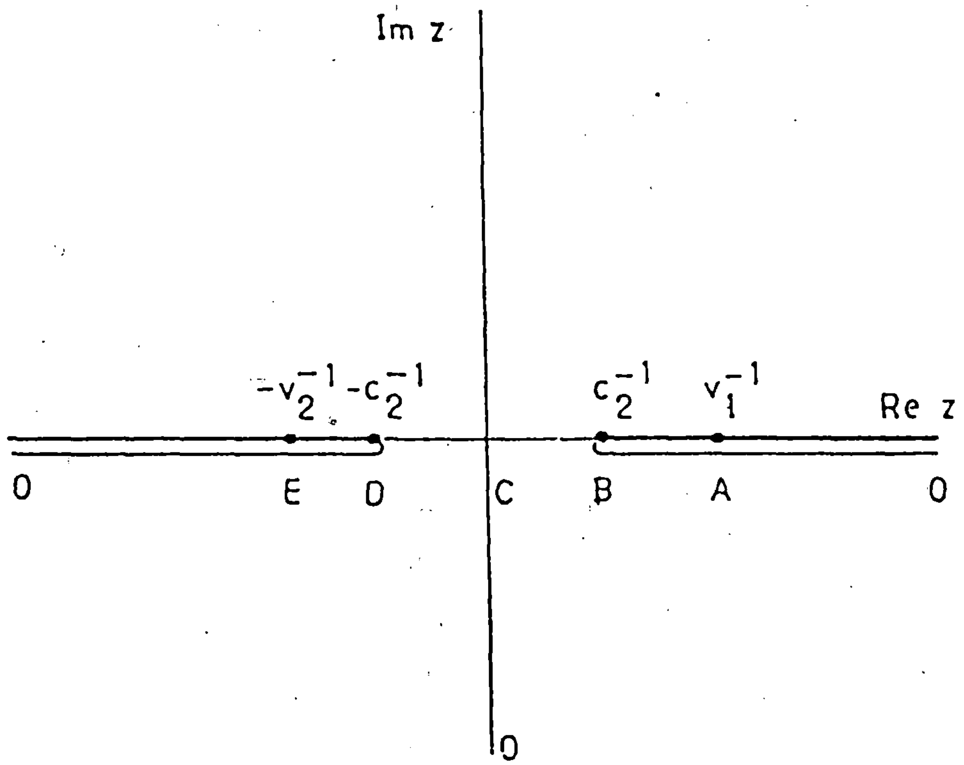


Fig. 1(b). Mapping of the interior of the semi-circle OABCDE in x–y plane on the lower half of the complex z-plane.

In view of the equations (5) and (6) we find

$$\frac{\partial \sigma_{yz}}{\partial t} = \mu \mathrm{Re}\left[\phi_o'(z)\ \frac{\partial z}{\partial y}\right]$$

so that

$$\frac{\partial \sigma_{yz}(x,0,t)}{\partial t} = \frac{1}{t}\ \mathrm{Re}\left[\ -\mu z \phi_o'(z)\sqrt{c_2^{-2}-z^2}\ \right] \tag{9}$$

Therefore the boundary conditions (3) are converted to the following conditions in z-plane

$$\mathrm{Im}z=0, \quad -v_2^{-1}<\mathrm{Re}z<v_1^{-1}\ ,\ \mathrm{Re}\ \phi_o(z) = 0 \tag{10}$$

$$\mathrm{Im}z=0, \quad \mathrm{Re}z<-v_2^{-1}, \mathrm{Re}z>v_1^{-1},\ \mathrm{Im}\ \phi_o'(z)=0 \tag{11}$$

In order to determine the analytic function $\phi_o(z)$ subject to the conditions (10),(11) it is necessary to know the behavior of the function $\phi_o(z)$ when $z \longrightarrow v_1^{-1}, -v_2^{-1}$ and $z \longrightarrow \infty$ .The infinite point of the z-plane corresponds to the origin of the coördinate of the physical plane where the displacement W is limited. Hence taking the representation (5) into account, we obtain

$$\mathrm{Re}\ \phi_o(z) = O(1) \quad \text{as } z \longrightarrow \infty \tag{12}$$

Further the condition (11) after integration with respect to z may be put in the form

$$\mathrm{Im}z=0, \quad \mathrm{Re}z<-v_2^{-1}, \mathrm{Re}z>v_1^{-1},\ \mathrm{Im}\ \phi_o(z)=0 \tag{13}$$

Moreover, the displacement derivative $\partial W/\partial t$ near the crack tips $x = v_1 t, -v_2 t$ should show square root singularities so that at $z \longrightarrow v_1^{-1}, -v_2^{-1}$

$$\phi_o(z) = O(\ 1/\sqrt{z-v_1^{-1}}\ ),\ O(\ 1/\sqrt{z+v_2^{-1}}\ ) \tag{14}$$

respectively.

The above boundary conditions given by (10) and (13) together with the consideration (12) and (14) suggest that

$$\phi_o(z) = \frac{Az + B}{\sqrt{(z-v_1^{-1})(z+v_2^{-1})}} \tag{15}$$

where A and B are unknown constants to be determined.

Integrating (9) with respect to t it can be easily shown that for $x>0$

$$\sigma_{yz}(x,0,t) = -\mu \; \mathrm{Re}\left\{ \left[\phi_o(z)\sqrt{c_2^{-2}-z^2} \right]_{c_2^{-1}}^{t/x} + \int_{c_2^{-1}}^{t/x} \frac{z\phi_o(z)dz}{\sqrt{c_2^{-2}-z^2}} \right\}$$

$$\sigma_{yz}(-x,0,t) = -\mu \; \mathrm{Re}\left\{ \left[\phi_o(z)\sqrt{c_2^{-2}-z^2} \right]_{-c_2^{-1}}^{-t/x} + \int_{-c_2^{-1}}^{-t/x} \frac{z\phi_o(z)dz}{\sqrt{c_2^{-2}-z^2}} \right\} \tag{16a,b}$$

Next using the boundary conditions that

$$\sigma_{yz}(x,0,t) = -p_o, \qquad 0 \le x < v_1 t$$

$$\sigma_{yz}(-x,0,t) = -p_o, \qquad -v_2 t < -x \le 0$$

in equation (16a,b) respectively we obtain two linear equations in A and B viz;

$$A I_2(v_1^{-1}, v_2^{-1}) + B I_1(v_1^{-1}, v_2^{-1}) = \frac{p_o}{\mu}$$

$$A I_2(v_2^{-1}, v_1^{-1}) - B I_1(v_2^{-1}, v_1^{-1}) = \frac{p_o}{\mu} \tag{17a,b}$$

where

$$I_p(u,v) = \int_{c_2^{-1}}^{u} \frac{z^p\,dz}{\sqrt{(z^2-c_2^{-2})(u-z)(v+z)}}, \qquad (p = 1, 2)$$

The stress intensity factors at the crack tips $|x| = v_1 t, y=0$ and $|x| = v_2 t, y=0$ defined by

$$N_{o1} = \underset{x \to v_1 t+}{\mathrm{lt}} \sqrt{x - v_1 t}\; \sigma_{yz}(x,0,t)$$

$$N_{o2} = \underset{x \to v_2 t+}{\mathrm{lt}} \sqrt{x - v_2 t}\; \sigma_{yz}(-x,0,t)$$

respectively are obtained with the help of the equations (15) and (16) as

$$N_{o1} = \frac{\mu}{c_2}\sqrt{\frac{v_2 t}{v_1}}\sqrt{\frac{c_2^2 - v_1^2}{v_1 + v_2}}\,(A+Bv_1)$$

$$N_{o2} = \frac{\mu}{c_2}\sqrt{\frac{v_1 t}{v_2}}\sqrt{\frac{c_2^2 - v_2^2}{v_1 + v_2}}\,(A-Bv_2) \qquad (18a,b)$$

The rate of energy flux into the extending crack edges defined by $\frac{dE}{dt}$ is given by Achenbah and Brock (1971)

$$\frac{1}{2}\frac{dE}{dt} = \int_{-\infty}^{\infty} \sigma_{yz}\frac{\partial W}{\partial t}\,dx \qquad (19)$$

which is obtained with the aid of (5),(15) and (16) for this case as

$$\frac{dE_1}{dt} = -\frac{\mu\pi t}{c_2(v_1+v_2)}\left[v_2\sqrt{c_2^2 - v_1^2}\,(A+Bv_1)^2 + v_1\sqrt{c_2^2 - v_2^2}\,(A-Bv_2)^2\right] \qquad (20)$$

where while carrying on the integration (19) the following result (1972)

$$\frac{H(v)}{\sqrt{v}}\,\frac{H(-v)}{\sqrt{-v}} = \frac{\pi}{2}\,\delta(v) \qquad (21)$$

has been used.

4. Problem Of Linearly Increasing Shear Traction With Time On The Crack Faces

For the case of shear tractions on the faces of the crack increasing linearly with time, the boundary conditions are

$$y=0,\ -v_2t<x<v_1t\ :\ \sigma_{yz} = -p_1 t \qquad (22)$$

$$y=0,\ x>v_1t,\ x<-v_2t,\ W = 0 \qquad (23)$$

where $p_1 = 2A_oc_2$.

The second order derivative $\frac{\partial^2 W}{\partial t^2}$ now shows dynamic similarity which can be taken as the real part of the analytic function $\phi_1(z)$ so that

$$\frac{\partial^2 W}{\partial t^2} = \mathrm{Re}\ \phi_1(z) \tag{24}$$

which implies $$\frac{\partial^2 \sigma_{yz}(x,0,t)}{\partial t^2} = \frac{1}{t}\ \mathrm{Re}\left[\ -\mu z \phi_1'(z)\sqrt{c_2^{-2}-z^2}\ \right] \tag{25}$$

where z is given by (8) and $\phi_1(z)$ satisfies the conditions

$$\mathrm{Im} z=0, \quad -v_2^{-1}<\mathrm{Re} z<v_1^{-1}\ , \quad \mathrm{Re}\ \phi_1(z) = 0 \tag{26}$$

$$\mathrm{Im} z=0, \quad \mathrm{Re} z<-v_2^{-1}, \mathrm{Re} z>v_1^{-1}, \quad \mathrm{Im}\ \phi_1'(z) = 0 \tag{27}$$

Integrating (24), we obtain

$$W = \frac{x^2}{2}\mathrm{Re}\int_{v_1^{-1}}^{z} (z-\tau)^2\ \phi_1'(\tau)\ d\tau \tag{28}$$

$$= \frac{x^2}{2}\ \mathrm{Re}\int_{v_1^{-1}}^{z} 2(z-\tau)\ \phi_1(\tau)\ d\tau \tag{29}$$

so that $$\frac{d^2}{dz^2}\left\{\frac{W}{x^2}\right\} = \mathrm{Re}\ \phi_1(z) \tag{30}$$

Taking into consideration the facts that near the crack tips $x=v_1 t, -v_2 t$; $y=0$ the displacement W varies in direct proportion to the factors $\sqrt{v_1 t-x}$, $\sqrt{v_2 t+x}$ respectively and that as $z \longrightarrow \infty$,

$$\mathrm{Re}\ \phi_1(z) = O(1)$$

we have in view of the conditions (26), (27) and also the equation (30), the result that

$$\phi_1(z) = \frac{d^2}{dz^2}\left[(Cz+D)\sqrt{(z-v_1^{-1})(z+v_2^{-1})}\ \right] \tag{31}$$

where the constants C,D are to be determined from the condition that on the crack surface stress $\sigma_{yz} = -p_1 t$.

From (25) after integration, we derive for $x>0$

$$\sigma_{yz}(x,0,t) = -\mu x \ \mathrm{Re} \int_{c_2^{-1}}^{t/x} \left\{ \sqrt{c_2^{-2} - \tau^2} + \frac{\tau(t/x-\tau)}{\sqrt{c_2^{-2} - \tau^2}} \right\} \phi_1(\tau)d\tau$$

$$\sigma_{yz}(-x,0,t) = \mu x \ \mathrm{Re} \int_{-c_2^{-1}}^{-t/x} \left\{ \sqrt{c_2^{-2} - \tau^2} - \frac{\tau(t/x+\tau)}{\sqrt{c_2^{-2} - \tau^2}} \right\} \phi_1(\tau)d\tau \qquad (32a,b)$$

Therefore, using the boundary conditions that

$$\sigma_{yz}(x,0,t) = -p_1 t, \qquad 0 \le x < v_1 t$$

$$\sigma_{yz}(-x,0,t) = -p_1 t, \qquad -v_2 t < -x \le 0$$

we obtain by the help of (32a,b) after simplification

$$CJ_1(v_1^{-1},v_2^{-1}) + DJ_2(v_1^{-1},v_2^{-1}) = \frac{p_1}{\mu}$$

$$CJ_1(v_2^{-1},v_1^{-1}) - DJ_2(v_2^{-1},v_1^{-1}) = \frac{p_1}{\mu} \qquad (33a,b)$$

where

$$J_1(v_1^{-1},v_2^{-1}) = \int_{c_2^{-1}}^{v_1^{-1}} \Big[\left\{8\tau + 3(v_2^{-1} - v_1^{-1})\right\} M(\tau,v_1^{-1},v_2^{-1}) + N(\tau,v_1^{-1},v_2^{-1}) . \left\{4\tau^2 + 3\tau(v_2^{-1} - v_1^{-1}) - 2(v_1 v_2)^{-1}\right\} \Big] d\tau$$

$$J_2(v_1^{-1},v_2^{-1}) = \int_{c_2^{-1}}^{v_1^{-1}} \Big[2M(\tau,v_1^{-1},v_2^{-1}) + N(\tau,v_1^{-1},v_2^{-1}) \left\{2\tau + (v_2^{-1} - v_1^{-1})\right\} \Big] \cdot d\tau$$

with $M(\tau,v_1^{-1},v_2^{-1}) = \dfrac{v_1 \tau \sqrt{v_1^{-1} - \tau}}{2\sqrt{(\tau + v_2^{-1})(\tau^2 - c_2^{-2})}}$

$$N(\tau, v_1^{-1}, v_2^{-1}) = \frac{\tau v_1}{4\sqrt{\tau^2 - c_2^{-2}}}\left[\frac{3}{\sqrt{(\tau+v_2^{-1})(v_1^{-1}-\tau)}} - \frac{\sqrt{v_1^{-1}-\tau}}{(\tau+v_2^{-1})^{3/2}}\right]$$

The stress intensity factors at the crack tips defined by

$$N_{11} = \lim_{x\rightarrow v_1 t+} \sqrt{x-v_1 t}\,\sigma_{yz}(x,0,t)$$

$$N_{12} = \lim_{x\rightarrow v_2 t+} \sqrt{x-v_2 t}\,\sigma_{yz}(-x,0,t)$$

are found to be

$$N_{11} = \frac{\mu t}{2c_2}\sqrt{\frac{t}{v_1 v_2}}\sqrt{(c_2^2 - v_1^2)(v_1 + v_2)}\ (C+Dv_1)$$

$$N_{12} = \frac{\mu t}{2c_2}\sqrt{\frac{t}{v_1 v_2}}\ \sqrt{(c_2^2 - v_2^2)(v_1 + v_2)}\ (C-Dv_2) \qquad (34a,b)$$

and in this case the rate of energy flux $\frac{dE_2}{dt}$ into the crack edges defined by (19) is obtained as

$$\frac{dE_2}{dt} = -\frac{\pi\mu t^3(v_1+v_2)}{4c_2}\left[v_2^{-1}\sqrt{c_2^2 - v_1^2}\ (C+Dv_1)^2 + v_1^{-1}\sqrt{c_2^2 - v_2^2}\ (C-Dv_2)^2\right] \qquad (35)$$

where while carrying on the integration (19) the use of the result (21) has again been made.

5. Problem Of Linearly Varying Shear Traction With Distance Along The Crack Plane.

Consider the initially undisturbed half-space $y\geq 0$ subjected to the shear traction $-p_2 x$ over $y=0, -v_2 t<x<v_1 t$. The boundary conditions are

$$y=0,\ -v_2 t<x<v_1 t\ :\ \sigma_{yz} = -p_2 x$$

$$y=0,\ x>v_1 t,\ x<-v_2 t,\ W = 0 \qquad (36a,b)$$

where $p_2 = 2A_0 \cos\theta_0$.

In this case, $\frac{\partial^2 W}{\partial x \partial t}$ shows dynamic similarity. So we take keeping (8) in mind,

$$\frac{\partial^2 W}{\partial x \partial t} = \text{Re}\ \phi_2(z)$$

with

$$\frac{\partial^2 \sigma_{yz}}{\partial x \partial t} = -\frac{\mu}{t}\ \text{Re}\left[z\phi_2'(z)\sqrt{c_2^{-2}-z^2} \right] \qquad (37a,b)$$

where $\phi_2(z)$ satisfies the conditions

$$\text{Im}z=0,\quad -v_2^{-1}<\text{Re}z<v_1^{-1},\ \text{Re}\ \phi_2(z) = 0$$

$$\text{Im}z=0,\quad \text{Re}z<-v_2^{-1}, \text{Re}z>v_1^{-1},\ \text{Im}\ \phi_2'(z)=0 \qquad (38a,b)$$

From equation (37a) after integration it is found that

$$W = -x^2 \text{Re} \int_{v_1^{-1}}^{z} \tau^{-1}(z-\tau)\ \phi_2(\tau)\ d\tau$$

so that

$$-z^2 \frac{d}{dz}\left\{ \frac{1}{t}\frac{\partial W}{\partial t} \right\} = \text{Re}\ \phi_2(z)$$

Since $\frac{\partial W}{\partial t}$ near the crack tips should show square root singularity and also since $\text{Re}\ \phi_2(z) = O(1)$ as $z \to \infty$, we have in view of the conditions (38)

$$\phi_2(z) = z^2 \frac{d}{dz}\left[\frac{Rz^{-1}+L}{\sqrt{(z-v_1^{-1})(z+v_2^{-1})}} \right] \qquad (39)$$

where the constants R, L are to be determined.

Equation (37b) can be integrated to obtain for x>0

$$\sigma_{yz}(x,0,t) = \mu x \ \mathrm{Re} \int_{c_2^{-1}}^{t/x} \left\{ \frac{t}{x\tau^2} \sqrt{c_2^{-2} - \tau^2} + \frac{t - \tau x}{x\sqrt{c_2^{-2} - \tau^2}} \right\} \phi_2(\tau)\,d\tau$$

$$\sigma_{yz}(-x,0,t) = \mu x \ \mathrm{Re} \int_{-c_2^{-1}}^{-t/x} \left\{ \frac{t}{x\tau^2} \sqrt{c_2^{-2} - \tau^2} + \frac{t + \tau x}{x\sqrt{c_2^{-2} - \tau^2}} \right\} \phi_2(\tau)\,d\tau \qquad (40a,b)$$

So using the boundary conditions that

$$\sigma_{yz}(x,0,t) = -p_2 x, \qquad 0 \le x < v_1 t$$

$$\sigma_{yz}(-x,0,t) = p_2 x, \qquad -v_2 t < -x \le 0$$

it is found by the help of equations (39),(40)

$$-RK_1(v_1^{-1},v_2^{-1}) + LK_2(v_1^{-1},v_2^{-1}) = \frac{p_2}{\mu}$$

$$RK_1(v_2^{-1},v_1^{-1}) + LK_2(v_2^{-1},v_1^{-1}) = \frac{p_2}{\mu}, \qquad (41)$$

where

$$K_1(v_1^{-1},v_2^{-1}) = \int_{c_2^{-1}}^{v_1^{-1}} \left[P(\tau,v_1^{-1},v_2^{-1}) - \tau^{-1}Q(\tau,v_1^{-1},v_2^{-1}) \right] d\tau$$

$$K_2(v_1^{-1},v_2^{-1}) = \int_{c_2^{-1}}^{v_1^{-1}} Q(\tau,v_1^{-1},v_2^{-1})\,d\tau$$

$$P(\tau,v_1^{-1},v_2^{-1}) = -\frac{\sqrt{v_1^{-1} - \tau}}{\sqrt{(\tau + v_2^{-1})(\tau^2 - c_2^{-2})}}$$

and $$Q(\tau, v_1^{-1}, v_2^{-1}) = \frac{\tau^2}{\sqrt{\tau^2 - c_2^{-2}}}\left[\frac{\sqrt{v_1^{-1} - \tau}}{2(\tau + v_2^{-1})^{3/2}} - \frac{(2v_1^{-1} + \tau)}{2\tau\sqrt{(\tau + v_2^{-1})(v_1^{-1} - \tau)}}\right]$$

In this case, the stress intensity factors are obtained as

$$N_{21} = \operatorname*{lt}_{x \longrightarrow v_1 t+} \sqrt{x - v_1 t}\,\sigma_{yz}(x, 0, t) = -\frac{\mu t}{c_2}\sqrt{\frac{v_2 t}{v_1}}\sqrt{\frac{c_2^2 - v_1^2}{v_1 + v_2}}\,(Rv_1^2 + Lv_1)$$

$$N_{22} = \operatorname*{lt}_{x \longrightarrow v_2 t+} \sqrt{x - v_2 t}\,\sigma_{yz}(-x, 0, t) = -\frac{\mu t}{c_2}\sqrt{\frac{v_1 t}{v_2}}\sqrt{\frac{c_2^2 - v_2^2}{v_1 + v_2}}\,(Rv_2^2 - Lv_2) \qquad (42a,b)$$

The rate of energy flux $\frac{dE_3}{dt}$ into the extending crack edges is found to be

$$\frac{dE_3}{dt} = 2\int_{-\infty}^{\infty} \sigma_{yz}\frac{\partial W}{\partial t}\,dx = -\frac{\mu\pi t^3}{c_2(v_1 + v_2)}\left[v_2\sqrt{c_2^2 - v_1^2}\,(Rv_1^2 + Lv_1)^2 + v_1\sqrt{c_2^2 - v_2^2}\,.\,(Rv_2^2 - Lv_2)^2\right] \qquad (43)$$

where the result (21) has been used.

6. Particular Case: $v_1 = v_2$

If we set $v_1 = v_2 = v$ in all the cases solved above, the following results are obtained

(i) For the case of constant traction $\sigma_{yz} = -p_0$ on the crack faces, we find from (17) that

$B = 0$, $A = \frac{vp_0}{\mu E(q)}$, where E(q) is the complete Elliptic integral of second kind and $q = \sqrt{1 - v^2/c_2^2}$. Equations (18) yield the stress intensity factor at the crack tips as

$$N_0 = N_{01} = N_{02} = \frac{A\mu\sqrt{t}}{c_2}\sqrt{\frac{c_2^2 - v^2}{2v}}$$

Also from (20) we obtain

$$\frac{dE_1}{dt} = -\frac{\mu\pi t}{c_2}\sqrt{c_2^2 - v^2}\quad A^2$$

(ii) For the case of shear traction $\sigma_{yz} = -p_1 t$ on the crack faces increasing linearly with time, it is found from equation (39) that

$$D = 0, \quad C = \frac{p_1 v}{\mu\, I}$$

where

$$I = 2E(q) - F(q) + \frac{2c_2^2}{(v+c_2)(v^2-c_2^2)}\left\{2v\Pi(r^2, r) + (v+c_2)F(r)\right\}$$

$F(r)$, $\Pi(r^2, r)$ are complete Elliptic integrals of first and third kind respectively and $r = (c_2 - v)/(c_2 + v)$.

In this case, the stress intensity factors and the rate of energy flux into the extending crack tips given by (34) and (35) can be simplified to

$$N_1 = N_{11} = N_{12} = \frac{C\mu t}{c_2}\sqrt{\frac{t}{2v}}\sqrt{c_2^2 - v^2}$$

and

$$\frac{dE_2}{dt} = -\frac{\mu\pi t^3}{c_2}\sqrt{c_2^2 - v^2}\quad C^2$$

(iii) For the case of shear traction $\sigma_{yz} = -p_2 x$ on the crack faces, it is obvious from equations (41) that

$$R = 0, \quad L = \frac{p_2 v}{\mu\, J}$$

where

$$J = \frac{2c_2^2}{(v+c_2)(v^2-c_2^2)}\left\{2v\Pi(r^2, r) + (v+c_2)F(r)\right\} - E(q) - F(q)$$

and it is found from equations (42),(43) that stress intensity factors at the crack tips and the rate of energy flux into the extending crack edges in this case are given by

$$N_2 = N_{21} = -N_{22} = \frac{-\mu t L}{c_2}\sqrt{\frac{vt}{2}}\sqrt{c_2^2 - v^2}$$

and
$$\frac{dE_3}{dt} = -\frac{\mu\pi t^3}{c_2}\sqrt{c_2^2 - v^2}\;L^2 v^2$$

7. Numerical Results And Discussions.

The solution of the original crack problem is obtained by taking $p_o = \sigma + 2A_o c_2 t_o$, $p_1 = 2A_o c_2$ and $p_2 = 2A_o \cos\theta_o$ and superposing the results obtained in sections 3-5 with the stress fields given by (1). Taking together the results obtained in the sections 3-5 it is possible to write the stress intensity factors at the edges of the crack and the rate of energy flux into the extending crack edges as

$$S_1 = \frac{N_{01} + N_{11} + N_{21}}{\sigma\sqrt{v_1 t_o}} = \sqrt{\frac{u_2\tau}{u_1 + u_2}}\;\mu H_+(u_1, u_2, \tau)$$

$$S_2 = \frac{N_{02} + N_{12} + N_{22}}{\sigma\sqrt{v_1 t_o}} = \sqrt{\frac{u_2\tau}{u_1 + u_2}}\;\mu H_-(u_2, u_1, \tau) \qquad (44a,b)$$

and

$$En = \frac{\mu}{t_o c_2^2 \sigma^2}\,\frac{d}{dt}\left(E_1 + E_2 + E_3\right) = -\frac{\pi u_2 \mu^2}{u_1 + u_2}\left[G_+(u_1, u_2, \tau) + \frac{u_1}{u_2}\,G_-(u_2, u_1, \tau)\right] \qquad (45)$$

where

$$H_\pm(u_1, u_2, \tau) = \sqrt{1 - u_1^2}\left[\frac{1+\Delta}{p_o}\left(\frac{A}{c_2 u_1} \pm B\right) + \Delta\tau\left\{\frac{u_1 + u_2}{2p_1 u_2}\left(\frac{C}{c_2 u_1} \pm D\right) - \frac{u_1\cos\theta_o}{p_2}\left(\pm\frac{L}{c_2 u_1} + R\right)\right\}\right]$$

$$G_{\pm}(u_1,u_2,\tau)=\tau\sqrt{1-u_1^2}\left[\left(\frac{1+\Delta}{p_o}\right)^2\left(\frac{A}{c_2}\pm Bu_1\right)^2+(\Delta\tau)^2\left\{\frac{(u_1+u_2)^2}{4p_1^2u_2^2}\left(\frac{C}{c_2}\pm Du_1\right)+\right.\right.$$

$$\left.\left.+\frac{u_1^2\cos^2\theta_o}{p_2^2}\left(\frac{L}{c_2}\pm Ru_1\right)^2\right\}\right]$$

and the parameter $\tau = \frac{t}{t_o}-1$ is the non-dimensionalized time after crack initiation and $\Delta = \frac{2A_o c_2 t_o}{\sigma}$ is the ratio at x=y=0 at initiation of the crack plane stress due to the plane waves and the pre-stress.

Also u_1, u_2 are the non-dimensional crack tip velocities given by $u_1 = v_1/c_2$ and $u_2 = v_2/c_2$.

The variations of stress intensity factors and energy flux rate given by (44) and (45) respectively with

(i) v_1/c_2 for different values of v_2/c_2 and

(ii) τ for different values of v_1/c_2 and Δ

have been presented in Figs. 2-4. It has been shown in Fig.2 that Stress intensity factors at the edge $x=v_1t'$, $y=0$ decreases with the increase in the values of v_1/c_2 but increases with the increase in the values of v_2/c_2 and for $v_1/c_2<0.45$, the stress intensity factor at the edge $x=v_2t'$, $y=0$ increases as v_2/c_2 increases but for $v_1/c_2>0.45$, the variation of stress intensity factor at that edge shows an opposite character. It has also been depicted in Fig.2 that the value of energy flux rate $|En|$ increases with the increase in the value of v_1/c_2, shows maximum at $v_1/c_2 = 0.8$ after which it decreases with the increase in the value of v_1/c_2.

In Fig.3. the variations of S_1, S_2 and $|En|$ with τ for various values of $v_1/c_2 \le v_2/c_2$ have been depicted. It may be observed from this figure that $S_1, S_2, |En|$ all increase rapidly with the increase in the value of τ. It may be noted further that for fixed value of v_2/c_2, values of stress intensity factors at the crack tips decrease with the increase in the value of v_1/c_2 whereas energy flux rate $|En|$ increases with the gradual increase in the value of v_1/c_2.

In Fig.4, S_1, S_2 and $|En|$ are again plotted vs τ but in this case, crack tip velocities are kept fixed whereas Δ is assumed to vary. It may be seen that increase in the values of Δ produce marked increase in the value of S_1, S_2 and $|En|$ for any fixed value of τ.

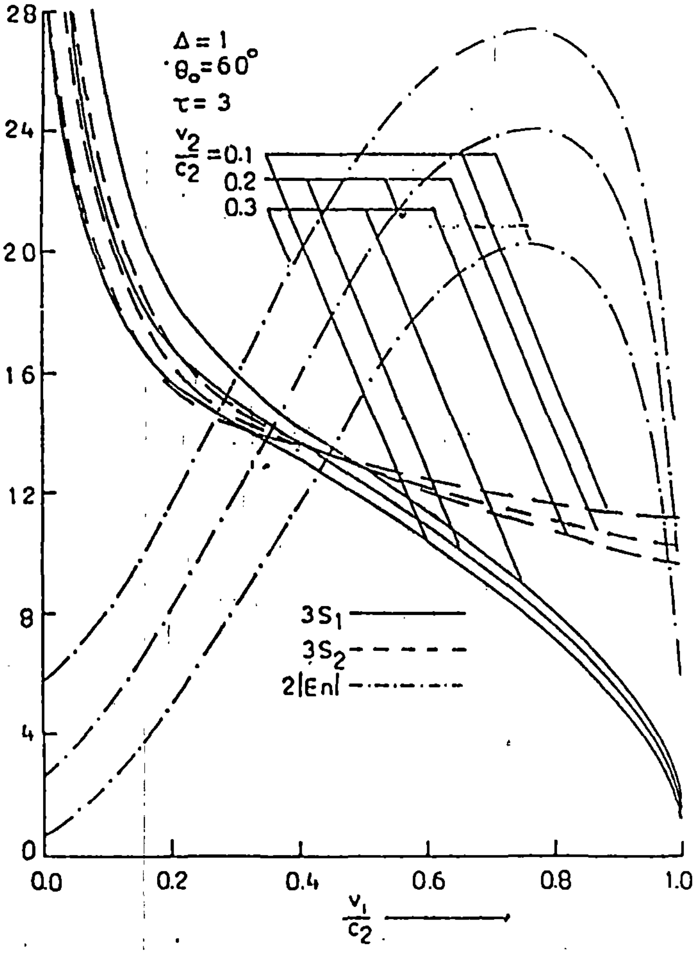


Fig. 2. Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional speed v_1/c_2.

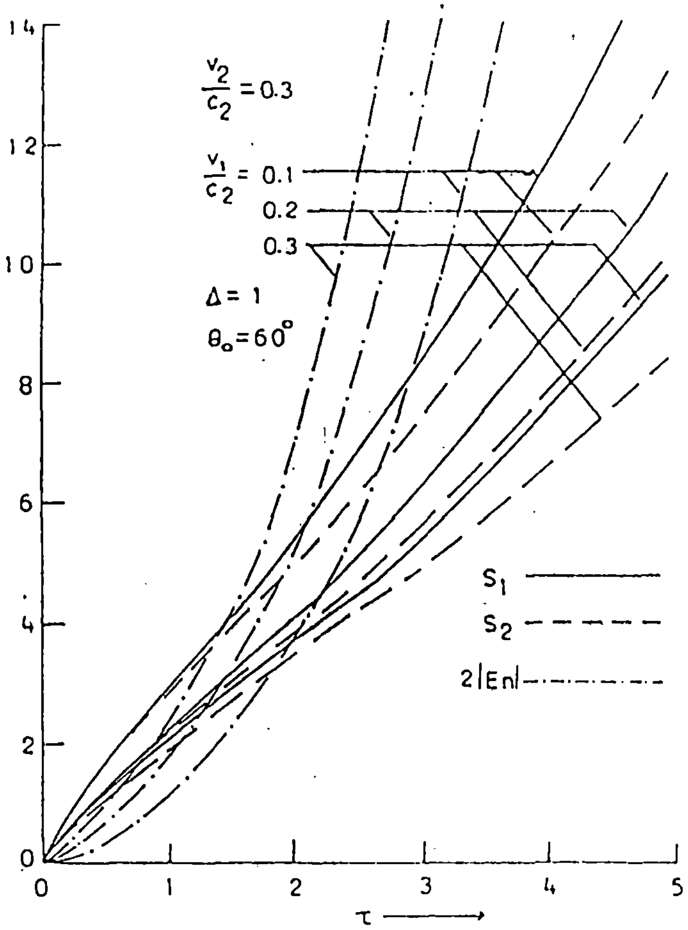


Fig. 3. Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional time after fracture initiation τ.

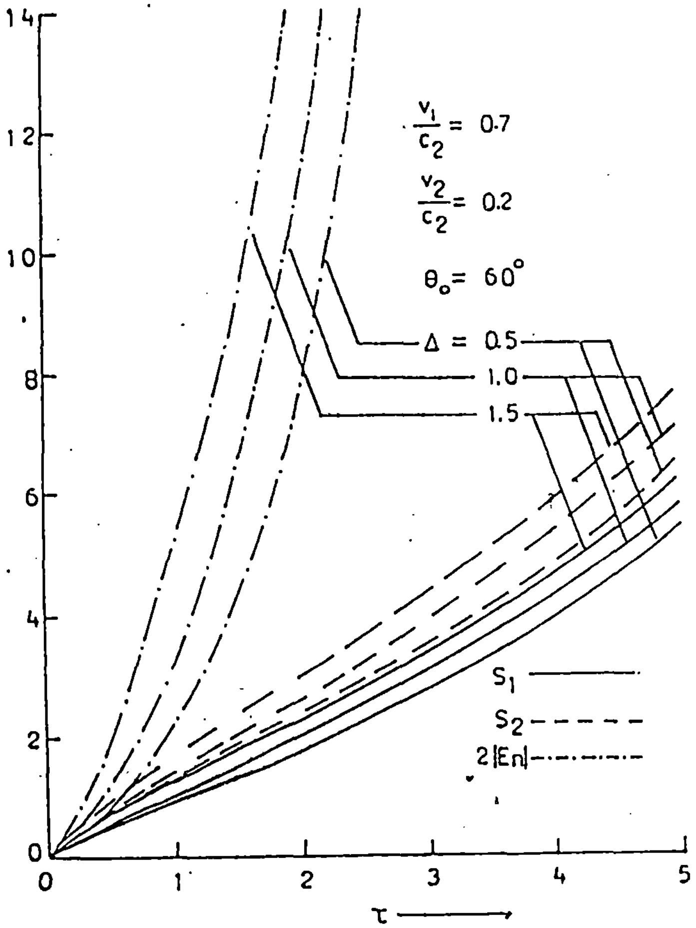


Fig. 4. Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional time after fracture initiation τ.

EXTENSION OF A CRACK DUE TO PLANE SH WAVE IN A PRE-STRESSED INFINITE ELASTIC MEDIUM

1. Introduction

Since Broberg's (1960) investigation of the solution of crack expanding symmetrically with constant velocity under conditions of plane stress or strain in a homogeneous isotropic elastic medium in a field of spatially and time invariant tensile stress, a number of papers have appeared analyzing different geometrical conditions. The problem of Broberg (1960) **was** also solved by Craggs (1963)using the method of homogeneous function and the corresponding anti-plane problem was examined by Achenbach and Brock (1971). All the problems mentioned above are however self similar ones with index (0,0) and are concerned with symmetric extension of a crack.

Problems involving non-symmetric extension of cracks under uniform loading along the crack surface are not found much in the literature perhaps due to severe mathematical complexity encountered in solving such problems. Following the method of homogeneous function Brock (1976,1975) solved the problems of non-symmetric extension of a small flaw into a plane crack. Recently, the elastodynamic problem of non-uniform expansion of a crack under homogeneous polynomial form loading has been solved by Georgiadis (1991) by means of complex variable method.

The problem of non-symmetric extension of an infinitesimal flaw into a plane crack at a constant rate due to the action of two non-parallel plane SH-waves having different amplitudes propagating towards each other in an infinite isotropic elastic medium which is initially in a state of uniform anti-plane shear has been analyzed in this paper. A finite time after the crossing of the plane wave fronts, a fracture is assumed to initiate along the line where the wave fronts crossed and the crack is then assumed to extend

---

Accepted for publication in " International Journal Of Engineering Science", 1993.

non-symmetrically along the trace of the wave intersection. Superposition considerations allow the original problem to be separated into three self-similar problems with (0,0) (0,1) and (1,0) as the index of similarity. The dynamic similarity of certain field variable in each problem suggests application of the method of homogeneous functions. Expressions for the stress intensity factors and the rate of energy flux into the extending crack tips have been derived. Finally, the nature of the variation of the stress intensity factors at the crack tips and also of the rate of energy flux into the crack edges with velocities of the crack edges and also with the time after crack initiation have been depicted by means of graphs.

2. Formulation Of The Problem

Let two identical plane waves defined by

$$\sigma_{yz} = S_\pm W_\pm H(W_\pm), \quad \sigma_{xz} = S_\pm \cot\theta_0 W_\pm H(W_\pm) \qquad (1a,b)$$

referred to coordinate system (x,y,z) where

$$W_\pm = t \pm y\sin\theta_0/c_2 + x\cos\theta_0/c_2$$

and H() is Heaviside's unit function, propagate through the infinite solid which is pre-stressed such that

$$\sigma^0_{yz} = \sigma, \quad \sigma^0_{xz} = 0 \qquad (2a,b)$$

Let us assume that at t=0 the non-parallel plane waves intersect along the line y=0. A micro crack is assumed to appear at $t=t_0$ at x=y=0 which starts to extend bilaterally along the line y=0 with uniform velocities V_R and V_L. The expanding crack, the circular wave front associated with its motion and the plane wave fronts are shown in Fig.1.

In effect crack extension occurs by removing the stresses which would be generated in the crack plane by the combined applied static and dynamic fields if no cracks were present. So, both the crack faces are subjected to shear tractions equal to $-\sigma - (S_+ + S_-)t - (S_+ + S_-)x\cos\theta_0/c_2$.

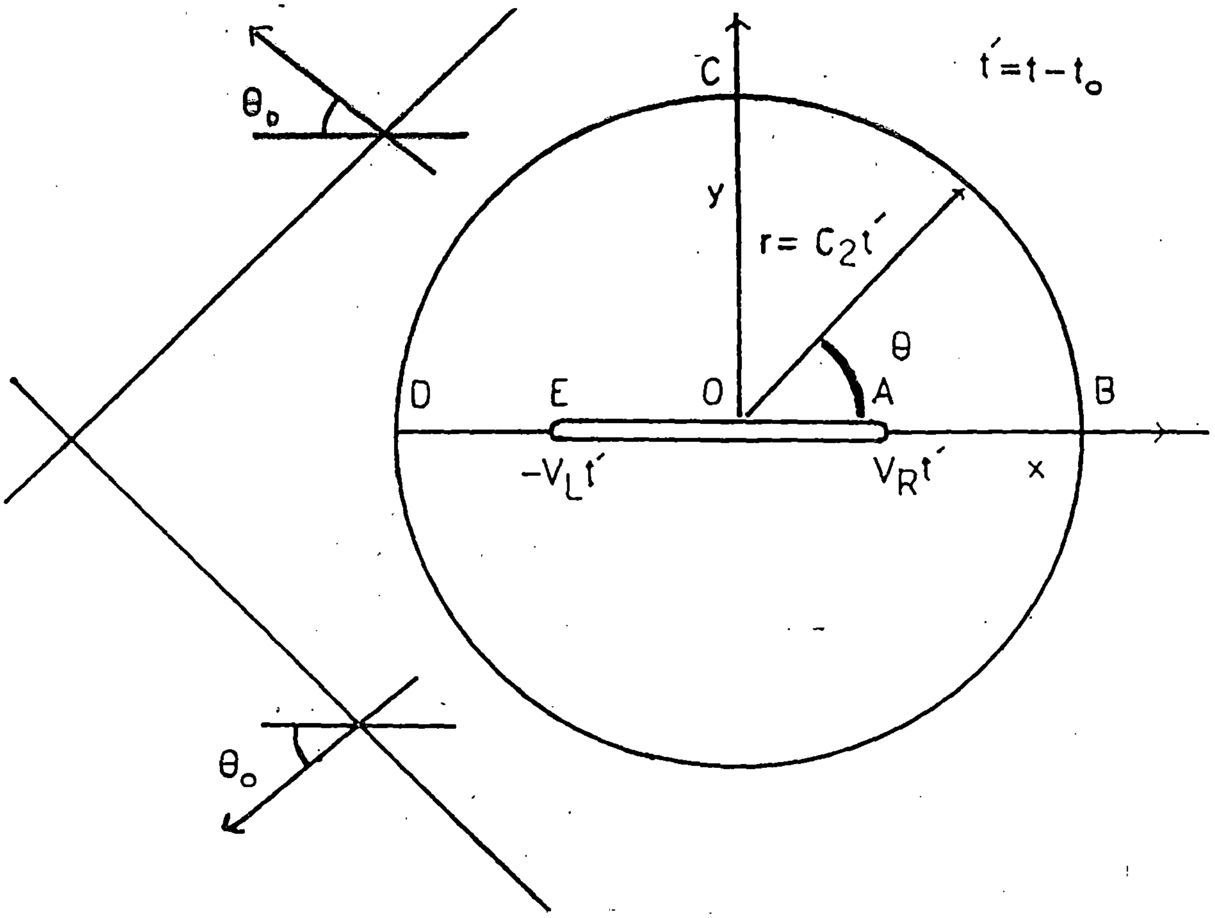


Fig.1: The x-y plane.

The anti-symmetry of this loading about the crack plane implies that, it is sufficient to consider the half-plane $y>0$ with bounding surface $y=0$. The boundary conditions for this half-plane are then given by

$$y=0,\quad -V_L t' < x < V_R t' : \sigma_{yz} = -\sigma-(S_+ + S_-)t_o-(S_+ + S_-)t' - (S_+ + S_-)x\cos\theta_o/c_2$$

$$y=0,\quad x > V_R t',\ x < -V_L t' : W = 0 \qquad (3a,b)$$

where $t' = t - t_o$

Equation (3a) shows that invoking superposition principle the proposed problem can be divided into three separate problems of a constant shear traction and a shearing stress linearly varying with time and a shear linearly varying with distance along the crack plane.

The two dimensional wave equation in polar coordinates r,θ and t where $r = (x^2+y^2)^{1/2}$ and $\theta = \tan^{-1}(y/x)$, for a field variable $\phi(r,\theta,t)$ is

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial\phi}{\partial r}\right\} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{1}{c_2^2}\frac{\partial^2\phi}{\partial t^2} \qquad (4)$$

where $c_2 = (\mu/\rho)^{1/2}$, μ is the shear modulus and ρ is the material density.

The absence of the characteristic length in the formulation of the problem suggests that the solution of (4) will be dynamically similar i.e. depends on r/t and θ rather on r,θ,t separately. Introducing the variable

$$s = r/t$$

we see that $\phi(s,\theta)$ satisfies the equation

$$s^2\left[1 - \frac{s^2}{c^2}\right]\frac{\partial^2\phi}{\partial s^2} + s\left[1 - 2\frac{s^2}{c^2}\right]\frac{\partial\phi}{\partial s} + \frac{\partial^2\phi}{\partial\theta^2} = 0 \qquad (5)$$

For $s<c$, the Chaplygin's transformation

$$\beta = \cosh^{-1}(c/s) \tag{6}$$

reduces (5) to Laplace equation

$$\frac{\partial^2\phi}{\partial\beta^2} + \frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{7}$$

and maps the interior of semi circular region in the upper half of the physical plane into a semi-infinite strip in θ-β plane.

A convenient method to solve the equation (7) is to express $\phi(s,\theta)$ as the real part of an analytic function and to construct an appropriate analytic function of the complex variable $\beta+i\theta$. Superposition in (3a) is invoked to consider the problems separately in next three sections.

3. Constant Shear Traction In The Crack Faces.

The wave motion generated by constant shear tractions on the faces of the crack defined by $y=0$, $-V_Lt<x<V_Rt$ has been considered in this section and for simplicity t instead of t′ has been used. The boundary conditions are

$$y = 0,\ -V_Lt<x<V_Rt \ :\ \sigma_{yz} = -p_o$$

$$y = 0,\ x<-V_Lt,\ x>V_Rt \ :\ W = 0 \tag{8a,b}$$

where $p_o = \sigma + (S_-+S_+)t_o$

From the conditions (8) we observe that $\frac{\partial W}{\partial x}$, $\frac{\partial W}{\partial y}$ and $\frac{\partial W}{\partial t}$ show dynamic similarity. We can choose $\frac{\partial W}{\partial t}$ to take place of ϕ in (4) — (7).

Considering $y\geq0$, the boundary conditions (8) are converted to the following conditions in β-θ plane

$$\theta = 0,\ V_R<s<c_2 \ :\ \Omega(s,\theta) = 0$$

$$\theta = \pi,\ -c_2<s<-V_L :\ \Omega(s,\theta) = 0$$

$$\theta = 0,\ 0<s<V_R \ :\ \frac{\partial\Omega(s,\theta)}{\partial\theta} = 0$$

$$\theta = \pi,\ -V_L<s<0 \ :\ \frac{\partial\Omega(s,\theta)}{\partial\theta} = 0 \tag{9a-d}$$

where $\frac{\partial W}{\partial t} = \Omega(s,\theta)$ and s is related to β by (6). (10)

Further, $\frac{\partial W}{\partial t}$ vanishes at the wave front, $r = c_2 t$, which yields

$$s = c, \quad 0 \le \theta \le \pi, \quad \Omega(s,\theta) = 0 \tag{11}$$

The derivative $\frac{\partial W}{\partial t}$ may be written as real part of an analytic function which can be obtained by mapping the interior of the strip in θ-β plane, see Fig.2, on the lower half-plane of the ζ-plane by means of the conformal transformation

$$\zeta = \xi + i\eta = \mathrm{Sech}(\beta + i\theta) \tag{12}$$

The mappings of the various points are indicated in Figs 1-3. In the ζ-plane we take

$$\frac{\partial W}{\partial t} = \mathrm{Re}\ \phi_o(\zeta) \tag{13}$$

In view of (12) and (13) we find

$$\frac{\partial \sigma_{yz}(x,0,t)}{\partial t} = \frac{\mu}{x}\ \mathrm{Im}\left[\zeta\sqrt{1-\zeta^2}\ \phi_o'(\zeta)\right] \tag{14}$$

Therefore, the boundary conditions given by (9) and (11) are converted into the following conditions in complex ζ-plane

$$\eta = 0, \quad -\infty < \xi < -V_L/c_2, \quad V_R/c_2 < \xi < \infty \ : \ \mathrm{Re}\phi_o(\zeta) = 0$$

$$\eta = 0, -V_L/c_2 < \xi < V_R/c_2 \quad : \ \mathrm{Im}\ \phi_o'(\zeta) = 0 \tag{15a,b}$$

In order to determine the analytic function $\phi_o(\zeta)$ subject to the conditions (15) it is necessary to know the behaviour of the function $\phi_o(\zeta)$ when $\zeta \to -V_L/c_2, V_R/c_2$ and $\zeta \to \infty$. The infinite point in the ζ-plane corresponds to the point $x=0$, $y=c_2 t$ in the physical plane where $\partial W/\partial t$ is limited. Hence taking (13) into account, we obtain

$$\mathrm{Re}\ \phi_o(\zeta) = O(1) \text{ as } \zeta \to \infty \tag{16}$$

Further, the condition (15b) after integration with respect to ζ may

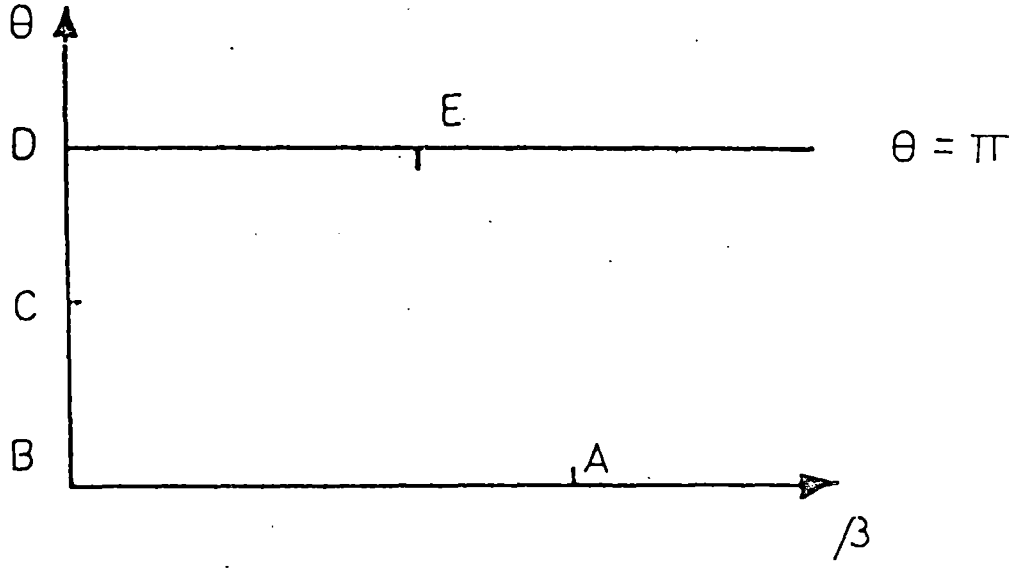


Fig. 2: The θ-β plane.

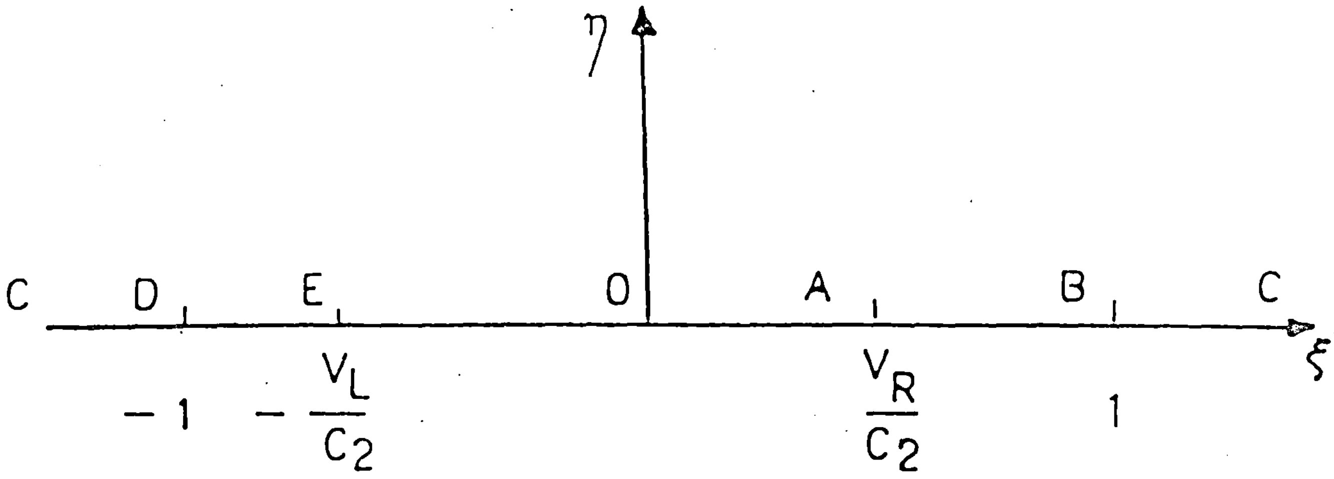


Fig. 3: The ξ-η plane.

be put in the form

$$\eta = 0, -V_L/c_2<\xi<V_R/c_2 \quad : \text{Im}\,\phi_o(\zeta) = 0 \tag{17}$$

Now from (13) we obtain

$$\text{Re}\,\phi_o(\zeta) = \zeta^2 \frac{d}{d\zeta}\left[-\frac{c_2}{x}W\right] \tag{18}$$

Taking into the consideration the fact that near the crack tips $x = -V_Lt, V_Rt$, $y = 0$ the displacement W varies in direct proportion to the factors $\sqrt{V_Rt- x}$ and $\sqrt{V_Lt+ x}$ respectively we have in view of the equation (18), the result that

$$\phi_o(\zeta) = \frac{A\zeta+B}{\sqrt{(V_R/c_2-\zeta)(V_L/c_2+\zeta)}} \tag{19}$$

where the constants A and B are to be determined

Integrating (14) with respect to t it can be easily shown that for $x > 0$

$$\sigma_{yz}(x,0,t) = \frac{\mu}{c_2}\,\text{Im}\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\,\phi_o(\zeta)\Bigg|_{x/c_2t}^{1} + \int_{x/c_2t}^{1}\frac{\phi_o(\zeta)}{\zeta^2\sqrt{1-\zeta^2}}\,d\zeta\right],$$

$$\sigma_{yz}(-x,0,t) = \frac{\mu}{c_2}\,\text{Im}\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\,\phi_o(\zeta)\Bigg|_{-x/c_2t}^{-1} + \int_{-x/c_2t}^{-1}\frac{\phi_o(\zeta)}{\zeta^2\sqrt{1-\zeta^2}}\,d\zeta\right]$$

(20a,b)

Next using the boundary conditions

$$\sigma_{yz}(x,0,t) = -p_o, \qquad 0\le x<V_Rt$$

$$\sigma_{yz}(-x,0,t) = -p_o, \qquad -V_Lt<-x\le 0$$

in equations (20a,b) respectively we obtain two linear equation in A and B viz;

$$A I_1\left(\frac{V_R}{c_2},\frac{V_L}{c_2}\right) + B I_2\left(\frac{V_R}{c_2},\frac{V_L}{c_2}\right) = \frac{c_2 p_o}{\mu}$$

$$-A I_1\left(\frac{V_L}{c_2},\frac{V_R}{c_2}\right) + B I_2\left(\frac{V_L}{c_2},\frac{V_R}{c_2}\right) = \frac{c_2 p_o}{\mu} \qquad (21a,b)$$

where

$$I_p(u,v) = \int_u^1 \frac{z^{-p}\,dz}{\sqrt{(1-z^2)(z-u)(z+v)}}, \quad (p=1,2)$$

The stress intensity factors at the crack tips $|x| = V_R t$, $y = 0$ and $|x| = V_L t$, $y = 0$ defined by

$$N_{01} = \underset{x \to V_R t+}{\mathrm{Lt}} \sqrt{x - V_R t}\; \sigma_{yz}(x,0,t)$$

$$N_{02} = \underset{x \to V_L t+}{\mathrm{Lt}} \sqrt{x - V_L t}\; \sigma_{yz}(-x,0,t)$$

respectively are obtained with the help of equations (20a,b) as

$$N_{01} = \frac{\mu}{c_2}\sqrt{\frac{(c_2^2 - V_R^2)t}{V_R + V_L}}\left[A + B\frac{c_2}{V_R}\right]$$

$$N_{02} = \frac{\mu}{c_2}\sqrt{\frac{(c_2^2 - V_L^2)t}{V_R + V_L}}\left[-A + B\frac{c_2}{V_L}\right] \qquad (21c,d)$$

The rate of energy flux into the extending crack edges defined by dE/dt is given by Achenbach and Brock (1971)

$$\frac{1}{2}\frac{dE}{dt} = -\int_{-\infty}^{\infty} \sigma_{yz}\frac{\partial w}{\partial t}\,dx \qquad (22)$$

which is obtained with the aid of (13),(19) and (20) for this case as

$$\frac{1}{2}\frac{dE_1}{dt} = -\frac{\mu\pi}{2c_2}\frac{t}{V_R+V_L}\left[(AV_R+Bc_2)^2\sqrt{\frac{c_2^2}{V_R^2}-1} + (AV_L-Bc_2)^2\sqrt{\frac{c_2^2}{V_L^2}-1}\right] \tag{23}$$

where while carrying on the integration (22) the following result (1972)

$$\frac{H(v)}{\sqrt{v}}\,\frac{H(-v)}{\sqrt{-v}} = \frac{\pi}{2}\,\delta(v) \tag{24}$$

has been used.

4. Linearly Increasing Shear Traction With Time On The Crack Faces

For the case of shear traction on the crack faces increasing linearly with time, the boundary conditions are

$$y = 0,\ -V_L t<x<V_R t\ :\ \sigma_{yz} = -p_1 t$$

$$y = 0,\ x<-V_L t,\ x>V_R t\ :\ W = 0 \tag{25a,b}$$

where $p_1 = (S_-+S_+)$.

The second order derivatives $\frac{\partial^2 W}{\partial x^2}$, $\frac{\partial^2 W}{\partial y^2}$, $\frac{\partial^2 W}{\partial x\partial y}$ and $\frac{\partial^2 W}{\partial t^2}$ now show dynamic similarity. We select $\frac{\partial^2 W}{\partial t^2}$ to take the place of ϕ in (4)-(7). Accordingly we assume

$$\frac{\partial^2 W}{\partial t^2} = \mathrm{Re}\ \phi_1(\zeta) \tag{26}$$

so that

$$\frac{\partial^2 \sigma_{yz}(x,0,t)}{\partial t^2} = \frac{\mu}{x}\ \mathrm{Im}\left[\zeta\sqrt{1-\zeta^2}\ \phi_1'(\zeta)\right] \tag{27}$$

where $\phi_1(\zeta)$ satisfies

$$\eta = 0,\ -\infty<\xi<-V_L/c_2,\ V_R/c_2<\xi<\infty\ :\ \mathrm{Re}\ \phi_1(\zeta) = 0$$

$$\eta = 0,\ -V_L/c_2<\xi<V_R/c_2 \quad : \ \mathrm{Im}\ \phi_1'(\zeta) = 0 \qquad (28a,b)$$

From (26) we find that

$$\frac{\partial W}{\partial t} = -\frac{x}{c_2}\,\mathrm{Re}\int_{\frac{V_R}{c_2}}^{\zeta} \frac{\phi_1(\tau)}{\tau^2}\,d\tau$$

so that

$$\mathrm{Re}\ \phi_1(\zeta) = \zeta^2\frac{d}{d\zeta}\left[-\frac{c_2}{x}\frac{\partial W}{\partial t}\right] \qquad (29)$$

Since $\partial W/\partial t$ near the crack tips should show square root singularity and also since Re $\phi_1(\zeta) \longrightarrow O(1)$ as $\zeta \longrightarrow \infty$ we have in view of the conditions (28) and the equation (29)

$$\phi_1(\zeta) = \zeta^2\frac{d}{d\zeta}\left[\frac{C+D/\zeta}{\sqrt{(V_R/c_2-\zeta)(V_L/c_2+\zeta)}}\right] \qquad (30)$$

where the constants C and D are to be determined from the condition that on the crack surface $\sigma_{yz} = -p_1 t$.

Integrating (27), we derive for x>0

$$\sigma_{yz}(x,0,t) = -\frac{\mu x}{c_2^2}\,\mathrm{Im}\int_{x/c_2 t}^{1}\left[\frac{(2\zeta-\tau)\sqrt{1-\tau^2}}{\zeta\tau^3} - \frac{(\tau-\zeta)}{\zeta\tau\sqrt{1-\tau^2}}\right]\phi_1(\tau)\,d\tau$$

$$\sigma_{yz}(-x,0,t) = \frac{\mu x}{c_2^2}\,\mathrm{Im}\int_{-x/c_2 t}^{-1}\left[\frac{(2\zeta-\tau)\sqrt{1-\tau^2}}{\zeta\tau^3} - \frac{(\tau-\zeta)}{\zeta\tau\sqrt{1-\tau^2}}\right]\phi_1(\tau)\,d\tau$$

(31a,b)

Therefore, using the boundary condition

$$\sigma_{yz}(x,0,t) = -p_1 t, \qquad 0\le x<V_R t$$

$$\sigma_{yz}(-x,0,t) = -p_1 t, \qquad -V_L t < -x \leq 0$$

in equations (31a,b) respectively we obtain after simplification

$$CJ_1\left(\frac{V_R}{c_2},\frac{V_L}{c_2}\right) + DJ_2\left(\frac{V_R}{c_2},\frac{V_L}{c_2}\right) = -\frac{c_2 p_1}{\mu}$$

$$-CJ_1\left(\frac{V_L}{c_2},\frac{V_R}{c_2}\right) + DJ_2\left(\frac{V_L}{c_2},\frac{V_R}{c_2}\right) = -\frac{c_2 p_1}{\mu} \qquad (32a,b)$$

where

$$J_1\left(\frac{V_R}{c_2},\frac{V_L}{c_2}\right) = \int_{V_R/c_2}^{1} M\left(\tau,\frac{V_R}{c_2},\frac{V_L}{c_2}\right)d\tau$$

$$J_2\left(\frac{V_R}{c_2},\frac{V_L}{c_2}\right) = \int_{V_R/c_2}^{1} \tau^{-1}\left[M\left(\tau,\frac{V_R}{c_2},\frac{V_L}{c_2}\right) + N\left(\tau,\frac{V_R}{c_2},\frac{V_L}{c_2}\right) \right]d\tau$$

with

$$M\left(\tau,\frac{V_R}{c_2},\frac{V_L}{c_2}\right) = \left(\frac{2V_R}{c_2\tau^2} - \frac{\tau}{2}\right)\frac{1}{\sqrt{(1-\tau^2)(\tau-V_R/c_2)(\tau+V_L/c_2)}} + \frac{\tau\sqrt{\tau-V_R/c_2}}{2\sqrt{(1-\tau^2)(\tau+V_L/c_2)^3}}$$

$$N\left(\tau,\frac{V_R}{c_2},\frac{V_L}{c_2}\right)\Big] = \frac{\sqrt{\tau - V_R/c_2}}{\sqrt{(1-\tau^2)(\tau+V_L/c_2)}}$$

The stress intensity factors at the crack tips defined by

$$N_{11} = \underset{x \to V_R t+}{Lt} \sqrt{x - V_R t}\;\sigma_{yz}(x,0,t)$$

$$N_{12} = \underset{x \to V_L t+}{Lt} \sqrt{x - V_L t}\, \sigma_{yz}(-x,0,t)$$

are found to be

$$N_{11} = -\frac{\mu t^{3/2}}{V_R c_2}\sqrt{\frac{c_2^2 - V_R^2}{V_R + V_L}}\,(CV_R + Dc_2)$$

$$N_{12} = \frac{\mu t^{3/2}}{V_L c_2}\sqrt{\frac{c_2^2 - V_L^2}{V_R + V_L}}\,(CV_L - Dc_2) \qquad (33a,b)$$

and in this case the rate of energy flux $\frac{1}{2}\frac{dE_2}{dt}$ into the crack edges defined by (22) is obtained as

$$\frac{1}{2}\frac{dE_2}{dt} = -\frac{\mu\pi}{2c_2}\frac{t^3}{V_R+V_L}\left[(CV_R+Dc_2)^2\sqrt{\frac{c_2^2}{V_R^2} - 1} + (CV_L-Dc_2)^2\sqrt{\frac{c_2^2}{V_L^2} - 1}\right] \qquad (34)$$

where while carrying on the integration (22) the use of the result (24) has again been made.

5. Problem Of Linearly varying shear Traction With Distance Along The Crack Plane

Consider the initially undisturbed half-space $y \geq 0$ subjected to the shear traction $-p_2 x$ over $y=0$, $-V_L t < x < V_R t$. The boundary conditions are

$$y = 0,\ -V_L t < x < V_R t : \sigma_{yz} = -p_2 x$$

$$y = 0,\ x < -V_L t,\ x > V_R t : W = 0 \qquad (35a,b)$$

where $p_2 = (S_+ + S_-)\cos\theta_0 / c_2$

In this case we select $\dfrac{\partial^2 W}{\partial x \partial t}$ to take place of ϕ in (4)-(7). So we take

$$\frac{\partial^2 W}{\partial x \partial t} = \operatorname{Re} \phi_2(\zeta)$$

with

$$\frac{\partial^2 \sigma_{yz}}{\partial x \partial t} = \frac{\mu}{x} \operatorname{Im} \left[\zeta \sqrt{1 - \zeta^2}\, \phi_2'(\zeta) \right] \qquad \text{(36a,b)}$$

where $\phi_2(\zeta)$ satisfies

$$\eta = 0, \ -\infty < \xi < -V_L/c_2, \ V_R/c_2 < \xi < \infty \ : \ \operatorname{Re} \phi_2(\zeta) = 0$$

$$\eta = 0, \ -V_L/c_2 < \xi < V_R/c_2 \ : \ \operatorname{Im} \phi_2'(\zeta) = 0 \qquad \text{(37a,b)}$$

From (36a) we have

$$\frac{\partial W}{\partial t} = c_2 t \operatorname{Re} \int_{\frac{V_R}{c_2}}^{\zeta} \phi_2(\tau)\, d\tau$$

so that

$$\operatorname{Re} \phi_2(\zeta) = \frac{d}{d\zeta} \left(\frac{1}{c_2 t} \frac{\partial W}{\partial t} \right). \qquad \text{(38)}$$

Taking into consideration the fact that near the crack tips $x = V_R t$, $-V_L t$, $y=0$ the displacement derivative $\partial W/\partial t$ varies in inverse proportion to the factors $\sqrt{V_R t - x}$, $\sqrt{V_L t + x}$ respectively and as $\zeta \to \infty$ $\operatorname{Re} \phi_2(\zeta) = O(1)$. We can take, keeping (37)and (38) in mind

$$\phi_2(\zeta) = \frac{d}{d\zeta} \left(\frac{(R\zeta + L)\, \zeta}{\sqrt{(V_R/c_2 - \zeta)(V_L/c_2 + \zeta)}} \right) \qquad \text{(39)}$$

where the constants R and L are to be determined.

Equation (36b) can be integrated to derive for x>0

$$\sigma_{yz}(x,0,t) = \mu t \; \mathrm{Im} \int_{x/c_2 t}^{1} \left[\frac{\zeta\sqrt{1-\tau^2}}{\tau^2} - \frac{(\tau-\zeta)}{\sqrt{1-\tau^2}} \right] \phi_2(\tau)\, d\tau$$

$$\sigma_{yz}(-x,0,t) = \mu t \; \mathrm{Im} \int_{-x/c_2 t}^{-1} \left[\frac{\zeta\sqrt{1-\tau^2}}{\tau^2} - \frac{(\tau-\zeta)}{\sqrt{1-\tau^2}} \right] \phi_2(\tau)\, d\tau \qquad (40a,b)$$

Therefore, using the boundary condition

$$\sigma_{yz}(x,0,t) = -p_2 x, \qquad 0 \le x < V_R t$$

$$\sigma_{yz}(-x,0,t) = p_2 x, \qquad -V_L t < -x \le 0$$

in equations (40a,b) respectively we obtain after simplification

$$RK_1\left(\frac{V_R}{c_2}, \frac{V_L}{c_2}\right) + LK_2\left(\frac{V_R}{c_2}, \frac{V_L}{c_2}\right) = \frac{V_R p_2}{\mu}$$

$$-RK_1\left(\frac{V_L}{c_2}, \frac{V_R}{c_2}\right) + LK_2\left(\frac{V_L}{c_2}, \frac{V_R}{c_2}\right) = \frac{V_L p_2}{\mu} \qquad (41a,b)$$

where

$$K_1\left(\frac{V_R}{c_2}, \frac{V_L}{c_2}\right) = \int_{V_R/c_2}^{1} \tau \left[P\left(\tau, \frac{V_R}{c_2}, \frac{V_L}{c_2}\right) - 2Q\left(\tau, \frac{V_R}{c_2}, \frac{V_L}{c_2}\right) \right] d\tau$$

$$K_2\left(\frac{V_R}{c_2}, \frac{V_L}{c_2}\right) = \int_{V_R/c_2}^{1} \left[P\left(\tau, \frac{V_R}{c_2}, \frac{V_L}{c_2}\right) - Q\left(\tau, \frac{V_R}{c_2}, \frac{V_L}{c_2}\right) \right] d\tau$$

with $$P\left(\tau, \frac{V_R}{c_2}, \frac{V_L}{c_2}\right) = \left(\frac{2V_R}{c_2 \tau^2} + \frac{\tau}{2} - \frac{V_R}{c_2} \right) \frac{1}{\sqrt{(1-\tau^2)(\tau - V_R/c_2)(\tau + V_L/c_2)}} +$$

$$+\frac{\tau\sqrt{\tau-V_R/c_2}}{2\sqrt{(1-\tau^2)(\tau+V_L/c_2)^3}}$$

$$Q\Big[\tau,\frac{V_R}{c_2},\frac{V_L}{c_2}\Big)\Big] = \frac{\sqrt{\tau-V_R/c_2}}{\sqrt{(1-\tau^2)(\tau+V_L/c_2)}}$$

In this case, the stress intensity factors at the crack tips are obtained as

$$N_{21} = \underset{x\to V_R t+}{\mathrm{Lt}} \sqrt{x-V_R t}\,\sigma_{yz}(x,0,t) = \frac{\mu t^{3/2}}{c_2}\sqrt{\frac{c_2^2-V_R^2}{V_R+V_L}}\,(RV_R+Lc_2)$$

$$N_{22} = \underset{x\to V_L t+}{\mathrm{Lt}} \sqrt{x-V_L t}\,\sigma_{yz}(-x,0,t) = \frac{\mu t^{3/2}}{c_2}\sqrt{\frac{c_2^2-V_L^2}{V_R+V_L}}\,(RV_L-Lc_2) \qquad (42a,b)$$

and in this case the rate of energy flux $\frac{1}{2}\frac{dE_2}{dt}$ into the crack edges defined by (22) is obtained as

$$\frac{dE_3}{dt} = \frac{\mu\pi}{c_2}\,\frac{t^3}{V_R+V_L}\left[V_R^2(RV_R+Lc_2)^2\sqrt{\frac{c_2^2}{V_R^2}-1} + V_L^2(RV_L-Lc_2)^2\sqrt{\frac{c_2^2}{V_L^2}-1}\right] \qquad (43)$$

where while carrying on the integration (22) the use of the result (24) has again been made.

