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TWO COPLANAR GRIFFITH CRACKS MOVING IN A STRIP UNDER ANTIPLANE SHEAR STRESS

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1. Introduction

In fracture mechanics, the problem of diffraction of elastic waves by cracks of finite dimensions in a strip of elastic material has been investigated by several authors. SIH and CHEN [1] investigated the problem of propagation of a crack of finite length in a strip under plane extension. The resulting mixed bounadry value problem was reduced to the solution of a Fredholm integral equation of second kind, which was solved numerically. Closed form solution for a finite length crack moving in a strip under antiplane shear stress was also obtained by SINGH *et al.* [2]. As regards the dynamic crack problem, research has been restricted mainly to the case of a single crack because of the severe mathematical complexity encountered in finding solutions of two or more cracks. However, using finite Hilbert transform techniques developed by SRIVASTAVA and LOWENGRUB [3], LOWENGRUB and SRIVASTAVA [4] solved the statical problem of distribution of stress in an infinitely long elastic strip containing two coplanar Griffith cracks. The scattering of time-harmonic normally incident plane waves by two parallel and coplanar Griffith cracks in an infinite elastic medium has been studied by JAIN and KANWAL [5] and more recently by ITOU [6]. The problem of diffraction of elastic waves by two coplanar cracks moving steadily along the interface of two bonded dissimilar elastic media has recently been studied by DAS and GHOSH [7] using Hilbert transform technique.

In this paper we have considered the problem of propagation of two complanar YOFFE [8] cracks moving steadily in an infinitely long finite width strip. Employing Fourier transform and finite Hilbert transform technique, closed-form solutions are obtained for two cases of practical interest. Firstly, the case when the rigidly clamped edges are pulled apart in opposite directions is considered. Secondly, we have treated the case when the lateral boundaries are subjected to shearing stresses. Exact expressions for the crack opening displacement and the stress intensity factors have been derived in both the cases. Finally, numerical results for stress intensity factors are presented graphically to show their variation with crack speed for different values of the lengths of the cracks.

2. Formulation of the Problem

We consider two cracks of finite lengths placed on the X-axis from $-b$ to $-a$ and from a to b with reference to the rectangular coordinate system (x, y, z) which, referred to a fixed coordinate system (X, Y, Z), is moving with constant velocity v along X-direction within the strip of elastic material occupying the region $-h' \leq Y \leq h'$, as shown in Fig. 1.

In dynamic problem of antiplane shear, the non-vanishing component of displacement W directed in the Z-direction satisfies the equation of motion

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} = \frac{1}{c_2^2}\frac{\partial^2 W}{\partial t^2}, \tag{2.1}$$

where $c_2 = (\mu/\rho)^{1/2}$ is the shear wave velocity and ρ is the density of the material. The non-vanishing components of stress are

$$\sigma_{xz} = \mu\frac{\partial W}{\partial X}, \qquad \sigma_{yz} = \mu\frac{\partial W}{\partial Y}. \tag{2.2}$$

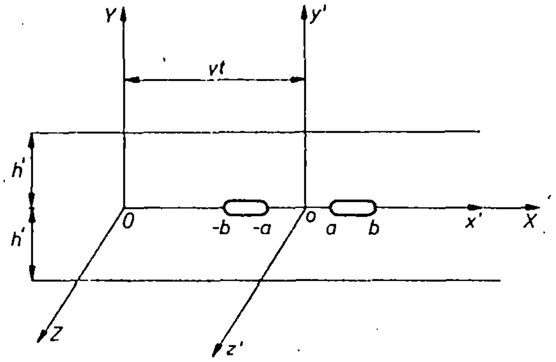

FIG. 1. Moving cracks in a strip under antiplane shear.

Using Galilean transformation $x' = X - vt$, $y' = Y$, $z' = Z$, $t' = t$, where (x', y', z') is the moving coordinate system shown in Fig. 1 and, next, introducing the dimensionless coordinates x, y, z such that $x' = xb$, $y' = yb$, $z' = zb$, $h' = hb$, Eq. (2.1) reduces to

$$s^2\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = 0, \tag{2.3}$$

with

$$s^2 = 1 - v^2/c_2^2. \tag{2.4}$$

3. Boundary Conditions

We consider two basic problems of practical interest with different boundary conditions.

PROBLEM 1. The edges of the strip $y = \pm h$ are assumed to be rigidly clamped and displaced laterally in opposite directions by an equal distance w_0, where w_0 is a constant. As a result, antiplane shear motion takes place in z-direction, whereas cracks move in the x-direction and the boundary conditions are

$$W(x, \pm h) = \pm w_0, \quad -\infty < x < \infty, \tag{3.1}$$

$$\sigma_{yz}(x, 0) = 0, \quad d < |x| < 1, \tag{3.2}$$

$$W(x, 0) = 0, \quad 0 \le |x| < d, \quad |x| > 1, \tag{3.3}$$

where $d = a/b$.

In order to apply the integral transform technique it is necessary to solve a different but equilvalent problem which can be obtained from the problem of a clamped strip (without any crack) subject to a uniform strain. The equivalent stress condition on the crack are

$$\sigma_{yz}(x,0) = -\frac{\mu w_0}{h}, \quad d < |x| < 1 \tag{3.4}$$

and the displacement must satisfy

$$W(x,0) = 0, \quad 0 \leq |x| < d, \quad |x| > 1, \tag{3.5}$$

$$W(x,\pm h) = 0, \quad -\infty < x < \infty. \tag{3.6}$$

PROBLEM 2. In this case uniform shearing stress p_0 is applied to the upper and lower boundaries $y = \pm h$ of the strip. The equivalent problem in this case involves the application of the shear stress $-p_O$ to the crack faces at $y = 0$. Accordingly, the boundary conditions are

$$\sigma_{yz}(x,\pm h) = 0, \quad -\infty < x < \infty, \tag{3.7}$$

$$\sigma_{yz}(x,0) = -p_0, \quad d < |x| < 1, \tag{3.8}$$

$$W(x,0) = 0, \quad 0 \leq |x| < d, \quad |x| > 1. \tag{3.9}$$

4.Solutions of the Problems

Due to the symmetry about (x,z)-plane we need to consider the region $0 < y < h$ only. Employing

$$F_c[A(\xi) : \xi \to x] = \sqrt{\frac{2}{\pi}} \int_0^\infty A(\xi)\cos(\xi x)\, d\xi, \tag{4.1}$$

and

$$F_s[A(\xi) : \xi \to x] = \sqrt{\frac{2}{\pi}} \int_0^\infty A(\xi)\sin(\xi x)\, d\xi, \tag{4.2}$$

we obtain the solution of Eq. (2.3) as

$$W(x,y) = F_c[A_1(\xi)\exp(-\xi ys) + A_2(\xi)\exp(\xi ys) : \xi \to x], \tag{4.3}$$

with

$$\sigma_{yz}(x,y) = \mu s F_c[\xi\{-A_1(\xi)\exp(-\xi ys) + A_2(\xi)\exp(\xi ys)\} : \xi \to x]. \tag{4.4}$$

PROBLEM 1. Using the expression for $W(x,y)$ given in Eq. (4.3) in Eq. (3.6) we get

$$A_1(\xi) = \frac{A(\xi)}{1 - \exp(-2\xi hs)},$$

$$A_2(\xi) = \frac{-A(\xi)\exp(-2\xi hs)}{1 - \exp(-2\xi hs)},$$

where $A(\xi)$ is to be determined.

From Eqs. (3.4) and (3.5) we find that $A(\xi)$ satisfies the set of triple integral equations

$$F_c[\xi A(\xi)\,\mathrm{cth}\,(\xi hs) : \xi \to x] = \frac{w_0}{hs}, \quad d < x < 1, \tag{4.5}$$

$$F_c[A(\xi) : \xi \to x] = 0, \quad 0 \le x < d, \quad x > 1. \tag{4.6}$$

Let us take

$$A(\xi) = \frac{1}{\xi}\sqrt{\frac{\pi}{2}} \int_d^1 g_1(\tau)\,\mathrm{Sech}^2\,(c\tau)\sin(\xi\tau)\,d\tau, \tag{4.7}$$

It is clear that the above choice of $A(\xi)$ satisfies Eq. (4.6) if and only if

$$\int_d^1 g_1(\tau)\,\mathrm{Sech}^2\,(c\tau)\,d\tau = 0. \tag{4.8}$$

Equation (4.5) can be written as

$$\frac{d}{dx} F_s[A(\xi)\,\mathrm{cth}\,(\xi hs) : \xi \to x] = \frac{w_0}{hs}, \quad d < x < 1. \tag{4.9}$$

Inserting Eq. (4.7) in Eq. (4.9) and using the result [9]

$$\int_0^\infty \frac{\mathrm{cth}\,(\xi hs)\sin(\xi\tau)\sin(\xi x)}{\xi}\,d\xi = \frac{1}{2}\log\left|\frac{\mathrm{th}\,(cx) + \mathrm{th}\,(c\tau)}{\mathrm{th}\,(cx) - \mathrm{th}\,(c\tau)}\right|, \tag{4.10}$$

where $c = \pi/2hs$, we obtain

$$\int_d^1 \frac{c g_1(\tau)\,\mathrm{Sech}^2\,(c\tau)\,\mathrm{th}\,(c\tau)}{\mathrm{th}^2\,(c\tau) - \mathrm{th}^2\,(cx)}\,d\tau = \frac{w_0}{hs\,\mathrm{Sch}^2\,(cx)}, \quad d < x < 1. \tag{4.11}$$

Substituting $\mathrm{th}\,(c\tau) = T_1$, Eq. (4.11) is found to reduce to the form

$$\int_{D_1}^{I_1} \frac{T_1 A(T_1^2)\,dT_1}{T_1^2 - X_1^2} = \frac{w_0}{hs(1 - X_1^2)}, \quad D_1 < X_1 < I_1, \tag{4.12}$$

where $D_1 = \mathrm{th}\,(cd)$, $I_1 = \mathrm{th}\,(c)$, $X_1 = \mathrm{th}\,(cx)$ and $A(T_1^2) = g_1(\tau)$.

Using finite Hilbert transform [3], the solution of Eq. (4.12) is

$$g_1(\tau) = A(T_1^2) = \frac{2w_0\,\mathrm{ch}\,(cd)}{\pi hs(1 - T_1^2)\,\mathrm{ch}\,(c)}\sqrt{\frac{T_1^2 - D_1^2}{I_1^2 - T_1^2}} + \\ + \frac{K_1}{[(T_1^2 - D_1^2)(I_1^2 - T_1^2)]^{1/2}}, \quad d < \tau < 1, \tag{4.13}$$

where the constant K_1 determined from Eq. (4.8) is found to be equal to

$$K_1 = \frac{2w_0\,\mathrm{ch}\,(cd)}{\pi hs\,\mathrm{ch}\,(c)} D_1^2 \cdot \left(1 - \frac{\Pi\left(\frac{\pi}{2}, \frac{I_1^2 - D_1^2}{I_1^2(1 - D_1^2)}, q\right)}{F\left(\frac{\pi}{2}, q\right)}\right), \tag{4.14}$$

where $q = (I_1^2 - D_1^2)^{1/2}/I_1$ and $F(\phi, k)$, $\Pi(\phi, n, k)$ are elliptic integrals of the first and third kind, respectively.

The expressions of displacement and shear stress on the plane of the crack are expressed as

$$W(x,0) = \frac{\pi}{2}\int_x^1 g_1(\tau)\,\mathrm{Sech}^2(c\tau)\,d\tau, \quad d < x < 1 \tag{4.15}$$

and

$$\sigma_{yz}(x,0) = \mu s c \int_d^1 \frac{g_1(\tau)\,\mathrm{Sech}^2(c\tau)\,\mathrm{th}(c\tau)\,\mathrm{Sech}(cx)}{\mathrm{th}^2(cx) - \mathrm{th}^2(c\tau)}\,d\tau, \quad 0 \le x < d, \quad x > 1 \tag{4.16}$$

Now, inserting Eq. (4.13) in Eqs. (4.15) and (4.16), we obtain

$$W(x,0) = -\frac{w_0\,\mathrm{ch}(cd)}{hsc\,\mathrm{sh}(c)}\left[F(\lambda,q)\left\{1 - D_1^2\left(1 - \frac{\Pi\left(\frac{\pi}{2}, \frac{I_1^2 - D_1^2}{I_1^2(1-D_1^2)}, q\right)}{F\left(\frac{\pi}{2}, q\right)}\right)\right\} + \right.$$
$$\left. + \frac{\mathrm{ch}^2(c)}{\mathrm{ch}^2(cd)}\Pi\left(\lambda, \frac{I_1^2 - D_1^2}{1 - I_1^2}, q\right)\right], \quad d < x < 1, \tag{4.17}$$

where

$$\mathrm{Sin}\,\lambda = \sqrt{\frac{I_1^2 - \mathrm{th}^2(cx)}{I_1^2 - D_1^2}},$$

$$\sigma_{yz}(x,0) = \frac{\mu w_0\,\mathrm{ch}(cd)}{h\,\mathrm{ch}(c)}\left[\sqrt{\frac{\mathrm{th}^2(cx) - D_1^2}{\mathrm{th}^2(cx) - I_1^2}} - \frac{\mathrm{ch}(c)}{\mathrm{ch}(cd)} + \right.$$
$$\left. + \left(1 - \frac{\Pi\left(\frac{\pi}{2}, \frac{I_1^2 - D_1^2}{I_1^2(1-D_1^2)}, q\right)}{F\left(\frac{\pi}{2}, q\right)}\right)\frac{D_1^2\,\mathrm{Sech}^2(cx)}{\sqrt{[\mathrm{th}^2(cx) - D_1^2][\mathrm{th}^2(cx) - I_1^2]}}\right], \quad x > 1, \tag{4.18}$$

$$\sigma_{yz}(x,0) = \frac{\mu w_0\,\mathrm{ch}(cd)}{h\,\mathrm{ch}(c)}\left[\sqrt{\frac{D_1^2 - \mathrm{th}^2(cx)}{I_1^2 - \mathrm{th}^2(cx)}} - \frac{\mathrm{ch}(c)}{\mathrm{ch}(cd)} - \right.$$
$$\left. - \left[1 - \frac{\Pi\left(\frac{\pi}{2}, \frac{I_1^2 - D_1^2}{I_1^2(1-D_1^2)}, q\right)}{F\left(\frac{\pi}{2}, q\right)}\right]\frac{D_1^2\,\mathrm{Sech}^2(cx)}{\sqrt{[D_1^2 - \mathrm{th}^2(cx)][I_1^2 - \mathrm{th}^2(cx)]}}\right], \quad 0 < x < d, \tag{4.19}$$

where we have used the result

$$(4.20)\qquad \int_d^1 \frac{1}{\sqrt{(t^2-d^2)(1-t^2)}}\frac{t\,dt}{t^2-x^2} = \begin{cases} \dfrac{\pi}{2\sqrt{(d^2-x^2)(1-x^2)}}, & 0<x<d,\\ 0, & d<x<1,\\ \dfrac{-\pi}{2\sqrt{(x^2-d^2)(x^2-1)}}, & x>1. \end{cases}$$

The stress intensity factor at $x = 1$ is given by

$$(4.21)\quad S_{11} = \lim_{x\to 1}\sqrt{2(x-1)}\sigma_{yz}(x,0) = \frac{\mu w_0}{h\,\mathrm{Sech}\,(cd)}\left[\sqrt{\frac{I_1^2-D_1^2}{cI_1}} + \frac{D_1^2(1-I_1^2)}{\sqrt{cI_1(I_1^2-D_1^2)}}\cdot\right.$$

$$\left.\cdot\left[1-\frac{\Pi\left(\frac{\pi}{2},\frac{I_1^2-D_1^2}{I_1^2(1-D_1^2)},q\right)}{F\left(\frac{\pi}{2},q\right)}\right]\right],$$

and the stress intensity factor at $x = d$ is given by

$$(4.22)\quad S_{1d} = \lim_{x\to d}\sqrt{2(d-x)}\sigma_{yz}(x,0) = -\frac{\mu w_0\sqrt{D_1^3(1-I_1^2)}}{h\sqrt{c(I_1^2-D_1^2)}}\cdot$$

$$\cdot\left[1-\frac{\Pi\left(\frac{\pi}{2},\frac{I_1^2-D_1^2}{I_1^2(1-D_1^2)},q\right)}{F\left(\frac{\pi}{2},q\right)}\right].$$

Letting $d = a/b = 0$ in the expressions for displacement, stress and stress intensity factors, it can be easily shown that the results coincide with the corresponding expressions given by SINGH *et al.* [2].

PROBLEM 2. In this case again we take the general solution of Eq. (2.3) as

$$(4.23)\qquad W(x,y) = F_c[C_1(\xi)\exp(-\xi ys) + C_2(\xi)\exp(\xi ys) : \xi \to x],$$

and inserting it in Eq. (3.7) we find that

$$C_1(\xi) = \frac{D(\xi)}{1+\exp(-2\xi hs)},$$

$$C_2(\xi) = \frac{D(\xi)\exp(-2\xi hs)}{1+\exp(-2\xi hs)}.$$

From Eqs. (3.8) and (3.9) it is determined that $D(\xi)$ satisfies the following set of triple integral equation

$$(4.24)\qquad F_c[\xi D(\xi).\mathrm{th}\,(\xi hs) : \xi \to x] = \frac{p_0}{\mu_s},\quad d<x<1,$$

$$(4.25)\qquad F_c[D(\xi) : \xi \to x] = 0,\quad 0\le x<d,\quad x>1.$$

Proceeding as in problem 1, we consider a trial solution

$$D(\xi) = \frac{1}{\xi}\sqrt{\frac{\pi}{2}} \int_d^1 g_2(\tau)\,\mathrm{ch}\,(c\tau)\sin(\xi\tau)\,d\tau. \tag{4.26}$$

With this choice of $D(\xi)$, Eq. (4.25) will be satisfied provided the unknown function $g_2(\tau)$ in Eq. (4.26) satisfies

$$\int_d^1 g_2(\tau)\,\mathrm{ch}\,(c\tau)\,d\tau = 0. \tag{4.27}$$

Now Eq. (4.24) can be written as

$$\frac{d}{dx}F_s[D(\xi)\,\mathrm{th}\,(\xi hs) : \xi \to x] = \frac{p_0}{\mu_s}, \quad d < x < 1. \tag{4.28}$$

Insertion of Eq. (4.26) in (4.28) and application of the result [9]

$$\int_0^\infty \frac{\mathrm{th}\,(\xi hs)\sin(\xi\tau)\sin(\xi x)}{\xi}\,d\xi = \frac{1}{2}\log\left|\frac{\mathrm{sh}(cx) + \mathrm{sh}(c\tau)}{\mathrm{sh}(cx) - \mathrm{sh}(c\tau)}\right|, \tag{4.29}$$

where $c = \pi/2hs$, gives

$$\int_d^1 \frac{cg_2(\tau)\,\mathrm{Sh}\,(2c\tau)}{\mathrm{sh}^2(c\tau) - \mathrm{sh}^2(cx)}\,d\tau = \frac{p_0}{\mu s\,\mathrm{ch}\,(cx)}, \quad d < x < 1. \tag{4.30}$$

Substituting $T_2 = \mathrm{sh}\,(c\tau)$, $I_2 = \mathrm{sh}\,(c)$, $D_2 = \mathrm{sh}\,(cd)$ and $X_2 = \mathrm{Sh}\,(cx)$ and proceeding as in problem 1, we obtain the solution of Eq. (4.30) as

$$g_2(\tau) = -\frac{4p_0}{\pi^2\mu s}\sqrt{\frac{T_2^2 - D_2^2}{I_2^2 - T_2^2}} \times \frac{1}{\sqrt{1 + I_2^2}}\left[\Pi\left(\frac{\pi}{2}, \frac{I_2^2 - D_2^2}{I_2^2 - T_2^2}, q''\right) - F\left(\frac{\pi}{2}, q''\right)\right] + \\ + \frac{K_2}{\sqrt{(T_2^2 - D_2^2)(I_2^2 - T_2^2)}}, \tag{4.31}$$

where $q' = (I_2^2 - D_2^2)^{1/2}/I_2$, $q'' = q' \cdot \mathrm{th}(c)$ and the constant K_2, as determined from Eq. (4.27), is

$$K_2 = \frac{4p_0\,\mathrm{th}\,(c)}{\pi^2\mu s F\left(\frac{\pi}{2}, q'\right)} \int_{D_2}^{I_2} \sqrt{\frac{T_2^2 - D_2^2}{I_2^2 - T_2^2}}\left(\Pi\left(\frac{\pi}{2}, \frac{I_2^2 - D_2^2}{I_2^2 - T_2^2}, q''\right) - F\left(\frac{\pi}{2}, q''\right)\right) dT_2. \tag{4.32}$$

The relevant displacement and stress components in the plane of the cracks may be written as

$$W(x, 0) = \frac{\pi}{2}\int_x^1 g_2(\tau)\,\mathrm{ch}(c\tau)\,d\tau, \quad d < x < 1, \tag{4.33}$$

and

$$\sigma_{yz}(x,0) = \frac{\mu s c}{2} \int_d^1 \frac{g_2(\tau)\,\mathrm{sh}\,(2c\tau)\,\mathrm{ch}\,(cx)}{\mathrm{sh}^2(cx) - \mathrm{sh}^2(c\tau)}\, d\tau, \quad 0 \le x < d, \quad x > 1. \tag{4.34}$$

Now using Eq. (4.31) in Eqs. (4.33) and (4.34) we obtain

$$W(x,0) = -\frac{2p_0}{\pi\mu s\,\mathrm{ch}\,(c)}\Bigg[\int_x^1 \sqrt{\frac{\mathrm{sh}^2(c\tau) - \mathrm{sh}^2(cd)}{\mathrm{sh}^2(c) - \mathrm{sh}^2(c\tau)}} \times \times \left\{ \Pi\left(\frac{\pi}{2}, \frac{I_2^2 - D_2^2}{I_2^2 - T_2^2}, q''\right) - F\left(\frac{\pi}{2}, q''\right)\right\} \mathrm{ch}\,(c\tau)\, d\tau\Bigg] + \frac{K_2 F(\lambda', q')}{c I_2}, \tag{4.35}$$

where

$$\sin\lambda' = \sqrt{\frac{I_2^2 - X_2^2}{I_2^2 - D_2^2}},$$

$$\sigma_{yz}(x,0) = -\frac{2p_0\,\mathrm{ch}(cx)}{\pi}\sqrt{\frac{\mathrm{sh}^2(cx) - D_2^2}{\mathrm{sh}^2(cx) - I_2^2}} \int_{D_2}^{I_2} \sqrt{\frac{I_2^2 - T_2^2}{T_2^2 - D_2^2}} \times \frac{1}{T_2^2 - \mathrm{sh}^2(cx)} \times \times \frac{T_2 dT_2}{\sqrt{1 + T_2^2}} + \frac{\pi\mu s\,\mathrm{ch}\,(cx) K_2}{2\sqrt{(\mathrm{sh}^2(cx) - I_2^2)(\mathrm{sh}^2(cx) - D_2^2)}}, \quad \text{for} \quad x > 1, \tag{4.36}$$

$$\sigma_{yz}(x,0) = -\frac{2p_0\,\mathrm{ch}\,(cx)}{\pi}\sqrt{\frac{D_2^2 - \mathrm{sh}^2(cx)}{I_2^2 - \mathrm{sh}^2(cx)}} \int_{D_2}^{I_2} \sqrt{\frac{I_2^2 - T_2^2}{T_2^2 - D_2^2}} \times \frac{1}{T_2^2 - \mathrm{sh}^2(cx)} \times \times \frac{T_2 dT_2}{\sqrt{1 + T_2^2}} - \frac{\pi\mu s\,\mathrm{ch}\,(cx) K_2}{2\sqrt{[I_2^2 - \mathrm{sh}^2(cx)][D_2^2 - \mathrm{sh}^2(cx)]}}, \quad \text{for} \quad 0 < x < d. \tag{4.37}$$

The stress intensity factor as $x = 1$ is given by

$$S_{21} = \lim_{x\to 1} \sqrt{2(x-1)}\,\sigma_{yz}(x,0) = \frac{2p_0}{\pi}\sqrt{\frac{I_2^2 - D_2^2}{c I_2\,\mathrm{ch}\,(c)}} \times F\left(\frac{\pi}{2}, q''\right) + + \frac{\pi\mu s K_2}{2\sqrt{c\,\mathrm{th}\,(c)[I_2^2 - D_2^2]}} \tag{4.38}$$

and the stress intensity factor at $x = d$ is given by

$$S_{2d} = \lim_{x\to d} \sqrt{2(d-x)}\,\sigma_{yz}(x,0) = \frac{-\pi\mu s K_2}{2\sqrt{c\,\mathrm{th}\,(cd)[I_2^2 - D_2^2]}}. \tag{4.39}$$

Again letting $d = 0$ in the expressions for displacement,stress and stress intensity factors, we obtain the corresponding results given by SINGH *et al.* [2].