6.Particular case - I : $V_R=V_L$

If we set $V_R=V_L=V$ in all the cases solved above, the following results are obtained

(i) For the case of constant shear traction $\sigma_{yz} = -p_0$ on the crack faces, we find from (21a,b) that

$A=0, B = \frac{Vmp_0}{\mu E(q)}$, where $E(q)$ is the complete Elliptic integral of second kind and $m = V/c_2$, $q = \sqrt{1-m^2}$. Equations

(21c,d) yield the stress intensity factors at the crack tips as

$$N_o = N_{o1} = N_{o2} = \mu B\sqrt{t}\sqrt{\frac{c_2^2 - V^2}{2v^3}}$$

Also from (23) we obtain

$$\frac{dE_1}{dt} = -\pi\mu tqB^2/m^2$$

(ii) For the case of shear traction $\sigma_{yz} = -p_1 t$ on the crack surface increasing linearly with time, it is found from equation (32) that

$$C = 0, \; D = -p_1c_2/\mu I$$

where

$$I = m^{-2}\left[\frac{\pi}{2} + m^2F(q) - \frac{2m^3}{(1-m^2)(m+1)}\left\{2\Pi(r^4, r^2) - (m+1)F(r^2)\right\}\right]$$

$F(r^2)$, $\Pi(r^4, r^2)$ are the complete Elliptic integral of first and third kind respectively and $r = \sqrt{(1-m)/(1+m)}$.

In this case the stress intensity factors and the rate of energy flux into the extending crack edges given by (33) and (34) can be simplified to

$$N_1 = N_{11} = N_{12} = -\mu Dt^{3/2}\sqrt{\frac{c_2^2 - V^2}{2v^3}}$$

and

$$\frac{dE_2}{dt} = -\pi\mu t^3qD^2/m^2$$

(iii) For the case of shear traction $\sigma_{yz} = -p_2 x$ on the crack faces, it is obvious form equation (41) that

$$R = 0, \; L = p_2V/\mu J$$

where

$$J = \frac{2}{m}E(q) - m\,F(q) + \frac{2m^2}{(1-m^2)(m+1)}\left\{2\Pi(r^4, r^2) - (m+1)F(r^2)\right\}$$

and it is found from equations (42),(43) that the stress intensity factors and the rate of energy flux into the extending crack edges in this case are given by

$$N_2 = N_{21} = -N_{22} = \mu L t^{3/2} \sqrt{\frac{c_2^2 - V^2}{2v}}$$

and

$$\frac{dE_2}{dt} = \pi\mu t^3 q L^2 c_2^2$$

7. Particular Case - II: $V_L = 0$, $V_R = V$.

If we set $V_R = V$ and $V_L = 0$ in all the cases solved above , the following results are obtained

(i) For the case of constant shear traction $\sigma_{yz} = -p_o$ on the crack faces, we find from (21d) that

$$B=0, A = \frac{c_2 p_o}{\mu I_1},$$

where

$$I_1 = \frac{2}{\sqrt{m+1}} \left[2\Pi(r^2, r) - F(r) \right]$$

(21c,d) yield the stress intensity factors at the crack tips as

$$N_{01} = \mu A q \sqrt{t/v}$$

$$N_{02} = -\mu A \sqrt{t/v}$$

Also from (23) we obtain

$$\frac{dE_1}{dt} = -\pi\mu t q A^2$$

(ii) For the case of shear traction $\sigma_{yz} = -p_1 t$ increasing linearly with time on the crack surface , it is found from equation (33b) that

$$D = 0, \; C = -p_1 c_2/\mu J_1$$

where

$$J_1 = \frac{8}{3}\frac{\sqrt{m+1}}{m}E(r) + \frac{m-8}{3\sqrt{m+1}}F(r)$$

In this case the stress intensity factors and the rate of energy flux into the extending crack edges given by (33) and (34) can be simplified to

$$N_{11} = -\mu C t^{3/2} q/\sqrt{\bar{v}}$$

$$N_{12} = \mu C t^{3/2}/\sqrt{\bar{v}}$$

and

$$\frac{dE_2}{dt} = -\pi\mu t^3 q C^2$$

(iii) For the case of shear traction $\sigma_{yz} = -p_2 x$ on the crack faces, it is obvious form equation (41) that

$$L = 0, \quad R = p_2 V/\mu K_1$$

where

$$K_1 = \frac{m}{\sqrt{m+1}}\left[8\Pi(r^2,r)+2\Pi(-r^2,r) - 5F(r)\right] - G(m)$$

with

$$G(m) = \int_m^1 \frac{x^{3/2}}{\sqrt{(1-x^2)(x-m)}}dx$$

and it is found from equations (42),(43) that the stress intensity factors and the rate of energy flux into the extending crack edges in this case are given by

$$N_{21} = \mu R t^{3/2} q/\sqrt{\bar{v}}$$

$$N_{22} = 0$$

and

$$\frac{dE_2}{dt} = \pi\mu t^3 q R^2 v^2$$

8. Numerical Results and Discussions

The solution of the original problem is obtained by taking $p_o = \sigma + (S_+ + S_-)t_o$, $p_1 = (S_+ + S_-)$ $p_2 = (S_+ + S_-)\cos\theta_o / c_2$ and superposing the results obtained in sections 3-5 with the stress fields given by (1) and (2). Taking together the results obtained in the sections 3-5 it is possible to write the stress intensity factors at the crack edges and the rate of energy flux into the extending crack edges as

$$S_1 = \frac{N_{01} + N_{11} + N_{21}}{\sigma \sqrt{c_2 t_o}} = H_+(v_1, v_2, \tau)$$

$$S_2 = \frac{N_{02} + N_{12} + N_{22}}{\sigma \sqrt{c_2 t_o}} = H_-(v_2, v_1, \tau) \qquad (44a,b)$$

and

$$En = \frac{\mu}{t_o c_2^2 \sigma^2}\left[\frac{dE_1}{dt} + \frac{dE_2}{dt} + \frac{dE_3}{dt} \right] = -\pi\tau\left(\frac{\mu}{c_2}\right)^2 \left[G_+(v_1, v_2, \tau) + G_-(v_2, v_1, \tau) \right] \qquad (45)$$

where

$$H_{\pm}(v_1, v_2, \tau) = \frac{\mu}{c_2}\sqrt{\frac{\tau\ (1-v_1^2)}{(v_1+v_2)}}\left[\frac{1+\ \Delta(1+S)}{p_o}\left[\pm A + B/v_1\right] - \Delta\tau(1+S)\left\{\frac{1}{p_1}\left[\pm C + D/v_1\right] - \right.\right.$$

$$\left.\left. - \frac{\cos\theta_o}{p_2}\left[Rv_1 \pm L\right]\right\}\right]$$

$$G_{\pm}(v_1, v_2, \tau) = \frac{\sqrt{(1-v_1^2)}}{v_1+v_2} v_1 \left[\left[\frac{1+\ \Delta(1+S)}{p_o}\right]^2 \left[\pm A + B/v_1\right]^2 + \left[\Delta\tau(1+S)\right]^2 \left\{\frac{1}{p_1^2}\left[\pm C + D/v_1\right]^2 - \right.\right.$$

$$\left.\left. - \frac{\cos^2\theta_o}{p_2^2}\left[Rv_1 \pm L\right]^2\right\}\right]$$

and the parameter $\tau = t/t_o - 1$ is the non-dimensionalized time after crack initiation and $\Delta = S_+t_o/\sigma$ is the ratio at x=y=0 at initiation of the crack plane stress due to one of the plane waves and the pre-stress and $S = S_-/S_+$ is the ratio of the stresses due to plane waves.

Also v_1, v_2 are the non-dimensional crack tip velocities given by $v_1 = V_R/c_2$, $v_2 = V_L/c_2$.

The variations of stress intensity factors and energy flux rate given by (44) and (45) respectively with (i) V_R/c_2 for different values of V_L/c_2, S and with (ii) V_L/c_2 for different values of V_R/c_2 have been presented in Figs.4-6. It has been shown in Fig.4 and Fig.5 that the stress intensity factor at the edge $x = V_Rt$, $y = 0$ increases slowly with the increase in the values of V_R/c_2, shows maximum at 0.4 after which it decreases gradually with the increase in the values of V_R/c_2 and also increases with the increase in the values of V_L/c_2 while the stress intensity factor at the other edge increases with the increase in the values of V_R/c_2, V_L/c_2. It has also been depicted in Fig.4 & 6 that |En| increases with the increase in the values of V_R/c_2, showing its maximum value at $V_R/c_2 = 0.8$ after which it decreases with the increase in the value of V_R/c_2. The variations shown in Fig.4 & 5 are expected from physical stand point. Fig.6 shows that S_1, S_2 and |En| also increase with the increase in the values of S.

In Fig.7, the variations of S_1, S_2 and |En| with τ for various values of Δ have been depicted. It may be observed from this figure that S_1, S_2, |En| all increase rapidly with the increase in the value of τ. It may be noted further that for fixed value of τ values S_1, S_2 and |En| increase with the increase in the values Δ.

In Fig.8, S_1, S_2 and |En| are again plotted Vs τ but in this case, Δ is kept fixed whereas S is assumed to vary. It may be seen that increase in the values of S produces marked increase in the value of S_1, S_2 and |En| for any fixed value of τ.

9. Conclusions

Up-to-now the Chaplygin's technique is the most simple and descriptive formulation among different similarity techniques and has been employed in several Elastodynamic problems on crack extension. However, this technique presents some disadvantages, especially in the plane-stress strain cases, during the final steps of the analysis where the appropriate form of the complex functions are sought to accomplish the solution.

But the method of determining the complex function presented in this paper is the correct one and ,therefore, there is no possibility of losing features of the solution.

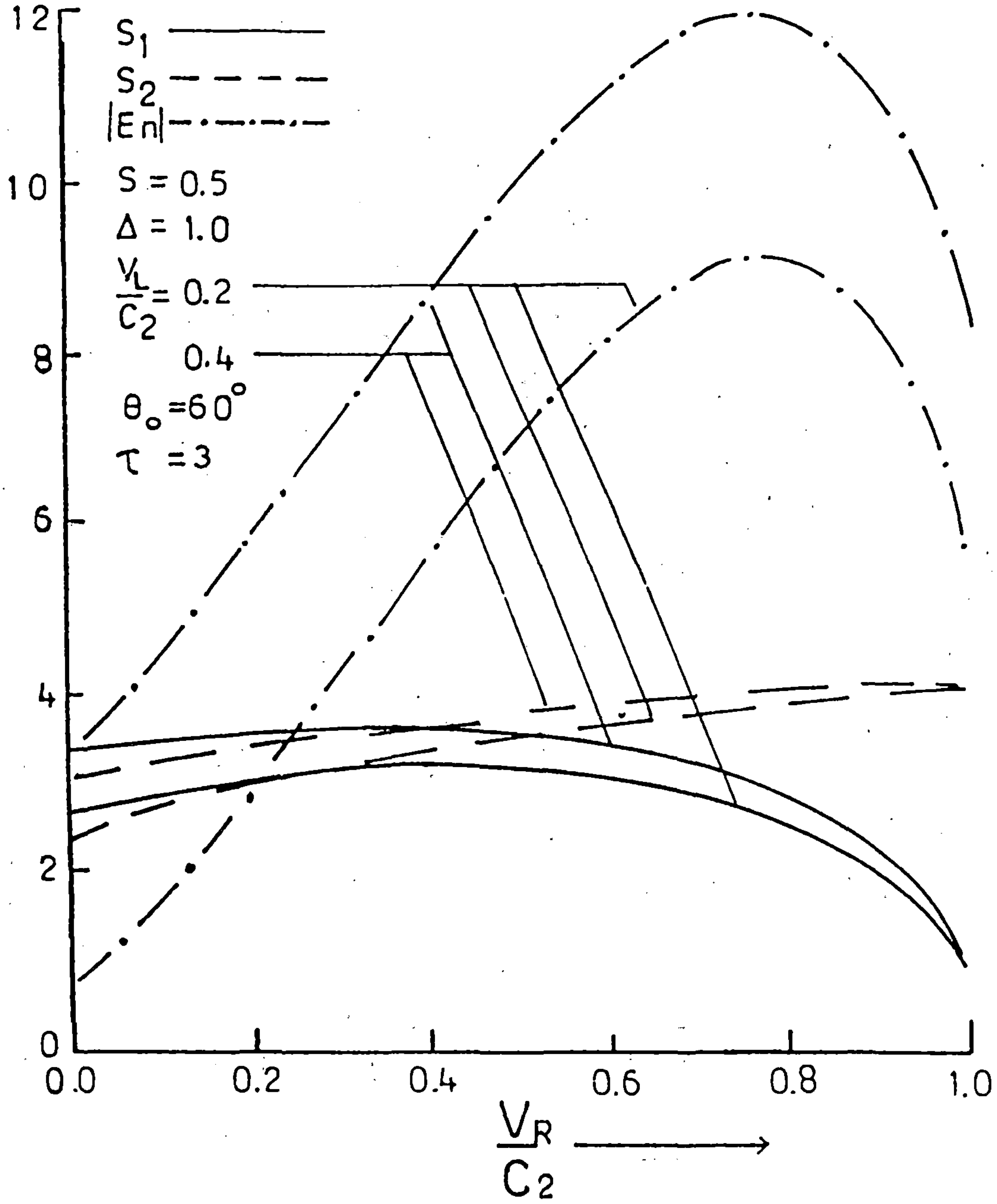


Fig. 4: Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional speed v_R/c_2.

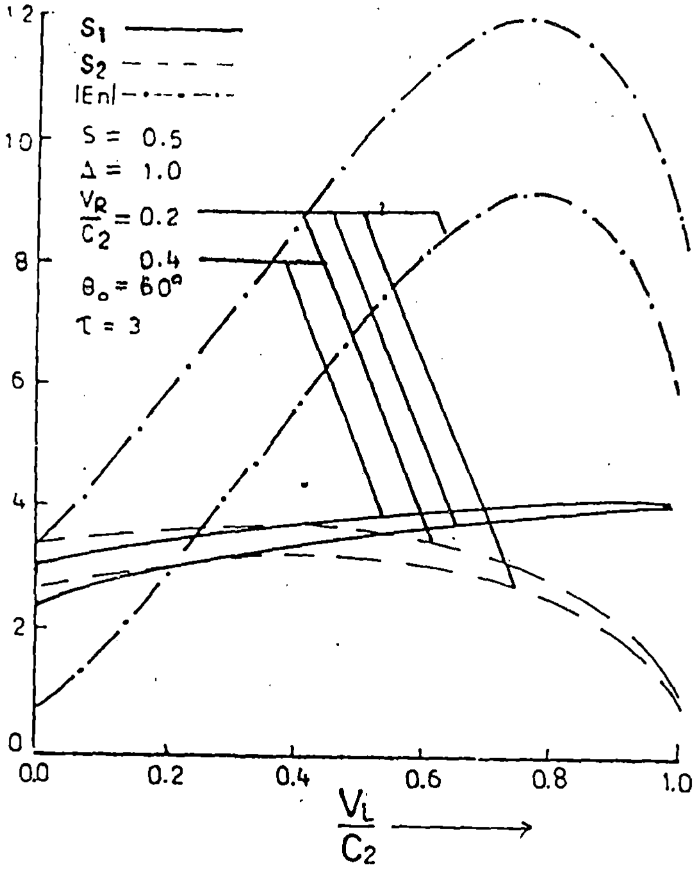


Fig. 5: Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate |En| with non-dimensional speed v_L/c_2.

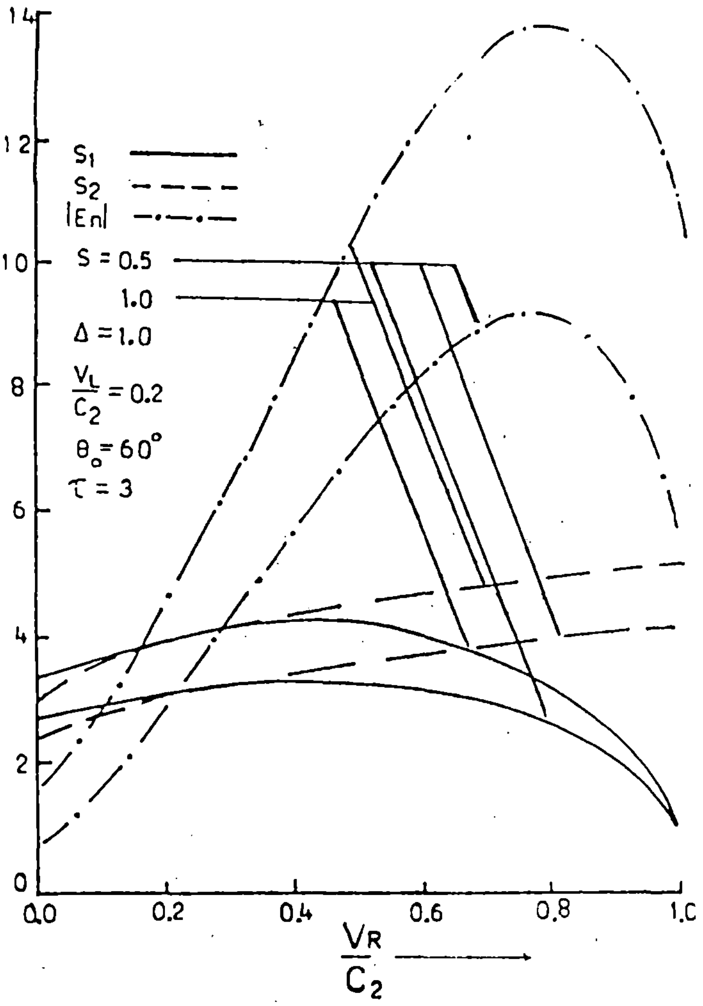


Fig. 6: Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional speed v_R/c_2.

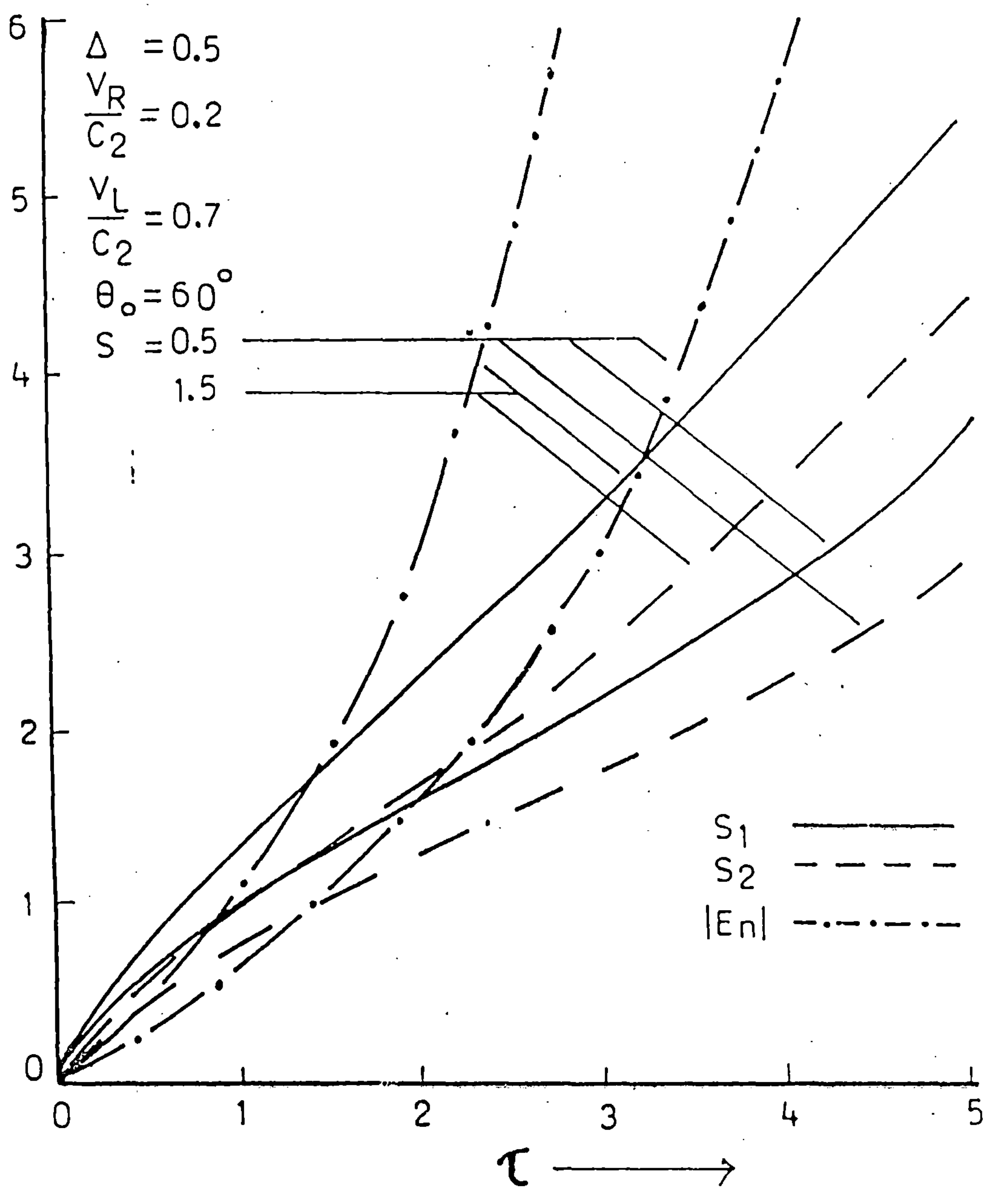


Fig. 7: Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional time after fracture initiation, τ.

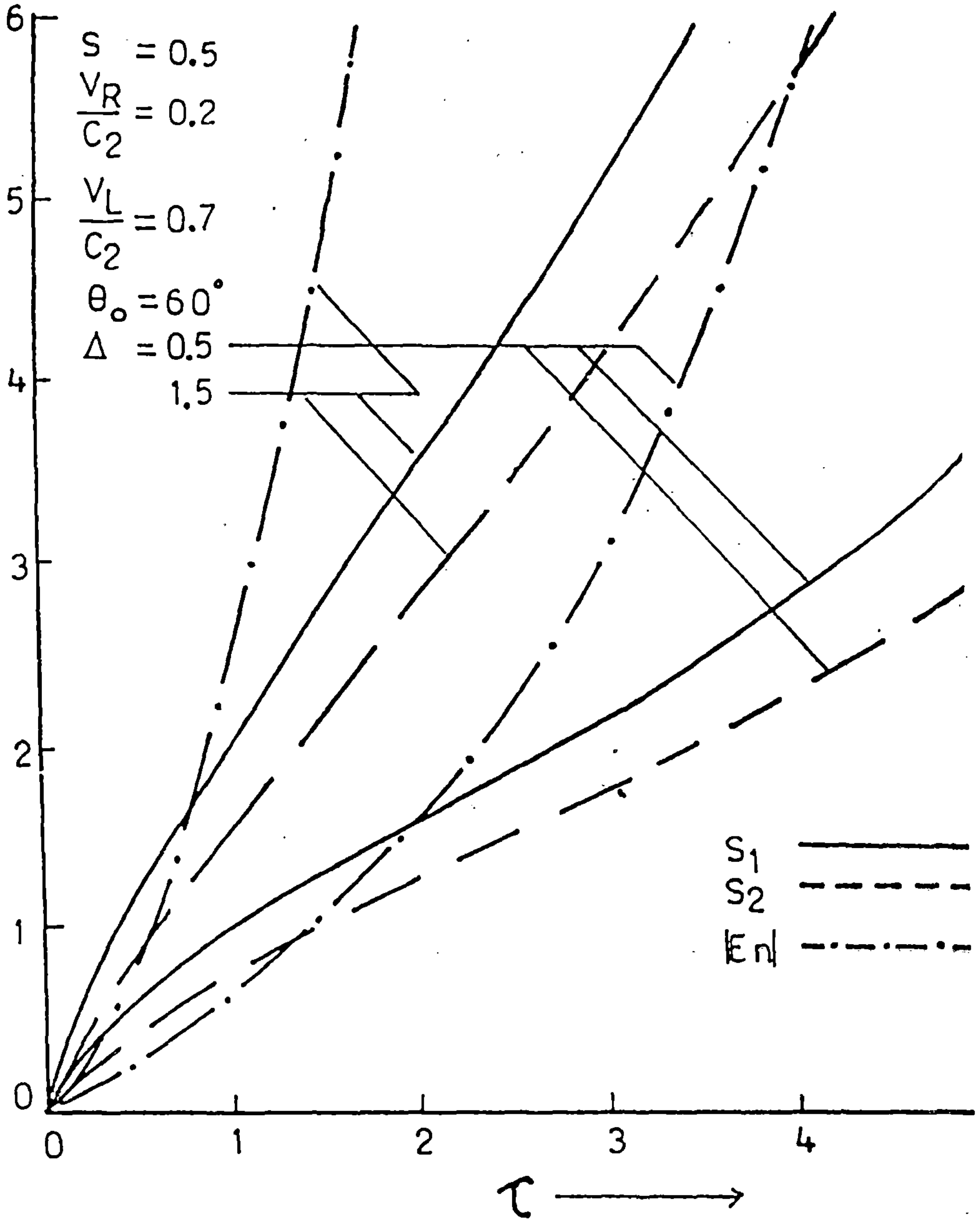


Fig. 8: Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate |En| with non-dimensional time after fracture initiation, τ.

BIFURCATION OF A CRACK DUE TO PLANE SH-WAVES IN AN INFINITE ELASTIC MEDIUM

1. Introduction

Several investigations on symmetric or non-symmetric extension of crack in its own plane in an infinite elastic medium have been carried out up-till-now. But when the extension of the crack occurs under an arbitrary angle with its own plane (which leads that a primary crack may bifurcate) the study becomes more relevant. Solutions for dynamic crack bifurcation in anti-plane strain for two special cases were solved by Burgers and Dempsey (1982). Corrected results for mode III kinking of crack under an arbitrary angle was given by Dempsey et. al.(1982). A numerical approach for the study of dynamic propagation of a kinked or bifurcated crack in anti-plane strain and also the dynamic kinking of a crack in plane strain have been given by Burgers(1982,1983). Recently, Achenbach et. al. (1984) have developed a method based on superposition principle to derive approximate expressions for the elastodynamic stress intensity factors of the kinked crack.

In this paper, the dynamic anti-plane problem of bifurcation of a semi-infinite crack due to the incidence of two linearly varying plane SH-waves with non-parallel wave fronts in an infinite elastic medium has been considered. The semi-infinite crack is assumed to bifurcate when the plane waves intersect the crack tip. For constant crack tip velocities the shear stress and particle velocity are self-similar which allow Chaplygin's transformation to reduce the problem to the solution of Laplace equation in semi-infinite strip containing a slit. The Schwarz-Christoffel transformation is employed to map the semi-infinite strip on a half-space. Expressions for shear stress in the planes of the cracks and stress intensity factors in the vicinity of the crack tips have been derived. Finally, numerical results for stress intensity factors have been presented graphically to show its variations with angle of skew for different values of the angle of incidence and the crack tip velocity.

2. Statement Of The Problem

Let two identical plane waves defined by

$$w^{\pm}_{inc} = \mp \frac{c\sigma}{\mu} \; \tau_{\pm} H(\tau_{\pm}) \tag{1}$$

referred to the coordinate system (r,θ,z) where

$$\tau_{\pm} = t + r\cos(\theta \mp \theta_o)/c, \quad 0 \le \theta_o \le \pi/2$$

and H() is the Heaviside step function, strike the tip of a stationary semi-infinite crack at $t=0$ and causes the crack to bifurcate symmetrically from the tip under an angle $k\pi$ with the plane of the crack and each of the branches starts to extend with velocity $v(<c)$. Thus,at time $t>0$, crack tips are defined by $r=vt$, $\theta=\pm k\pi$.The expanding crack, the circular wave front associated with its motion and the plane wave fronts are shown in Fig.1.The shear stress component $\sigma^{o}_{\theta z}$ corresponding to incident waves is

$$\sigma^{o}_{\theta z} = \sigma\left[\sin(\theta-\theta_o)H(\tau_+) - \sin(\theta+\theta_o)H(\tau_-) \right] \tag{2}$$

Superposing the fields due to incident waves and scattered wave we see that the conditions on the crack faces due to the scattered wave are

$$\theta=\pm\pi;\ r>0 : \sigma_{\theta z} = -2\sigma \sin\theta_o H(t - r\cos\theta_o/c)$$

$$\theta=\pm k\pi;\ 0 \le r < vt : \sigma_{\theta z} = 2\sigma \sin\theta_o \cos k\pi \tag{3a,b}$$

The shear traction given by (3a) generate the plane waves with constant particle velocity, i.e., of magnitude $\mp 2c\sigma/\mu$.Since stresses and velocities are continuous across the cylindrical wave front, on the cylindrical wave front the conditions in particle velocity are

$$\pi-\theta_o < \theta < \pi,\ r=ct : \dot{w} = -2c\sigma/\mu$$

$$-\pi < \theta < -\pi+\theta_o,\ r=ct : \dot{w} = 2c\sigma/\mu$$

and

$$-\pi+\theta_o < \theta < \pi-\theta_o,\ r=ct : \dot{w} = 0 \tag{4a-c}$$

The problem is obviously anti-symmetric about x- axis with respect to particle velocity, so only a half-plane need be considered.

In polar coordinate (r,θ); the two dimensional anti-plane wave motions are governed by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}=\frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \tag{5}$$

where $w(r,\theta,t)$ is the displacement in the z- direction and $c=\sqrt{\mu/\rho}$ is the velocity of transverse waves.

Absence of any characteristic length in the geometrical configuration of the problem and the boundary conditions (3),(4) suggest that the particle velocity $\dot{w}$ is self-similar, implying thereby that depends on $r/t,\theta$ rather than on r,θ,t separately.

Introducing the variable

$$s=r/t$$

it is found that $\dot{w}(s,\theta)$ satisfies the equation

$$s^2(1-s^2/c^2)\frac{\partial^2\dot{w}}{\partial s^2}+s(1-2s^2/c^2)\frac{\partial\dot{w}}{\partial s}+\frac{\partial^2\dot{w}}{\partial\theta^2}=0 \tag{6}$$

Within the half-circular region ABEMDCA, see Fig.1, the boundary conditions on $\dot{w}(s,\theta)$ are

$\theta=\pi,\ s\le c\ :\ \dfrac{\partial\dot{w}}{\partial\theta}=0$

$\pi-\theta_o<\theta<\pi,\ s=c\ :\ \dot{w}=-\dfrac{2c\sigma}{\mu}$

$0<\theta<\pi-\theta_o,\ s=c\ :\ \dot{w}=0$

$\theta=0,\ 0\le s\le c:\quad \dot{w}=0$

$\theta=k\pi\pm\varepsilon,\ 0\le s<v:\ \dfrac{\partial\dot{w}}{\partial\theta}=0$ (7a-e)

For s<c, the Chaplygin's transformation

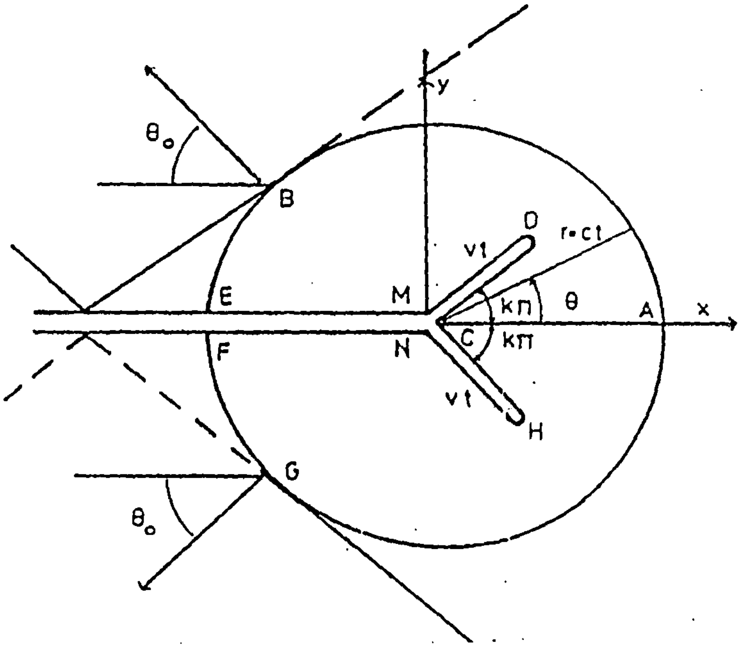


Fig.1: Pattern of incident, reflected and diffracted waves.

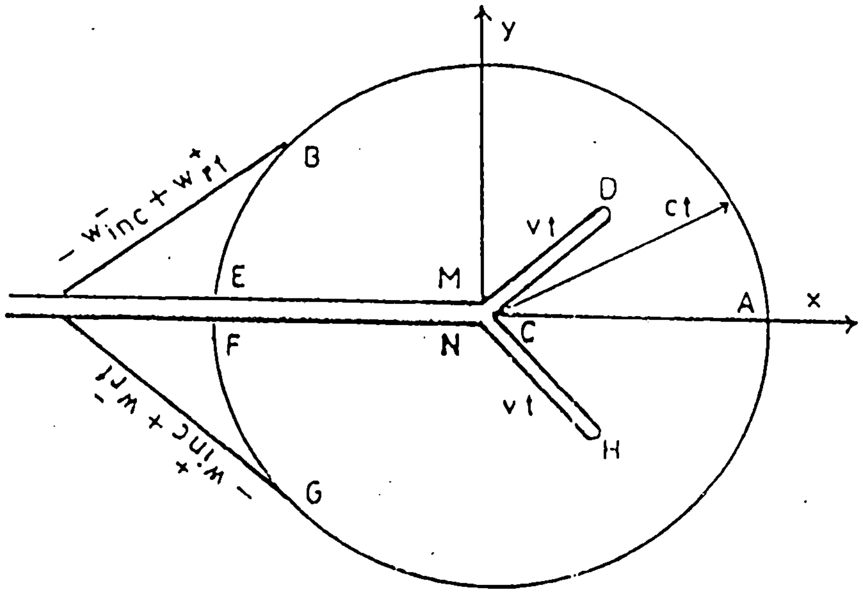


Fig.2: Pattern of waves for the superposition problem.

$$\beta = \cosh^{-1}(c/s) \tag{8}$$

reduces equation (6) to Laplace equation

$$\frac{\partial^2 \dot{w}}{\partial \beta^2} + \frac{\partial^2 \dot{w}}{\partial \theta^2} = 0 \tag{9}$$

and maps the interior of the half-circular region $[0 \le \theta \le \pi, s \le c]$ of the physical plane into a semi-infinite strip $[0 \le \theta \le \pi, 0 \le \beta < \infty]$ in θ-β plane as shown in Fig 3.

A convenient method to solve the equation (9) is to express $\dot{w}(s,\theta)$ as the real part of an analytic function and to construct an appropriate analytic function of the complex variable $\beta + i\theta$.

The domain in the γ-plane can be related to the upper half-plane of the ζ-plane by means of Schwarz-Christoffel transformation

$$\gamma = \omega(\zeta), \quad \zeta = \xi + i\eta$$

An appropriate transformation is

$$\gamma = C_o \int_1^{\zeta} \frac{u \, du}{(u+\xi_C)(u-\xi_M)\sqrt{1-u^2}} + i\pi \tag{10}$$

where C_o is an arbitrary complex constant. The ζ-plane is shown in Fig.4. The transformation given by (10) implies that the points E, A and D are mapped into $\zeta = 1$, $\zeta = -1$ and $\zeta = 0$ respectively.

Equation (10) may be integrated to yield

$$\gamma = -\frac{\xi_C}{\xi_M + \xi_C} \frac{C_o}{\sqrt{1-\xi_C^2}} \left[\ln\left\{ \sqrt{(1-\xi_C^2)(1-\zeta^2)} + \zeta\xi_C + 1 \right\} - \ln(\zeta + \xi_C) \right] -$$

$$- \frac{\xi_M}{\xi_M + \xi_C} \frac{C_o}{\sqrt{1-\xi_M^2}} \left[\ln\left\{ \sqrt{(1-\xi_M^2)(1-\zeta^2)} - \zeta\xi_M + 1 \right\} - \ln(\zeta - \xi_M) \right] + i\pi \tag{11}$$

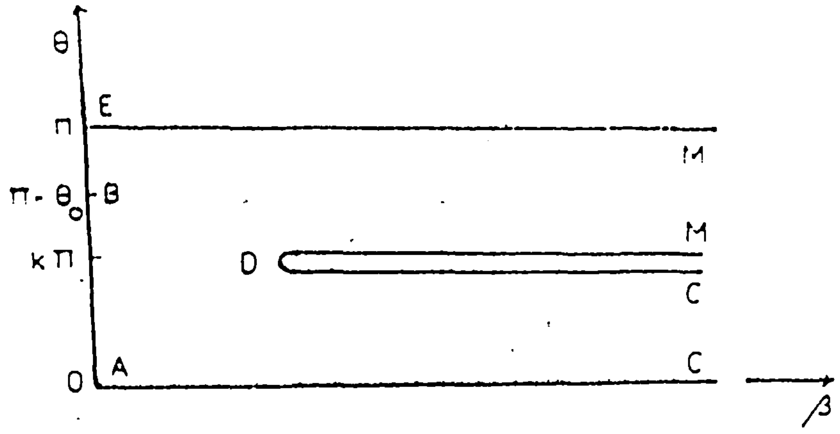


Fig.3: The θ-β plane.

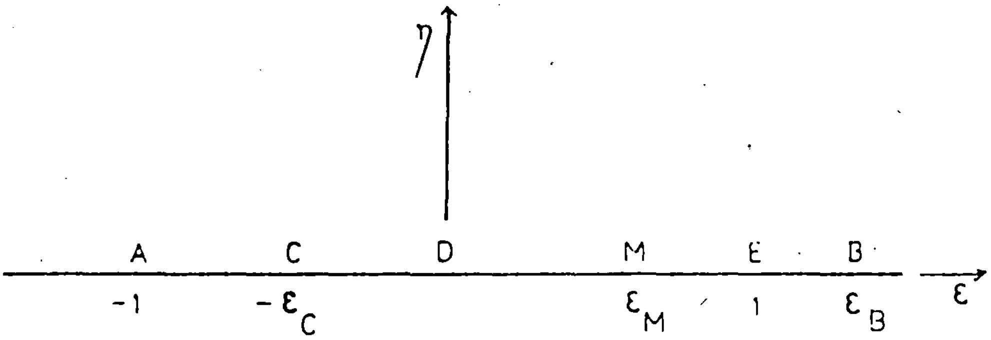


Fig. 4: The ξ-η plane.

Considering the change in imaginary parts in M and C, we obtain

$$-\frac{\xi_M}{\xi_M+\xi_C}\frac{C_o}{\sqrt{1-\xi_M^2}} = 1-k \tag{12}$$

and

$$-\frac{\xi_C}{\xi_M+\xi_C}\frac{C_o}{\sqrt{1-\xi_C^2}} = k \tag{13}$$

respectively. Thus the result (11) becomes with the aid of (12) and (13)

$$\gamma = k\left[\ln\left\{\sqrt{(1-\xi_C^2)(1-\zeta^2)} + \zeta\xi_C+1\right\} - \ln(\zeta+\xi_C)\right] + (1-k)\left[\ln\left\{\sqrt{(1-\xi_M^2)(1-\zeta^2)} -\zeta\xi_M+1\right\} - \ln(\zeta-\xi_M)\right] + i\pi \tag{14}$$

Comparing the coordinate of the point D in the γ-plane and ζ-plane we obtain

$$\gamma_D = \cosh^{-1}\frac{c}{v} + ik\pi = k\ \ln\frac{1+\sqrt{1-\xi_C^2}}{\xi_C} + (1-k)\ln\frac{1+\sqrt{1-\xi_M^2}}{\xi_M} + ik\pi \tag{15}$$

Comparison of the coordinate of the point B in γ- plane and ζ- plane results in the relation

$$k\ \sin^{-1}\frac{1+\xi_C\xi_B}{\xi_B+\xi_C} - (1-k)\ \sin^{-1}\frac{\xi_B\xi_M-1}{\xi_B-\xi_M} = \pi/2 - \theta_o \tag{16}$$

Equations (12), (13) (15) and (16) can be used to solve for C_o, ξ_M, ξ_C, ξ_B.

The boundary conditions given by (7a-e) turn into the following conditions in the ζ-plane

$$\eta=0,\quad -\infty<\xi\le-\xi_C \;:\; \dot{w} = 0$$

$$\eta=0,\quad -\xi_C<\xi<1 \;:\; \frac{\partial\dot{w}}{\partial\eta} = 0$$

$$\eta=0,\quad 1\le\xi\le\xi_B \;:\; \dot{w} = -\frac{2c\sigma}{\mu}$$

$$\eta=0,\quad \xi_B<\xi<\infty \;:\; \dot{w} = 0 \qquad (17a\text{-}d)$$

Before we proceed to construct an analytic function which satisfies the conditions (17a-d), we will investigate the relations between small distances from the point D in the physical plane, in the γ-plane and in the ζ- plane. In the physical plane we consider the distance r-vt in the crack plane $\theta=k\pi$. For $(r-vt)/r<<1$ we easily establish that

$$\frac{r-vt}{r} = \frac{v}{c}\left[\text{Cosh}\beta_D - \text{Cosh}\beta\right] \approx \left[1-v^2/c^2\right]^{1/2}(\beta_D-\beta)$$

In the ζ- plane we find for $|\zeta|<<1$

$$\gamma-\gamma_D = \frac{1}{2}\,\omega_o\zeta^2$$

i.e.,
$$\zeta^2 = -\frac{2}{\omega_o}\left[1-v^2/c^2\right]^{-1/2}(r-vt)/vt \qquad (18)$$

where
$$\omega_o = k\,\frac{\sqrt{1-\xi_C^2}}{\xi_C^2} + (1-k)\,\frac{\sqrt{1-\xi_M^2}}{\xi_M^2} \qquad (19)$$

Equation (18) has been derived by expanding (14) and maintaining terms of $O(\zeta^2)$. The terms $O(\zeta)$ in the expansion is found to vanish with the aid of the results (12), (13). Further the equation (18) suggests that $\zeta=i\eta$

Mhere
$$\eta = \left[\frac{2}{\omega_o}\left[1-v^2/c^2\right]^{-1/2}\frac{c}{v}\,\frac{r-vt}{ct}\right]^{1/2} \qquad (20)$$

Next, if we take $\dot{w}=\text{Re}\,F(\zeta)$ then in view of the conditions given by (17) it is found convenient to work with $F'(\zeta)$. Accordingly, we consider

$$F'(\zeta) = F_1'(\zeta) + F_2'(\zeta) \tag{21}$$

where

$$F_1'(\zeta) = \frac{A}{(\zeta-\xi_B)\sqrt{(1-\zeta)(\zeta+\xi_C)}} \tag{22}$$

and

$$F_2'(\zeta) = \frac{1}{\sqrt{(1-\zeta)(\zeta+\xi_C)}}\left[\frac{B}{\zeta} + \frac{C}{\zeta^2}\right] \tag{23}$$

Integrating (21) with respect to ζ and using the condition that $\dot{w}$ possesses a jump discontinuity at $\xi=\xi_B$ as seen from (17c) and (17d) we find that

$$A = 2\frac{c\sigma}{\pi\mu}\sqrt{(\xi_B+\xi_C)(\xi_B-1)} \tag{24}$$

Integrating (23) we obtain

$$F_2(\zeta) = \xi_C^{-1}\left[-C\zeta^{-1}\sqrt{(\zeta+\xi_C)(1-\zeta)} + \xi_C^{-1/2}\left\{(1-\xi_C)C-2B\xi_C\right\}\ln\left\{\frac{\sqrt{\xi_C(1-\zeta)}+\sqrt{\zeta+\xi_C}}{\sqrt{\zeta\xi_C}}\right\}\right] \tag{25}$$

Since the term involving logarithm gives rise to a logarithmic singularity at $\zeta=0$ which is not acceptable, we require

$$B = (1-\xi_C)C/2\xi_C \tag{26}$$

The shear stress at $r>vt$, $\theta=k\pi$ can be obtained using the relation

$$\frac{\partial \dot{w}}{\partial \theta} = -\mathrm{Im}\left[F'(\zeta)\,\frac{d\zeta}{d\gamma}\right]$$

as

$$\tau_{\theta z} = -\frac{\mu}{r}\,\mathrm{Im}\int_{r/c}^{t} F'(\zeta)\,\frac{d\zeta}{d\gamma}\,dt \tag{27}$$

As for all values of k, ζ can not be expressed in terms of t explicitly. Hence for $k \neq 0$ the integration (27) is to be carried out numerically.

In order to extract the singular term we change the integration of $F_2'(\zeta)$ over t in (27) to an integration over ζ as follows

$$I_2 = -\frac{\mu}{c}\,\mathrm{Im}\int_{\Gamma} \sinh\beta(\zeta)\; F_2'(\zeta)\; d\zeta \tag{28}$$

where Γ is the corresponding contour in the ζ-plane.

Integrating (28) by parts and then changing the variable in the integration over $F_1'(\zeta)$ and $F_2(\zeta)$ to the variable s we obtain from (27)

$$\tau_{\theta z} = -\frac{\mu}{s}\left(1-\frac{s^2}{c^2}\right)^{1/2}\mathrm{Im}\;F_2(\zeta) - \mu I \tag{29}$$

where

$$I = I_3 + C I_4 \tag{30}$$

with

$$I_3 = 2\frac{c\sigma}{\pi\mu}\sqrt{(\xi_B+\xi_C)(\xi_B-1)}\;\mathrm{Im}\int_s^c s^{-2}\,\frac{d\zeta}{d\gamma}\,\frac{ds}{(\zeta-\xi_B)\sqrt{(1-\zeta)(\zeta+\xi_C)}} \tag{31}$$

and

$$I_4 = \xi_C^{-1}\;\mathrm{Im}\int_s^c \zeta^{-1}\sqrt{(\zeta+\xi_C)(1-\zeta)}\;\frac{ds}{s^2\sqrt{1-s^2/c^2}} \tag{32}$$

The stress intensity factor N, at the crack tip defined by r=vt, $\theta=k\pi$ is obtained using (20),(25) and (29) as

$$N = \underset{r\to vt}{\mathrm{Lt}}\sqrt{2\pi(r-vt)}\;\tau_{\theta z} = -\left[\frac{\pi\omega_0}{vc}\left(1-\frac{v^2}{c^2}\right)^{3/2}\right]^{1/2}\mu C\sqrt{ct/\xi_C} \tag{33}$$

From the asymptotic analysis of the deformation field about a dynamically extending crack tip, we see that if $\tau_{\theta z}\to\tau$ as $r\to vt-0$ on

$\theta=k\pi$, then the regular term in $\tau_{\theta z}$ as $r\rightarrow vt+0, \theta=k\pi$ should also be equal to τ. So, we require

$$I_3\big|_{s=v}+CI_4\big|_{s=v} = -\frac{2\sigma}{\mu}\sin\theta_0\cos k\pi \qquad (34)$$

where I_3 and I_4 are given by (31) and (32) respectively. Equation (34) gives the value of C.

3. Numerical Results and Discussions

In this section numerical results for the dimensionless stress intensity factor S where $S = N/\sigma\sqrt{ct}$ and N is defined by (33) have been plotted in Fig.5 versus the parameter k which defines the angle of skew, for different values of v/c and θ_0. It has been shown in Fig.5 that for fixed value of v/c the values of S decreases with the increase in the value of k and as the value of v/c increases the values of S is found to decrease which is expected from physical stand point. Again it is to be noted from Fig.4 that the values of S increases with the increase in the value of the angle of incidence of the plane waves.

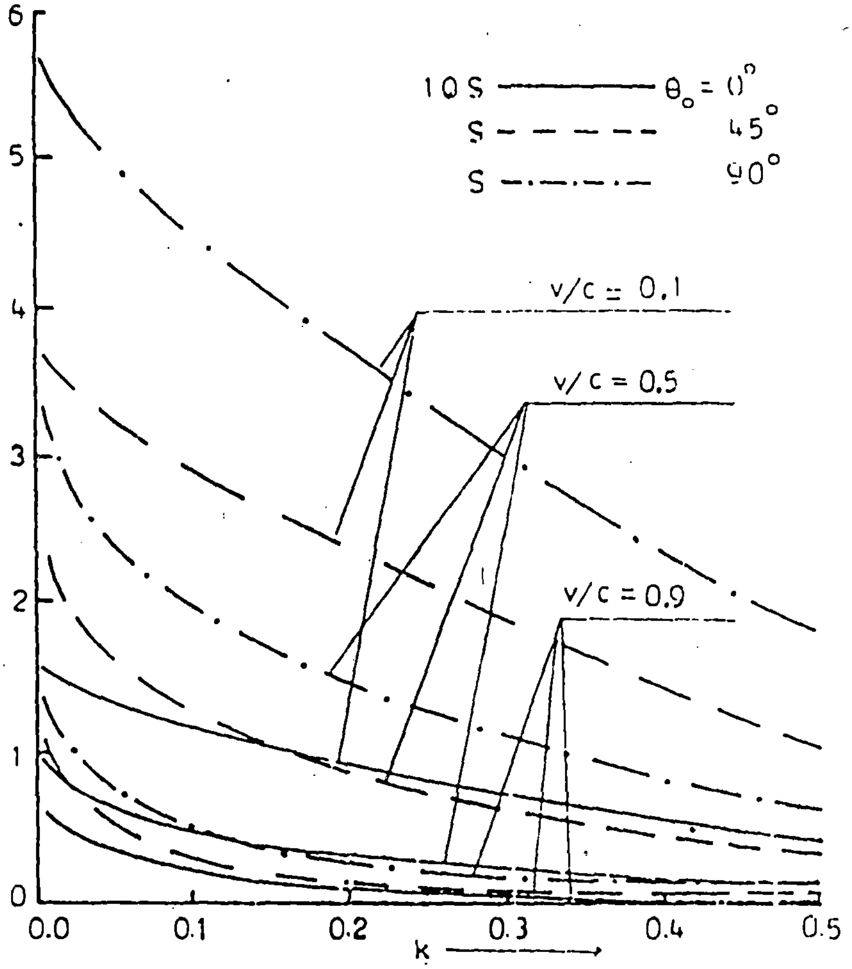


Fig.5: Variation of stress intensity factor with k.

CHAPTER III

PROBLEMS ON PROPAGATION OF ELASTIC WAVES IN THE PRESENCE OF TOPOGRAPHICAL IRREGULARITY

Paper 10: Time step SH- wave transmission across a rectangular step.

Paper 11: SH- wave propagation across a vertical step in two joined elastic half- spaces

---

TIME STEP SH- WAVE TRANSMISSION ACROSS A RECTANGULAR STEP

1. Introduction

In recent past, several papers have appeared on propagation on elastic wave in materials with free surface having irregularities. Wolf (1967) studied the propagation of Love wave in layers with irregular boundaries using perturbation method. Adopting the representation theorem due to Knopoff (1956) to obtain a perturbed solution of the problem, Knopoff and Hudson (1964) studied the transmission of Love wave past a continental margin considering the crust to have an abrupt increase in thickness on the continental side. A series of problems involving the scattering of elastic waves by two dimensional and three dimensional topographical irregularities have been solved by Sanchez- Sesma (1979, 1982, 1983, 1985) by using a newly developed boundary method. The diffracted wave fields are constructed with linear combinations of solutions which form c-complete families for the wave equations and boundary conditions are then satisfied in a least square sense. Bose (1975) was also solved the transmission of SH- wave across a step like irregularity in the surface of an elastic half- space by a different method. He considered a time harmonic plane SH- wave in the form exp(i(ωt-kX)) propagating in the direction perpendicular to the step and solving the resulting integral equation asymptotically, transmitted wave at distance far away form the step was obtained. Dutta and Mitra (1974) have presented the SH- wave motion in an elastic layer of different shear wave velocity.

In this paper, we studied the problem of transmission of time step SH- wave motion across a step like irregularity in the surface of an elastic half- space consisting of two quarter spaces of same material joined along the common boundary. Considering the incident wave in the form H(T-X/c) where H() is the Heaviside step function the problem is reduced to an integral equation by using integral transform and Green's function technique and finally using Cagniard- Dehoop's method of finding inverse Laplace transform, , transmitted field at any distances form the step on the free surface has been determined using iterative procedure. Numerical results have been presented in the form of graph to illustrate the nature of transmission.

2. Formulation Of Problem

The transmission of SH- wave across a step like irregularity in the surface of an elastic half-space consisting of two quarter spaces of same material joined along the common boundary X=0 has been considered here. We introduce the axes of coordinates as shown in Fig.1. Denoting the coordinates of a point in the X-Z plane by (X,Z), we take the incident plane SH- wave as H(T-X/c), where H(x) is the Heaviside's step function, so that the propagation is from higher side to the lower side of the step.

Let V_1, V_2 be the Y- components of the displacement in the two media I and II respectively. The field equations are wave equations in two media and boundary conditions are that the shearing stress vanishes everywhere on the outer boundaries $Z=0, X<0,; X=0, 0\le Z\le H$ and $Z=H, X>0$ the displacement and stress are continuous on the interface $X=0, Z>H$.

Taking Laplace transform of these wave equations and boundary conditions with respect to time T(with parameter p), we get

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} - \frac{p^2}{c^2}\right]\bar{V}_1 = 0 \qquad (1)$$

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} - \frac{p^2}{c^2}\right]\bar{V}_2 = 0 \qquad (2)$$

$$\left[\frac{\partial \bar{V}_2}{\partial Z}\right]_{Z=0} = 0 \qquad \text{for } X<0$$

$$\left[\frac{\partial \bar{V}_1}{\partial Z}\right]_{Z=H} = 0 \qquad \text{for } X>0$$

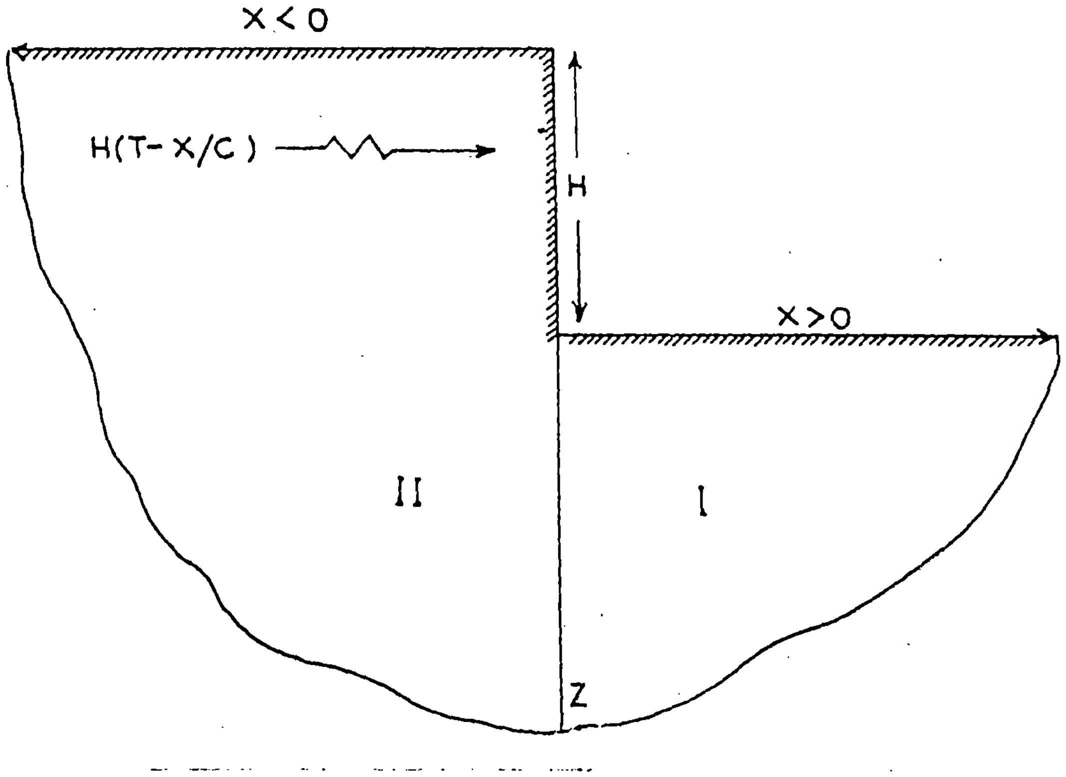


Fig.1: Geometry and coordinate system.

$$\left[\frac{\partial \overline{V}_2}{\partial X} \right]_{X=0} = 0 \qquad \text{for} \quad 0 \le Z \le H \qquad (3a\text{-}c)$$

$$\left[\frac{\partial \overline{V}_2}{\partial X} \right]_{X=0} = \left[\frac{\partial \overline{V}_1}{\partial X} \right]_{X=0} \qquad \text{for } Z>H$$

$$\overline{V}_2(0,Z) = \overline{V}_1(0,Z) \quad \text{for } Z>H \qquad (4a,b)$$

where H is the height of the step.

We represent transverse displacement in two domains X<0 and X>0 in the form

$$V_2 = H(T-X/c) + H(T+X/c) + V_2'(X,Z,T), \quad X<0,\ Z>0$$

$$V_1 = V_1(X,Z,T), \qquad X>0,\ Z>H. \qquad (5)$$

Obviously $\overline{V}_2'$ which is Laplace transform of $V_2'(X,Z,T)$ satisfies the the equation

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} - \frac{p^2}{c^2} \right] \overline{V}_2' = 0 \qquad (6)$$

3. Reduction to integral equation and its solution

We introduce Green's functions $G_1(X,Z:R,S)$ and $G_2(X,Z:U,W)$ in medium I and II respectively such that $G_2(X,Z:U,W)$ is the solution of

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} - \frac{p^2}{c^2} \right] G_2(X,Z:U,W) = -4\pi\delta(X-U)\delta(Z-W) \qquad (7)$$

for medium II with vanishing normal derivative at X<0, Z=0 and at X=0, Z>0. Similarly, $G_1(X,Z:R,S)$ defined with reference to medium I is the solution of

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} - \frac{p^2}{c^2} \right] G_1(X,Z:R,S) = -4\pi\delta(X-R)\delta(Z-S) \qquad (8)$$

and satisfies the vanishing of normal derivative at X>0, Z=H and at X=0, Z>H.

Now multiplying equation (7) by $\overline{V}'_2$ and equation (6) by G_2, then subtracting the resulting second equation from the first, we obtain

$$\overline{V}'_2\nabla^2 G_2 - G_2\nabla^2\overline{V}'_2 = -4\pi\ \overline{V}'_2(X,Z)\ \delta(X-U)\ \delta(Z-W)$$

where ∇^2 is the Laplacian operator in the X-Z plane.

Integration of this over the region of medium II yields

$$-4\pi\ \overline{V}'_2(U,W) = \iint \left[\overline{V}'_2\nabla^2 G_2 - G_2\nabla^2\overline{V}'_2 \right] dX\ dZ$$

$$= \int_\Gamma \left\{ \overline{V}'_2 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial \overline{V}'_2}{\partial n} \right\} d\Gamma$$

where Γ is the boundary of the medium II and n is outward drawn normal to it.

Therefore,
$$4\pi\ \overline{V}'_2(U,W) = \int_H^\infty G_2(0,Z:U,W) \left(\frac{\partial \overline{V}'_2}{\partial X}\right)_{X=0} dZ \qquad (9)$$

and a similar application of Green's theorem to medium I yields

$$4\pi\ \overline{V}_1(R,S) = -\int_H^\infty G_1(0,Z:R,S) \left(\frac{\partial \overline{V}_1}{\partial X}\right)_{X=0} dZ \qquad (10)$$

Substitution of (9) and (10) into the equation (4b) yields with the aid of (4a) and (5) the integral equation

$$\int_H^\infty \left[G_2(0,Z:0,W) + G_1(0,Z:0,W)\right] \left(\frac{\partial \overline{V}_1}{\partial X}\right)_{X=0} dZ = -8\pi/p \qquad (11)$$

To evaluate the Green's Function for the medium I and II Fourier cosine transform with respect to Z has been taken. With this application of Fourier cosine transform, equation (7) reduces to ordinary equation viz

$$d^2G_2^c/dX^2 - (\alpha^2+p^2/c^2)G_2^c = -4\pi\cos(\alpha W)\ \delta(X-U)$$

where

$$G_2^c = \int_0^\infty G_2(X,Z:U,W)\ \cos(\alpha Z)\ dZ.$$

Solution of the above differential equation can be taken as

$$G_2^c = A\ \cosh(\beta X)\ e^{\beta U}, \qquad X\geq U$$

$$= A\ \cosh(\beta U)\ e^{\beta X}, \qquad X\leq U$$

where the unknown constant A is determined as follows

$$A\beta e^{\beta U}\ [\sinh(\beta U) - \cosh(\beta U)] = -4\pi\ \cos(\alpha W)$$

or $\quad A = 4\pi\ \cos(\alpha W)/\beta$

Hence taking inverse Fourier cosine transform we find

$$G_2(X,Z:U,W) = 8\int_0^\infty \frac{e^{\beta U}}{\beta}\cosh(\beta X)\ \cos(\alpha W)\ \cos(\alpha Z)\ d\alpha \qquad 0\geq X\geq U$$

$$= 8\int_0^\infty \frac{e^{\beta X}}{\beta}\cosh(\beta U)\ \cos(\alpha W)\ \cos(\alpha Z)\ d\alpha \qquad -\infty<X\leq U \tag{12}$$

where $\beta^2 = \alpha^2+p^2/c^2$.

Again introducing Fourier cosine transform defined by

$$G_1^c = \int_0^\infty G_1(X,Z:R,S)\ \cos\alpha(Z-H)\ d(Z-H)$$

in equation (8), we obtain,

$$G_1(X,Z:R,S) = 8\int_0^\infty \frac{e^{-\beta R}}{\beta} \cosh(\beta X) \cos\alpha(S-H) \cos\alpha(Z-H)\, d\alpha \qquad 0\le X\le R$$

$$= 8\int_0^\infty \frac{e^{-\beta X}}{\beta} \cosh(\beta R) \cos\alpha(S-H) \cos\alpha(Z-H)\, d\alpha \qquad R\le X<\infty \qquad (13)$$

On substitution of these values of Green's functions in the integral equation (11), it takes the form

$$\int_H^\infty \left(\frac{\partial \bar{V}_1}{\partial X}\right)_{X=0} dZ \int_0^\infty \left[\frac{\cos\alpha(W-H)\cos\alpha(Z-H)}{\beta} + \frac{\cos\alpha W \ \cos\alpha Z}{\beta}\right] d\alpha = -\pi/p \qquad (14)$$

Substituting W=wH, Z=zH, we obtain form (14)

$$H\int_1^\infty \left(\frac{\partial \bar{V}_1}{\partial X}\right)_{X=0} dz \int_0^\infty \frac{\cos\alpha H(w-1)\cos\alpha H(z-1)}{\beta}\, d\alpha = -\pi/p - H\int_1^\infty \left(\frac{\partial \bar{V}_1}{\partial X}\right)_{X=0} dz$$

$$\times\int_0^\infty \frac{\cos(\alpha Hw)\ \cos(\alpha Hz)}{\beta}\, d\alpha$$

On inversion with respect to α, it takes the form

$$H\int_1^\infty \left(\frac{\partial \bar{V}_1}{\partial X}\right)_{X=0} \frac{\cos\alpha H(z-1)}{\beta}\, dz = -\frac{2H}{p}\int_1^\infty \cos\alpha H(w-1)\, dw - \frac{2H^2}{\pi}\int_1^\infty \left(\frac{\partial \bar{V}_1}{\partial X}\right)_{X=0} dz$$

$$\times\int_0^\infty \frac{\cos(\alpha Hz)}{\beta(\tau)}\, d\tau \int_1^\infty \cos(\tau Hw)\ \cos\alpha H(w-1)\, dw \qquad (15)$$

where $\beta(\tau)$ is obtained form β by replacing α by τ,

Next using the results

$$\int_1^\infty \cos\alpha H(w-1)\, dw = \pi\delta(\alpha)/H \qquad (16)$$

and
$$\int_1^\infty \sin\alpha H(w-1)\, dw = 1/\alpha H \tag{17}$$

it can be easily shown that

$$\int_1^\infty \cos\tau Hw \cos\alpha H(w-1)dw = \frac{\pi}{2}\cos(\tau H)\Big[\delta\{(\tau+\alpha)H\}+\delta\{(\tau-\alpha)H\}\Big] - \frac{\tau \sin(\tau H)}{H(\tau^2-\alpha^2)} \tag{18}$$

where $\delta(x)$ is the dirac δ- function.

Using equation (18) in equation (15) and on simplifaication it reduces to the form

$$H\int_1^\infty \left(\frac{\partial\overline{V}_1}{\partial X}\right)_{X=0} \frac{\cos\alpha H(z-1)}{\beta}\, dz = -\frac{\pi\delta(\alpha)}{p} + \frac{H}{2}\int_1^\infty \left(\frac{\partial\overline{V}_1}{\partial X}\right)_{X=0} \frac{\sin(\alpha Hz)\sin(\alpha H)}{\beta} dz$$

$$+\frac{H}{\pi}\int_1^\infty \left(\frac{\partial\overline{V}_1}{\partial X}\right)_{X=0} \int_0^\infty \frac{\cos(\tau Hz)\sin(\tau H)}{\beta(\tau)(\tau^2-\alpha^2)}\,\tau\, d\tau \tag{19}$$

4. Evaluation of Displacement

Substituting the value of $G_1(0,Z:R,S)$ form equation (13) in equation (10) we obtain

$$4\pi\overline{V}_1(X,S) = -8H\int_0^\infty e^{-\beta X}\cos\alpha H(s-1)\, d\alpha \int_1^\infty \left(\frac{\partial\overline{V}_1}{\partial X}\right)_{X=0} \frac{\cos\alpha H(z-1)}{\beta}\, dz$$

where $S=sH$. Inserting the result (19) in the above expression we find

$$\overline{V}_1(X,S) = \frac{e^{-pX/c}}{p} - \frac{H}{\pi}\int_0^\infty e^{-\beta X}\cos\alpha H(s-1)d\alpha \int_1^\infty \left(\frac{\partial\overline{V}_1}{\partial X}\right)_{X=0} \frac{\cos\alpha H(z-1)}{\beta}\, dz\ +$$

$$+\frac{2H^2}{\pi^2}\int_0^\infty e^{-\beta X}\cos\alpha H(s-1)\,d\alpha\int_0^\infty\frac{d\tau}{\beta(\tau)}\int_0^\infty\cos(\tau Hw)\cos\alpha H(w-1)\,dw\times$$

$$\times\int_0^\infty\left(\frac{\partial\overline{V}_1}{\partial X}\right)_{X=0}\cos(\tau Hz)dz \qquad (20)$$

We can compute $\overline{V}_1(X,S)$ form equation (20) by iterative process. We take the first iterate of (20) as

$$\overline{V}_1(X,S) = \frac{e^{-pX/c}}{p} \qquad (21)$$

Deriving $\left(\frac{\partial\overline{V}_1}{\partial X}\right)_{X=0}$ form (21) and substituting it in the second term on the right hand side of (20), the second term, with the aid of (16) becomes $\frac{e^{-pX/c}}{2p}$.

Similarly, with the aid of (16) and (17), the third term on the right hand side of (20) becomes

$$-\frac{e^{-pX/c}}{2p}+\frac{2H^2}{\pi^2 c}\int_1^\infty dw\int_0^\infty e^{-\beta X}\cos\alpha H(s-1)\cos\alpha H(w-1)\,d\alpha\int_0^\infty\frac{\cos(\tau Hw)\cos(\tau H)}{\tau\beta(\tau)}d\tau$$

Thus the second iterate is

$$\overline{V}_1(X,S) = \frac{e^{-pX/c}}{p}+\frac{2H^2}{\pi^2 c}\int_1^\infty dw\int_0^\infty e^{-\beta X}\cos\alpha H(s-1)\cos\alpha H(w-1)\,d\alpha\,\times$$

$$\times\int_0^\infty\frac{\cos(\tau Hw)\cos(\tau H)}{\tau\beta(\tau)}d\tau \qquad (22)$$

In order to find the displacement on the free surface, the next task is to put s=1 and obtain Laplace inversion of the right hand side of (22).

Now $$\int_0^\infty \frac{\cos(\tau Hw)\sin(\tau H)}{\tau\beta(\tau)}\,d\tau = \frac{1}{4ip}\int_{-\infty}^{\infty} \frac{e^{ipH(w+1)v} - e^{ipH(w-1)v}}{v\sqrt{v^2+1/c^2}}\,dv$$

where $\tau = pv$.

Deforming the path of integration to the path parallel to the imaginary v- axis round the branch point $v=1/c$, the above integral can be reduced to the form

$$\frac{H}{2p}\left[(w-1)\int_{H(w-1)/c}^{\infty} \frac{e^{-pT}\,dT}{T\sqrt{T^2-H^2(w-1)^2/c^2}} - (w+1)\int_{H(w+1)/c}^{\infty} \frac{e^{-pT}\,dT}{T\sqrt{T^2-H^2(w+1)^2/c^2}}\right] \tag{23}$$

Further

$$\int_0^\infty e^{-\beta X}\cos\alpha H(w-1)\,d\alpha = \frac{1}{2}\int_{-\infty}^{\infty} p\,e^{-p[\sqrt{u^2+c^{-2}}\,X - iuH(w-1)]}\,du$$

which by the well known Cagniard- Dehoop transformation reduces to the form

$$pX\int_{R/c}^{\infty} \frac{T\,e^{-pT}}{R^2\sqrt{T^2-R^2/c^2}}\,dT \tag{24}$$

where $R^2 = X^2 + H^2(w-1)^2$.

Substituting the results (23) and (24) on the right hand side of (22) and taking the Laplace inverse transform, we obtain

$$V_1(X,H,T) = H(T-X/C) + \frac{XH^2}{\pi^2 c}\left[H(T-X/C)\int_1^{1+\frac{C^2T^2-X^2}{2HCT}} \frac{w-1}{R^2}\,dw\ \times\right.$$

$$\times \int_{H(w-1)/c}^{T-R/C} \frac{(T-\sigma)\ d\sigma}{\sigma\sqrt{[(T-\sigma)^2-R^2/C^2][\sigma^2-H^2(w-1)^2/c^2]}} - H(T-\frac{X+2H}{C}) \times$$

$$\times \int_1^{1+\frac{(CT-2H)^2-X^2}{2H(CT-2H)}} \frac{w+1}{R^2}\ dw \int_{H(w+1)/c}^{T-R/C} \frac{(T-\sigma)\ d\sigma}{\sigma\sqrt{[(T-\sigma)^2-R^2/C^2][\sigma^2-H^2(w+1)^2/c^2]}} \tag{25}$$

which represents the displacement due to transmitted wave in the free surface of the medium 1.

Introducing $X=Hx$, $T=Ht/c$, $\sigma=H\sigma'/c$ and writing $A=t+\sqrt{x^2+(w-1)^2}$, $B=t-\sqrt{x^2+(w-1)^2}$, $D=w-1$, $E=-D$ so that $A>B>D>E$, we get from (25)

$$V_1(x,1,t) = H(t-x) + \frac{x}{\pi^2}\left[H(t-x)\int_1^{1+\frac{t^2-x^2}{2t}} \frac{w-1}{x^2+(w-1)^2}\ dw \left\{ \frac{2}{\sqrt{\alpha_1\beta_1}} \times \right.\right.$$

$$\left.\times \left(\frac{2t}{D}\,\Pi(\pi/2,-\delta_1/\beta_1,q_1) - (\frac{t}{D}+1)F(\pi/2,q_1)\right)\right\} - H(t-\overline{x+2}) \times$$

$$\times \int_1^{1+\frac{(t-2)^2-x^2}{2(t-2)}} \frac{w+1}{x^2+(w-1)^2}\ dw \left\{ \frac{2}{\sqrt{\alpha_2\beta_2}} \left(\frac{2t}{D}\,\Pi(\pi/2,-\delta_2/\beta_2,q_2) - (\frac{t}{D}+1) \times \right.\right.$$

$$\left.\left.\left. \times F(\pi/2,q_2)\right)\right\}\ dw \right] \tag{26}$$

where

$$A-D = \alpha_1, \quad B-E = \beta_1, \quad A-E = \tau_1, \quad B-D = \delta_1$$

$$A-\bar{D} = \alpha_2, \quad B-\bar{E} = \beta_2, \quad A-\bar{E} = \tau_2, \quad B-\bar{D} = \delta_2$$

$$q_i = \sqrt{\frac{\delta_i \tau_i}{\alpha_i \beta_i}}, \quad i=1,2$$

and $\bar{D}, \bar{E}$ are obtained from D,E respectively by replacing w-1 by w+1 and $\Pi(\phi, n, k)$, $F(\phi, k)$ are elliptic integral of third and first kind respectively.

5. Numerical results.

To show the nature of the motion, we have evaluated numerically the displacement curve for transmitted wave. The results shown in Fig.2 depicted the variation of transmitted wave versus dimensionless time t for different values of x. Due to the presemce of the step, the value of the displacement is found to increase initially at the arrival of the displacement an abrupt decrease after a small interval of time. Also, as expected from physical stand point, the initial increament in the value of the displacement is forund to decrease gradually with the increase in distance from the step.

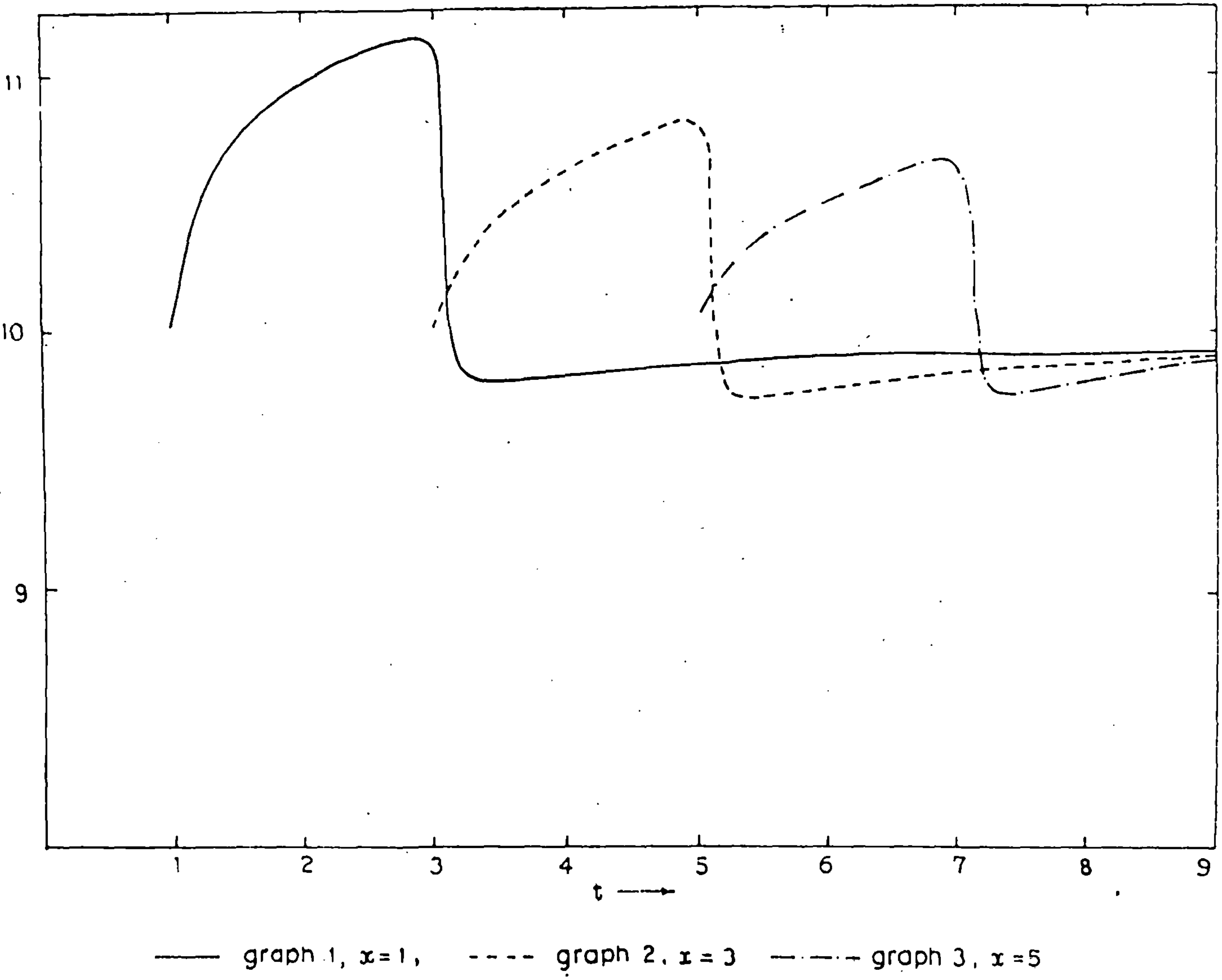


Fig. 2: Variation of transverse displacement with time.

SH WAVE PROPAGATION ACROSS A VERTICAL STEP IN TWO JOINED ELASTIC HALF SPACES

1. Introduction

The problem of propagation of elastic waves in the presence of surface irregularities has been studied by several investigators. Abubakar (1963) studied the effect of an irregular surface with an isolated irregularity like a tough or ditch on incident harmonic P- and SV-waves. Propagation of Love wave in an elastic layer having an irregular boundary overlying a rigid half-space has been treated by Wolf (1967) using perturbation technique. The transmission of elastic waves across a step like irregularity in the surface of an elastic half-space is of great importance in seismology in connection with the propagation of waves from ocean basins to continental regions and vice versa. Knopoff and Hudson (1964) studied the transmission of Love waves past a continental margin considering the crust to have an abrupt increase in thickness on the continental side. The transmission of SH-waves across a step like irregularity in the surface of an elastic half space was also considered by Bose (1975). Sato (1961) discussed the problem of propagation of Love wave in an elastic layer of variable thickness overlying a semi-infinite elastic medium. Approximate expressions for the transmission and reflection factors are obtained by the application of a method based on Wiener-Hopf technique.

In this paper, we consider the propagation of SH- wave in a medium consisting of two welded quarter spaces of different materials and having a step change in elevation at the vertical interface. The problem reduces to an integral equation by using transform and Green's function method and finally applying the method of steepest descent, transmitted and reflected fields at large distance from the step have been determined. It may be mentioned in this connection that the problem of transient shear wave in a half-space composed of two joined elastic quarter spaces of different

---

Published in " Journal Of Technical Physics " Vol 33, 3-4, pp 411-420, 1992.

materials are subjected to time varying shear tractions at the free surface, parallel to plane of juncture has been solved by Achenbach (1969). Datta and Mitra (1974) also considered SH-wave propagation in a composite elastic medium consisting of an elastic quarter space in welded contact with a uniform layer of different shear wave velocity.

2. Formulation Of The Problem

We consider two quarter spaces of different materials joined along the common boundary $X=0$ in such a way that there is a step change in elevation at the free surface. We consider the axes of coordinate as shown in Fig 1. Denoting the coordinate of a point in the X-Z plane by (X,Z), we take the incident plane SH-wave as $\exp[i(\omega t-K_2X)]$ where $K_2=\omega/c_2$, so that the propagation is from higher side to the lower side of the step.

The boundaries $Z=0$, $X<0$; $Z=H, X>0$ and $X=0$, $0\le Z\le H$ are assumed to be stress free. Omitting the time factor $\exp(i\omega t)$, let $V_1(X,z), V_2(X,Z)$ be the SH-wave displacement component in two media (I) and (II)respectively in Y-direction which is perpendicular to the plane of the paper.

The field equations are wave equations in the two media and boundary conditions are that (i) The outer boundary is stress free and (ii) the displacement and stress are continuous on the interface $X=0$, $Z>H$. μ, ρ, c are assumed to be the modulus of rigidity, density and shear wave velocity with appropriate suffix for each of the two media.

Introducing the dimensionless quantities

$$x = \frac{X}{H}, \quad z = \frac{Z}{H}$$

We get from the wave equations and boundary conditions

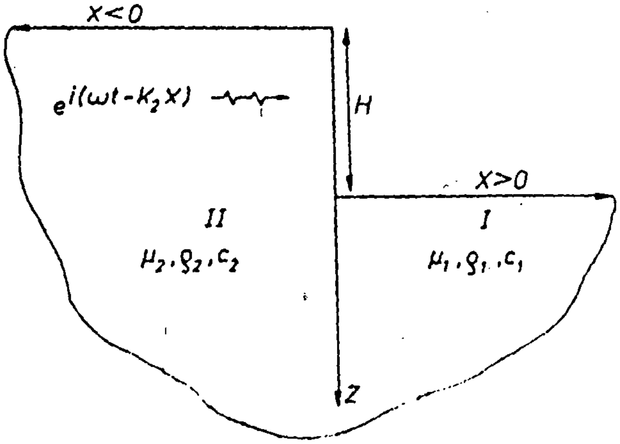


Fig.1: Geometry and coordinate system.

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_1^2 \right] v_1 = 0$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_2^2 \right] v_2 = 0 \qquad (2.1)$$

$$\mu_2 \frac{\partial v_2}{\partial x} \bigg|_{x=0} = 0 \text{ for } 0<z<1$$

$$\mu_2 \frac{\partial v_2}{\partial z} \bigg|_{z=0} = 0 \text{ for } x<0 \qquad (2.2)$$

$$\mu_1 \frac{\partial v_1}{\partial z} \bigg|_{z=1} = 0 \text{ for } x>0 \qquad (2.3)$$

$$\mu_1 \frac{\partial v_1}{\partial x} \bigg|_{x=0} = \mu_2 \frac{\partial v_2}{\partial x} \bigg|_{x=0} = 0 \text{ for } z>1$$

$$v_1(0,z) = v_2(0,z) \qquad \text{for } z>1 \qquad (2.4)$$

where H is the height of the step and $k_i^2 = \omega^2 H^2/c_i^2$, $V_i(X,Z)=v_i(x,z)$, $i=1,2$. We represent transverse displacement in the two domains $x<0$ and $x>0$ in the form

$$\begin{aligned} v_2 &= 2\cos k_2 x + v_2'(x,z) \qquad && x<0, z>0 \\ v_1 &= v_1(x,z) && x.0, z>1 \qquad (2.5) \end{aligned}$$

3. Reduction To Integral Equation and Its Solution

We introduce Green's functions $G_1(x,z:r,s)$ and $G_2(x,z:u,v)$ for the medium (I) and (II) respectively such that $G_2(x,z:u,v)$ is the solution of

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_2^2 \right] G_2(x,z:u,v) = -4\pi\delta(x-u)\delta(z-v) \qquad (3.1)$$

for medium (II) with vanishing normal derivative at $x<0, z=0$ and at

$x=0, z>0$. Similarly, $G_1(x,z:r,s)$ is the solution of

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+k_1^2\right]G_1(x,z:r,s) = -4\pi\delta(x-r)\delta(z-s) \qquad (3.2)$$

and satisfies the vanishing of the normal derivative at $x>0, z=1$ and at $x=0, z>1$.

From equation (2.1b) and (3.1) we obtain by applying Green's theorem to the medium (II) and using appropriate boundary condition

$$4\pi v_2'(u,v) = \int_1^\infty G_2(0,z:u,v)\left[\frac{\partial v_2'}{\partial x}\right]_{x=0} dz \qquad (3.3)$$

and similar application of Green's theorem to the medium (I) yields

$$4\pi v_1(r,s) = -\int_1^\infty G_1(0,z:r,s)\left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz \qquad (3.4)$$

Substitution of (3.3) and (3.4) into the equation (2.4b) yields with the aid of (2.4a) and equation (2.5) the integral equation

$$\int_1^\infty \left[G_1(0,z:0,v)+\frac{\mu_1}{\mu_2}G_2(0,z:0,v)\right]\left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz = -8\pi \qquad (3.5)$$

The expression of the Green's function for the medium (II) will now be derived using Fourier cosine transform with respect to z; which reduces the determination of $G_2(x,z:u,v)$ to that of a Green's function for an ordinary differential equation. Accordingly taking Fourier cosine transform defined by

$G_2^c = \int_0^\infty G_2(x,z:u,v)\cos(\alpha z)dz$, we obtain from the equation (3.1)

$$\frac{d^2G_2^c}{dx^2} - (\alpha^2 - k_2^2)\ G_2^c = -4\pi\cos(\alpha v)\delta(x-u)$$

from which we obtain in a straight forward manner

$$G_2(x,z:u,v) = 8\int_0^\infty \frac{e^{\beta_2 u}}{\beta_2} \cosh(\beta_2 x)\cos(\alpha v)\cos(\alpha z)\,d\alpha \qquad u\le x\le 0$$

$$= 8\int_0^\infty \frac{e^{\beta_2 x}}{\beta_2} \cosh(\beta_2 u)\cos(\alpha v)\cos(\alpha z)\,d\alpha \qquad -\infty<x\le u \tag{3.6}$$

where $\beta_2^2 = \alpha^2 - k_2^2$.

Again introducing Fourier cosine transform defined by

$G_1^c = \int_0^\infty G_1(x,z:r,s)\cos\alpha(z-1)\,d(z-1)$, we obtain from equation (3.2)

$$\frac{d^2 G_1^c}{dx^2} - (\alpha^2 - k_1^2)\, G_1^c = -4\pi \cos\alpha(s-1)\, \delta(x-r)$$

from which we obtain easily

$$G_1(x,z:r,s) = 8\int_0^\infty \frac{e^{-\beta_1 r}}{\beta_1} \cosh(\beta_1 x)\cos\alpha(s-1)\cos\alpha(z-1)\,d\alpha \qquad 0\le x\le r$$

$$= 8\int_0^\infty \frac{e^{-\beta_1 x}}{\beta_1} \cosh(\beta_1 r)\cos\alpha(s-1)\cos\alpha(z-1)\,d\alpha \qquad r\le x\le\infty \tag{3.7}$$

where $\beta_1^2 = \alpha^2 - k_1^2$.

On substituting the values of $G_1(0,z:0,v)$ and $G_2(0,z:0,v)$ from equations (3.7) and (3.6) in the equation (3.5) we obtain

$$\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz \int_0^\infty \left[\frac{\cos\alpha(v-1)\cos\alpha(z-1)}{\beta_1} + \frac{\mu_1}{\mu_2}\,\frac{\cos(\alpha v)\cos(\alpha z)}{\beta_2}\right] d\alpha = -\pi$$

$$\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz \int_0^\infty \frac{\cos\alpha(v-1)\cos\alpha(z-1)}{\beta_1}\,d\alpha = -\pi - \frac{\mu_1}{\mu_2}\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz \times$$

$$\times \int_0^\infty \frac{\cos(\alpha v)\cos(\alpha z)}{\beta_2}\,d\alpha$$

Taking Fourier cosine inverse transform with respect to α, we get

$$\int_1^{\infty}\left[\frac{\partial v_1}{\partial x}\right]_{x=0} \frac{\cos\alpha(v-1)}{\beta_1}\, dz = -2\int_1^{\infty}\cos\alpha(v-1)\, dv - \frac{2\mu_1}{\pi\mu_2}\int_1^{\infty}\left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz$$

$$\times \int_0^{\infty}\frac{\cos(\tau z)}{\beta_2(\tau)}\, d\tau \int_1^{\infty}\cos(\tau v)\ \cos\alpha(v-1)\ dv \tag{3.8}$$

where $\beta_2(\tau)$ is obtained from β_2 by replacing α by τ.

Next using the results that

$$\int_1^{\infty}\cos\alpha(v-1)\ dv = \pi\delta(\alpha) \tag{3.9}$$

$$\int_1^{\infty}\sin\alpha(v-1)\ dv = \frac{1}{\alpha} \tag{3.10}$$

it can be easily shown that

$$\int_1^{\infty}\cos(\tau v)\cos\alpha(v-1)\ dv = \frac{\pi}{2}\cos\tau\left[\delta(\tau+\alpha) + \delta(\tau-\alpha)\right] - \frac{\tau\sin\tau}{\tau^2-\alpha^2} \tag{3.11}$$

where $\delta(x)$ is the Dirac δ-function.

Using these results and after a little algebraic manipulation it can be easily shown that equation (3.8) reduces to the form

$$\int_1^{\infty}\left[\frac{\partial v_1}{\partial x}\right]_{x=0}\cos\alpha(z-1)dz = -\frac{2\pi\mu_2\beta_1\beta_2\delta(\alpha)}{\mu_1\beta_1 + \mu_2\beta_2} + \frac{\mu_1\beta_1}{\mu_1\beta_1 + \mu_2\beta_2}\int_1^{\infty}\left[\frac{\partial v_1}{\partial x}\right]_{x=0}$$

$$\times\ \sin(\alpha z)\sin\alpha\ dz + \frac{2\mu_1\beta_1\beta_2}{\pi(\mu_1\beta_1 + \mu_2\beta_2)}\int_1^{\infty}\left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz \int_0^{\infty}\frac{\tau\cos(\tau z)\sin\tau}{\beta_2(\tau)(\tau^2-\alpha^2)}\, d\tau \tag{3.12}$$

4. Evaluation Of Displacement

Substituting the value of $G_1(0,z:r,s)$ from equation (3.7) in equation (3.4) and then using the result (3.12), the displacement in the medium I is obtained in the form

$$v_1(x,s) = \frac{2\mu_2 k_2}{\mu_1 k_1+\mu_2 k_2}\, e^{-ik_1 x} - \frac{2\mu_1}{\pi}\int_0^\infty \frac{e^{-\beta_1 x}\cos\alpha(s-1)}{\mu_1\beta_1+\mu_2\beta_2}\, d\alpha \;\times$$

$$\times \int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} \sin(\alpha z)\sin\alpha \, dz - \frac{4\mu_1}{\pi^2}\int_0^\infty \frac{\beta_2 e^{-\beta_1 x}\cos\alpha(s-1)}{\mu_1\beta_1+\mu_2\beta_2}\, d\alpha \int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz$$

$$\times \int_0^\infty \frac{\tau\cos(\tau z)\sin\tau}{\beta_2(\tau)(\tau^2-\alpha^2)}\, d\tau \qquad (4.1)$$

We can compute $v_1(x,s)$ iteratively solving equation (4.1) and using asymptotic values of integrals arising in the right hand side of (4.1) for large values of x.

The first iterate is

$$v_1(x,s) = \frac{2\mu_2 k_2}{\mu_1 k_1+\mu_2 k_2}\, e^{-ik_1 x} \qquad (4.2)$$

which is obviously the displacement in medium (I) in the absence of step change in elevation.

Deriving $\left[\frac{\partial v_1}{\partial x}\right]_{x=0}$ from the first iterate given in (4.2) and using this in the right hand side of (4.1) with the aid of (3.10) it can be easily show that second term in the right hand side of (4.1) takes the form.

$$I_1 = \frac{2ik_1 k_2\mu_1\mu_2}{\pi(\mu_1 k_1+\mu_2 k_2)}\int_0^\infty \frac{e^{-\beta_1 x}\cos\alpha(s-1)}{\alpha(\mu_1\beta_1+\mu_2\beta_2)}\sin 2\alpha \, d\alpha \qquad (4.3)$$

Which, for large values of x, can be evaluated asymptotically by the method of steepest descent. therefore, for large x, we find

$$I_1 \sim \frac{4\mu_1 k_1 \mu_2 k_2}{(\mu_1 k_1 + \mu_2 k_2)^2} \left[\frac{k_1}{2\pi x} \right]^{1/2} \exp(\pi i/4 - ik_1 x) \tag{4.4}$$

Similarly, with the aid of (4.2) and the result given in (3.9), the third term in the right hand side of (4.1) reduces to the form

$$I_2 = \frac{8i\mu_1 k_1 \mu_2 k_2}{\pi^2 (\mu_1 k_1 + \mu_2 k_2)} \int_0^\infty \frac{\sin^2\tau}{\beta_2(\tau)} d\tau \int_0^\infty \frac{\beta_2 e^{-\beta_1 x} \cos\alpha(s-1)}{(\mu_1\beta_1 + \mu_2\beta_2)(\alpha^2 - \tau^2)} d\alpha \tag{4.5}$$

In order to evaluate asymptotically for large values of x, integrals of the type

$$I'_1 = \int_0^\infty \frac{f(\alpha)}{(\alpha^2 - \tau^2)} e^{-\beta_1 x} d\alpha$$

we have to take into account the residue at the singularity $\alpha = \tau$ in addition to the integral along the steepest descent. Thus we get

$$I'_1 \sim \pi i \frac{f(\tau)}{2\tau} e^{-\beta_1(\tau)x} - \left[\frac{\pi k_1}{2x} \right]^{1/2} \frac{f(0)}{\tau^2} \exp(\pi i/4 - ik_1 x) \tag{4.6}$$

Using this result in(4.5). for large values of x, we obtain

$$I_2 \sim - \frac{4\mu_1 k_1 \mu_2 k_2}{(\mu_1 k_1 + \mu_2 k_2)^2} \left[\frac{k_1}{2\pi x} \right]^{1/2} M \exp(\pi i/4 - ik_1 x) \tag{4.7}$$

where

$$M = \frac{2ik_2}{\pi}\int_0^\infty \frac{\sin^2\tau}{\tau^2\beta_2(\tau)}\,d\tau \tag{4.8}$$

$$= \frac{2}{\pi k_2}\int_0^1 \frac{\sin^2 k_2 t}{t^2\sqrt{1-t^2}}\,dt + \frac{2i}{\pi k_2}\int_1^\infty \frac{\sin^2 k_2 t}{t^2\sqrt{t^2-1}}\,dt$$

$$= J - iY$$

Following Bose (1975) it can be shown that

$$J = \int_0^{2k_2} J_0(z)\,dz - J_1(2k_2) \quad \text{and} \quad Y = \int_0^{2k_2} Y_0(z)\,dz - Y_1(2k_2) - \frac{1}{\pi k_2}$$

Thus from (4.1),(4.4) and (4.7), the second iterate, for large x, is

$$v_1(x,z) = \frac{2\mu_2 k_2}{\mu_1 k_1+\mu_2 k_2}\left[1 + \frac{2(1-M)\mu_1 k_1}{\mu_1 k_1+\mu_2 k_2}\left\{\frac{k_1}{2\pi x}\right\}^{1/2} e^{\pi i/4}\right] e^{-ik_1 x} \tag{4.9}$$

If we neglect terms of order 1/x, the higher order iterates yield the same expression.

Now in order to find out displacement component due to reflected wave in the medium (II), we rewrite the equation (3.12) with the aid of equation (2.4a) and (2.5a) as

$$\int_1^\infty \left[\frac{\partial v'_2}{\partial x}\right]_{x=0} \cos\alpha(z-1)\,dz = -\frac{2\pi\mu_1\beta_1\beta_2\delta(\alpha)}{\mu_1\beta_1+\mu_2\beta_2} + \frac{\mu_1\beta_1}{\mu_1\beta_1+\mu_2\beta_2}\int_1^\infty \left[\frac{\partial v'_2}{\partial x}\right]_{x=0}$$

$$\times \sin(\alpha z)\sin\alpha\,dz + \frac{2\mu_1\beta_1\beta_2}{\pi(\mu_1\beta_1+\mu_2\beta_2)}\int_1^\infty \left[\frac{\partial v'_2}{\partial x}\right]_{x=0} dz \int_0^\infty \frac{\tau\cos(\tau z)\sin\tau}{\beta_2(\tau)(\tau^2-\alpha^2)}\,d\tau$$

Taking inverse Fourier cosine transform with respect to z, we get

$$\left[\frac{\partial v'_2}{\partial x}\right]_{x=0} = -\frac{2i\mu_1 k_1 k_2}{\mu_1 k_1 + \mu_2 k_2} + \frac{2}{\pi}\int_0^{\infty}\frac{\mu_1\beta_1}{\mu_1\beta_1 + \mu_2\beta_2}\cos\alpha(z-1)\,d\alpha \times$$

$$\int_1^{\infty}\left[\frac{\partial v'_2}{\partial x}\right]_{x=0}\sin(\alpha u)\sin\alpha\,du + \frac{4}{\pi^2}\int_0^{\infty}\frac{\mu_1\beta_1\beta_2}{(\mu_1\beta_1 + \mu_2\beta_2)}\cos\alpha(z-1)\,d\alpha\int_1^{\infty}\left[\frac{\partial v'_2}{\partial x}\right]_{x=0}du$$

$$\times\int_0^{\infty}\frac{\tau\cos(\tau u)\sin\tau}{\beta_2(\tau)(\tau^2-\alpha^2)}\,d\tau \qquad (4.10)$$

Thus substitution of equation (4.10) in equation (3.3) with the aid of equation (3.6) yields

$$v'_2(x,v) = \frac{-2\mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2}e^{ik_2x} + \frac{4i\mu_1 k_1 k_2}{\pi(\mu_1 k_1 + \mu_2 k_2)}\int_0^{\infty}\frac{e^{\beta_2 x}\cos(\alpha v)\sin\alpha}{\alpha\,\beta_2}\,d\alpha$$

$$+\frac{1}{\pi}\int_0^{\infty}\frac{\mu_1\beta_1 e^{\beta_2 x}\cos(\alpha v)}{\beta_2(\mu_1\beta_1 + \mu_2\beta_2)}\sin 2\alpha\,d\alpha\int_1^{\infty}\left[\frac{\partial v'_2}{\partial x}\right]_{x=0}\sin(\alpha u)\,du - \frac{4}{\pi^2}\times$$

$$\int_0^{\infty}\frac{\alpha\,e^{\beta_2 x}\cos(\alpha v)\sin\alpha}{\beta_2}\,d\alpha\int_0^{\infty}\frac{\beta'_1\mu_1\sin\alpha'}{(\mu_1\beta'_1 + \mu_2\beta'_2)}\frac{d\alpha'}{(\alpha^2-\alpha'^2)}\int_1^{\infty}\left[\frac{\partial v'_2}{\partial x}\right]_{x=0}\sin(\alpha' u)\,du\ +$$

$$+\frac{4\mu_1}{\pi^2}\int_0^{\infty}\frac{\beta_1 e^{\beta_2 x}\cos(\alpha v)}{(\mu_1\beta_1 + \mu_2\beta_2)}\cos\alpha\,d\alpha\int_0^{\infty}\frac{\tau\sin\tau}{\beta_2(\tau)(\tau^2-\alpha^2)}\,d\tau\int_1^{\infty}\left[\frac{\partial v'_2}{\partial x}\right]_{x=0}\cos(\tau u)\,du -$$

$$-\frac{8\mu_1}{\pi^3}\int_0^{\infty}\frac{\alpha\,e^{\beta_2 x}\cos(\alpha v)\sin\alpha}{\beta_2}\,d\alpha\int_0^{\infty}\frac{\beta'_1\beta'_2}{(\mu_1\beta'_1 + \mu_2\beta'_2)}\frac{d\alpha'}{(\alpha^2-\alpha'^2)}\int_0^{\infty}\frac{\tau\sin\tau}{\beta_2(\tau)(\tau^2-\alpha'^2)}\,d\tau\ \times$$

$$\times\int_1^\infty\left[\frac{\partial v'_2}{\partial x}\right]_{x=0}\cos(\tau u)\,du \qquad (4.11)$$

where β'_1 and β'_2 are obtained from β_1 and β_2 by replacing α by α' .

Now to solve equation (4.11) iteratively, we take the first iteration as

$$v'_2(x,v)=\frac{-2\mu_1 k_1}{\mu_1 k_1+\mu_2 k_2}\,e^{ik_2 x}+\frac{4i\mu_1 k_1 k_2}{\pi(\mu_1 k_1+\mu_2 k_2)}\int_0^\infty\frac{e^{\beta_2 x}\cos(\alpha v)\sin\alpha}{\alpha\,\beta_2}\,d\alpha \qquad (4.12)$$

$$\text{i.e.,}\quad\left[\frac{\partial v'_2}{\partial x}\right]_{x=0}=-\frac{2ik_2\mu_1 k_1}{\mu_1 k_1+\mu_2 k_2} \qquad (4.13)$$

The asymptotic evaluation of (4.12) for large values of x by the method of steepest descent yield the first iterate as

$$v'_2(x,v)=\frac{-2\mu_1 k_1}{\mu_1 k_1+\mu_2 k_2}\left[1-\left\{\frac{2k_2}{-\pi x}\right\}^{1/2}e^{\pi i/4}\right]e^{ik_2 x} \qquad (4.14)$$

The second iterate is obtained by inserting the value of $\left[\frac{\partial v'_2}{\partial x}\right]_{x=0}$ given in the equation (4.13) on the right hand side of equation (4.11) and using the results

$$\int_1^\infty\sin(\alpha z)\,dz=\frac{\cos\alpha}{\alpha}\quad\text{and}\quad\int_1^\infty\cos(\alpha z)\,dz=\pi\delta(\alpha)-\frac{\sin\alpha}{\alpha}$$

and then evaluating the integrals on the right hand side of (4.11) by the method steepest descent for large value of x. Thus we obtain the second iterate as

$$v'_2(x,v)=\frac{-2\mu_1 k_1}{\mu_1 k_1+\mu_2 k_2}\left[1-2\left(1-\frac{(1-M)\mu_1 k_1}{\mu_1 k_1+\mu_2 k_2}\right)\left\{\frac{k_2}{-2\pi x}\right\}^{1/2}e^{\pi i/4}\right]e^{ik_2 x} \qquad (4.15)$$

where M is given in (4.8)

Thus from equation (2.5a) and (4.15) we get

$$v_2(x,v)=e^{-ik_2x}+\left[\frac{\mu_2k_2-\mu_1k_1}{\mu_1k_1+\mu_2k_2}+\frac{4\mu_1k_1}{\mu_1k_1+\mu_2k_2}\left(1-\frac{(1-M)\mu_1k_1}{\mu_1k_1+\mu_2k_2}\right)\sqrt{\frac{k_2}{-2\pi x}}\,e^{\pi i/4}\right]e^{ik_2x} \tag{4.16}$$

5. Numerical Results and Discussions

To investigate the nature of the motion, we have evaluated numerically the increment in amplitude due to the step for both transmitted wave and reflected waves. The results are shown in the form of graphs showing the variation of $\sqrt{x}\Delta V_{1T}$ with k_2 for different values of μ_2/μ_1 in Fig.2 for transmitted wave and the variation $\sqrt{-x}\ \Delta V_{2R}$ with k_2 in Fig.3 for reflected wave, where

$$\Delta V_{1T} = |\ v_1\ | - \frac{2\mu_2k_2}{\mu_1k_1+\mu_2k_2} \quad \text{and} \quad \Delta V_{2R} = |\ v_{2R}\ | - \frac{\mu_2k_2-\mu_1k_1}{\mu_1k_1+\mu_2k_2}$$

where v_{2R} is the reflected part of v_2 .

The value of $\sqrt{x}\ \Delta V_{1T}$ is found to increase gradually with the increase in the value of μ_2/μ_1 and for all values of μ_2/μ_1, it is found that the maximum value of $\sqrt{x}\ \Delta V_{1T}$ occurs at $k_2 = 0.75$. It is also observed from Fig.2 that $\sqrt{x}\ \Delta V_{1T}$ is positive for all values of k_2 and μ_2/μ_1, that is the amplitude of transmitted wave is always greater than that of transmitted wave in the absence of of the step. Moreover, with the increase in the values of k_2 the graphs show an undulating character with decreasing amplitude.

From Fig.3, we see that the value of $\sqrt{-x}\ \Delta V_{2R}$ gradually decreases with the increase in the value of μ_2/μ_1 and shows a gradual increase as the value of k_2 increases.

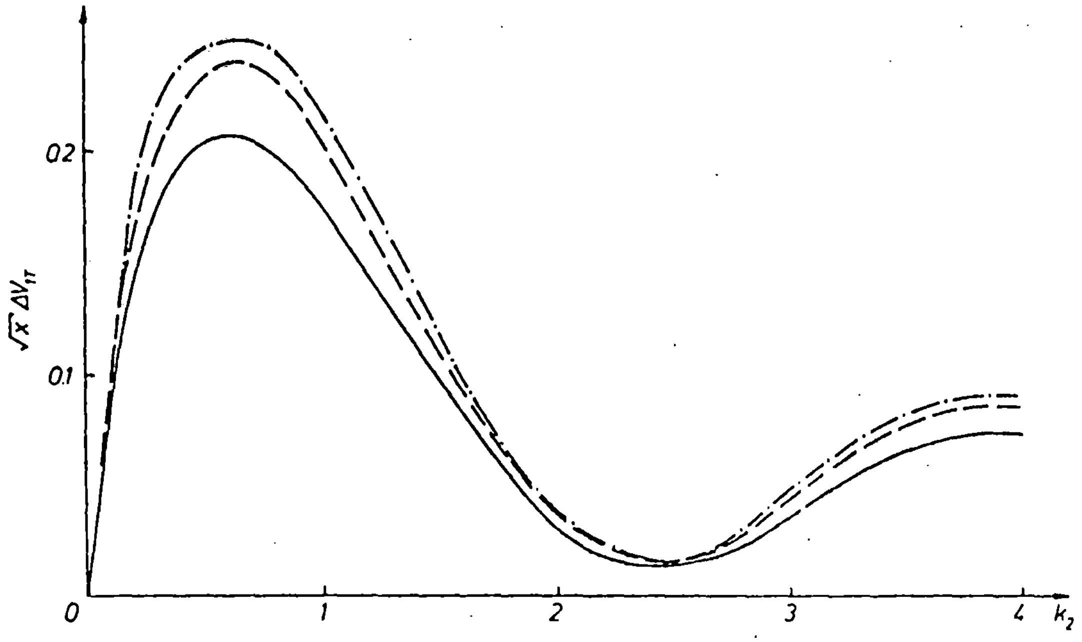


Fig. 2: Variation of $\sqrt{x}\,\Delta V_{1T}$ with k_2.

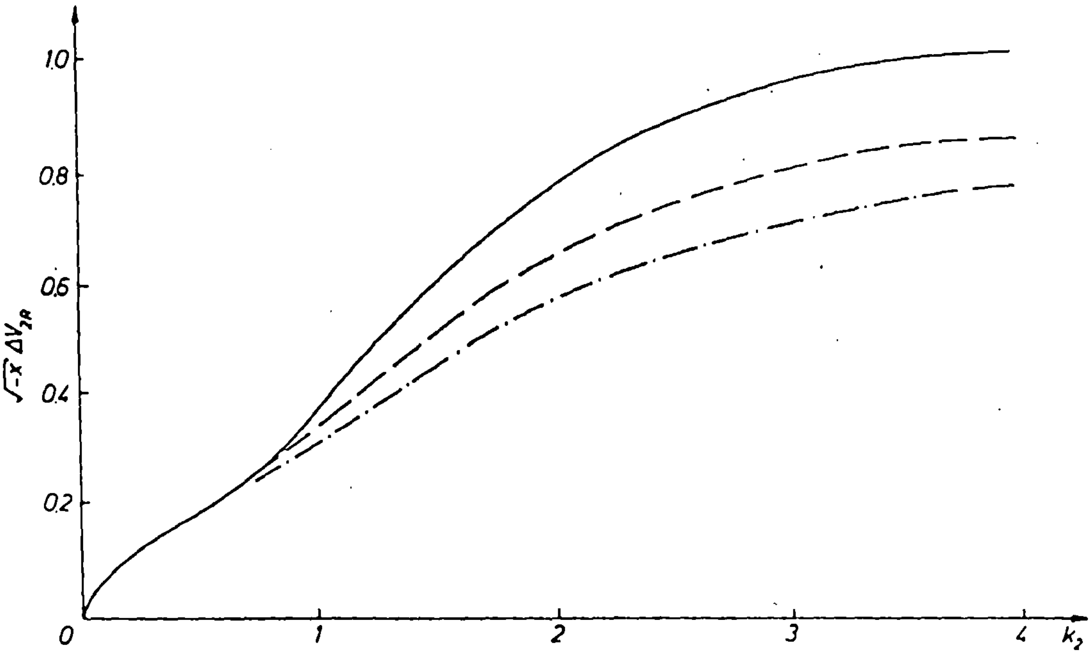


Fig. 3: Variation of $\sqrt{-x}\,\Delta V_{2R}$ with k_2.

BIBLIOGRAPHY

1. Abubakar, I. (1963):
Scattering of plane waves at rough surface, Proc. Camb. Phil. Soc., Vol 59, p 231.

2. Ang,D.D. and Williams, M.L. (1959):
The dynamic stress field due to an extensional dislocation, in Proc. of 4th annual conference on solid mechanics; The university of Texas, p 36.

3. Achenbach, J.D. (1969):
Transient shear waves in two joined elastic quarter spaces, J. Appl. Mech., Vol 36, p 491.

4. Achenbach, J.D. and Brock, L.M. (1971):
Rapid extension of a crack, J. Elasticity. Vol 1, p 51.

5. Achenbach, J.D. (1972):
Dynamic effects in brittle fracture, Mechanics Today, Vol 1, ed Nemat- Nasser, Pergamon Press, New York.

6. Achenbach, J.D. and Varatharajulu, V.K. (1974):
Skew crack propagation at the diffraction of a transient stress wave, Quar. Appl. Math. XXXII, p 123.

7. Achenbach, J.D. (1974):
Skew crack propagation under stress wave loading, Mech. Res. Comm., Vol 1, p 325.

8. Achenbach, J.D. (1975):
Wave propagation in elastic solids. North-Holland publishing company, Amsterdam- Oxford, American Elsevier publishing company, INC., New York.

9. Achenbach, J.D. (1975):
Bifurcation of a running crack in anti- plane strain, Int. J. Solids Structures, Vol 11, p 1301.

10. Achenbach, J.D. and Bazant, Z.P. (1975b):
Elastodynamic near tip stress and displacement fields for rapidly propagating cracks in orthotropic materials, J. Appl. Mech., Vol 42, p 183.

11. Achenbach, J.D. (1976):
Wave propagation, elastodynamic stress intensity factors and fracture, Theoretical and Applied Mechanics, ed W.T. Koiter; Proc. 14th IUTAM Congress, North-Holland Publishing Co. Amsterdam.

12. Achenbach, J.D., Kuo, M.K. and Dempsey, J.P. (1984):
Mode III and mixed mode I-II crack kinking under stress wave loading. Int. J. Solids Structures, Vol 20, p 395.

13. Atkinson, C. (1965):
Int. J. Engng. Sci., Vol 3 ,p 77.

14. Atkinson, C. (1968):
On axially symmetric expanding boundary value problems in classical elasticity, Int. J. Engng. Sci., Vol 6, p 27.

15. Atkinson, C. (1974):
Some aspects of dynamic crack propagation: A review and some generations, prospects of fracture mechanics, ed G.C.Sih, Noordhoff International publishing, p 337.

16. Atkinson, C. (1974a):
On the dynamic stress and displacement field associated with a crack propagating across the interface between two media, Int. J. Engng. Sci., Vol 12, p 491.

17. Angel, Y.C. and Achenbach, J.D. (1985):
Journal Of Elasticity, Vol 15, p89.

18. Afanasev, E.P. and Cherepanov, G.P. (1973):
Some dynamic problems of the theory of elasticity, Appl. Math. Mech. Vol 37, p 584.

19. Burridge, R. and Willis, J. R. (1963):
Proc. Camb. Phil. Soc., Vol 66, p443.

20. Bostrom, A. (1987):
Elastic wave scattering from an interface crack, J. Appl. Mech., Vol 54, p 503.

21. Baker, B.R. (1962):
Dynamic stress created by a moving crack. J. Appl. Mech. Vol 15, p 449.

22. Broberg, K.B. (1960):
The propagation of a brittle crack, Arkiv för Fysik, Vol 18, p 159.

23. Brock, L.M. and Achenbach, J.D. (1973):
Extension of an interface flaw under the influence of transient waves. Int. J. Solids Structures, Vol 9, P 53.

24. Brock, L.M. and Achenbach, J.D. (1973a):
Dynamic and quasi static extension of a small flaw, Development in Mechanics, Proc. 13th Midwestern Mechanics Conference, Vol 7, p 595.

25. Brock, L.M. and Achenbach, J.D. (1974):
Rapid tearing along the interface, ZAMP, Vol 25, p331.

26. Brock, L.M. (1974):
Dynamic stress intensity factors for an interface flaw extending at a non- uniform rate, J. Elasticity, Vol 4, p 51.

27. Brock, L.M. (1975):
Non-symmetric extension of plane crack due to tensile pre-stress and dilatational waves, Int. J. Engng. Sci., Vol 13, p 951.

28. Brock, L.M. (1976):
Non-symmetric extension of a small flaw into a plane crack

at a constant rate under polynomial form of loading. Int. J. Engng. Sci., Vol 14, p 181.

29. Brock, L.M. and Deng, Y.C. (1985): *Dynamic studies of running half plane and cruciform cracks,* Int. J. Engng. Sci., Vol 23, p 163.

30. Barenblatt, G.I., Salganik, R.L. and Cherepanov, G.P. (1962): *On the non- steady motion of cracks.* Appl. Math. Mech. Vol 26, p 469.

31. Bergkvist, H. (1973): *The motion of brittle crack,* J. Mech. Phys. Solids, Vol 21, p 229.

32. Brekhovskikh, L.M. (1960): *Waves in layered media,* Academic Press, New York.

33. Bose, S.K. (1975): *Transmission of SH- waves across a rectangular step,* Bull. Seis. Soc. Am. Vol 65, p 1779.

34. Burgers, P. and Dempsey, J.P. (1982): *Two analytical solutions for dynamic crack bifurcation in anti- plane strain,* J. Appl. Mech., Vol 49, p 366.

35. Burgers, P. (1982): *Dynamic propagation of kinked or bifurcated crack in anti-plane strain,* J. Appl. Mech., Vol 49, p 371.

36. Burgers, P. (1983): *Dynamic kinking of a crack in- plane strain,* Int. J. Solids Structures, Vol 19, p 735.

37. Cagniard, L. (1962): *Reflection and refraction of progressive waves,* translated and revised by E.A.Flinn and C.H.Dix, McGraw- Hill, New York.

38. Cherepanov, G.P and Afanasev, E.P. (1974):
Some dynamic problems of the theory of elasticity:A review. Int. J. Engng. Sci, Vol 12, p665.

39. Cherepanov, G.P. (1979):
Mechanics of brittle fracture. McGraw- Hill, New York.

40. Chen, E.P. and Sih, G.C. (1973):
Running crack in an incident wave field. Int. J. Solids Structures, Vol 9, p 897.

41. Chen, E.P. and Sih, G.C. (1977):
Transient response of cracks to impact loads: Elastodynamic crack problems, Mechanics of Fracture IV, ed G.C.Sih, Noordhoff International Publishing, Lyden, p 1.

42. Chen, E.P. (1978):
Sudden appearance of a crack in a stretched finite strip. J. Appl. Mech., Vol 45, p 277.

43. Cooke, J.C. (1963):
Some further triple integral equation's solutions, Proc. Camb. Phil. Soc., Vol 13, p 303.

44. Cooke, J.C. (1970):
The solutions of some integral equations and their connection with dual integral equations and trigonometric series. Glass. Math. J., Vol 11, p 9.

45. Craggs, J.W. (1960):
On the propagation of a crack in an elastic brittle solid. J. Mech. Phys. Solids., Vol 8, p 66

46. Craggs, J.W. (1963):
Fracture of solids, John Wiley and Sons. New York.

47. Chakroborty, M., Chattyopadhyay, A. and Dey, S. (1983):
The effects of initial stress and irregularity on the propagation of SH- waves, Ind. J. Pure. Appl. Math., Vol 14,

p 850.

48. De, J. and Patra, B. (1990):
Moving Griffith crack in an orthotropic strip, Int. J. Engng. Sci. Vol 28, p 809.

49. Dhawan, G.K. and Dhaliwal, R.S. (1978):
On three coplanar cracks in an infinite transversely isotropic medium, Int. J. Engng. Sci. , Vol 16, p 253.

50. Dutta, A. and Mitra, M. (1974):
SH- wave propagation in a composite elastic medium, Pure and Appl. Geoph. Vol 112, p 339.

51. Dempsey, J.P., Kuo, M.K. and Achenbach, J.D. (1982):
Mode III crack kinking under stress wave loading, Wave Motion, Vol 4, p 181.

52. Dempsey,J.P. and Smith, E.B. (1985):
The rapid tearing of a half- plane, J. Appl. Mech., Vol 52, p 51.

53. Dempsey,J.P., Kuo, M.K. and Bently,D.L. (1986):
Dynamic effects in mode III crack bifurcation, Int J. Solids Structures, Vol 22, p 333.

54. Das, A.N. (1992):
Three coplanar moving Griffith cracks in an infinite elastic strip, Arch. Mech., Vol 44, p 465.

55. Das, A.N. (1993):
Three coplanar moving Griffith cracks in an infinite elastic medium, Int. J. Frac. Vol 59, p 291.

56. Das, A.N. (1993a):
Extension of a plane crack due to plane SH- waves in a pre-stressed infinite elastic medium. Arch. Mech.,Vol 45,p 183.

57. Das, A.N. (1993b):
Extension of a plane crack in a pre- stressed infinite elastic medium. Engng. Frac. Mech. (In press)

58. Das, A. N. (1993c):
Non-symmetric extensionn of a crack, Acta Mech. (In Press)

59. Das, A. N. (1993d):
Non-symmetric extensionn of a crack in an infinite elastic medium under anti-plane stress, Int. J. Engng. Sci. (In Press)

60. Das,A.N. and Sarkar,J. (1994)
An elastic strip with three coplanar moving Griffith cracks, Engng. Frac. Mech. (In Press)

61. Das, A. N. (1993e):
Indentation of an elastic layer by four moving punches. Engng. Frac. Mech. (In Press)

62. Eshelby, J.D. (1960):
In " *Progress in Solid Mechanics*" (I.N.Sneddon and R.Hill eds.), Vol 2, North- Holland, Amsterdam.

63. Eshelby, J.D. (1969):
The elastic field of crack extending non- uniformly under general anti- plane loading, J. Mech. Phys. Solids, Vol 17, p 177.

64. Ewing, W.M., Jardetzky, W.S. and Press, F. (1957):
Elastic waves in layered media. McGraw- Hill , New York.

65. Erdogan, F. (1968):
Crack propagation theories, Fracture II, ed H.Liebowitz, Academic Press, p 497.

66. Eringen, A.C. and Suhubi, E.S. (1975):
Elastodynamics, Vol II, Linear theory,Academic Press, New York.

67. Erdelyi, A., ed. (1954):
"Tables of integral transform", Bateman Manuscript project, Calif, Inst. of Technol. 2 Vols, McGraw- Hill, New York.

68. Freund, L.B. (1971):
Int. J. Solids Structures, Vol 7, p 1199.

69. Freund, L.B. (1972):
Crack propagation in an elastic solid subjected to general loading II., Non- uniform rate of extension, J. Mech. Phys. Solids, Vol 20, p 141.

70. Freund, L.B. (1972a):
Energy flux into the tip of an extending crack in an elastic solid, J. Elasticity, Vol 2, p 341.

71. Freund, L.B. and Clifton, J.F. (1974):
On the uniqueness of elastodynamic solution for running crack, J. Elasticity, Vol 4, p 293.

72. Freund, L.B. (1976):
Dynamic crack propagation, *The mechanics of fracture*, Ed. F.Erdogan., ASME AMD. Vol 19. New York.

73. Freund, L.B. (1976a):
The analysis of elastodynamic crack tip stress fields, Mechanics Today, Vol III (ed S. Nemat-Nasser), Pergamon Press, New York.

74. Griffith, A.A. (1920):
The phenomena of rupture and flow in solids, Phil. Trans. Roy. Soc., Vol 221, p 163.

75. Glenie, E.B. and Willis, J.R. (1971):
An examination of the effects of some idealized models of fracture on accelerating cracks, J. Mech. Phys. Solids, Vol 9, p 11.

76. Gradhsteyn, I.S. and Ryzhik, I.M. (1965):
Tables of integrals, series and products, Academeic press, New York.

77. Graff,K.F.(1973):
Wave motions in elastic solids, Ohio state university press.

78. Georgiadis, H.G. and Theocaries, P.S. (1986):
Int. J. Engng. Sci. Vol 24, p 1135.

79. Georgiadis, H.G.(1987):
Transient crack propagation in asymmetric cruciform paths, Acta. Mech., Vol 68, p 265.

80. Georgiadis., H.G. (1991):
An integral equation approach to self- similar plane-elastodynamic crack problems, J. Elasticity, Vol 25, p 17.

81. Gakhov, F.D. (1966):
Boundary value problems, New York, Pergamon Press.

82. Green, A.E. and Zerna, W. (1968):
Theoretical elasticity.2nd ed.,Oxford univ.press(Claredon), London and New York.

83. Gel'fand. I.M. and Shilov, G.E.(1964):
Generalized function, Vol 1, Academic Press, New York.

84. Gol'dshtein, R.V. (1966):
On the steady motion of a crack along a straight line boundary between two joined materials, Inzh, Zn. M.T.T., Vol 15, p 93.

85. Gol'dshtein, R.V. (1967):
On surface waves in joined elastic materials and their relation to crack propagation along the junction. Appl. Math. Mech. Vol 31, p 496.

86. Hudson, J.A. (1980):
The excitation and propagation of elastic waves, Camb. Univ. Press, Cambridge.

87. Homma, S. (1952):
Love waves in a surface layer of varying thickness, Geophys. Mag., Vol 24, p 9.

88. Hayashi, K. and Nemmat Nasser, S. (1981):
Energy release rate and crack kinking, Int J. Solids Structures, Vol 17, p 107.

89. Itou, S. (1979):
Three dimensional problem of running crack, Int. J. Engng. Sci., Vol 17, p 59.

90. Itou, S. (1980):
Diffraction of anti- plane shear waves by two coplanar Griffith cracks in an infinite elastic medium. Int. J. solids and Structures, Vol 16, P 1147.

91. Irwin, G.R. (1957):
Relation of stresses near a crack to the crack extension force, in " Actes. IX Cong. Int. Mech. Appl., University of Bruxelles, p 245.

92. Jain, D.L. and Kanwal, R.P. (1972):
Diffraction of elastic waves by two coplanar Griffith cracks in an infinite elastic medium, Int. J. Solids and Structure, Vol 8, p 961.

93. Kanwal, R.P. (1971):
Linear integral equations, Theory and Technique, Academic Press, New York.

94. Kolosky, H. (1953):
Stress waves in solids. Oxford univ. press,(Claredon), London and New York.

95. Knopoff, L. and Hudson, J.A. (1964):
Transmission of Love waves past a continental margin, J. Geophys. Res., Vol 69, p 1649.

96. Kostorov, B.V. (1964):
Self-similar problems of propagation of shear cracks. Appl. Math. Mech., Vol 28, p1077.

97. Kostorov, B.V. (1964a):
The axisymmetric problem of propagation of a tension crack. Appl. Math. Mech., Vol 28, p 793.

98. Kassir, M.K. and Tse, S. (1983):
Moving Griffith crack in an orthotropic material. Int. J. Engng. Sci., Vol 21, p315.

99. Kim, K.S. (1979):
Dynamic propagation of a finite crack, Int. J. Solids Structures, Vol 15, p 685.

100. Kundu, T. (1986):
Transient response of an interface crack in a layered plate, J. Appl. Mech., Vol 53, p 579.

101. Kundu, T. (1987):
The transient response of two cracks at the interface of a layered Half- space, Int. J. Engng. Sci. Vol 25, p 1427.

102. Lo, K.K. (1978):
Analysis of branched cracks, J. Appl. Mech., Vol 45, p 797.

103. Love, A.E.H. (1944):
A treatise on the mathematical theory of elasticity, 4th ed., Dover publications, INC., New York.

104. Lowengrub, L. and Srivastava, K.N. (1968):
On two coplanar Griffith cracks in an infinite elastic medium. Int. J. Engng. Sci., Vol 6, p 359.

105. Miklowitz, J. (1966):
Elastic wave propagation in " Applied Mechanics Surveys" (H.M.Abranson, H.Liebowitz, T.M.Crowle and S. Juhasz, eds), Spartan Books, Washington, D.C.

106. Mal, A.K. (1970):
Interaction of elastic waves with a Griffith crack. Int. J. Engng. Sci., Vol 8, p 369.

107. Morse, P.M. and Feshbach, H. (1953):
Methods of Theoretical Physics, Part I and II, McGraw-Hill, New York.

108. Miles, J.W. (1960):
Homogeneous solutions in elastic wave propagation. Quar. Appl. Math., Vol 18, p 37.

109. McClintock, F.M. and Sukhatme, S.P.(1960):
Travelling cracks in elastic materials under longitudinal shear, J. Mech. Phys. Solids., Vol 8, p 187.

110. Matczynski, M. (1973):
Motion of a crack in anti- plane strain of an elastic strip, Arch. Mech., Vol 25, p 823.

111. Mann, K.K. and Deshwal, P.S. (1986):
Rayleigh wave scattering by a plane barrier in a shallow ocean, Ind. J. Pure Appl. Math., Vol 17, p 1056.

112. Neerhopf, F.L. (1979):
Diffraction of Love waves by a stress free crack of finite width in the plane interface of a layered composite, Appl. Sci. Res., Vol 35, p 237.

113. Nishida, Y., Shindo, Y. and Atsumi, A. (1984):
Diffraction of horizontal shear waves by a moving interface crack. Acta. Mech., Vol 54, p 23.

114. Norwood, F.R. (1973):
Similarity solutions in plane elastodynamics, Int. J. Solids Structures, Vol 9, p 789.

115. Norwood, F.R. (1976):
Correction and extension of Broberg's result on brittle crack propagation. Int. J. Engng. Sci., Vol 14, p 477.

116. Nilsson, F. (1972):
Dynamic stress intensity factor for finite strip problems. Int. J. Frac., Vol 8, p 403.

117. Nilsson, F. (1974):
A note on the stress singularity at a non- uniformly moving crack tip, J. Elasticity, Vol 4, p 73.

118. Nuismer, R.J. and Achenbach, J.D. (1972):
Dynamically induced Fracture, J. Mech. Phys. Solids., Vol 20, p 203.

119. Ong, M.K. and Srivastava, R.P. (1985):
A cruciform crack opened by integral pressure varying with time, Int. J. Engng. Sci., Vol 23, p 999.

120. Parihar, K.S. and Sowdamini, H. (1985):
Three collinear cracks in an infinite elastic medium. Int. J. Engng. Sci., Vol 23, p 151.

121. Piva, A. and Radi, E. (1991):
J. Appl. Mech., Vol 58, p 982.

122. Patron, V.Z. and Morozov, E.M. (1978):
Elastic plastic fracture mechanics, Moscow, MIR.

123. Patron, V.Z. and Perlin, P.I. (1984):
Mathematical methods of the theory of elasticity, Moscow, MIR.

124. Pilant, W.L. (1979):
Elastic waves in the earth. Elsevier Scientific Publishing Company, Amsterdam-Oxford-New York.

125. Smirnoff, V. and Sobolev, S. (1932):
Trudy Seismologichesky Instituta, No. 20.

126. Sanchez- Sesma, F.J. and Esquivel, A. (1979):
Ground motion on alluvial valleys under incident plane SH-waves. Bull. Seis. Soc. Am., Vol 69, p 1107.

127. Sanchez- Sesma, F.J., Herrera, I. and Aviles, J. (1982):
A boundary method for elastic wave diffraction: Application to scattering SH- waves by surface irregularities, Bull. Seis. Soc. Am., Vol 72, p 473.

128. Sanchez- Sesma, F.J. (1983):
Diffraction of elastic waves by three dimensional surface irregularities, Bull. Seis. Soc. Am., Vol 73, p 1621.

129. Sanchez- Sesma, F.J., Bravo, M.A. and Herrera, I. (1985):
Surface motion of topographical irregularities for a incident P, SV and Rayleigh waves, Bull. Seis. Soc. Am., Vol 75, p 263.

130. Sarkar, J., Ghosh, M.L. and Mandal, S.C. (1991):
Scattering of anti- plane shear wave by a propagating crack at the interface of two dissimilar elastic media. ,Proc. Ind. Acad. Sci. (Math. Sci.) Vol 101, p 183.

131. Shindo, Y., Nozaki, H. and Higaki, H. (1986):
Impact response of a finite crack in an orthotropic strip., Acta. Mech., Vol 62. p 87.

132. Shindo, Y. (1988):
Impact response of a crack in a semi- infinite body with a surface layer under longitudinal shear. Acta. Mech., Vol 73, p 147.

133. Sih, G.C. (1968):
Some elastodynamic problems of cracks, Int. J. Frac., Vol 1, p 51.

134. Sih, G.C. and Liebowitz, H. (1968):
Mathematical theory of brittle fracture, in "Fracture" (H.Liebowitz ed), Vol II, Academic Press, New York.

135. Sih, G.C. and Chen, E.P. (1970):
Moving crack in a strip under tearing action, J. Frank. Inst., Vol 290. p 25.

136. Sih, G.C. and Chen, E.P. (1972):
Crack propagating in a strip of elastic material under plane extension, Int. J. Engng. Sci., Vol 10, p 537.

137. Sih, G.C. and Chen, E.P. (1980):
Normal and shear impact of layered composite with a crack: Dynamic stress intensification, J. Appl. Mech., Vol 47. p 351.

138. Sih, G.C. and Chen, E.P. (1982):
Moving cracks in layered composites, Int. J. engng. Sci., Vol 20, p 1181.

139. Sneddon, I.N. (1951):
Fourier transform, McGraw- Hill , New York.

140. Sneddon, I.N. (1966):
Mixed boundary value problems in potential theory. North-Holland. Amsterdam.

141. Sneddon, I.N. (1972):
The use of integral transform, McGraw- Hill, New York.

142. Srivastava, K.N. and Lowengrub, M. (1968):
Finite Hilbert transform technique for triple integral equations with trigonometric kernels, Proc. Roy. Soc. Edin. p 309.

143. Sato, R. (1961):
Propagation of Love waves in a elastic layer of variable thickness, J. Phys. Earth, Vol 9, p 19.

144. Singh, B.M., Moodie, T.B. and Haddow, J.B. (1981):
Closed -form solutions for a finite length crack moving in a strip under anti-plane shear stress, Acta. Mech., Vol 38, p 99.

145. Singh, B.M., Dhaliwal, R.S. and Vrbik, J.(1983):
Diffraction of SH- waves by a moving crack, Acta. Mech., Vol 48, p 71.

146. Sinha, T.K. (1980):
Propagation of Love waves in elastic layer of variable thickness, J. Tech. Phys., Vol 21, p 275.

147. Tricomi, F.G. (1957):
Integral equations, Wiley (Interscience), New York.

148. Tranter, C.J. (1951):
Integral transforms in mathematical physics, Methuen, London.

149. Thau, S.A. and Lu, T.H. (1971):
Transient stress intensity factors for a crack in an elastic solid caused by a dilatational wave. Int. J. Solids Structures, Vol 7, p 731.

150. Radok, J.R.M. (1956):
On the solutions of problems of dynamic plane elasticity, Quar. Appl. Math, Vol 14, p 289.

151. Rice, J.R. (1968):
Mathematical analysis in the mechanics of fracture, in "Fracture", Vol II (H.Liebowitz ed), Academic Press, New York.

152. Tait, R.J. and Moodie, T.B. (1981):
Complex variable methods and closed form solutions to dynamic crack and punch problems in the classical theory of elasticity. Int. J. Engng. Sci., Vol 19, p 221.

153. Willis, J.R. (1967):
A comparison of the fracture criterion of Griffith and Barenblatt, J. Mech. Phys. solids, Vol 15, p 151.

154. Willis, J.R. (1973):
Self- similar problems in elastodynamic, Phil, Trans. Roy. Soc. London, Vol 274, p 435.

155. Williams, M.L. (1957):
On the stress distribution at the base of a stationary crack, J. Appl. Mech,., Vol 24, p 109.

156. Wolf, B. (1967):
Propagation of Love waves in surface layer of varying thickness, Pure Appl, Geoph., Vol 67, p 76.

157. Wolf, B. (1970):
Propagation of Love waves in surface layer with irregular boundaries,Pure Appl, Geoph., Vol 78, p 48.

158. Yoffe, E.H. (1951):
The moving Griffith crack, Phil. Mag., Vol42, p 739.

159. Zaman, F.D.,Asghar, S. and Ahmed, M. (1987):
Diffraction of SH- waves in a layered plate. J. Tech. Phys., Vol 28, p143.

Journal of Technical Physics, J. Tech. Phys., 32, 3–4, 479–489, 1991.
Polish Academy of Sciences, Institute of Fundamental Technological Research, Warszawa.

TWO COPLANAR GRIFFITH CRACKS MOVING IN A STRIP UNDER ANTIPLANE SHEAR STRESS

A. N. DAS and M. L. GHOSH (DARJEELING)

1. Introduction

In fracture mechanics, the problem of diffraction of elastic waves by cracks of finite dimensions in a strip of elastic material has been investigated by several authors. SIH and CHEN [1] investigated the problem of propagation of a crack of finite length in a strip under plane extension. The resulting mixed bounadry value problem was reduced to the solution of a Fredholm integral equation of second kind, which was solved numerically. Closed form solution for a finite length crack moving in a strip under antiplane shear stress was also obtained by SINGH *et al.* [2]. As regards the dynamic crack problem, research has been restricted mainly to the case of a single crack because of the severe mathematical complexity encountered in finding solutions of two or more cracks. However, using finite Hilbert transform techniques developed by SRIVASTAVA and LOWENGRUB [3], LOWENGRUB and SRIVASTAVA [4] solved the statical problem of distribution of stress in an infinitely long elastic strip containing two coplanar Griffith cracks. The scattering of time-harmonic normally incident plane waves by two parallel and coplanar Griffith cracks in an infinite elastic medium has been studied by JAIN and KANWAL [5] and more recently by ITOU [6]. The problem of diffraction of elastic waves by two coplanar cracks moving steadily along the interface of two bonded dissimilar elastic media has recently been studied by DAS and GHOSH [7] using Hilbert transform technique.

In this paper we have considered the problem of propagation of two complanar YOFFE [8] cracks moving steadily in an infinitely long finite width strip. Employing Fourier transform and finite Hilbert transform technique, closed-form solutions are obtained for two cases of practical interest. Firstly, the case when the rigidly clamped edges are pulled apart in opposite directions is considered. Secondly, we have treated the case when the lateral boundaries are subjected to shearing stresses. Exact expressions for the crack opening displacement and the stress intensity factors have been derived in both the cases. Finally, numerical results for stress intensity factors are presented graphically to show their variation with crack speed for different values of the lengths of the cracks.

2. Formulation of the Problem

We consider two cracks of finite lengths placed on the X-axis from $-b$ to $-a$ and from a to b with reference to the rectangular coordinate system (x, y, z) which, referred to a fixed coordinate system (X, Y, Z), is moving with constant velocity v along X-direction within the strip of elastic material occupying the region $-h' \leq Y \leq h'$, as shown in Fig. 1.

In dynamic problem of antiplane shear, the non-vanishing component of displacement W directed in the Z-direction satisfies the equation of motion

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} = \frac{1}{c_2^2}\frac{\partial^2 W}{\partial t^2}, \tag{2.1}$$

where $c_2 = (\mu/\rho)^{1/2}$ is the shear wave velocity and ρ is the density of the material. The non-vanishing components of stress are

$$\begin{aligned} \sigma_{xz} &= \mu\frac{\partial W}{\partial X}, \\ \sigma_{yz} &= \mu\frac{\partial W}{\partial Y}. \end{aligned} \tag{2.2}$$

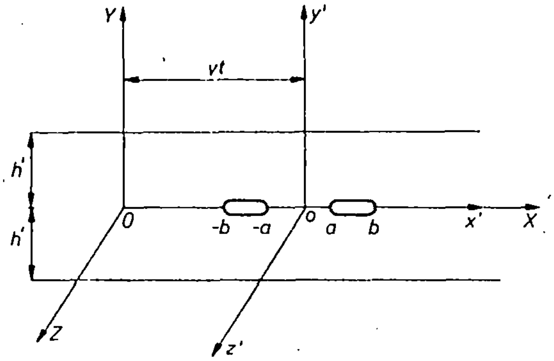


FIG. 1. Moving cracks in a strip under antiplane shear.