5. Numerical Results

In this section we present the variation of stress intensity factors with ratio of crack speed v to shear wave speed c_2 for both problems. The crack length dependence of the stress intensity factors and its variations with v/c_2 have been shown in Figs. 2–5. Figures 2 and 3 illustrate the fact that in Problem 1, the stress intensity factors at both the crack tips decrease with the increase in the distance between the cracks.

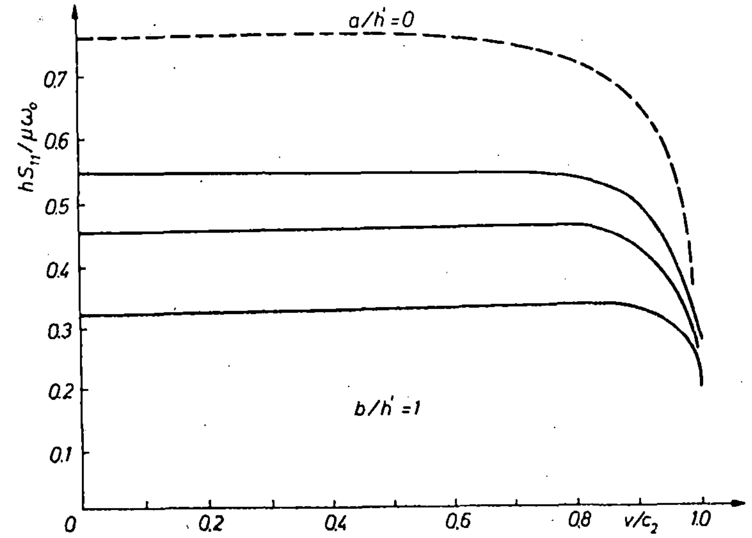

FIG. 2. Stress intensity factor at the outer edge $vs. v/c_2$ for Problem 1.

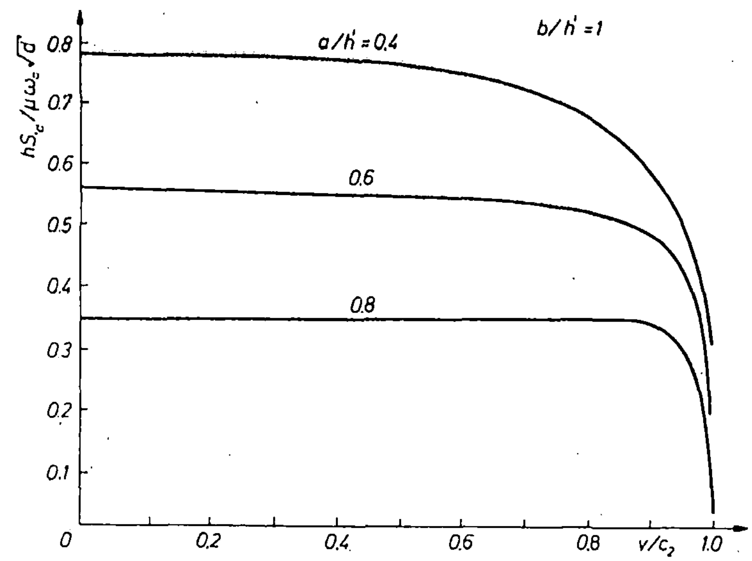

FIG. 3. Stress intensity factor at the inner edge $vs. vs/c_2$ for Problem 1.

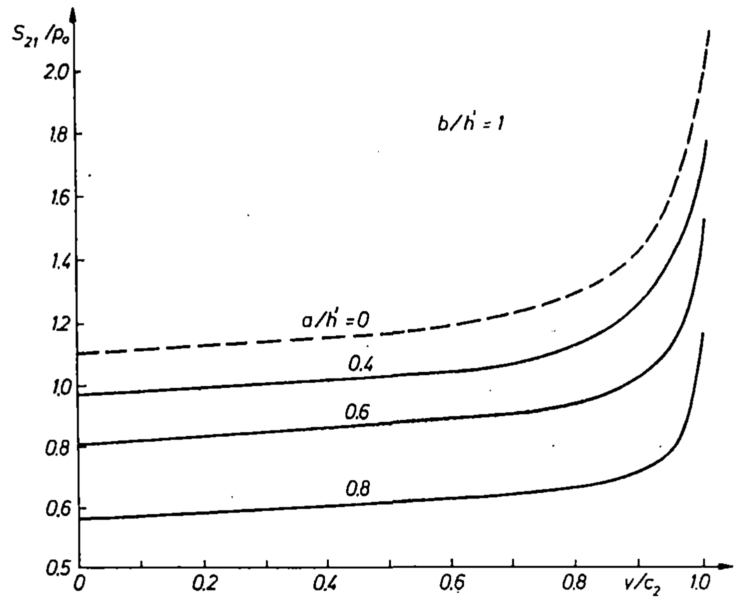

FIG. 4. Stress intensity factor at the outer edge $vs. v/c_2$ for Problem 2.

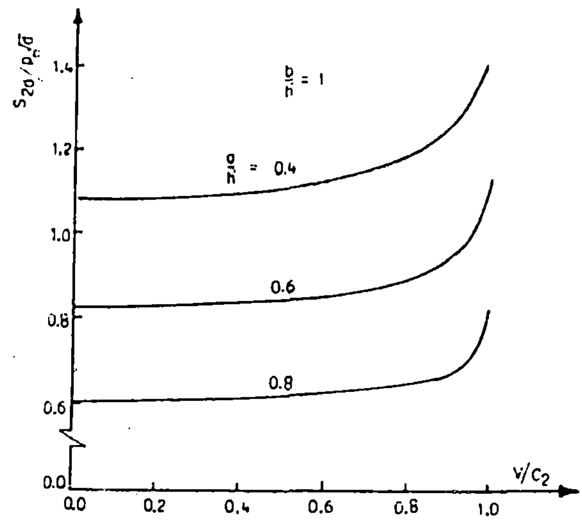

FIG. 5. Stress intensity factor at the inner edge $vs. v/c_2$ for Problem 2.

But for the Problem 2, as seen from Figs. 4 and 5, it is found that the behaviour of the stress intensity factors at the crack tips is of different nature.* In the Problem 1, the stress intensity factors at both the crack edges decrease with the increase in the value of v/c_2 and approach zero as $v/c_2 \to 1$. But in Problem 2, the stress intensity factor at both the ~~outer~~ edges increase gradually with the increase in the value of v/c_2 and approaches infinity as $v/c_2 \to 1$;** ~~whereas the corresponding value at the inner edge decreases gradually to zero with the increase in the value of v/c_2~~. The dashed line in Fig. 2 and Fig. 4 corresponding

* as compared to the corresponding nature of problem I.

** In problem II it is found that the stress intensity factors at both the edges decrease with the increase in the values of to the stress intensity factors at the tip of a single crack as given by SINGH *et al.* [2] for the case $b/h' = 1$.

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Streszczenie

RUCH DWÓCH WSPÓŁLINIOWYCH SZCZELIN GRIFFITHA W PAŚMIE W WARUNKACH ANTYPŁASKIEGO STANU ODKSZTAŁCENIA

Rozważono przypadek ustalonego ruchu dwóch szczelin o jednakowych długościach $b-a$ i wzajemnej odległości $2a$ poruszających się z ustaloną prędkością w płaszczyźnie symetrii $y = 0$ pasma sprężystego ograniczonego płaszczyznami $y = \pm h'$. Rozważono przypadki, gdy płaszczyzny te są utwierdzone lub swobodne od naprężeń. Rozwiązania uzyskano w postaci zamkniętej. Przedyskutowano i zilustrowano wykresami zależności współczynników intensywności naprężenia w wierzchołkach szczelin od predkości ich propagacji i parametrów geometrycznych zadania.

Резюме

ДВИЖЕНИЕ ДВУХ КОЛЛИНЕАРНЫХ ТРЕЩИН ГРИФФИТА В ПОЯСЕ В УСЛОВИЯХ АНТИПЛОСКОГО ДЕФОРМИРОВАННОГО СОСТОЯНИЯ

Рассматривался случай фиксированного движения двух трещин одинаковой длины $b-a$ и расположенных на расстоянии $2a$, передвигающихся с определенной скоростю в плоскости симметрии $y = 0$ упругого пояса, ограниченного плоскостями $y = \pm h'$. Рассматривался случай, когда эти плоскости закреплены или свободны от напряжений. Рещения были получены в замкнутом виде. Зависимость коеффициентов интенсивности напряжения в вицрщинах трещин от скорости их распространения и геомоетрических параметров задачи подлегала обсуждению, иллюстрировалась граффиками.

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the separating distance between the cracks.

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TWO CO-PLANAR GRIFFITH CRACKS MOVING ALONG THE INTERFACE OF TWO DISSIMILAR ELASTIC MEDIA

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Abstract—In this paper, the distribution of stress and displacement due to propagation of two parallel and co-planar Griffith cracks with constant velocity under antiplane shear stress at the interface of two dissimilar elastic media are presented. In the first case, cracks are assumed to propagate along the interface of two dissimilar infinite elastic half-spaces. In the second case, the problem of propagation of two co-planar Griffith cracks with uniform velocity at the interface of a layered composite has been treated. Cracks are assumed to be moving at the interface of a layer of thickness h and a semi-infinite substrate of different material. By the use of Fourier transform the problems have been reduced to the solution of a set of triple integral equations which have been solved by using the finite Hilbert transform technique. In the second problem, analytical solutions up to the order h^{-4}, where $h \gg 1$, have been derived for both the crack opening displacement and the stress intensity factors. Numerical results are also shown graphically.

1. INTRODUCTION

SCATTERING of elastic waves by cracks located in a homogeneous, isotropic medium has important applications in geophysics and seismology. If the cracks are located at the interfaces of layered media, the study becomes more relevant. Scattering of elastic waves from an interface crack under antiplane strain was solved by Bostrom[1]. Srivastava *et al.*[2] solved the problem of interaction of an antiplane shear wave by an interface crack. The problem of diffraction of Love waves by a crack of finite width in the plane interface of a layered composite has been solved by Neerhoff[3]. As regards the dynamic crack problem, research has been restricted mainly to the cases of a single crack because of the severe mathematical complexity encountered in finding solutions of problems involving two or more cracks. The diffraction of an antiplane shear wave by two co-planar Griffith cracks in an infinite elastic medium has been treated by Itou[4]. Lowengrub and Srivastava[5] treated the static problem of stress distribution in the presence of two co-planar Griffith cracks in an infinite elastic strip. The scattering of time harmonic normally incident plane waves by two co-planar Griffith cracks was also solved by Jain and Kanwal[6].

To our knowledge, the diffraction of elastic waves by two cracks moving along the interface of bonded dissimilar elastic media has not been investigated so far. In this paper we consider the problem of determining the distribution of shear stress in the neighbourhood of the cracks, moving along the interface of two bonded dissimilar elastic media. Two cases of practical importance have been considered here. Firstly, the case of two co-planar Griffith cracks moving along the interface of two semi-infinite dissimilar elastic media has been treated; secondly, the problem of propagation of two co-planar Griffith cracks along the interface of an elastic layer overlying a semi-infinite medium of different elastic properties has been considered. Employing Fourier transform we reduced these problems to solving a set of triple integral equations with cosine karnel and weight functions. These equations are solved using the finite Hilbert transform technique. In the second problem, analytical expressions are retained up to the order h^{-4}, where h is the thickness of the upper layer, for deriving the dynamic stress intensity factors and crack opening displacement. Numerical results are also presented graphically.

2. FORMULATION OF THE PROBLEM

We consider two cracks of finite length to be placed along the X-axis from -1 to $-c$ and c to 1 with reference to a set of rectangular coordinates (x, y, z) which, referred to a fixed coordinate system (X, Y, Z), are moving with constant velocity v along the X-axis, as shown in Fig. 1.

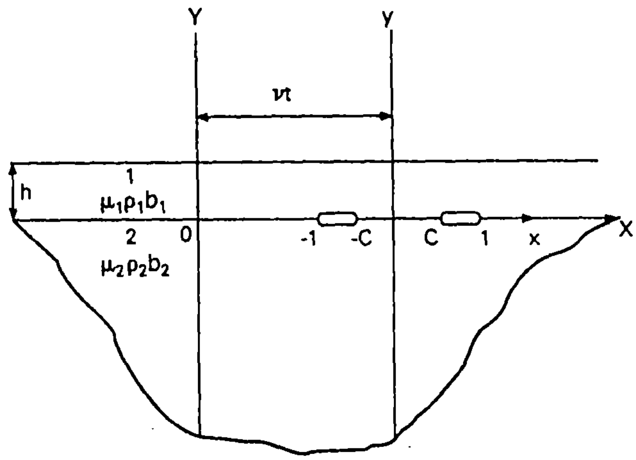

Fig. 1. Geometry and coordinate system.

The coordinates are regarded as dimensionless, referring to the outer edge of the crack. In the dynamic problem of antiplane shear, there exists a single non-vanishing component of displacement in the Z-direction, $W_i = W_i(X, Y, t)$, $i = 1, 2$, where W_1 and W_2 are the displacement components along the Z-direction in media $Y > 0$ and $Y < 0$ respectively. In the absence of body force the equation of motion is

$$\frac{\partial^2 W_i}{\partial X^2} + \frac{\partial^2 W_i}{\partial Y^2} = \frac{1}{b_i^2}\frac{\partial^2 W_i}{\partial t^2} \tag{1}$$

where $b_i = (\mu_i/\rho_i)^{1/2}$ $(i = 1, 2)$ are the shear wave speeds; ρ_i are the densities of the materials and μ_i are the shear moduli.

Using Galilean transformation $x = X - vt$, $y = Y$, $z = Z$, $t' = t$, where (x, y, z) represents the translating coordinate system shown in Fig. 1, eq. (1) becomes independent of t and reduces to

$$s_i^2 \frac{\partial^2 W_i}{\partial x^2} + \frac{\partial^2 W_i}{\partial y^2} = 0 \tag{2}$$

with

$$s_i^2 = 1 - v^2/b_i^2. \tag{3}$$

3. BOUNDARY CONDITIONS

Problem I

In this case the cracks are placed along the interface of two joined dissimilar elastic half-spaces and are moving along the interface with constant velocity v. The x-axis is taken along the interface of these media. The cracks are excited by a normally incident antiplane shear wave. The boundary conditions are

$$\left.\begin{aligned} [\tau_{yz}(x,0)]_1 &= [\tau_{yz}(x,0)]_2 = -p, && c < |x| < 1 \\ [\tau_{yz}(x,0)]_1 &= [\tau_{yz}(x,0)]_2, && 0 \leqslant |x| < c, \quad |x| > 1 \\ W_1(x,0) &= W_2(x,0), && 0 \leqslant |x| < c, \quad |x| > 1 \end{aligned}\right\}. \tag{4a–c}$$

Problem I now consists of solving eq. (2) together with the conditions (4).

Problem II

In this case two co-planar Griffith cracks of finite width are assumed to be moving with uniform velocity under antiplane shear stress along the interface of an elastic layer kept in welded contact with a semi-infinite medium of different elastic properties. The boundary conditions of this dynamic antiplane problem are

$$\left.\begin{array}{ll}[\tau_{yz}(x,0)]_1=[\tau_{yz}(x,0)]_2=-p, & c<|x|<1\\ [\tau_{yz}(x,0)]_1=[\tau_{yz}(x,0)]_2, & 0\leqslant|x|<c,\quad |x|>1\\ W_1(x,0)=W_2(x,0), & 0\leqslant|x|<c,\quad |x|>1\\ [\tau_{yz}(x,h)]_1=0 & -\infty<x<\infty\end{array}\right\}, \tag{5a–d}$$

Problem II now consists of solving eq. (2) together with the conditions (5).

4. SOLUTION OF PROBLEM I

Employing Fourier cosine transform, namely

$$f_c(\xi,y)=\int_0^\infty f(x,y)\cos(\xi x)\,dx \quad \text{and} \quad f(x,y)=\frac{2}{\pi}\int_0^\infty f_c(\xi,y)\cos(\xi x)\,d\xi,$$

we obtain the solution of eq. (2) as

$$W_1(x,y)=\frac{2}{\pi}\int_0^\infty A_1(\xi)\exp(-s_1\xi y)\cos(\xi x)\,d\xi, \quad \text{for } y>0 \tag{6}$$

$$W_2(x,y)=\frac{2}{\pi}\int_0^\infty A_2(\xi)\exp(s_2\xi y)\cos(\xi x)\,d\xi, \quad \text{for } y<0 \tag{7}$$

where s_i is the positive root of (3) and $A_i(\xi)$ are unknown functions to be determined.

From (6) and (7) we obtain

$$[\tau_{yz}(x,y)]_1=-\frac{2\mu_1 s_1}{\pi}\int_0^\infty \xi A_1(\xi)\exp(-s_1\xi y)\cos(\xi x)\,d\xi, \quad \text{for } y>0 \tag{8}$$

$$[\tau_{yz}(x,y)]_2=\frac{2\mu_2 s_2}{\pi}\int_0^\infty \xi A_2(\xi)\exp(s_2\xi y)\cos(\xi x)\,d\xi, \quad \text{for } y<0. \tag{9}$$

Using (4a) and (4b) we get

$$A_2(\xi)=-\frac{\mu_1 s_1}{\mu_2 s_2}A_1(\xi). \tag{10}$$

Therefore, the crack opening displacement $\Delta w(x)$ is obtained as

$$\begin{aligned}\Delta w(x)&=W_1(x,0^+)-W_2(x,0^-)\\ &=\frac{2L}{\pi}\int_0^\infty A_1(\xi)\cos(\xi x)\,d\xi, \quad c\leqslant x\leqslant 1\\ &=0, \qquad 0\leqslant x<c, \quad x>1\end{aligned} \tag{11}$$

where

$$L=\frac{\mu_1 s_1+\mu_2 s_2}{\mu_2 s_2}. \tag{12}$$

From (8) and (4a),

$$\int_0^\infty \xi A_1(\xi)\cos(\xi x)\,d\xi=\frac{p\pi}{2\mu_1 s_1}. \tag{13}$$

Let us take

$$A_1(\xi)=\frac{1}{\xi}\int_c^1 h(t^2)\sin(\xi t)\,dt. \tag{14}$$

Substituting (14) in (11) we see that this choice of $A_1(\xi)$ leads to

$$\int_c^1 h(t^2)\,dt = 0. \tag{15}$$

Inserting (14) in (13) we obtain

$$\int_0^\infty \frac{th(t^2)\,dt}{t^2 - x^2} = \frac{p\pi}{2\mu_1 s_1}, \quad c < x < 1. \tag{16}$$

Using the finite Hilbert transform, the solution of (16) is

$$h(t^2) = -\frac{2p}{\pi\mu_1 s_1}\sqrt{\frac{t^2 - c^2}{1 - t^2}}\int_c^1 \sqrt{\frac{1 - x^2}{x^2 - c^2}} \times \frac{x\,dx}{x^2 - t^2} + \frac{K'}{\sqrt{(t^2 - c^2)(1 - t^2)}} \tag{17}$$

where the unknown constant K', determined from (15), is

$$K' = \frac{p}{\mu_1 s_1}(c^2 - E/F), \tag{18}$$

where $F = F(\pi/2, q)$ and $E = E(\pi/2, q)$ are complete elliptic integrals of the first and second kind respectively and $q = (1 - c^2)^{1/2}$.

The relevant expressions for the crack opening displacement and stress component at the interface are

$$\Delta w(x) = L\int_x^1 h(t^2)\,dt, \quad c < x < 1 \tag{19}$$

$$[\tau_{yz}(x, 0)]_1 = \frac{2\mu_1 s_1}{\pi}\int_c^1 \frac{th(t^2)\,dt}{x^2 - t^2}, \quad 0 \leqslant x < c, \quad x > 1. \tag{20}$$

Substituting the values of $h(t^2)$ from (17) in (19) and (20) we obtain

$$\Delta w(x) = \frac{Lp}{\mu_1 s_1}\left[E(\lambda, q) - \frac{E}{F}F(\lambda, q)\right], \tag{21}$$

where

$$\sin\lambda = \sqrt{\frac{1 - x^2}{1 - c^2}} \tag{22}$$

and

$$[\tau_{yz}(x, 0)]_1 = p\left[\sqrt{\frac{x^2 - c^2}{x^2 - 1}} - 1 - \frac{E/F - c^2}{\sqrt{(x^2 - c^2)(x^2 - 1)}}\right], \quad \text{for } x > 1 \tag{23}$$

$$= p\left[\sqrt{\frac{c^2 - x^2}{1 - x^2}} - 1 + \frac{E/F - c^2}{\sqrt{(c^2 - x^2)(1 - x^2)}}\right], \quad \text{for } x < c, \tag{24}$$

where we have used

$$\int_c^1 \frac{t\,dt}{(x^2 - t^2)\sqrt{(t^2 - c^2)(1 - t^2)}} = \begin{cases} -\dfrac{\pi}{2\sqrt{(c^2 - x^2)(1 - x^2)}}, & \text{for } 0 < x < c \\ 0, & \text{for } c < x < 1 \\ \dfrac{\pi}{2\sqrt{(x^2 - c^2)(x^2 - 1)}}, & \text{for } x > 1. \end{cases} \tag{25}$$

The stress intensity factors at the tips of the cracks $x = 1$ and $x = c$ respectively are given by

$$K_1 = \lim_{x\to 1} \sqrt{2(x-1)}[\tau_{yz}(x,0)]_1 = \frac{p(1-E/F)}{\sqrt{1-c^2}} \tag{26}$$

$$K_c = \lim_{x\to c} \sqrt{2(c-x)}[\tau_{yz}(x,0)]_1 = \frac{p(E/F-c^2)}{\sqrt{c(1-c^2)}}. \tag{27}$$

5. SOLUTION OF PROBLEM II

Employing Fourier cosine transform the solutions of the problem are sought in the form

$$W_1(x,y) = \frac{2}{\pi}\int_0^\infty [A_1(\xi)\exp(-s_1\xi y) + A_2(\xi)\exp(s_1\xi y)]\cos(\xi x)\,d\xi, \quad \text{for } 0 \leqslant y \leqslant h$$

$$W_2(x,y) = \frac{2}{\pi}\int_0^\infty A_3(\xi)\exp(s_2\xi y)\cos(\xi x)\,d\xi, \quad \text{for } y < 0. \tag{28}$$

Using (28) we obtain the stress components as

$$[\tau_{yz}(x,y)]_1 = \frac{2\mu_1 s_1}{\pi}\int_0^\infty \xi[-A_1(\xi)\exp(-s_1\xi y) + A_2(\xi)\exp(s_1\xi y)]\cos(\xi x)\,d\xi, \quad \text{for } 0 \leqslant y \leqslant h$$

$$[\tau_{yz}(x,y)]_2 = \frac{2\mu_2 s_2}{\pi}\int_0^\infty \xi A_3(\xi)\exp(s_2\xi y)\cos(\xi x)\,d\xi, \quad \text{for } y < 0. \tag{29}$$

Applying (5a), (5b) and (5d) we obtain

$$A_3(\xi) = \frac{\mu_1 s_1}{\mu_2 s_2}[A_2(\xi) - A_1(\xi)] \tag{30}$$

and

$$A_2(\xi) = A_1(\xi)\exp(-2\xi h s_1). \tag{31}$$

We define the crack opening displacement $\Delta w(x)$ as

$$\begin{aligned} \Delta w(x) &= W_1(x,0^+) - W_2(x,0^-) \\ &= \frac{2L}{\pi}\int_0^\infty f(\xi)\cos(\xi x)\,d\xi, \quad c \leqslant x \leqslant 1 \\ &= 0, \qquad 0 \leqslant x \leqslant c, \quad x > 1 \end{aligned} \tag{32}$$

where

$$f(\xi) = A_1(\xi)\left[1 + \frac{\mu_2 s_2 - \mu_1 s_1}{\mu_2 s_2 + \mu_1 s_1}\exp(-2\xi h s_1)\right]. \tag{33}$$

Therefore, by (5c) and (5a), $f(\xi)$ is found to be the solution of the following triple integral equations:

$$\int_0^\infty f(\xi)\cos(\xi x)\,d\xi = 0, \quad 0 \leqslant x < c, \quad x > 1 \tag{34}$$

$$\int_0^\infty \xi f(\xi)[1 + M(\xi h)]\cos(\xi x)\,d\xi = \frac{p\pi}{2\mu_1 s_1}, \quad c < x < 1 \tag{35}$$

with

$$M(\xi h) = -\frac{1-\tanh(\xi h s_1)}{\left[1 + \frac{\mu_1 s_1}{\mu_2 s_2}\tanh(\xi h s_1)\right]}. \tag{36}$$

Assuming

$$f(\xi)=\frac{1}{\xi}\int_c^1 h(t^2)\sin(\xi t)\,\mathrm{d}t, \tag{37}$$

it is found from (35) and (36) that $h(x^2)$ is the solution of the following Fredholm integral equation:

$$h(x^2)+\int_c^1 h(t^2)K(x^2,t)\,\mathrm{d}t=F(x^2),\quad c<x<1 \tag{38}$$

satisfying the condition

$$\int_c^1 h(x^2)\,\mathrm{d}x=0 \tag{39}$$

where

$$K(x^2,t)=-\frac{4}{\pi^2}\sqrt{\frac{x^2-c^2}{1-x^2}}\int_c^1\sqrt{\frac{1-y^2}{y^2-c^2}}\times\frac{yK_1(y,t)}{y^2-x^2}\,\mathrm{d}y \tag{40}$$

with

$$K_1(y,t)=\int_0^\infty M(\xi h)\cos(\xi y)\sin(\xi t)\,\mathrm{d}\xi \tag{41}$$

and

$$F(x^2)=-\frac{2p}{\pi\mu_1 s_1}\sqrt{\frac{x^2-c^2}{1-x^2}}\int_c^1\sqrt{\frac{1-y^2}{y^2-c^2}}\frac{y\,\mathrm{d}y}{y^2-x^2}+\frac{K''}{\sqrt{(x^2-c^2)(1-x^2)}}, \tag{42}$$

K'' being an arbitrary constant determined by the condition (39). If we take $h \gg 1$, then, by substituting $\eta=\xi h$ and expanding $\cos(\eta y/h)$, $\sin(\eta y/h)$ we may write (41) in the form

$$K_1(y,t)=\frac{I_0 t}{h^2}+\frac{I_1 t}{h^4}(t^2+3y^2)+O(h^{-6}) \tag{43}$$

where

$$I_j=\frac{(-1)^j}{(2j+1)!}\int_0^\infty \eta^{2j+1}M(\eta)\,\mathrm{d}\eta\quad(j=0,1) \tag{44}$$

and hence

$$K(x^2,t)=\frac{2}{\pi}\sqrt{\frac{x^2-c^2}{1-x^2}}\left[\frac{I_0 t}{h^2}+\frac{I_1 t}{h^4}(t^2+3x^2-\tfrac{3}{2}k^2)\right]+O(h^{-6}) \tag{45}$$

where $k^2=1-c^2$.

Integrating both sides of (38) with respect to x from c to 1 and using (39), we obtain

$$K''=\frac{p}{\mu_1 s_1}\left[c^2-\frac{E}{F}\right]+\frac{1}{F}\int_c^1 h(t^2)K(t)\,\mathrm{d}t \tag{46}$$

with

$$K(t)=\frac{2}{\pi}\left[\frac{I_0 t}{h^2}(E-c^2F)+\frac{I_1 t}{h^4}\{(t^2-\tfrac{3}{2}k^2)(E-c^2F)-c^2(E+F)+2E\}\right]+O(h^{-6})$$

where E and F defined by $E=E(\pi/2,q)$ and $F=F(\pi/2,q)$ with $q=k$ are known as elliptic integrals of the first and second kind respectively.

Hence, $h(x^2)$ must satisfy the integral equation

$$h(x^2)+\int_c^1 h(t^2)M(x^2,t)\,\mathrm{d}t=S(x^2) \tag{47}$$

where

$$M(x^2, t) = \frac{2t}{\pi\sqrt{(x^2-c^2)(1-x^2)}}\left[\frac{I_0}{h^2}\left(x^2-\frac{E}{F}\right)+\frac{I_1}{h^4}\left\{(t^2+\tfrac{3}{2}k^2)\left(x^2-\frac{E}{F}\right)\right.\right.$$
$$\left.\left.+3x^2(x^2-1)+\frac{E}{F}+c^2-\frac{2c^2E}{F}\right\}\right]+O(h^{-6}) \quad (48)$$

and

$$S(x^2) = \frac{p\left[x^2-\dfrac{E}{F}\right]}{\mu_1 s_1\sqrt{(x^2-c^2)(1-x^2)}}. \quad (49)$$

Since $h \gg 1$, and $|M(x^2, t)| < 1$, the solution of (47) may be written in the form

$$h(x^2) = h_0(x^2)+\frac{1}{h^2}h_1(x^2)+\frac{1}{h^4}h_2(x^2)+O(h^{-6}) \quad (50)$$

where

$$h_0(x^2) = \frac{p\left[x^2-\dfrac{E}{F}\right]}{\mu_1 s_1\sqrt{(x^2-c^2)(1-x^2)}} \quad (51)$$

$$h_1(x^2) = \frac{-I_0C_0p\left[x^2-\dfrac{E}{F}\right]}{2\mu_1 s_1\sqrt{(x^2-c^2)(1-x^2)}} \quad (52)$$

$$h_2(x^2) = \frac{pC_0}{4\mu_1 s_1\sqrt{(x^2-c^2)(1-x^2)}}\left[I_0^2C_0\left\{x^2-\frac{E}{F}\right\}-2I_1(3x^4+C_1x^2+C_2)\right] \quad (53)$$

with

$$C_0 = 1+c^2-2\frac{E}{F}$$
$$C_1 = k^4/4C_0-(1+c^2)$$
$$C_2 = c^2+\frac{E}{F}\left\{C_1-\frac{k^4}{2C_0}\right\}.$$

The relevant crack opening displacement and stress component at the interface are

$$\Delta w(x) = L\int_x^1 h(t^2)\,dt, \quad c \leqslant x \leqslant 1 \quad (54)$$

$$[\tau_{yz}(x,0)]_1 = -\frac{2\mu_1 s_1}{\pi}\left[\int_c^1\frac{th(t^2)\,dt}{t^2-x^2}+\int_c^1 h(t^2)K_1(x,t)\,dt\right], \quad 0\leqslant x<c, \quad x>1 \quad (55)$$

where $K_1(x, t)$ is given in (41).

Using (43) and eqs (50)–(53), we find that

$$\int_c^1 h(t^2)K_1(x,t)\,dt = \frac{p\pi}{8\mu_1 s_1}\left[\frac{2I_0C_0}{h^2}-\frac{I_0^2C_0^2}{h^4}+\frac{2I_1C_0}{h^4}\{3x^2+C_1+\tfrac{3}{2}(1+c^2)\}\right]+0(h^{-6}). \quad (56)$$

Using the results given by (25), we get for $0 < x < c$

$$\int_c^1 \frac{th(t^2)\,dt}{t^2 - x^2} = \frac{p\pi}{2\mu_1 s_1}\left[\left\{1 - \frac{I_0 C_0}{2h^2} + \frac{I_0^2 C_0^2}{4h^4}\right\} \times \left\{\frac{x^2 - E/F}{X_1} + 1\right\} - \frac{I_1 C_0}{2h^4} \times \left\{\frac{3x^4 + C_1 x^2 + C_2}{X_1} + 3\left(\frac{1+c^2}{2} + x^2\right) + C_1\right\}\right] + O(h^{-6}), \quad (57)$$

and for $x > 1$

$$\int_c^1 \frac{th(t^2)\,dt}{t^2 - x^2} = \frac{p\pi}{2\mu_1 s_1}\left[\left\{1 - \frac{I_0 C_0}{2h^2} + \frac{I_0^2 C_0^2}{4h^4}\right\} \times \left\{\frac{E/F - x^2}{X_2} + 1\right\} + \frac{I_1 C_0}{2h^4} \times \left\{\frac{3x^4 + C_1 x^2 + C_2}{X_2} - 3\left(\frac{1+c^2}{2} + x^2\right) - C_1\right\}\right] + O(h^{-6}), \quad (58)$$

where

$$X_1 = \sqrt{(c^2 - x^2)(1 - x^2)}$$

$$X_2 = \sqrt{(x^2 - c^2)(x^2 - 1)}.$$

Using eqs (50)–(53) we obtain from (54), after integration, the crack opening displacement as

$$\Delta w(x) = \frac{pL}{\mu_1 s_1}\left[\left\{1 - \frac{I_0 C_0}{2h^2} + \frac{I_0^2 C_0^2 + 2I_1 C_0(C_1 - k^4/2C_0)}{4h^4}\right\} \times \left\{E(\lambda, q) - \frac{E}{F}F(\lambda, q)\right\} - \frac{2I_1 C_0}{4h^4} x\sqrt{(1-x^2)(x^2-c^2)}\right] + O(h^{-6}) \quad (59)$$

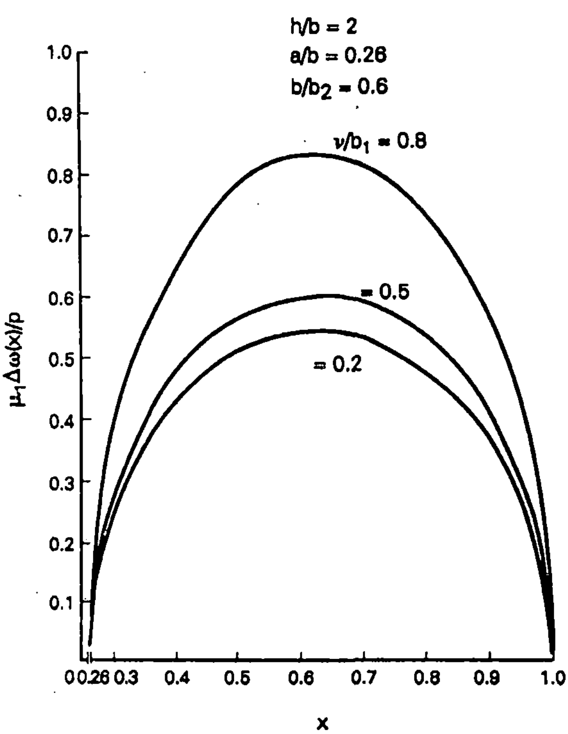

Fig. 2. Variation of crack opening displacement with x for problem II.

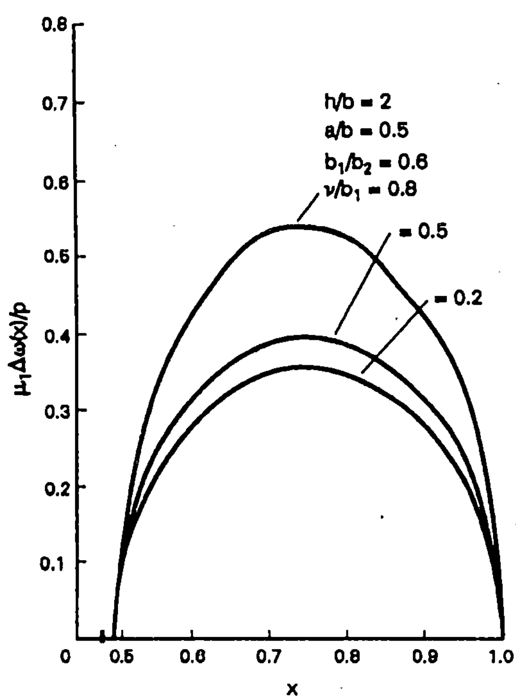

Fig. 3. Variation of crack opening displacement with x for problem II.

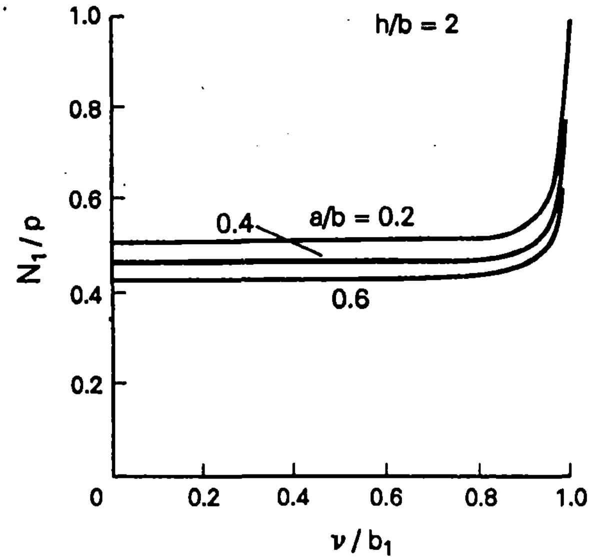

Fig. 4. Variation of stress intensity factor at the outer edge with v/b_1 for problem II.

where

$$\sin \lambda = \sqrt{\frac{1-x^2}{1-c^2}}.$$

Substituting the results obtained in (56), (57) and (58) on the right hand side of (55) the stress in the plane of the crack can be derived and from it stress intensity factors at the crack tips can easily be found.

We find that the stress intensity factor at $x = 1$ is given by

$$\begin{aligned} N_1 &= \lim_{x\to 1} \sqrt{2(x-1)}[\tau_{yz}(x,0)]_1 \\ &= \frac{-p}{\sqrt{1-c^2}}\left[(E/F-1)\left\{1-\frac{I_0 C_0}{2h^2}+\frac{I_0^2 C_0^2}{4h^4}\right\}+\frac{I_1 C_0}{2h^4}(3+C_1+C_2)\right]+O(h^{-6}) \end{aligned} \tag{60}$$

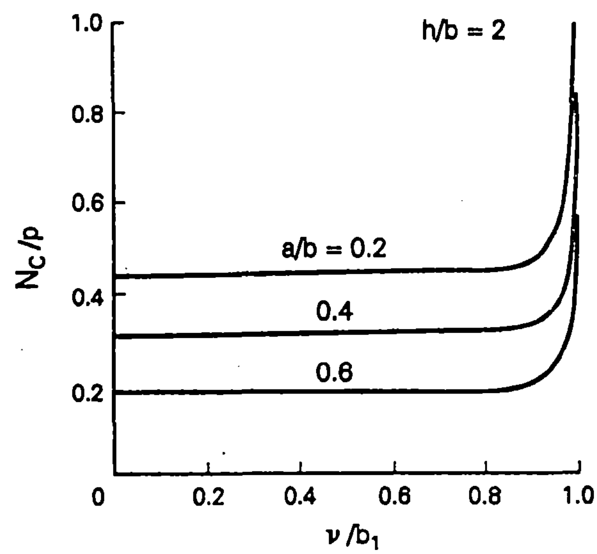

Fig. 5. Variation of stress intensity factor at the inner edge with v/b_1 for problem II.

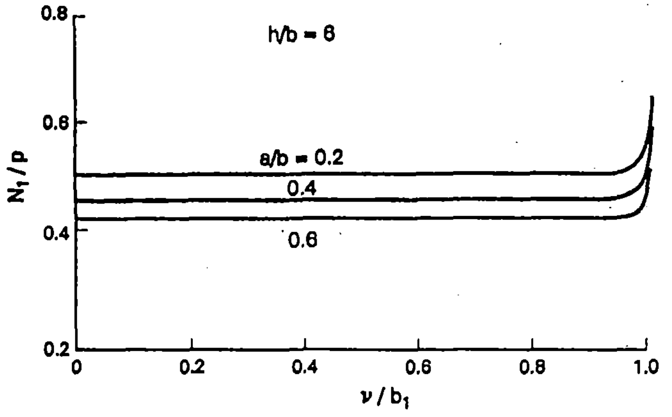

Fig. 6. Variation of stress intensity factor at the outer edge with v/b_1 for problem II.