Using Galilean transformation $x' = X - vt$, $y' = Y$, $z' = Z$, $t' = t$, where (x', y', z') is the moving coordinate system shown in Fig. 1 and, next, introducing the dimensionless coordinates x, y, z such that $x' = xb$, $y' = yb$, $z' = zb$, $h' = hb$, Eq. (2.1) reduces to

$$s^2\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = 0, \tag{2.3}$$

with

$$s^2 = 1 - v^2/c_2^2. \tag{2.4}$$

3. Boundary Conditions

We consider two basic problems of practical interest with different boundary conditions.

PROBLEM 1. The edges of the strip $y = \pm h$ are assumed to be rigidly clamped and displaced laterally in opposite directions by an equal distance w_0, where w_0 is a constant. As a result, antiplane shear motion takes place in z-direction, whereas cracks move in the x-direction and the boundary conditions are

$$W(x, \pm h) = \pm w_0, \quad -\infty < x < \infty, \tag{3.1}$$

$$\sigma_{yz}(x, 0) = 0, \quad d < |x| < 1, \tag{3.2}$$

$$W(x, 0) = 0, \quad 0 \le |x| < d, \quad |x| > 1, \tag{3.3}$$

where $d = a/b$.

In order to apply the integral transform technique it is necessary to solve a different but equilvalent problem which can be obtained from the problem of a clamped strip (without any crack) subject to a uniform strain. The equivalent stress condition on the crack are

$$\sigma_{yz}(x,0) = -\frac{\mu w_0}{h}, \quad d < |x| < 1 \tag{3.4}$$

and the displacement must satisfy

$$W(x,0) = 0, \quad 0 \le |x| < d, \quad |x| > 1, \tag{3.5}$$

$$W(x,\pm h) = 0, \quad -\infty < x < \infty. \tag{3.6}$$

PROBLEM 2. In this case uniform shearing stress p_0 is applied to the upper and lower boundaries $y = \pm h$ of the strip. The equivalent problem in this case involves the application of the shear stress $-p_O$ to the crack faces at $y = 0$. Accordingly, the boundary conditions are

$$\sigma_{yz}(x,\pm h) = 0, \quad -\infty < x < \infty, \tag{3.7}$$

$$\sigma_{yz}(x,0) = -p_0, \quad d < |x| < 1, \tag{3.8}$$

$$W(x,0) = 0, \quad 0 \le |x| < d, \quad |x| > 1. \tag{3.9}$$

4.Solutions of the Problems

Due to the symmetry about (x,z)-plane we need to consider the region $0 < y < h$ only. Employing

$$F_c[A(\xi) : \xi \to x] = \sqrt{\frac{2}{\pi}} \int_0^\infty A(\xi)\cos(\xi x)\,d\xi, \tag{4.1}$$

and

$$F_s[A(\xi) : \xi \to x] = \sqrt{\frac{2}{\pi}} \int_0^\infty A(\xi)\sin(\xi x)\,d\xi, \tag{4.2}$$

we obtain the solution of Eq. (2.3) as

$$W(x,y) = F_c[A_1(\xi)\exp(-\xi ys) + A_2(\xi)\exp(\xi ys) : \xi \to x], \tag{4.3}$$

with

$$\sigma_{yz}(x,y) = \mu s F_c[\xi\{-A_1(\xi)\exp(-\xi ys) + A_2(\xi)\exp(\xi ys)\} : \xi \to x]. \tag{4.4}$$

PROBLEM 1. Using the expression for $W(x,y)$ given in Eq. (4.3) in Eq. (3.6) we get

$$A_1(\xi) = \frac{A(\xi)}{1 - \exp(-2\xi hs)},$$

$$A_2(\xi) = \frac{-A(\xi)\exp(-2\xi hs)}{1 - \exp(-2\xi hs)},$$

where $A(\xi)$ is to be determined.

From Eqs. (3.4) and (3.5) we find that $A(\xi)$ satisfies the set of triple integral equations

$$F_c[\xi A(\xi)\,\mathrm{cth}\,(\xi hs) : \xi \to x] = \frac{w_0}{hs}, \quad d < x < 1, \tag{4.5}$$

$$F_c[A(\xi) : \xi \to x] = 0, \quad 0 \le x < d, \quad x > 1. \tag{4.6}$$

Let us take

$$A(\xi) = \frac{1}{\xi}\sqrt{\frac{\pi}{2}} \int_d^1 g_1(\tau)\,\mathrm{Sech}^2(c\tau)\sin(\xi\tau)\,d\tau, \tag{4.7}$$

It is clear that the above choice of $A(\xi)$ satisfies Eq. (4.6) if and only if

$$\int_d^1 g_1(\tau)\,\mathrm{Sech}^2(c\tau)\,d\tau = 0. \tag{4.8}$$

Equation (4.5) can be written as

$$\frac{d}{dx} F_s[A(\xi)\,\mathrm{cth}\,(\xi hs) : \xi \to x] = \frac{w_0}{hs}, \quad d < x < 1. \tag{4.9}$$

Inserting Eq. (4.7) in Eq. (4.9) and using the result [9]

$$\int_0^\infty \frac{\mathrm{cth}\,(\xi hs)\sin(\xi\tau)\sin(\xi x)}{\xi}\,d\xi = \frac{1}{2}\log\left|\frac{\mathrm{th}\,(cx) + \mathrm{th}\,(c\tau)}{\mathrm{th}\,(cx) - \mathrm{th}\,(c\tau)}\right|, \tag{4.10}$$

where $c = \pi/2hs$, we obtain

$$\int_d^1 \frac{c g_1(\tau)\,\mathrm{Sech}^2(c\tau)\,\mathrm{th}\,(c\tau)}{\mathrm{th}^2(c\tau) - \mathrm{th}^2(cx)}\,d\tau = \frac{w_0}{hs\,\mathrm{Sch}^2(cx)}, \quad d < x < 1. \tag{4.11}$$

Substituting $\mathrm{th}\,(c\tau) = T_1$, Eq. (4.11) is found to reduce to the form

$$\int_{D_1}^{I_1} \frac{T_1 A(T_1^2)\,dT_1}{T_1^2 - X_1^2} = \frac{w_0}{hs(1 - X_1^2)}, \quad D_1 < X_1 < I_1, \tag{4.12}$$

where $D_1 = \mathrm{th}\,(cd)$, $I_1 = \mathrm{th}\,(c)$, $X_1 = \mathrm{th}\,(cx)$ and $A(T_1^2) = g_1(\tau)$.

Using finite Hilbert transform [3], the solution of Eq. (4.12) is

$$g_1(\tau) = A(T_1^2) = \frac{2w_0\,\mathrm{ch}\,(cd)}{\pi hs(1 - T_1^2)\,\mathrm{ch}\,(c)}\sqrt{\frac{T_1^2 - D_1^2}{I_1^2 - T_1^2}} + \\ + \frac{K_1}{[(T_1^2 - D_1^2)(I_1^2 - T_1^2)]^{1/2}}, \quad d < \tau < 1, \tag{4.13}$$

where the constant K_1 determined from Eq. (4.8) is found to be equal to

$$K_1 = \frac{2w_0\,\mathrm{ch}\,(cd)}{\pi hs\,\mathrm{ch}\,(c)} D_1^2 \cdot \left(1 - \frac{\Pi\left(\frac{\pi}{2}, \frac{I_1^2 - D_1^2}{I_1^2(1 - D_1^2)}, q\right)}{F\left(\frac{\pi}{2}, q\right)}\right), \tag{4.14}$$

where $q = (I_1^2 - D_1^2)^{1/2}/I_1$ and $F(\phi, k)$, $\Pi(\phi, n, k)$ are elliptic integrals of the first and third kind, respectively.

The expressions of displacement and shear stress on the plane of the crack are expressed as

$$W(x,0) = \frac{\pi}{2} \int_x^1 g_1(\tau)\,\mathrm{Sech}^2(c\tau)\,d\tau, \quad d < x < 1 \tag{4.15}$$

and

$$\sigma_{yz}(x,0) = \mu s c \int_d^1 \frac{g_1(\tau)\,\mathrm{Sech}^2(c\tau)\,\mathrm{th}(c\tau)\,\mathrm{Sech}(cx)}{\mathrm{th}^2(cx) - \mathrm{th}^2(c\tau)}\,d\tau, \quad 0 \le x < d, \quad x > 1 \tag{4.16}$$

Now, inserting Eq. (4.13) in Eqs. (4.15) and (4.16), we obtain

$$W(x,0) = -\frac{w_0\,\mathrm{ch}(cd)}{hsc\,\mathrm{sh}(c)}\left[F(\lambda,q)\left\{1 - D_1^2\left(1 - \frac{\Pi\left(\frac{\pi}{2}, \frac{I_1^2 - D_1^2}{I_1^2(1-D_1^2)}, q\right)}{F\left(\frac{\pi}{2}, q\right)}\right)\right\} + \frac{\mathrm{ch}^2(c)}{\mathrm{ch}^2(cd)}\Pi\left(\lambda, \frac{I_1^2 - D_1^2}{1 - I_1^2}, q\right)\right], \quad d < x < 1, \tag{4.17}$$

where

$$\mathrm{Sin}\,\lambda = \sqrt{\frac{I_1^2 - \mathrm{th}^2(cx)}{I_1^2 - D_1^2}},$$

$$\sigma_{yz}(x,0) = \frac{\mu w_0\,\mathrm{ch}(cd)}{h\,\mathrm{ch}(c)}\left[\sqrt{\frac{\mathrm{th}^2(cx) - D_1^2}{\mathrm{th}^2(cx) - I_1^2}} - \frac{\mathrm{ch}(c)}{\mathrm{ch}(cd)} + \left(1 - \frac{\Pi\left(\frac{\pi}{2}, \frac{I_1^2 - D_1^2}{I_1^2(1-D_1^2)}, q\right)}{F\left(\frac{\pi}{2}, q\right)}\right)\frac{D_1^2\,\mathrm{Sech}^2(cx)}{\sqrt{[\mathrm{th}^2(cx) - D_1^2][\mathrm{th}^2(cx) - I_1^2]}}\right], \quad x > 1, \tag{4.18}$$

$$\sigma_{yz}(x,0) = \frac{\mu w_0\,\mathrm{ch}(cd)}{h\,\mathrm{ch}(c)}\left[\sqrt{\frac{D_1^2 - \mathrm{th}^2(cx)}{I_1^2 - \mathrm{th}^2(cx)}} - \frac{\mathrm{ch}(c)}{\mathrm{ch}(cd)} - \left[1 - \frac{\Pi\left(\frac{\pi}{2}, \frac{I_1^2 - D_1^2}{I_1^2(1-D_1^2)}, q\right)}{F\left(\frac{\pi}{2}, q\right)}\right]\frac{D_1^2\,\mathrm{Sech}^2(cx)}{\sqrt{[D_1^2 - \mathrm{th}^2(cx)][I_1^2 - \mathrm{th}^2(cx)]}}\right], \quad 0 < x < d, \tag{4.19}$$

where we have used the result

$$(4.20)\qquad \int\limits_d^1 \frac{1}{\sqrt{(t^2-d^2)(1-t^2)}}\frac{t\,dt}{t^2-x^2} = \begin{cases} \dfrac{\pi}{2\sqrt{(d^2-x^2)(1-x^2)}}, & 0<x<d, \\ 0, & d<x<1, \\ \dfrac{-\pi}{2\sqrt{(x^2-d^2)(x^2-1)}}, & x>1. \end{cases}$$

The stress intensity factor at $x = 1$ is given by

$$(4.21)\quad S_{11} = \lim_{x\to 1}\sqrt{2(x-1)}\sigma_{yz}(x,0) = \frac{\mu w_0}{h\,\mathrm{Sech}\,(cd)}\left[\sqrt{\frac{I_1^2-D_1^2}{cI_1}} + \frac{D_1^2(1-I_1^2)}{\sqrt{cI_1(I_1^2-D_1^2)}}\cdot\right.$$
$$\left.\cdot\left[1-\frac{\Pi\left(\dfrac{\pi}{2},\dfrac{I_1^2-D_1^2}{I_1^2(1-D_1^2)},q\right)}{F\left(\dfrac{\pi}{2},q\right)}\right]\right],$$

and the stress intensity factor at $x = d$ is given by

$$(4.22)\quad S_{1d} = \lim_{x\to d}\sqrt{2(d-x)}\sigma_{yz}(x,0) = -\frac{\mu w_0\sqrt{D_1^3(1-I_1^2)}}{h\sqrt{c(I_1^2-D_1^2)}}\cdot$$
$$\cdot\left[1-\frac{\Pi\left(\dfrac{\pi}{2},\dfrac{I_1^2-D_1^2}{I_1^2(1-D_1^2)},q\right)}{F\left(\dfrac{\pi}{2},q\right)}\right].$$

Letting $d = a/b = 0$ in the expressions for displacement, stress and stress intensity factors, it can be easily shown that the results coincide with the corresponding expressions given by SINGH *et al.* [2].

PROBLEM 2. In this case again we take the general solution of Eq. (2.3) as

$$(4.23)\qquad W(x,y) = F_c[C_1(\xi)\exp(-\xi ys) + C_2(\xi)\exp(\xi ys) : \xi \to x],$$

and inserting it in Eq. (3.7) we find that

$$C_1(\xi) = \frac{D(\xi)}{1+\exp(-2\xi hs)},$$
$$C_2(\xi) = \frac{D(\xi)\exp(-2\xi hs)}{1+\exp(-2\xi hs)}.$$

From Eqs. (3.8) and (3.9) it is determined that $D(\xi)$ satisfies the following set of triple integral equation

$$(4.24)\qquad F_c[\xi D(\xi)\,\mathrm{th}\,(\xi hs) : \xi \to x] = \frac{p_0}{\mu_s},\quad d<x<1,$$

$$(4.25)\qquad F_c[D(\xi) : \xi \to x] = 0,\quad 0 \le x < d,\quad x > 1.$$

Proceeding as in problem 1, we consider a trial solution

$$D(\xi) = \frac{1}{\xi}\sqrt{\frac{\pi}{2}}\int_d^1 g_2(\tau)\,\mathrm{ch}\,(c\tau)\sin(\xi\tau)\,d\tau. \tag{4.26}$$

With this choice of $D(\xi)$, Eq. (4.25) will be satisfied provided the unknown function $g_2(\tau)$ in Eq. (4.26) satisfies

$$\int_d^1 g_2(\tau)\,\mathrm{ch}\,(c\tau)\,d\tau = 0. \tag{4.27}$$

Now Eq. (4.24) can be written as

$$\frac{d}{dx}F_s[D(\xi)\,\mathrm{th}\,(\xi hs):\xi\to x] = \frac{p_0}{\mu_s}, \quad d<x<1. \tag{4.28}$$

Insertion of Eq. (4.26) in (4.28) and application of the result [9]

$$\int_0^\infty \frac{\mathrm{th}\,(\xi hs)\sin(\xi\tau)\sin(\xi x)}{\xi}\,d\xi = \frac{1}{2}\log\left|\frac{\mathrm{sh}(cx)+\mathrm{sh}(c\tau)}{\mathrm{sh}(cx)-\mathrm{sh}(c\tau)}\right|, \tag{4.29}$$

where $c = \pi/2hs$, gives

$$\int_d^1 \frac{cg_2(\tau)\,\mathrm{Sh}\,(2c\tau)}{\mathrm{sh}^2(c\tau)-\mathrm{sh}^2(cx)}\,d\tau = \frac{p_0}{\mu s\,\mathrm{ch}\,(cx)}, \quad d<x<1. \tag{4.30}$$

Substituting $T_2 = \mathrm{sh}\,(c\tau)$, $I_2 = \mathrm{sh}\,(c)$, $D_2 = \mathrm{sh}\,(cd)$ and $X_2 = \mathrm{Sh}\,(cx)$ and proceeding as in problem 1, we obtain the solution of Eq. (4.30) as

$$g_2(\tau) = -\frac{4p_0}{\pi^2\mu s}\sqrt{\frac{T_2^2-D_2^2}{I_2^2-T_2^2}}\times\frac{1}{\sqrt{1+I_2^2}}\left[\Pi\left(\frac{\pi}{2},\frac{I_2^2-D_2^2}{I_2^2-T_2^2},q''\right)-F\left(\frac{\pi}{2},q''\right)\right]+ \\ +\frac{K_2}{\sqrt{(T_2^2-D_2^2)(I_2^2-T_2^2)}}, \tag{4.31}$$

where $q' = (I_2^2-D_2^2)^{1/2}/I_2$, $q'' = q'\cdot\mathrm{th}(c)$ and the constant K_2, as determined from Eq. (4.27), is

$$K_2 = \frac{4p_0\,\mathrm{th}\,(c)}{\pi^2\mu sF\left(\frac{\pi}{2},q'\right)}\int_{D_2}^{I_2}\sqrt{\frac{T_2^2-D_2^2}{I_2^2-T_2^2}}\left(\Pi\left(\frac{\pi}{2},\frac{I_2^2-D_2^2}{I_2^2-T_2^2},q''\right)-F\left(\frac{\pi}{2},q''\right)\right)dT_2. \tag{4.32}$$

The relevant displacement and stress components in the plane of the cracks may be written as

$$W(x,0) = \frac{\pi}{2}\int_x^1 g_2(\tau)\,\mathrm{ch}(c\tau)\,d\tau, \quad d<x<1, \tag{4.33}$$

and

$$\text{(4.34)} \qquad \sigma_{yz}(x,0) = \frac{\mu s c}{2} \int\limits_{d}^{1} \frac{g_2(\tau)\,\mathrm{sh}\,(2c\tau)\,\mathrm{ch}\,(cx)}{\mathrm{sh}^2\,(cx) - \mathrm{sh}^2\,(c\tau)}\, d\tau, \quad 0 \le x < d, \quad x > 1.$$

Now using Eq. (4.31) in Eqs. (4.33) and (4.34) we obtain

$$\text{(4.35)} \quad W(x,0) = -\frac{2p_0}{\pi \mu s\,\mathrm{ch}\,(c)} \Bigg[\int\limits_{x}^{1} \sqrt{\frac{\mathrm{sh}^2\,(c\tau) - \mathrm{sh}^2\,(cd)}{\mathrm{sh}^2\,(c) - \mathrm{sh}^2\,(c\tau)}} \times \\ \times \left\{ \Pi\left(\frac{\pi}{2}, \frac{I_2^2 - D_2^2}{I_2^2 - T_2^2}, q''\right) - F\left(\frac{\pi}{2}, q''\right) \right\} \mathrm{ch}\,(c\tau)\, d\tau \Bigg] + \frac{K_2 F(\lambda', q')}{c I_2},$$

where

$$\sin \lambda' = \sqrt{\frac{I_2^2 - X_2^2}{I_2^2 - D_2^2}},$$

$$\text{(4.36)} \quad \sigma_{yz}(x,0) = -\frac{2p_0\,\mathrm{ch}(cx)}{\pi} \sqrt{\frac{\mathrm{sh}^2\,(cx) - D_2^2}{\mathrm{sh}^2\,(cx) - I_2^2}} \int\limits_{D_2}^{I_2} \sqrt{\frac{I_2^2 - T_2^2}{T_2^2 - D_2^2}} \times \frac{1}{T_2^2 - \mathrm{sh}^2\,(cx)} \times \\ \times \frac{T_2 dT_2}{\sqrt{1 + T_2^2}} + \frac{\pi \mu s\,\mathrm{ch}\,(cx) K_2}{2\sqrt{(\mathrm{sh}^2\,(cx) - I_2^2)(\mathrm{sh}^2\,(cx) - D_2^2)}}, \quad \text{for} \quad x > 1,$$

$$\text{(4.37)} \quad \sigma_{yz}(x,0) = -\frac{2p_0\,\mathrm{ch}\,(cx)}{\pi} \sqrt{\frac{D_2^2 - \mathrm{sh}^2\,(cx)}{I_2^2 - \mathrm{sh}^2\,(cx)}} \int\limits_{D_2}^{I_2} \sqrt{\frac{I_2^2 - T_2^2}{T_2^2 - D_2^2}} \times \frac{1}{T_2^2 - \mathrm{sh}^2\,(cx)} \times \\ \times \frac{T_2 dT_2}{\sqrt{1 + T_2^2}} - \frac{\pi \mu s\,\mathrm{ch}\,(cx) K_2}{2\sqrt{[I_2^2 - \mathrm{sh}^2\,(cx)][D_2^2 - \mathrm{sh}^2\,(cx)]}}, \quad \text{for} \quad 0 < x < d.$$

The stress intensity factor as $x = 1$ is given by

$$\text{(4.38)} \quad S_{21} = \lim_{x \to 1} \sqrt{2(x-1)}\,\sigma_{yz}(x,0) = \frac{2p_0}{\pi} \sqrt{\frac{I_2^2 - D_2^2}{c I_2\,\mathrm{ch}\,(c)}} \times F\left(\frac{\pi}{2}, q''\right) + \\ + \frac{\pi \mu s K_2}{2\sqrt{c\,\mathrm{th}\,(c)[I_2^2 - D_2^2]}}$$

and the stress intensity factor at $x = d$ is given by

$$\text{(4.39)} \qquad S_{2d} = \lim_{x \to d} \sqrt{2(d-x)}\,\sigma_{yz}(x,0) = \frac{-\pi \mu s K_2}{2\sqrt{c\,\mathrm{th}\,(cd)[I_2^2 - D_2^2]}}.$$

Again letting $d = 0$ in the expressions for displacement,stress and stress intensity factors, we obtain the corresponding results given by SINGH *et al.* [2].

5. Numerical Results

In this section we present the variation of stress intensity factors with ratio of crack speed v to shear wave speed c_2 for both problems. The crack length dependence of the stress intensity factors and its variations with v/c_2 have been shown in Figs. 2–5. Figures 2 and 3 illustrate the fact that in Problem 1, the stress intensity factors at both the crack tips decrease with the increase in the distance between the cracks.

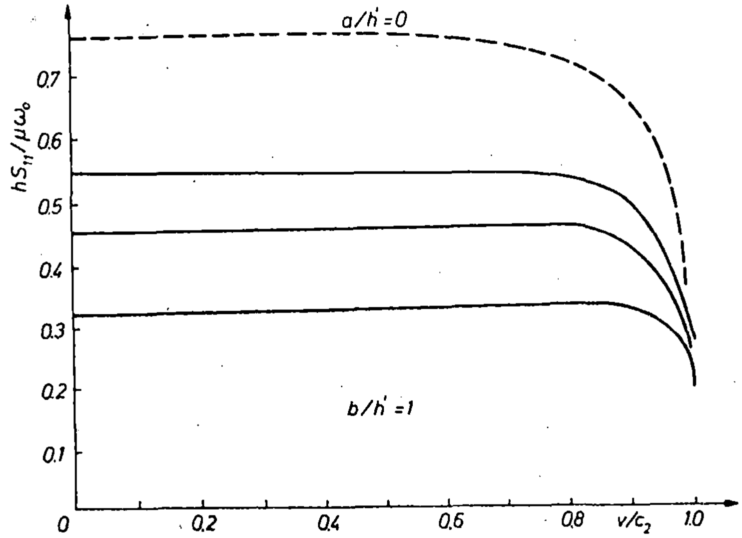


FIG. 2. Stress intensity factor at the outer edge *vs.*v/c_2 for Problem 1.

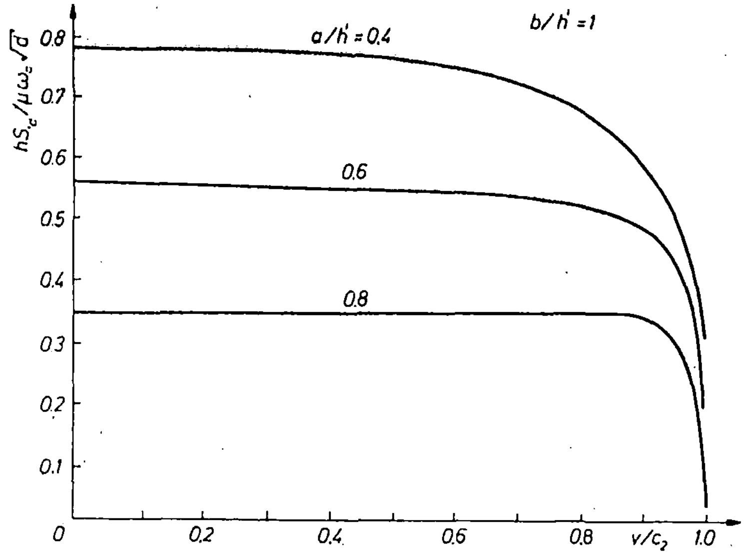


FIG. 3. Stress intensity factor at the inner edge *vs.vs*$/c_2$ for Problem 1.

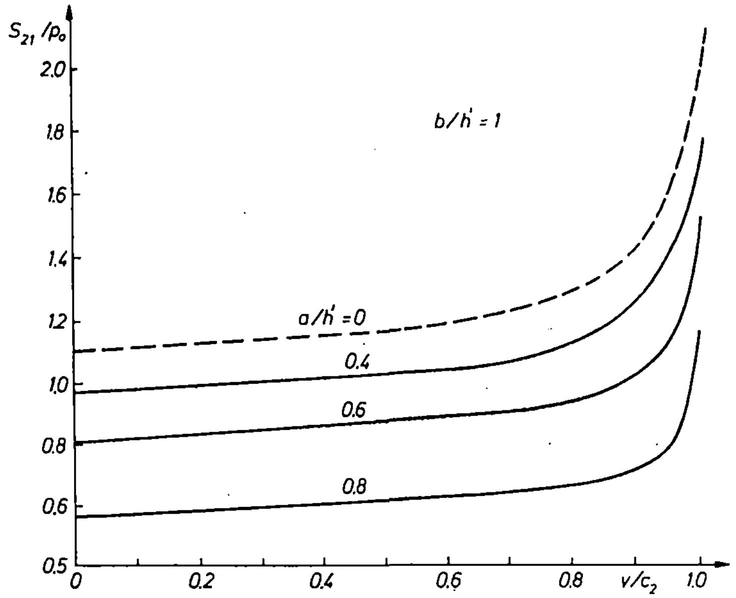


FIG. 4. Stress intensity factor at the outer edge $vs. v/c_2$ for Problem 2.

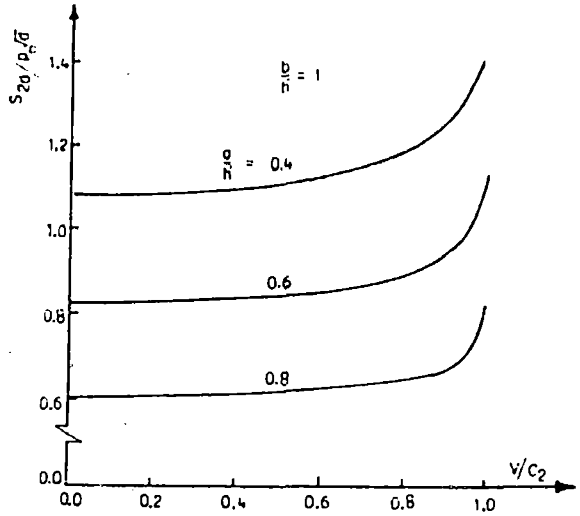


FIG. 5. Stress intensity factor at the inner edge $vs. v/c_2$ for Problem 2.

But for the Problem 2, as seen from Figs. 4 and 5, it is found that the behaviour of the stress intensity factors at the crack tips is of different nature.* In the Problem 1, the stress intensity factors at both the crack edges decrease with the increase in the value of v/c_2 and approach zero as $v/c_2 \to 1$. But in Problem 2, the stress intensity factor at both the ~~outer~~ edges increase gradually with the increase in the value of v/c_2 and approaches infinity as $v/c_2 \to 1$;** ~~whereas the corresponding value at the inner edge decreases gradually to zero with the increase in the value of v/c_2~~. The dashed line in Fig. 2 and Fig. 4 corresponding

* as compared to the corresponding nature of problem I.

** In problem II it is found that the stress intensity factors at both the edges decrease with the increase in the values of to the stress intensity factors at the tip of a single crack as given by SINGH *et al.* [2] for the case $b/h' = 1$.

References

1. G. C. SIH, E. P. CHEN, *Cracks propagating in a strip of material under plane extension*, Int. J. Eng. Sci., 10, 537–551, 1972.
2. B. M. SINGH, T. B. MOODIE, B. J. HADDOW, *Closed-form solutions for finite length crack moving in a strip under anti-plane shear stress*, Acta Mech., **38**, 99–109, 1981.
3. K. N. SRIVASTAVA, M. LOWENGRUB, *Finite Hilbert transform technique for triple integral equations with trigonometric kernels*, Proc. R. Soc. Edinb. 1968.
4. M. LOWENGRUB, K. N. SRIVASTAVA, *Two coplanar Griffith cracks in an infinitely long elastic strip*, Int. J. Sci., **6**, 425–434, 1968.
5. D. L. JAIN, R. P. KANWAL, *Diffraction of elastic waves by two coplanar Griffith cracks in an infinite elastic medium*, Int. J. Solids Structure, **8**. 961–975, 1972.
6. S. ITOU, *Diffraction of an antiplane shear wave by two coplanar Griffith cracks in an infinite elastic medium*, Int. J. Solids Structure, **16**, 1147–1153, 1980.
7. A. N. DAS, M. L. GHOSH, *Two coplanar Griffith cracks moving along the interface of two dissimilar elastic media*, Engng. Frac. Mech. (to appear)
8. E. H. YOFFE, *The moving Griffith crack*, Phil. Mag., **42**, 739–750, 1951.
9. I. S. GRADSHTEYN, I. M. RYZHIK, *Tables of integrals, series and products*, New York, Academic Press 1965.

Streszczenie

RUCH DWÓCH WSPÓŁLINIOWYCH SZCZELIN GRIFFITHA W PAŚMIE W WARUNKACH ANTYPŁASKIEGO STANU ODKSZTAŁCENIA

Rozważono przypadek ustalonego ruchu dwóch szczelin o jednakowych długościach $b - a$ i wzajemnej odległości $2a$ poruszających się z ustaloną prędkością w płaszczyźnie symetrii $y = 0$ pasma sprężystego ograniczonego płaszczyznami $y = \pm h'$. Rozważono przypadki, gdy płaszczyzny te są utwierdzone lub swobodne od naprężeń. Rozwiązania uzyskano w postaci zamkniętej. Przedyskutowano i zilustrowano wykresami zależności współczynników intensywności naprężenia w wierzchołkach szczelin od predkości ich propagacji i parametrów geometrycznych zadania.

Резюме

ДВИЖЕНИЕ ДВУХ КОЛЛИНЕАРНЫХ ТРЕЩИН ГРИФФИТА В ПОЯСЕ В УСЛОВИЯХ АНТИПЛОСКОГО ДЕФОРМИРОВАННОГО СОСТОЯНИЯ

Рассматривался случай фиксированного движения двух трещин одинаковой длины $b-a$ и расположенных на расстоянии $2a$, передвигающихся с определенной скоростю в плоскости симмиетрии $y = 0$ упругого пояса, ограниченного плоскостями $y = \pm h'$. Рассматривался случай, когда эти плоскости закреплены или свободны от напряжений. Рещения были получены в замкнутом виде. Зависимость коеффициентов интенсивности напряжения в виерщинах трещин от скорости их распространения и геомоетрических параметров задачи подлегала обсуждению, иллюстрировалась граффиками.

DEPARTMENT OF MATHEMATICS
NORTH BENGAL UNIVERSITY DARJEELING, WEST BENGAL, INDIA.

Received May 31, 1991.

the separating distance between the cracks.

Engineering Fracture Mechanics Vol. 41, No. 1, pp. 59–69, 1992
Printed in Great Britain.

0013-7944/92 $5.00 + 0.00
Pergamon Press plc.

TWO CO-PLANAR GRIFFITH CRACKS MOVING ALONG THE INTERFACE OF TWO DISSIMILAR ELASTIC MEDIA

A. N. DAS and M. L. GHOSH
Department of Mathematics, North Bengal University, Darjeeling, West Bengal 734430, India

Abstract—In this paper, the distribution of stress and displacement due to propagation of two parallel and co-planar Griffith cracks with constant velocity under antiplane shear stress at the interface of two dissimilar elastic media are presented. In the first case, cracks are assumed to propagate along the interface of two dissimilar infinite elastic half-spaces. In the second case, the problem of propagation of two co-planar Griffith cracks with uniform velocity at the interface of a layered composite has been treated. Cracks are assumed to be moving at the interface of a layer of thickness h and a semi-infinite substrate of different material. By the use of Fourier transform the problems have been reduced to the solution of a set of triple integral equations which have been solved by using the finite Hilbert transform technique. In the second problem, analytical solutions up to the order h^{-4}, where $h \gg 1$, have been derived for both the crack opening displacement and the stress intensity factors. Numerical results are also shown graphically.

1. INTRODUCTION

SCATTERING of elastic waves by cracks located in a homogeneous, isotropic medium has important applications in geophysics and seismology. If the cracks are located at the interfaces of layered media, the study becomes more relevant. Scattering of elastic waves from an interface crack under antiplane strain was solved by Bostrom[1]. Srivastava *et al.*[2] solved the problem of interaction of an antiplane shear wave by an interface crack. The problem of diffraction of Love waves by a crack of finite width in the plane interface of a layered composite has been solved by Neerhoff[3]. As regards the dynamic crack problem, research has been restricted mainly to the cases of a single crack because of the severe mathematical complexity encountered in finding solutions of problems involving two or more cracks. The diffraction of an antiplane shear wave by two co-planar Griffith cracks in an infinite elastic medium has been treated by Itou[4]. Lowengrub and Srivastava[5] treated the static problem of stress distribution in the presence of two co-planar Griffith cracks in an infinite elastic strip. The scattering of time harmonic normally incident plane waves by two co-planar Griffith cracks was also solved by Jain and Kanwal[6].

To our knowledge, the diffraction of elastic waves by two cracks moving along the interface of bonded dissimilar elastic media has not been investigated so far. In this paper we consider the problem of determining the distribution of shear stress in the neighbourhood of the cracks, moving along the interface of two bonded dissimilar elastic media. Two cases of practical importance have been considered here. Firstly, the case of two co-planar Griffith cracks moving along the interface of two semi-infinite dissimilar elastic media has been treated; secondly, the problem of propagation of two co-planar Griffith cracks along the interface of an elastic layer overlying a semi-infinite medium of different elastic properties has been considered. Employing Fourier transform we reduced these problems to solving a set of triple integral equations with cosine karnel and weight functions. These equations are solved using the finite Hilbert transform technique. In the second problem, analytical expressions are retained up to the order h^{-4}, where h is the thickness of the upper layer, for deriving the dynamic stress intensity factors and crack opening displacement. Numerical results are also presented graphically.

2. FORMULATION OF THE PROBLEM

We consider two cracks of finite length to be placed along the X-axis from -1 to $-c$ and c to 1 with reference to a set of rectangular coordinates (x, y, z) which, referred to a fixed coordinate system (X, Y, Z), are moving with constant velocity v along the X-axis, as shown in Fig. 1.

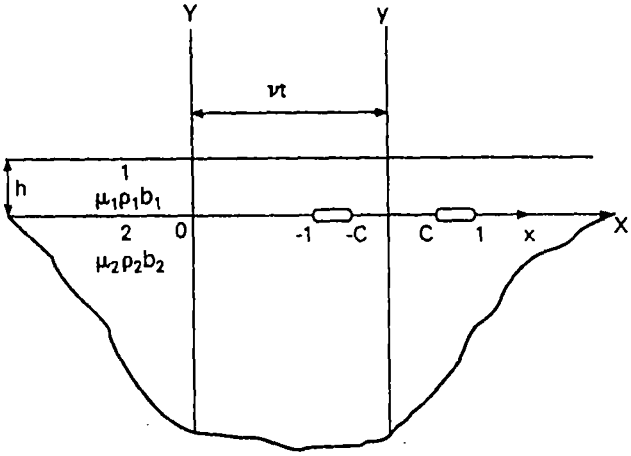


Fig. 1. Geometry and coordinate system.

The coordinates are regarded as dimensionless, referring to the outer edge of the crack. In the dynamic problem of antiplane shear, there exists a single non-vanishing component of displacement in the Z-direction, $W_i = W_i(X, Y, t)$, $i = 1, 2$, where W_1 and W_2 are the displacement components along the Z-direction in media $Y > 0$ and $Y < 0$ respectively. In the absence of body force the equation of motion is

$$\frac{\partial^2 W_i}{\partial X^2} + \frac{\partial^2 W_i}{\partial Y^2} = \frac{1}{b_i^2}\frac{\partial^2 W_i}{\partial t^2} \tag{1}$$

where $b_i = (\mu_i/\rho_i)^{1/2}$ $(i = 1, 2)$ are the shear wave speeds; ρ_i are the densities of the materials and μ_i are the shear moduli.

Using Galilean transformation $x = X - vt$, $y = Y$, $z = Z$, $t' = t$, where (x, y, z) represents the translating coordinate system shown in Fig. 1, eq. (1) becomes independent of t and reduces to

$$s_i^2\frac{\partial^2 W_i}{\partial x^2} + \frac{\partial^2 W_i}{\partial y^2} = 0 \tag{2}$$

with

$$s_i^2 = 1 - v^2/b_i^2. \tag{3}$$

3. BOUNDARY CONDITIONS

Problem I

In this case the cracks are placed along the interface of two joined dissimilar elastic half-spaces and are moving along the interface with constant velocity v. The x-axis is taken along the interface of these media. The cracks are excited by a normally incident antiplane shear wave. The boundary conditions are

$$\left.\begin{array}{ll} [\tau_{yz}(x,0)]_1 = [\tau_{yz}(x,0)]_2 = -p, & c < |x| < 1 \\ [\tau_{yz}(x,0)]_1 = [\tau_{yz}(x,0)]_2, & 0 \leqslant |x| < c, \quad |x| > 1 \\ W_1(x,0) = W_2(x,0), & 0 \leqslant |x| < c, \quad |x| > 1 \end{array}\right\}. \tag{4a–c}$$

Problem I now consists of solving eq. (2) together with the conditions (4).

Problem II

In this case two co-planar Griffith cracks of finite width are assumed to be moving with uniform velocity under antiplane shear stress along the interface of an elastic layer kept in welded contact with a semi-infinite medium of different elastic properties. The boundary conditions of this dynamic antiplane problem are

$$\left.\begin{array}{ll} [\tau_{yz}(x,0)]_1=[\tau_{yz}(x,0)]_2=-p, & c<|x|<1 \\ [\tau_{yz}(x,0)]_1=[\tau_{yz}(x,0)]_2, & 0\leqslant|x|<c,\quad |x|>1 \\ W_1(x,0)=W_2(x,0), & 0\leqslant|x|<c,\quad |x|>1 \\ [\tau_{yz}(x,h)]_1=0 & -\infty<x<\infty \end{array}\right\}, \tag{5a--d}$$

Problem II now consists of solving eq. (2) together with the conditions (5).

4. SOLUTION OF PROBLEM I

Employing Fourier cosine transform, namely

$$f_c(\xi,y)=\int_0^\infty f(x,y)\cos(\xi x)\,\mathrm{d}x \quad \text{and} \quad f(x,y)=\frac{2}{\pi}\int_0^\infty f_c(\xi,y)\cos(\xi x)\,\mathrm{d}\xi,$$

we obtain the solution of eq. (2) as

$$W_1(x,y)=\frac{2}{\pi}\int_0^\infty A_1(\xi)\exp(-s_1\xi y)\cos(\xi x)\,\mathrm{d}\xi, \quad \text{for } y>0 \tag{6}$$

$$W_2(x,y)=\frac{2}{\pi}\int_0^\infty A_2(\xi)\exp(s_2\xi y)\cos(\xi x)\,\mathrm{d}\xi, \quad \text{for } y<0 \tag{7}$$

where s_i is the positive root of (3) and $A_i(\xi)$ are unknown functions to be determined.

From (6) and (7) we obtain

$$[\tau_{yz}(x,y)]_1=-\frac{2\mu_1 s_1}{\pi}\int_0^\infty \xi A_1(\xi)\exp(-s_1\xi y)\cos(\xi x)\,\mathrm{d}\xi, \quad \text{for } y>0 \tag{8}$$

$$[\tau_{yz}(x,y)]_2=\frac{2\mu_2 s_2}{\pi}\int_0^\infty \xi A_2(\xi)\exp(s_2\xi y)\cos(\xi x)\,\mathrm{d}\xi, \quad \text{for } y<0. \tag{9}$$

Using (4a) and (4b) we get

$$A_2(\xi)=-\frac{\mu_1 s_1}{\mu_2 s_2}A_1(\xi). \tag{10}$$

Therefore, the crack opening displacement $\Delta w(x)$ is obtained as

$$\begin{aligned} \Delta w(x) &= W_1(x,0^+)-W_2(x,0^-) \\ &= \frac{2L}{\pi}\int_0^\infty A_1(\xi)\cos(\xi x)\,\mathrm{d}\xi, \quad c\leqslant x\leqslant 1 \\ &= 0, \qquad\qquad 0\leqslant x<c,\quad x>1 \end{aligned} \tag{11}$$

where

$$L=\frac{\mu_1 s_1+\mu_2 s_2}{\mu_2 s_2}. \tag{12}$$

From (8) and (4a),

$$\int_0^\infty \xi A_1(\xi)\cos(\xi x)\,\mathrm{d}\xi=\frac{p\pi}{2\mu_1 s_1}. \tag{13}$$

Let us take

$$A_1(\xi)=\frac{1}{\xi}\int_c^1 h(t^2)\sin(\xi t)\,\mathrm{d}t. \tag{14}$$

Substituting (14) in (11) we see that this choice of $A_1(\xi)$ leads to

$$\int_c^1 h(t^2)\,dt = 0. \tag{15}$$

Inserting (14) in (13) we obtain

$$\int_0^\infty \frac{th(t^2)\,dt}{t^2 - x^2} = \frac{p\pi}{2\mu_1 s_1}, \quad c < x < 1. \tag{16}$$

Using the finite Hilbert transform, the solution of (16) is

$$h(t^2) = -\frac{2p}{\pi\mu_1 s_1}\sqrt{\frac{t^2 - c^2}{1 - t^2}} \int_c^1 \sqrt{\frac{1 - x^2}{x^2 - c^2}} \times \frac{x\,dx}{x^2 - t^2} + \frac{K'}{\sqrt{(t^2 - c^2)(1 - t^2)}} \tag{17}$$

where the unknown constant K', determined from (15), is

$$K' = \frac{p}{\mu_1 s_1}(c^2 - E/F), \tag{18}$$

where $F = F(\pi/2, q)$ and $E = E(\pi/2, q)$ are complete elliptic integrals of the first and second kind respectively and $q = (1 - c^2)^{1/2}$.

The relevant expressions for the crack opening displacement and stress component at the interface are

$$\Delta w(x) = L \int_x^1 h(t^2)\,dt, \quad c < x < 1 \tag{19}$$

$$[\tau_{yz}(x, 0)]_1 = \frac{2\mu_1 s_1}{\pi} \int_c^1 \frac{th(t^2)\,dt}{x^2 - t^2}, \quad 0 \leqslant x < c, \quad x > 1. \tag{20}$$

Substituting the values of $h(t^2)$ from (17) in (19) and (20) we obtain

$$\Delta w(x) = \frac{Lp}{\mu_1 s_1}\left[E(\lambda, q) - \frac{E}{F}F(\lambda, q)\right], \tag{21}$$

where

$$\sin\lambda = \sqrt{\frac{1 - x^2}{1 - c^2}} \tag{22}$$

and

$$[\tau_{yz}(x, 0)]_1 = p\left[\sqrt{\frac{x^2 - c^2}{x^2 - 1}} - 1 - \frac{E/F - c^2}{\sqrt{(x^2 - c^2)(x^2 - 1)}}\right], \quad \text{for } x > 1 \tag{23}$$

$$= p\left[\sqrt{\frac{c^2 - x^2}{1 - x^2}} - 1 + \frac{E/F - c^2}{\sqrt{(c^2 - x^2)(1 - x^2)}}\right], \quad \text{for } x < c, \tag{24}$$

where we have used

$$\int_c^1 \frac{t\,dt}{(x^2 - t^2)\sqrt{(t^2 - c^2)(1 - t^2)}} = \begin{cases} -\dfrac{\pi}{2\sqrt{(c^2 - x^2)(1 - x^2)}}, & \text{for } 0 < x < c \\ 0, & \text{for } c < x < 1 \\ \dfrac{\pi}{2\sqrt{(x^2 - c^2)(x^2 - 1)}}, & \text{for } x > 1. \end{cases} \tag{25}$$

The stress intensity factors at the tips of the cracks $x = 1$ and $x = c$ respectively are given by

$$K_1 = \lim_{x\to 1} \sqrt{2(x-1)}[\tau_{yz}(x,0)]_1 = \frac{p(1-E/F)}{\sqrt{1-c^2}} \tag{26}$$

$$K_c = \lim_{x\to c} \sqrt{2(c-x)}[\tau_{yz}(x,0)]_1 = \frac{p(E/F-c^2)}{\sqrt{c(1-c^2)}}. \tag{27}$$

5. SOLUTION OF PROBLEM II

Employing Fourier cosine transform the solutions of the problem are sought in the form

$$W_1(x,y) = \frac{2}{\pi}\int_0^\infty [A_1(\xi)\exp(-s_1\xi y) + A_2(\xi)\exp(s_1\xi y)]\cos(\xi x)\,d\xi, \quad \text{for } 0 \leqslant y \leqslant h$$

$$W_2(x,y) = \frac{2}{\pi}\int_0^\infty A_3(\xi)\exp(s_2\xi y)\cos(\xi x)\,d\xi, \qquad \text{for } y < 0. \tag{28}$$

Using (28) we obtain the stress components as

$$[\tau_{yz}(x,y)]_1 = \frac{2\mu_1 s_1}{\pi}\int_0^\infty \xi[-A_1(\xi)\exp(-s_1\xi y) + A_2(\xi)\exp(s_1\xi y)]\cos(\xi x)\,d\xi, \quad \text{for } 0 \leqslant y \leqslant h$$

$$[\tau_{yz}(x,y)]_2 = \frac{2\mu_2 s_2}{\pi}\int_0^\infty \xi A_3(\xi)\exp(s_2\xi y)\cos(\xi x)\,d\xi, \qquad \text{for } y < 0. \tag{29}$$

Applying (5a), (5b) and (5d) we obtain

$$A_3(\xi) = \frac{\mu_1 s_1}{\mu_2 s_2}[A_2(\xi) - A_1(\xi)] \tag{30}$$

and

$$A_2(\xi) = A_1(\xi)\exp(-2\xi h s_1). \tag{31}$$

We define the crack opening displacement $\Delta w(x)$ as

$$\begin{aligned}\Delta w(x) &= W_1(x,0^+) - W_2(x,0^-)\\ &= \frac{2L}{\pi}\int_0^\infty f(\xi)\cos(\xi x)\,d\xi, \quad c \leqslant x \leqslant 1\\ &= 0, \qquad 0 \leqslant x \leqslant c, \quad x > 1\end{aligned} \tag{32}$$

where

$$f(\xi) = A_1(\xi)\left[1 + \frac{\mu_2 s_2 - \mu_1 s_1}{\mu_2 s_2 + \mu_1 s_1}\exp(-2\xi h s_1)\right]. \tag{33}$$

Therefore, by (5c) and (5a), $f(\xi)$ is found to be the solution of the following triple integral equations:

$$\int_0^\infty f(\xi)\cos(\xi x)\,d\xi = 0, \qquad 0 \leqslant x < c, \quad x > 1 \tag{34}$$

$$\int_0^\infty \xi f(\xi)[1 + M(\xi h)]\cos(\xi x)\,d\xi = \frac{p\pi}{2\mu_1 s_1}, \quad c < x < 1 \tag{35}$$

with

$$M(\xi h) = -\frac{1-\tanh(\xi h s_1)}{\left[1 + \frac{\mu_1 s_1}{\mu_2 s_2}\tanh(\xi h s_1)\right]}. \tag{36}$$

Assuming

$$f(\xi) = \frac{1}{\xi}\int_c^1 h(t^2)\sin(\xi t)\,\mathrm{d}t, \tag{37}$$

it is found from (35) and (36) that $h(x^2)$ is the solution of the following Fredholm integral equation:

$$h(x^2) + \int_c^1 h(t^2)K(x^2, t)\,\mathrm{d}t = F(x^2), \quad c < x < 1 \tag{38}$$

satisfying the condition

$$\int_c^1 h(x^2)\,\mathrm{d}x = 0 \tag{39}$$

where

$$K(x^2, t) = -\frac{4}{\pi^2}\sqrt{\frac{x^2-c^2}{1-x^2}}\int_c^1 \sqrt{\frac{1-y^2}{y^2-c^2}} \times \frac{yK_1(y, t)}{y^2-x^2}\,\mathrm{d}y \tag{40}$$

with

$$K_1(y, t) = \int_0^\infty M(\xi h)\cos(\xi y)\sin(\xi t)\,\mathrm{d}\xi \tag{41}$$

and

$$F(x^2) = -\frac{2p}{\pi\mu_1 s_1}\sqrt{\frac{x^2-c^2}{1-x^2}}\int_c^1 \sqrt{\frac{1-y^2}{y^2-c^2}}\,\frac{y\,\mathrm{d}y}{y^2-x^2} + \frac{K''}{\sqrt{(x^2-c^2)(1-x^2)}}, \tag{42}$$

K'' being an arbitrary constant determined by the condition (39). If we take $h \gg 1$, then, by substituting $\eta = \xi h$ and expanding $\cos(\eta y/h)$, $\sin(\eta y/h)$ we may write (41) in the form

$$K_1(y, t) = \frac{I_0 t}{h^2} + \frac{I_1 t}{h^4}(t^2 + 3y^2) + O(h^{-6}) \tag{43}$$

where

$$I_j = \frac{(-1)^j}{(2j+1)!}\int_0^\infty \eta^{2j+1}M(\eta)\,\mathrm{d}\eta \quad (j = 0, 1) \tag{44}$$

and hence

$$K(x^2, t) = \frac{2}{\pi}\sqrt{\frac{x^2-c^2}{1-x^2}}\left[\frac{I_0 t}{h^2} + \frac{I_1 t}{h^4}(t^2 + 3x^2 - \tfrac{3}{2}k^2)\right] + O(h^{-6}) \tag{45}$$

where $k^2 = 1 - c^2$.

Integrating both sides of (38) with respect to x from c to 1 and using (39), we obtain

$$K'' = \frac{p}{\mu_1 s_1}\left[c^2 - \frac{E}{F}\right] + \frac{1}{F}\int_c^1 h(t^2)K(t)\,\mathrm{d}t \tag{46}$$

with

$$K(t) = \frac{2}{\pi}\left[\frac{I_0 t}{h^2}(E - c^2F) + \frac{I_1 t}{h^4}\{(t^2 - \tfrac{3}{2}k^2)(E - c^2F) - c^2(E + F) + 2E\}\right] + O(h^{-6})$$

where E and F defined by $E = E(\pi/2, q)$ and $F = F(\pi/2, q)$ with $q = k$ are known as elliptic integrals of the first and second kind respectively.

Hence, $h(x^2)$ must satisfy the integral equation

$$h(x^2) + \int_c^1 h(t^2)M(x^2, t)\,\mathrm{d}t = S(x^2) \tag{47}$$

where

$$M(x^2, t) = \frac{2t}{\pi\sqrt{(x^2 - c^2)(1 - x^2)}}\left[\frac{I_0}{h^2}\left(x^2 - \frac{E}{F}\right) + \frac{I_1}{h^4}\left\{(t^2 + \tfrac{3}{2}k^2)\left(x^2 - \frac{E}{F}\right)\right.\right.$$
$$\left.\left. + 3x^2(x^2 - 1) + \frac{E}{F} + c^2 - \frac{2c^2E}{F}\right\}\right] + O(h^{-6}) \quad (48)$$

and

$$S(x^2) = \frac{p\left[x^2 - \dfrac{E}{F}\right]}{\mu_1 s_1\sqrt{(x^2 - c^2)(1 - x^2)}}. \quad (49)$$

Since $h \gg 1$, and $|M(x^2, t)| < 1$, the solution of (47) may be written in the form

$$h(x^2) = h_0(x^2) + \frac{1}{h^2}h_1(x^2) + \frac{1}{h^4}h_2(x^2) + O(h^{-6}) \quad (50)$$

where

$$h_0(x^2) = \frac{p\left[x^2 - \dfrac{E}{F}\right]}{\mu_1 s_1\sqrt{(x^2 - c^2)(1 - x^2)}} \quad (51)$$

$$h_1(x^2) = \frac{-I_0 C_0 p\left[x^2 - \dfrac{E}{F}\right]}{2\mu_1 s_1\sqrt{(x^2 - c^2)(1 - x^2)}} \quad (52)$$

$$h_2(x^2) = \frac{pC_0}{4\mu_1 s_1\sqrt{(x^2 - c^2)(1 - x^2)}}\left[I_0^2 C_0\left\{x^2 - \frac{E}{F}\right\} - 2I_1(3x^4 + C_1x^2 + C_2)\right] \quad (53)$$

with

$$C_0 = 1 + c^2 - 2\frac{E}{F}$$

$$C_1 = k^4/4C_0 - (1 + c^2)$$

$$C_2 = c^2 + \frac{E}{F}\left\{C_1 - \frac{k^4}{2C_0}\right\}.$$

The relevant crack opening displacement and stress component at the interface are

$$\Delta w(x) = L\int_x^1 h(t^2)\,\mathrm{d}t, \quad c \leqslant x \leqslant 1 \quad (54)$$

$$[\tau_{yz}(x, 0)]_1 = -\frac{2\mu_1 s_1}{\pi}\left[\int_c^1 \frac{th(t^2)\,\mathrm{d}t}{t^2 - x^2} + \int_c^1 h(t^2)K_1(x, t)\,\mathrm{d}t\right], \quad 0 \leqslant x < c, \quad x > 1 \quad (55)$$

where $K_1(x, t)$ is given in (41).

Using (43) and eqs (50)–(53), we find that

$$\int_c^1 h(t^2)K_1(x, t)\,\mathrm{d}t = \frac{p\pi}{8\mu_1 s_1}\left[\frac{2I_0C_0}{h^2} - \frac{I_0^2C_0^2}{h^4} + \frac{2I_1C_0}{h^4}\{3x^2 + C_1 + \tfrac{3}{2}(1 + c^2)\}\right] + 0(h^{-6}). \quad (56)$$

Using the results given by (25), we get for $0 < x < c$

$$\int_c^1 \frac{th(t^2)\,dt}{t^2 - x^2} = \frac{p\pi}{2\mu_1 s_1}\left[\left\{1 - \frac{I_0 C_0}{2h^2} + \frac{I_0^2 C_0^2}{4h^4}\right\} \times \left\{\frac{x^2 - E/F}{X_1} + 1\right\} - \frac{I_1 C_0}{2h^4} \times \left\{\frac{3x^4 + C_1 x^2 + C_2}{X_1} + 3\left(\frac{1+c^2}{2} + x^2\right) + C_1\right\}\right] + O(h^{-6}), \quad (57)$$

and for $x > 1$

$$\int_c^1 \frac{th(t^2)\,dt}{t^2 - x^2} = \frac{p\pi}{2\mu_1 s_1}\left[\left\{1 - \frac{I_0 C_0}{2h^2} + \frac{I_0^2 C_0^2}{4h^4}\right\} \times \left\{\frac{E/F - x^2}{X_2} + 1\right\} + \frac{I_1 C_0}{2h^4} \times \left\{\frac{3x^4 + C_1 x^2 + C_2}{X_2} - 3\left(\frac{1+c^2}{2} + x^2\right) - C_1\right\}\right] + O(h^{-6}), \quad (58)$$

where

$$X_1 = \sqrt{(c^2 - x^2)(1 - x^2)}$$

$$X_2 = \sqrt{(x^2 - c^2)(x^2 - 1)}.$$

Using eqs (50)–(53) we obtain from (54), after integration, the crack opening displacement as

$$\Delta w(x) = \frac{pL}{\mu_1 s_1}\left[\left\{1 - \frac{I_0 C_0}{2h^2} + \frac{I_0^2 C_0^2 + 2I_1 C_0 (C_1 - k^4/2C_0)}{4h^4}\right\} \times \left\{E(\lambda, q) - \frac{E}{F} F(\lambda, q)\right\} - \frac{2I_1 C_0}{4h^4} x\sqrt{(1 - x^2)(x^2 - c^2)}\right] + O(h^{-6}) \quad (59)$$

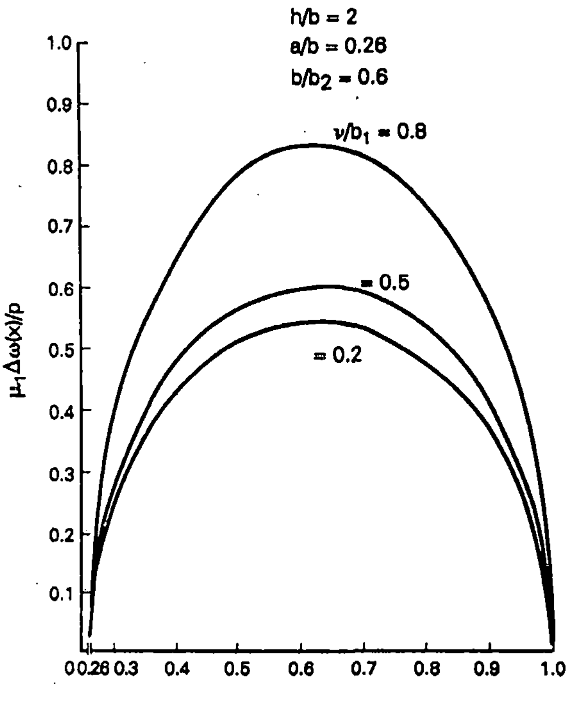


Fig. 2. Variation of crack opening displacement with x for problem II.

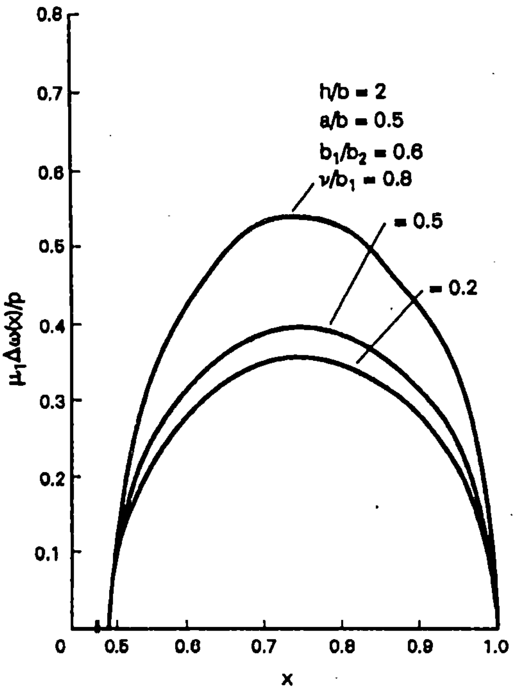


Fig. 3. Variation of crack opening displacement with x for problem II.

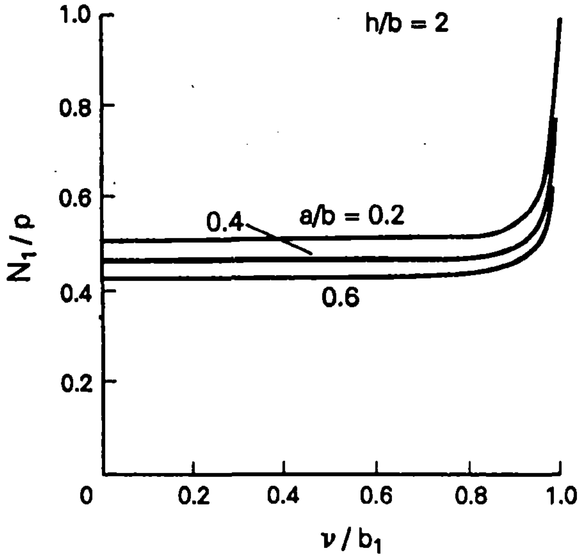


Fig. 4. Variation of stress intensity factor at the outer edge with v/b_1 for problem II.

where

$$\sin\lambda = \sqrt{\frac{1-x^2}{1-c^2}}.$$

Substituting the results obtained in (56), (57) and (58) on the right hand side of (55) the stress in the plane of the crack can be derived and from it stress intensity factors at the crack tips can easily be found.

We find that the stress intensity factor at $x = 1$ is given by

$$N_1 = \lim_{x\to 1}\sqrt{2(x-1)}[\tau_{yz}(x,0)]_1$$

$$= \frac{-p}{\sqrt{1-c^2}}\left[(E/F-1)\left\{1-\frac{I_0C_0}{2h^2}+\frac{I_0^2C_0^2}{4h^4}\right\}+\frac{I_1C_0}{2h^4}(3+C_1+C_2)\right]+O(h^{-6}) \tag{60}$$

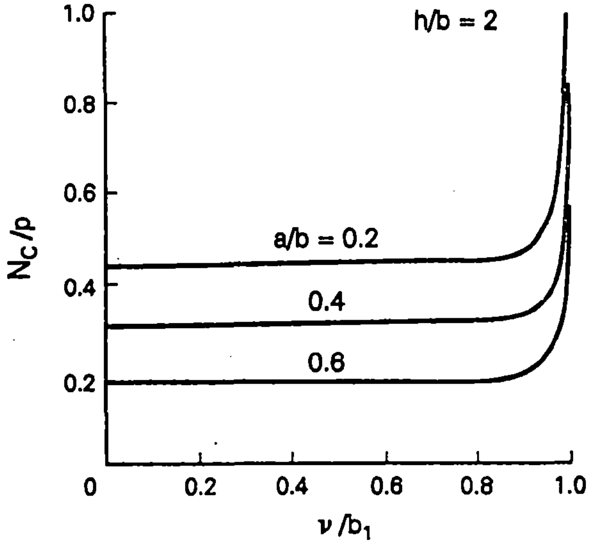


Fig. 5. Variation of stress intensity factor at the inner edge with v/b_1 for problem II.

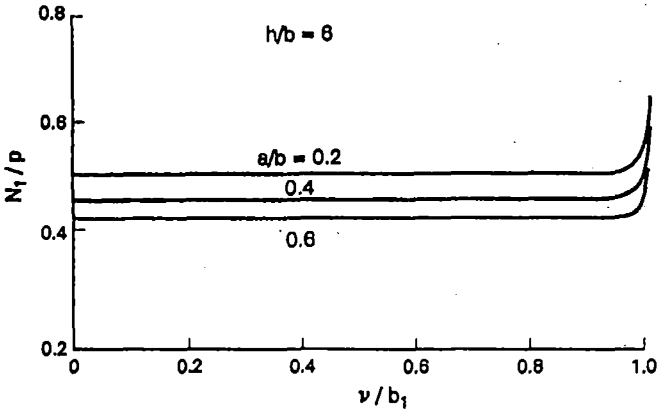


Fig. 6. Variation of stress intensity factor at the outer edge with v/b_1 for problem II.

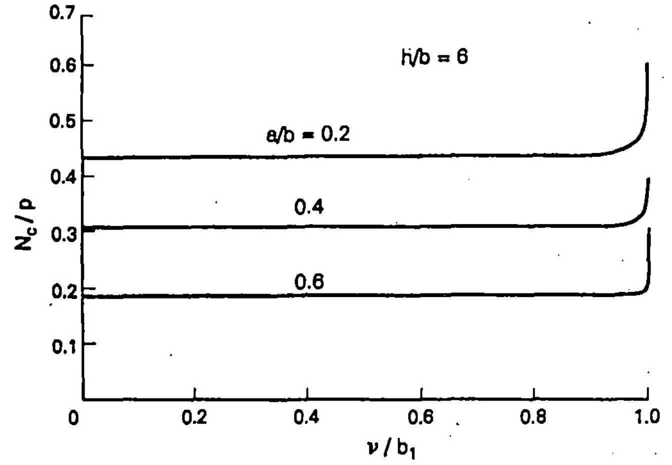


Fig. 7. Variation of stress intensity factor at the inner edge with v/b_1 for problem II.

and the stress intensity factor at $x = c$ is found to be

$$\begin{aligned} N_c &= \lim_{x \to c} \sqrt{2(c-x)}\,[\tau_{yz}(x,0)]_1 \\ &= \frac{-p}{\sqrt{c(1-c^2)}}\left[(c^2 - E/F)\left\{1 - \frac{I_0 C_0}{2h^2} + \frac{I_0^2 C_0^2}{4h^4}\right\} - \frac{I_1 C_0}{2h^4}(3c^4 + C_1 c^2 + C_2)\right] + O(h^{-6}). \end{aligned} \tag{61}$$

6. NUMERICAL RESULTS

In this section we present numerical results for the stress intensity factors at the crack tips and also the crack opening displacements for different values of the layer thickness and the crack speed and for $b_1/b_2 = 0.6$. The crack opening displacement is found to increase gradually with the increase in the value of the crack speed. Further, for a fixed crack speed, the crack opening displacement increases with the decrease in the value of the separating distance between the cracks.

Variation of the stress intensity factors at both crack tips with crack speed is depicted in Figs 4–7. It is interesting to note that stress intensity factors at both the crack tips increase very slowly at the onset with the increase in the value of v/b_1 but change rapidly and go to infinity as v/b_1 approaches 1. This fact becomes prominent as the layer thickness becomes large.

REFERENCES

[1] A. Bostrom, Elastic wave scattering from an interface crack. *J. appl. Mech.* **54**, 503 (1987).
[2] K. N. Srivastava, R. M. Paliya and D. S. Karaluia, Interaction of antiplane shear waves by a Griffith crack at the interface of two bonded dissimilar elastic half-spaces. *Int. J. Fracture* **16**(4), 349 (1980).
[3] F. L. Neerhoff, Diffraction of Love waves by a stress-free crack of finite width in the plane interface of a layered composite. *Appl. Sci. Res.* **35**, 237 (1979).
[4] S. Itou, Diffraction of an antiplane shear wave by two coplanar Griffith cracks in an infinite elastic medium. *Int. J. Solids Structures* **16**, 1147 (1980).
[5] M. Lowngrub and K. N. Srivastava, Two coplanar Griffith cracks in an infinitely long elastic strip. *Int. J. Engng Sci.* **6**, 425 (1968).
[6] D. L. Jain and R. P. Kanwal, Diffraction of elastic waves by two coplanar Griffith cracks in an infinite elastic medium. *Int. J. Solids Structures* **8**, 961 (1972).

(*Received* 26 *October* 1990)

International Journal of Fracture 56: 279–298, 1992.

Problem of two coplanar Griffith cracks running steadily under three-dimensional loading

A.N. DAS and M.L. GHOSH
Department of Mathematics, North Bengal University, Darjeeling, West Bengal – 734430, India

Received 1 December 1990; accepted in revised form 4 December 1991

Abstract. In this paper, the three-dimensional problem of two coplanar Griffith cracks propagating uniformly in an elastic medium has been considered. Equal and opposite tractions which are triaxial in nature are applied to the crack surfaces. The two-dimensional Fourier transforms have been used to reduce the mixed boundary value problem to the solution of triple integral equations. In order to solve the problem, the transformed surface displacement has been expanded in a series of Chebyshev polynomials which is automatically zero outside the cracks and also satisfies the edge conditions. Finally Schmidt method has been used to determine the unknown constants occurring in the series. Numerical calculations are carried out to obtain the crack opening displacement and also the stress intensity factors for different values of the parameters.

1. Introduction

Yoffe [1] considered the problem of propagation of a crack of fixed length at a constant speed through a stretched isotropic elastic solid of infinite extent. In recent years, Yoffe's investigation was extended to include different types of materials and different material geometries. Sih and Chen [2] considered the problem of a uniformly propagating finite crack in a strip of isotropic elastic material. Recently Kassir and Tse [3] solved the plane stress problem of a moving Griffith crack in an infinite orthotropic stressed medium by using integral transform technique and the same technique has been employed by De and Patra [4] to solve Yoffe's problem in a stressed orthotropic strip of finite thickness.

However all the problems mentioned above have been solved using the dynamic equations of elasticity in two dimensions. But practically in most instances, cracks are subjected to a state of stress that is triaxial in nature. Crack problems involving three-dimensional loading have generally not been attempted so far.