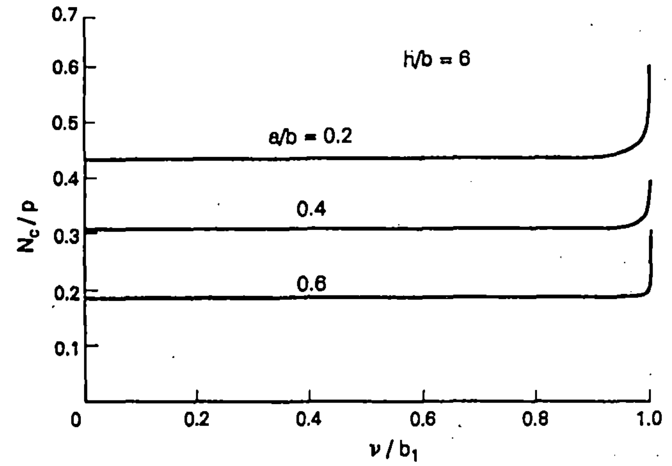

Fig. 7. Variation of stress intensity factor at the inner edge with v/b_1 for problem II.

and the stress intensity factor at $x = c$ is found to be

$$N_c = \lim_{x \to c} \sqrt{2(c-x)}\,[\tau_{yz}(x,0)]_1$$

$$= \frac{-p}{\sqrt{c(1-c^2)}}\left[(c^2 - E/F)\left\{1 - \frac{I_0 C_0}{2h^2} + \frac{I_0^2 C_0^2}{4h^4}\right\} - \frac{I_1 C_0}{2h^4}(3c^4 + C_1 c^2 + C_2)\right] + O(h^{-6}). \tag{61}$$

6. NUMERICAL RESULTS

In this section we present numerical results for the stress intensity factors at the crack tips and also the crack opening displacements for different values of the layer thickness and the crack speed and for $b_1/b_2 = 0.6$. The crack opening displacement is found to increase gradually with the increase in the value of the crack speed. Further, for a fixed crack speed, the crack opening displacement increases with the decrease in the value of the separating distance between the cracks.

Variation of the stress intensity factors at both crack tips with crack speed is depicted in Figs 4–7. It is interesting to note that stress intensity factors at both the crack tips increase very slowly at the onset with the increase in the value of v/b_1 but change rapidly and go to infinity as v/b_1 approaches 1. This fact becomes prominent as the layer thickness becomes large.

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Problem of two coplanar Griffith cracks running steadily under three-dimensional loading

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Abstract. In this paper, the three-dimensional problem of two coplanar Griffith cracks propagating uniformly in an elastic medium has been considered. Equal and opposite tractions which are triaxial in nature are applied to the crack surfaces. The two-dimensional Fourier transforms have been used to reduce the mixed boundary value problem to the solution of triple integral equations. In order to solve the problem, the transformed surface displacement has been expanded in a series of Chebyshev polynomials which is automatically zero outside the cracks and also satisfies the edge conditions. Finally Schmidt method has been used to determine the unknown constants occurring in the series. Numerical calculations are carried out to obtain the crack opening displacement and also the stress intensity factors for different values of the parameters.

1. Introduction

Yoffe [1] considered the problem of propagation of a crack of fixed length at a constant speed through a stretched isotropic elastic solid of infinite extent. In recent years, Yoffe's investigation was extended to include different types of materials and different material geometries. Sih and Chen [2] considered the problem of a uniformly propagating finite crack in a strip of isotropic elastic material. Recently Kassir and Tse [3] solved the plane stress problem of a moving Griffith crack in an infinite orthotropic stressed medium by using integral transform technique and the same technique has been employed by De and Patra [4] to solve Yoffe's problem in a stressed orthotropic strip of finite thickness.

However all the problems mentioned above have been solved using the dynamic equations of elasticity in two dimensions. But practically in most instances, cracks are subjected to a state of stress that is triaxial in nature. Crack problems involving three-dimensional loading have generally not been attempted so far.

Recently Angel and Achenbach [5] derived the elastodynamics stress intensity factor for three-dimensional loading of a cracked half-space. Freund [6] also solved the three dimensional problem of the oblique reflection of a Rayleigh wave from the edge of a semi-infinite crack employing a Wiener-Hopf technique. The problem of a uniformly propagating finite crack in an elastic medium has been solved by Itou [7] using dynamic equations of elasticity in three dimensions.

Regarding the dynamic crack problem, research has been restricted mainly to a single crack because of severe mathematical complexity encountered in finding the solutions for two or more cracks. Recently Jain and Kanwal [8] presented the low-frequency solution of diffraction of normally incident longitudinal waves by two coplanar Griffith cracks in an infinite isotropic elastic medium. They used the finite Hilbert transform technique developed by Srivastava and Lowengrub [9] to solve the mixed boundary value problem. Using a completely different technique Itou [10] solved the diffraction problem of elastic waves by two coplanar Griffith cracks in an infinite elastic medium.

In this paper we have considered the problem of propagation of two coplanar Griffith cracks propagating steadily with uniform velocity under three-dimensional loading. The application of two-dimensional Fourier transforms reduced this problem to that of solving triple integral equations in which the double Fourier transforms of the crack opening displacement appear as the unknown. In an attempt to solve the problem the transformed surface displacement has been expanded in a series of a function which is automatically zero outside the cracks. Finally the Schmidt method [11] has been employed to solve the integral equations. The dynamic stress intensity factors and the crack opening displacement have been evaluated numerically for various values of crack speed and distance between the cracks.

2. Formulation of the problem

Let (X, Y, Z) be a fixed rectangular coordinate system. Two coplanar Griffith cracks of infinite length but finite width located in the XZ-plane, the Z-axis being in the direction of the length of the cracks, are assumed to be moving steadily with velocity U in the direction of the X-axis. It is convenient to introduce Galilean transform $x = X - UT$, $y = Y$, $z = Z$, $t = T$ where (x, y, z) represents the translating coordinate system shown in Fig. 1. Referred to this moving system of the coordinate the cracks are assumed to occupy the positions $b < |x| < a$, $y = 0$, $|z| < \infty$.

The equations of motion in the absence of body force are

$$(\lambda + \mu)\frac{\partial}{\partial X}\left(\frac{\partial u^*}{\partial X} + \frac{\partial v^*}{\partial Y} + \frac{\partial w^*}{\partial Z}\right) + \mu\left(\frac{\partial^2 u^*}{\partial X^2} + \frac{\partial^2 u^*}{\partial Y^2} + \frac{\partial^2 u^*}{\partial Z^2}\right) = \rho\frac{\partial^2 u^*}{\partial T^2},$$

$$(\lambda + \mu)\frac{\partial}{\partial Y}\left(\frac{\partial u^*}{\partial X} + \frac{\partial v^*}{\partial Y} + \frac{\partial w^*}{\partial Z}\right) + \mu\left(\frac{\partial^2 v^*}{\partial X^2} + \frac{\partial^2 v^*}{\partial Y^2} + \frac{\partial^2 v^*}{\partial Z^2}\right) = \rho\frac{\partial^2 v^*}{\partial T^2}, \tag{2.1}$$

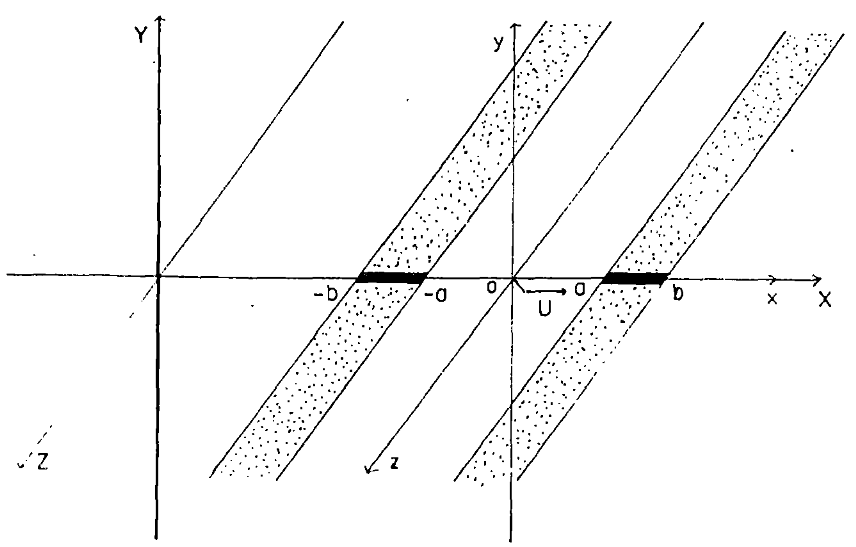

Fig. 1.

$$(\lambda + \mu)\frac{\partial}{\partial Z}\left(\frac{\partial u^*}{\partial X} + \frac{\partial v^*}{\partial Y} + \frac{\partial w^*}{\partial Z}\right) + \mu\left(\frac{\partial^2 u^*}{\partial X^2} + \frac{\partial^2 w^*}{\partial Y^2} + \frac{\partial^2 w^*}{\partial Z^2}\right) = \rho\frac{\partial^2 w^*}{\partial T^2},$$

where u^*, v^*, w^* are the displacement components, λ and μ are Lame's constants and ρ is the material density. Using Galilean transformation

$$x = X - UT, \quad y = Y, \quad z = Z, \quad t = T$$

(2.1) reduces to

$$(\lambda + \mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho U^2\frac{\partial^2 u}{\partial x^2},$$

$$(\lambda + \mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y} + \frac{\partial^2 v}{\partial z^2}\right) = \rho U^2\frac{\partial^2 v}{\partial x^2}, \tag{2.2}$$

$$(\lambda + \mu)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho U^2\frac{\partial^2 w}{\partial x^2}.$$

The stress components for the three dimensional problem are

$$\sigma_x = (\lambda + 2\mu)\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right); \quad \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{2.3.1}$$

$$\sigma_y = (\lambda + 2\mu)\frac{\partial v}{\partial y} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right); \quad \tau_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \tag{2.3.2}$$

$$\sigma_z = (\lambda + 2\mu)\frac{\partial w}{\partial z} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right); \quad \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right). \tag{2.3.3}$$

The boundary conditions are

$$\frac{\sigma_y}{2\mu} = -p(x, z), \quad \text{for } y = 0, \quad a \leqslant |x| \leqslant b, \quad |z| < \infty,$$

$$v = 0, \quad \text{for } y = 0, \quad |x| > b, \quad |x| < a, \quad |z| < \infty, \tag{2.4}$$

$$\tau_{xy} = 0 = \tau_{yz}, \quad \text{for } y = 0, \quad |x| < \infty, \quad |z| < \infty.$$

3. Solution of the problem

Using Fourier transformations viz.

$$\bar{g}(\xi, y, \zeta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y, z)\, e^{i(\xi x + \zeta z)}\, dx\, dz,$$

and

$$g(x, y, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{g}(\xi, y, \zeta)\, e^{-i(\xi x + \zeta z)}\, d\xi\, d\zeta, \tag{3.1}$$

(2.2) reduces to

$$\begin{aligned}
&\left\{\frac{d^2}{dy^2} - (\alpha^2 - M^2)\xi^2 - \zeta^2\right\}\bar{u} - i(\alpha^2 - 1)\xi\frac{d\bar{v}}{dy} - (\alpha^2 - 1)\xi\zeta\,\bar{w} = 0,\\
&-i\xi(\alpha^2 - 1)\frac{d\bar{u}}{dy} + \left\{\alpha^2\frac{d^2}{dy^2} - (1 - M^2)\xi^2 - \zeta^2\right\}\bar{v} - i(\alpha^2 - 1)\xi\frac{d\bar{w}}{dy} = 0,\\
&-(\alpha^2 - 1)\xi\zeta\,\bar{u} - i(\alpha^2 - 1)\zeta\frac{d\bar{v}}{dy} + \left\{\frac{d^2}{dy^2} - (1 - M^2)\xi^2 - \alpha^2\zeta^2\right\}\bar{w} = 0,
\end{aligned} \tag{3.2}$$

with $\alpha^2 = (\lambda + 2\mu)/\mu$, $\beta^2 = \rho/\mu$ and $M^2 = \beta^2 U^2$.

Due to symmetry given in (2.4), we need to consider the region $y \geqslant 0$ only. The solutions of (3.2) in the region $y \geqslant 0$ can easily be found to be of the form

$$\begin{aligned}
\bar{u} &= A_1 e^{-s_1 y} + B_1 e^{-s_2 y},\\
\bar{v} &= A_2 e^{-s_1 y} + B_2 e^{-s_2 y},\\
\bar{w} &= A_3 e^{-s_1 y} + B_3 e^{-s_2 y},
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
s_1 &= \sqrt{(1 - M^2/\alpha^2)\xi^2 + \zeta^2},\\
s_2 &= \sqrt{(1 - M^2)\xi^2 + \zeta^2},
\end{aligned} \tag{3.4}$$

and

$$A_1 = i\xi A_2/s_1, \quad A_3 = i\zeta A_2/s_1, \quad B_2 = -i(\xi B_1 + \zeta B_3)/s_2. \tag{3.5}$$

The transformed stress components $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}_{xy}$, $\bar{\tau}_{yz}$ obtained from (3.3), (3.5) and (2.3) are

$$\begin{aligned}
\frac{\bar{\sigma}_x}{2\mu} &= \frac{1}{2s_1}[\xi^2 M^2(1 - 2/\alpha^2) + 2\xi^2]A_2 e^{-s_1 y} - i\xi B_1 e^{-s_2 y},\\
\frac{\bar{\sigma}_y}{2\mu} &= \frac{1}{2s_1}[\xi^2 M^2 - 2(\xi^2 + \zeta^2)]A_2 e^{-s_1 y} + i(\xi B_1 + \zeta B_3) e^{-s_2 y},
\end{aligned} \tag{3.6}$$

$$\frac{\bar{\tau}_{xy}}{2\mu} = -i\xi A_2 e^{-s_1 y} - \frac{1}{2s_2}[\xi\zeta B_3 + (s_2^2 + \xi^2)B_1]e^{-s_2 y},$$

$$\frac{\bar{\tau}_{yz}}{2\mu} = -i\zeta A_2 e^{-s_1 y} - \frac{1}{2s_2}[\xi\zeta B_1 + (s_2^2 + \zeta^2)B_3]e^{-s_2 y}.$$

Using the conditions (2.4.3) B_1 and B_3 can be expressed in terms of A_2 as

$$B_1 = \frac{-2i\xi s_2 A_2}{(2 - M^2)\xi^2 + 2\zeta^2},$$
$$B_3 = \frac{-2i\zeta s_2 A_2}{(2 - M^2)\xi^2 + 2\zeta^2}. \tag{3.7}$$

Hence we find that all the components of stress and displacement can be expressed in terms of the unknown function $A_2(\xi, \zeta)$. Now insertion of (3.5) and (3.7) in $\bar{v}$ given in (3.3) yields

$$A_2 = -\frac{(2 - M^2)\xi^2 + 2\zeta^2}{M^2\xi^2}\bar{v}_0, \tag{3.8}$$

where $\bar{v}_0$ is the transformed displacement on $y = 0$.

Using (3.7) and (3.8) we obtain from (3.6)

$$\frac{\bar{\sigma}_x}{2\mu} = -\bar{v}_0\left[\{2 - M^2)\xi^2 + 2\zeta^2\}\{2 + M^2(1 - 2/\alpha^2)\}\frac{e^{-s_1 y}}{2M^2 s_1} - \frac{2s_2 e^{-s_2 y}}{M^2}\right],$$

$$\frac{\bar{\sigma}_y}{2\mu} = \bar{v}_0\left[\{2 - M^2)\xi^2 + 2\zeta^2\}^2\frac{e^{-s_1 y}}{2M^2\xi^2 s_1} - \frac{2(\xi^2 + \zeta^2)s_2 e^{-s_2 y}}{\xi^2 M^2}\right],$$

$$\frac{\bar{\tau}_{xy}}{2\mu} = i\xi\bar{v}_0\{(2 - M^2)\xi^2 + 2\zeta^2\}\frac{e^{-s_1 y} - e^{-s_2 y}}{M^2\xi^2},$$

$$\frac{\bar{\tau}_{yz}}{2\mu} = i\zeta\bar{v}_0\{(2 - M^2)\xi^2 + 2\zeta^2\}\frac{e^{-s_1 y} - e^{-s_2 y}}{M^2\xi^2}. \tag{3.9}$$

Using the conditions (2.4.1) and (2.4.2) we obtain the following triple integral equations

$$\frac{\bar{\sigma}_y}{2\mu} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{v}_0 G(\xi, \zeta)e^{-i\xi x}d\xi = -\bar{p}(x, \zeta), \quad \text{for } a < |x| < b$$

and

$$\bar{v}_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{v}_0 e^{-i\xi x}d\xi = 0, \quad \text{for } |x| > b, \quad |x| < a \tag{3.10}$$

with

$$G(\xi, \zeta) = \frac{1}{2M^2\xi^2 s_1}[\{(2 - M^2)\xi^2 + 2\zeta^2\}^2 - 4(\xi^2 + \zeta^2)s_1 s_2]. \tag{3.11}$$

Taking $p(x, z)$ as the even function of x, the solution may be assumed as

$$\bar{v}_0(x, \zeta) = \sum_{n=1}^{\infty} c_n(\zeta)\frac{(-1)^{n+1}}{n}\sin\left[n\cos^{-1}\left\{\frac{a + b - 2|x|}{b - a}\right\}\right], \quad \text{for } a \leqslant |x| \leqslant b$$

$$= 0, \quad \text{for } 0 \leqslant |x| < a, \quad |x| > b, \tag{3.12}$$

where $c_n(\zeta)$ are the unknown functions to be determined.

Applying Fourier transformation on (3.12) we obtain

$$\bar{v}_0(\xi, \zeta) = \frac{2\pi}{\xi}\sum_{n=1}^{\infty} c_n(\zeta)\sin\left(\frac{a + b}{2}\xi - \frac{n\pi}{2}\right)J_n\left(\frac{b - a}{2}\xi\right), \tag{3.13}$$

where $J_n(\ )$ are Bessel functions.

Insertion of the expression (3.13) in the first equation of (3.10) yields

$$2\sum_{n=1}^{\infty} c_n(\zeta)\int_0^{\infty}\frac{G(\xi, \zeta)}{\xi}\sin\left(\frac{a + b}{2}\xi - \frac{n\pi}{2}\right)J_n\left(\frac{b - a}{2}\xi\right)\cos(\xi x)\,d\xi = -\bar{p}(x, \zeta), \quad \text{for } a < x < b. \tag{3.14}$$

Using the following results [12]

$$\int_0^{\infty}\cos(a_1\xi)J_n(a_2\xi)\,d\xi = \frac{\cos(n\varepsilon)}{\sqrt{a_2^2 - a_1^2}}, \quad \text{for } a_2 > a_1 > 0$$

$$= \frac{a_2^n\sin(n\pi/2)}{\sqrt{a_1^2 - a_2^2}\,[a_1 + \sqrt{a_1^2 - a_2^2}]^n}, \quad \text{for } a_1 > a_2 > 0 \tag{3.15}$$

and

$$\int_0^{\infty}\sin(a_1\xi)J_n(a_2\xi)\,d\xi = \frac{\sin(n\varepsilon)}{\sqrt{a_2^2 - a_1^2}}, \quad \text{for } a_2 > a_1 > 0$$

$$= \frac{a_2^n\cos(n\pi/2)}{\sqrt{a_1^2 - a_2^2}\,[a_1 + \sqrt{a_1^2 - a_2^2}]^n}, \quad \text{for } a_1 > a_2 > 0, \tag{3.15}$$

where

$$\varepsilon = \sin^{-1}(a_1/a_2)$$

in (3.14) we obtain,

$$\sum_{n=1}^{\infty} c_n(\zeta)\Bigg[\Bigg[\int_0^{\infty}\left\{\frac{G(\xi,\zeta)}{\xi}-\frac{G(\delta,\zeta)}{\delta}\right\}\Bigg[\cos\left(\frac{n\pi}{2}\right)\left\{\sin\left(\frac{a+b+2x}{2}\xi\right)+\sin\left(\frac{a+b-2x}{2}\xi\right)\right\}$$

$$-\sin\left(\frac{n\pi}{2}\right)\left\{\cos\left(\frac{a+b+2x}{2}\xi\right)+\cos\left(\frac{a+b-2x}{2}\xi\right)\right\}\Bigg]J_n\left(\frac{b-a}{2}\xi\right)\mathrm{d}\xi+\frac{G(\delta,\zeta)}{\delta}$$

$$\times\Bigg[\left(\frac{b-a}{2}\right)^n\Bigg/\Bigg[\sqrt{\left(\frac{a+b+2x}{2}\right)^2-\left(\frac{b-a}{2}\right)^2}\left\{\frac{a+b+2x}{2}\right.$$

$$\left.+\sqrt{\left(\frac{a+b+2x}{2}\right)^2-\left(\frac{b-a}{2}\right)^2}\right\}^n\Bigg]+\sin\left\{n\sin^{-1}\left(\frac{a+b-2x}{b-a}\right)-\frac{n\pi}{2}\right\}\Bigg/$$

$$\sqrt{\left(\frac{b-a}{2}\right)^2-\left(\frac{a+b-2x}{2}\right)^2}\Bigg]\Bigg]=-\bar{p}(x,\zeta), \tag{3.16}$$

where

$$\frac{G(\delta,\zeta)}{\delta}=\operatorname*{Lt}_{\xi\to\infty}\frac{G(\xi,\zeta)}{\xi}=\{(2-M^2)^2-4\sqrt{1-M^2}\cdot\sqrt{1-M^2/\alpha^2}\}/2M^2\sqrt{1-M^2/\alpha^2}. \tag{3.17}$$

Since the function $G(\xi,\zeta)/\xi - G(\delta,\zeta)/\delta$ behaves as ξ^{-2} for large ξ, the semi-infinite integral on the left hand side of (3.16) can easily be evaluated by Filon's method.

To solve (3.16) for unknown coefficients $c_n(\zeta)$ we adopt the Schmidt method [11] and write (3.14) as

$$\sum_{n=1}^{\infty} c_n(\zeta)F_n(\zeta,x)=-f(\zeta,x),\quad \text{for } a<x<b, \tag{3.18}$$

where $F_n(\zeta,x)$ and $f(\zeta,x)=\bar{p}(\zeta,x)$ are known functions. Let $H_n(\zeta,x)$'s be a set of orthogonal functions which satisfy

$$\int_a^b H_n(\zeta,x)H_m(\zeta,x)\,\mathrm{d}x=N_n\delta_{mn},$$

where

$$N_n=\int_a^b H_n^2(\zeta,x)\,\mathrm{d}x. \tag{3.19}$$

Then $H_n(\zeta,x)$'s can be constructed from the functions $F_n(\zeta,x)$ in the following way

$$H_n(\zeta,x)=\sum_{i=1}^{\infty}\frac{C_{in}}{C_{nn}}F_i(\zeta,x) \tag{3.20}$$

with C_{in} as the cofactor of the element e_{in} of D_n which is defined as

$$D_n = \begin{vmatrix} e_{11}e_{12}\cdots\cdots e_{1n} \\ e_{21}\cdots\cdots\cdots \\ \vdots \qquad\qquad \vdots \\ e_{n1}\cdots\cdots\cdots e_{nn} \end{vmatrix}, \quad e_{in} = \int_a^b F_n(\zeta, x) F_i(\zeta, x)\,\mathrm{d}x. \tag{3.21}$$

Now in terms of the set of orthogonal functions $H_n(\zeta, x)$, the function $-f(\zeta, x)$ can be expressed as

$$-f(\zeta, x) = \sum_{i=1}^{\infty} h_i H_i(\zeta, x). \tag{3.22}$$

Substituting values of $H_n(\zeta, x)$ from (3.20) into (3.22), we obtain from (3.18) after some rearrangement

$$\sum_{n=1}^{\infty} c_n(\zeta) F_n(\zeta, x) = \sum_{n=1}^{\infty} F_n(\zeta, x) \sum_{i=n}^{\infty} h_i \frac{C_{ni}}{C_{ii}}. \tag{3.23}$$

Comparing the coefficients of $F_n(\zeta, x)$ from both sides of (3.23) we find

$$c_n = \sum_{i=n}^{\infty} h_i \frac{C_{ni}}{C_{ii}}, \tag{3.24}$$

where

$$h_i = -\frac{1}{N_i} \int_a^b f(\zeta, x) H_i(\zeta, x)\,\mathrm{d}x. \tag{3.25}$$

4. Stress intensity factors and crack opening displacement

To evaluate the stress intensity factors at the vicinity of the crack ends we put $x = b + r\cos\theta$, $y = r\sin\theta$ for the stress intensity factor at the outer edge and $x = a - r\cos\theta$, $y = r\sin\theta$ for the stress intensity factor at the inner edge.

The required stress σ_θ given by

$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta\cos\theta \tag{4.1}$$

is to be evaluated for small values of r.

Using asymptotic values of

$$J_n\left(\frac{b-a}{2}\xi\right)$$

for large values of ξ, it is found that for small values of r

$$\int_0^\infty e^{-\sqrt{1-q^2}\xi y}\sin\left(\frac{a+b}{2}\xi-\frac{n\pi}{2}\right)J_n\left(\frac{b-a}{2}\xi\right)\cos(\xi x)\,d\xi=-\frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4r(b-a)}}$$

$$\times\left[\cos\left(\frac{n\pi}{2}\right)\sqrt{\frac{(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right.$$

$$\left.+\sin\left(\frac{n\pi}{2}\right)\sqrt{\frac{-(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right]+O(r^0),\quad \text{for } x>b \tag{4.2}$$

$$=\frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4r(b-a)}}\times\left[\cos\left(\frac{n\pi}{2}\right)\sqrt{\frac{(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right.$$

$$\left.-\sin\left(\frac{n\pi}{2}\right)\sqrt{\frac{-(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right]+O(r^0),\quad \text{for } x<a \tag{4.3}$$

and

$$\int_0^\infty e^{-\sqrt{1-q^2}\xi y}\sin\left(\frac{a+b}{2}\xi-\frac{n\pi}{2}\right)J_n\left(\frac{b-a}{2}\xi\right)\sin(\xi x)\,d\xi=\frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4r(b-a)}}$$

$$\times\left[\cos\left(\frac{n\pi}{2}\right)\sqrt{\frac{-(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right.$$

$$\left.-\sin\left(\frac{n\pi}{2}\right)\sqrt{\frac{(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right]+O(r^0),\quad \text{for } x>b \tag{4.4}$$

$$=\frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4r(b-a)}}\times\left[\cos\left(\frac{n\pi}{2}\right)\sqrt{\frac{-(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right.$$

$$\left.+\sin\left(\frac{n\pi}{2}\right)\sqrt{\frac{(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right]+O(r^0),\quad \text{for } x<a. \tag{4.5}$$

Inserting (3.13) into (3.9) and taking inverse Fourier transform of (3.9) we obtain the stress intensity factor at $x = b$ with the aid of (4.2)–(4.5) as

$$K_b = \frac{\sigma_0}{2\mu}\sqrt{r}|_{r\to 0} = \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{b-a}}\left[\frac{2-M^2}{2M^2\sqrt{1-M^2/\alpha^2}} Q_1^+ \{(2+M^2(1-2/\alpha^2))\sin^2\theta\right.$$

$$-(2-M^2)\cos^2\theta\} + \frac{2\sqrt{1-M^2}\cos 2\theta}{M^2} Q_2^+ - \left.\frac{2-M^2}{M^2}(P_1^- - P_2^-)\sin 2\theta\right]$$

$$\times \frac{1}{2\pi}\int_{-\infty}^{\infty} c_n(\zeta)\,e^{-i\zeta z}\,d\zeta \tag{4.6}$$

and also the stress intensity factor at $x = a$ is found to be

$$K_a = \frac{\sigma_0}{2\mu}\sqrt{r}|_{r\to 0} = \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{b-a}}\left[\frac{-(2-M^2)}{2M^2\sqrt{1-M^2/\alpha^2}} Q_1^- \{(2+M^2(1-2/\alpha^2))\sin^2\theta\right.$$

$$-(2-M^2)\cos^2\theta\} - \frac{2\sqrt{1-M^2}\cos 2\theta}{M^2} Q_2^- - \left.\frac{2-M^2}{M^2}(P_1^+ - P_2^+)\sin 2\theta\right]$$

$$\times \frac{1}{2\pi}\int_{-\infty}^{\infty} c_n(\zeta)\,e^{-i\zeta z}\,d\zeta \tag{4.7}$$

where

$$\left.\begin{aligned} Q_i^{\pm} &= \left[\cos\left(\frac{n\pi}{2}\right)\sqrt{q_i + (-1)^n\cos\theta} \pm \sin\left(\frac{n\pi}{2}\right)\sqrt{q_i - (-1)^n\cos\theta}\right]\Big/ q_i \\ P_i^{\pm} &= \left[\cos\left(\frac{n\pi}{2}\right)\sqrt{q_i - (-1)^n\cos\theta} \pm \sin\left(\frac{n\pi}{2}\right)\sqrt{q_i + (-1)^n\cos\theta}\right]\Big/ q_i \end{aligned}\right\} \quad (i = 1, 2)$$

and

$$q_1 = \sqrt{1 - \frac{M^2}{\alpha^2}\sin^2\theta},$$

$$q_2 = \sqrt{1 - M^2\sin^2\theta}.$$

It is to be noted that in (4.6) $\theta = \tan^{-1} y/(x-b)$ whereas in (4.7) it is given by $\theta = \tan^{-1} y/(a-x)$.

Taking Fourier inversion of (3.12) we obtain the crack surface displacement as

$$v_0(x, z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left[n \cos^{-1}\left\{\frac{a + b - 2|x|}{b - a}\right\}\right] \frac{1}{2\pi} \int_{-\infty}^{\infty} c_n(\zeta)\, e^{-i\zeta z}\, d\zeta,$$

$$\text{for } a \leqslant |x| \leqq b \tag{4.8}$$

5. Numerical discussions

In order to evaluate the stress intensity factors and crack surface displacement we take the function $p(x, z)$ as

$$p(x, z) = \frac{P}{1 + d^2 z^2},$$

where d governs the distribution of the applied force and P is a constant. Numerical calculations have been done taking $\lambda = \mu$ and $d = 1$. The semi-infinite integral in (3.16) is evaluated by Filon's method as the integral converges rapidly because of the rapid decay of the function

$$\left\{\frac{G(\xi, \zeta)}{\xi} - \frac{G(\delta, \zeta)}{\delta}\right\}$$

with the increase in ξ. Adopting the first seven terms of the infinite series given in the left hand side of (3.18) we used the Schmidt method to determine the coefficients $c_n(\zeta)$. For the check of accuracy the values of $\sum_{n=1}^{7} c_n(\zeta)\, F_n(\zeta, x)/Pb$ and $-f(\zeta, x)/Pb$ are given in Table 1 for $\zeta b = 0.0$, 0.2, $M = 0.4$ and for $a/b = 0.3$, 0.4.

From Table 1 it is clear that the Schmidt method is carried out satisfactorily. The values of $c_n(\zeta)$ are given in Table 2 for $M = 0.4$, $a/b = 0.4$.

The variation of stress intensity factor at the outer edge and at the inner edge with M is shown in Fig. 2 and Fig. 3 respectively for $\theta = 0°$, 18°, 36° and $a/b = 0.2$, 0.3, 0.4. Figure 2 depicts the fact that the value of stress intensity factor at the outer edge decreases with the increase in the values of a/b, whereas from Fig. 3 it is evident that the stress intensity factor at the inner edge is of an opposite character. It increases with the increase in the values of a/b.

The variations of stress intensity factor both at the inner edge and outer edge with z have been presented in Figs. 4–7 for different values of a/b, M and θ. The values of stress intensity factor in all the cases are found to decrease gradually with the increase in the values of z, which is expected from the physical standpoint.

The variation of stress intensity factor corresponding to the circumferential stress σ_θ given by (4.1) with θ at both the crack tips has been shown in Figs. 8–12 for different values of a/b and M.

It is known that there are several factors which contribute to crack curving and branching. One factor, of course, is based upon the criterion that a crack may propagate in a direction normal to the maximum tensile stress and it is interesting to note from Fig. 8 and Fig. 10, there is the possibility of curving and branching of the cracks at the outer edge at very low velocities

Table 1.

ζ/b	a/b	x/b	$\sum_{n=1}^{7} c_n(\zeta)F_n(\zeta,x)/Pb$	$f(\zeta,x)/Pb$
0.0	0.3	0.3	−3.140993	−3.140994
		0.4	−3.140995	
		0.5	−3.140993	
		0.6	−3.140996	
		0.7	−3.140991	
		0.8	−3.140994	
		0.9	−3.140993	
		1.0	−3.140992	
	0.4	0.4	−3.140995	
		0.5	−3.140994	
		0.6	−3.140994	
		0.7	−3.140994	
		0.8	−3.140994	
		0.9	−3.140995	
		1.0	−3.140994	
0.2	0.3	0.3	−2.572111	−2.572113
		0.4	−2.572113	
		0.5	−2.572111	
		0.6	−2.572116	
		0.7	−2.572110	
		0.8	−2.572113	
		0.9	−2.572108	
		1.0	−2.572106	
	0.4	0.4	−2.572114	
		0.5	−2.572114	
		0.6	−2.572114	
		0.7	−2.572113	
		0.8	−2.572113	
		0.9	−2.572113	
		1.0	−2.572113	

Table 2.

ζ/b	$c_1(\zeta)$	$c_2(\zeta)$	$c_7(\zeta)$
0.0	-0.165871×10^{1}	-0.923569×10^{-4}	-0.759039×10^{-8}
0.2	-0.135194×10^{1}	-0.734980×10^{-4}	0.105638×10^{-6}
0.4	-0.109342×10^{1}	-0.556495×10^{-4}	0.357814×10^{-6}
3.0	-0.578184×10^{-3}	-0.601254×10^{-7}	0.114694×10^{-5}
4.0	-0.182994×10^{-3}	0.883491×10^{-7}	0.659423×10^{-6}
5.0	-0.573139×10^{-4}	0.489839×10^{-7}	0.342023×10^{-6}
9.6	0.366305×10^{-5}	-0.816894×10^{-8}	-0.907244×10^{-7}
9.8	0.362848×10^{-5}	-0.829789×10^{-8}	-0.938769×10^{-7}
10.0	0.358409×10^{-5}	-0.843117×10^{-8}	-0.967438×10^{-7}

of the cracks whereas from Fig. 9, Fig. 11 and Fig. 12 it is clear that for $a/b = 0.3$, the crack tends to become curved at the inner edge for values of M about 0.65.

Finally the crack opening displacement in the plane $z = 0$ has been shown by means of graphs in Figs. 15–16 for different values of a/b and M. The variation of crack opening displacement with z for some fixed x for different values of M and a/b has been depicted in Figs. 13–14.

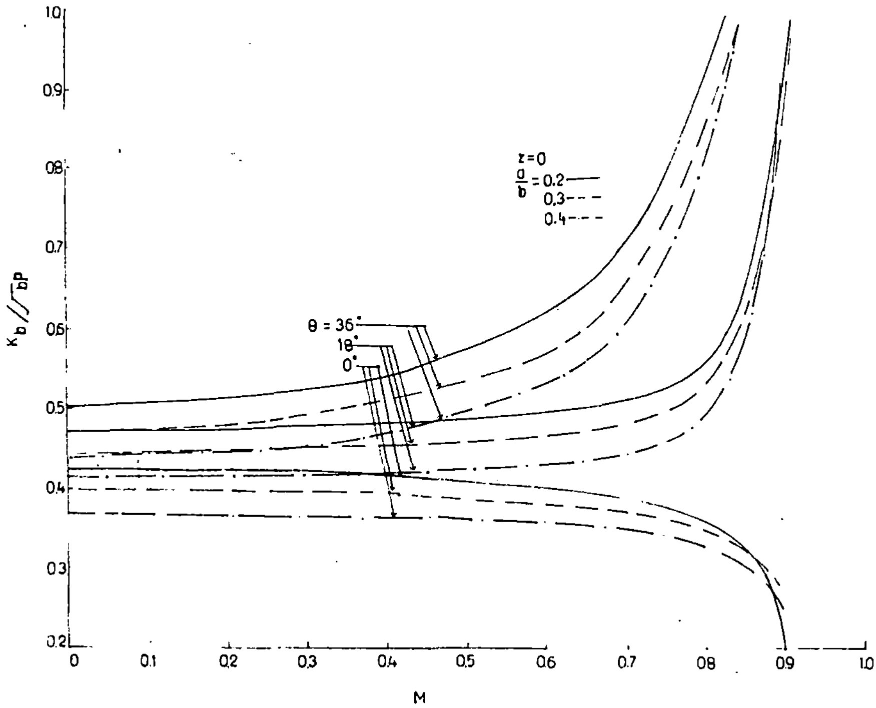
z=0
a/b = 0.2
0.3
0.4
θ = 36°
18°
0°
$K_b/\sqrt{bP}$
M

Fig. 2.

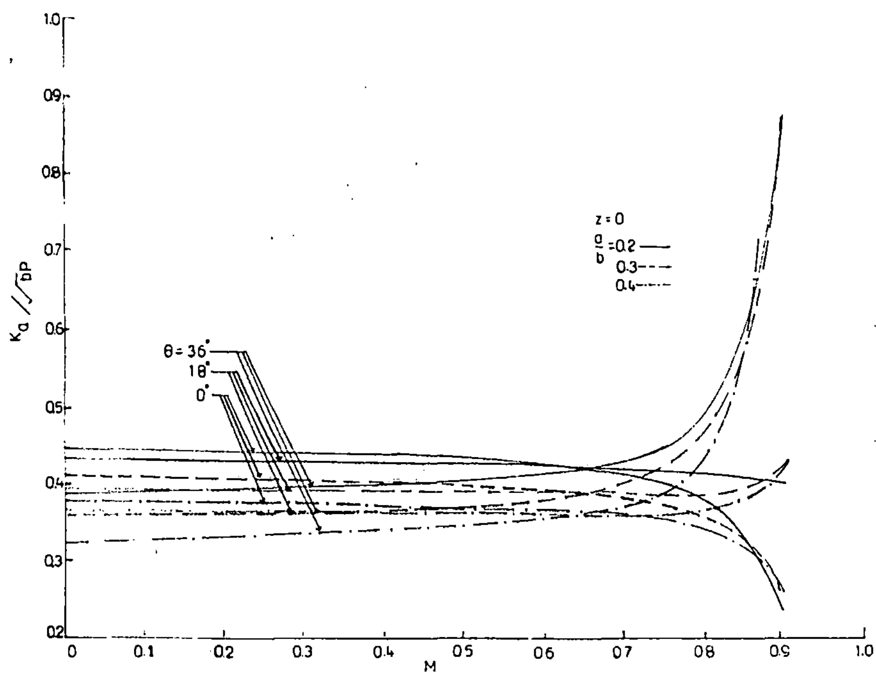
z=0
a/b = 0.2
0.3
0.4
θ = 36°
18°
0°
$K_a/\sqrt{bP}$
M

Fig. 3.

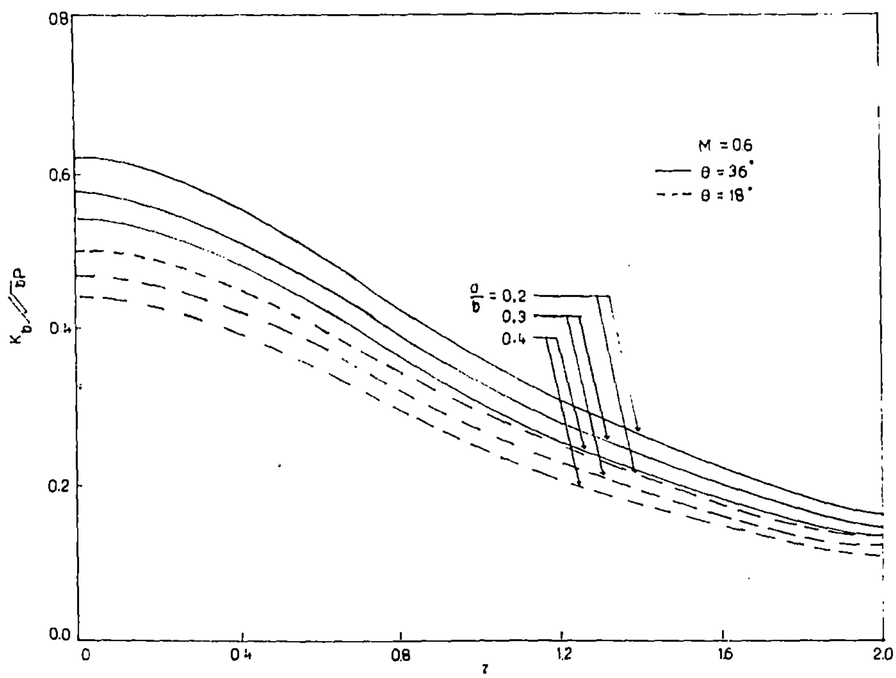
M = 0.6
θ = 36°
θ = 18°
a/b = 0.2
0.3
0.4
$K_b/\sqrt{bP}$
z

Fig. 4.

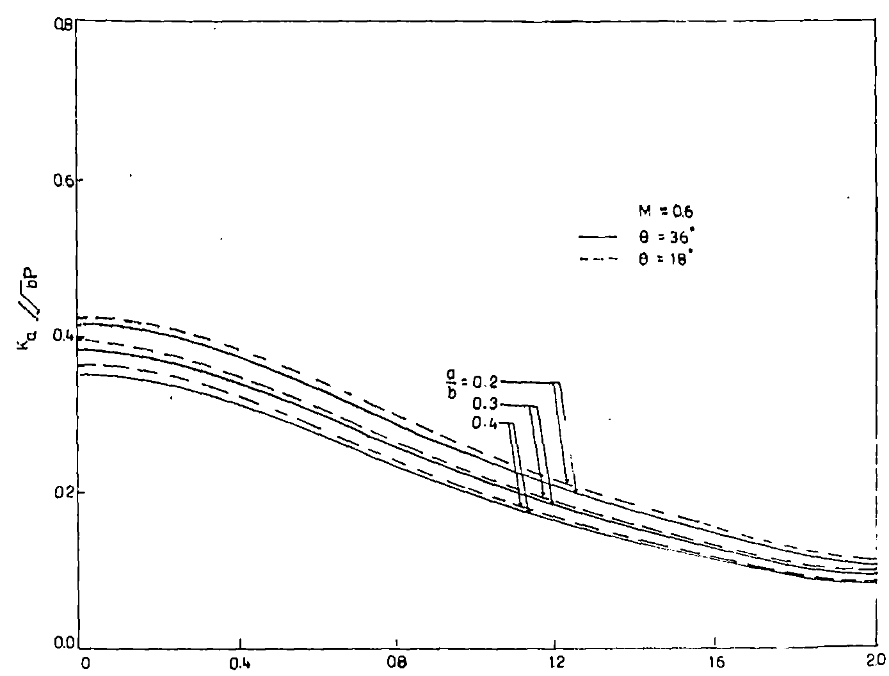
M = 0.6
θ = 36°
θ = 18°
a/b = 0.2
0.3
0.4
$K_a/\sqrt{bP}$
z

Fig. 5.