Recently Angel and Achenbach [5] derived the elastodynamics stress intensity factor for three-dimensional loading of a cracked half-space. Freund [6] also solved the three dimensional problem of the oblique reflection of a Rayleigh wave from the edge of a semi-infinite crack employing a Wiener-Hopf technique. The problem of a uniformly propagating finite crack in an elastic medium has been solved by Itou [7] using dynamic equations of elasticity in three dimensions.

Regarding the dynamic crack problem, research has been restricted mainly to a single crack because of severe mathematical complexity encountered in finding the solutions for two or more cracks. Recently Jain and Kanwal [8] presented the low-frequency solution of diffraction of normally incident longitudinal waves by two coplanar Griffith cracks in an infinite isotropic elastic medium. They used the finite Hilbert transform technique developed by Srivastava and Lowengrub [9] to solve the mixed boundary value problem. Using a completely different technique Itou [10] solved the diffraction problem of elastic waves by two coplanar Griffith cracks in an infinite elastic medium.

In this paper we have considered the problem of propagation of two coplanar Griffith cracks propagating steadily with uniform velocity under three-dimensional loading. The application of two-dimensional Fourier transforms reduced this problem to that of solving triple integral equations in which the double Fourier transforms of the crack opening displacement appear as the unknown. In an attempt to solve the problem the transformed surface displacement has been expanded in a series of a function which is automatically zero outside the cracks. Finally the Schmidt method [11] has been employed to solve the integral equations. The dynamic stress intensity factors and the crack opening displacement have been evaluated numerically for various values of crack speed and distance between the cracks.

2. Formulation of the problem

Let (X, Y, Z) be a fixed rectangular coordinate system. Two coplanar Griffith cracks of infinite length but finite width located in the XZ-plane, the Z-axis being in the direction of the length of the cracks, are assumed to be moving steadily with velocity U in the direction of the X-axis. It is convenient to introduce Galilean transform $x = X - UT$, $y = Y$, $z = Z$, $t = T$ where (x, y, z) represents the translating coordinate system shown in Fig. 1. Referred to this moving system of the coordinate the cracks are assumed to occupy the positions $b < |x| < a$, $y = 0$, $|z| < \infty$.

The equations of motion in the absence of body force are

$$(\lambda + \mu)\frac{\partial}{\partial X}\left(\frac{\partial u^*}{\partial X} + \frac{\partial v^*}{\partial Y} + \frac{\partial w^*}{\partial Z}\right) + \mu\left(\frac{\partial^2 u^*}{\partial X^2} + \frac{\partial^2 u^*}{\partial Y^2} + \frac{\partial^2 u^*}{\partial Z^2}\right) = \rho\frac{\partial^2 u^*}{\partial T^2},$$

$$(\lambda + \mu)\frac{\partial}{\partial Y}\left(\frac{\partial u^*}{\partial X} + \frac{\partial v^*}{\partial Y} + \frac{\partial w^*}{\partial Z}\right) + \mu\left(\frac{\partial^2 v^*}{\partial X^2} + \frac{\partial^2 v^*}{\partial Y^2} + \frac{\partial^2 v^*}{\partial Z^2}\right) = \rho\frac{\partial^2 v^*}{\partial T^2}, \quad (2.1)$$

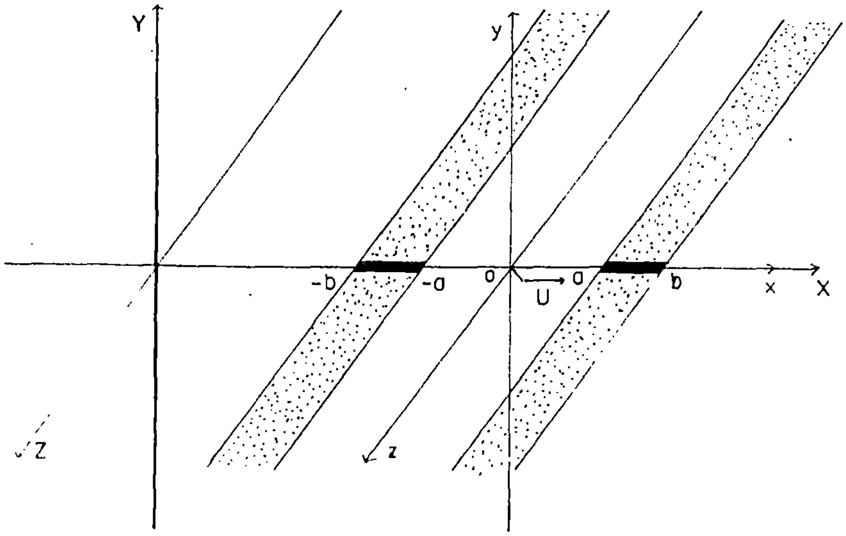


Fig. 1.

$$(\lambda + \mu)\frac{\partial}{\partial Z}\left(\frac{\partial u^*}{\partial X} + \frac{\partial v^*}{\partial Y} + \frac{\partial w^*}{\partial Z}\right) + \mu\left(\frac{\partial^2 u^*}{\partial X^2} + \frac{\partial^2 w^*}{\partial Y^2} + \frac{\partial^2 w^*}{\partial Z^2}\right) = \rho\frac{\partial^2 w^*}{\partial T^2},$$

where u^*, v^*, w^* are the displacement components, λ and μ are Lame's constants and ρ is the material density. Using Galilean transformation

$$x = X - UT, \quad y = Y, \quad z = Z, \quad t = T$$

(2.1) reduces to

$$(\lambda + \mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho U^2\frac{\partial^2 u}{\partial x^2},$$

$$(\lambda + \mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y} + \frac{\partial^2 v}{\partial z^2}\right) = \rho U^2\frac{\partial^2 v}{\partial x^2}, \tag{2.2}$$

$$(\lambda + \mu)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho U^2\frac{\partial^2 w}{\partial x^2}.$$

The stress components for the three dimensional problem are

$$\sigma_x = (\lambda + 2\mu)\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right); \quad \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{2.3.1}$$

$$\sigma_y = (\lambda + 2\mu)\frac{\partial v}{\partial y} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right); \quad \tau_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \tag{2.3.2}$$

$$\sigma_z = (\lambda + 2\mu)\frac{\partial w}{\partial z} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right); \quad \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right). \tag{2.3.3}$$

The boundary conditions are

$$\frac{\sigma_y}{2\mu} = -\,p(x, z), \quad \text{for } y = 0, \quad a \leqslant |x| \leqslant b, \quad |z| < \infty,$$

$$v = 0, \quad \text{for } y = 0, \quad |x| > b, \quad |x| < a, \quad |z| < \infty, \tag{2.4}$$

$$\tau_{xy} = 0 = \tau_{yz}, \quad \text{for } y = 0, \quad |x| < \infty, \quad |z| < \infty.$$

3. Solution of the problem

Using Fourier transformations viz.

$$\bar{g}(\xi, y, \zeta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y, z)\, e^{i(\xi x + \zeta z)}\, \mathrm{d}x\, \mathrm{d}z,$$

and

$$g(x, y, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{g}(\xi, y, \zeta)\, \mathrm{e}^{-i(\xi x + \zeta z)}\, d\xi\, d\zeta, \tag{3.1}$$

(2.2) reduces to

$$\begin{aligned}
&\left\{\frac{d^2}{dy^2} - (\alpha^2 - M^2)\xi^2 - \zeta^2\right\}\bar{u} - i(\alpha^2 - 1)\xi \frac{d\bar{v}}{dy} - (\alpha^2 - 1)\xi\zeta\,\bar{w} = 0,\\
&-i\xi(\alpha^2 - 1)\frac{d\bar{u}}{dy} + \left\{\alpha^2 \frac{d^2}{dy^2} - (1 - M^2)\xi^2 - \zeta^2\right\}\bar{v} - i(\alpha^2 - 1)\xi \frac{d\bar{w}}{dy} = 0,\\
&-(\alpha^2 - 1)\xi\zeta\,\bar{u} - i(\alpha^2 - 1)\zeta \frac{d\bar{v}}{dy} + \left\{\frac{d^2}{dy^2} - (1 - M^2)\xi^2 - \alpha^2\zeta^2\right\}\bar{w} = 0,
\end{aligned} \tag{3.2}$$

with $\alpha^2 = (\lambda + 2\mu)/\mu$, $\beta^2 = \rho/\mu$ and $M^2 = \beta^2 U^2$.

Due to symmetry given in (2.4), we need to consider the region $y \geqslant 0$ only. The solutions of (3.2) in the region $y \geqslant 0$ can easily be found to be of the form

$$\begin{aligned}
\bar{u} &= A_1 \mathrm{e}^{-s_1 y} + B_1 \mathrm{e}^{-s_2 y},\\
\bar{v} &= A_2 \mathrm{e}^{-s_1 y} + B_2 \mathrm{e}^{-s_2 y},\\
\bar{w} &= A_3 \mathrm{e}^{-s_1 y} + B_3 \mathrm{e}^{-s_2 y},
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
s_1 &= \sqrt{(1 - M^2/\alpha^2)\xi^2 + \zeta^2},\\
s_2 &= \sqrt{(1 - M^2)\xi^2 + \zeta^2},
\end{aligned} \tag{3.4}$$

and

$$A_1 = i\xi A_2/s_1, \quad A_3 = i\zeta A_2/s_1, \quad B_2 = -i(\xi B_1 + \zeta B_3)/s_2. \tag{3.5}$$

The transformed stress components $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}_{xy}$, $\bar{\tau}_{yz}$ obtained from (3.3), (3.5) and (2.3) are

$$\begin{aligned}
\frac{\bar{\sigma}_x}{2\mu} &= \frac{1}{2s_1}[\xi^2 M^2(1 - 2/\alpha^2) + 2\xi^2]A_2 \mathrm{e}^{-s_1 y} - i\xi B_1 \mathrm{e}^{-s_2 y},\\
\frac{\bar{\sigma}_y}{2\mu} &= \frac{1}{2s_1}[\xi^2 M^2 - 2(\xi^2 + \zeta^2)]A_2 \mathrm{e}^{-s_1 y} + i(\xi B_1 + \zeta B_3)\,\mathrm{e}^{-s_2 y},
\end{aligned} \tag{3.6}$$

$$\frac{\bar{\tau}_{xy}}{2\mu} = -i\xi A_2 e^{-s_1 y} - \frac{1}{2s_2}[\xi\zeta B_3 + (s_2^2 + \xi^2)B_1]\, e^{-s_2 y},$$

$$\frac{\bar{\tau}_{yz}}{2\mu} = -i\zeta A_2 e^{-s_1 y} - \frac{1}{2s_2}[\xi\zeta B_1 + (s_2^2 + \zeta^2)B_3]\, e^{-s_2 y}.$$

Using the conditions (2.4.3) B_1 and B_3 can be expressed in terms of A_2 as

$$B_1 = \frac{-2i\xi s_2 A_2}{(2 - M^2)\xi^2 + 2\zeta^2},$$

$$B_3 = \frac{-2i\zeta s_2 A_2}{(2 - M^2)\xi^2 + 2\zeta^2}. \tag{3.7}$$

Hence we find that all the components of stress and displacement can be expressed in terms of the unknown function $A_2(\xi, \zeta)$. Now insertion of (3.5) and (3.7) in $\bar{v}$ given in (3.3) yields

$$A_2 = -\frac{(2 - M^2)\xi^2 + 2\zeta^2}{M^2\xi^2}\bar{v}_0, \tag{3.8}$$

where $\bar{v}_0$ is the transformed displacement on $y = 0$.

Using (3.7) and (3.8) we obtain from (3.6)

$$\frac{\bar{\sigma}_x}{2\mu} = -\bar{v}_0\left[\{2 - M^2)\xi^2 + 2\zeta^2\}\{2 + M^2(1 - 2/\alpha^2)\}\frac{e^{-s_1 y}}{2M^2 s_1} - \frac{2s_2 e^{-s_2 y}}{M^2}\right],$$

$$\frac{\bar{\sigma}_y}{2\mu} = \bar{v}_0\left[\{2 - M^2)\xi^2 + 2\zeta^2\}^2\frac{e^{-s_1 y}}{2M^2\xi^2 s_1} - \frac{2(\xi^2 + \zeta^2)s_2 e^{-s_2 y}}{\xi^2 M^2}\right],$$

$$\frac{\bar{\tau}_{xy}}{2\mu} = i\xi\bar{v}_0\{(2 - M^2)\xi^2 + 2\zeta^2\}\frac{e^{-s_1 y} - e^{-s_2 y}}{M^2\xi^2},$$

$$\frac{\bar{\tau}_{yz}}{2\mu} = i\zeta\bar{v}_0\{(2 - M^2)\xi^2 + 2\zeta^2\}\frac{e^{-s_1 y} - e^{-s_2 y}}{M^2\xi^2}. \tag{3.9}$$

Using the conditions (2.4.1) and (2.4.2) we obtain the following triple integral equations

$$\frac{\bar{\sigma}_y}{2\mu} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{v}_0 G(\xi, \zeta)\, e^{-i\xi x}\, d\xi = -\bar{p}(x, \zeta), \quad \text{for } a < |x| < b$$

and

$$\bar{v}_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{v}_0\, e^{-i\xi x}\, d\xi = 0, \quad \text{for } |x| > b, \quad |x| < a \tag{3.10}$$

with

$$G(\xi, \zeta) = \frac{1}{2M^2\xi^2 s_1}[\{(2 - M^2)\xi^2 + 2\zeta^2\}^2 - 4(\xi^2 + \zeta^2)s_1 s_2]. \tag{3.11}$$

Taking $p(x, z)$ as the even function of x, the solution may be assumed as

$$\begin{aligned} \bar{v}_0(x, \zeta) &= \sum_{n=1}^{\infty} c_n(\zeta)\frac{(-1)^{n+1}}{n}\sin\left[n\cos^{-1}\left\{\frac{a+b-2|x|}{b-a}\right\}\right], \quad \text{for } a \leqslant |x| \leqslant b \\ &= 0, \quad \text{for } 0 \leqslant |x| < a, \quad |x| > b, \end{aligned} \tag{3.12}$$

where $c_n(\zeta)$ are the unknown functions to be determined.

Applying Fourier transformation on (3.12) we obtain

$$\bar{v}_0(\xi, \zeta) = \frac{2\pi}{\xi}\sum_{n=1}^{\infty} c_n(\zeta)\sin\left(\frac{a+b}{2}\xi - \frac{n\pi}{2}\right)J_n\left(\frac{b-a}{2}\xi\right), \tag{3.13}$$

where $J_n(\ )$ are Bessel functions.

Insertion of the expression (3.13) in the first equation of (3.10) yields

$$2\sum_{n=1}^{\infty} c_n(\zeta)\int_0^{\infty}\frac{G(\xi,\zeta)}{\xi}\sin\left(\frac{a+b}{2}\xi - \frac{n\pi}{2}\right)J_n\left(\frac{b-a}{2}\xi\right)\cos(\xi x)\,\mathrm{d}\xi = -\bar{p}(x, \zeta), \quad \text{for } a < x < b. \tag{3.14}$$

Using the following results [12]

$$\begin{aligned} \int_0^{\infty}\cos(a_1\xi)J_n(a_2\xi)\,\mathrm{d}\xi &= \frac{\cos(n\varepsilon)}{\sqrt{a_2^2 - a_1^2}}, \quad \text{for } a_2 > a_1 > 0 \\ &= \frac{a_2^n\sin(n\pi/2)}{\sqrt{a_1^2 - a_2^2}[a_1 + \sqrt{a_1^2 - a_2^2}]^n}, \quad \text{for } a_1 > a_2 > 0 \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \int_0^{\infty}\sin(a_1\xi)J_n(a_2\xi)\,\mathrm{d}\xi &= \frac{\sin(n\varepsilon)}{\sqrt{a_2^2 - a_1^2}}, \quad \text{for } a_2 > a_1 > 0 \\ &= \frac{a_2^n\cos(n\pi/2)}{\sqrt{a_1^2 - a_2^2}[a_1 + \sqrt{a_1^2 - a_2^2}]^n}, \quad \text{for } a_1 > a_2 > 0, \end{aligned} \tag{3.15}$$

where

$$\varepsilon = \sin^{-1}(a_1/a_2)$$

in (3.14) we obtain,

$$\sum_{n=1}^{\infty} c_n(\zeta)\left[\left[\int_0^{\infty}\left\{\frac{G(\xi,\zeta)}{\xi}-\frac{G(\delta,\zeta)}{\delta}\right\}\left[\cos\left(\frac{n\pi}{2}\right)\left\{\sin\left(\frac{a+b+2x}{2}\xi\right)+\sin\left(\frac{a+b-2x}{2}\xi\right)\right\}\right.\right.\right.$$
$$\left.-\sin\left(\frac{n\pi}{2}\right)\left\{\cos\left(\frac{a+b+2x}{2}\xi\right)+\cos\left(\frac{a+b-2x}{2}\xi\right)\right\}\right]J_n\left(\frac{b-a}{2}\xi\right)\mathrm{d}\xi+\frac{G(\delta,\zeta)}{\delta}$$
$$\times\left[\left(\frac{b-a}{2}\right)^n\Big/\left[\sqrt{\left(\frac{a+b+2x}{2}\right)^2-\left(\frac{b-a}{2}\right)^2}\left\{\frac{a+b+2x}{2}\right.\right.\right.$$
$$\left.\left.+\sqrt{\left(\frac{a+b+2x}{2}\right)^2-\left(\frac{b-a}{2}\right)^2}\right\}^n\right]+\sin\left\{n\sin^{-1}\left(\frac{a+b-2x}{b-a}\right)-\frac{n\pi}{2}\right\}\Big/$$
$$\left.\left.\sqrt{\left(\frac{b-a}{2}\right)^2-\left(\frac{a+b-2x}{2}\right)^2}\right]\right]=-\bar{p}(x,\zeta), \tag{3.16}$$

where

$$\frac{G(\delta,\zeta)}{\delta}=\operatorname*{Lt}_{\xi\to\infty}\frac{G(\xi,\zeta)}{\xi}=\{(2-M^2)^2-4\sqrt{1-M^2}\cdot\sqrt{1-M^2/\alpha^2}\}/2M^2\sqrt{1-M^2/\alpha^2}. \tag{3.17}$$

Since the function $G(\xi,\zeta)/\xi - G(\delta,\zeta)/\delta$ behaves as ξ^{-2} for large ξ, the semi-infinite integral on the left hand side of (3.16) can easily be evaluated by Filon's method.

To solve (3.16) for unknown coefficients $c_n(\zeta)$ we adopt the Schmidt method [11] and write (3.14) as

$$\sum_{n=1}^{\infty} c_n(\zeta)F_n(\zeta,x)=-f(\zeta,x), \quad \text{for } a<x<b, \tag{3.18}$$

where $F_n(\zeta,x)$ and $f(\zeta,x)=\bar{p}(\zeta,x)$ are known functions. Let $H_n(\zeta,x)$'s be a set of orthogonal functions which satisfy

$$\int_a^b H_n(\zeta,x)H_m(\zeta,x)\,\mathrm{d}x=N_n\delta_{mn},$$

where

$$N_n=\int_a^b H_n^2(\zeta,x)\,\mathrm{d}x. \tag{3.19}$$

Then $H_n(\zeta,x)$'s can be constructed from the functions $F_n(\zeta,x)$ in the following way

$$H_n(\zeta,x)=\sum_{i=1}^{\infty}\frac{C_{in}}{C_{nn}}F_i(\zeta,x) \tag{3.20}$$

with C_{in} as the cofactor of the element e_{in} of D_n which is defined as

$$D_n = \begin{vmatrix} e_{11}e_{12}\cdots\cdots e_{1n} \\ e_{21}\cdots\cdots\cdots \\ \vdots \qquad\qquad \vdots \\ e_{n1}\cdots\cdots\cdots e_{nn} \end{vmatrix}, \quad e_{in} = \int_a^b F_n(\zeta, x)F_i(\zeta, x)\,\mathrm{d}x. \tag{3.21}$$

Now in terms of the set of orthogonal functions $H_n(\zeta, x)$, the function $-f(\zeta, x)$ can be expressed as

$$-f(\zeta, x) = \sum_{i=1}^{\infty} h_i H_i(\zeta, x). \tag{3.22}$$

Substituting values of $H_n(\zeta, x)$ from (3.20) into (3.22), we obtain from (3.18) after some rearrangement

$$\sum_{n=1}^{\infty} c_n(\zeta)F_n(\zeta, x) = \sum_{n=1}^{\infty} F_n(\zeta, x) \sum_{i=n}^{\infty} h_i \frac{C_{ni}}{C_{ii}}. \tag{3.23}$$

Comparing the coefficients of $F_n(\zeta, x)$ from both sides of (3.23) we find

$$c_n = \sum_{i=n}^{\infty} h_i \frac{C_{ni}}{C_{ii}}, \tag{3.24}$$

where

$$h_i = -\frac{1}{N_i}\int_a^b f(\zeta, x)H_i(\zeta, x)\,\mathrm{d}x. \tag{3.25}$$

4. Stress intensity factors and crack opening displacement

To evaluate the stress intensity factors at the vicinity of the crack ends we put $x = b + r\cos\theta$, $y = r\sin\theta$ for the stress intensity factor at the outer edge and $x = a - r\cos\theta$, $y = r\sin\theta$ for the stress intensity factor at the inner edge.

The required stress σ_θ given by

$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta \tag{4.1}$$

is to be evaluated for small values of r.

Using asymptotic values of

$$J_n\left(\frac{b-a}{2}\xi\right)$$

for large values of ξ, it is found that for small values of r

$$\int_0^\infty e^{-\sqrt{1-q^2}\xi y}\sin\left(\frac{a+b}{2}\xi-\frac{n\pi}{2}\right)J_n\left(\frac{b-a}{2}\xi\right)\cos(\xi x)\,d\xi=-\frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4r(b-a)}}$$

$$\times\left[\cos\left(\frac{n\pi}{2}\right)\sqrt{\frac{(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right.$$

$$\left.+\sin\left(\frac{n\pi}{2}\right)\sqrt{\frac{-(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right]+O(r^0),\quad\text{for } x>b \tag{4.2}$$

$$=\frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4r(b-a)}}\times\left[\cos\left(\frac{n\pi}{2}\right)\sqrt{\frac{(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right.$$

$$\left.-\sin\left(\frac{n\pi}{2}\right)\sqrt{\frac{-(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right]+O(r^0),\quad\text{for } x<a \tag{4.3}$$

and

$$\int_0^\infty e^{-\sqrt{1-q^2}\xi y}\sin\left(\frac{a+b}{2}\xi-\frac{n\pi}{2}\right)J_n\left(\frac{b-a}{2}\xi\right)\sin(\xi x)\,d\xi=\frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4r(b-a)}}$$

$$\times\left[\cos\left(\frac{n\pi}{2}\right)\sqrt{\frac{-(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right.$$

$$\left.-\sin\left(\frac{n\pi}{2}\right)\sqrt{\frac{(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right]+O(r^0),\quad\text{for } x>b \tag{4.4}$$

$$=\frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4r(b-a)}}\times\left[\cos\left(\frac{n\pi}{2}\right)\sqrt{\frac{-(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right.$$

$$\left.+\sin\left(\frac{n\pi}{2}\right)\sqrt{\frac{(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right]+O(r^0),\quad\text{for } x<a. \tag{4.5}$$

Inserting (3.13) into (3.9) and taking inverse Fourier transform of (3.9) we obtain the stress intensity factor at $x = b$ with the aid of (4.2)–(4.5) as

$$K_b = \frac{\sigma_\theta}{2\mu}\sqrt{r}|_{r\to 0} = \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{b-a}} \left[\frac{2-M^2}{2M^2\sqrt{1-M^2/\alpha^2}} Q_1^+ \{(2+M^2(1-2/\alpha^2))\sin^2\theta\right.$$

$$\left. -(2-M^2)\cos^2\theta\} + \frac{2\sqrt{1-\mathrm{M}^2}\cos 2\theta}{M^2} Q_2^+ - \frac{2-M^2}{\mathrm{M}^2}(P_1^- - \mathrm{P}_2^-)\sin 2\theta\right]$$

$$\times \frac{1}{2\pi}\int_{-\infty}^{\infty} c_n(\zeta)\,\mathrm{e}^{-i\zeta z}\,\mathrm{d}\zeta \tag{4.6}$$

and also the stress intensity factor at $x = a$ is found to be

$$K_a = \frac{\sigma_\theta}{2\mu}\sqrt{r}|_{r\to 0} = \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{b-a}} \left[\frac{-(2-M^2)}{2M^2\sqrt{1-M^2/\alpha^2}} Q_1^- \{(2+M^2(1-2/\alpha^2))\sin^2\theta\right.$$

$$\left. -(2-M^2)\cos^2\theta\} - \frac{2\sqrt{1-M^2}\cos 2\theta}{M^2} Q_2^- - \frac{2-M^2}{M^2}(P_1^+ - P_2^+)\sin 2\theta\right]$$

$$\times \frac{1}{2\pi}\int_{-\infty}^{\infty} c_n(\zeta)\,\mathrm{e}^{-i\zeta z}\,\mathrm{d}\zeta \tag{4.7}$$

where

$$\left.\begin{aligned} Q_i^{\pm} &= \left[\cos\left(\frac{n\pi}{2}\right)\sqrt{q_i + (-1)^n\cos\theta} \pm \sin\left(\frac{n\pi}{2}\right)\sqrt{q_i - (-1)^n\cos\theta}\right]\Big/ q_i \\ P_i^{\pm} &= \left[\cos\left(\frac{n\pi}{2}\right)\sqrt{q_i - (-1)^n\cos\theta} \pm \sin\left(\frac{n\pi}{2}\right)\sqrt{q_i + (-1)^n\cos\theta}\right]\Big/ q_i \end{aligned}\right\} \quad (i = 1, 2)$$

and

$$q_1 = \sqrt{1 - \frac{M^2}{\alpha^2}\sin^2\theta},$$

$$q_2 = \sqrt{1 - M^2\sin^2\theta}.$$

It is to be noted that in (4.6) $\theta = \tan^{-1} y/(x - b)$ whereas in (4.7) it is given by $\theta = \tan^{-1} y/(a - x)$.

Taking Fourier inversion of (3.12) we obtain the crack surface displacement as

$$v_0(x, z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left[n \cos^{-1}\left\{\frac{a+b-2|x|}{b-a}\right\}\right] \frac{1}{2\pi} \int_{-\infty}^{\infty} c_n(\zeta)\, e^{-i\zeta z}\, d\zeta,$$

$$\text{for } a \leqslant |x| \leqq b \tag{4.8}$$

5. Numerical discussions

In order to evaluate the stress intensity factors and crack surface displacement we take the function $p(x, z)$ as

$$p(x, z) = \frac{P}{1 + d^2 z^2},$$

where d governs the distribution of the applied force and P is a constant. Numerical calculations have been done taking $\lambda = \mu$ and $d = 1$. The semi-infinite integral in (3.16) is evaluated by Filon's method as the integral converges rapidly because of the rapid decay of the function

$$\left\{\frac{G(\xi, \zeta)}{\xi} - \frac{G(\delta, \zeta)}{\delta}\right\}$$

with the increase in ξ. Adopting the first seven terms of the infinite series given in the left hand side of (3.18) we used the Schmidt method to determine the coefficients $c_n(\zeta)$. For the check of accuracy the values of $\sum_{n=1}^{7} c_n(\zeta) F_n(\zeta, x)/Pb$ and $-f(\zeta, x)/Pb$ are given in Table 1 for $\zeta b = 0.0$, 0.2, $M = 0.4$ and for $a/b = 0.3, 0.4$.

From Table 1 it is clear that the Schmidt method is carried out satisfactorily. The values of $c_n(\zeta)$ are given in Table 2 for $M = 0.4$, $a/b = 0.4$.

The variation of stress intensity factor at the outer edge and at the inner edge with M is shown in Fig. 2 and Fig. 3 respectively for $\theta = 0°$, $18°$, $36°$ and $a/b = 0.2, 0.3, 0.4$. Figure 2 depicts the fact that the value of stress intensity factor at the outer edge decreases with the increase in the values of a/b, whereas from Fig. 3 it is evident that the stress intensity factor at the inner edge is of an opposite character. It increases with the increase in the values of a/b.

The variations of stress intensity factor both at the inner edge and outer edge with z have been presented in Figs. 4–7 for different values of a/b, M and θ. The values of stress intensity factor in all the cases are found to decrease gradually with the increase in the values of z, which is expected from the physical standpoint.

The variation of stress intensity factor corresponding to the circumferential stress σ_θ given by (4.1) with θ at both the crack tips has been shown in Figs. 8–12 for different values of a/b and M.

It is known that there are several factors which contribute to crack curving and branching. One factor, of course, is based upon the criterion that a crack may propagate in a direction normal to the maximum tensile stress and it is interesting to note from Fig. 8 and Fig. 10, there is the possibility of curving and branching of the cracks at the outer edge at very low velocities

Table 1.

ζ/b	a/b	x/b	$\sum_{n=1}^{7} c_n(\zeta)F_n(\zeta, x)/Pb$	$f(\zeta, x)/Pb$
0.0	0.3	0.3	−3.140993	−3.140994
		0.4	−3.140995	
		0.5	−3.140993	
		0.6	−3.140996	
		0.7	−3.140991	
		0.8	−3.140994	
		0.9	−3.140993	
		1.0	−3.140992	
	0.4	0.4	−3.140995	
		0.5	−3.140994	
		0.6	−3.140994	
		0.7	−3.140994	
		0.8	−3.140994	
		0.9	−3.140995	
		1.0	−3.140994	
0.2	0.3	0.3	−2.572111	−2.572113
		0.4	−2.572113	
		0.5	−2.572111	
		0.6	−2.572116	
		0.7	−2.572110	
		0.8	−2.572113	
		0.9	−2.572108	
		1.0	−2.572106	
	0.4	0.4	−2.572114	
		0.5	−2.572114	
		0.6	−2.572114	
		0.7	−2.572113	
		0.8	−2.572113	
		0.9	−2.572113	
		1.0	−2.572113	

Table 2.

ζb	$c_1(\zeta)$	$c_2(\zeta)$	$c_7(\zeta)$
0.0	-0.165871×10^{1}	-0.923569×10^{-4}	-0.759039×10^{-8}
0.2	-0.135194×10^{1}	-0.734980×10^{-4}	0.105638×10^{-6}
0.4	-0.109342×10^{1}	-0.556495×10^{-4}	0.357814×10^{-6}
3.0	-0.578184×10^{-3}	-0.601254×10^{-7}	0.114694×10^{-5}
4.0	-0.182994×10^{-3}	0.883491×10^{-7}	0.659423×10^{-6}
5.0	-0.573139×10^{-4}	0.489839×10^{-7}	0.342023×10^{-6}
9.6	0.366305×10^{-5}	-0.816894×10^{-8}	-0.907244×10^{-7}
9.8	0.362848×10^{-5}	-0.829789×10^{-8}	-0.938769×10^{-7}
10.0	0.358409×10^{-5}	-0.843117×10^{-8}	-0.967438×10^{-7}

of the cracks whereas from Fig. 9, Fig. 11 and Fig. 12 it is clear that for $a/b = 0.3$, the crack tends to become curved at the inner edge for values of M about 0.65.

Finally the crack opening displacement in the plane $z = 0$ has been shown by means of graphs in Figs. 15–16 for different values of a/b and M. The variation of crack opening displacement with z for some fixed x for different values of M and a/b has been depicted in Figs. 13–14.

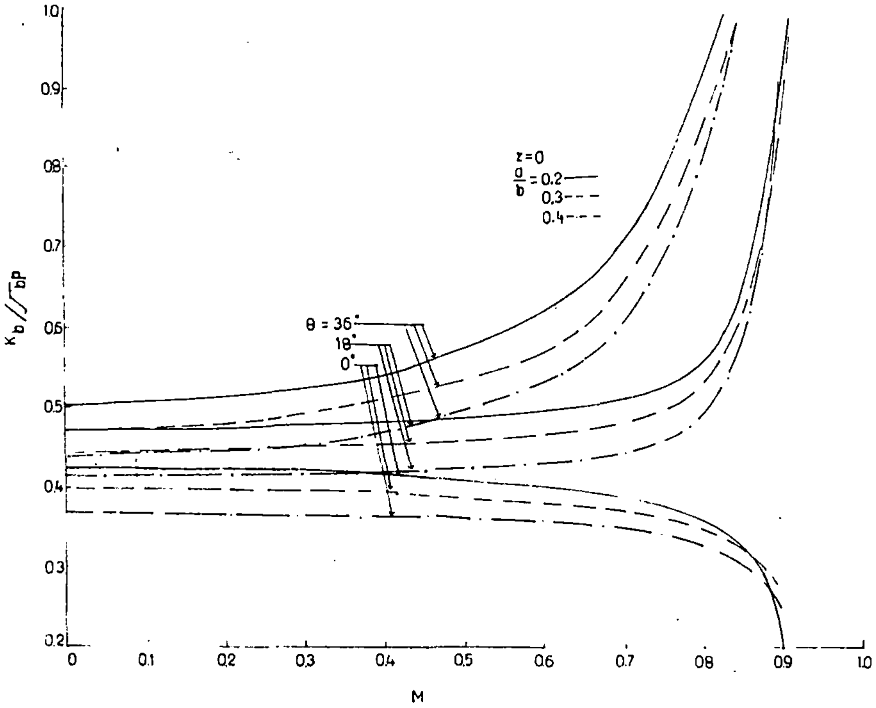

1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
$K_b/\sqrt{bP}$
z=0
$\frac{a}{b}$=0.2
0.3
0.4
θ = 36°
18°
0°
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
M

Fig. 2.

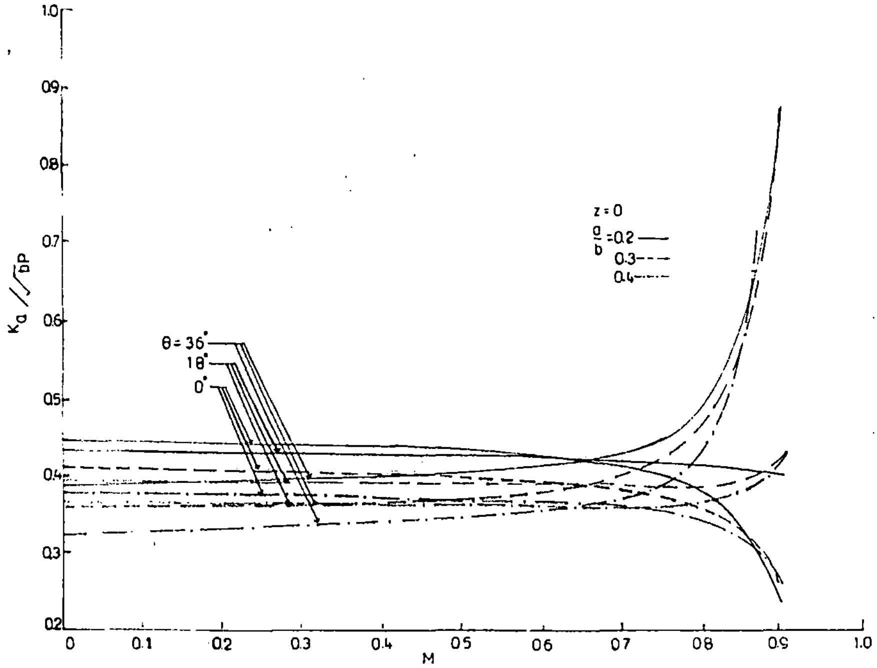

1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
$K_a/\sqrt{bP}$
z=0
$\frac{a}{b}$=0.2
0.3
0.4
θ=36°
18°
0°
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
M

Fig. 3.

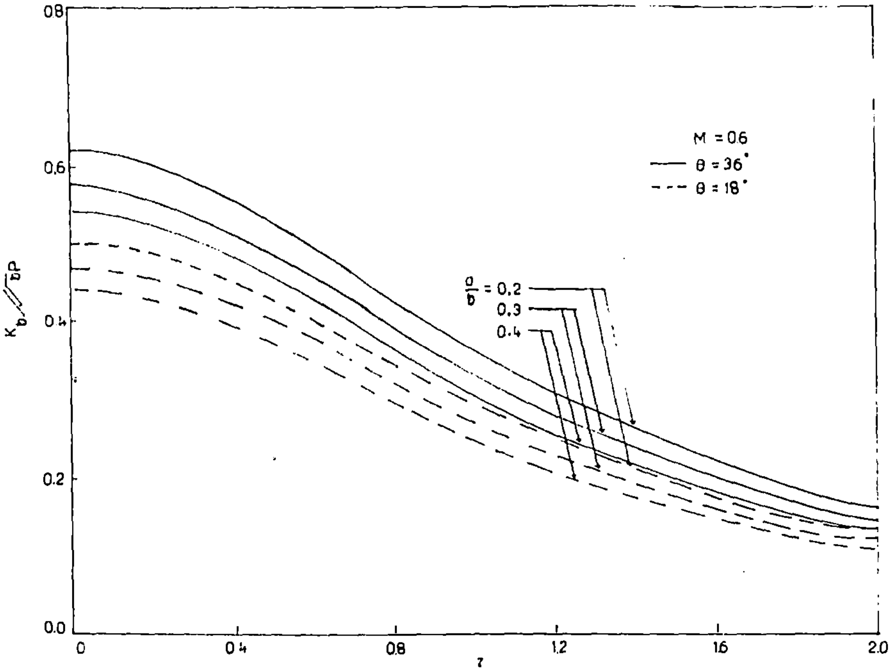

0.8
0.6
0.4
0.2
0.0
$K_b/\sqrt{bP}$
0
0.4
0.8
1.2
1.6
2.0
z
M = 0.6
θ = 36°
θ = 18°
$\frac{a}{b}$ = 0.2
0.3
0.4

Fig. 4.

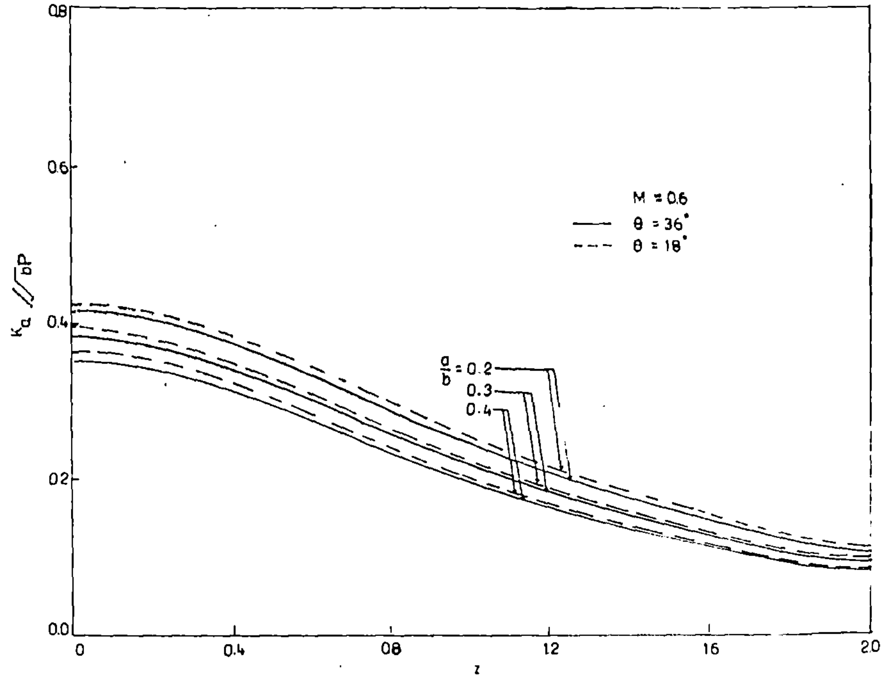

0.8
0.6
0.4
0.2
0.0
$K_a/\sqrt{bP}$
0
0.4
0.8
1.2
1.6
2.0
z
M = 0.6
θ = 36°
θ = 18°
$\frac{a}{b}$ = 0.2
0.3
0.4

Fig. 5.

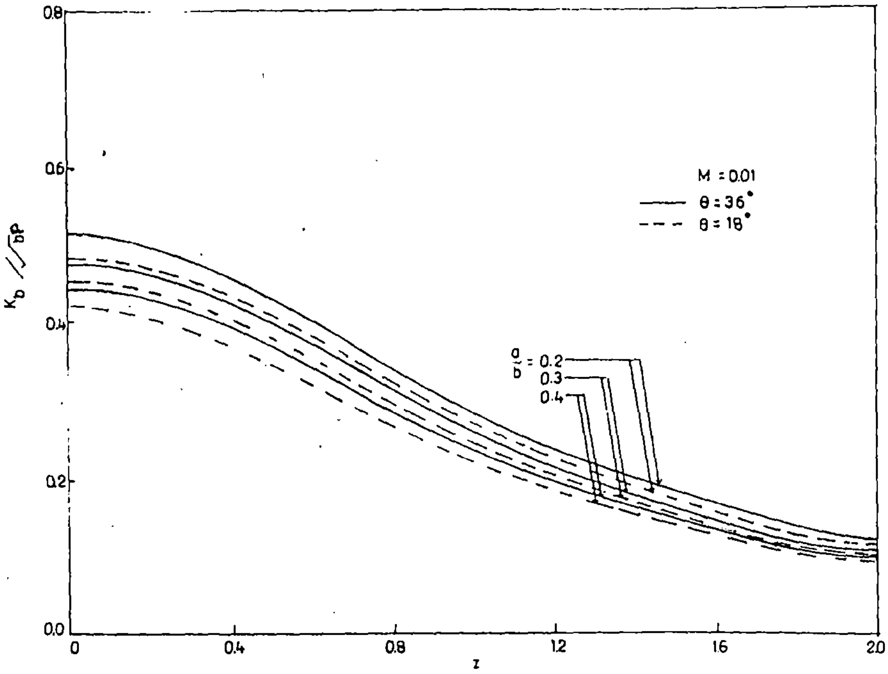

M = 0.01
θ = 36°
θ = 18°
a/b = 0.2
0.3
0.4
$K_b/\sqrt{bP}$
z
0.0
0.2
0.4
0.6
0.8
0
0.4
0.8
1.2
1.6
2.0

Fig. 6.

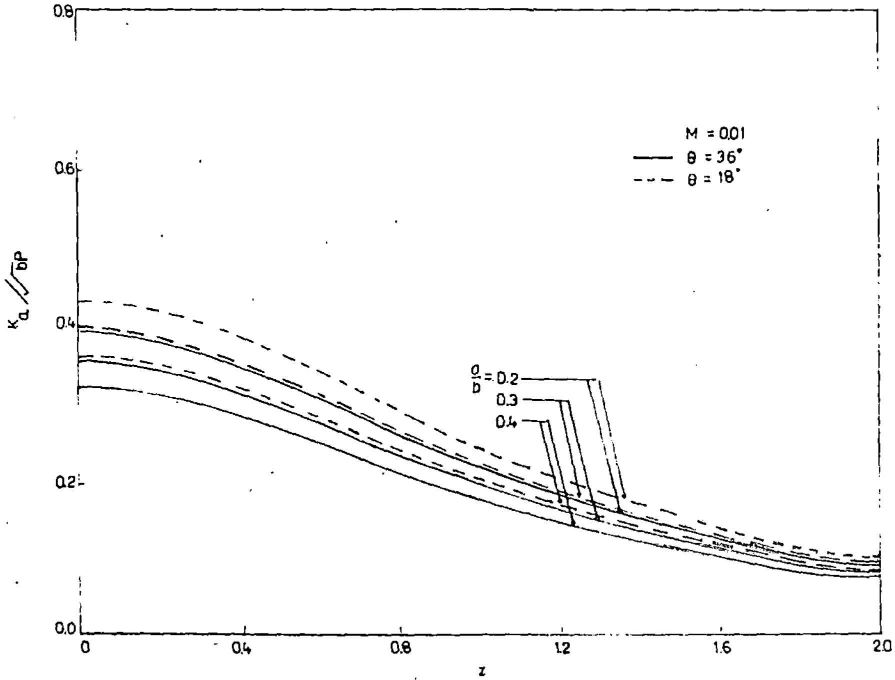

M = 0.01
θ = 36°
θ = 18°
a/b = 0.2
0.3
0.4
$K_a/\sqrt{bP}$
z
0.0
0.2
0.4
0.6
0.8
0
0.4
0.8
1.2
1.6
2.0

Fig. 7.

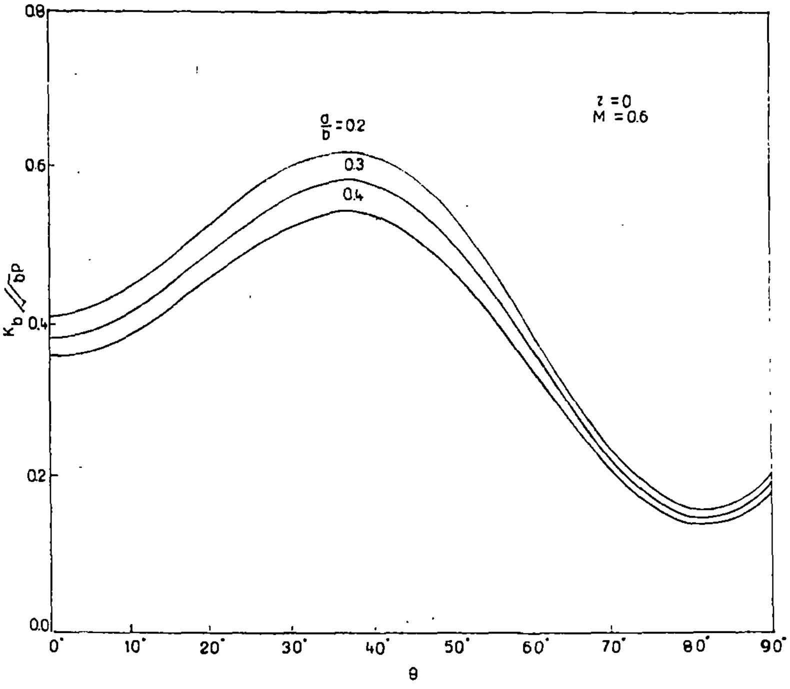

$K_b/\sqrt{bP}$
0.8
0.6
0.4
0.2
0.0
0°
10°
20°
30°
40°
50°
60°
70°
80°
90°
θ
$\frac{a}{b}$ = 0.2
0.3
0.4
z = 0
M = 0.6

Fig. 8.

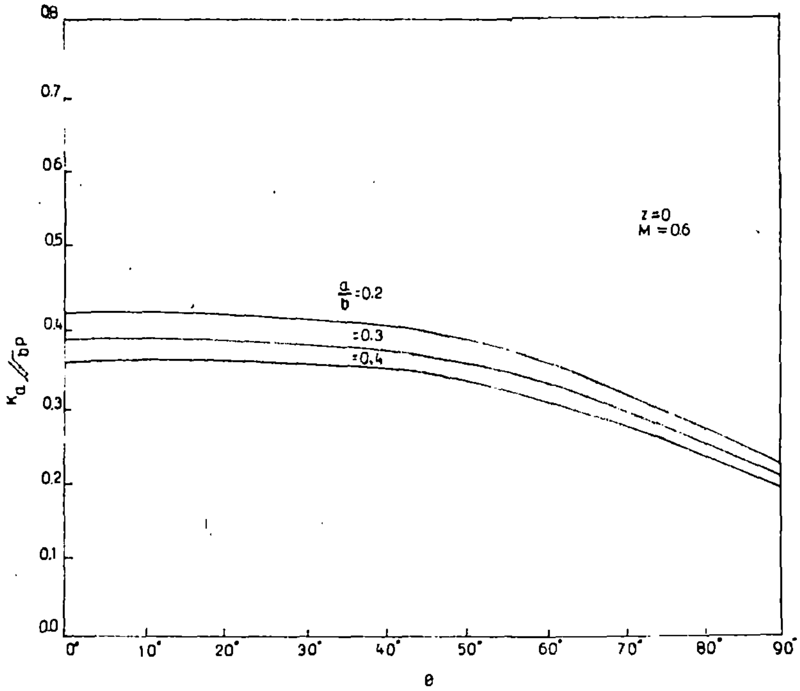

$K_a/\sqrt{bP}$
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0°
10°
20°
30°
40°
50°
60°
70°
80°
90°
θ
$\frac{a}{b}$ = 0.2
= 0.3
= 0.4
z = 0
M = 0.6

Fig. 9.

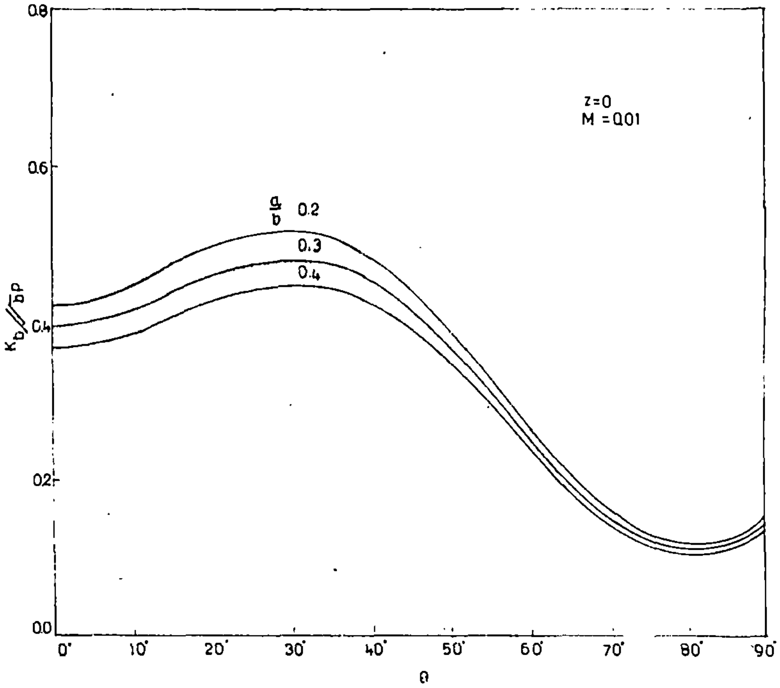

z=0
M =0.01
a/b 0.2
0.3
0.4
$K_b/\sqrt{b}P$
0.0
0.2
0.4
0.6
0.8
0°
10°
20°
30°
40°
50°
60°
70°
80°
90°
θ

Fig. 10.

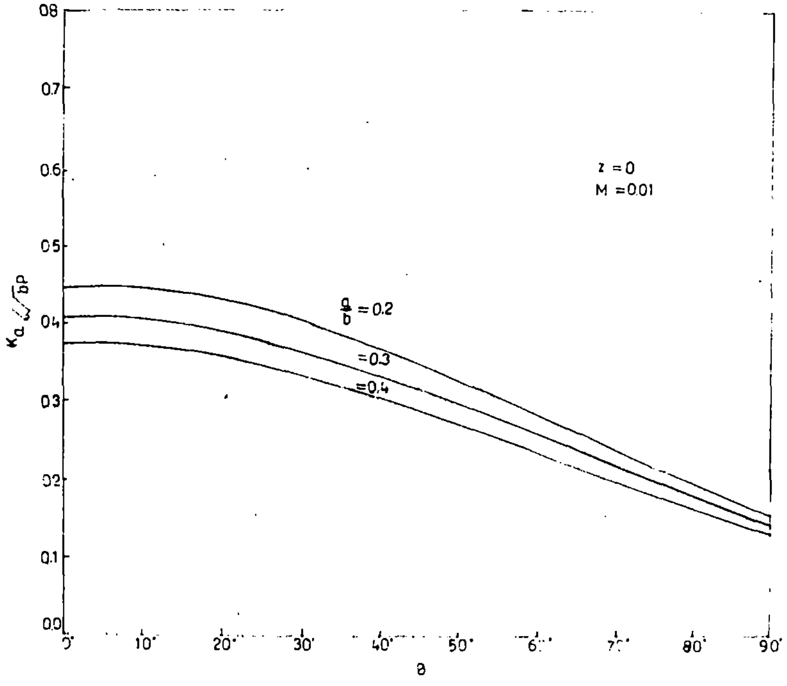
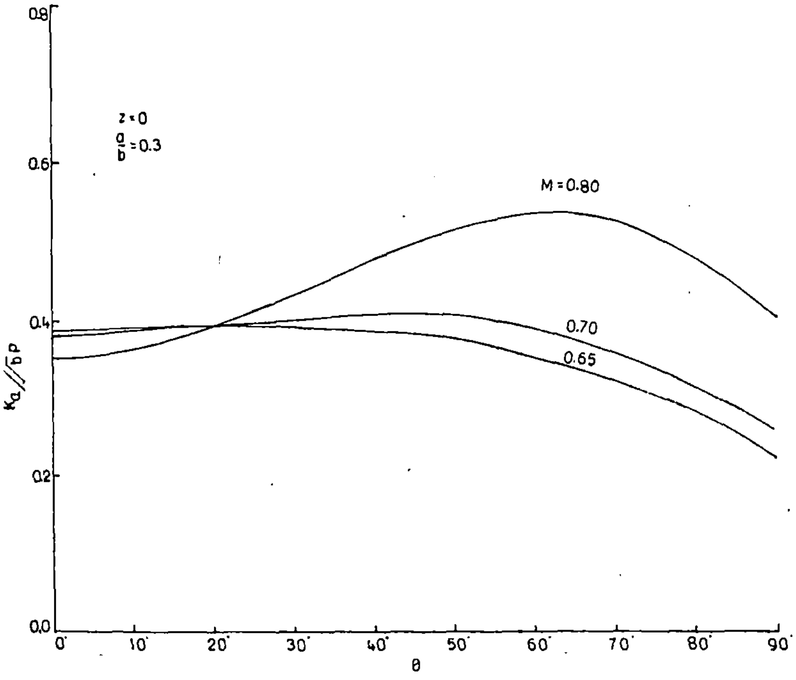
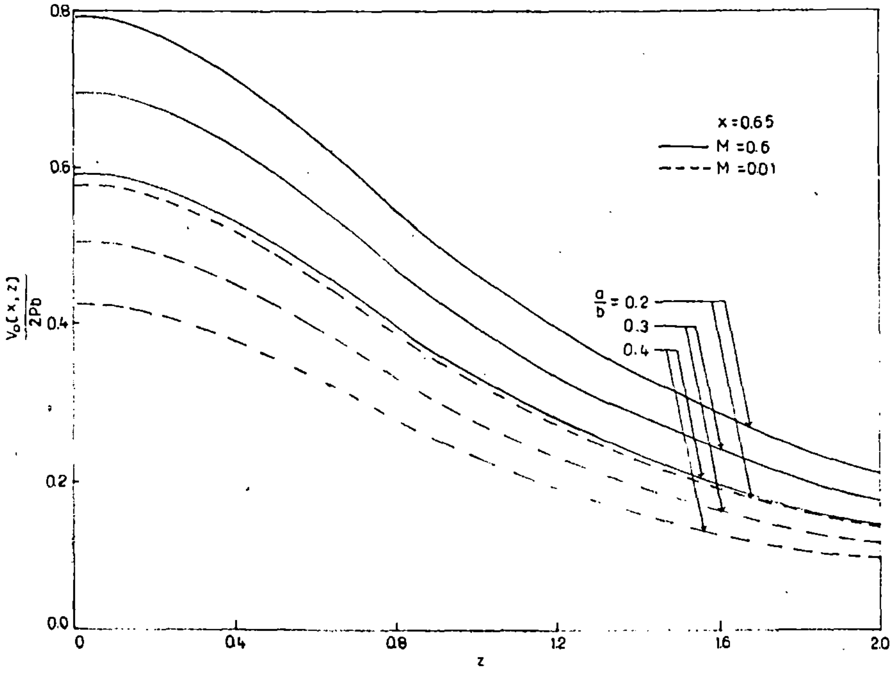

z =0
M =0.01
a/b =0.2
=0.3
=0.4
$K_a/\sqrt{b}P$
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0°
10°
20°
30°
40°
50°
60°
70°
80°
90°
θ

Fig. 11.



0.8
0.6
0.4
0.2
0.0
$K_a/\sqrt{b}p$
$z=0$
$\frac{a}{b}=0.3$
M = 0.80
0.70
0.65
0°
10°
20°
30°
40°
50°
60°
70°
80°
90°
θ

Fig. 12.



0.8
0.6
0.4
0.2
0.0
$\frac{V_0(x,z)}{2Pb}$
x = 0.65
M = 0.6
M = 0.01
$\frac{a}{b}$ = 0.2
0.3
0.4
0
0.4
0.8
1.2
1.6
2.0
z

Fig. 13.

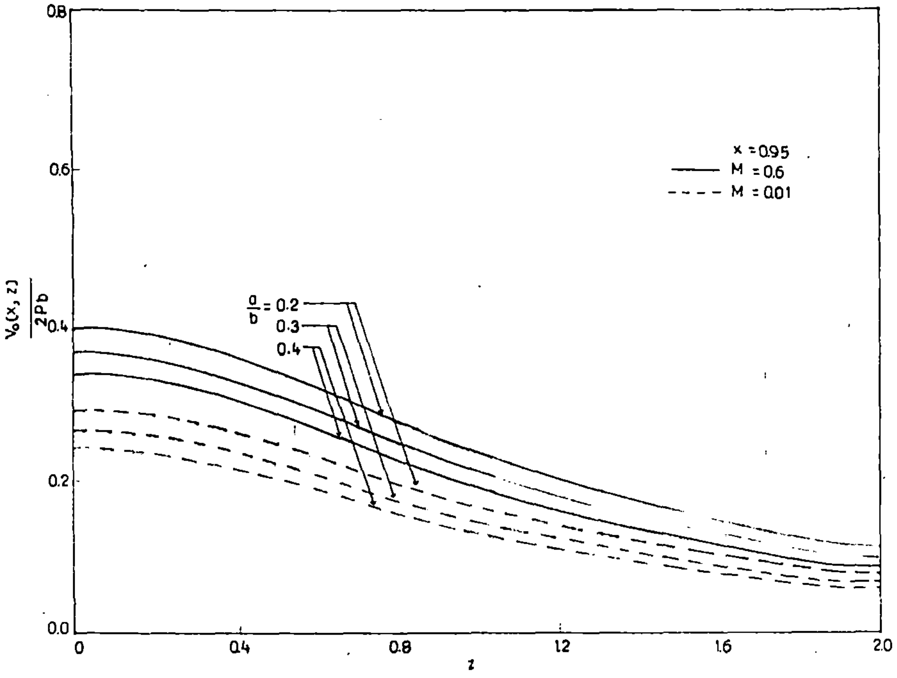

0.8
0.6
0.4
0.2
0.0
$\frac{V_0(x,z)}{2Pb}$
0
0.4
0.8
1.2
1.6
2.0
z
x = 0.95
M = 0.6
M = 0.01
$\frac{a}{b}$ = 0.2
0.3
0.4

Fig. 14.

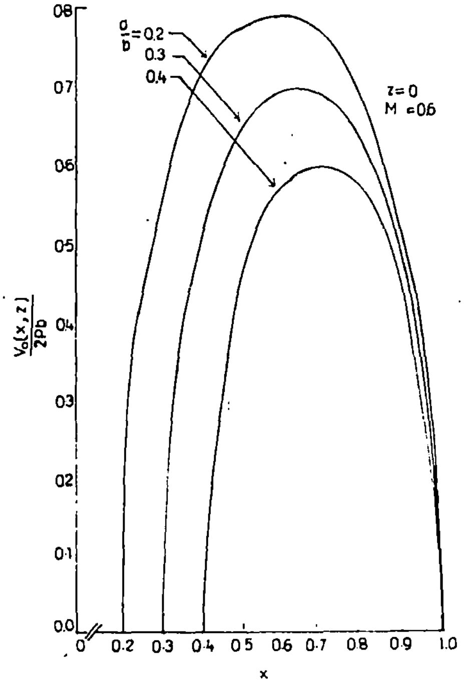

0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
$\frac{V_0(x,z)}{2Pb}$
0
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
x
$\frac{a}{b}$ = 0.2
0.3
0.4
z = 0
M = 0.6

Fig. 15.

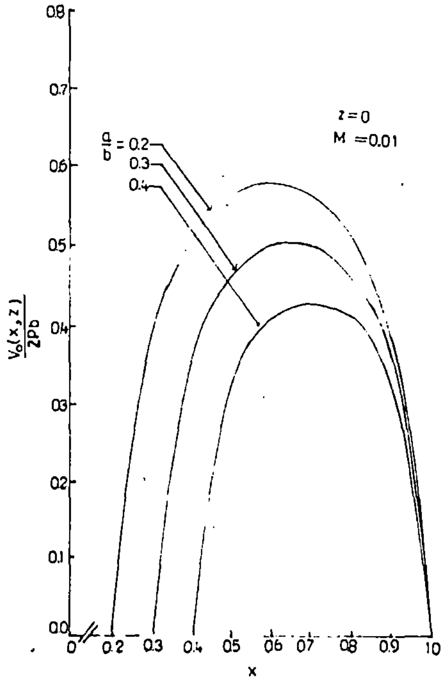


Fig. 16.

References

1. E.H. Yoffe, *Philosophical Magazine* 42 (1951) 739.
2. G.C. Sih and E.P. Chen, *International Journal of Engineering Science* 10 (1972) 537.
3. M.K. Kassir and S. Tse, *International Journal of Engineering Science* 21 (1983) 315.
4. J. De and B. Patra, *International Journal of Engineering Science* 28 (1990) 809.
5. Y.C. Angel and J.D. Achenbach, *Journal of Elasticity* 15 (1985) 89.
6. L.B. Freund, *International Journal of Solids and Structures* 7 (1971) 1199.
7. S. Itou, *International Journal of Engineering Science* 17 (1979) 59.
8. D.L. Jain and R.P. Kanwal, *International Journal of Solids and Structures* 8 (1972) 961.
9. K.N. Srivastava and M. Lowengrub, *Proceedings of Royal Society of Edinburgh* (1968).
10. S. Itou, *International Journal of Solids and Structures* 16 (1980) 1147.
11. P.M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Vol. 1, McGraw-Hill, New York (1958).
12. I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press (1965).

International Journal of Fracture **59**: 291–306, 1993.
© 1993 *Kluwer Academic Publishers. Printed in the Netherlands.*

Three co-planar moving Griffith cracks in an infinite elastic medium

A.N. DAS
Department of Mathematics, North Bengal University, Darjeeling, West Bengal-734 430, India

Received 18 February 1992; accepted in revised form 20 August 1992

Abstract. The dynamic in-plane problem of determining the stress and displacement due to three co-planar Griffith cracks moving steadily at a subsonic speed in a fixed direction in an infinite, isotropic, homogeneous medium under normal stress has been treated. The static problem of determining the stress and displacement around three co-planar Griffith cracks in an infinite isotropic elastic medium has also been considered. In both the cases, employing Fourier integral transform, the problems have been reduced to solving a set of four integral equations. These integral equations have been solved using finite Hilbert transform technique and Cook's result [16] to obtain the exact form of crack opening displacement and stress intensity factors which are presented in the form of graphs.

1. Introduction

In fracture mechanics, scattering of elastic waves by cracks of finite dimension in an infinite elastic medium has been examined by several investigators. The problem of scattering of elastic waves from an interface crack was solved by Bostrom [1]. Srivastava et al. [2] solved the problem of interaction of an anti-plane shear wave by an interface crack. The problem of diffraction of Love waves by a crack of finite width in the plane interface of a layered composite has been solved by Neerhoff [3]. Itou [4] solved the problem of diffraction of an anti-plane shear wave by two co-planar Griffith cracks in an infinite elastic medium. The scattering of a time harmonic normally incident plane wave by two co-planar Griffith cracks was solved by Jain and Kanwal [5]. Itou [6] also solved the problem of stress concentration around two co-planar Griffith cracks in an infinite elastic medium. Yoffe [7] considered the problem of propagation of a crack of fixed length at a constant speed through a stretched isotropic elastic solid of infinite extent. The problem of diffraction of horizontal shear waves by a moving interface crack has been solved by Nishida et al. [8]. Recently Kassir and Tse [9] have solved the plane stress problem of a moving Griffith crack in an infinite orthotropic stressed medium by using integral transform technique and the same technique has been employed by De and Patra [10] to solve Yoffe's problem in a stressed orthotropic strip of finite thickness. Several problems on two moving co-planar Griffith cracks have been solved by Das and Ghosh [11–13].

As regards the crack problem, research has been restricted mainly to the case of the single crack or a pair of cracks because of severe mathematical complexity encountered in solving the problems of three or more cracks. Recently, Dhawan and Dhaliwal [14] solved the statical problem of determining the stress distribution in an infinite transversely isotropic medium containing three co-planar cracks.

To the best knowledge of the author, the problem of stress distribution around three co-planar moving Griffith cracks in an infinite isotropic elastic medium has not been investigated so far. In this paper, two cases regarding stress distribution around three co-planar Griffith cracks in an infinite homogeneous, isotropic medium have been investigated. In the first case, cracks are assumed to be moving steadily along a fixed direction with constant velocity V. In the second case, the statical problem of determining the stress and displacement in an infinite homogeneous, isotropic medium weakened by three co-planar Griffith cracks has been considered. Using Fourier intergral transform both the problems have been reduced to solving a set of four integral equations. Employing finite Hilbert transform technique [15] and Cook's result [16] the integral equations have been solved to derive crack opening displacement and stress intensity factors which are presented in the form of graphs.

2. Statement of Problem I and its formulation

Consider an infinite homogeneous isotropic material weakened by three co-planar Griffith cracks, moving steadily at a constant velocity V in the X-direction referred to a fixed coordinate system (X, Y, Z) as shown in the Fig. 1. In the absence of body force equations of motion in terms of displacement are

$$(\lambda + 2\mu)[u_{,XX} + v_{,XY}] + \mu[u_{,YY} - v_{,XY}] = \rho u_{,TT}, \tag{2.1}$$

$$(\lambda + 2\mu)[u_{,XY} + v_{,YY}] + \mu[v_{,XX} - u_{,XY}] = \rho v_{,TT}, \tag{2.2}$$

where u, v denote the displacement components in X and Y directions and λ, μ are Lame's constants and $u_{,X}$ represents partial derivatives of u with respect to X.

For cracks moving steadily with constant velocity V in the X-direction it is convenient to introduce the Galilean transformation

$$x = X - VT, \quad y = Y, \quad z = Z, \quad t = T, \tag{2.3}$$

where (x, y, z) represents the translating coordinate system as shown in Fig. 1.

Let the positions of the coplanar Griffith cracks referred to the translating coordinates (x, y, z) be $-a < x < a$, $-c < x < -b$, and $b < x < c$ on $y = 0$.

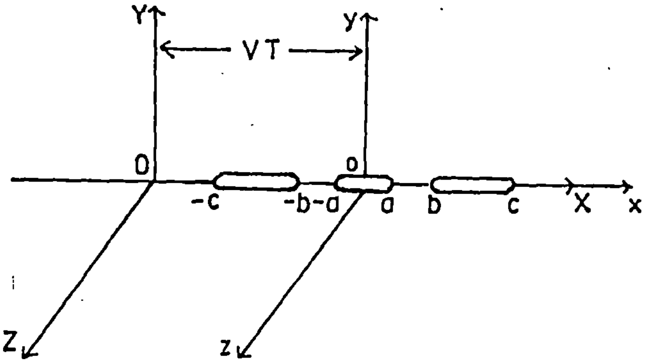


Fig. 1. Geometry and coordinate system.

In the moving coordinates, the equations of motion (2.1) and (2.2) become independent of time and take the form

$$\begin{aligned} &(\lambda + 2\mu - \rho V^2)u_{,xx} + (\lambda + \mu)v_{,xy} + \mu u_{,yy} = 0 \\ &(\lambda + 2\mu)v_{,yy} + (\mu - \rho V^2)v_{,xx} + (\lambda + \mu)u_{,xy} = 0. \end{aligned} \tag{2.4}$$

The cracks are assumed to be moving steadily in an infinite medium subjected to a homogeneous stress such that the state of stress at infinity is given by $\sigma_{yy}^{\infty} = p$, $\sigma_{xx}^{\infty} = \sigma_{xy}^{\infty} = 0$.

For symmetry about the x-axis, only a half-plane need be considered.

The stress conditions at $y = \infty$ can all be made zero by superposing the simple static problem $\sigma_{yy}^{\infty} = -p$, $\sigma_{xx}^{\infty} = \sigma_{xy}^{\infty} = 0$.