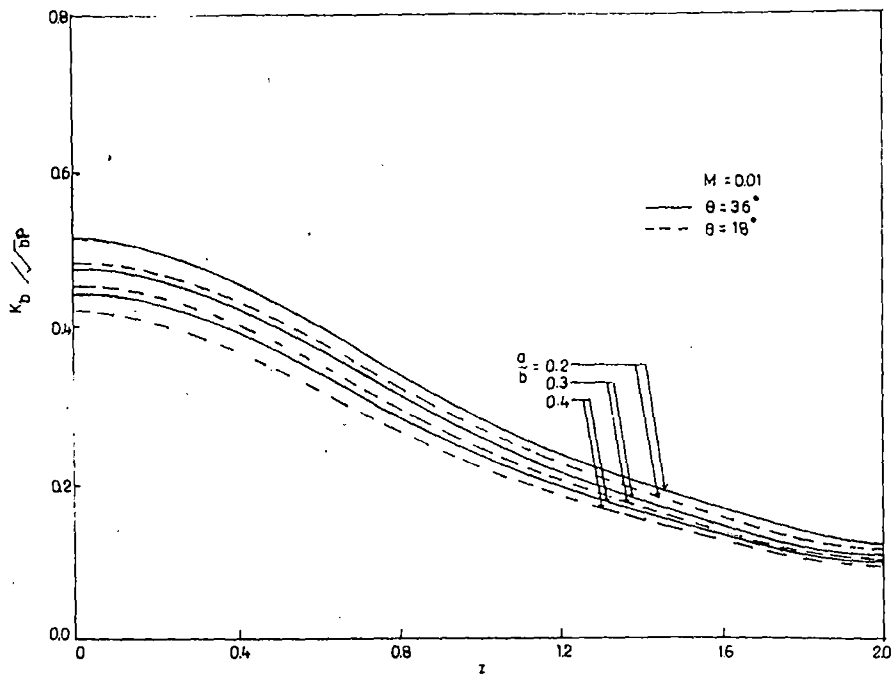
M = 0.01
θ = 36°
θ = 18°
a/b = 0.2
0.3
0.4
$K_b/\sqrt{bP}$
z
0.0
0.2
0.4
0.6
0.8
0
0.4
0.8
1.2
1.6
2.0

Fig. 6.

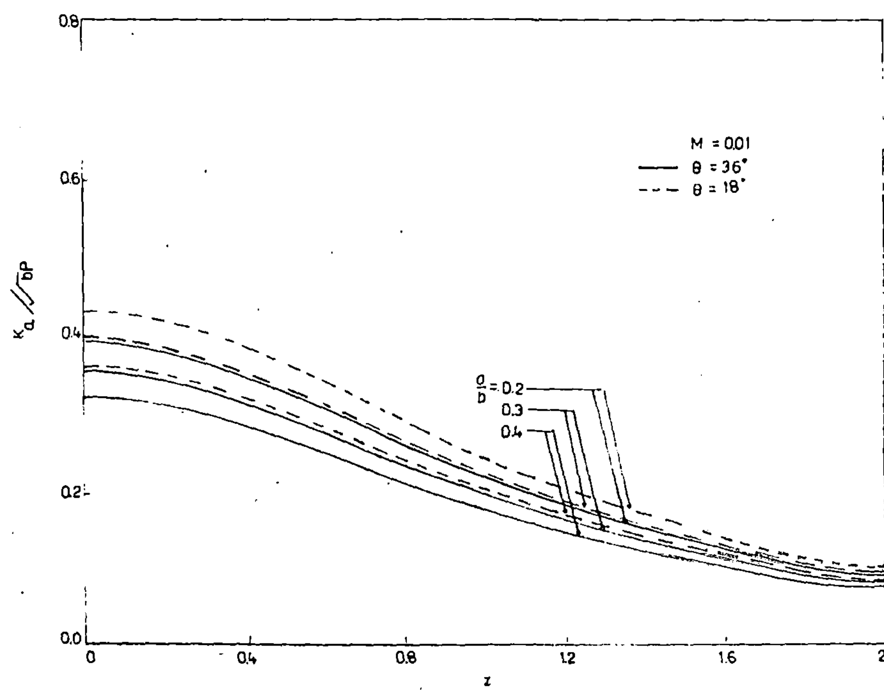
M = 0.01
θ = 36°
θ = 18°
a/b = 0.2
0.3
0.4
$K_a/\sqrt{bP}$
z
0.0
0.2
0.4
0.6
0.8
0
0.4
0.8
1.2
1.6
2.0

Fig. 7.

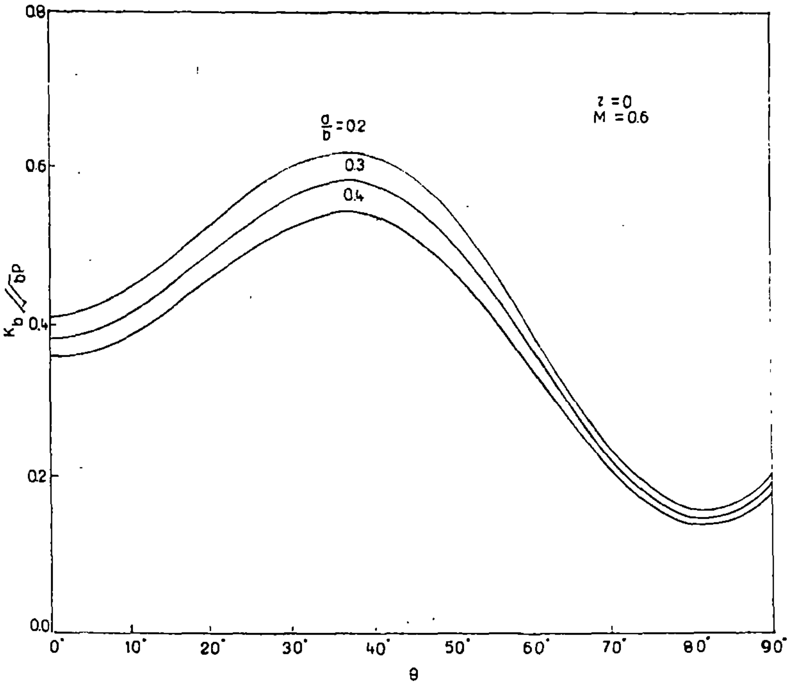
0.8
0.6
0.4
0.2
0.0
$K_b/\sqrt{b}P$
a/b = 0.2
0.3
0.4
z = 0
M = 0.6
0° 10° 20° 30° 40° 50° 60° 70° 80° 90°
θ

Fig. 8.

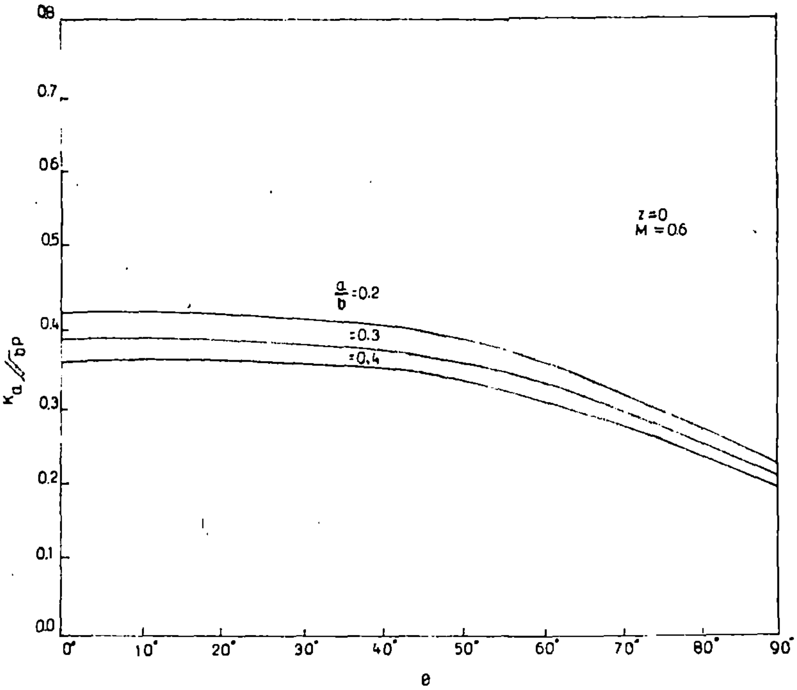
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
$K_a/\sqrt{b}P$
a/b = 0.2
= 0.3
= 0.4
z = 0
M = 0.6
0° 10° 20° 30° 40° 50° 60° 70° 80° 90°
θ

Fig. 9.

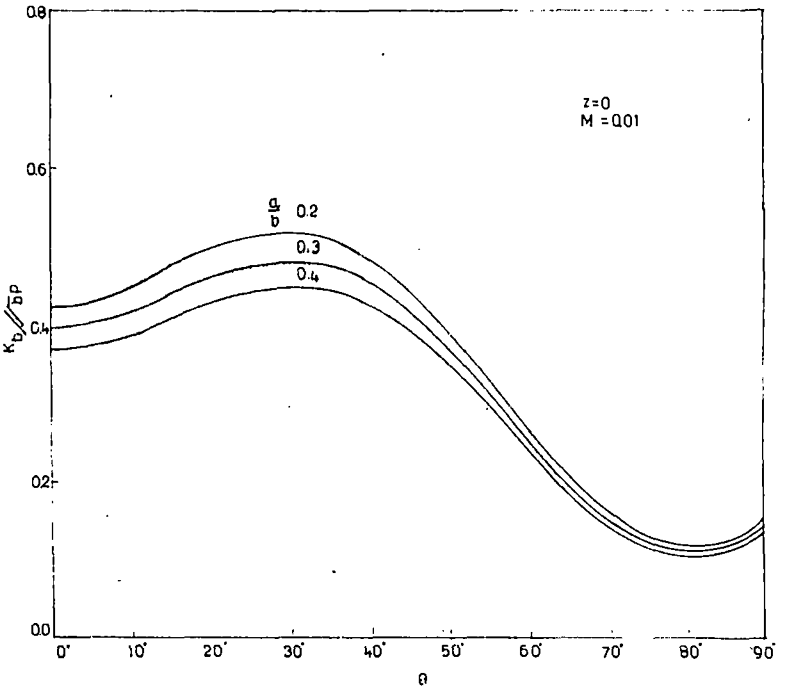
z=0
M = 0.01
$\frac{a}{b}$ 0.2
0.3
0.4
0.8
0.6
0.4
0.2
0.0
$K_b/\sqrt{b}P$
0° 10° 20° 30° 40° 50° 60° 70° 80° 90°
θ

Fig. 10.

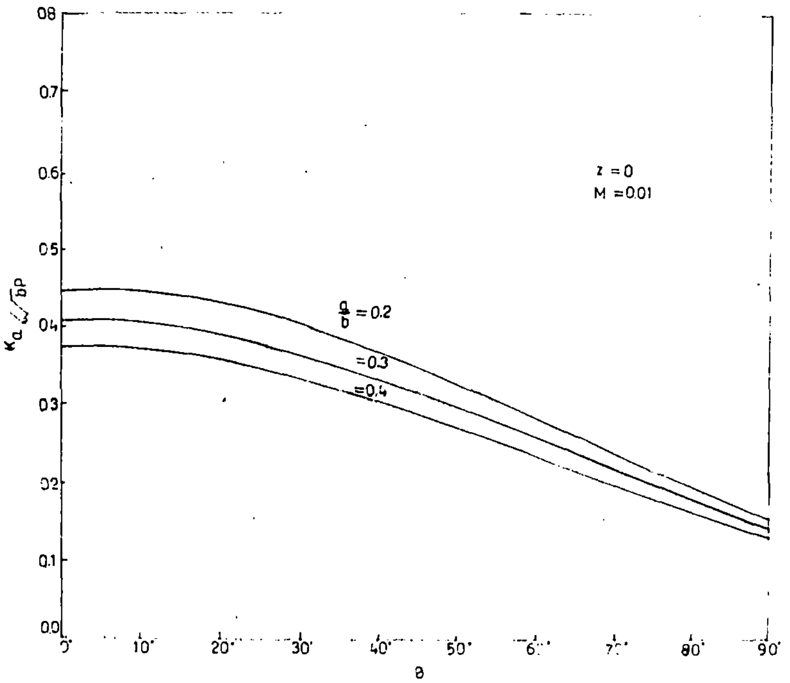
z = 0
M = 0.01
$\frac{a}{b}$ = 0.2
= 0.3
= 0.4
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
$K_a/\sqrt{b}P$
0° 10° 20° 30° 40° 50° 60° 70° 80° 90°
θ

Fig. 11.

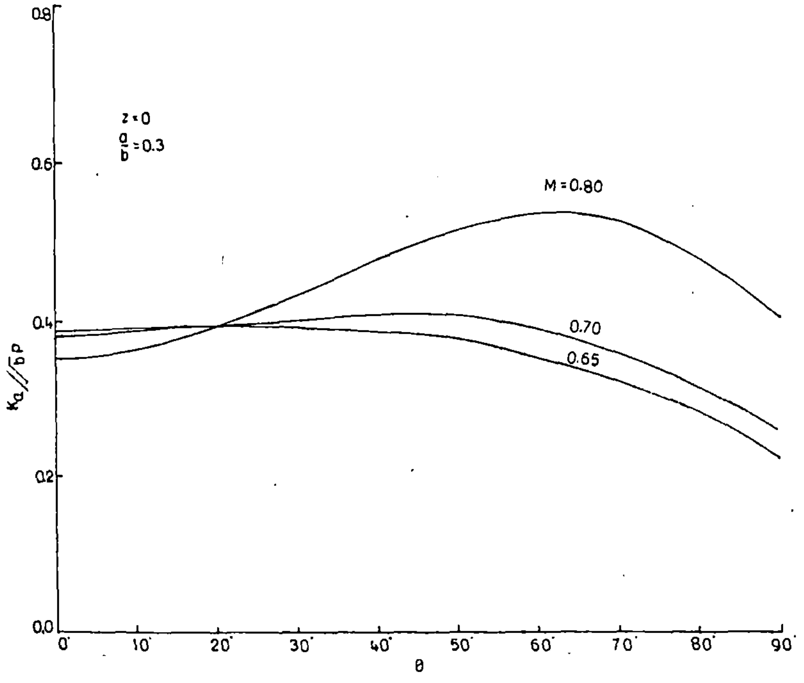

Fig. 12.

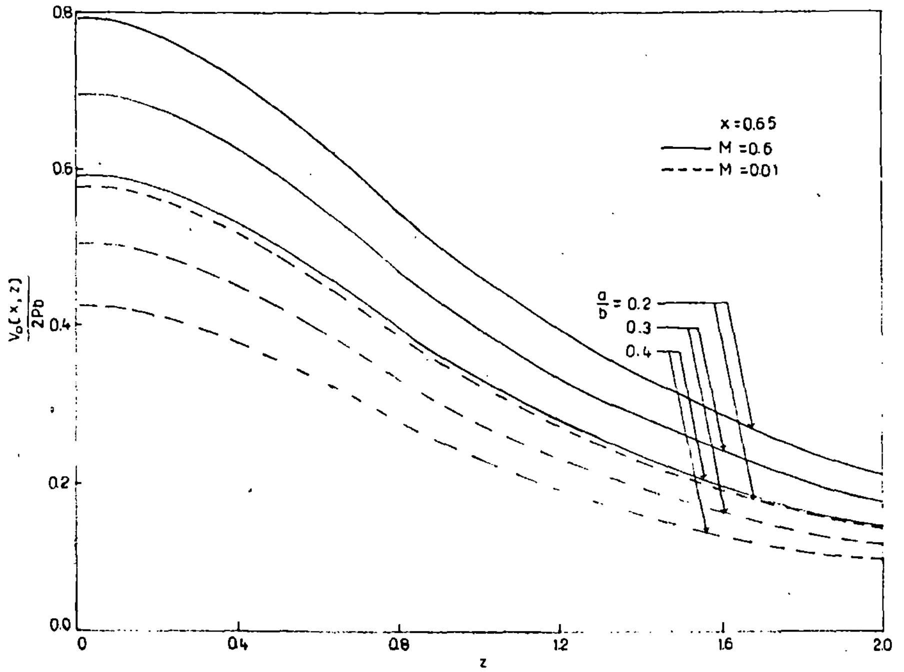

Fig. 13.

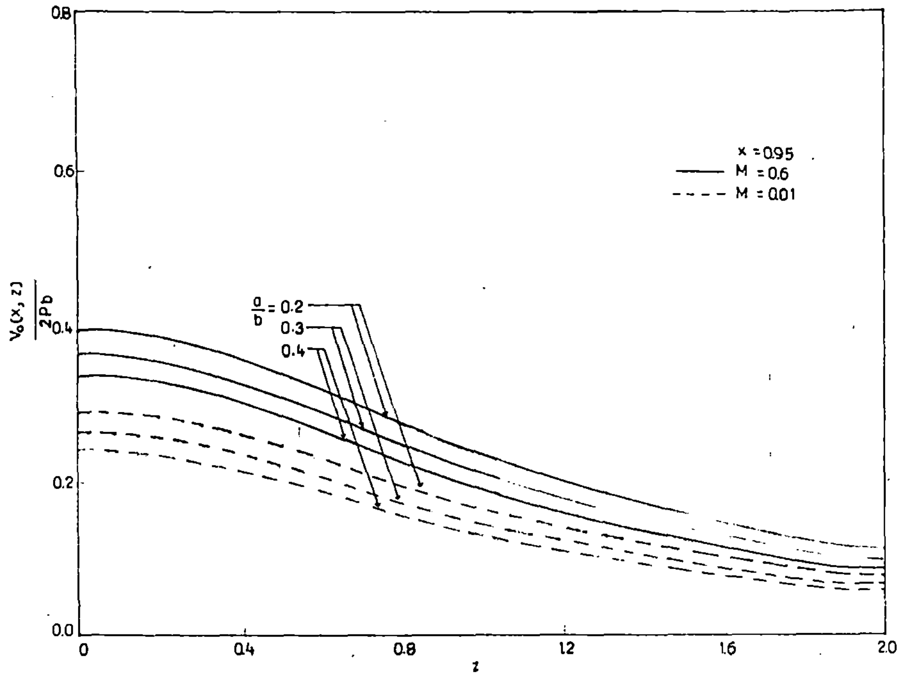
0.8
0.6
0.4
0.2
0.0
0
0.4
0.8
1.2
1.6
2.0
z
$\frac{V_0(x,z)}{2Pb}$
x = 0.95
M = 0.6
M = 0.01
$\frac{a}{b}$ = 0.2
0.3
0.4

Fig. 14.

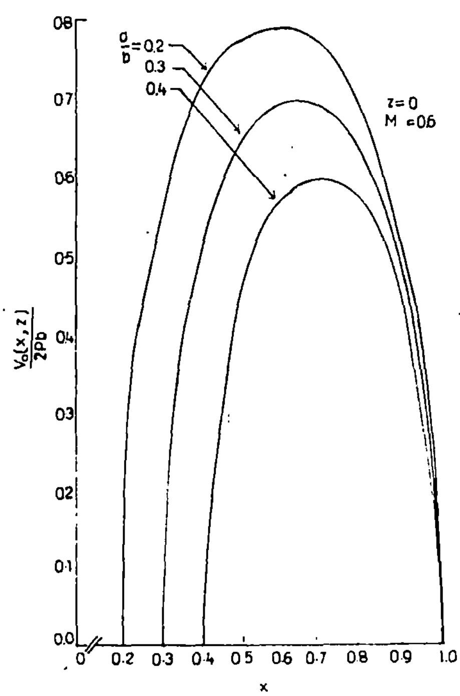
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
x
$\frac{V_0(x,z)}{2Pb}$
$\frac{a}{b}$ = 0.2
0.3
0.4
z = 0
M = 0.6

Fig. 15.

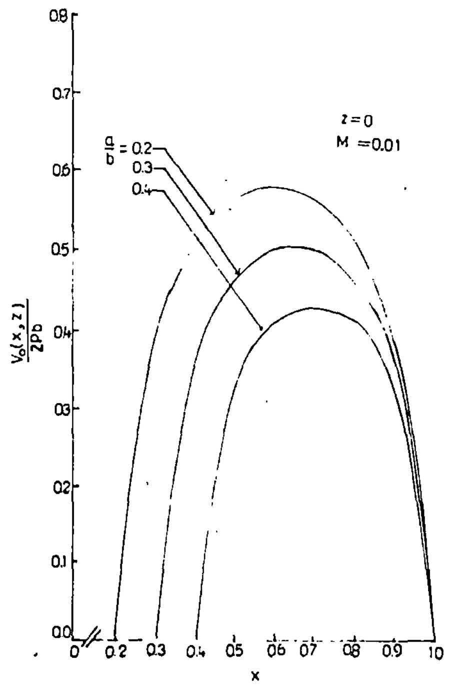

Fig. 16.

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Three co-planar moving Griffith cracks in an infinite elastic medium

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Abstract. The dynamic in-plane problem of determining the stress and displacement due to three co-planar Griffith cracks moving steadily at a subsonic speed in a fixed direction in an infinite, isotropic, homogeneous medium under normal stress has been treated. The static problem of determining the stress and displacement around three co-planar Griffith cracks in an infinite isotropic elastic medium has also been considered. In both the cases, employing Fourier integral transform, the problems have been reduced to solving a set of four integral equations. These integral equations have been solved using finite Hilbert transform technique and Cook's result [16] to obtain the exact form of crack opening displacement and stress intensity factors which are presented in the form of graphs.

1. Introduction

In fracture mechanics, scattering of elastic waves by cracks of finite dimension in an infinite elastic medium has been examined by several investigators. The problem of scattering of elastic waves from an interface crack was solved by Bostrom [1]. Srivastava et al. [2] solved the problem of interaction of an anti-plane shear wave by an interface crack. The problem of diffraction of Love waves by a crack of finite width in the plane interface of a layered composite has been solved by Neerhoff [3]. Itou [4] solved the problem of diffraction of an anti-plane shear wave by two co-planar Griffith cracks in an infinite elastic medium. The scattering of a time harmonic normally incident plane wave by two co-planar Griffith cracks was solved by Jain and Kanwal [5]. Itou [6] also solved the problem of stress concentration around two co-planar Griffith cracks in an infinite elastic medium. Yoffe [7] considered the problem of propagation of a crack of fixed length at a constant speed through a stretched isotropic elastic solid of infinite extent. The problem of diffraction of horizontal shear waves by a moving interface crack has been solved by Nishida et al. [8]. Recently Kassir and Tse [9] have solved the plane stress problem of a moving Griffith crack in an infinite orthotropic stressed medium by using integral transform technique and the same technique has been employed by De and Patra [10] to solve Yoffe's problem in a stressed orthotropic strip of finite thickness. Several problems on two moving co-planar Griffith cracks have been solved by Das and Ghosh [11–13].