The boundary conditions of the resulting dynamic problem are in terms of moving coordinates.

$$\begin{aligned} &v = 0; && y = 0, \quad a \leqslant |x| \leqslant b, \quad |x| \geqslant c, \\ &\sigma_{xy} = 0; && y = 0, \quad |x| < \infty, \\ &\sigma_{yy} = -p; && y = 0, \quad |x| < a, \qquad b < |x| < c. \end{aligned} \tag{2.5}$$

In view of the symmetry of the proposed problem with respect to y-axis, we introduce

$$\bar{u}_s(\xi, y) = \int_0^{\infty} u(x, y)\sin(\xi x)\,dx,$$

$$\bar{v}_c(\xi, y) = \int_0^{\infty} v(x, y)\cos(\xi x)\,dx$$

and

$$u(x, y) = \frac{2}{\pi}\int_0^{\infty} \bar{u}_s(\xi, y)\sin(\xi x)\,d\xi,$$

$$v(x, y) = \frac{2}{\pi}\int_0^{\infty} \bar{v}_c(\xi, y)\cos(\xi x)\,d\xi$$

in (2.4) so that equations given by (2.4) reduce to

$$\begin{aligned} &\mu\bar{u}_{s,yy} - \xi(\lambda + \mu)\bar{v}_{c,y} - \xi^2(\lambda + 2\mu - \rho V^2)\bar{u}_s = 0, \\ &(\lambda + 2\mu)\bar{v}_{c,yy} + \xi(\lambda + \mu)\bar{u}_{s,y} - \xi^2(\mu - \rho V^2)\bar{v}_c = 0. \end{aligned} \tag{2.6}$$

Elimination of $\bar{u}_s$ from (2.6) yields the following ordinary differential equation

$$\left[\left\{\frac{d^2}{dy^2} - (1 - M^2k^2)\xi^2\right\}\left\{\frac{d^2}{dy^2} - (1 - M^2)\xi^2\right\}\right]\bar{v}_c = 0, \tag{2.7}$$

where $M = V/c_2$, $k = c_2/c_1$.

The solution of the differential equation given by (2.7), for $y \geqslant 0$, is

$$\bar{v}_s(\xi, y) = A(\xi)\, \mathrm{e}^{-\xi y\sqrt{1-M^2k^2}} + B(\xi)\, \mathrm{e}^{-\xi y\sqrt{1-M^2}}, \tag{2.8}$$

where the unknown functions $A(\xi)$ and $B(\xi)$ are to be determined using the boundary conditions of the proposed problem.

Employing (2.8) in (2.6) it can be shown that

$$\bar{u}_s(\xi, y) = \frac{A(\xi)}{\sqrt{1-M^2k^2}}\, \mathrm{e}^{-\xi y\sqrt{1-M^2k^2}} + \sqrt{1-M^2}\, B(\xi)\, \mathrm{e}^{-\xi y\sqrt{1-M^2}}, \quad y \geqslant 0. \tag{2.9}$$

Therefore, the stress components given by

$$\begin{aligned} \sigma_{yy} &= \lambda(u_{,x} + v_{,y}) + 2\mu v_{,y}, \\ \sigma_{xy} &= \mu(u_{,y} + v_{,x}) \end{aligned} \tag{2.10}$$

become

$$\begin{aligned} \sigma_{yy}(x, y) = -\frac{2\mu}{\pi}\int_0^\infty \xi\Bigg[&\frac{2-M^2}{\sqrt{1-M^2k^2}} A(\xi)\, \mathrm{e}^{-\xi y\sqrt{1-M^2k^2}} \\ &+ 2\sqrt{1-M^2}\, B(\xi)\, \mathrm{e}^{-\xi y\sqrt{1-M^2}}\Bigg]\cos(\xi x)\, \mathrm{d}\xi, \end{aligned} \tag{2.11}$$

$$\sigma_{xy}(x, y) = -\frac{2\mu}{\pi}\int_0^\infty \xi\left[2A(\xi)\, \mathrm{e}^{-\xi y\sqrt{1-M^2k^2}} + (2-M^2)B(\xi)\, \mathrm{e}^{-\xi y\sqrt{1-M^2}}\right]\sin(\xi x)\, \mathrm{d}\xi,$$

with

$$u(x, y) = \frac{2}{\pi}\int_0^\infty \left[\frac{A(\xi)}{\sqrt{1-M^2k^2}}\, \mathrm{e}^{-\xi y\sqrt{1-M^2k^2}} + \sqrt{1-M^2}\, B(\xi)\, \mathrm{e}^{-\xi y\sqrt{1-M^2}}\right]\sin(\xi x)\, \mathrm{d}\xi,$$

and

$$v(x, y) = \frac{2}{\pi}\int_0^\infty \left[A(\xi)\, \mathrm{e}^{-\xi y\sqrt{1-M^2k^2}} + B(\xi)\, \mathrm{e}^{-\xi y\sqrt{1-M^2}}\right]\cos(\xi x)\, \mathrm{d}\xi. \tag{2.12}$$

On account of symmetry with respect to the y-axis the boundary conditions (2.5) can be rewritten as

$$v(x, 0) = 0, \quad x \in I_2, I_4, \tag{2.13}$$

$$\sigma_{xy}(x, 0) = 0, \quad 0 < x < \infty, \tag{2.14}$$

$$\sigma_{yy}(x, 0) = -p, \quad x \in I_1, I_3, \tag{2.15}$$

where $I_1 = (0, a)$, $I_2 = (a, b)$, $I_3 = (b, c)$, $I_4 = (c, \infty)$.

Using the condition (2.14) in (2.11.2) it is found that $A(\xi)$, $B(\xi)$ are related by

$$B(\xi) = -\frac{2}{2 - M^2} A(\xi). \tag{2.16}$$

With the help of the boundary condition (2.13), Eqn. (2.12.2) reduces to

$$\int_0^\infty A(\xi)\cos(\xi x)\,\mathrm{d}\xi = 0, \quad x \in I_2, I_4. \tag{2.17}$$

Substitution of (2.11.1) in (2.15) yields, with the aid of (2.16)

$$\int_0^\infty \xi A(\xi)\cos(\xi x)\,\mathrm{d}\xi = \frac{P\pi}{2\mu}, \quad x \in I_1, I_3, \tag{2.18}$$

where

$$P = \frac{p}{K}, \quad K = \frac{(2 - M^2)^2 - 4\sqrt{(1 - M^2k^2)(1 - M^2)}}{(2 - M^2)\sqrt{1 - M^2k^2}}.$$

3. Method of solution

In order to solve the set of four integral equations given in (2.17) and (2.18) let us take

$$A(\xi) = \frac{1}{\xi}\int_0^a h(s)\sin(\xi s)\,\mathrm{d}s + \frac{1}{\xi}\int_b^c g(t^2)\sin(\xi t)\,\mathrm{d}t, \tag{3.1}$$

where $h(s)$ and $g(t^2)$ are unknown functions to be determined from the boundary conditions.

Inserting the value of $A(\xi)$ from (3.1) in (2.17) and using the following result [17]

$$\int_0^\infty \frac{\sin(\xi x)\cos(\xi y)}{\xi}\,\mathrm{d}\xi = \begin{cases} \frac{1}{2}\pi, & x > y > 0 \\ \frac{1}{4}\pi, & x = y > 0, \\ 0, & y > x > 0 \end{cases}$$

it is found that this choice of $A(\xi)$ leads to the equation

$$\int_b^c g(t^2)\,\mathrm{d}t = 0. \tag{3.2}$$

Further substitution of $A(\xi)$ from (3.1) in (2.18.1) yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^a h(s)\log\left|\frac{s+x}{s-x}\right|\mathrm{d}s+\frac{\mathrm{d}}{\mathrm{d}x}\int_b^c g(t^2)\log\left|\frac{t+x}{t-x}\right|\mathrm{d}t=\frac{\pi P}{\mu},\quad x\in I_1.$$

Rewriting this equation as

$$\int_0^a h(s)\log\left|\frac{s+x}{s-x}\right|\mathrm{d}s=\pi F(x),\quad x\in I_1,$$

where

$$F(x)=\int_0^x\left[\frac{P}{\mu}-\frac{2}{\pi}\int_c^d\frac{tg(t^2)}{t^2-x'^2}\mathrm{d}t\right]\mathrm{d}x',$$

and using Cook's result [16] it is found that

$$h(s)=\frac{P}{\mu}\frac{s}{\sqrt{a^2-s^2}}-\frac{2}{\pi}\frac{s}{\sqrt{a^2-s^2}}\int_b^c\frac{\sqrt{t^2-a^2}g(t^2)}{t^2-s^2}\mathrm{d}t, \tag{3.3}$$

where the result

$$\int_0^a\sqrt{a^2-x^2}\,\frac{\mathrm{d}x}{(s^2-x^2)(t^2-x^2)}=\tfrac{1}{2}\pi\frac{\sqrt{t^2-a^2}}{t}\,\frac{1}{t^2-s^2}$$

has been used.

Substituting the value of $h(s)$ from (3.3) in (3.1) and using the resulting value of $A(\xi)$ in (2.18.2) and using the result

$$\int_0^a\frac{1}{\sqrt{a^2-s^2}}\,\frac{s^2\,\mathrm{d}s}{(s^2-x^2)(t^2-s^2)}=\tfrac{1}{2}\pi\left[\frac{t}{\sqrt{t^2-a^2}}-\frac{x}{\sqrt{x^2-a^2}}\right]\frac{1}{t^2-x^2},\quad\text{for } x\in I_3,$$

it can be shown that $g(t^2)$ is the solution of the singular integral equation

$$\int_b^c\frac{\sqrt{t^2-a^2}}{t^2-x^2}g(t^2)\,\mathrm{d}t=\frac{\pi P}{2\mu},\quad x\in I_3.$$

Using finite Hilbert transform technique [15] the solution of this integral equation is obtained as

$$g(t^2)=\frac{P}{\mu}\sqrt{\frac{t^2(t^2-b^2)}{(t^2-a^2)(c^2-t^2)}}+\frac{tC_1}{\sqrt{(t^2-a^2)(t^2-b^2)(c^2-t^2)}}, \tag{3.4}$$

the constant C_1 is to be determined using the condition given by (3.2).

Next substituting the value of $g(t^2)$ from (3.4) in (3.3) and finally using the following results

$$\int_b^c \sqrt{\frac{t^2-b^2}{c^2-t^2}}\,\frac{t\,\mathrm{d}t}{(t^2-s^2)} = \tfrac{1}{2}\pi\left[1-\sqrt{\frac{b^2-s^2}{c^2-s^2}}\right]$$

$$\int_b^c \frac{t\,\mathrm{d}t}{(t^2-s^2)\sqrt{(t^2-b^2)(c^2-t^2)}} = \frac{\pi}{2\sqrt{(c^2-s^2)(b^2-s^2)}}, \quad \text{for } s\in I_1,$$

$h(s)$ is derived in the form

$$h(s) = \frac{P}{\mu}\sqrt{\frac{s^2(b^2-s^2)}{(a^2-s^2)(c^2-s^2)}} - \frac{sC_1}{\sqrt{(a^2-s^2)(b^2-s^2)(c^2-s^2)}}. \tag{3.5}$$

Now insertion of (3.4) in condition (3.2) yields

$$C_1 = -\frac{P}{\mu}\left[(c^2-a^2)\frac{E(\frac{1}{2}\pi,l)}{F(\frac{1}{2}\pi,l)} - (b^2-a^2)\right], \tag{3.6}$$

where $F(\phi,l)$ and $E(\phi,l)$ are elliptic integrals of first kind and second kind respectively and $l=\sqrt{(c^2-b^2)/(c^2-a^2)}$.

The relevant displacement and stress components in the plane of the crack can now be shown to be given by

$$\begin{aligned} v(x,0) &= \int_x^a h(s)\,\mathrm{d}s, \quad 0\leqslant x\leqslant a,\\ &= \int_x^c g(t^2)\,\mathrm{d}t, \quad b\leqslant x\leqslant c \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} [\sigma_{yy}(x,0)]_{a<x<b} &= \frac{2\mu K}{\pi}\left[\int_0^a \frac{sh(s)}{x^2-s^2}\,\mathrm{d}s - \int_b^c \frac{tg(t^2)}{t^2-x^2}\,\mathrm{d}t\right],\\ [\sigma_{yy}(x,0)]_{x>c} &= \frac{2\mu K}{\pi}\left[\int_0^a \frac{sh(s)}{x^2-s^2}\,\mathrm{d}s + \int_b^c \frac{tg(t^2)}{x^2-t^2}\,\mathrm{d}t\right]. \end{aligned} \tag{3.8}$$

Insertion of the values of $h(s)$ and $g(t^2)$ as given by (3.5) and (3.4) in the expressions (3.8) yields after some algebraic manipulation,

$$\begin{aligned} [\sigma_{yy}(x,0)]_{a<x<b} &= \frac{2\mu K}{\pi}[F_1(x)-F_2(x)+F_3(x)-F_5(x)-F_6(x)],\\ [\sigma_{yy}(x,0)]_{x>c} &= \frac{2\mu K}{\pi}[F_1(x)-F_2(x)+F_4(x)-F_5(x)+F_6(x)], \end{aligned} \tag{3.9}$$

where

$$F_1(x) = \left[\frac{P}{\mu}(b^2 - a^2) - C_1\right]\left[\sqrt{\frac{x^2}{x^2 - a^2}} - 1\right]\frac{\pi}{2\sqrt{(c^2 - a^2)(b^2 - a^2)}},$$

$$F_2(x) = \int_0^a \left[\frac{P}{\mu}(c^2 - b^2) - C_1\,\frac{2u^2 - b^2 - c^2}{b^2 - u^2}\right]\frac{g_1(u, x)}{c^2 - u^2}\,\mathrm{d}u,$$

$$F_{3,4}(x) = \left\{\frac{P}{\mu}\left[\sqrt{\frac{b^2 - x^2}{c^2 - x^2}} - 1\right] \mp \frac{C_1}{\sqrt{(c^2 - x^2)(b^2 - x^2)}}\right\}\frac{\pi c}{2\sqrt{c^2 - a^2}},$$

$$F_5(x) = \frac{P}{\mu}a^2\int_b^c\left[\tan^{-1}\sqrt{\frac{v^2 - b^2}{c^2 - v^2}} - \sqrt{\frac{b^2 - x^2}{c^2 - x^2}}\tan^{-1}\sqrt{\frac{(c^2 - x^2)(v^2 - b^2)}{(b^2 - x^2)(c^2 - v^2)}}\right]\frac{\mathrm{d}v}{\sqrt{(v^2 - a^2)^3}},$$

$$F_6(x) = \frac{a^2 C_1}{\sqrt{(c^2 - x^2)(b^2 - x^2)}}\int_b^c \frac{\tan^{-1}\sqrt{\dfrac{(u^2 - b^2)(x^2 - c^2)}{(c^2 - u^2)(x^2 - b^2)}}}{\sqrt{(u^2 - a^2)^3}}\,\mathrm{d}u,$$

$$g_1(u, x) = \frac{u}{\sqrt{(b^2 - u^2)(c^2 - u^2)}}\left[\sin^{-1}\left(\frac{u}{a}\right) - \frac{x}{\sqrt{x^2 - a^2}}\tan^{-1}\sqrt{\frac{(x^2 - a^2)u^2}{(a^2 - u^2)x^2}}\right]. \tag{3.10}$$

The dynamic stress intensity factors are given by

$$\begin{aligned} N_a &= \operatorname*{Lt}_{x \to a+} \sqrt{2(x - a)}[\sigma_{yy}(x, 0)]_{a<x<b}, \\ N_b &= \operatorname*{Lt}_{x \to b-} \sqrt{2(b - x)}[\sigma_{yy}(x, 0)]_{a<x<b}, \\ N_c &= \operatorname*{Lt}_{x \to c+} \sqrt{2(x - c)}[\sigma_{yy}(x, 0)]_{x>c}. \end{aligned} \tag{3.11}$$

Employing (3.9) in (3.11) it can be shown that

$$N_a = p\sqrt{a}\sqrt{\frac{c^2 - a^2}{b^2 - a^2}}\,\frac{E(\frac{1}{2}\pi, l)}{F(\frac{1}{2}\pi, l)},$$

$$N_b = \frac{p\sqrt{b}}{\sqrt{(c^2 - b^2)(b^2 - a^2)}}\left[(c^2 - a^2)\frac{E(\frac{1}{2}\pi, l)}{F(\frac{1}{2}\pi, l)} - (b^2 - a^2)\right],$$

$$N_c = p\sqrt{c}\sqrt{\frac{c^2 - a^2}{c^2 - b^2}}\left[1 - \frac{E(\frac{1}{2}\pi, l)}{F(\frac{1}{2}\pi, l)}\right].$$

Now using the values of $h(s)$ and $g(t^2)$ from (3.5) and (3.4) in the expressions given by (3.7) displacements on the cracks are obtained as

$$[v(x,0)]_{0\leqslant x\leqslant a} = \frac{P}{\mu}\sqrt{c^2 - a^2}F(\beta, l)\left[\frac{E(\frac{1}{2}\pi, l)}{F(\frac{1}{2}\pi, l)} - \frac{E(\beta, l)}{F(\beta, l)}\right] + \frac{P}{\mu}\frac{\sqrt{(c^2 - x^2)(a^2 - x^2)}}{\sqrt{b^2 - x^2}},$$

$$[v(x,0)]_{b\leqslant x\leqslant c} = \frac{P}{\mu}\sqrt{c^2 - a^2}F(\lambda, l)\left[\frac{E(\lambda, l)}{F(\lambda, l)} - \frac{E(\frac{1}{2}\pi, l)}{F(\frac{1}{2}\pi, l)}\right],$$

where

$$\sin\lambda = \sqrt{\frac{c^2 - x^2}{c^2 - b^2}} \quad \text{and} \quad \sin\beta = \sqrt{\frac{a^2 - x^2}{b^2 - x^2}}.$$

It is interesting to note that the crack opening displacements depend on the crack velocity V but in the plane of the cracks the stresses and stress intensity factors are independent of the velocity of the moving cracks in an infinite elastic medium.

4. Statement of Problem II and its formulation

In this case, consider an infinite homogeneous isotropic material with three coplanar Griffith cracks, located at $Y = 0$, $-a \leqslant X \leqslant a$, $b \leqslant |X| \leqslant c$ and subjected to uniform internal pressure q. In the absence of body force equations of equilibrium in terms of displacement are

$$(\lambda + 2\mu)[u_{,XX} + v_{,XY}] + \mu[u_{,YY} - v_{,XY}] = 0$$

and

$$(\lambda + 2\mu)[u_{,XY} + v_{,YY}] + \mu[v_{,XX} - u_{,XY}] = 0. \tag{4.1}$$

Since the problem exhibits a state of symmetry abut $Y = 0$, attention can be made to a single half-space occupying the region $Y \geqslant 0$.

Equations (4.1) are to be solved subject to the boundary conditions

$$v(X,0) = 0, \qquad a \leqslant |X| \leqslant b, \quad |X| \geqslant c, \tag{4.2}$$

$$\sigma_{XY}(X,0) = 0, \qquad -\infty < X < \infty, \tag{4.3}$$

$$\sigma_{YY}(X,0) = -q, \quad |X| \leqslant a, \quad b \leqslant |X| \leqslant c. \tag{4.4}$$

In view of the boundary conditions, appropriate integral solutions of (4.1) are

$$u(X,Y) = \frac{2}{\pi}\int_0^\infty \left[C(\xi) + D(\xi)\left\{Y - \frac{1}{\xi}\frac{\lambda + 3\mu}{\lambda + \mu}\right\}\right]e^{-\xi Y}\sin(\xi X)\,d\xi$$

and

$$v(X, Y) = \frac{2}{\pi} \int_0^\infty [C(\xi) + Y D(\xi)] \, e^{-\xi Y} \cos(\xi X) \, d\xi. \tag{4.5}$$

Therefore,

$$\sigma_{YY}(X, Y) = -\frac{4\mu}{\pi} \int_0^\infty \left[\xi C(\xi) + \left\{ Y\xi - \frac{\mu}{\lambda + \mu} \right\} D(\xi) \right] e^{-\xi Y} \cos(\xi X) \, d\xi,$$

$$\sigma_{XY}(X, Y) = -\frac{4\mu}{\pi} \int_0^\infty \left[\xi C(\xi) + \left\{ Y\xi - \frac{\lambda + 2\mu}{\lambda + \mu} \right\} D(\xi) \right] e^{-\xi Y} \sin(\xi X) \, d\xi. \tag{4.6}$$

It may be noted that the displacement and stress components given by (4.5) and (4.6) can not be derived from the corresponding expressions of the dynamic problem given in (2.12) and (2.11) on setting $M = 0$.

The functions $C(\xi)$ and $D(\xi)$ are to be determined from the boundary conditions (4.2)–(4.4), which yield

$$C(\xi) = \frac{1}{\xi} \frac{\lambda + 2\mu}{\lambda + \mu} D(\xi) \tag{4.7}$$

and the following set of four integral equations

$$\int_0^\infty C(\xi) \cos(\xi X) \, d\xi = 0, \quad X \in I_2, I_4, \tag{4.8}$$

$$\int_0^\infty \xi C(\xi) \cos(\xi X) \, d\xi = \frac{Q\pi}{2\mu}, \quad X \in I_1, I_3, \tag{4.9}$$

where $Q = ((\lambda + 2\mu)/2(\lambda + \mu))q$ and I_j $(j = 1, 2, 3, 4)$ are the intervals defined earlier in Problem I.

5. Method of solution and quantities of physical interest

Integral equations given by (4.8) and (4.9) are found to be the same as given by (2.17) and (2.18) with the exception that P is replaced by Q. Therefore, the same technique as that used in Problem I can be employed to obtain

$$[v(X,0)]_{0 \leq X \leq a} = \frac{Q}{\mu} \sqrt{c^2 - a^2} F(\beta', l) \left[\frac{E(\frac{1}{2}\pi, l)}{F(\frac{1}{2}\pi, l)} - \frac{E(\beta', l)}{F(\beta', l)} \right] + \frac{Q}{\mu} \frac{\sqrt{(c^2 - X^2)(a^2 - X^2)}}{\sqrt{b^2 - X^2}},$$

$$[v(X,0)]_{b \leq X \leq c} = \frac{Q}{\mu} \sqrt{c^2 - a^2} F(\lambda', l) \left[\frac{E(\lambda', l)}{F(\lambda', l)} - \frac{E(\frac{1}{2}\pi, l)}{F(\frac{1}{2}\pi, l)} \right], \tag{5.1}$$

where

$$\sin \lambda' = \sqrt{\frac{c^2 - X^2}{c^2 - b^2}} \quad \text{and} \quad \sin \beta' = \sqrt{\frac{a^2 - X^2}{b^2 - X^2}}.$$

Stresses in the regions $a < X < b$, $X > c$ are found to be the same as that given in (3.9), the only change being that P is to be replaced by Q.

6. Numerical results and discussions

Numerical results for the stress intensity factors and crack opening displacement, defined as $\Delta v(x, 0) = v(x, 0^+) - v(x, 0^-)$, for different values of the parameters and $\lambda = \mu$ are presented in this section. Numerical calculations have been carried out for both the dynamic and static problems. As the crack velocity is less than Rayleigh wave velocity, it is reasonable to take the value of M less than 0.9194.

Problem I: Variations of crack opening displacement for different values of crack speed, crack lengths and the separating distance between the cracks have been plotted in Figs. 2–4. It is interesting to note from Fig. 2 that crack opening displacement on both the cracks decreases with the increase in the value of M at the onset and takes its minimum value at $M = 0.7415$, after which it increases with the increase in the value of M. It has also been depicted in Figs. 3–4 that on each of the cracks, crack opening displacement decreases as the crack length decreases.

It has been mentioned earlier that the stress intensity factors at the crack tips are independent of crack speed and are found to depend on the crack lengths and the separating distance between the cracks. Variation of stress intensity factors with a/b for different values of c/b, and that with b/a for different values c/a are plotted in Fig. 5 and Fig. 6 respectively.

It has been found from these graphs that when the separating distance between the inner crack and outer pair of cracks decreases the variations of stress intensity factors at the tips $x = a$ and $x = b$ become more prominent than at the edge $x = c$. Figure 7 shows that the stress intensity factors at the edges of the inner crack and outer pair of cracks increases as the length of the outer pair of cracks increases, keeping the separating distance between the inner crack and outer pair of cracks fixed.

Problem II: Figure 8 shows the variations of crack opening displacement for different values of the parameters a/b, c/b. They exhibit that crack opening displacement on a crack of fixed length increases with the increase in the length of the other crack as expected from the physical standpoint.

Acknowledgment

The author takes this opportunity to thank the referees for their valuable suggestions for the improvement of the paper and the author also gratefully acknowledges the financial support of U.G.C. to conduct this research work.

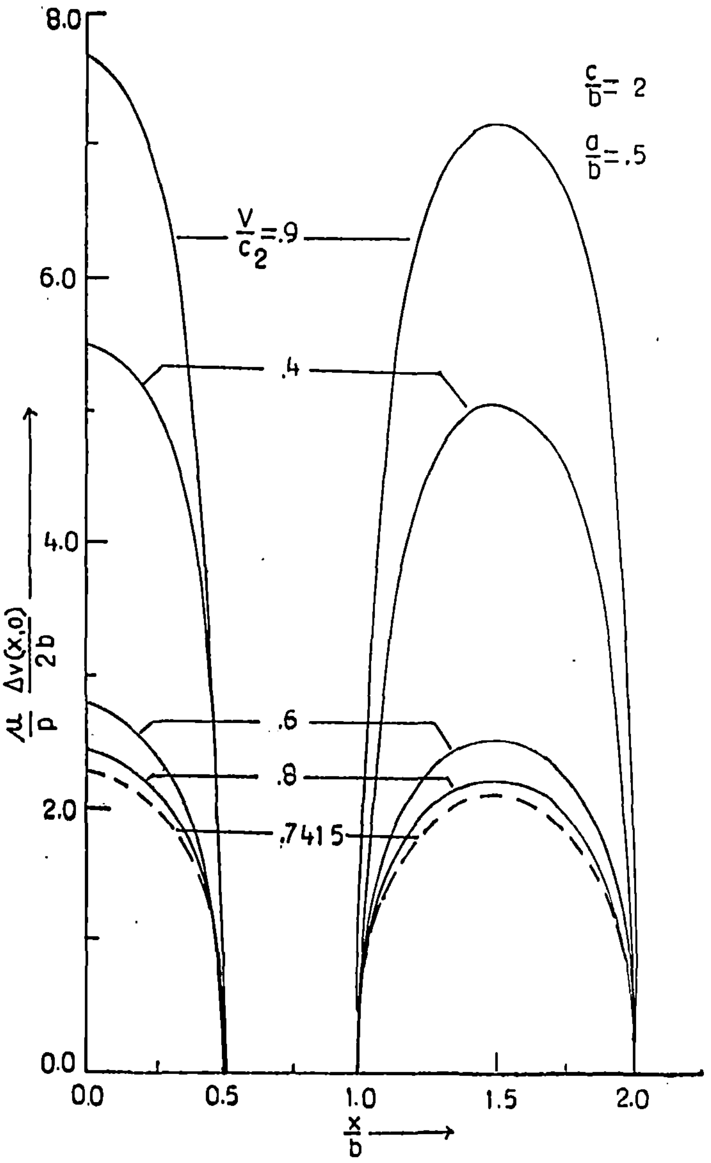


Fig. 2. Variation of crack opening displacement with x/b on both the cracks for the problem I.

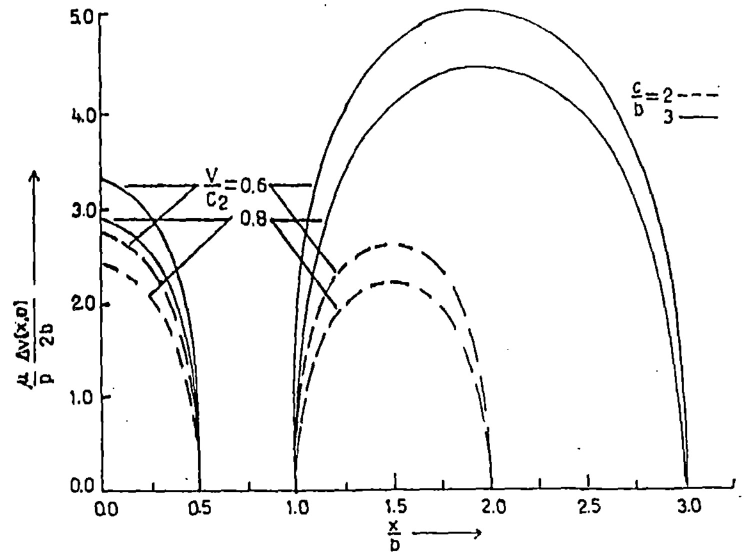


Fig. 3. Variation of crack opening displacement with x/b on both the cracks for the problem I.

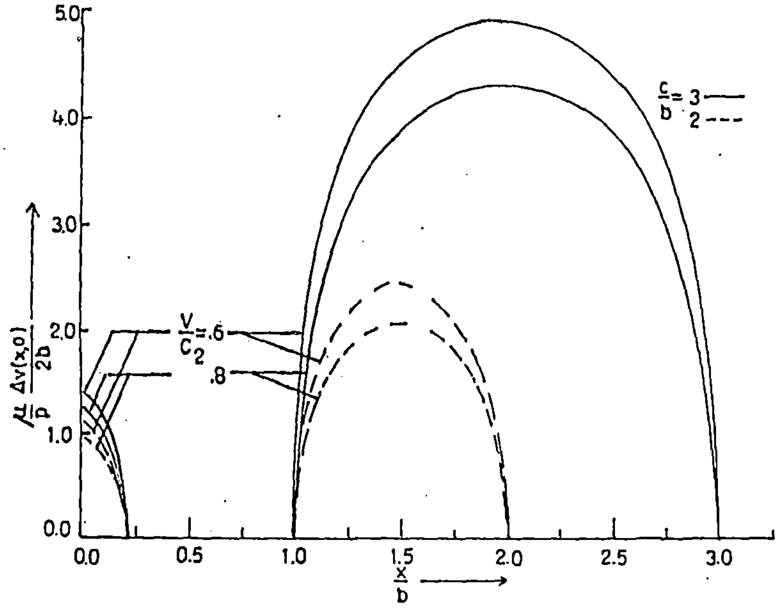


Fig. 4. Variation of crack opening displacement with x/b on both the cracks for the problem I.

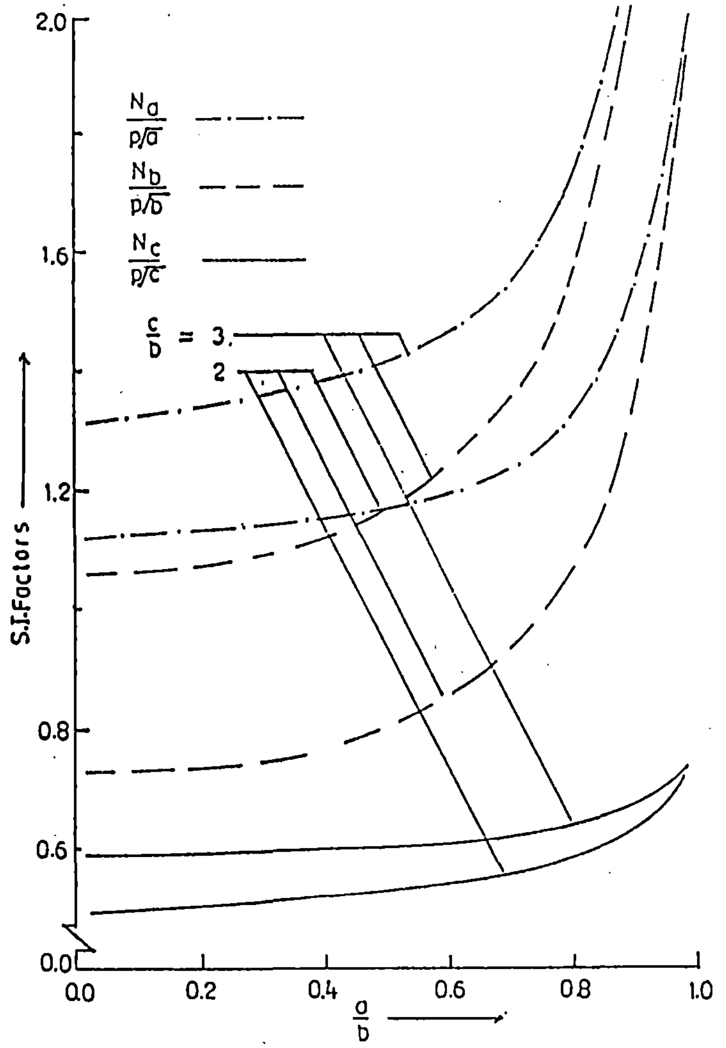


Fig. 5. Stress intensity factors Vs. *a/b*.

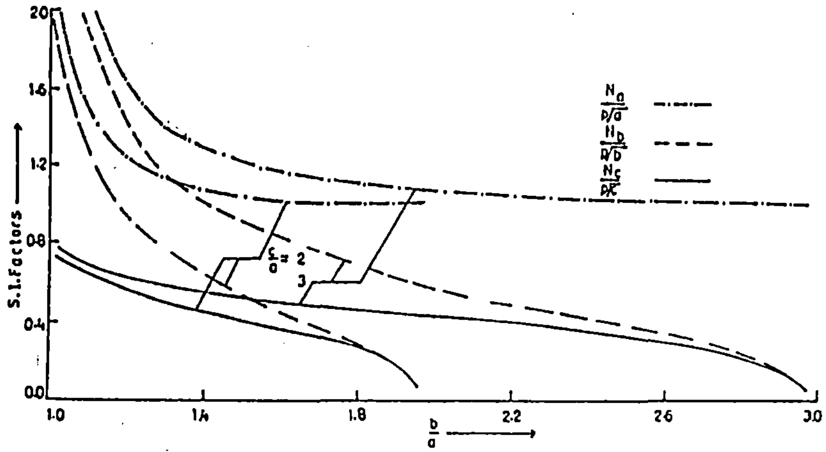


Fig. 6. Stress intensity factors Vs. *b/a*.

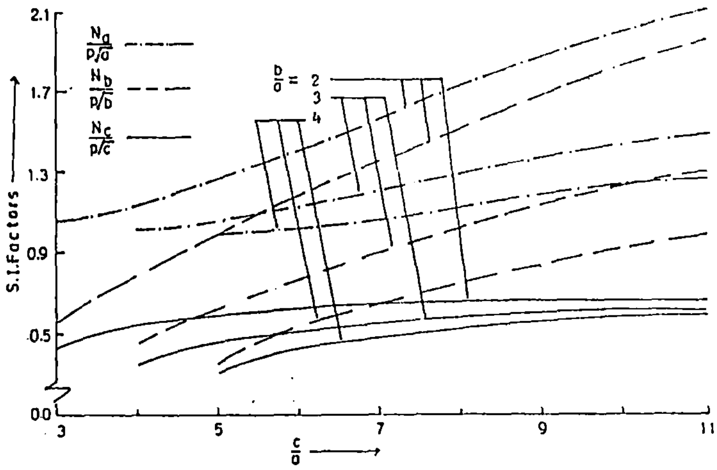


Fig. 7. Stress intensity factors Vs. c/a.

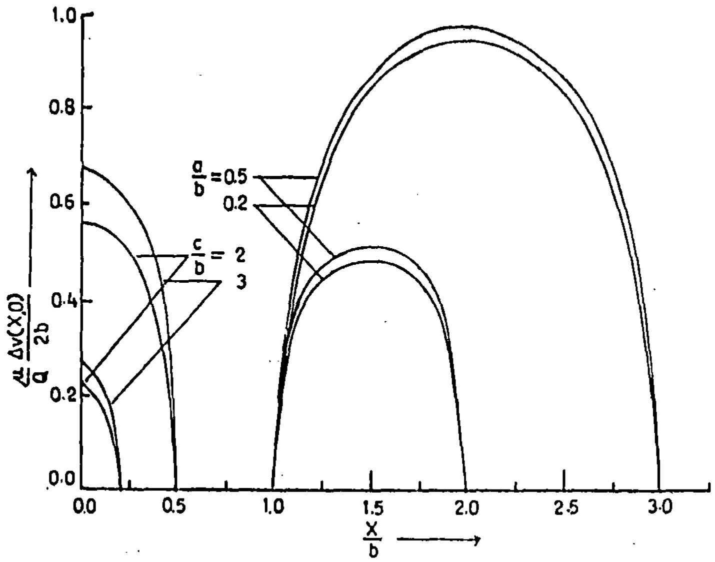


Fig. 8. Variation of crack opening displacement with X/b on both the cracks for the problem II.

References

1. A. Bostrom, *Journal of Applied Mechanics* 54 (1987) 503.
2. K.N. Srivastava, R.M. Palaiya and D.S. Karaulia, *International Journal of Fracture* 16(4) (1980) 349.
3. F.L. Neerhoff, *Applied Scientific Research* 35 (1979) 237.
4. S. Itou, *International Journal of Solids and Structures* 16 (1980) 1147.
5. D.L. Jain and R.P. Kanwal, *International Journal of Solids and Structures* 8 (1972) 961.
6. S. Itou, *Journal of Applied Mechanics* 45(4) (1978) 803.
7. E.H. Yoffe, *Philosophical Magazine* 42 (1951) 739.
8. Y. Nishida, Y. Shindo and A. Atsumi, *Acta Mechanica* 54 (1984) 23.
9. M.K. Kassir and S. Tse, *International Journal of Engineering Science* 21 (1983) 315.
10. J. De and B. Patra, *International Journal of Engineering Science* 28 (1990) 809.
11. A.N. Das and M.L. Ghosh, *Journal of Technical Physics* 32 (3–4) (1991) 479.

12. A.N. Das and M.L. Ghosh, *Engineering Fracture Mechanics* 41(1) (1992) 59.
13. A.N. Das and M.L. Ghosh, *International Journal of Fracture* 56 (1992) 279.
14. G.K. Dhawan and R.S. Dhaliwal, *International Journal of Engineering Science* 16 (1978) 253.
15. K.N. Srivastava and M. Lowengrub, *Proceedings of Royal Society of Edinburgh* (1968) 309.
16. J.C. Cook, *Glasgow Mathematical Journal* 11 (1970) 9.
17. I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, New York (1965).

Arch. Mech., 44, 4, pp. 463–474, Warszawa 1992

Three coplanar moving Griffith cracks in an infinite elastic strip

A. N. DAS (DARJEELING)

THE DYNAMIC anti-plane problem of determining stress and displacement due to three coplanar Griffith cracks moving steadily at a subsonic speed in an infinite elastic strip has been considered. Employing Fourier integral transform, the problem when the lateral boundaries are subjected to shearing stress, has been reduced to solving a set of four integral equations. These integral equations have been solved using finite Hilbert transform technique and Cook's result [9] to obtain the exact form of crack opening displacement and stress intensity factors. Numerical results for stress intensity factors have been presented in the form of graphs.

1. Introduction

IN FRACTURE MECHANICS, the problem of diffraction of elastic waves by cracks of finite dimension in a strip of elastic material has been investigated by several investigators. Sih and Chen [1] investigated the problem of propagation of a crack of finite length in a strip under plane extension. Closed-form solutions for a finite length crack moving in a strip under anti-plane shear stress was obtained by SINGH *et al.* [2]. Using finite Hilbert transform technique developed by SRIVASTAVA and LOWENGRUB [3], LOWENGRUB and SRIVASTAVA [4] solved the statical problem of distribution of stress and displacement in an infinitely long elastic strip containing two coplanar Griffith cracks. Several dynamic problems of determining stress and displacement due to two coplanar moving Griffith cracks have been solved by DAS and GHOSH [5–7].

As regards the crack problem, research has been restricted mainly to the case of a single crack or a pair of cracks because of severe mathematical complexity encountered in solving the problems of three or more cracks. Recently, DHAWAN and DHALIWAL [8] solved the statical problem of determining the stress distribution in an infinite transversely isotropic medium containing three coplanar Griffith cracks.

To the best knowledge of the author, the problem of stress distribution around three coplanar moving Griffith cracks in an infinite elastic strip has not been investigated so far. In this paper, the problem of propagation of three coplanar Griffith cracks in a fixed direction with constant velocity V in an infinitely long elastic strip of finite width has been considered. Employing Fourier integral transform, the problem when the lateral boundaries are subjected to shearing stress, has been reduced to solving a set of four integral equations using finite Hilbert transform technique [3] and COOK'S result [9] to derive the exact form of stress intensity factors and the crack opening displacement. Numerical results for the stress intensity factors are presented graphically to show their variations with crack speed, crack lengths and the separation distance between the cracks.

2. Statement of the problem

Consider an infinitely long elastic strip occupying the region $-h \leq Y \leq h$, weakened by three coplanar Griffith cracks moving steadily at a constant velocity V in the X-direction referred to a fixed coordinate system (X, Y, Z) as shown in the Fig. 1.

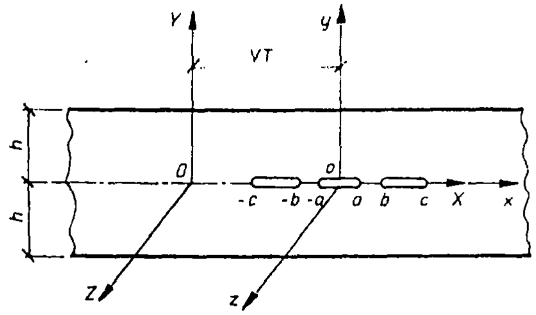


FIG. 1. Geometry and coordinate system.

In dynamic problem of anti-plane shear, the non-vanishing component of displacement W in the Z-direction satisfies the equation of motion

$$W_{,XX} + W_{,YY} = \frac{1}{C_2^2} W_{,TT}, \tag{2.1}$$

where $C_2 = (\mu/\rho)^{1/2}$ is the shear wave velocity, ρ is the material density and $W_{,X}$ represents partial derivatives of W with respect to X.

For cracks moving at constant velocity V in the X-direction it is convenient to introduce the Galilean transformation

$$x = X - VT, \quad y = Y, \quad z = Z, \quad t = T, \tag{2.2}$$

where (x, y, z) represents the moving coordinate system as shown in the Fig. 1.

Let the positions of the coplanar Griffith cracks referred to the coordinates (x, y, z) be $-a < x < a$, $-c < x < -b$ and $b < x < c$ on $y = 0$, and let the uniform shearing stress p be applied to the lateral boundaries $y = \pm h$ of the strip. The equivalent problem involves the application of shear stress $-p$ to the crack faces at $y = 0$. Accordingly, the boundary conditions of the proposed problem are

$$\sigma_{yz}(x, 0) = -p, \quad |x| < a, \quad b < |x| < c, \tag{2.3}$$

$$\sigma_{yz}(x, \pm h) = 0, \quad -\infty < x < \infty, \tag{2.4}$$

$$W(x, 0) = 0, \quad a < |x| < b, \quad |x| > c. \tag{2.5}$$

In the moving coordinate system, the equation of motion becomes independent of time and takes the form

$$s^2 W_{,xx} + W_{,yy} = 0, \tag{2.6}$$

with

$$s = \sqrt{1 - V^2/C_2^2}. \tag{2.7}$$

Due to the symmetry about x, z-plane we need to consider the region $0 < y < h$ only. Introducing the Fourier transforms

$$(2.8)\qquad \begin{aligned} \overline{W}_C(\xi, y) &= \int_0^\infty W(x, y)\cos(\xi x)\,dx, \\ W(x, y) &= \frac{2}{\pi}\int_0^\infty \overline{W}_C(\xi, y)\cos(\xi x)\,d\xi, \end{aligned}$$

in Eq. (2.6), the solution of Eq. (2.6) is obtained as

$$(2.9)\qquad W(x, y) = \frac{2}{\pi}\int_0^\infty [C_1(\xi)e^{-\xi ys} + C_3(\xi)e^{\xi ys}]\cos(\xi x)\,d\xi,$$

with

$$(2.10)\qquad \sigma_{yz}(x, y) = -\frac{2\mu s}{\pi}\int_0^\infty \xi[C_1(\xi)e^{-\xi ys} - C_3(\xi)e^{\xi ys}]\cos(\xi x)\,d\xi.$$

Using the expression for $\sigma_{yz}(x, y)$ given by Eq. (2.10) in Eq. (2.4), it has been found that

$$\begin{aligned} C_1(\xi) &= \frac{C(\xi)}{1 + e^{-2\xi hs}}, \\ C_3(\xi) &= \frac{C(\xi)e^{-2\xi hs}}{1 + e^{-2\xi hs}}, \end{aligned}$$

where the unknown function $C(\xi)$ is to be determined. From conditions (2.3) and (2.5) it is found that $C(\xi)$ satisfies the following quadruple integral equations:

$$(2.11)\qquad \int_0^\infty \xi C(\xi hs)\,\mathrm{th}(\xi hs)\cos(\xi x)\,d\xi = \frac{\pi p}{2\mu s}, \qquad x \in I_1, I_3,$$

and

$$(2.12)\qquad \int_0^\infty C(\xi)\cos(\xi x)d\xi = 0, \qquad x \in I_2, I_4,$$

where

$$I_1 = (0, a), \qquad I_2 = (a, b), \qquad I_3 = (b, c), \qquad I_4 = (c, \infty).$$

3. Method of solution

In order to solve the quadruple integral equations (2.11) and (2.12), let us take

$$(3.1)\qquad C(\xi) = \frac{1}{\xi}\int_0^a h(u)\sin(\xi u)\,du + \frac{1}{\xi}\int_b^c g(v^2)\,\mathrm{ch}(cv)\sin(\xi v)\,dv,$$

where $h(u)$ and $g(v^2)$ are the unknown functions to be determined from the boundary conditions of the problem considered. Substituting the value of $C(\xi)$ given by Eq. (3.1) into Eq. (2.12) and using the well-known result

$$\int_0^\infty \frac{\sin(x\xi)\cos(y\xi)}{\xi}\,d\xi = \begin{cases} \dfrac{\pi}{2}, & x > y > 0,\\[4pt] \dfrac{\pi}{4}, & x = y > 0,\\[4pt] 0, & y > x > 0 \end{cases}$$

it is found that this choice of $C(\xi)$ leads to the condition

$$\int_b^c g(v^2)\,\mathrm{ch}(ev)\,dv = 0. \tag{3.2}$$

Rewritting Eq. $(2.11)_1$ in the form

$$\frac{d}{dx}\int_0^\infty C(\xi)\,\mathrm{th}(\xi hs)\sin(\xi x)\,d\xi = \frac{\pi p}{2\mu s}, \qquad x \in I_1 \tag{3.3}$$

and inserting the value of $C(\xi)$ from Eq. (3.1) in (3.3) it is found that $h(u)$ is the solution of the following singular integral equation:

$$\int_0^a h(u)\log\left|\frac{\mathrm{sh}(ex)+\mathrm{sh}(eu)}{\mathrm{sh}(ex)-\mathrm{sh}(eu)}\right| du = \pi f(x), \qquad x \in I_1 \tag{3.4}$$

with

$$f(x) = \int_0^x \left[\frac{p}{\mu s} - \frac{1}{\pi}\int_b^c \frac{eg(v^2)\,\mathrm{ch}(ex')\,\mathrm{sh}(2ev)}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ex')}\,dv\right] dx',$$

where the following result [10] has been used:

$$\int_0^\infty \mathrm{th}(\xi hs)\frac{\sin(\xi x)\sin(\xi u)}{\xi}\,d\xi = \frac{1}{2}\log\left|\frac{\mathrm{sh}(ex)+\mathrm{sh}(eu)}{\mathrm{sh}(ex)-\mathrm{sh}(eu)}\right|, \qquad e = \frac{\pi}{2hs}. \tag{3.5}$$

Now using the Cook's result [9], the solution of Eq. (3.4) has been obtained with the aid of the formula

$$\int_0^a \frac{\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(ex)}\,e\,\mathrm{ch}(ex)\,dx}{[\mathrm{sh}^2(ex)-\mathrm{sh}^2(eu)][\mathrm{sh}^2(ev)-\mathrm{sh}^2(ex)]} = -\frac{\pi}{2\,\mathrm{sh}(ev)}\frac{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eu)}$$

for $u \in I_1$ and $v \in I_3$,

$$h(u) = \frac{-e\,\mathrm{sh}(2eu)}{\pi\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}\left[\frac{p}{\mu s}\int_0^a \sqrt{\frac{\mathrm{sh}^2(ea)-\mathrm{sh}^2(ex)}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eu)}}\,dx + \int_b^c \frac{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eu)}\,g(v^2)\,\mathrm{ch}(ev)\,dv\right]. \tag{3.6}$$

Substitute now the resulting value of $C(\xi)$, obtained by inserting Eqs. (3.6) into Eq. (3.1), in condition $(2.11)_2$, and make use of the following results:

$$\int_0^a \frac{e\,\mathrm{sh}^2(eu)\,\mathrm{ch}(eu)\,du}{[\mathrm{sh}^2(eu)-\mathrm{sh}^2(ex)][\mathrm{sh}^2(ev)-\mathrm{sh}^2(eu)]\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}$$
$$= \frac{\pi}{2[\mathrm{sh}^2(ev)-\mathrm{sh}^2(ex)]}\left[\frac{\mathrm{sh}(ev)}{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}} - \frac{\mathrm{sh}(ex)}{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}\right],$$

$$\int_0^a \frac{e\,\mathrm{sh}^2(eu)\,\mathrm{ch}(eu)\,du}{[\mathrm{sh}^2(eu)-\mathrm{sh}^2(ex)][\mathrm{sh}^2(ey')-\mathrm{sh}^2(eu)]\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}$$
$$= \frac{\pi}{2[\mathrm{sh}^2(ex)-\mathrm{sh}^2(ey')]}\frac{\mathrm{sh}(ex)}{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}, \quad \text{for} \quad x, v \in I_3 \quad \text{and} \quad y' \in I_1.$$

It can be shown that $g(v^2)$ is the solution of the following singular integral equation

$$\int_b^c \frac{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ex)} e g(v^2)\,\mathrm{ch}(ev)\,dv = \frac{\pi p}{\mu s}\left[\frac{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}{\mathrm{sh}(2ex)} + \frac{1}{\pi}\int_0^a \frac{\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(ey')}}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ey')}\,dy'\right], \quad \text{for} \quad x \in I_3. \tag{3.7}$$

Using finite Hilbert transform technique [3] and the formula

$$\int_b^c \sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eb)}}\frac{\mathrm{sh}(2ex)\,dx}{[\mathrm{sh}^2(ex)-\mathrm{sh}^2(ey')][\mathrm{sh}^2(ex)-\mathrm{sh}^2(ev)]}$$
$$= -\frac{\pi}{e[\mathrm{sh}^2(ev)-\mathrm{sh}^2(ey')]}\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ey')}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ey')}},$$

the solution of Eq. (3.7) is found to be

$$g(v^2) = -\frac{2cp}{\mu\pi s}\frac{\mathrm{sh}(ev)\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}}{\sqrt{[\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)][\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)]}}\left[\int_b^c \sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eb)}} \times \frac{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ev)}\,dx - \int_0^a \sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ey')}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ey')}}\frac{\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(ey')}}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ey')}\,dy'\right]$$
$$+\frac{C_1\,\mathrm{sh}(ev)}{\sqrt{[\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)][\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)][\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)]}}. \tag{3.8}$$

Next, substitution of $g(v^2)$ from Eq. (3.8) in Eq. (3.6) and finally application of the formula

$$\int_b^c \sqrt{\frac{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}}\frac{\mathrm{sh}(2ev)\,dv}{[\mathrm{sh}^2(ev)-\mathrm{sh}^2(eu)][\mathrm{sh}^2(ex')-\mathrm{sh}^2(ev)]}$$
$$= \frac{\pi}{e[\mathrm{sh}^2(eu)-\mathrm{sh}^2(ex')]}\left[\sqrt{\frac{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}} - \sqrt{\frac{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ex')}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex')}}\right], \quad \text{for} \quad u, x' \in I_1$$

yields $h(u)$ in the form

$$(3.9)\quad h(u) = -\frac{2ep}{\mu\pi s}\frac{\mathrm{ch}(eu)\,\mathrm{sh}(eu)\sqrt{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}}{\sqrt{[\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)][\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)]}}\left[\int\limits_0^a\sqrt{\frac{\mathrm{sh}^2(ea)-\mathrm{sh}^2(ey')}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ey')}}\right.$$

$$\left.\times\frac{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ey')}}{\mathrm{sh}^2(ey')-\mathrm{sh}^2(eu)}dy' - \int\limits_b^c\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eb)}}\frac{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eu)}\,dx\right]$$

$$-\frac{C_1\,\mathrm{sh}(eu)\,\mathrm{ch}(eu)}{\sqrt{[\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)][\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)][\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)]}}.$$

Substitution of the value of $g(v^2)$ from Eq. (3.8) in the condition (3.2) yields

$$(3.10)\quad C_1 = -\frac{2ep}{\pi\mu s}\left[\int\limits_b^c\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eb)}}\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}\left\{\frac{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}\right.\right.$$

$$\left.\times\Pi\left\{\frac{\pi}{2},\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)},q\right\}\Big/F\left(\frac{\pi}{2},q\right)+1\right\}dx$$

$$+\int\limits_0^a\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(es)}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(es)}}\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(es)}$$

$$\times\left\{1-\frac{\mathrm{sh}^2(eb)-\mathrm{sh}^2(es)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(es)}\Pi\left\{\frac{\pi}{2},\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(es)},q\right\}\Big/F\left(\frac{\pi}{2},q\right)\right\}ds,$$

where $F(\phi,q)$ and $\Pi(\phi,n,q)$ are elliptic integrals of the first and third kind, respectively, and $q=\sqrt{\dfrac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}$.

The relevant displacement and stress components in the plane of the crack can now be shown to be given by

$$(3.11)\qquad \begin{aligned} W(x,0) &= \int\limits_x^a h(u)\,du, && 0\le x\le a,\\ &= \int\limits_x^c g(v^2)\,\mathrm{ch}(ev)\,dv, && b\le x\le c, \end{aligned}$$

and

$$[\sigma_{yz}(x,0)]_{a<x<b} = \frac{2\mu s}{\pi}\left[\int\limits_0^a\frac{eh(u)\,\mathrm{sh}(eu)\,du}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eu)} - \int\limits_b^c\frac{eg(v^2)\,\mathrm{sh}(ev)\,\mathrm{ch}(ev)}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ex)}\,dv\right]\mathrm{ch}(ex),$$

(3.12)

$$[\sigma_{yz}(x,0)]_{x>c} = \frac{2\mu s}{\pi}\left[\int\limits_0^a\frac{eh(u)\,\mathrm{sh}(eu)\,du}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eu)} + \int\limits_b^c\frac{eg(v^2)\,\mathrm{sh}(ev)\,\mathrm{ch}(ev)}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ev)}\,dv\right]\mathrm{ch}(ex).$$

Now, insertion of the values of $h(u)$ and $g(v^2)$, as given by Eqs. (3.9) and (3.8), in the expressions (3.12) yields (after some algebraic manipulations)

$$[\sigma_{yz}(x,0)]_{a<x<b} = \frac{2pe}{\pi}\Bigg[-\sqrt{\frac{\operatorname{sh}^2(eb)-\operatorname{sh}^2(ea)}{\operatorname{sh}^2(ec)-\operatorname{sh}^2(ea)}}\frac{\operatorname{sh}(ex)}{\sqrt{\operatorname{sh}^2(ex)-\operatorname{sh}^2(ea)}}$$

$$\Bigg\{\int_0^a F_2(u,x)\,du+\int_b^c F_2(v,x)dv\Bigg\}-\frac{2e[\operatorname{sh}^2(ec)-\operatorname{sh}^2(eb)]}{\pi}\Bigg\{\int_0^a F_2(u',x)\,du'\int_0^a F_4(c,u)$$

$$\times F_3(0,x,u)\,du+\int_b^c F_2(v,x)dv\int_0^a F_4(c,u)F_3(v,x,u)\,du\Bigg\}+\frac{\mu s}{ep}C_1\Bigg\{\frac{\pi}{2}$$

$$\times\frac{1-\operatorname{sh}(ex)/\sqrt{\operatorname{sh}^2(ex)-\operatorname{sh}^2(ea)}}{\sqrt{[\operatorname{sh}^2(eb)-\operatorname{sh}^2(ea)][\operatorname{sh}^2(ec)-\operatorname{sh}^2(ea)]}}+e\int_0^a F_4(c,u)F_5(u,x)\,du\Bigg\}$$

$$+\frac{e[\operatorname{sh}^2(eb)-\operatorname{sh}^2(ea)]}{\pi}\Bigg\{\int_b^c F_2(v',x)dv'\int_b^c F_4(a,v)F_6(v',x,v)dv+\int_0^a F_2(u,x)\,du$$

$$\times\int_b^c F_4(a,v)F_6(u,x,v)dv-\frac{\operatorname{sh}^2(ec)-\operatorname{sh}^2(eb)}{\operatorname{sh}^2(eb)-\operatorname{sh}^2(ea)}\int_0^a F_1(u,x)du\int_0^a F_4(c,u')F_9(u,u')du'\Bigg\}$$

$$-\frac{\mu s}{pe}\frac{C_1}{X_1}\Bigg\{\frac{\pi}{2}\frac{\operatorname{sh}(ec)}{\sqrt{\operatorname{sh}^2(ec)-\operatorname{sh}^2(ea)}}+e\operatorname{sh}^2(ea)\int_b^c F_7(x,v)\,dv\Bigg\}\Bigg]\operatorname{ch}(ex),$$

$$[\sigma_{yz}(x,0)]_{x>c} = \frac{2pe}{\pi}\Bigg[-\sqrt{\frac{\operatorname{sh}^2(eb)-\operatorname{sh}^2(ea)}{\operatorname{sh}^2(ec)-\operatorname{sh}^2(ea)}}\frac{\operatorname{sh}(ex)}{\sqrt{\operatorname{sh}^2(ex)-\operatorname{sh}^2(ea)}} \tag{3.13}$$

$$\times\Bigg\{\int_0^a F_2(u,x)\,du+\int_b^c F_2(v,x)\,dv\Bigg\}-\frac{2e[\operatorname{sh}^2(ec)-\operatorname{sh}^2(eb)]}{\pi}\Bigg\{\int_0^a F_2(u',x)\,du'\int_0^a F_4(c,u)$$

$$\times F_3(0,x,u)\,du+\int_b^c F_2(v,x)\,dv\int_0^a F_4(c,u)F_3(v,x,u)\,du\Bigg\}+\frac{\mu s}{ep}C_1\Bigg\{\frac{\pi}{2}$$

$$\times\frac{1-\operatorname{sh}(ex)/\sqrt{\operatorname{sh}^2(ex)-\operatorname{sh}^2(ea)}}{\sqrt{[\operatorname{sh}^2(ec)-\operatorname{sh}^2(ea)][\operatorname{sh}^2(eb)-\operatorname{sh}^2(ea)]}}+e\int_0^a F_4(c,u)F_5(u,x)\,du\Bigg\}$$

$$-\frac{e[\operatorname{sh}^2(eb)-\operatorname{sh}^2(ea)]}{\pi}\Bigg\{\int_b^c F_2(v',x)\,dv'\int_b^c F_4(a,v)F_8(v',v,x)\,dv+\int_0^a F_2(u,x)\,du$$

$$\times\int_b^c F_4(a,v)F_8(u,v,x)dv+\frac{\operatorname{sh}^2(ec)-\operatorname{sh}^2(eb)}{\operatorname{sh}^2(eb)-\operatorname{sh}^2(ea)}\int_0^a F_1(u,x)\,du\int_0^a F_4(c,u')F_9(u,u')du'\Bigg\}$$

$$+\frac{\mu sC_1}{peX_1}\Bigg\{\frac{\pi}{2}\frac{\operatorname{sh}(ec)}{\sqrt{\operatorname{sh}^2(ec)-\operatorname{sh}^2(ea)}}+e\operatorname{sh}^2(ea)\int_b^c F_7(x,v)dv\Bigg\}-\sqrt{\frac{\operatorname{sh}^2(ec)-\operatorname{sh}^2(eb)}{\operatorname{sh}^2(ec)-\operatorname{sh}^2(ea)}}$$

$$\times\frac{\operatorname{sh}(ex)}{\sqrt{\operatorname{sh}^2(ex)-\operatorname{sh}^2(ec)}}\Bigg\{\int_0^a F_2(u,x)\,du+\int_b^c F_2(v,x)\,dv\Bigg\}\Bigg]\operatorname{ch}(ex).$$

In the above formulae

$$F_1(u,x) = \sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}}\,\frac{\mathrm{sh}(eu)}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eu)},$$

$$F_2(v,x) = \sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}}\,\frac{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ex)},$$

$$F_3(v,x,u) = \frac{\mathrm{sh}(ex)}{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}\tan^{-1}\left\{\frac{\mathrm{sh}(eu)}{\mathrm{sh}(ex)}\sqrt{\frac{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}\right\}$$

$$-\frac{\mathrm{sh}(ev)}{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}}\tan^{-1}\left\{\frac{\mathrm{sh}(eu)}{\mathrm{sh}(ev)}\sqrt{\frac{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}\right\},$$

$$F_4(\omega,u) = \frac{\mathrm{ch}(eu)\,\mathrm{sh}(eu)}{\sqrt{[\mathrm{sh}^2(e\omega)-\mathrm{sh}^2(eu)]^3[\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)]}},$$

$$F_5(u,x) = [2\,\mathrm{sh}^2(eu)-\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)]\left\{\sin^{-1}\left(\frac{\mathrm{sh}(eu)}{\mathrm{sh}(ea)}\right)-F_3(0,x,u)\right\}, \tag{3.14}$$

$$F_6(u,x,v) = \frac{\mathrm{sh}(ex)}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}}$$

$$\times\log\left|\frac{\mathrm{sh}(ex)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}+\mathrm{sh}(ev)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}}{\mathrm{sh}(ex)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}-\mathrm{sh}(ev)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}}\right|-\frac{\mathrm{sh}(eu)}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}}$$

$$\times\log\left|\frac{\mathrm{sh}(eu)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}+\mathrm{sh}(ev)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}}{\mathrm{sh}(eu)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}-\mathrm{sh}(ev)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}}\right|,$$

$$F_7(x,v) = \tan^{-1}\left(\frac{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}}\,\frac{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}}{\sqrt{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ex)}}\right)$$

$$\times\frac{\mathrm{ch}(ev)}{\sqrt{[\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)]^3}},$$

$$F_8(u,v,x) = -\frac{2\,\mathrm{sh}(ex)}{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ec)}}\tan^{-1}\left\{\frac{\mathrm{sh}(ev)}{\mathrm{sh}(ex)}\sqrt{\frac{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ec)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}}\right\}$$

$$+\frac{\mathrm{sh}(eu)}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}}\log\left|\frac{\mathrm{sh}(eu)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}+\mathrm{sh}(ev)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}}{\mathrm{sh}(eu)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}-\mathrm{sh}(ev)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}}\right|,$$

$$F_9(u,u') = \log\left|\frac{\mathrm{sh}(eu)\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu')}+\mathrm{sh}(eu')\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}{\mathrm{sh}(eu)\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu')}-\mathrm{sh}(eu')\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}\right|$$

and

$$X_1 = \sqrt{[\mathrm{sh}^2(eb) - \mathrm{sh}^2(ex)][\mathrm{sh}^2(ec) - \mathrm{sh}^2(ex)]}.$$

The dynamic stress intensity factors are defined by

$$\begin{aligned} N_a &= \lim_{x\to a^+} \sqrt{2(x-a)}[\sigma_{yz}(x,0)]_{a<x<b}, \\ N_b &= \lim_{x\to b^-} \sqrt{2(b-x)}[\sigma_{yz}(x,0)]_{a<x<b}, \\ N_c &= \lim_{x\to c^+} \sqrt{2(x-c)}[\sigma_{yz}(x,0)]_{x>c}. \end{aligned} \tag{3.15}$$

Substitution of the results given by Eqs. (3.13) in expressions (3.15) yields

$$\begin{aligned} N_a &= \sqrt{\frac{\mathrm{sh}(2ea)}{e}}\Bigg[-\sqrt{\frac{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}\frac{2pe}{\pi}\left\{\int_0^a F_2(u,a)\,du + \int_b^c F_2(v,a)\,dv\right\} \\ &\qquad - \frac{\mu s C_1}{\sqrt{[\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)][\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)]}}\Bigg], \\ N_b &= -\frac{\mu s C_1}{\sqrt{[\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)][\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)]}}\sqrt{\frac{\mathrm{sh}(2eb)}{e}}, \\ N_c &= \sqrt{\frac{\mathrm{sh}(2ec)}{e}}\Bigg[-\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}\frac{2pe}{\pi}\left\{\int_0^a F_2(u,c)du + \int_b^c F_2(v,c)\,dv\right\} \\ &\qquad + \frac{\mu s C_1}{\sqrt{[\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)][\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)]}}\Bigg]. \end{aligned} \tag{3.16}$$

Again, insertion of the values of $h(u)$ and $g(v^2)$, given by Eqs. (3.8) and (3.9), in the expressions for displacements given by Eqs. (3.11) yields

$$\begin{aligned} [W(x,0)]_{0\le x\le a} &= -\frac{p}{\mu\pi s}\Bigg[\frac{2[\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)]}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}\Bigg\{\int_b^c \Pi\left\{\lambda, \frac{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}, q\right\} \\ &\quad\times\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}}\frac{dv}{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}} - \int_0^a \Pi\left\{\lambda, \frac{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}, q\right\} \\ &\quad\times\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}}\frac{du}{\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}\Bigg\}\Bigg] - \frac{C_1 F(\lambda,q)}{e\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}, \end{aligned}$$

and

$$\begin{aligned} [W(x,0)]_{b\le x\le c} &= \Bigg[\frac{2p}{\mu\pi s}\Bigg(\int_b^c \sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}}\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}\Bigg\{F(\lambda',q) \\ &\quad + \frac{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}\Pi\left\{\lambda', \frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}, q\right\}\Bigg\}\,dv + \int_0^a \sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}} \end{aligned}$$

$$\times\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}\Big\{F(\lambda',q)-\frac{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}\Pi\Big\{\lambda',\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)},q\Big\}\Big\}\,du\Big)$$

$$+\frac{C_1}{e}F(\lambda',q)\Big]\frac{1}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}},$$

where

$$\sin\lambda=\sqrt{\frac{\mathrm{sh}^2(ea)-\mathrm{sh}^2(ex)}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ex)}},\qquad \sin\lambda'=\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}}$$

and $F(\phi,q)$, $\Pi(\phi,n,q)$, and q have been defined earlier.

On putting $b=c$ and simplifying, it may be noted that the results $(3.16)_1$ and $(3.17)_1$ become those given by Eqs. (4.18) and (4.19) of SINGH *et al.* [2], and for $a=0$ the results given by Eqs. $(3.16)_2$, $(3.16)_3$ and $(3.17)_2$ coincide with those given by Eqs. (4.38), (4.39) and (4.35) of DAS and GHOSH [5].

4. Numerical results and discussions

Numerical results for stress intensity factors at the tips of the cracks for different values of crack speed, crack lengths and the separating distance between the cracks have been presented in this section. The dependence of the stress intensity factors on crack lengths and their variations with V/C_2 have been shown in Figs. 2–5. It is seen in Figs. 2–3 that stress intensity factors at the edges of the cracks increase rapidly when $V/C_2 \to 1$, and variation of stress intensity factors at the edge $x=a$ is greater than that at the tips $x=b$ and $x=c$ when the length of the inner crack increases.

Variations of stress intensity factors at the edges of the cracks with a/b for different values of c/b and that with b/a for different values of c/a are plotted in Figs. 4–5, respectively. It has been found that when the distance between the inner crack and the outer pair of cracks decreases, the stress intensity factors at the tips $x=a$ and $x=b$ become greater than that at the edge $x=c$.

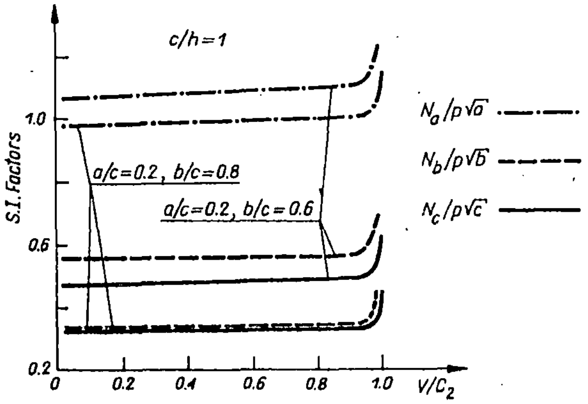


FIG. 2. Variations of stress intensity factors with V/C_2.

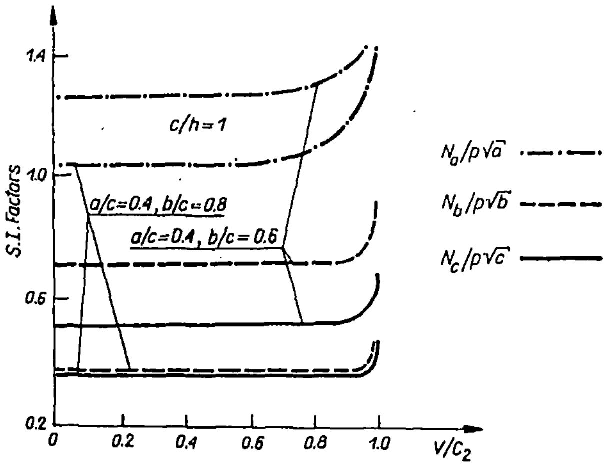


FIG. 3. Variations of stress intensity factors with V/C_2.