As regards the crack problem, research has been restricted mainly to the case of the single crack or a pair of cracks because of severe mathematical complexity encountered in solving the problems of three or more cracks. Recently, Dhawan and Dhaliwal [14] solved the statical problem of determining the stress distribution in an infinite transversely isotropic medium containing three co-planar cracks.

To the best knowledge of the author, the problem of stress distribution around three co-planar moving Griffith cracks in an infinite isotropic elastic medium has not been investigated so far. In this paper, two cases regarding stress distribution around three co-planar Griffith cracks in an infinite homogeneous, isotropic medium have been investigated. In the first case, cracks are assumed to be moving steadily along a fixed direction with constant velocity V. In the second case, the statical problem of determining the stress and displacement in an infinite homogeneous, isotropic medium weakened by three co-planar Griffith cracks has been considered. Using Fourier intergral transform both the problems have been reduced to solving a set of four integral equations. Employing finite Hilbert transform technique [15] and Cook's result [16] the integral equations have been solved to derive crack opening displacement and stress intensity factors which are presented in the form of graphs.

2. Statement of Problem I and its formulation

Consider an infinite homogeneous isotropic material weakened by three co-planar Griffith cracks, moving steadily at a constant velocity V in the X-direction referred to a fixed coordinate system (X, Y, Z) as shown in the Fig. 1. In the absence of body force equations of motion in terms of displacement are

$$(\lambda + 2\mu)[u_{,XX} + v_{,XY}] + \mu[u_{,YY} - v_{,XY}] = \rho u_{,TT}, \tag{2.1}$$

$$(\lambda + 2\mu)[u_{,XY} + v_{,YY}] + \mu[v_{,XX} - u_{,XY}] = \rho v_{,TT}, \tag{2.2}$$

where u, v denote the displacement components in X and Y directions and λ, μ are Lame's constants and $u_{,X}$ represents partial derivatives of u with respect to X.

For cracks moving steadily with constant velocity V in the X-direction it is convenient to introduce the Galilean transformation

$$x = X - VT, \quad y = Y, \quad z = Z, \quad t = T, \tag{2.3}$$

where (x, y, z) represents the translating coordinate system as shown in Fig. 1.

Let the positions of the coplanar Griffith cracks referred to the translating coordinates (x, y, z) be $-a < x < a$, $-c < x < -b$, and $b < x < c$ on $y = 0$.

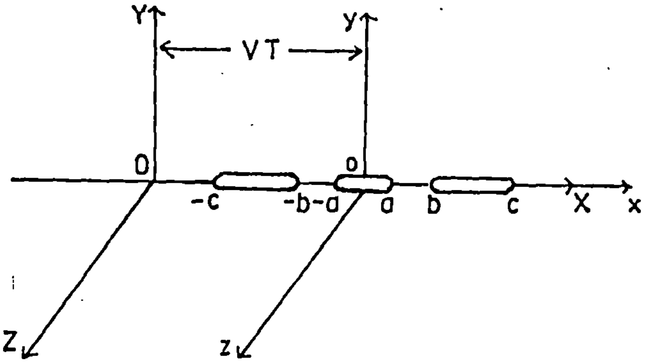

Fig. 1. Geometry and coordinate system.

In the moving coordinates, the equations of motion (2.1) and (2.2) become independent of time and take the form

$$\begin{aligned} &(\lambda + 2\mu - \rho V^2)u_{,xx} + (\lambda + \mu)v_{,xy} + \mu u_{,yy} = 0 \\ &(\lambda + 2\mu)v_{,yy} + (\mu - \rho V^2)v_{,xx} + (\lambda + \mu)u_{,xy} = 0. \end{aligned} \tag{2.4}$$

The cracks are assumed to be moving steadily in an infinite medium subjected to a homogeneous stress such that the state of stress at infinity is given by $\sigma_{yy}^{\infty} = p$, $\sigma_{xx}^{\infty} = \sigma_{xy}^{\infty} = 0$.

For symmetry about the x-axis, only a half-plane need be considered.

The stress conditions at $y = \infty$ can all be made zero by superposing the simple static problem $\sigma_{yy}^{\infty} = -p$, $\sigma_{xx}^{\infty} = \sigma_{xy}^{\infty} = 0$.

The boundary conditions of the resulting dynamic problem are in terms of moving coordinates.

$$\begin{aligned} &v = 0; && y = 0, \quad a \leqslant |x| \leqslant b, \quad |x| \geqslant c, \\ &\sigma_{xy} = 0; && y = 0, \quad |x| < \infty, \\ &\sigma_{yy} = -p; && y = 0, \quad |x| < a, \qquad b < |x| < c. \end{aligned} \tag{2.5}$$

In view of the symmetry of the proposed problem with respect to y-axis, we introduce

$$\bar{u}_s(\xi, y) = \int_0^{\infty} u(x, y)\sin(\xi x)\,\mathrm{d}x,$$

$$\bar{v}_c(\xi, y) = \int_0^{\infty} v(x, y)\cos(\xi x)\,\mathrm{d}x$$

and

$$u(x, y) = \frac{2}{\pi}\int_0^{\infty} \bar{u}_s(\xi, y)\sin(\xi x)\,\mathrm{d}\xi,$$

$$v(x, y) = \frac{2}{\pi}\int_0^{\infty} \bar{v}_c(\xi, y)\cos(\xi x)\,\mathrm{d}\xi$$

in (2.4) so that equations given by (2.4) reduce to

$$\begin{aligned} &\mu\bar{u}_{s,yy} - \xi(\lambda + \mu)\bar{v}_{c,y} - \xi^2(\lambda + 2\mu - \rho V^2)\bar{u}_s = 0, \\ &(\lambda + 2\mu)\bar{v}_{c,yy} + \xi(\lambda + \mu)\bar{u}_{s,y} - \xi^2(\mu - \rho V^2)\bar{v}_c = 0. \end{aligned} \tag{2.6}$$

Elimination of $\bar{u}_s$ from (2.6) yields the following ordinary differential equation

$$\left[\left\{\frac{d^2}{\mathrm{d}y^2} - (1 - M^2k^2)\xi^2\right\}\left\{\frac{d^2}{\mathrm{d}y^2} - (1 - M^2)\xi^2\right\}\right]\bar{v}_c = 0, \tag{2.7}$$

where $M = V/c_2$, $k = c_2/c_1$.

The solution of the differential equation given by (2.7), for $y \geq 0$, is

$$\bar{v}_s(\xi, y) = A(\xi)\,e^{-\xi y\sqrt{1-M^2k^2}} + B(\xi)\,e^{-\xi y\sqrt{1-M^2}}, \tag{2.8}$$

where the unknown functions $A(\xi)$ and $B(\xi)$ are to be determined using the boundary conditions of the proposed problem.

Employing (2.8) in (2.6) it can be shown that

$$\bar{u}_s(\xi, y) = \frac{A(\xi)}{\sqrt{1-M^2k^2}}\,e^{-\xi y\sqrt{1-M^2k^2}} + \sqrt{1-M^2}\,B(\xi)\,e^{-\xi y\sqrt{1-M^2}}, \quad y \geq 0. \tag{2.9}$$

Therefore, the stress components given by

$$\begin{aligned} \sigma_{yy} &= \lambda(u_{,x} + v_{,y}) + 2\mu v_{,y}, \\ \sigma_{xy} &= \mu(u_{,y} + v_{,x}) \end{aligned} \tag{2.10}$$

become

$$\begin{aligned} \sigma_{yy}(x, y) = -\frac{2\mu}{\pi}\int_0^\infty \xi\bigg[&\frac{2-M^2}{\sqrt{1-M^2k^2}}A(\xi)\,e^{-\xi y\sqrt{1-M^2k^2}} \\ &+ 2\sqrt{1-M^2}\,B(\xi)\,e^{-\xi y\sqrt{1-M^2}}\bigg]\cos(\xi x)\,d\xi, \end{aligned} \tag{2.11}$$

$$\sigma_{xy}(x, y) = -\frac{2\mu}{\pi}\int_0^\infty \xi\left[2A(\xi)\,e^{-\xi y\sqrt{1-M^2k^2}} + (2-M^2)B(\xi)\,e^{-\xi y\sqrt{1-M^2}}\right]\sin(\xi x)\,d\xi,$$

with

$$u(x, y) = \frac{2}{\pi}\int_0^\infty \left[\frac{A(\xi)}{\sqrt{1-M^2k^2}}\,e^{-\xi y\sqrt{1-M^2k^2}} + \sqrt{1-M^2}\,B(\xi)\,e^{-\xi y\sqrt{1-M^2}}\right]\sin(\xi x)\,d\xi,$$

and

$$v(x, y) = \frac{2}{\pi}\int_0^\infty \left[A(\xi)\,e^{-\xi y\sqrt{1-M^2k^2}} + B(\xi)\,e^{-\xi y\sqrt{1-M^2}}\right]\cos(\xi x)\,d\xi. \tag{2.12}$$

On account of symmetry with respect to the y-axis the boundary conditions (2.5) can be rewritten as

$$v(x, 0) = 0, \quad x \in I_2, I_4, \tag{2.13}$$

$$\sigma_{xy}(x, 0) = 0, \quad 0 < x < \infty, \tag{2.14}$$

$$\sigma_{yy}(x,0) = -p, \quad x \in I_1, I_3, \tag{2.15}$$

where $I_1 = (0, a)$, $I_2 = (a, b)$, $I_3 = (b, c)$, $I_4 = (c, \infty)$.

Using the condition (2.14) in (2.11.2) it is found that $A(\xi)$, $B(\xi)$ are related by

$$B(\xi) = -\frac{2}{2 - M^2} A(\xi). \tag{2.16}$$

With the help of the boundary condition (2.13), Eqn. (2.12.2) reduces to

$$\int_0^\infty A(\xi)\cos(\xi x)\,\mathrm{d}\xi = 0, \quad x \in I_2, I_4. \tag{2.17}$$

Substitution of (2.11.1) in (2.15) yields, with the aid of (2.16)

$$\int_0^\infty \xi A(\xi)\cos(\xi x)\,\mathrm{d}\xi = \frac{P\pi}{2\mu}, \quad x \in I_1, I_3, \tag{2.18}$$

where

$$P = \frac{p}{K}, \quad K = \frac{(2 - M^2)^2 - 4\sqrt{(1 - M^2k^2)(1 - M^2)}}{(2 - M^2)\sqrt{1 - M^2k^2}}.$$

3. Method of solution

In order to solve the set of four integral equations given in (2.17) and (2.18) let us take

$$A(\xi) = \frac{1}{\xi}\int_0^a h(s)\sin(\xi s)\,\mathrm{d}s + \frac{1}{\xi}\int_b^c g(t^2)\sin(\xi t)\,\mathrm{d}t, \tag{3.1}$$

where $h(s)$ and $g(t^2)$ are unknown functions to be determined from the boundary conditions.

Inserting the value of $A(\xi)$ from (3.1) in (2.17) and using the following result [17]

$$\int_0^\infty \frac{\sin(\xi x)\cos(\xi y)}{\xi}\,\mathrm{d}\xi = \begin{cases} \frac{1}{2}\pi, & x > y > 0 \\ \frac{1}{4}\pi, & x = y > 0, \\ 0, & y > x > 0 \end{cases}$$

it is found that this choice of $A(\xi)$ leads to the equation

$$\int_b^c g(t^2)\,\mathrm{d}t = 0. \tag{3.2}$$

Further substitution of $A(\xi)$ from (3.1) in (2.18.1) yields

$$\frac{d}{dx}\int_0^a h(s)\log\left|\frac{s+x}{s-x}\right|ds + \frac{d}{dx}\int_b^c g(t^2)\log\left|\frac{t+x}{t-x}\right|dt = \frac{\pi P}{\mu}, \quad x \in I_1.$$

Rewriting this equation as

$$\int_0^a h(s)\log\left|\frac{s+x}{s-x}\right|ds = \pi F(x), \quad x \in I_1,$$

where

$$F(x) = \int_0^x \left[\frac{P}{\mu} - \frac{2}{\pi}\int_c^d \frac{tg(t^2)}{t^2 - x'^2}\,dt\right]dx',$$

and using Cook's result [16] it is found that

$$h(s) = \frac{P}{\mu}\frac{s}{\sqrt{a^2 - s^2}} - \frac{2}{\pi}\frac{s}{\sqrt{a^2 - s^2}}\int_b^c \frac{\sqrt{t^2 - a^2}g(t^2)}{t^2 - s^2}\,dt, \tag{3.3}$$

where the result

$$\int_0^a \sqrt{a^2 - x^2}\,\frac{dx}{(s^2 - x^2)(t^2 - x^2)} = \tfrac{1}{2}\pi\frac{\sqrt{t^2 - a^2}}{t}\,\frac{1}{t^2 - s^2}$$

has been used.

Substituting the value of $h(s)$ from (3.3) in (3.1) and using the resulting value of $A(\xi)$ in (2.18.2) and using the result

$$\int_0^a \frac{1}{\sqrt{a^2 - s^2}}\,\frac{s^2\,ds}{(s^2 - x^2)(t^2 - s^2)} = \tfrac{1}{2}\pi\left[\frac{t}{\sqrt{t^2 - a^2}} - \frac{x}{\sqrt{x^2 - a^2}}\right]\frac{1}{t^2 - x^2}, \quad \text{for } x \in I_3,$$

it can be shown that $g(t^2)$ is the solution of the singular integral equation

$$\int_b^c \frac{\sqrt{t^2 - a^2}}{t^2 - x^2}\,g(t^2)\,dt = \frac{\pi P}{2\mu}, \quad x \in I_3.$$

Using finite Hilbert transform technique [15] the solution of this integral equation is obtained as

$$g(t^2) = \frac{P}{\mu}\sqrt{\frac{t^2(t^2 - b^2)}{(t^2 - a^2)(c^2 - t^2)}} + \frac{tC_1}{\sqrt{(t^2 - a^2)(t^2 - b^2)(c^2 - t^2)}}, \tag{3.4}$$

the constant C_1 is to be determined using the condition given by (3.2).

Next substituting the value of $g(t^2)$ from (3.4) in (3.3) and finally using the following results

$$\int_b^c \sqrt{\frac{t^2-b^2}{c^2-t^2}}\,\frac{t\,dt}{(t^2-s^2)} = \tfrac{1}{2}\pi\left[1-\sqrt{\frac{b^2-s^2}{c^2-s^2}}\right]$$

$$\int_b^c \frac{t\,dt}{(t^2-s^2)\sqrt{(t^2-b^2)(c^2-t^2)}} = \frac{\pi}{2\sqrt{(c^2-s^2)(b^2-s^2)}}, \quad \text{for } s\in I_1,$$

$h(s)$ is derived in the form

$$h(s) = \frac{P}{\mu}\sqrt{\frac{s^2(b^2-s^2)}{(a^2-s^2)(c^2-s^2)}} - \frac{sC_1}{\sqrt{(a^2-s^2)(b^2-s^2)(c^2-s^2)}}. \tag{3.5}$$

Now insertion of (3.4) in condition (3.2) yields

$$C_1 = -\frac{P}{\mu}\left[(c^2-a^2)\frac{E(\frac{1}{2}\pi, l)}{F(\frac{1}{2}\pi, l)} - (b^2-a^2)\right], \tag{3.6}$$

where $F(\phi, l)$ and $E(\phi, l)$ are elliptic integrals of first kind and second kind respectively and $l = \sqrt{(c^2-b^2)/(c^2-a^2)}$.

The relevant displacement and stress components in the plane of the crack can now be shown to be given by

$$\begin{aligned} v(x,0) &= \int_x^a h(s)\,ds, \quad 0 \leqslant x \leqslant a,\\ &= \int_x^c g(t^2)\,dt, \quad b \leqslant x \leqslant c \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} [\sigma_{yy}(x,0)]_{a<x<b} &= \frac{2\mu K}{\pi}\left[\int_0^a \frac{sh(s)}{x^2-s^2}\,ds - \int_b^c \frac{tg(t^2)}{t^2-x^2}\,dt\right],\\ [\sigma_{yy}(x,0)]_{x>c} &= \frac{2\mu K}{\pi}\left[\int_0^a \frac{sh(s)}{x^2-s^2}\,ds + \int_b^c \frac{tg(t^2)}{x^2-t^2}\,dt\right]. \end{aligned} \tag{3.8}$$

Insertion of the values of $h(s)$ and $g(t^2)$ as given by (3.5) and (3.4) in the expressions (3.8) yields after some algebraic manipulation,

$$\begin{aligned} [\sigma_{yy}(x,0)]_{a<x<b} &= \frac{2\mu K}{\pi}[F_1(x) - F_2(x) + F_3(x) - F_5(x) - F_6(x)],\\ [\sigma_{yy}(x,0)]_{x>c} &= \frac{2\mu K}{\pi}[F_1(x) - F_2(x) + F_4(x) - F_5(x) + F_6(x)], \end{aligned} \tag{3.9}$$

where

$$F_1(x) = \left[\frac{P}{\mu}(b^2 - a^2) - C_1\right]\left[\sqrt{\frac{x^2}{x^2 - a^2}} - 1\right]\frac{\pi}{2\sqrt{(c^2 - a^2)(b^2 - a^2)}},$$

$$F_2(x) = \int_0^a \left[\frac{P}{\mu}(c^2 - b^2) - C_1 \frac{2u^2 - b^2 - c^2}{b^2 - u^2}\right]\frac{g_1(u, x)}{c^2 - u^2}\,\mathrm{d}u,$$

$$F_{3,4}(x) = \left\{\frac{P}{\mu}\left[\sqrt{\frac{b^2 - x^2}{c^2 - x^2}} - 1\right] \mp \frac{C_1}{\sqrt{(c^2 - x^2)(b^2 - x^2)}}\right\}\frac{\pi c}{2\sqrt{c^2 - a^2}},$$

$$F_5(x) = \frac{P}{\mu}a^2 \int_b^c \left[\tan^{-1}\sqrt{\frac{v^2 - b^2}{c^2 - v^2}} - \sqrt{\frac{b^2 - x^2}{c^2 - x^2}}\tan^{-1}\sqrt{\frac{(c^2 - x^2)(v^2 - b^2)}{(b^2 - x^2)(c^2 - v^2)}}\right]\frac{\mathrm{d}v}{\sqrt{(v^2 - a^2)^3}},$$

$$F_6(x) = \frac{a^2 C_1}{\sqrt{(c^2 - x^2)(b^2 - x^2)}}\int_b^c \frac{\tan^{-1}\sqrt{\dfrac{(u^2 - b^2)(x^2 - c^2)}{(c^2 - u^2)(x^2 - b^2)}}}{\sqrt{(u^2 - a^2)^3}}\,\mathrm{d}u,$$

$$g_1(u, x) = \frac{u}{\sqrt{(b^2 - u^2)(c^2 - u^2)}}\left[\sin^{-1}\left(\frac{u}{a}\right) - \frac{x}{\sqrt{x^2 - a^2}}\tan^{-1}\sqrt{\frac{(x^2 - a^2)u^2}{(a^2 - u^2)x^2}}\right]. \tag{3.10}$$

The dynamic stress intensity factors are given by

$$\begin{aligned} N_a &= \operatorname*{Lt}_{x \to a+} \sqrt{2(x - a)}[\sigma_{yy}(x, 0)]_{a<x<b}, \\ N_b &= \operatorname*{Lt}_{x \to b-} \sqrt{2(b - x)}[\sigma_{yy}(x, 0)]_{a<x<b}, \\ N_c &= \operatorname*{Lt}_{x \to c+} \sqrt{2(x - c)}[\sigma_{yy}(x, 0)]_{x>c}. \end{aligned} \tag{3.11}$$

Employing (3.9) in (3.11) it can be shown that

$$N_a = p\sqrt{a}\sqrt{\frac{c^2 - a^2}{b^2 - a^2}}\frac{E(\frac{1}{2}\pi, l)}{F(\frac{1}{2}\pi, l)},$$

$$N_b = \frac{p\sqrt{b}}{\sqrt{(c^2 - b^2)(b^2 - a^2)}}\left[(c^2 - a^2)\frac{E(\frac{1}{2}\pi, l)}{F(\frac{1}{2}\pi, l)} - (b^2 - a^2)\right],$$

$$N_c = p\sqrt{c}\sqrt{\frac{c^2 - a^2}{c^2 - b^2}}\left[1 - \frac{E(\frac{1}{2}\pi, l)}{F(\frac{1}{2}\pi, l)}\right].$$

Now using the values of $h(s)$ and $g(t^2)$ from (3.5) and (3.4) in the expressions given by (3.7) displacements on the cracks are obtained as

$$[v(x,0)]_{0\leqslant x\leqslant a} = \frac{P}{\mu}\sqrt{c^2-a^2}F(\beta,l)\left[\frac{E(\frac{1}{2}\pi,l)}{F(\frac{1}{2}\pi,l)} - \frac{E(\beta,l)}{F(\beta,l)}\right] + \frac{P}{\mu}\frac{\sqrt{(c^2-x^2)(a^2-x^2)}}{\sqrt{b^2-x^2}},$$

$$[v(x,0)]_{b\leqslant x\leqslant c} = \frac{P}{\mu}\sqrt{c^2-a^2}F(\lambda,l)\left[\frac{E(\lambda,l)}{F(\lambda,l)} - \frac{E(\frac{1}{2}\pi,l)}{F(\frac{1}{2}\pi,l)}\right],$$

where

$$\sin\lambda = \sqrt{\frac{c^2-x^2}{c^2-b^2}} \quad \text{and} \quad \sin\beta = \sqrt{\frac{a^2-x^2}{b^2-x^2}}.$$

It is interesting to note that the crack opening displacements depend on the crack velocity V but in the plane of the cracks the stresses and stress intensity factors are independent of the velocity of the moving cracks in an infinite elastic medium.