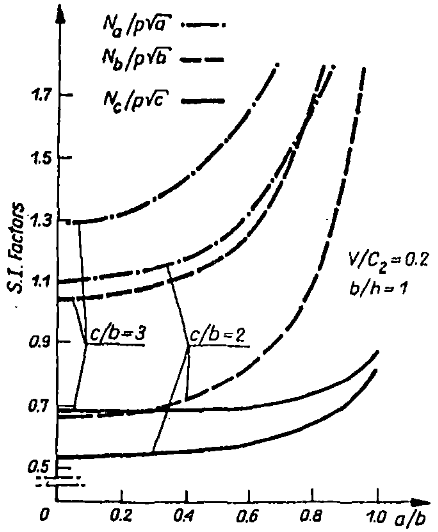


FIG. 4. Stress intensity factors Vs. a/b.

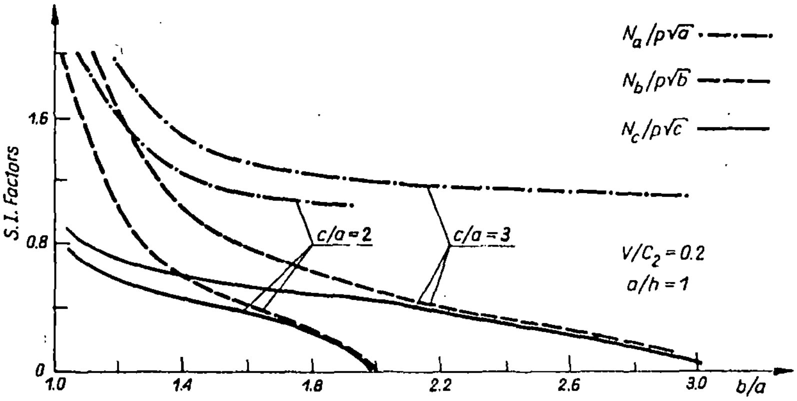


FIG. 5. Stress intensity factors Vs. b/a.

References

1. G. C. SIH and E. P. CHEN, *Cracks propagating in a strip of material under plane extension*, Int. J. Engng. Sci., **10**, 537, 1972.
2. B. M. SINGH, T. B. MOODIE and J. B. HADDOW, *Closed-form solutions for finite length crack moving in a strip under anti-plane shear stress*, Acta Mechanica, **38**, 99, 1981.
3. K. N. SRIVASTAVA and M. LOWENGRUB, *Finite Hilbert transform technique for triple integral equations with trigonometric kernels*, Proc. Roy. Soc. Edin., 309, 1968.
4. M. LOWENGRUB and K. N. SRIVASTAVA, *Two coplanar Griffith cracks in an infinitely long elastic strip*, Int. J. Engng. Sci., **6**, 625, 1968.
5. A. N. DAS and M. L. GHOSH, *Two coplanar Griffith cracks moving in a strip under anti-plane shear stress*, J. Tech. Phys., **32**, 3–4, 165, 1991.
6. A. N. DAS and M. L. GHOSH, *Two coplanar Griffith cracks moving along the interface of two dissimilar elastic media*, Engng. Frac. Mech., **41**, 1, 59, 1991.
7. A. N. DAS and M. L. GHOSH, *Problem of two coplanar Griffith cracks running steadily under three-dimensional loading*, Int. J. Frac., 1992 [in press].
8. G. K. DHAWAN and R. S. DHALIWAL, *On three coplanar Griffith cracks in an infinite transversely isotropic medium*, Int. J. Engng. Sci., **16**, 253, 1978.
9. J. C. COOK, *The solution of some integral equations and their connection with dual integral equations and trigonometric series*, Glas. Math. J., **11**, 9, 1970.
10. I. S. GRADSHTEYN and I. M. RYZHIK, *Tables of integrals, series and products*, Academic Press 1965.

DEPARTMENT OF MATHEMATICS
NORTH BENGAL UNIVERSITY, DARJEELING, WEST BENGAL, INDIA.

Received July 16, 1992.

Engineering Fracture Mechanics Vol. 43, No. 6, pp. 941–955, 1992
Printed in Great Britain.

0013-7944/92 $5.00 + 0.00

FOUR CO-PLANAR GRIFFITH CRACKS IN AN INFINITE ELASTIC MEDIUM

A. N. DAS and M. L. GHOSH
Department of Mathematics, North Bengal University, Darjeeling, West Bengal 734430, India

Abstract—The dynamic in-plane problem of determining the stress and displacement due to four co-planar Griffith cracks moving steadily at a subsonic speed in a fixed direction in an infinite, isotropic, homogeneous medium under normal stress has been treated. The static problem of determining the stress and displacement in an infinite isotropic elastic medium has also been considered. In both cases, employing the Fourier integral transform, the problems have been reduced to solving a set of five integral equations. These integral equations have been solved using the finite Hilbert transform technique to obtain the exact form of crack opening displacement and stress intensity factors which are presented in the form of graphs.

INTRODUCTION

IN FRACTURE mechanics, scattering of elastic waves by cracks of finite dimension in an infinite elastic medium has been investigated by several investigators. The problem of scattering of elastic waves from an interface crack was solved by Bostrom [1]. Srivastava *et al.* [2] solved the problem of the interaction of an anti-plane shear wave with an interface crack. The problem of diffraction of Love waves by a crack of finite width in the plane interface of a layered composite has been solved by Neerhoff [3]. Itou [4] solved the problem of diffraction of an anti-plane shear wave by two co-planar Griffith cracks in an infinite elastic medium. The scattering of a time harmonic normally incident plane wave by two co-planar Griffith cracks was solved by Jain and Kanwal [5]. Itou [6] also solved the problem of stress concentration around two co-planar Griffith cracks in an infinite elastic medium. Problems on two co-planar Griffith cracks moving along the interface of a layered infinite half-space have also been solved by Das and Ghosh [7] recently.

As regards the crack problem, research has been restricted mainly to the case of a single crack or a pair of cracks because of the severe mathematical complexity encountered in solving the problems of three or more cracks. Recently, Dhawan and Dhaliwal [8] solved the static problem of determining the stress distribution in an infinite transversely isotropic medium containing three co-planar cracks.

To the best knowledge of the authors, the problem of stress distribution around four co-planar Griffith cracks has not been investigated so far. In this paper, we consider two cases regarding the stress distribution around four co-planar Griffith cracks in an infinite homogeneous, isotropic medium. In the first case, cracks are assumed to be moving steadily along a fixed direction with constant velocity V. In the second case, the static problem of determining the stress and displacement in an infinite homogeneous, isotropic medium weakened by four co-planar Griffith cracks has been considered. Using Fourier integral transform both problems have been reduced to solving a set of five integral equations. Employing the finite Hilbert transform technique [9], the integral equations have been solved to derive crack opening displacement and stress intensity factors, which are presented in the form of graphs.

STATEMENT OF PROBLEM I AND ITS FORMULATION

Consider an infinite homogeneous, isotropic material weakened by four co-planar Griffith cracks, moving steadily at a constant velocity V in the X-direction referred to a fixed coordinate system (X, Y, Z), as shown in Fig. 1. In the absence of body force, the equations of motion in terms of displacement are

$$(\lambda + 2\mu)[u_{,XX} + v_{,XY}] + \mu[u_{,YY} - v_{,XY}] = \rho u_{,TT}$$

and

$$(\lambda + 2\mu)[u_{,XY} + v_{,YY}] + \mu[v_{,XX} - u_{,XY}] = \rho u_{,TT} \qquad \text{(1a, b)}$$

where u and v denote the displacement components in the X- and Y-directions, λ and μ are Lamé's constants and $u_{,X}$ represents partial derivatives of u with respect to X.

For cracks moving with constant velocity V in the X-direction it is convenient to introduce the Galilean transformation

$$x = X - VT, \quad y = Y, \quad z = Z, \quad t = T \tag{2}$$

where (x, y, z) represents the translating coordinate system as shown in Fig. 1.

In the moving coordinates, the equations of motion (1) become independent of time and take the form

$$\begin{gathered}(\lambda + 2\mu - \rho V^2)u_{,xx} + (\lambda + \mu)v_{,xy} + \mu u_{,yy} = 0 \\ (\lambda + 2\mu)v_{,yy} + (\mu - \rho V^2)v_{,xx} + (\lambda + \mu)u_{,xy} = 0.\end{gathered} \tag{3a, b}$$

Introducing

$$\begin{gathered}\bar{u}_s(\xi, y) = \int_0^\infty u(x, y)\sin(\xi x)\,\mathrm{d}x \\ \bar{v}_c(\xi, y) = \int_0^\infty v(x, y)\cos(\xi x)\,\mathrm{d}x\end{gathered} \tag{4a, b}$$

and

$$\begin{gathered}u(x, y) = \frac{2}{\pi}\int_0^\infty \bar{u}_s(\xi, y)\sin(\xi x)\,\mathrm{d}\xi \\ v(x, y) = \frac{2}{\pi}\int_0^\infty \bar{v}_c(\xi, y)\cos(\xi x)\,\mathrm{d}\xi,\end{gathered} \tag{5a, b}$$

in eq. (3) we obtain

$$\begin{gathered}\mu\bar{u}_{s,yy} - \xi(\lambda + \mu)\bar{v}_{c,y} - \xi^2(\lambda + 2\mu - \rho V^2)\bar{u}_s = 0 \\ (\lambda + 2\mu)\bar{v}_{c,yy} + \xi(\lambda + \mu)\bar{u}_{s,y} - \xi^2(\mu - \rho V^2)\bar{v}_c = 0.\end{gathered} \tag{6a, b}$$

Elimination of $\bar{u}_s$ from (6a, b) yields the following ordinary differential equation:

$$\left[\left\{\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (1 - M^2k^2)\xi^2\right\}\left\{\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (1 - M^2)\xi^2\right\}\right]\bar{v}_c = 0 \tag{7}$$

where $M = V/c_2$, $k = c_2/c_1$.

The solution of the differential equation given by (7), for $y \geqslant 0$, is

$$\bar{v}_c(\xi, y) = A(\xi)\mathrm{e}^{-\xi y\sqrt{(1 - M^2k^2)}} + B(\xi)\mathrm{e}^{-\xi y\sqrt{(1 - M^2)}} \tag{8}$$

where the unknown functions $A(\xi)$ and $B(\xi)$ are to be determined using the boundary conditions of the proposed problem.

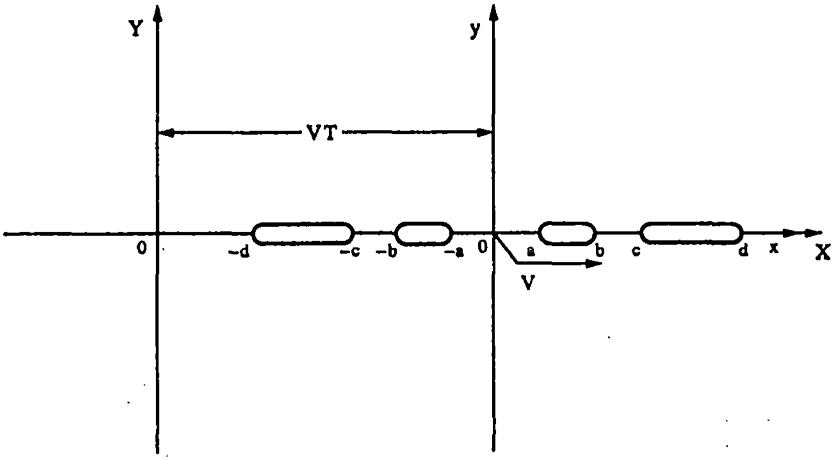


Fig. 1. Geometry and coordinate system.

Employing (8) in eqs (6a, b), we obtain

$$\bar{u}_s(\xi, y) = \frac{A(\xi)}{\sqrt{(1-M^2k^2)}} \mathrm{e}^{-\xi y\sqrt{(1-M^2k^2)}} + \sqrt{(1-M^2)}B(\xi)\,\mathrm{e}^{-\xi y\sqrt{(1-M^2)}}, \quad y \geqslant 0. \tag{9}$$

Therefore, the stress components given by

$$\begin{aligned} \sigma_{yy} &= \lambda(u_{,x} + v_{,y}) + 2\mu v_{,y} \\ \sigma_{xy} &= \mu(u_{,y} + v_{,x}) \end{aligned} \tag{10a, b}$$

become

$$\sigma_{yy}(x, y) = -\frac{2\mu}{\pi}\int_0^\infty \xi\left[\frac{2-M^2}{\sqrt{(1-M^2k^2)}}A(\xi)\,\mathrm{e}^{-\xi y\sqrt{(1-M^2k^2)}} + 2\sqrt{(1-M^2)}B(\xi)\,\mathrm{e}^{-\xi y\sqrt{(1-M^2)}}\right]\cdot\cos(\xi x)\,\mathrm{d}\xi$$

$$\sigma_{xy}(x, y) = -\frac{2\mu}{\pi}\int_0^\infty \xi[2A(\xi)\,\mathrm{e}^{-\xi y\sqrt{(1-M^2k^2)}} + (2-M^2)B(\xi)\,\mathrm{e}^{-\xi y\sqrt{(1-M^2)}}]\sin(\xi x)\,\mathrm{d}\xi \tag{11a, b}$$

with

$$u(x, y) = \frac{2}{\pi}\int_0^\infty\left[\frac{A(\xi)}{\sqrt{(1-M^2k^2)}}\mathrm{e}^{-\xi y\sqrt{(1-M^2k^2)}} + \sqrt{(1-M^2)}B(\xi)\,\mathrm{e}^{-\xi y\sqrt{(1-M^2)}}\right]\sin(\xi x)\,\mathrm{d}\xi$$

and

$$v(x, y) = \frac{2}{\pi}\int_0^\infty [A(\xi)\,\mathrm{e}^{-\xi y\sqrt{(1-M^2k^2)}} + B(\xi)\,\mathrm{e}^{-\xi y\sqrt{(1-M^2)}}]\cos(\xi x)\,\mathrm{d}\xi. \tag{12a, b}$$

Let four co-planar Griffith cracks of finite length located along the X-axis be moving steadily with velocity V in the X-direction so that their positions, referred to translating coordinates (x, y, z), are $a \leqslant |x| \leqslant b$, $c \leqslant |x| \leqslant d$ on $y = 0$.

The boundary conditions of the proposed problem on account of the symmetry with respect to the y-axis are

$$v(x, 0) = 0, \quad x \in I_1, I_3, I_5 \tag{13a–c}$$

$$\sigma_{xy}(x, 0) = 0, \quad 0 < x < \infty \tag{14}$$

$$\sigma_{yy}(x, 0) = -p, \quad x \in I_2, I_4 \tag{15a, b}$$

where $I_1 = (0, a)$, $I_2 = (a, b)$, $I_3 = (b, c)$, $I_4 = (c, d)$, $I_5 = (d, \infty)$.

Using the condition (14) in (11b) we find that $A(\xi)$ and $B(\xi)$ are related by

$$B(\xi) = -\frac{2}{2-M^2}A(\xi). \tag{16}$$

With the help of the boundary condition (13), we obtain from (12b)

$$\int_0^\infty A(\xi)\cos(\xi x)\,\mathrm{d}\xi = 0, \quad x \in I_1, I_3, I_5. \tag{17a–c}$$

Substitution of (11a) in (15) yields, with the aid of (16)

$$\int_0^\infty \xi A(\xi)\cos(\xi x)\,\mathrm{d}\xi = \frac{P\pi}{2\mu}, \quad x \in I_2, I_4 \tag{18a, b}$$

where

$$P = \frac{p}{K}, \quad K = \frac{(2-M^2)^2 - 4\sqrt{[(1-M^2k^2)(1-M^2)]}}{(2-M^2)\sqrt{(1-M^2k^2)}}.$$

METHOD OF SOLUTION

In order to solve the set of five integral equations given in eqs (17) and (18), we assume

$$A(\xi)=\frac{1}{\xi}\int_a^b h(s^2)\sin(\xi s)\,\mathrm{d}s+\frac{1}{\xi}\int_c^d g(t^2)\sin(\xi t)\,\mathrm{d}t \tag{19}$$

where $h(s^2)$ and $g(t^2)$ are unknown functions to be determined from the boundary conditions.

Inserting the value of $A(\xi)$ from eq. (19) in eq. (17), it is found that this choice of $A(\xi)$ leads to the equations

$$\int_a^b h(s^2)\,\mathrm{d}s=0 \quad\text{and}\quad \int_c^d g(t^2)\,\mathrm{d}t=0. \tag{20a, b}$$

Further substituting $A(\xi)$ from eq. (19) in (18a), we obtain

$$\int_a^b \frac{sh(s^2)}{s^2-x^2}\,\mathrm{d}s+\int_c^d \frac{tg(t^2)}{t^2-x^2}\,\mathrm{d}t=\frac{\pi P}{2\mu},\quad x\in I_2.$$

Rewriting this equation as

$$\int_a^b \frac{sh(s^2)}{s^2-x^2}\,\mathrm{d}s=\frac{\pi}{2}F(x),\quad x\in I_2$$

where

$$F(x)=\frac{P}{\mu}-\frac{2}{\pi}\int_c^d \frac{tg(t^2)}{t^2-x^2}\,\mathrm{d}t$$

and using finite Hilbert transform technique [9], we obtain

$$h(s^2)=\frac{P}{\mu}\sqrt{\left(\frac{s^2-a^2}{b^2-s^2}\right)}-\frac{2}{\pi}\sqrt{\left(\frac{s^2-a^2}{b^2-s^2}\right)}\int_c^d\sqrt{\left(\frac{t^2-b^2}{t^2-a^2}\right)}\frac{tg(t^2)}{t^2-s^2}\,\mathrm{d}t+\frac{C_1}{\sqrt{[(s^2-a^2)(b^2-s^2)]}}, \tag{21}$$

where we have used

$$\int_a^b\sqrt{\left(\frac{b^2-x^2}{x^2-a^2}\right)}\frac{x\,\mathrm{d}x}{(s^2-x^2)(t^2-x^2)}=\frac{\pi}{2}\sqrt{\left(\frac{t^2-b^2}{t^2-a^2}\right)}\frac{1}{t^2-s^2}.$$

The constant C_1 is to be determined from eq. (20).

Substituting the value of $h(s^2)$ from (21) in (19) and using the resulting value of $A(\xi)$ in the boundary condition (18b) we obtain, using the results

$$\int_a^b\sqrt{\left(\frac{s^2-a^2}{b^2-s^2}\right)}\frac{s\,\mathrm{d}s}{(s^2-x^2)(t^2-s^2)}=\frac{\pi}{2}\left[\sqrt{\left(\frac{t^2-a^2}{t^2-b^2}\right)}-\sqrt{\left(\frac{x^2-a^2}{x^2-b^2}\right)}\right]\frac{1}{t^2-x^2}$$

and

$$\int_a^b\frac{s\,\mathrm{d}s}{(s^2-x^2)\sqrt{[(s^2-a^2)(b^2-s^2)]}}=-\frac{\pi}{2\sqrt{[(x^2-a^2)(x^2-b^2)]}}\quad\text{for } x\in I_4,$$

the singular integral equation

$$\int_c^d\sqrt{\left(\frac{t^2-b^2}{t^2-a^2}\right)}\frac{tg(t^2)}{t^2-x^2}\,\mathrm{d}t=\frac{\pi}{2}\left[\frac{P}{\mu}+\frac{C_1}{x^2-a^2}\right],\quad x\in I_4.$$

Again using the finite Hilbert transform technique [9], we obtain

$$g(t^2)=\frac{P}{\mu}\sqrt{\left[\frac{(t^2-a^2)(t^2-c^2)}{(t^2-b^2)(d^2-t^2)}\right]}+\sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)}\frac{C_1\sqrt{(t^2-c^2)}}{\sqrt{[(t^2-a^2)(t^2-b^2)(d^2-t^2)]}}+\frac{C_2\sqrt{(t^2-a^2)}}{\sqrt{[(t^2-b^2)(t^2-c^2)(d^2-t^2)]}}, \tag{22}$$

where we have used

$$\int_c^d\sqrt{\left(\frac{d^2-x^2}{x^2-c^2}\right)}\frac{x\,\mathrm{d}x}{(x^2-a^2)(x^2-t^2)}=-\frac{\pi}{2}\sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)}\frac{1}{t^2-a^2}$$

and the constant C_2 is to be determined using the condition given by eq. (20).

Next, substituting the value of $g(t^2)$ from (22) in eq. (21) and finally using the following results:

$$\int_c^d \sqrt{\left(\frac{t^2-c^2}{d^2-t^2}\right)}\frac{t\,\mathrm{d}t}{(t^2-a^2)(t^2-s^2)}=\frac{\pi}{2}\left[\sqrt{\left(\frac{c^2-a^2}{d^2-a^2}\right)}-\sqrt{\left(\frac{c^2-s^2}{d^2-s^2}\right)}\right]\frac{1}{s^2-a^2}$$

$$\int_c^d \frac{t\,\mathrm{d}t}{(t^2-s^2)\sqrt{[(t^2-c^2)(d^2-t^2)]}}=\frac{\pi}{2\sqrt{[(c^2-s^2)(d^2-s^2)]}}\quad \text{for } s\in I_2,$$

$h(s^2)$ is derived in the form

$$h(s^2)=\frac{P}{\mu}\sqrt{\left[\frac{(s^2-a^2)(c^2-s^2)}{(b^2-s^2)(d^2-s^2)}\right]}+\sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)}\frac{C_1\sqrt{(c^2-s^2)}}{\sqrt{[(s^2-a^2)(b^2-s^2)(d^2-s^2)]}}$$
$$-\frac{C_2\sqrt{(s^2-a^2)}}{\sqrt{[(b^2-s^2)(c^2-s^2)(d^2-s^2)]}}. \qquad (23)$$

To determine the values of the unknown constants C_1 and C_2, we substitute $g(t^2)$ and $h(s^2)$ given by (22) and (23) in (20) and obtain

$$C_1=\frac{K^{c,d}_{a,b}I^{a,b}_{c,d}+K^{a,b}_{c,d}J^{c,d}_{a,b}}{I^{c,d}_{a,b}I^{a,b}_{c,d}+J^{c,d}_{a,b}J^{a,b}_{c,d}}\frac{P}{\mu}\sqrt{\left(\frac{c^2-a^2}{d^2-a^2}\right)}$$

$$C_2=\frac{K^{a,b}_{c,d}I^{c,d}_{a,b}-K^{c,d}_{a,b}J^{a,b}_{c,d}}{I^{c,d}_{a,b}I^{a,b}_{c,d}+J^{c,d}_{a,b}J^{a,b}_{c,d}}\frac{P}{\mu}$$

where

$$I^{r,s}_{p,q}=\int_p^q \frac{\sqrt{(x^2-r^2)}\,\mathrm{d}x}{\sqrt{[(x^2-p^2)(x^2-q^2)(s^2-x^2)]}}$$

$$J^{r,s}_{p,q}=\int_p^q \frac{\sqrt{(x^2-p^2)}\,\mathrm{d}x}{\sqrt{[(x^2-q^2)(x^2-r^2)(s^2-x^2)]}}$$

$$K^{r,s}_{p,q}=-\int_p^q \frac{\sqrt{[(x^2-p^2)(x^2-r^2)]}}{\sqrt{[(x^2-q^2)(s^2-x^2)]}}\,\mathrm{d}x.$$

The relevant displacement and stress components in the plane of the crack can now be shown to be given by

$$v(x,0)=\int_x^b h(s^2)\,\mathrm{d}s,\quad a\leqslant x\leqslant b$$
$$=\int_x^d g(t^2)\,\mathrm{d}t,\quad c\leqslant x\leqslant d \qquad (24\text{a, b})$$

and

$$[\sigma_{yy}(x,0)]_{0<x<a}=-\frac{2\mu K}{\pi}\left[\int_a^b \frac{sh(s^2)}{s^2-x^2}\,\mathrm{d}s+\int_c^d \frac{tg(t^2)}{t^2-x^2}\,\mathrm{d}t\right]$$

$$[\sigma_{yy}(x,0)]_{b<x<c}=\frac{2\mu K}{\pi}\left[\int_a^b \frac{sh(s^2)}{x^2-s^2}\,\mathrm{d}s-\int_c^d \frac{tg(t^2)}{t^2-x^2}\,\mathrm{d}t\right]$$

$$[\sigma_{yy}(x,0)]_{x>d}=\frac{2\mu K}{\pi}\left[\int_a^b \frac{sh(s^2)}{x^2-s^2}\,\mathrm{d}s-\int_c^d \frac{tg(t^2)}{t^2-x^2}\,\mathrm{d}t\right]. \qquad (25\text{a–c})$$

Insertion of the values of $h(s^2)$ and $g(t^2)$ as given by eqs (22) and (23) in the expressions (25) yields, after some algebraic manipulation,

$$[\sigma_{yy}(x,0)]_{0<x<a}=-\frac{2\mu K}{\pi}[F_1(x)+F_2(x)+F_3(x)+F_4(x)+F_6(x)+F_7(x)]$$

$$[\sigma_{yy}(x,0)]_{b<x<c}=-\frac{2\mu K}{\pi}[F_1(x)+F_2(x)+F_3(x)+F_4(x)-F_5(x)-F_8(x)]$$

$$[\sigma_{yy}(x,0)]_{x>d}=-\frac{2\mu K}{\pi}[F_1(x)+F_2(x)+F_3(x)+F_4(x)-F_6(x)-F_7(x)] \qquad (26\text{a–c})$$

where

$$F_1(x)=\left[\frac{P}{\mu}(c^2-a^2)-C_2\right]\left[1-\sqrt{\left(\frac{a^2-x^2}{b^2-x^2}\right)}\right]\frac{\pi}{2\sqrt{[(c^2-a^2)(d^2-a^2)]}}$$

$$F_2(x)=\int_a^b\left[\frac{P}{\mu}(d^2-c^2)-C_2\frac{2u^2-d^2-c^2}{c^2-u^2}\right]\frac{g_1(u,x)}{d^2-u^2}\,\mathrm{d}u$$

$$F_3(x)=\left[\frac{P}{\mu}(c^2-a^2)+C_1\sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)}\right]\left[1-\sqrt{\left(\frac{c^2-x^2}{d^2-x^2}\right)}\right]\frac{\pi}{2\sqrt{[(c^2-a^2)(c^2-b^2)]}}$$

$$F_4(x)=\int_c^d\left[\frac{P}{\mu}(b^2-a^2)+C_1\sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)}\frac{2u^2-a^2-b^2}{u^2-a^2}\right]\frac{g_2(u,x)}{u^2-b^2}\,\mathrm{d}u$$

$$F_{5,6}(x)=\frac{\pi}{2}\sqrt{\left(\frac{d^2-a^2}{d^2-b^2}\right)}\left[\frac{C_1}{X_1}\sqrt{\left(\frac{c^2-b^2}{c^2-a^2}\right)}\mp\frac{C_2}{X_2}\right]$$

$$F_{7,8}(x)=\frac{C_1}{X_1}\sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)}L^{a,b}_{c,d}(x)\mp\frac{C_2}{X_2}L^{c,d}_{a,b}(x)$$

$$g_1(u,x)=\frac{u}{\sqrt{[(d^2-u^2)(c^2-u^2)]}}\left[\sqrt{\left(\frac{a^2-x^2}{b^2-x^2}\right)}\tan^{-1}\sqrt{\left[\frac{(a^2-x^2)(b^2-u^2)}{(b^2-x^2)(u^2-a^2)}\right]}-\tan^{-1}\sqrt{\left(\frac{b^2-u^2}{u^2-a^2}\right)}\right]$$

$$g_2(u,x)=\frac{u}{\sqrt{[(u^2-b^2)(u^2-a^2)]}}\left[\sqrt{\left(\frac{c^2-x^2}{d^2-x^2}\right)}\tan^{-1}\sqrt{\left[\frac{(c^2-x^2)(d^2-u^2)}{(d^2-x^2)(u^2-c^2)}\right]}-\tan^{-1}\sqrt{\left(\frac{d^2-u^2}{u^2-c^2}\right)}\right]$$

$$X_1=\sqrt{[(x^2-a^2)(x^2-b^2)]}$$

$$X_2=\sqrt{[(x^2-c^2)(x^2-d^2)]}$$

$$L^{p,q}_{r,s}(x)=\int_p^q\frac{(s^2-r^2)u\tan^{-1}\sqrt{\left[\frac{(u^2-p^2)(x^2-q^2)}{(q^2-u^2)(x^2-p^2)}\right]}}{\sqrt{[(s^2-u^2)^3(r^2-u^2)]}}\,\mathrm{d}u. \qquad (27\text{a–k})$$

STRESS INTENSITY FACTOR

The dynamic stress intensity factors are given by

$$N_a=\lim_{x\to a^-}\sqrt{[2(a-x)]}[\sigma_{yy}(x,0)]_{0<x<a}$$

$$N_b=\lim_{x\to b^+}\sqrt{[2(x-b)]}[\sigma_{yy}(x,0)]_{b<x<c}$$

$$N_c=\lim_{x\to c^-}\sqrt{[2(c-x)]}[\sigma_{yy}(x,0)]_{b<x<c}$$

$$N_d=\lim_{x\to d^+}\sqrt{[2(x-d)]}[\sigma_{yy}(x,0)]_{x>d}. \qquad (28\text{a–d})$$

Employing (26) in (28) we obtain

$$N_a=-\frac{\mu KC_1}{\sqrt{[a(b^2-a^2)]}}$$

$$N_b=\mu K\left[\frac{P}{\mu}\sqrt{\left[\frac{(b^2-a^2)(c^2-b^2)}{b(d^2-b^2)}\right]}+C_1\sqrt{\left[\frac{(d^2-a^2)(c^2-b^2)}{b(b^2-a^2)(d^2-b^2)(c^2-a^2)}\right]}\right.$$
$$\left.-C_2\sqrt{\left[\frac{(b^2-a^2)}{b(c^2-b^2)(d^2-b^2)}\right]}\right]$$

$$N_c = -\frac{\mu K C_2}{\sqrt{[c(d^2-c^2)]}}\sqrt{\left(\frac{c^2-a^2}{c^2-b^2}\right)}$$

$$N_d = \mu K\left[\frac{P}{\mu}\sqrt{\left[\frac{(d^2-a^2)(d^2-c^2)}{d(d^2-b^2)}\right]} + C_1\sqrt{\left[\frac{(d^2-c^2)}{d(c^2-a^2)(d^2-b^2)}\right]} + C_2\sqrt{\left[\frac{(d^2-a^2)}{d(d^2-c^2)(d^2-b^2)}\right]}\right] \qquad (29a\text{–}d)$$

It is interesting to note that the crack opening displacements depend on the crack velocity V, but in the plane of the cracks the stresses and stress intensity factors are independent of the velocity of the moving cracks in an infinite elastic medium.

STATEMENT OF PROBLEM II AND ITS FORMULATION

In this case, we consider an infinite homogeneous isotropic material with four co-planar Griffith cracks located at $Y = 0$, $a \leqslant |X| \leqslant b$, $c \leqslant |X| \leqslant d$ and subjected to uniform internal pressure q. In the absence of body force, the equations of equilibrium in terms of displacement are

$$(\lambda + 2\mu)[u_{,XX} + v_{,XY}] + \mu[u_{,YY} - v_{,XY}] = 0$$

and

$$(\lambda + 2\mu)[u_{,XY} + v_{,YY}] + \mu[v_{,XX} - u_{,XY}] = 0. \qquad (30a, b)$$

Since the problem exhibits a state of symmetry about $Y = 0$, we can restrict our attention to a single half-space occupying the region $Y \geqslant 0$.

The equations (30) are to be solved subject to the boundary conditions

$$v(X, 0) = 0, \quad |X| \leqslant a, \quad b \leqslant |X| \leqslant c, \quad |X| \geqslant d \qquad (31a\text{–}c)$$

$$\sigma_{XY}(X, 0) = 0, \quad -\infty < X < \infty \qquad (32)$$

$$\sigma_{YY}(X, 0) = -q, \quad a \leqslant |X| \leqslant b, \quad c \leqslant |X| \leqslant d. \qquad (33a, b)$$

In view of the boundary conditions, the appropriate integral solutions of eq. (30) are

$$u(X, Y) = \frac{2}{\pi}\int_0^\infty \left[C(\xi) + D(\xi)\left\{Y - \frac{1}{\xi}\frac{\lambda + 3\mu}{\lambda + \mu}\right\}\right] e^{-\xi Y}\sin(\xi X)\,d\xi$$

and

$$v(X, Y) = \frac{2}{\pi}\int_0^\infty [C(\xi) + YD(\xi)]\,e^{-\xi Y}\cos(\xi X)\,d\xi. \qquad (34a, b)$$

Therefore,

$$\sigma_{YY}(X, Y) = -\frac{4\mu}{\pi}\int_0^\infty \left[\xi C(\xi) + \left\{Y\xi - \frac{\mu}{\lambda + \mu}\right\}D(\xi)\right] e^{-\xi Y}\cos(\xi X)\,d\xi$$

$$\sigma_{XY}(X, Y) = -\frac{4\mu}{\pi}\int_0^\infty \left[\xi C(\xi) + \left\{Y\xi - \frac{\lambda + 2\mu}{\lambda + \mu}\right\}D(\xi)\right] e^{-\xi Y}\sin(\xi X)\,d\xi. \qquad (35a, b)$$

It may be noted that the displacement and stress components given by (34) and (35) cannot be derived from the corresponding expressions of the dynamic problem given in (11) and (12) on setting $M = 0$.

The functions $C(\xi)$ and $D(\xi)$ are to be determined from the boundary conditions (31)–(33), which yield

$$C(\xi) = \frac{1}{\xi}\frac{\lambda + 2\mu}{\lambda + \mu}D(\xi) \qquad (36)$$

and the following set of five integral equations

$$\int_0^\infty C(\xi)\cos(\xi X)\,d\xi = 0, \quad X \in I_1, I_3, I_5 \qquad (37a\text{–}c)$$

$$\int_0^\infty \xi C(\xi)\cos(\xi X)\,d\xi = \frac{Q\pi}{2\mu}, \quad X \in I_2, I_4, \qquad (38a, b)$$

where $Q = (\lambda + 2\mu)/2(\lambda + \mu)q$ and I_j $(j = 1, 2, \ldots, 5)$ are the intervals defined earlier in problem I.

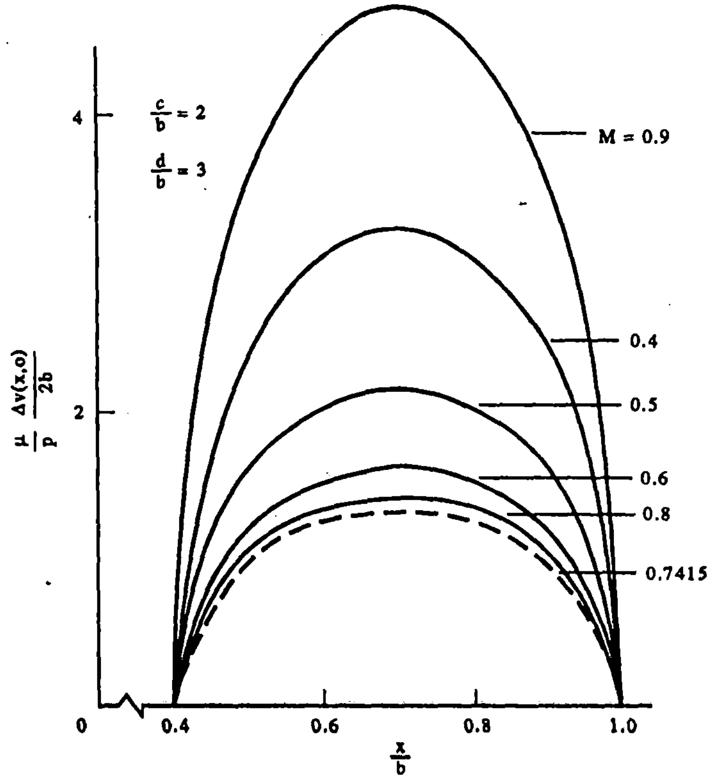


Fig. 2. Variation of crack opening displacement with x/b on the crack of the outer pair for problem I.

METHOD OF SOLUTION AND QUANTITIES OF PHYSICAL INTEREST

The integral equations given by (37) and (38) are found to be the same as those given by eqs (17) and (18) with the exception that P is replaced by Q. Therefore, the same technique as that used in problem I can be employed to obtain

$$v(X,0)=\int_X^b\left[\frac{Q}{\mu}\sqrt{\left[\frac{(s^2-a^2)(c^2-s^2)}{(b^2-s^2)(d^2-s^2)}\right]}+\sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)}\frac{C_1\sqrt{(c^2-s^2)}}{\sqrt{[(s^2-a^2)(b^2-s^2)(d^2-s^2)]}}\right.$$

$$\left.-\frac{C_2\sqrt{(s^2-a^2)}}{\sqrt{[(b^2-s^2)(c^2-s^2)(d^2-s^2)]}}\right]\mathrm{d}s,\quad a\leqslant X\leqslant b$$

$$=\int_X^d\left[\frac{Q}{\mu}\sqrt{\left[\frac{(t^2-a^2)(t^2-c^2)}{(t^2-b^2)(d^2-t^2)}\right]}+\sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)}\frac{C_1\sqrt{(t^2-c^2)}}{\sqrt{[(t^2-a^2)(t^2-b^2)(d^2-t^2)]}}\right.$$

$$\left.+\frac{C_2\sqrt{(t^2-a^2)}}{\sqrt{[(t^2-b^2)(t^2-c^2)(d^2-t^2)]}}\right]\mathrm{d}t,\quad c\leqslant X\leqslant d. \qquad (39a,b)$$

Stresses in the regions $0<X<a$, $b<X<c$, $X>d$ are found to be the same as that given in (26), the only change being that P is replaced by Q.

The amounts of energy in opening the cracks $a \leqslant |X| \leqslant b$, $c \leqslant |X| \leqslant d$ are given by $E = 2E_1 + 2E_2$, where

$$E_1 = 2\left|\int_a^b [\sigma_{YY}(X,0)v(X,0)]\,\mathrm{d}X\right|$$

$$E_2 = 2\left|\int_c^d [\sigma_{YY}(X,0)v(X,0)]\,\mathrm{d}X\right|. \tag{40a, b}$$

Equations (40) can be simplified, with the aid of (33) and (39), to

$$E_1 = -2q\left[\frac{Q}{\mu}M_{a,b}^{c,d} + (c^2-b^2)L_1 \Pi\left\{\frac{\pi}{2},\ \frac{b^2-a^2}{c^2-a^2}, r\right\} + \frac{(c^2-a^2)C_2 - c^4\dfrac{Q}{\mu}}{\sqrt{[(d^2-b^2)(c^2-a^2)]}}F\left(\frac{\pi}{2}, r\right)\right]$$

$$E_2 = 2q\left[\frac{Q}{\mu}M_{c,d}^{a,b} - (d^2-a^2)L_2 \Pi\left\{\frac{\pi}{2},\ \frac{c^2-d^2}{c^2-a^2}, r\right\} - \frac{\sqrt{[(c^2-a^2)(d^2-a^2)]}C_1 + a^4\dfrac{Q}{\mu}}{\sqrt{[(d^2-c^2)(c^2-a^2)]}}F\left(\frac{\pi}{2}, r\right)\right]$$

where

$$L_{1,2} = \frac{\left[(a^2+c^2)\dfrac{Q}{\mu} \mp C_1\sqrt{\left(\dfrac{d^2-a^2}{c^2-a^2}\right)} \mp C_2\right]}{\sqrt{[(d^2-b^2)(c^2-a^2)]}},$$

$$r = \sqrt{\left[\frac{(d^2-c^2)(b^2-a^2)}{(d^2-b^2)(c^2-a^2)}\right]}, \quad 2M_{p,q}^{r,s} = \int_{p^2}^{q^2} \frac{z^2\,\mathrm{d}z}{\sqrt{[(z-p^2)(z-q^2)(z-r^2)(s^2-z)]}}$$

and $F(\phi, r)$, $\Pi(\phi, n, r)$ are the elliptic integrals of the first and third kinds respectively.

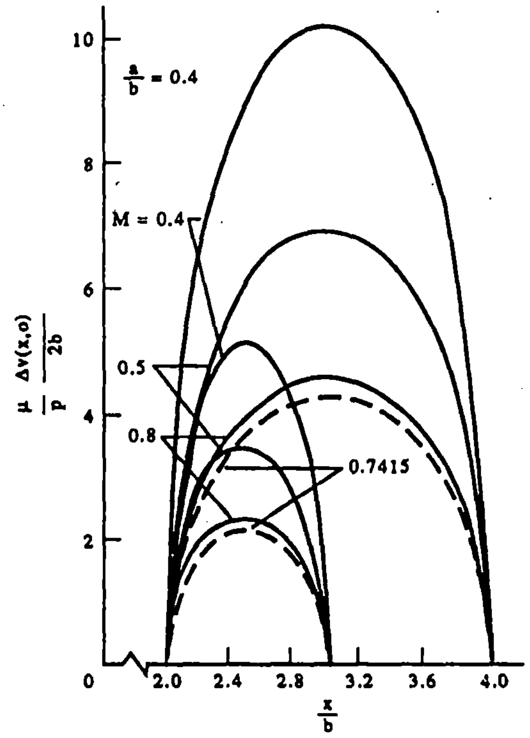


Fig. 3. Variation of crack opening displacement with x/b on the crack of the inner pair for problem I.

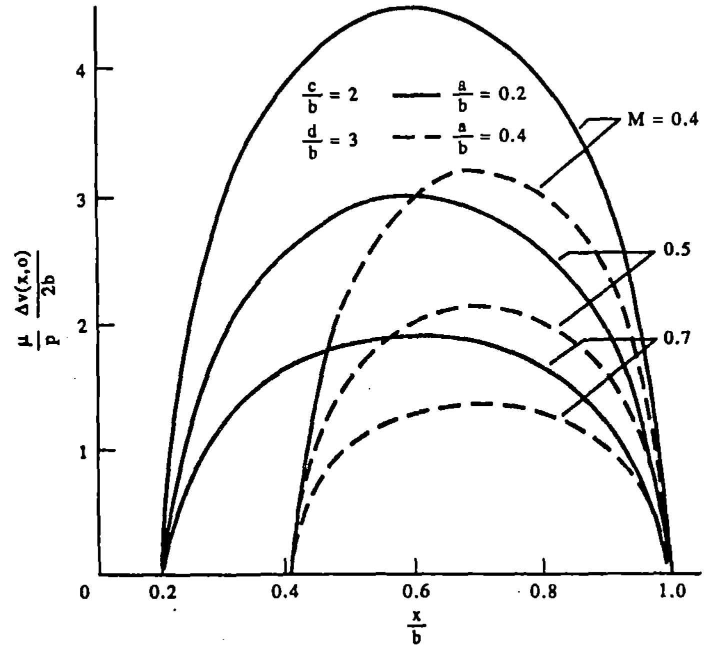


Fig. 4. Variation of crack opening displacement with x/b on the crack of the inner pair for problem I.

NUMERICAL RESULTS AND DISCUSSION

Numerical results for the stress intensity factors and crack opening displacement, defined as $\Delta v(x, 0) = v(x, 0^+) - v(x, 0^-)$, for different values of the parameters are presented in this section. Numerical calculations have been carried out for both the dynamic and static problems. As the crack velocity is less than the Rayleigh wave velocity, it is reasonable to take the value of M as less than 0.9194.

Problem I

Variations of crack opening displacement for different values of crack speed, crack lengths and the separating distance between the cracks have been plotted in Figs 2–4. It is interesting to note from these graphs that the crack opening displacement on both the cracks decreases with increase in the value of M at the onset and takes its minimum value at $M = 0.7415$, after which it increases

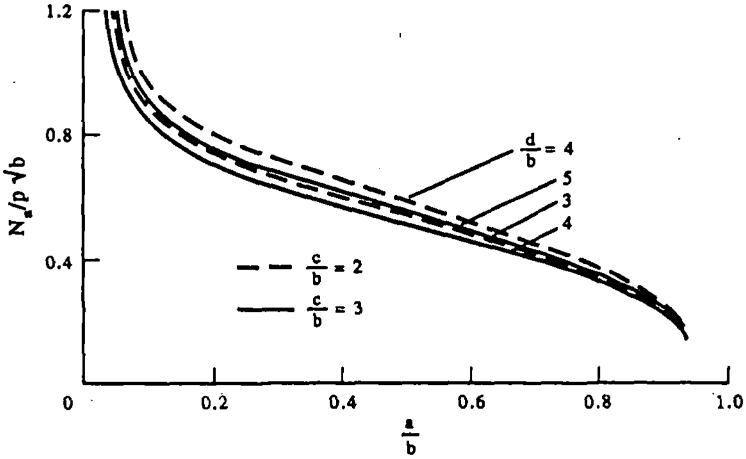


Fig. 5. Stress intensity factor vs a/b at the edge $x = a$.

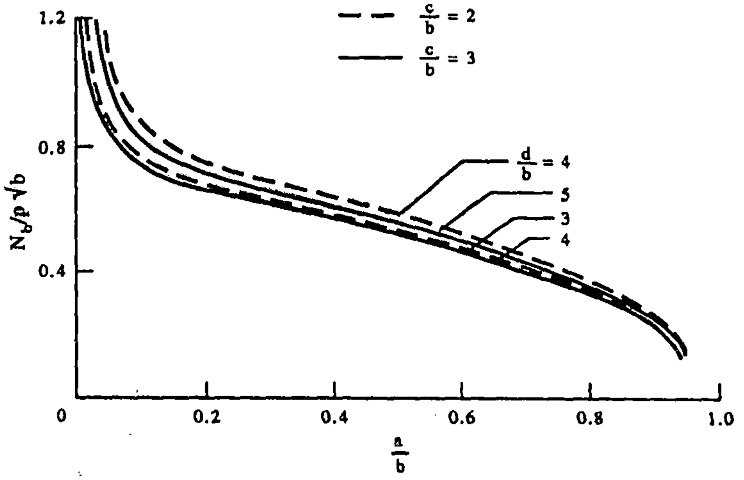


Fig. 6. Stress intensity factor vs a/b at the edge $x = b$.

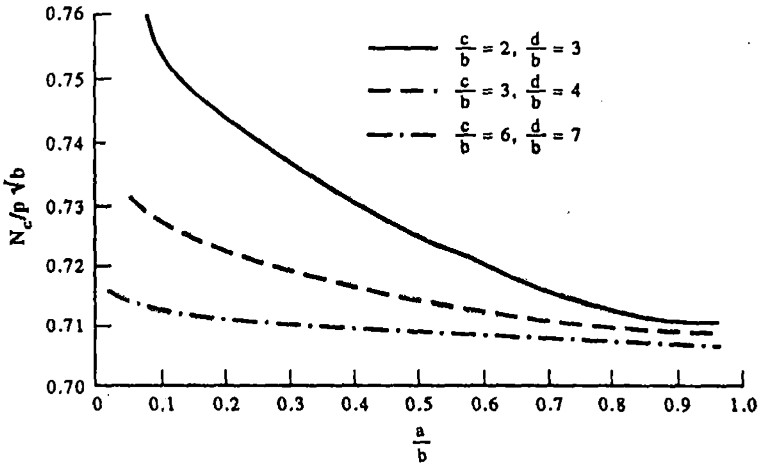


Fig. 7. Stress intensity factor vs a/b at the edge $x = c$.

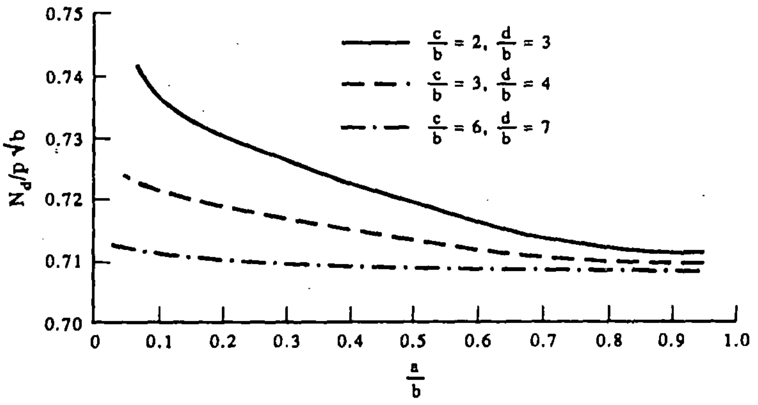


Fig. 8. Stress intensity factor vs a/b at the edge $x = d$.

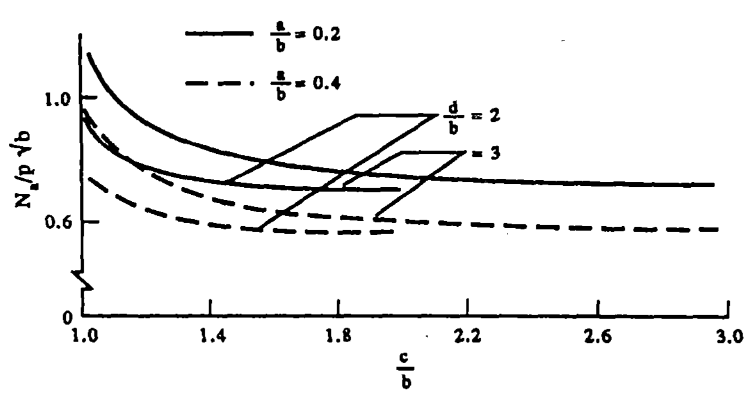


Fig. 9. Stress intensity factor vs *c*/*b* at the edge $x = a$.

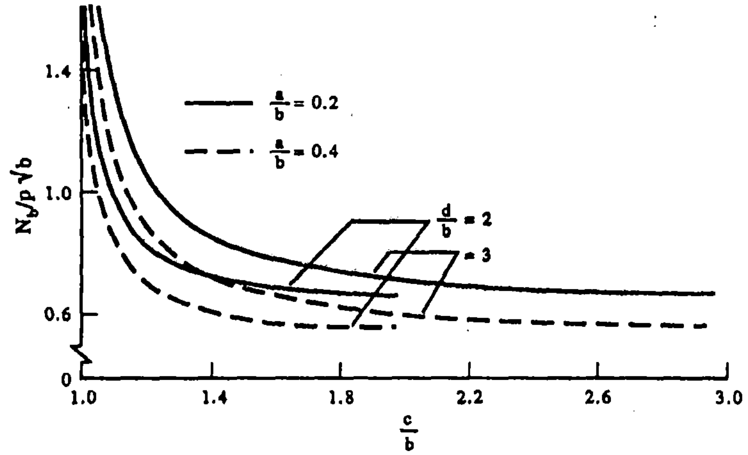


Fig. 10. Stress intensity factor vs *c*/*b* at the edge $x = b$.

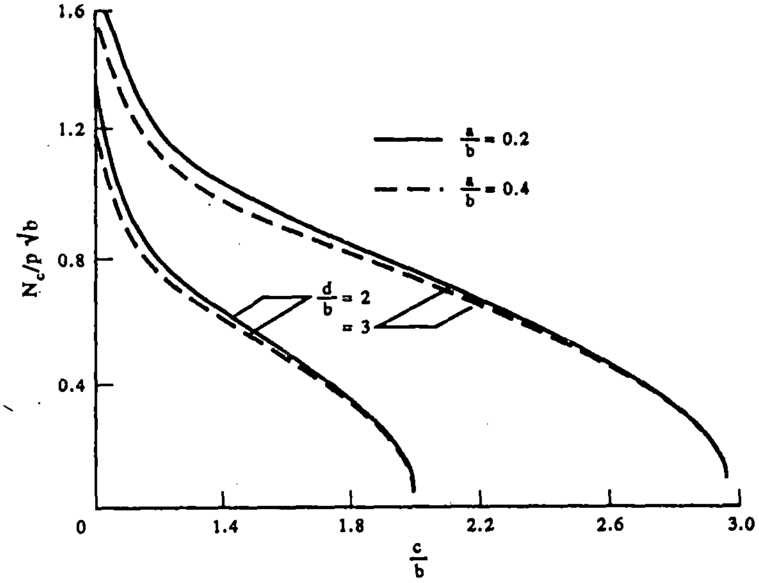


Fig. 11. Stress intensity factor vs *c*/*b* at the edge $x = c$.

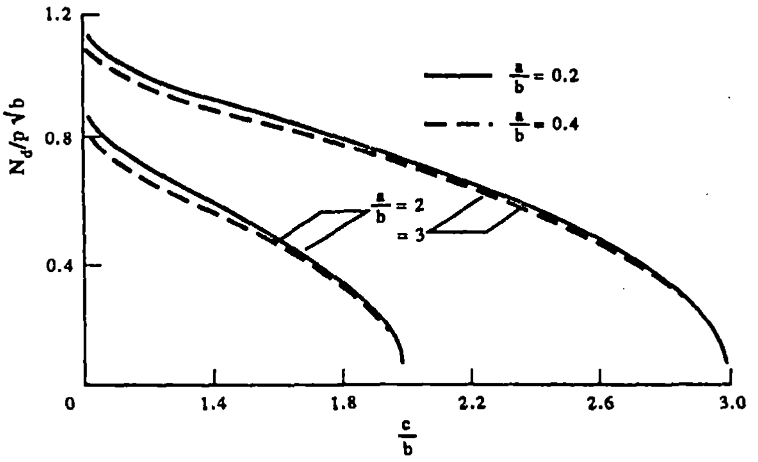


Fig. 12. Stress intensity factor vs c/b at the edge $x = d$.

with increase in the value of M. It has also been depicted in Figs 3 and 4 that on each of the cracks, crack opening displacement decreases as the crack length decreases.

It has been mentioned earlier that the stress intensity factors at the crack tips are independent of crack speed and are found to depend on the crack lengths and the separating distance between the cracks. Variation of stress intensity factors with a/b for different values of c/b, d/b and that with c/b for different values of a/b and d/b are plotted in Figs 5–8 and Figs 9–12 respectively.

It has been found that the effect of variation of the length of either the inner or the outer pair of cracks is more prominent on the stress intensity factors at the edges of the cracks whose lengths are varying compared to its effect on the stress intensity factors at the tips of the cracks whose lengths are kept fixed.

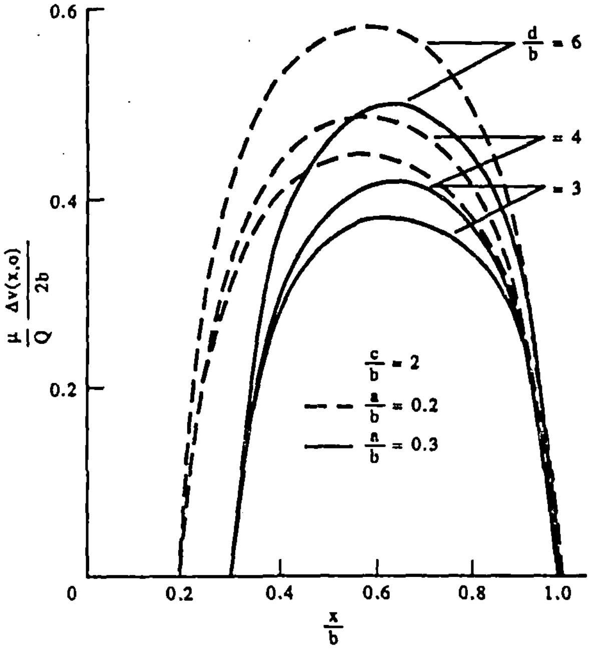


Fig. 13. Variation of crack opening displacement with X/b on the crack of the inner pair for problem II.

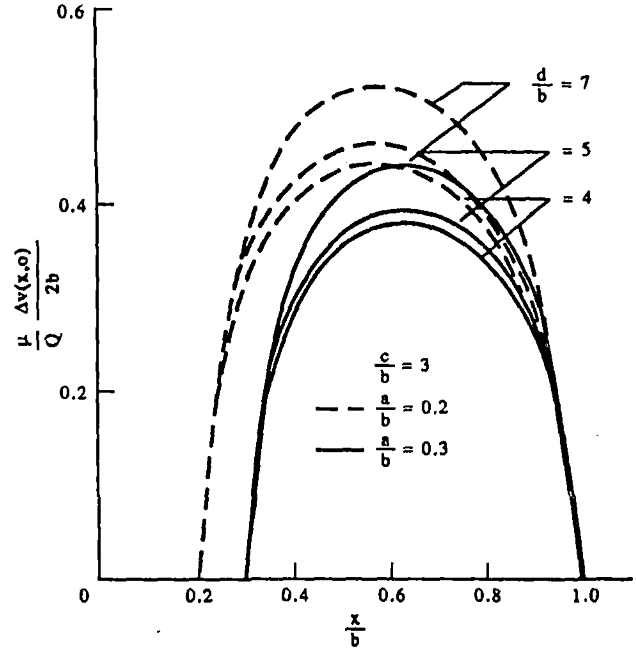


Fig. 14. Variation of crack opening displacement with X/b on the crack of the inner pair for problem II.

Problem II

Figures 13–15 show the variations of crack opening displacement for different values of the parameters a/b, c/b and d/b. They show that crack opening displacement on a crack of fixed length increases with increase in the length of the other crack, as expected from the physical standpoint.

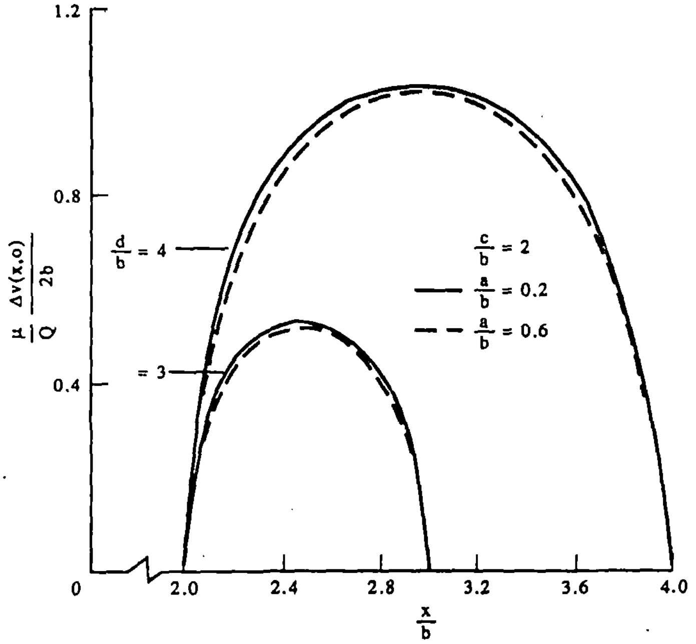


Fig. 15. Variation of crack opening displacement with X/b on the crack of the outer pair for problem II.

REFERENCES

[1] A. Bostrom, Elastic wave scattering from an interface crack. *J. appl. Mech.* **54**, 503 (1987).
[2] K. N. Srivastava, R. M. Paliya and D. S. Karaluia, Interaction of an anti-plane shear wave by a Griffith crack at the interface of two bonded dissimilar elastic half-spaces. *Int. J. Fracture* **16**, 349 (1980).
[3] F. L. Neerhoff, Diffraction of Love waves by a stress-free crack of finite width in the plane interface of a layered composite. *Appl. Sci. Res.* **35**, 237 (1979).
[4] S. Itou, Diffraction of an anti-plane shear wave by two co-planar Griffith cracks in an infinite elastic medium. *Int. J. Solids Structures* **16**, 1147 (1980).
[5] D. L. Jain and R. P. Kanwal, Diffraction of elastic waves by two co-planar Griffith cracks in an infinite elastic medium. *Int. J. Solids Structures* **8**, 961 (1972).
[6] S. Itou, Dynamic stress concentration around two co-planar Griffith cracks in an infinite elastic medium. *J. appl. Mech.* **45**, 803 (1978).
[7] A. N. Das and M. L. Ghosh, Two co-planar Griffith cracks moving along the interface of two dissimilar elastic media. *Engng Fracture Mech.* **41**, 59–69 (1992).
[8] G. K. Dhawan and R. S. Dhaliwal, On three co-planar Griffith cracks in an infinite transversely isotropic medium. *Int. J. Engng Sci.* 16, 253 (1978).
[9] K. N. Srivastava and M. Lowengrub, Finite Hilbert transform technique for triple integral equations with trigonometric kernels. *Proc. R. Soc. Edinb.* 309 (1968).

(*Received* 7 *January* 1992)

International Journal of Fracture **61**: 361–377, 1993.
© 1993 *Kluwer Academic Publishers. Printed in the Netherlands.*

Non-symmetric extension of a plane crack due to plane SH-waves in a prestressed infinite elastic medium

A.N. DAS and M.L. GHOSH
Department of Mathematics, North Bengal University, Darjeeling-734430, West Bengal, India.

Received 29 September 1992; accepted in revised form 15 January 1993

Abstract. In an infinite isotropic elastic medium initially in a state of uniform anti-plane shear, the problem of non-symmetric extension of an infinitesimal flaw into a plane shear crack due to two identical linearly varying plane SH-waves with non-parallel wave fronts has been analyzed. Fracture is assumed to initiate at a point a finite time after the waves intersect there and the crack is assumed to extend non-symmetrically along the trace of the wave intersection. Following Cherepanov [10], Cherepanov and Afanas'ev [11] the general solution of the problem has been derived in terms of analytic function of complex variable. Numerical results have been presented to illustrate the nature of variation of the stress intensity factors and the rate of energy flux into the crack edges with the speed of the crack tips and also with the time after fracture initiation.

1. Introduction

Since Broberg's [1] investigation of the solution of a crack expanding symmetrically with constant velocity under conditions of plane stress or strain in a homogeneous isotropic elastic medium in a field of spatially and time invariant tensile stress, a number of papers have appeared analyzing different geometrical situations. Craggs [2] later solved the same problem as that done by Broberg but he used the method of homogeneous function to obtain the solution. Achenbach and Brock [3] considered the wave motion generated by a uniformly extending shear crack in a body in a state of uniform anti-plane shear. The case of a crack expanding in an anisotropic medium was considered by Atkinson [4]. This work was later extended by Burridge and Willis [5], who solved the problem of a crack with elliptical cross-section expanding symmetrically with uniform speed in an anisotropic medium. All the problems mentioned above are however self-similar ones with index (0, 0) and are concerned with symmetric expansion of cracks.

Problems involving non-symmetric extension of cracks under uniform loading along the crack surface are seldom found in the literature perhaps due to severe mathematical complexity encountered in solving such problems. Following the method of homogeneous functions developed by Craggs [2], non-symmetric extension of a small flaw into a plane crack under polynomial form of loading was solved by Brock [6]. Following the same procedure, Brock [7] also solved the problem of non-symmetric extension of a crack due to incidence of plane dilatational waves. The problem of determining the dynamic stress field due to a plane dislocation moving in an infinite elastic medium was formulated by Ang and Williams [8] in terms of Fourier integral equation and solved in closed form. Recently, Georgiadis [9] has developed an integral equation approach to self-similar plane elastodynamic problems. He considered the elastodynamic problem of an expanding crack under homogeneous polynomial form loading and reduced it to the solution of a Cauchy integral equation.

In this paper, non-symmetric extension of an infinitesimal flaw into a plane shear crack at a constant rate due to the action of two identical non-parallel plane SH-waves propagating towards each other in an infinite isotropic elastic medium which is initially in a state of uniform anti-plane shear has been treated. A finite time after the crossing of the plane wave fronts, a fracture is assumed to initiate along the line where the wave fronts crossed and the crack edges are then assumed to travel non-symmetrically with different constants speeds. Superposition considerations allow the original problem to be separated into three self-similar problems with (0, 0), (0, 1) and (1, 0) as the index of self-similarity. Following Cherepanov [10], Cherepanov and Afanas'ev [11] the mentioned self-similar problems have all been formulated as some problems of Riemann and Hilbert for half-plane, which are solved easily. Out of all the existing similarity techniques, the method of Smirnov–Sobolev [12] which has been used extensively by Cherepanov [10], Cherepanov and Afanas'ev [11] being the most elegant and straight forward has been used to solve our problem. Analytical expressions for the dynamic stress intensity factors at the crack tips and also the rate of energy flux into the crack edges have been derived. Finally, the nature of the variation of the stress intensity factors and the energy flux rate at the crack tips with the velocities of the crack edges and also with the time after crack initiation have been depicted by means of graphs. The development of a crack initiating at a point being a physically realistic model from the point of view of modelling of earthquake sources, this problem also has application in seismology.

2. Formulation of the problem

Let two identical plane waves defined by

$$\sigma_{yz} = A_0 W_\pm H(W_\pm), \quad \sigma_{xz} = \pm A_0 \cot\theta_0 W_\pm H(W_\pm), \tag{1a,b}$$

referring to coordinate system (x, y, z) where

$$W_\pm = c_2 t \pm y \sin\theta_0 + x\cos\theta_0, \quad 0 \leqslant \theta_0 \leqslant \tfrac{1}{2}\pi$$

and $H(\ )$ is Heaviside's unit function, propagate through the infinite solid which is pre-stressed such that

$$\sigma^0_{yz} = \sigma, \quad \sigma^0_{xz} = 0. \tag{1c}$$

Let us assume that at $t = 0$ the non-parallel plane waves intersect at $x = y = 0$. A micro crack is assumed to appear at $t = t_0$ at $x = y = 0$ which starts to extend bilaterally along the trace of the wave intersection with uniform velocities v_1 and v_2. The expanding crack, the circular wave front associated with its motion and the plane wave front are shown in Fig. 1(a).

In effect crack extension occurs by removing the stresses which would be generated in the crack plane by the combined applied static and dynamic fields if no cracks were present.

Accordingly, both the crack faces are subjected to shear tractions equal to $-\sigma - 2A_0(c_2 t + x\cos\theta_0)$.

The anti-symmetry of this loading about the crack plane implies that it is sufficient to consider the half-plane $y > 0$ with bounding surface $y = 0$. The boundary conditions for this

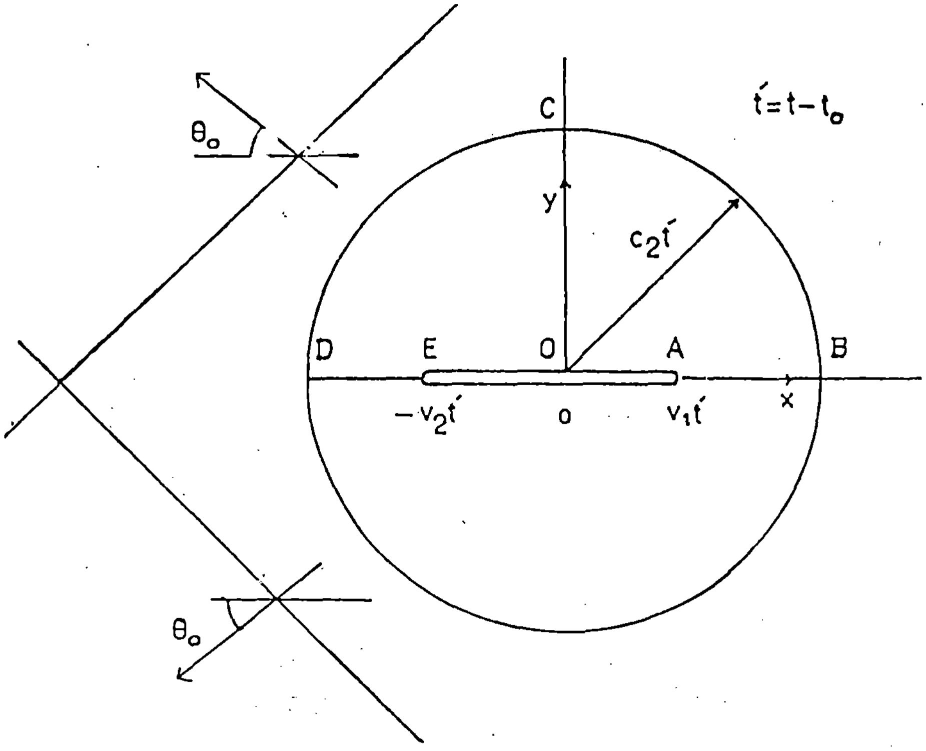


Fig. 1(a). The expanding crack and the pattern of wave front.

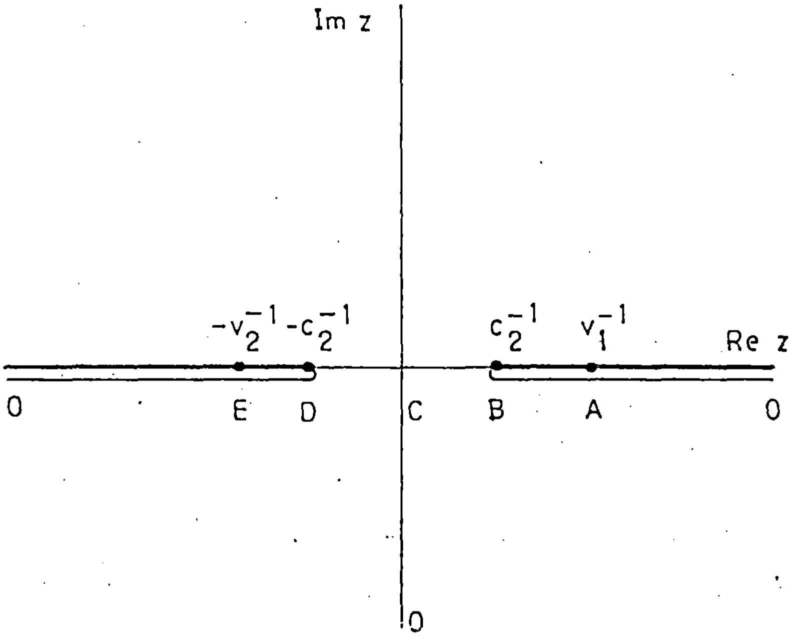


Fig. 1(b). Mapping of the interior of the semi-circle OABCDE in x–y plane on the lower half of the complex z-plane.

half-plane are then given by

$$
\begin{aligned}
&y = 0, \quad -v_2 t' < x < v_1 t'; \quad \sigma_{yz} = -\sigma - 2A_0 c_2 t_0 - 2A_0(c_2 t' + x\cos\theta_0),\\
&y = 0, \quad x > v_1 t', \quad x < -v_2 t'; \quad W = 0,
\end{aligned}
\tag{2a,b}
$$

where $t' = t - t_0$.

Equation (2a) shows that invoking superposition principle the proposed problem can be divided into three separate problems of a constant shear traction, a shearing stress linearly varying with time and shear linearly varying with distance along the crack plane.

3. Constant shear traction on the crack faces

The wave motion generated by constant shear tractions on the faces of the crack defined by $y = 0$, $-v_2 t < x < v_1 t$ has been considered in this section and for simplicity t instead of t' has been used. The boundary conditions are

$$
\begin{aligned}
&y = 0, \quad -v_2 t < x < v_1 t; \quad \sigma_{yz} = -p_0,\\
&y = 0, \quad x > v_1 t, \quad x < -v_2 t, \quad W = 0,
\end{aligned}
\tag{3a,b}
$$

where $p_0 = \sigma + 2A_0 c_2 t_0$.

The displacement W which satisfies the wave equation

$$
\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = \frac{1}{c_2^2}\frac{\partial^2 W}{\partial t^2}
\tag{4}
$$

is to be determined subject to the boundary conditions given by (3). From the boundary conditions we observe that $\partial W/\partial t$ shows dynamic similarity and is a homogeneous function of degree zero in x/t and y/t. Therefore, by the functionally invariant method of Smirnoff and Sobolev [12] we can write

$$
\frac{\partial W}{\partial t} = \operatorname{Re}\phi_0(z),
\tag{5}
$$

where

$$
t - xz + y\sqrt{c_2^{-2} - z^2} = 0.
\tag{6}
$$

The sign of the radical is to be fixed by the condition that

$$
\text{as } z \to \infty, \quad \sqrt{c_2^{-2} - z^2} = \mathrm{i}z + O(z^{-1}).
\tag{7}
$$

Equation (6) maps the semi-circular region of the cylindrical waves defined by OABCDE to the lower half of the complex cut z-plane given by

$$z = \frac{xt - iy\sqrt{t^2 - c_2^{-2}(x^2 + y^2)}}{x^2 + y^2}, \tag{8}$$

as shown in Fig. 1(b).

In view of (5) and (6) we find

$$\frac{\partial \sigma_{yz}}{\partial t} = \mu \operatorname{Re}\left[\phi_0'(z) \frac{\partial z}{\partial y} \right],$$

so that

$$\frac{\partial \sigma_{yz}}{\partial t}(x, 0, t) = \frac{1}{t} \operatorname{Re}[-\mu z \phi_0'(z)\sqrt{c_2^{-2} - z^2}]. \tag{9}$$

Therefore the boundary conditions (3) are converted to the following conditions in z-plane

$$\operatorname{Im} z = 0, \quad -v_2^{-1} < \operatorname{Re} z < v_1^{-1}, \quad \operatorname{Re} \phi_0(z) = 0, \tag{10}$$

$$\operatorname{Im} z = 0, \quad \operatorname{Re} z < -v_2^{-1}, \quad \operatorname{Re} z > v_1^{-1}, \quad \operatorname{Im} \phi_0'(z) = 0. \tag{11}$$

In order to determine the analytic function $\phi_0(z)$ subject to the conditions (10), (11) it is necessary to know the behavior of the function $\phi_0(z)$ when $z \to v_1^{-1}, -v_2^{-1}$ and $z \to \infty$. The infinite point of the z-plane corresponds to the origin of the coordinate of the physical plane where the displacement W is limited. Hence taking the representation (5) into account, we obtain

$$\operatorname{Re} \phi_0(z) = O(1) \quad \text{as} \quad z \to \infty. \tag{12}$$

Further the condition (11) after integration with respect to z may be put in the form

$$\operatorname{Im} z = 0, \quad \operatorname{Re} z < -v_2^{-1}, \quad \operatorname{Re} z > v_1^{-1}, \quad \operatorname{Im} \phi_0(z) = 0. \tag{13}$$

Moreover, the displacement derivative $\partial W/\partial t$ near the crack tips $x = v_1 t, -v_2 t$ should show square root singularities so that at $z \to v_1^{-1}, -v_2^{-1}$

$$\phi_0(z) = O(1/\sqrt{z - v_1^{-1}}), \quad O(1/\sqrt{z + v_2^{-1}}) \tag{14}$$

respectively.

The above boundary conditions given by (10) and (13) together with the consideration (12) and (14) suggest that

$$\phi_0(z) = \frac{Az + B}{\sqrt{(z - v_1^{-1})(z + v_2^{-1})}}, \tag{15}$$

where A and B are unknown constants to be determined.

Integrating (9) with respect to t it can be easily shown that for $x > 0$

$$\sigma_{yz}(x,0,t) = -\mu \,\mathrm{Re}\left\{[\phi_0(z)\sqrt{c_2^{-2} - z^2}]_{c_2^{-1}}^{t/x} + \int_{c_2^{-1}}^{t/x} \frac{z\phi_0(z)\,\mathrm{d}z}{\sqrt{c_2^{-2} - z^2}}\right\},$$

$$\sigma_{yz}(-x,0,t) = -\mu \,\mathrm{Re}\left\{[\phi_0(z)\sqrt{c_2^{-2} - z^2}]_{-c_2^{-1}}^{-t/x} + \int_{-c_2^{-1}}^{-t/x} \frac{z\phi_0(z)\,\mathrm{d}z}{\sqrt{c_2^{-2} - z^2}}\right\}. \qquad (16a,b)$$

Next using the boundary conditions that

$$\sigma_{yz}(x,0,t) = -p_0, \qquad 0 \leqslant x < v_1 t,$$
$$\sigma_{yz}(-x,0,t) = -p_0, \qquad -v_2 t < -x \leqslant 0,$$

in (16a, b) respectively we obtain two linear equations in A and B viz;

$$AI_2(v_1^{-1}, v_2^{-1}) + BI_1(v_1^{-1}, v_2^{-1}) = \frac{p_0}{\mu},$$

$$AI_2(v_2^{-1}, v_1^{-1}) - BI_1(v_2^{-1}, v_1^{-1}) = \frac{p_0}{\mu}, \qquad (17a,b)$$

where

$$I_p(u,v) = \int_{c_2^{-1}}^{u} \frac{z^p\,\mathrm{d}z}{\sqrt{(z^2 - c_2^{-2})(u - z)(v + z)}}, \quad (p = 1, 2).$$

The stress intensity factors at the crack tips $|x| = v_1 t, y = 0$ and $|x| = v_2 t, y = 0$ defined by

$$N_{01} = \mathop{\mathrm{lt}}_{x \to v_1 t} \sqrt{x - v_1 t}\,\sigma_{yz}(x,0,t),$$
$$N_{02} = \mathop{\mathrm{lt}}_{x \to v_2 t} \sqrt{x - v_2 t}\,\sigma_{yz}(-x,0,t),$$

respectively are obtained with the help of (15) and (16) as

$$N_{01} = \frac{\mu}{c_2}\sqrt{\frac{v_2 t}{v_1}}\sqrt{\frac{c_2^2 - v_1^2}{v_1 + v_2}}\,(A + Bv_1),$$

$$N_{02} = \frac{\mu}{c_2}\sqrt{\frac{v_1 t}{v_2}}\sqrt{\frac{c_2^2 - v_2^2}{v_1 + v_2}}\,(A - Bv_2). \qquad (18a,b)$$

The rate of energy flux into the extending crack edges defined by $\mathrm{d}E/\mathrm{d}t$ is given by [3]

$$\frac{1}{2}\frac{\mathrm{d}E}{\mathrm{d}t} = \int_{-\infty}^{\infty} \sigma_{yz}\frac{\partial W}{\partial t}\,\mathrm{d}x, \qquad (19)$$

which is obtained with the aid of (5), (15) and (16) for this case as

$$\frac{\mathrm{d}E_1}{\mathrm{d}t} = -\frac{\mu\pi t}{c_2(v_1 + v_2)}\left[v_2\sqrt{c_2^2 - v_1^2}(A + Bv_1)^2 + v_1\sqrt{c_2^2 - v_2^2}(A - Bv_2)^2\right], \tag{20}$$

where while carrying on the integration (19) the following result [13]

$$\frac{H(v)}{\sqrt{v}}\,\frac{H(-v)}{\sqrt{-v}} = \frac{\pi}{2}\delta(v) \tag{21}$$

has been used.

4. Problem of linearly increasing shear traction with time on the crack faces

For the case of shear tractions on the faces of the crack increasing linearly with time, the boundary conditions are

$$y = 0, \quad -v_2 t < x < v_1 t; \quad \sigma_{yz} = -p_1 t, \tag{22}$$

$$y = 0, \quad x > v_1 t, \quad x < -v_2 t, \quad W = 0, \tag{23}$$

where $p_1 = 2A_0 c_2$.

The second order derivative $\partial^2 W/\partial t^2$ now shows dynamic similarity which can be taken as the real part of the analytic function $\phi_1(z)$ so that

$$\frac{\partial^2 W}{\partial t^2} = \mathrm{Re}\,\phi_1(z), \tag{24}$$

which implies

$$\frac{\partial^2 \sigma_{yz}}{\partial t^2}(x, 0, t) = \frac{1}{t}\,\mathrm{Re}\left[-\mu z\phi_1'(z)\sqrt{c_2^{-2} - z^2}\right], \tag{25}$$

where z is given by (8) and $\phi_1(z)$ satisfies the conditions

$$\mathrm{Im}\, z = 0, \quad -v_2^{-1} < \mathrm{Re}\, z < v_1^{-1}, \quad \mathrm{Re}\,\phi_1(z) = 0, \tag{26}$$

$$\mathrm{Im}\, z = 0, \quad \mathrm{Re}\, z < -v_2^{-1}, \quad \mathrm{Re}\, z > v_1^{-1}, \quad \mathrm{Im}\,\phi_1'(z) = 0. \tag{27}$$

Integrating (24), we obtain

$$W = \tfrac{1}{2}x^2\,\mathrm{Re}\int_{v_1^{-1}}^{z} (z - \tau)^2 \phi_1'(\tau)\,\mathrm{d}\tau, \tag{28}$$

$$= \tfrac{1}{2}x^2\,\mathrm{Re}\int_{v_1^{-1}}^{z} 2(z - \tau)\phi_1(\tau)\,\mathrm{d}\tau, \tag{29}$$

so that

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left\{\frac{W}{x^2}\right\} = \operatorname{Re}\phi_1(z). \tag{30}$$

Taking into consideration the facts that near the crack tips $x = v_1 t,\ -v_2 t;\ y = 0$ the displacement W varies in direct proportion to the factors $\sqrt{v_1 t - x}$, $\sqrt{v_2 t + x}$ respectively and that as $z \to \infty$,

$$\operatorname{Re}\phi_1(z) = O(1),$$

we have in view of the conditions (26), (27) and also (30), the result that

$$\phi_1(z) = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[(Cz + D)\sqrt{(z - v_1^{-1})(z + v_2^{-1})}\right], \tag{31}$$

where the constants C, D are to be determined from the condition that on the crack surface stress $\sigma_{yz} = -p_1 t$.

From (25) after integration, we derive for $x > 0$.

$$\sigma_{yz}(x, 0, t) = -\mu x \operatorname{Re}\int_{c_2^{-1}}^{t/x}\left\{\sqrt{c_2^{-2} - \tau^2} + \frac{\tau(t/x - \tau)}{\sqrt{c_2^{-2} - \tau^2}}\right\}\phi_1(\tau)\,\mathrm{d}\tau,$$

$$\sigma_{yz}(-x, 0, t) = \mu x \operatorname{Re}\int_{-c_2^{-1}}^{-t/x}\left\{\sqrt{c_2^{-2} - \tau^2} - \frac{\tau(t/x + \tau)}{\sqrt{c_2^{-2} - \tau^2}}\right\}\phi_1(\tau)\,\mathrm{d}\tau. \tag{32a,b}$$

Therefore, using the boundary conditions that

$$\begin{aligned} &\sigma_{yz}(x, 0, t) = -p_1 t, \qquad 0 \leqslant x < v_1 t,\\ &\sigma_{yz}(-x, 0, t) = -p_1 t, \quad -v_2 t < -x \leqslant 0, \end{aligned}$$

we obtain by the help of (32a,b) after simplification

$$\begin{aligned} &C J_1(v_1^{-1}, v_2^{-1}) + D J_2(v_1^{-1}, v_2^{-1}) = \frac{p_1}{\mu},\\ &C J_1(v_2^{-1}, v_1^{-1}) - D J_2(v_2^{-1}, v_1^{-1}) = \frac{p_1}{\mu}, \end{aligned} \tag{33a,b}$$

where

$$\begin{aligned} J_1(v_1^{-1}, v_2^{-1}) = \int_{c_2^{-1}}^{v_1^{-1}} &[\{8\tau + 3(v_2^{-1} - v_1^{-1})\}M(\tau, v_1^{-1}, v_2^{-1}) + N(\tau, v_1^{-1}, v_2^{-1})\cdot\\ &\cdot\{4\tau^2 + 3\tau(v_2^{-1} - v_1^{-1}) - 2(v_1 v_2)^{-1}\}]\,\mathrm{d}\tau, \end{aligned}$$

$$J_2(v_1^{-1}, v_2^{-1}) = \int_{c_2^{-1}}^{v_1^{-1}} [2M(\tau, v_1^{-1}, v_2^{-1}) + N(\tau, v_1^{-1}, v_2^{-1})\{2\tau + (v_2^{-1} - v_1^{-1})\}]\,\mathrm{d}\tau,$$

with

$$M(\tau, v_1^{-1}, v_2^{-1}) = \frac{v_1 \tau \sqrt{v_1^{-1} - \tau}}{2\sqrt{(\tau + v_2^{-1})(\tau^2 - c_2^{-2})}},$$

$$N(\tau, v_1^{-1}, v_2^{-1}) = \frac{\tau v_1}{4\sqrt{\tau^2 - c_2^{-2}}}\left[\frac{3}{\sqrt{(\tau + v_2^{-1})(v_1^{-1} - \tau)}} - \frac{\sqrt{v_1^{-1} - \tau}}{(\tau + v_2^{-1})^{3/2}}\right].$$

The stress intensity factors at the crack tips defined by

$$N_{11} = \operatorname*{lt}_{x \to v_1 t} \sqrt{x - v_1 t}\,\sigma_{yz}(x, 0, t),$$

$$N_{12} = \operatorname*{lt}_{x \to v_2 t} \sqrt{x - v_2 t}\,\sigma_{yz}(-x, 0, t),$$

are found to be

$$N_{11} = \frac{\mu t}{2c_2}\sqrt{\frac{t}{v_1 v_2}}\sqrt{(c_2^2 - v_1^2)(v_1 + v_2)}(C + Dv_1),$$

$$N_{12} = \frac{\mu t}{2c_2}\sqrt{\frac{t}{v_1 v_2}}\sqrt{(c_2^2 - v_2^2)(v_1 + v_2)}(C - Dv_2) \qquad \text{(34a,b)}$$

and in this case the rate of energy flux $\mathrm{d}E_2/\mathrm{d}t$ into the crack edges defined by (19) is obtained as

$$\frac{\mathrm{d}E_2}{\mathrm{d}t} = -\frac{\pi\mu t^3(v_1 + v_2)}{4c_2}[v_2^{-1}\sqrt{c_2^2 - v_1^2}(C + Dv_1)^2 + v_1^{-1}\sqrt{c_2^2 - v_2^2}(C - Dv_2)^2], \qquad (35)$$

where while carrying on the integration (19) the use of the result (21) has again been made.

5. Problem of linearly varying shear traction with distance along the crack plane

Consider the initially undisturbed half-space $y \geqslant 0$ subjected to the shear traction $-p_2 x$ over $y = 0$, $-v_2 t < x < v_1 t$. The boundary conditions are

$$y = 0, \quad -v_2 t < x < v_1 t; \quad \sigma_{yz} = -p_2 x,$$

$$y = 0, \quad x > v_1 t, \quad x < -v_2 t, \quad W = 0, \qquad \text{(36a,b)}$$

where $p_2 = 2A_0 \cos\theta_0$.

In this case, $\partial^2 W/\partial x\,\partial t$ shows dynamic similarity. So, keeping (8) in mind,

$$\frac{\partial^2 W}{\partial x\,\partial t} = \operatorname{Re}\phi_2(z),$$

with

$$\frac{\partial^2 \sigma_{yz}}{\partial x\,\partial t} = -\frac{\mu}{t}\operatorname{Re}[z\phi_2'(z)\sqrt{c_2^{-2} - z^2}], \qquad (37a,b)$$

where $\phi_2(z)$ satisfies the conditions

$$\begin{aligned} &\operatorname{Im} z = 0, \quad -v_2^{-1} < \operatorname{Re} z < v_1^{-1}, \quad \operatorname{Re}\phi_2(z) = 0,\\ &\operatorname{Im} z = 0, \quad \operatorname{Re} z < -v_2^{-1}, \quad \operatorname{Re} z > v_1^{-1}, \quad \operatorname{Im}\phi_2'(z) = 0. \end{aligned} \qquad (38a,b)$$

From (37a) after integration it is found that

$$W = -x^2 \operatorname{Re}\int_{v_1^{-1}}^{z} \tau^{-1}(z-\tau)\phi_2(\tau)\,\mathrm{d}\tau,$$

so that

$$-z^2\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{1}{t}\frac{\partial W}{\partial t}\right\} = \operatorname{Re}\phi_2(z).$$

Since $\partial W/\partial t$ near the crack tips should show square root singularity and also since $\operatorname{Re}\phi_2(z) = O(1)$ as $z \to \infty$, we have in view of the conditions (38)

$$\phi_2(z) = z^2\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{Rz^{-1} + L}{\sqrt{(z - v_1^{-1})(z + v_2^{-1})}}\right], \qquad (39)$$

where the constants R, L are to be determined.

Equation (37b) can be integrated to obtain for $x > 0$

$$\sigma_{yz}(x,0,t) = \mu x \operatorname{Re}\int_{c_2^{-1}}^{t/x}\left\{\frac{t}{x\tau^2}\sqrt{c_2^{-2} - \tau^2} + \frac{t - \tau x}{x\sqrt{c_2^{-2} - \tau^2}}\right\}\phi_2(\tau)\,\mathrm{d}\tau,$$

$$\sigma_{yz}(-x,0,t) = \mu x \operatorname{Re}\int_{-c_2^{-1}}^{-t/x}\left\{\frac{t}{x\tau^2}\sqrt{c_2^{-2} - \tau^2} + \frac{t + \tau x}{x\sqrt{c_2^{-2} - \tau^2}}\right\}\phi_2(\tau)\,\mathrm{d}\tau. \qquad (40a, b)$$

So using the boundary conditions that

$$\begin{aligned} &\sigma_{yz}(x,0,t) = -p_2 x, \qquad 0 \leqslant x < v_1 t,\\ &\sigma_{yz}(-x,0,t) = p_2 x, \qquad -v_2 t < -x \leqslant 0, \end{aligned}$$

it is found by the help of (39), (40)

$$-RK_1(v_1^{-1}, v_2^{-1}) + LK_2(v_1^{-1}, v_2^{-1}) = \frac{p_2}{\mu},$$

$$RK_1(v_2^{-1}, v_1^{-1}) + LK_2(v_2^{-1}, v_1^{-1}) = \frac{p_2}{\mu}, \tag{41}$$

where

$$K_1(v_1^{-1}, v_2^{-1}) = \int_{c_2^{-1}}^{v_1^{-1}} [P(\tau, v_1^{-1}, v_2^{-1}) - \tau^{-1}Q(\tau, v_1^{-1}, v_2^{-1})]\,\mathrm{d}\tau,$$

$$K_2(v_1^{-1}, v_2^{-1}) = \int_{c_2^{-1}}^{v_1^{-1}} Q(\tau, v_1^{-1}, v_2^{-1})]\,\mathrm{d}\tau,$$

$$P(\tau, v_1^{-1}, v_2^{-1}) = -\frac{\sqrt{v_1^{-1} - \tau}}{\sqrt{(\tau + v_2^{-1})(\tau^2 - c_2^{-2})}}$$

and

$$Q(\tau, v_1^{-1}, v_2^{-1}) = \frac{\tau^2}{\sqrt{\tau^2 - c_2^{-2}}}\left[\frac{\sqrt{v_1^{-1} - \tau}}{2(\tau + v_2^{-1})^{3/2}} - \frac{(2v_1^{-1} + \tau)}{2\tau\sqrt{(\tau + v_2^{-1})(v_1^{-1} - \tau)}}\right].$$

In this case, the stress intensity factors are obtained as

$$N_{21} = \lim_{x \to v_1 t} \sqrt{x - v_1 t}\,\sigma_{yz}(x, 0, t) = -\frac{\mu t}{c_2}\sqrt{\frac{v_2 t}{v_1}}\sqrt{\frac{c_2^2 - v_1^2}{v_1 + v_2}}\,(Rv_1^2 + Lv_1),$$

$$N_{22} = \lim_{x \to v_2 t} \sqrt{x - v_2 t}\,\sigma_{yz}(-x, 0, t) = -\frac{\mu t}{c_2}\sqrt{\frac{v_1 t}{v_2}}\sqrt{\frac{c_2^2 - v_2^2}{v_1 + v_2}}\,(Rv_2^2 - Lv_2). \tag{42a,b}$$

The rate of energy flux $\mathrm{d}E_3/\mathrm{d}t$ into the extending crack edges is found to be

$$\frac{\mathrm{d}E_3}{\mathrm{d}t} = 2\int_{-t}^{t} \sigma_{yz}\frac{\partial W}{\partial t}\,\mathrm{d}x = -\frac{\mu\pi t^3}{c_2(v_1 + v_2)}$$

$$\times\,[v_2\sqrt{c_2^2 - v_1^2}(Rv_1^2 + Lv_1)^2 + v_1\sqrt{c_2^2 - v_2^2}(Rv_2^2 - Lv_2)^2], \tag{43}$$

where the result (21) has been used.

6. Particular case: $v_1 = v_2$

If we set $v_1 = v_2 = v$ in all the cases solved above, the following results are obtained

(i) For the case of constant traction $\sigma_{yz} = -p_0$ on the crack faces, we find from (17) that

$$B = 0, \quad A = \frac{vp_0}{\mu E(q)},$$

where $E(q)$ is the complete elliptic integral of second kind and $q = \sqrt{1 - v^2/c_2^2}$. Equations (18) yield the stress intensity factor at the crack tips as

$$N_0 = N_{01} = N_{02} = \frac{A\mu\sqrt{t}}{c_2}\sqrt{\frac{c_2^2 - v^2}{2v}}.$$

Also from (20) we obtain

$$\frac{\mathrm{d}E_1}{\mathrm{d}t} = -\frac{\mu\pi t}{c_2}\sqrt{c_2^2 - v^2}A^2.$$

(ii) For the case of shear traction $\sigma_{yz} = -p_1 t$ on the crack faces increasing linearly with time, it is found from (39) that

$$D = 0, \quad C = \frac{p_1 v}{\mu I},$$

where

$$I = 2E(q) - F(q) + \frac{2c_2^2}{(v + c_2)(v^2 - c_2^2)}\{2v\Pi(r^2, r) + (v + c_2)F(r)\},$$

$F(r), \Pi(r^2, r)$ are complete elliptic integrals of first and third kind respectively and $r = (c_2 - v)/(c_2 + v)$.

In this case, the stress intensity factors and the rate of energy flux into the extending crack tips given by (34) and (35) can be simplified to

$$N_1 = N_{11} = N_{12} = \frac{C\mu t}{c_2}\sqrt{\frac{t}{2v}}\sqrt{c_2^2 - v^2}$$

and

$$\frac{\mathrm{d}E_2}{\mathrm{d}t} = -\frac{\mu\pi t^3}{c_2}\sqrt{c_2^2 - v^2}C^2.$$

(iii) For the case of shear traction $\sigma_{yz} = -p_2 x$ on the crack faces, it is obvious from (41) that

$$R = 0, \quad L = \frac{p_2 v}{\mu J},$$

where

$$J = \frac{2c_2^2}{(v + c_2)(v^2 - c_2^2)}\{2v\Pi(r^2, r) + (v + c_2)F(r)\} - E(q) - F(q)$$

and it is found from (42), (43) that stress intensity factors at the crack tips and the rate of energy flux into the extending crack edges in this case are given by

$$N_2 = N_{21} = -N_{22} = \frac{-\mu t L}{c_2}\sqrt{\frac{vt}{2}}\sqrt{c_2^2 - v^2}$$

and

$$\frac{dE_3}{dt} = -\frac{\mu\pi t^3}{c_2}\sqrt{c_2^2 - v^2}L^2 v^2.$$

7. Numerical results and discussions

The solution of the original crack problem is obtained by taking $p_0 = \sigma + 2A_0 c_2 t_0$, $p_1 = 2A_0 c_2$ and $p_2 = 2A_0 \cos\theta_0$ and superposing the results obtained in Sections 3–5 with the stress fields given by (1). Taking together the results obtained in the Sections 3–5 it is possible to write the stress intensity factors at the edges of the crack and the rate of energy flux into the extending crack edges as

$$S_1 = \frac{N_{01} + N_{11} + N_{21}}{\sigma\sqrt{v_1 t_0}} = \sqrt{\frac{u_2\tau}{u_1 + u_2}}\,\mu H_+(u_1, u_2, \tau),$$

$$S_2 = \frac{N_{02} + N_{12} + N_{22}}{\sigma\sqrt{v_1 t_0}} = \sqrt{\frac{u_2\tau}{u_1 + u_2}}\,\mu H_-(u_2, u_1, \tau) \tag{44a,b}$$

and

$$En = \frac{\mu}{t_0 c_2^2 \sigma^2}\frac{d}{dt}(E_1 + E_2 + E_3) = -\frac{\pi u_2 \mu^2}{u_1 + u_2}\left[G_+(u_1, u_2, \tau) + \frac{u_1}{u_2}G_-(u_2, u_1, \tau)\right], \tag{45}$$

where

$$H_{\pm}(u_1,u_2,\tau)=\sqrt{1-u_1^2}\left[\frac{1+\Delta}{p_0}\left(\frac{A}{c_2u_1}\pm B\right)+\Delta\tau\left\{\frac{u_1+u_2}{2p_1u_2}\left(\frac{C}{c_2u_1}\pm D\right)-\frac{u_1\cos\theta_0}{p_2}\left(\pm\frac{L}{c_2u_1}+R\right)\right\}\right],$$

$$G_{\pm}(u_1,u_2,\tau)=\tau\sqrt{1-u_1^2}\left[\left(\frac{1+\Delta}{p_0}\right)^2\left(\frac{A}{c_2}\pm Bu_1\right)^2+(\Delta\tau)^2\left\{\frac{(u_1+u_2)^2}{4p_1^2u_2^2}\left(\frac{C}{c_2}\pm Du_1\right)^2+\frac{u_1^2\cos^2\theta_0}{p_2^2}\left(\frac{L}{c_2}\pm Ru_1\right)^2\right\}\right]$$

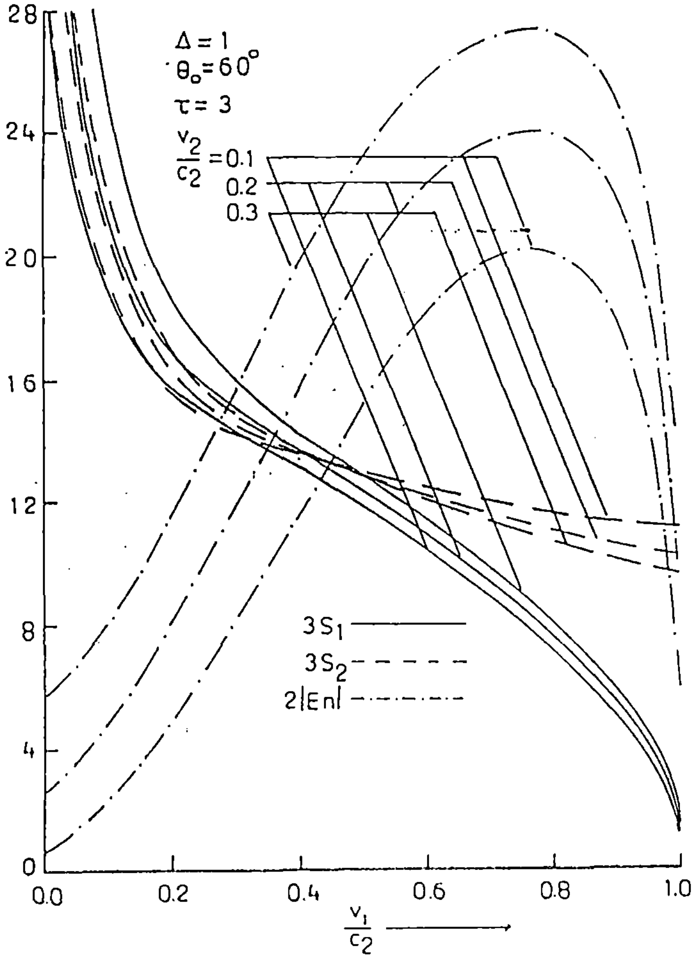


Fig. 2. Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional speed v_1/c_2.

and the parameter $\tau = (t/t_0) - 1$ is the non-dimensionalized time after crack initiation and $\Delta = 2A_0c_2t_0/\sigma$ is the ratio at $x = y = 0$ at initiation of the crack plane stress due to the plane waves and the prestress.

Also u_1, u_2 are the non-dimensional crack tip velocities given by $u_1 = v_1/c_2$ and $u_2 = v_2/c_2$.

The variations of stress intensity factors and energy flux rate given by (44) and (45) respectively with

(i) v_1/c_2 for different values of v_2/c_2 and with
(ii) τ for different values of v_1/c_2 and Δ have been presented in Figs. 2–4.

It has been shown in Fig. 2 that stress intensity factors at the edge $x = v_1t'$, $y = 0$ decrease with the increase in the values of v_1/c_2 but increase with the increase in the values of v_2/c_2 and for $v_1/c_2 < 0.45$, the stress intensity factor at the edge $x = v_2t'$, $y = 0$ increases as v_2/c_2 increases but for $v_1/c_2 > 0.45$, the variation of stress intensity factor at that edge shows an opposite character. It has also been shown in Fig. 2 that the value of energy flux rate $|En|$ increases with the increase

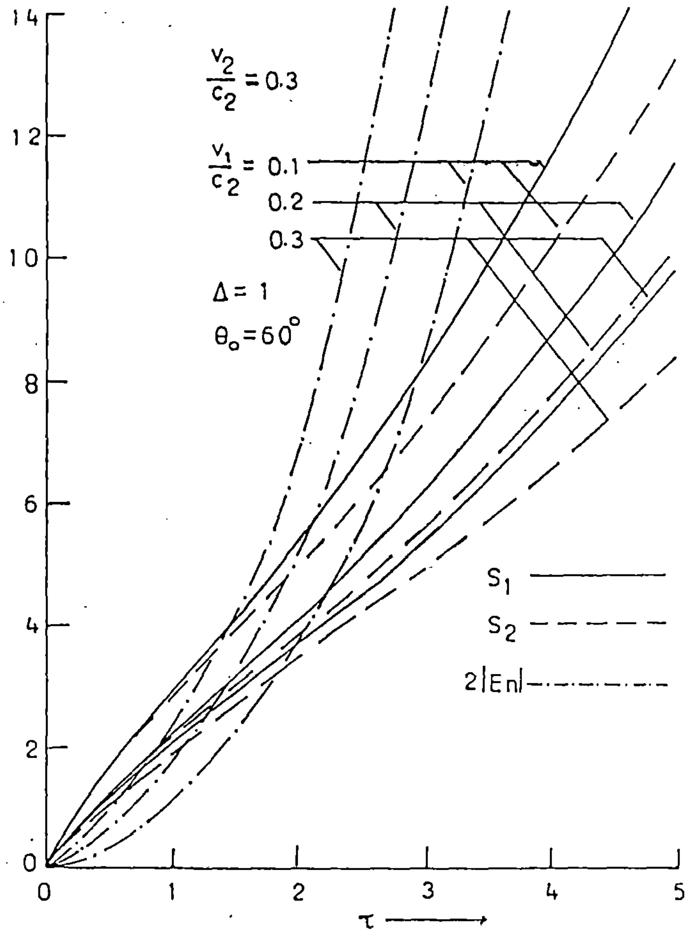


Fig. 3. Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional time after fracture initiation τ.

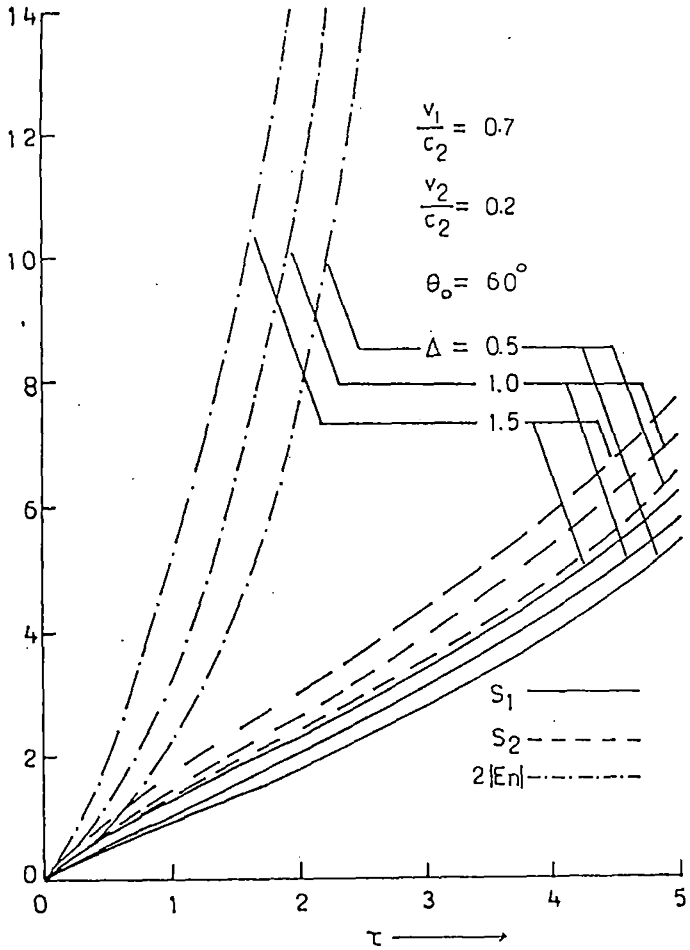


Fig. 4. Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional time after fracture initiation τ.

in the value of v_1/c_2, shows maximum at $v_1/c_2 = 0.8$ after which it decreases with the increase in the value of v_1/c_2.

In Fig. 3, the variations of S_1, S_2 and $|En|$ with τ for various values of $v_1/c_2 \leqslant v_2/c_2$ have been depicted. It may be observed from this figure that $S_1, S_2, |En|$ all increase rapidly with the increase in the value of τ. It may be noted further that for fixed value of v_2/c_2, values of stress intensity factors at the crack tips decrease with the increase in the value of v_1/c_2 whereas energy flux rate $|En|$ increases with the gradual increase in the value of v_1/c_2.

In Fig. 4, S_1, S_2 and $|En|$ are again plotted vs. τ but in this case, crack tip velocities are kept fixed whereas Δ is assumed to vary. It may be seen that increase in the values of Δ produce marked increase in the value of S_1, S_2 and $|En|$ for any fixed value of τ.

References

1. K.B. Broberg, *Arkiv för Fysik* 18 (1960) 159.
2. J.W. Craggs, *Fracture of Solids*, John Wiley and Sons, New York (1963).

3. J.D. Achenbach and L.M. Brock, *Journal of Elasticity* 1 (1971) 51.
4. C. Atkinson, *International Journal of Engineering Science* 3 (1965) 77.
5. R. Burridge and J.R. Willis, *Proceeding of Cambridge Philosophical Society* 66 (1963) 443.
6. L.M. Brock, *International Journal of Engineering Science* 14 (1976) 181.
7. L.M. Brock, *International Journal of Engineering Science* 13 (1975) 951.
8. D.D. Ang and M.L. Williams, in *Proceedings of the Fourth Annual Conference on Solid Mechanics*, The University of Texas (1959) 36.
9. H.G. Georgiadis, *Journal of Elasticity* 25 (1991) 17.
10. G.P. Cherepanov, *Mechanics of Brittle Fracture*, McGraw Hill (1979).
11. G.P. Cherepanov and E.F. Afanas'ev, *International Journal of Engineering Science* 12 (1974) 665.
12. V. Smirnoff and S. Sobolev, *Trudy Seysmologichesky Instituta*, No. 20 (1932).
13. J.D. Achenbach, *Mechanics Today* 1 (1972) 1.

Journal of Technical Physics, J. Tech. Phys., 33, 3–4, 411–420, 1992.
Polish Academy of Sciences, Institute of Fundamental Technological Research, Warszawa.

SH-WAVE PROPAGATION ACROSS A VERTICAL STEP IN TWO JOINED ELASTIC HALFSPACES

A. N. DAS and M. L. GHOSH (DARJEELING)

A SH-wave propagates in a medium consisting of two welded quarter-spaces of different materials and exhibiting a step-like change in elevation at the interface. The transmitted and reflected waves at a large distance from the step are determined by reducing the problem to an integral equation, using the Green function method and applying the method of steepest descent.

1. Introduction

The problem of propagation of elastic waves in the presence of surface irregularities has been studied by several investigators. ABUBAKAR [1] studied the effect of an irregular surface with an isolated irregularity like a trough or ditch on the incident harmonic P- and SV-waves. Propagation of a Love wave in a elastic layer having an irregular boundary overlying a rigid half-space has been treated by WOLF [2] using the perturbation technique. The transmission of elastic waves across a step-like irregularity in the surface of an elastic half-space is of great importance in seismology in connection with the propagation of waves from the ocean basins to continental regions and vice versa. KNOPOFF and HUDSON [3] studied the transmission of Love waves past a continental margin considering the crust to have an abrupt increase in thickness on the continental side. The transmission of SH-waves across a step-like irregularity at the surface of an elastic half-space was also considered by BOSE [4]. SATO [5] discussed the problem of propagation of Love waves in an elastic layer of variable thickness overlying a semi-infinite elastic medium. Approximate expressions for the transmission and reflection factors are obtained by the application of a method based on the Wiener–Hopf technique.

In this paper, we consider the propagation of SH-wave in a medium consisting of two welded quarter-spaces of different materials and having a step-like change in elevation at the vertical interface. The problem is reduced to an integral equation by using the Fourier transform and Green's function methods and, finally, by applying the method of steepest descent, the transmitted and reflected fields at large distance from the step have been determined. It may be mentioned in this connection that the problem of transient shear wave in a half-space composed of two elastic quarter-spaces of different materials which are subjected to time-dependent shear tractions at the free surface, parallel to plane of junction, has been solved by ACHENBACH [8]. DATTA and MITRA [7] also considered the SH-wave propagation in a composite elastic medium consisting of an elastic quarter-space welded to a uniform layer of different shear wave velocity. Recently, DAS and GHOSH [8] have solved the problem on transmission of time step SH-wave across a rectangular step using integral transform and Green's function technique.

2. Formulation of the Problem

We consider two quarter-spaces of different materials joined along the common boundary $X = 0$ in such a way that there is a step change in elevation at the free surface. We consider the coordinate axes as shown in Fig. 1. Denoting the coordinates of a point in the X-Z plane by (X, Z), we take the incident plane SH-wave as $\exp[i(\omega t - K_2 X)]$ where $K_2 = \omega / c_2$, so that the propagation proceeds from the higher side to the lower side of the step.

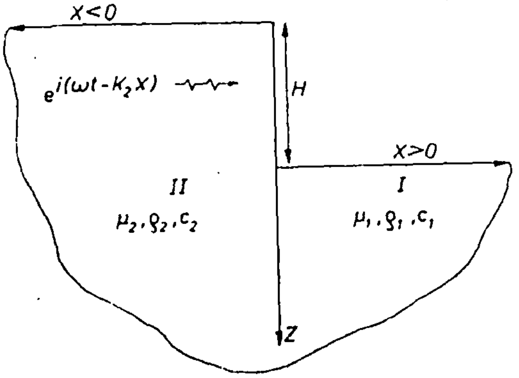


FIG. 1.

The boundaries $Z = 0$, $X < 0$, $Z = H$, $X > 0$ and $X = 0$, $0 \leq Z \leq H$ are assumed to be stress-free. Omitting the time factor $\exp(i\omega t)$ let $V_1(X, z)$, $V_2(X, Z)$ be the SH-wave displacement component in two media (I) and (II), respectively, in Y-direction which is perpendicular to the plane of the paper.

The field equations are wave equations in the two media and boundary conditions are the following: (i) — the outer boundary is stress-free, and (ii) — the displacements and stresses are continuous on the interface $X = 0$, $Z > H$. μ, ρ, c are assumed to be the modulus of rigidity, density and shear wave velocity with appropriate subscript for each of the two media.

Introducing the dimensionless quantities

$$x = \frac{X}{H}, \quad z = \frac{Z}{H},$$

we get from the wave equations and the boundary conditions

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_1^2\right] v_1 = 0, \tag{2.1}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_2^2\right] v_2 = 0;$$

$$\mu_2 \left.\frac{\partial v_2}{\partial x}\right|_{x=0} = 0 \quad \text{for } 0 < z < 1, \tag{2.2}$$

$$\mu_2 \left.\frac{\partial v_2}{\partial z}\right|_{z=0} = 0 \quad \text{for } x < 0;$$

$$(2.3)\qquad \mu_1 \frac{\partial v_1}{\partial z}\bigg|_{z=1} = 0 \quad \text{for } x > 0, \qquad \mu_1 \frac{\partial v_1}{\partial x}\bigg|_{x=0} = \mu_2 \frac{\partial v_2}{\partial x}\bigg|_{x=0} = 0 \quad \text{for } z > 1;$$

$$(2.4)\qquad v_1(0, z) = v_2(0, z) \quad \text{for } z > 1,$$

where H is the height of the step and $k_i^2 = \omega^2 H^2 / c_i^2$, $V_i(X, Z) = v_i(x, z)$. We represent transverse displacement in the two domains $x < 0$ and $x > 0$ in the form

$$(2.5)\qquad \begin{aligned} v_2 &= 2\cos k_2 x + v_2'(x, z), \; x < 0, \; z > 0, \\ v_1 &= v_1(x, z), \; x > 0, \; z > 1. \end{aligned}$$

3. Reduction to Integral Equation and Its Solution

We introduce Green's functions $G_1(x, z : r, s)$ and $G_2(x, z : u, v)$ for the medium (I) and (II), respectively, such that $G_2(x, z : u, v)$ is the solution of

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_2^2\right] G_2(x, z : u, v) = -4\pi\delta(x - u)\delta(z - v)$$

for medium (II) with vanishing normal derivative at $x < 0$, $z = 0$ and at $x = 0$, $z > 0$. Similarly, $G_1(x, z : r, s)$ is the solution of

$$(3.2)\qquad \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_1^2\right] G_1(x, z : r, s) = -4\pi\delta(x - r)\delta(z - s)$$

and satisfies the condition of vanishing of the normal derivative at $x > 0$, $z = 1$ and at $x = 0$, $z > 1$.

From Eqs. $(2.1)_2$ and (3.1) we obtain, by applying Green's theorem to the medium (II) and using appropriate boundary condition

$$(3.3)\qquad 4\pi v_2'(u, v) = \int_1^\infty G_2(0, z : u, v) \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} dz;$$

a similar application of Green's theorem to the medium (I) yields

$$(3.4)\qquad 4\pi v_1(r, s) = -\int_1^\infty G_1(0, z : r, s) \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz.$$

Substitution of Eqs. (3.3) and (3.4) into Eq. $(2.4)_2$ yields, with the aid of Eq. $(2.4)_1$ and Eq. (2.5), the integral equation

$$(3.5)\qquad \int_1^\infty \left[G_1(0, z : 0, v) + \frac{\mu_1}{\mu_2} G_2(0, z : 0, v)\right] \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz = -8\pi.$$

The expression of Green's function for the medium (II) will now be derived using the Fourier cosine transform with respect to z; this reduces the determination of $G_2(x, z : u, v)$ to that of a Green's function for an ordinary differential equation. Accordingly, taking the Fourier cosine transform defined by

$$G_2^c = \int_0^\infty G_2(x, z : u, v) \cos(\alpha z)\, dz,$$

we obtain from Eq. (3.1)

$$\frac{d^2G_2^c}{dx^2} - (\alpha^2 - k_2^2)G_2^c = -4\pi\cos(\alpha v)\delta(x - u)$$

from which we obtain in a straightforward manner

$$(3.6)\qquad G_2(x, z : u, v) = 8\int_0^\infty \frac{e^{\beta_2 u}}{\beta_2}\cosh(\beta_2 x)\cos(\alpha v)\cos(\alpha z)\,d\alpha, \qquad u \le x \le 0,$$

$$= 8\int_0^\infty \frac{e^{\beta_2 x}}{\beta_2}\cosh(\beta_2 u)\cos(\alpha v)\cos(\alpha z)d\alpha, \qquad -\infty < x \le u,$$

where

$$\beta_2^2 = \alpha^2 - k_2^2.$$

Again introducing the Fourier cosine transform defined by

$$G_1^c = \int_0^\infty G_1(x, z : r, s)\cos\alpha(z - 1)\,d(z - 1),$$

we obtain from Eq. (3.2)

$$\frac{d^2G_1^c}{dx^2} - (\alpha^2 - k_1^2)G_1^c = -4\pi\cos\alpha(s - 1)\delta(x - r),$$

from which it follows that

$$(3.7)\ G_1(x, z : r, s) = 8\int_0^\infty \frac{e^{-\beta_1 r}}{\beta_1}\cosh(\beta_1 x)\cos\alpha(s - 1)\cos\alpha(z - 1)\,d\alpha, \qquad 0 \le x \le r$$

$$= 8\int_0^\infty \frac{e^{-\beta_1 x}}{\beta_1}\cosh(\beta_1 r)\cos\alpha(s - 1)\cos\alpha(z - 1)d\alpha, \qquad r \le x \le \infty,$$

where

$$\beta_1^2 = \alpha^2 - k_1^2.$$

On substituting the values of $G_1(0, z : 0, v)$ and $G_2(0, z : 0, v)$ from Eqs. (3.7) and (3.6) in Eq. (3.5), we obtain

$$\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz \int_0^\infty \left[\frac{\cos\alpha(v - 1)\cos\alpha(z - 1)}{\beta_1} + \frac{\mu_1}{\mu_2}\frac{\cos(\alpha v)\cos(\alpha z)}{\beta_2}\right] d\alpha = -\pi,$$

$$\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz \int_0^\infty \frac{\cos\alpha(v - 1)\cos\alpha(z - 1)}{\beta_1}\,d\alpha = -\pi - \frac{\mu_1}{\mu_2}\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz \times$$

$$\times \int_0^\infty \frac{\cos(\alpha v)\cos(\alpha z)}{\beta_2}\,d\alpha.$$

Taking the inverse Fourier cosine transform with respect to α, we get

$$(3.8)\qquad \int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} \frac{\cos\alpha(z - 1)}{\beta_1}\,dz = -2\int_1^\infty \cos\alpha(v - 1)\,dv - \frac{2\mu_1}{\pi\mu_2}\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz$$

$$\times \int_0^\infty \frac{\cos(\tau z)}{\beta_2(\tau)}d\tau \int_1^\infty \cos(\tau v)\cos\alpha(v - 1)\,dv,$$

where $\beta_2(\tau)$ is obtained from β_2 by replacing α by τ. Next, using the formulae

$$\int_1^\infty \cos\alpha(v-1)\,dv = \pi\delta(\alpha), \tag{3.9}$$

$$\int_1^\infty \sin\alpha(v-1)\,dv = \frac{1}{\alpha} \tag{3.10}$$

it can be easily shown that

$$\int_1^\infty \cos(\tau v)\cos\alpha(v-1)\,dv = \frac{\pi}{2}\cos\tau[\delta(\tau+\alpha)+\delta(\tau-\alpha)] - \frac{\tau\sin\tau}{\tau^2-\alpha^2}, \tag{3.11}$$

where $\delta(x)$ is the Dirac δ-function.

Using these results and after a little algebraic manipulation it can be easily shown that (3.8) reduces to the form

$$\begin{aligned}\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} \cos\alpha(z-1)\,dz = &-\frac{2\pi\mu_2\beta_1\beta_2\delta(\alpha)}{\mu_1\beta_1+\mu_2\beta_2}+\\ &+\frac{\mu_1\beta_1}{\mu_1\beta_1+\mu_2\beta_2}\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} \sin(\alpha z)\sin\alpha\,dz+\\ &+\frac{2\mu_1\beta_1\beta_2}{\pi(\mu_1\beta_1+\mu_2\beta_2)}\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz \int_0^\infty \frac{\tau\cos(\tau z)\sin\tau}{\beta_2(\tau)(\tau^2-\alpha^2)}\,d\tau.\end{aligned} \tag{3.12}$$

4. Evaluation of Displacement

Substituting the value of $G_1(0, z : r, s)$ from Eq. (3.7) in Eq. (3.4) and then using the result (3.12), the displacement in the medium I is obtained in the form

$$\begin{aligned}v_1(x,s) = &\frac{2\mu_2k_2}{\mu_1k_1+\mu_2k_2}e^{-ik_1x} - \frac{2\mu_1}{\pi}\int_0^\infty \frac{e^{-\beta_1x}\cos\alpha(s-1)}{\mu_1\beta_1+\mu_2\beta_2}\,d\alpha\times\\ &\times\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} \sin(\alpha z)\sin\alpha\,dz - \frac{4\mu_1}{\pi^2}\int_0^\infty \frac{\beta_2e^{-\beta_1x}\cos\alpha(s-1)}{\mu_1\beta_1+\mu_2\beta_2}\,d\alpha\times\\ &\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz \int_0^\infty \frac{\tau\cos(\tau z)\sin\tau}{\beta_2(\tau)(\tau^2-\alpha^2)}\,d\tau.\end{aligned} \tag{4.1}$$

We can compute $v_1(x,s)$ iteratively solving equations (4.1) and using asymptotic values of integrals appearing the right-hand side of Eq. (4.1) for large values of x.

The first iteration yields

$$v_1(x,s) = \frac{2\mu_2k_2}{\mu_1k+\mu_2k_2}e^{-ik_1x}, \tag{4.2}$$

which is obviously the displacement in medium (I) in the absence of step change in elevation.

Deriving $\left[\frac{\partial v_1}{\partial x}\right]_{x=0}$ from the first iterate given in Eq. (4.2) and using this at the right-hand side of Eq. (4.1) with the aid of Eq. (3.10) it can be easily shown that the second term at the right-hand side of Eq. (4.1) takes the form

$$I_1 = \frac{2ik_1k_2\mu_1\mu_2}{\pi(\mu_1k_1 + \mu_2k_2)} \int_0^\infty \frac{e^{-\beta_1 x}\cos\alpha(s-1)}{\alpha(\mu_1\beta_1 + \mu_2\beta_2)} \sin 2\alpha d\alpha. \tag{4.3}$$

For large values of x it can be evaluated asymptotically by the method of steepest descent; therefore, for large x we find

$$I_1 \sim \frac{4\mu_1k_1\mu_2k_2}{(\mu_1k_1 + \mu_2k_2)^2}\left[\frac{k_1}{2\pi x}\right]^{1/2} \exp(\pi i/4 - ik_1x). \tag{4.4}$$

Similarly, with the aid of Eq. (4.2) and the result given in Eq. (3.9) the third term on the right-hand side of Eq. (4.1) reduces to the form

$$I_2 = \frac{8i\mu_1k_2\mu_2k_2}{\pi^2(\mu_1k_1 + \mu_2k_2)} \int_0^\infty \frac{\sin^2\tau}{\beta_2(\tau)} d\tau \int_0^\infty \frac{\beta_2 e^{-\beta_1 x}\cos\alpha(s-1)}{(\mu_1\beta_1 + \mu_2\beta_2)(\alpha^2 - \tau^2)} d\alpha. \tag{4.5}$$

In order to evaluate asymptotically, for large values of x, integrals of the type

$$I_1' = \int_0^\infty \frac{f(\alpha)}{(\alpha^2 - \tau^2)} e^{-\beta_1 x} d\alpha,$$

we have to take into account the residue at the singularity $\alpha = \tau$ in addition to the integral along the steepest descent. Thus we get

$$I_1' \sim \pi i \frac{f(\tau)}{2\tau} e^{-\beta_1(\tau)x} - \left[\frac{\pi k_1}{2x}\right]^{1/2} \frac{f(0)}{\tau^2} \exp(\pi i/4 - ik_1x). \tag{4.6}$$

Using this result in Eq. (4.5), for large values of x, we obtain

$$I_2 - \frac{4\mu_1k_1\mu_2k_2}{(\mu_1k_1 + \mu_2k_2)^2}\left[\frac{k_1}{2\pi x}\right]^{1/2} M \exp(\pi i/4 - ik_1x), \tag{4.7}$$

where

$$M = \frac{2ik_2}{\pi} \int_0^\infty \frac{\sin^2\tau}{\tau^2\beta_2(\tau)} d\tau = \frac{2}{\pi k_2} \int_0^1 \frac{\sin^2 k_2t}{t^2\sqrt{1-t^2}} dt + \frac{2i}{\pi k_2} \int_1^\infty \frac{\sin^2 k_2t}{t^2\sqrt{t^2-1}} dt = J - iY. \tag{4.8}$$

Following **Bose** [4] it can be shown that

$$J = \int_0^{2k_2} J_0(z)dz - J_1(2k_2) \quad \text{and} \quad Y = \int_0^{2k_2} Y_0(z)dz - Y_1(2k_2) - \frac{1}{\pi k_2}.$$

Thus from Eqs. (4.1), (4.4) and (4.7), the second iterate, for large x, is

$$v_1(x,z) = \frac{2\mu_2k_2}{\mu_1k_1 + \mu_2k_2}\left[1 + \frac{2(1-M)\mu_1k_1}{\mu_1k_1 + \mu_2k_2}\left\{\frac{k_1}{2\pi x}\right\}^{1/2} e^{\pi i/4}\right] e^{-ik_1x}. \tag{4.9}$$

If we neglect terms of order $1/x$, the higher order iterates yield the same expression.

Now, in order to find the displacement component due to the reflected wave in the medium (II), we rewrite Eqs. (3.12) with the aid of Eqs. $(2.4)_1$ and $(2.5)_1$ as

$$\int_1^\infty \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} \cos\alpha(z-1)dz = \frac{2\pi\mu_1\beta_1\beta_2\delta(\alpha)}{\mu_1\beta_1+\mu_2\beta_2} + \frac{\mu_1\beta_1}{\mu_1\beta_1+\mu_2\beta_2}\int_1^\infty \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} \times$$

$$\sin(\alpha z)\sin\alpha\, dz + \frac{2\mu_1\beta_1\beta_2}{\pi(\mu_1\beta_1+\mu_2\beta_2)}\int_1^\infty \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} dz \int_0^\infty \frac{\tau\cos(\tau z)\sin\tau}{\beta_2(\tau)(\tau^2-\alpha^2)}d\tau.$$

Taking the inverse Fourier cosine transform with respect to z, we get

$$(4.10)\quad \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} = -\frac{2i\mu_1k_1k_2}{\mu_1k_1+\mu_2k_2} + \frac{2}{\pi}\int_0^\infty \frac{\mu_1\beta_1}{\mu_1\beta_1+\mu_2\beta_2}\times$$

$$\times\cos\alpha(z-1)\,d\alpha \int_1^\infty \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} \sin(\alpha u)\sin\alpha\, du +$$

$$+\frac{4}{\pi^2}\int_0^\infty \frac{\mu_1\beta_1\beta_2}{(\mu_1\beta_1+\mu_2\beta_2)}\cos\alpha(z-1)\,d\alpha \int_1^\infty \left[\frac{dv_2'}{\partial x}\right]_{x=0} du \int_0^\infty \frac{\tau\cos(\tau u)\sin\tau}{\beta_2(\tau)(\tau^2-\alpha^2)}\,d\tau.$$

Thus substitution of (4.10) in (3.3) with the aid of (3.6) yields

$$(4.11)\quad v_2'(x,v) = \frac{-2\mu_1k_1}{\mu_1k_1+\mu_2k_2}e^{ik_2x} + \frac{4i\mu_1k_1k_2}{\pi(\mu_1k_1+\mu_2k_2)}\times$$

$$\times\int_0^\infty \frac{e^{\beta_2x}\cos(\alpha v)\sin\alpha}{\alpha\beta_2}\,d\alpha + \frac{1}{\pi}\int_0^\infty \frac{\mu_1\beta_1e^{\beta_2x}\cos(\alpha v)}{\beta_2(\mu_1\beta_1+\mu_2\beta_2)}\times$$

$$\times\sin 2\alpha\, d\alpha \int_1^\infty \left[\frac{dv_2'}{\partial x}\right]_{x=0} \sin(\alpha u)\,du - \frac{4}{\pi^2}\times$$

$$\times\int_0^\infty \frac{\alpha e^{\beta_2x}\cos(\alpha v)\sin\alpha}{\beta_2}\,d\alpha \int_0^\infty \frac{\beta_1'\mu_1\sin\alpha'}{(\mu_1\beta_1'+\mu_2\beta_2')}\frac{d\alpha'}{(\alpha^2-\alpha'^2)}\times$$

$$\times\int_1^\infty \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} \sin(\alpha'u)du + \frac{4\mu_1}{\pi^2}\int_0^\infty \frac{\beta_1e^{\beta_2x}\cos(\alpha v)}{(\mu_1\beta_1+\mu_2\beta_2)}\cos\alpha\times$$

$$\times d\alpha\int_0^\infty \frac{\tau\sin\tau}{\beta_2(\tau)(\tau^2-\alpha^2)}d\tau \int_1^\infty \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} \cos(\tau u)\,du -$$

$$-\frac{8\mu_1}{\pi^3}\int_0^\infty \frac{\alpha e^{\beta_2x}\cos(\alpha v)\sin\alpha}{\beta_2}\,d\alpha \int_0^\infty \frac{\beta_1'\beta_2'}{(\mu_1\beta_1'+\mu_2\beta_2')}\frac{d\alpha'}{(\alpha^2-\alpha'^2)}\times$$

$$\times\int_0^\infty \frac{\tau\sin\tau}{\beta_2(\tau)(\tau^2-\alpha'^2)}d\tau \int_1^\infty \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} \cos(\tau u)\,du,$$

where β_1' and β_2' are obtained from β_1 and β_2 by replacing α by α'.

Now, to solve equation (4.11) iteratively, we take the first iteration as

$$(4.12)\quad v_2'(x,v) = \frac{-2\mu_1k_1}{\mu_1k_1+\mu_2k_2}e^{ik_2x} + \frac{4i\mu k_1k_2}{\pi(\mu_1k_1+\mu_2k_2)}\int_0^\infty \frac{e^{\beta_2x}\cos(\alpha v)\sin\alpha}{\alpha\beta_2}\,d\alpha,$$

(4.13) i.e., $$\left[\frac{\partial v_2'}{\partial x}\right]_{x=0} = -\frac{2ik_2\mu_1k_1}{\mu_1k_1+\mu_2k_2}.$$

The asymptotic evaluation of Eq. (4.12) for large values of x by the method of steepest descent yield the first iterate as

$$v_2'(x,v) = \frac{-2\mu_1k_1}{\mu_1k_1+\mu_2k_2}\left[1-\left\{\frac{2k_2}{-\pi x}\right\}^{1/2} e^{\pi i/4}\right]e^{ik_2x}. \tag{4.14}$$

The second iterate is obtained by inserting the value of $\left[\frac{\partial v_2'}{\partial x}\right]_{x=0}$ given in Eq. (4.13) on the right-hand side of Eq. (4.11) and using the results

$$\int_1^\infty \sin(\alpha z)\,dz = \frac{\cos\alpha}{\alpha} \quad \text{and} \quad \int_1^\infty \cos(\alpha z)\,dz = \pi\delta(\alpha) - \frac{\sin\alpha}{\alpha}$$

and then evaluating the integrals on the right-hand side of Eq. (4.11) by the method steepest descent for large value of x. Thus we obtain the second iterate as

$$v_2'(x,v) = \frac{-2\mu_1k_1}{\mu_1k_1+\mu_2k_2}\left[1-2\left(1-\frac{(1-M)\mu_1k_1}{\mu_1k_1+\mu_2k_2}\right)\left\{\frac{k_2}{-2\pi x}\right\}^{1/2} e^{\pi i/4}\right]e^{ik_2x}, \tag{4.15}$$

where M is given in Eq. (4.8).

Thus from Eqs. (2.5) and (4.15) we get

$$\begin{aligned} v_2(x,v) = e^{-ik_2x} + \Bigg[&\frac{\mu_2k_2-\mu_1k_1}{\mu_1k_1+\mu_2k_2} + \\ &+\frac{4\mu_1k_1}{\mu_1k_1+\mu_2k_2}\left(1-\frac{(1-M)\mu_1k_1}{\mu_1k_1+\mu_2k_2}\right)\left\{\frac{k_2}{-2\pi x}\right\}^{1/2} e^{\pi i/4}\Bigg]e^{ik_2x}. \end{aligned} \tag{4.16}$$

5. Numerical Results and Discussion

To investigate the nature of the motion, we have evaluated numerically the increment in amplitude due to the step for both the transmitted and reflected waves. The results are shown in the form of graphs showing the variation of $\sqrt{x}\Delta V_{1T}$ with k_2 for different values of μ_2/μ_1 in Fig. 2 for the transmitted wave, and the variation $\sqrt{-x}\Delta V_{2R}$ with k_2 in Fig. 3 for the reflected wave, where

$$\Delta V_{1T} = |v_1| - \frac{2\mu_2k_2}{\mu_1k_1+\mu_2k_2} \quad \text{and} \quad \Delta V_{2R} = |v_{2R}| - \frac{\mu_2k_2-\mu_1k_1}{\mu_1k_1+\mu_2k_2},$$

and v_{2R} is the reflected part of v_2.

The value of $\sqrt{x}\Delta V_{1T}$ is found to increase gradually with the increase in the value of μ_2/μ_1, and for all values of μ_2/μ_1 it is found that the maximum value of $\sqrt{x}\Delta V_{1T}$ occurs at $k_2 = 0.75$. It is also observed from Fig. 2 that $\sqrt{x}\Delta V_{1T}$ is positive for all values of k_2 and μ_2/μ_1, what means that the amplitude of transmitted wave is always greater than that of the transmitted wave in the absence of the step. Moreover, with the increase in the values of k_2 the graphs show an undulating character and decreasing amplitude of the motion.

From Fig. 3 we see that the value of $\sqrt{-x}\Delta V_{2R}$ gradually decreases with the increase in the value of μ_2/μ_1 and shows a gradual increase as the value of k_2 increases.

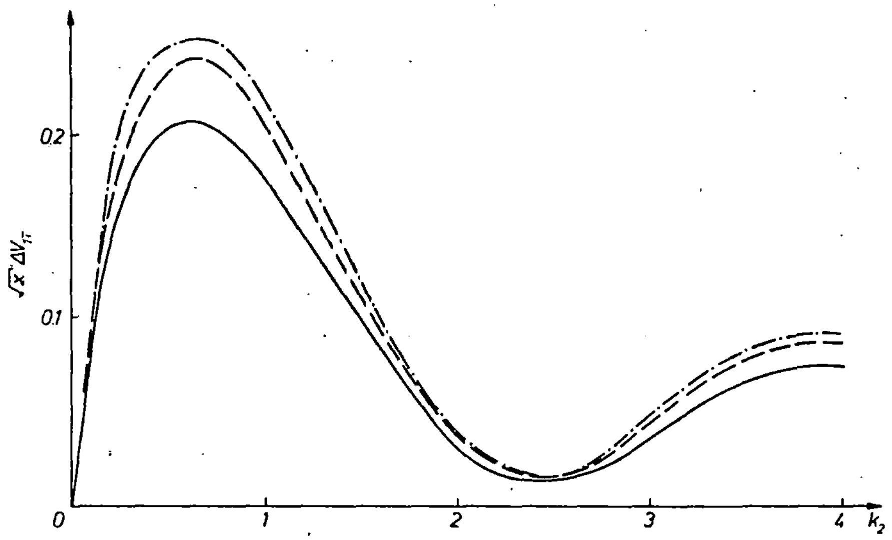


FIG. 2.

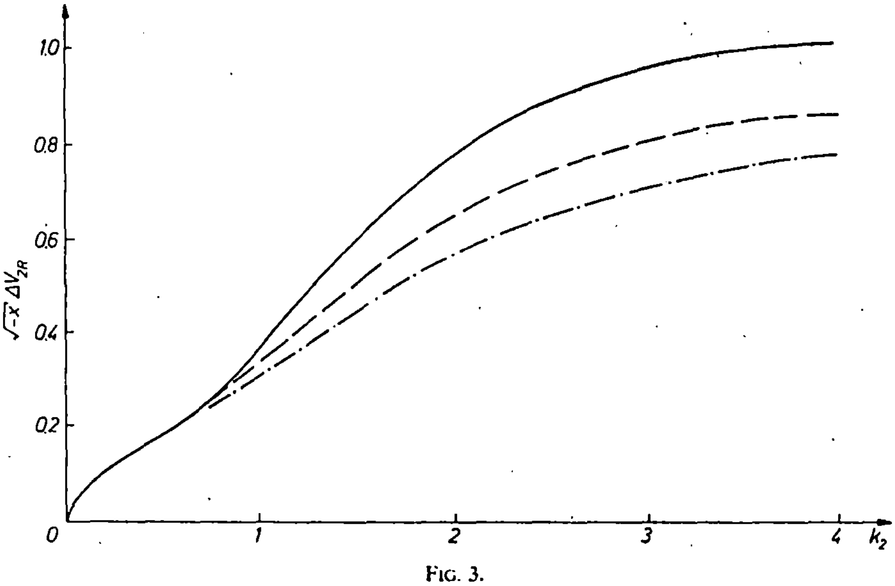


FIG. 3.

References

1. I. ABUBAKAR, *Scattering of plane elastic waves at rough surfaces*, Proc. Camb. Phil. Soc., 59, 231–248, 1963.
2. B. WOLF, *Propagation of Love waves in surface layer of varying thickness*, Pure and Appl. Geoph., 67, 76–83, 1967.
3. L. KNOPOFF and J. A. HUDSON, *Transmission of Love waves past a continental margin*, J. Geoph. Res., 69, 1649–1653, 1964.
4. S. K. BOSE, *Transmission of SH-wave across a rectangular step*, Bull. Seis. Soc. Amer., 65, 1779–1786, 1975.
5. R. SATO, *Propagation of Love waves in elastic layer of variable thickness*, J. Phys. Earth., 9, 19–31, 1961.

6. J. D. Achenbach, *Transient shear waves in two joined elastic quarter spaces*, J. App. Mech. ASME Trans., **36**, 491–496, 1969.
7. A. Dutta and A. Mitra, *SH-wave propagation in a composite elastic medium*, Pure and App. Geoph., **112**, 339–351, 1974.
8. A. N. Das and M. L. Ghosh, *Time step SH-wave transmission across a rectangular step*, Gerlands Beitrage Zur Geophysik 1992 (in press).

DEPARTMENT OF MATHEMATICS
NORTH BENGAL UNIVERSITY, DARJEELING,
WEST BENGAL, INDIA.

Received April 12, 1992.