4. Statement of Problem II and its formulation

In this case, consider an infinite homogeneous isotropic material with three coplanar Griffith cracks, located at $Y=0$, $-a\leqslant X\leqslant a$, $b\leqslant |X|\leqslant c$ and subjected to uniform internal pressure q. In the absence of body force equations of equilibrium in terms of displacement are

$$(\lambda+2\mu)[u_{,XX}+v_{,XY}] + \mu[u_{,YY}-v_{,XY}] = 0$$

and

$$(\lambda+2\mu)[u_{,XY}+v_{,YY}] + \mu[v_{,XX}-u_{,XY}] = 0. \tag{4.1}$$

Since the problem exhibits a state of symmetry abut $Y=0$, attention can be made to a single half-space occupying the region $Y\geqslant 0$.

Equations (4.1) are to be solved subject to the boundary conditions

$$v(X,0) = 0, \qquad a\leqslant |X|\leqslant b, \quad |X|\geqslant c, \tag{4.2}$$

$$\sigma_{XY}(X,0) = 0, \qquad -\infty < X < \infty, \tag{4.3}$$

$$\sigma_{YY}(X,0) = -q, \quad |X|\leqslant a, \quad b\leqslant |X|\leqslant c. \tag{4.4}$$

In view of the boundary conditions, appropriate integral solutions of (4.1) are

$$u(X,Y) = \frac{2}{\pi}\int_0^\infty \left[C(\xi) + D(\xi)\left\{Y - \frac{1}{\xi}\frac{\lambda+3\mu}{\lambda+\mu}\right\}\right] e^{-\xi Y}\sin(\xi X)\,d\xi$$

and

$$v(X, Y) = \frac{2}{\pi}\int_0^\infty [C(\xi) + YD(\xi)]\,e^{-\xi Y}\cos(\xi X)\,d\xi. \tag{4.5}$$

Therefore,

$$\sigma_{YY}(X, Y) = -\frac{4\mu}{\pi}\int_0^\infty \left[\xi C(\xi) + \left\{Y\xi - \frac{\mu}{\lambda + \mu}\right\}D(\xi)\right]e^{-\xi Y}\cos(\xi X)\,d\xi,$$

$$\sigma_{XY}(X, Y) = -\frac{4\mu}{\pi}\int_0^\infty \left[\xi C(\xi) + \left\{Y\xi - \frac{\lambda + 2\mu}{\lambda + \mu}\right\}D(\xi)\right]e^{-\xi Y}\sin(\xi X)\,d\xi. \tag{4.6}$$

It may be noted that the displacement and stress components given by (4.5) and (4.6) can not be derived from the corresponding expressions of the dynamic problem given in (2.12) and (2.11) on setting $M = 0$.

The functions $C(\xi)$ and $D(\xi)$ are to be determined from the boundary conditions (4.2)–(4.4), which yield

$$C(\xi) = \frac{1}{\xi}\frac{\lambda + 2\mu}{\lambda + \mu}D(\xi) \tag{4.7}$$

and the following set of four integral equations

$$\int_0^\infty C(\xi)\cos(\xi X)\,d\xi = 0, \quad X \in I_2, I_4, \tag{4.8}$$

$$\int_0^\infty \xi C(\xi)\cos(\xi X)\,d\xi = \frac{Q\pi}{2\mu}, \quad X \in I_1, I_3, \tag{4.9}$$

where $Q = ((\lambda + 2\mu)/2(\lambda + \mu))q$ and $I_j\ (j = 1, 2, 3, 4)$ are the intervals defined earlier in Problem I.

5. Method of solution and quantities of physical interest

Integral equations given by (4.8) and (4.9) are found to be the same as given by (2.17) and (2.18) with the exception that P is replaced by Q. Therefore, the same technique as that used in Problem I can be employed to obtain

$$[v(X,0)]_{0\le X\le a} = \frac{Q}{\mu}\sqrt{c^2 - a^2}F(\beta', l)\left[\frac{E(\frac{1}{2}\pi, l)}{F(\frac{1}{2}\pi, l)} - \frac{E(\beta', l)}{F(\beta', l)}\right] + \frac{Q}{\mu}\frac{\sqrt{(c^2 - X^2)(a^2 - X^2)}}{\sqrt{b^2 - X^2}},$$

$$[v(X,0)]_{b\le X\le c} = \frac{Q}{\mu}\sqrt{c^2 - a^2}F(\lambda', l)\left[\frac{E(\lambda', l)}{F(\lambda', l)} - \frac{E(\frac{1}{2}\pi, l)}{F(\frac{1}{2}\pi, l)}\right], \tag{5.1}$$

where

$$\sin\lambda' = \sqrt{\frac{c^2 - X^2}{c^2 - b^2}} \quad \text{and} \quad \sin\beta' = \sqrt{\frac{a^2 - X^2}{b^2 - X^2}}.$$

Stresses in the regions $a < X < b$, $X > c$ are found to be the same as that given in (3.9), the only change being that P is to be replaced by Q.

6. Numerical results and discussions

Numerical results for the stress intensity factors and crack opening displacement, defined as $\Delta v(x, 0) = v(x, 0^+) - v(x, 0^-)$, for different values of the parameters and $\lambda = \mu$ are presented in this section. Numerical calculations have been carried out for both the dynamic and static problems. As the crack velocity is less than Rayleigh wave velocity, it is reasonable to take the value of M less than 0.9194.

Problem I: Variations of crack opening displacement for different values of crack speed, crack lengths and the separating distance between the cracks have been plotted in Figs. 2–4. It is interesting to note from Fig. 2 that crack opening displacement on both the cracks decreases with the increase in the value of M at the onset and takes its minimum value at $M = 0.7415$, after which it increases with the increase in the value of M. It has also been depicted in Figs. 3–4 that on each of the cracks, crack opening displacement decreases as the crack length decreases.

It has been mentioned earlier that the stress intensity factors at the crack tips are independent of crack speed and are found to depend on the crack lengths and the separating distance between the cracks. Variation of stress intensity factors with a/b for different values of c/b, and that with b/a for different values c/a are plotted in Fig. 5 and Fig. 6 respectively.

It has been found from these graphs that when the separating distance between the inner crack and outer pair of cracks decreases the variations of stress intensity factors at the tips $x = a$ and $x = b$ become more prominent than at the edge $x = c$. Figure 7 shows that the stress intensity factors at the edges of the inner crack and outer pair of cracks increases as the length of the outer pair of cracks increases, keeping the separating distance between the inner crack and outer pair of cracks fixed.

Problem II: Figure 8 shows the variations of crack opening displacement for different values of the parameters a/b, c/b. They exhibit that crack opening displacement on a crack of fixed length increases with the increase in the length of the other crack as expected from the physical standpoint.

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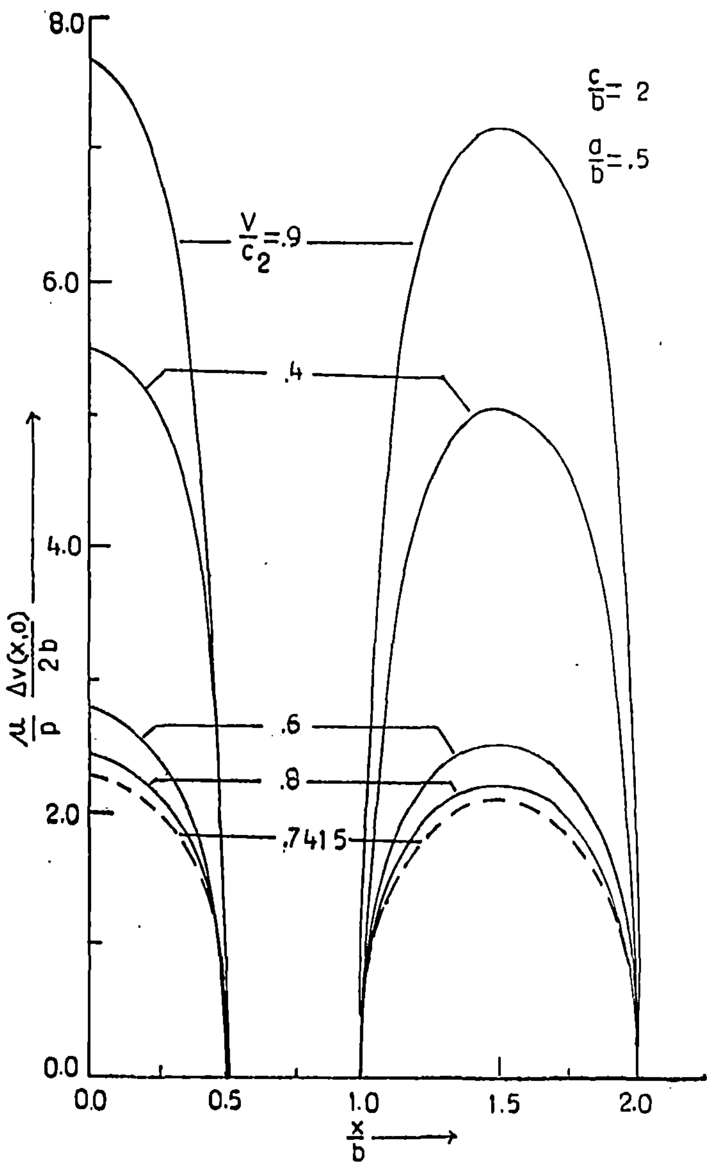

Fig. 2. Variation of crack opening displacement with x/b on both the cracks for the problem I.

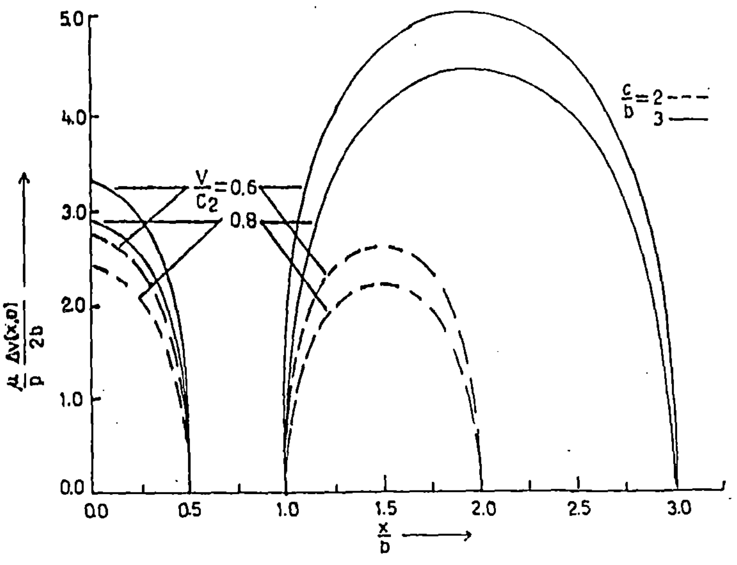

Fig. 3. Variation of crack opening displacement with x/b on both the cracks for the problem I.

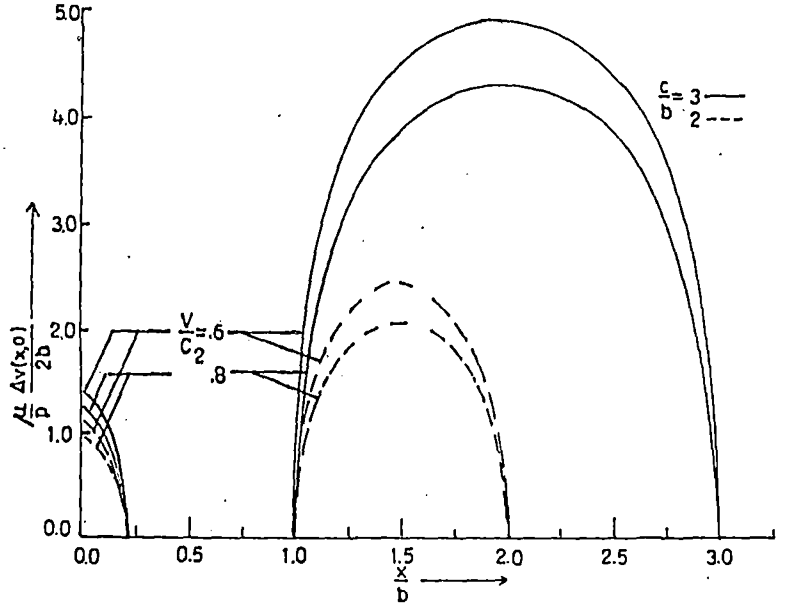

Fig. 4. Variation of crack opening displacement with x/b on both the cracks for the problem I.

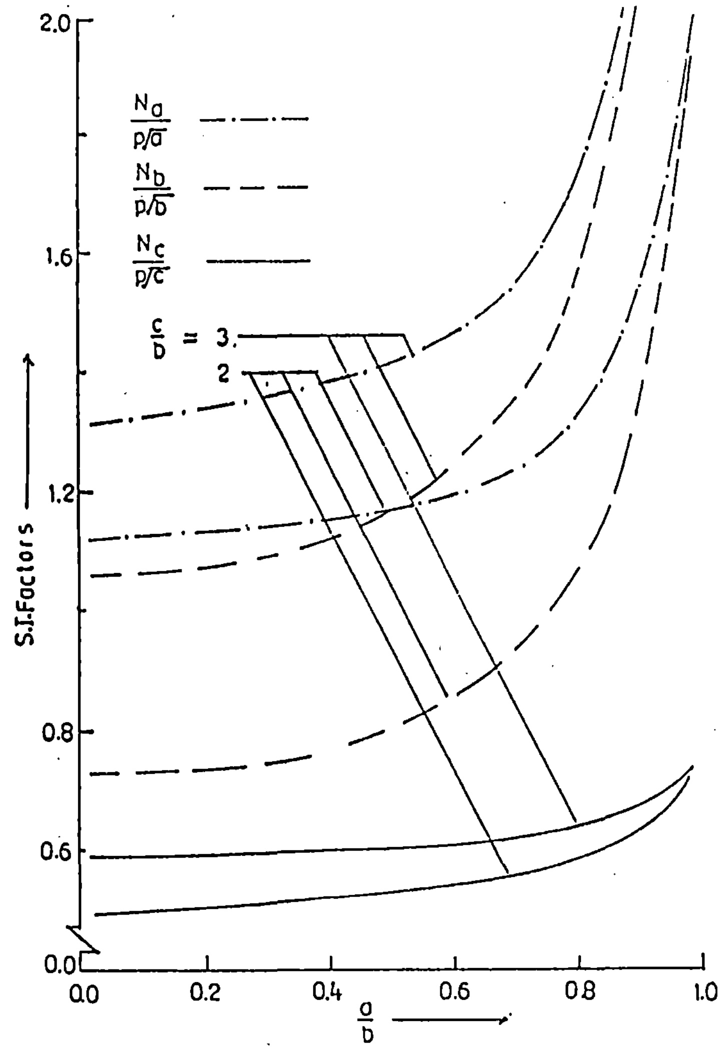

Fig. 5. Stress intensity factors Vs. a/b.

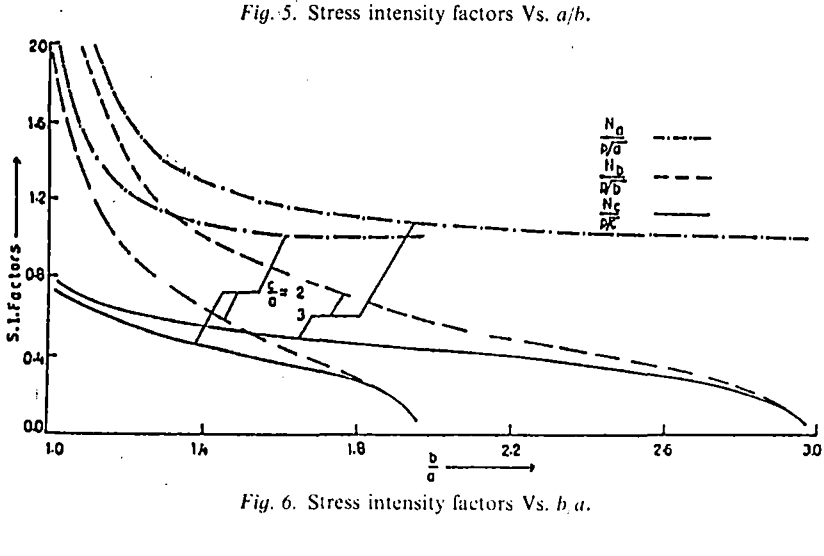

Fig. 6. Stress intensity factors Vs. b/a.

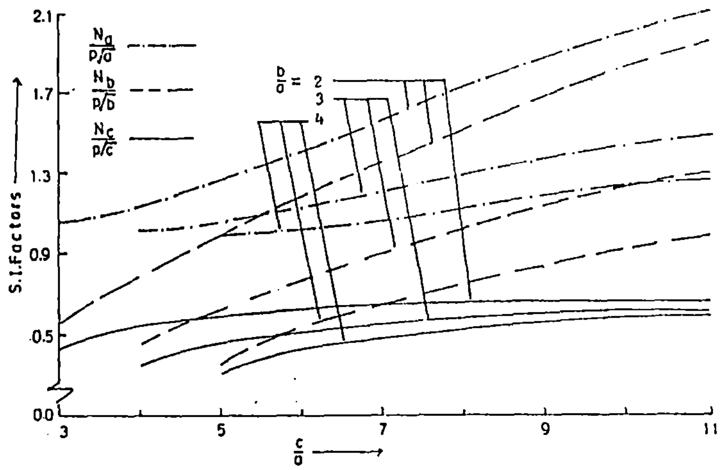

Fig. 7. Stress intensity factors Vs. c/a.

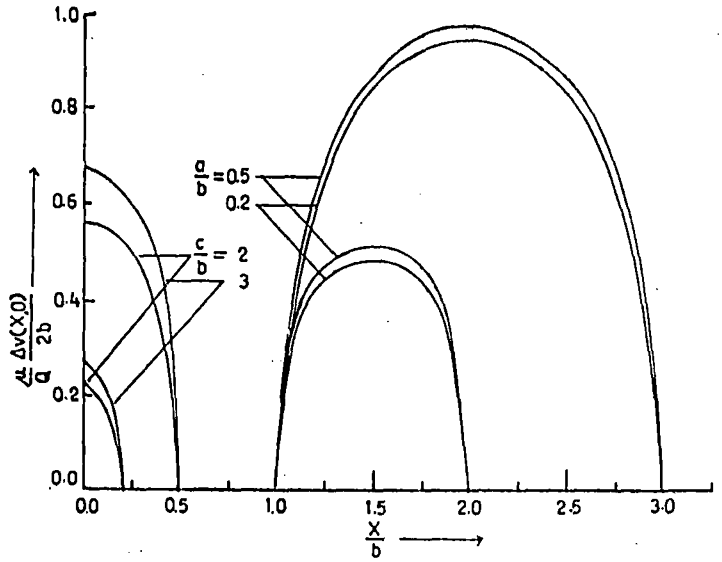

Fig. 8. Variation of crack opening displacement with X/b on both the cracks for the problem II.

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Three coplanar moving Griffith cracks in an infinite elastic strip

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THE DYNAMIC anti-plane problem of determining stress and displacement due to three coplanar Griffith cracks moving steadily at a subsonic speed in an infinite elastic strip has been considered. Employing Fourier integral transform, the problem when the lateral boundaries are subjected to shearing stress, has been reduced to solving a set of four integral equations. These integral equations have been solved using finite Hilbert transform technique and Cook's result [9] to obtain the exact form of crack opening displacement and stress intensity factors. Numerical results for stress intensity factors have been presented in the form of graphs.

1. Introduction

IN FRACTURE MECHANICS, the problem of diffraction of elastic waves by cracks of finite dimension in a strip of elastic material has been investigated by several investigators. Sih and Chen [1] investigated the problem of propagation of a crack of finite length in a strip under plane extension. Closed-form solutions for a finite length crack moving in a strip under anti-plane shear stress was obtained by SINGH *et al.* [2]. Using finite Hilbert transform technique developed by SRIVASTAVA and LOWENGRUB [3], LOWENGRUB and SRIVASTAVA [4] solved the statical problem of distribution of stress and displacement in an infinitely long elastic strip containing two coplanar Griffith cracks. Several dynamic problems of determining stress and displacement due to two coplanar moving Griffith cracks have been solved by DAS and GHOSH [5–7].

As regards the crack problem, research has been restricted mainly to the case of a single crack or a pair of cracks because of severe mathematical complexity encountered in solving the problems of three or more cracks. Recently, DHAWAN and DHALIWAL [8] solved the statical problem of determining the stress distribution in an infinite transversely isotropic medium containing three coplanar Griffith cracks.

To the best knowledge of the author, the problem of stress distribution around three coplanar moving Griffith cracks in an infinite elastic strip has not been investigated so far. In this paper, the problem of propagation of three coplanar Griffith cracks in a fixed direction with constant velocity V in an infinitely long elastic strip of finite width has been considered. Employing Fourier integral transform, the problem when the lateral boundaries are subjected to shearing stress, has been reduced to solving a set of four integral equations using finite Hilbert transform technique [3] and COOK'S result [9] to derive the exact form of stress intensity factors and the crack opening displacement. Numerical results for the stress intensity factors are presented graphically to show their variations with crack speed, crack lengths and the separation distance between the cracks.

2. Statement of the problem

Consider an infinitely long elastic strip occupying the region $-h \le Y \le h$, weakened by three coplanar Griffith cracks moving steadily at a constant velocity V in the X-direction referred to a fixed coordinate system (X, Y, Z) as shown in the Fig. 1.

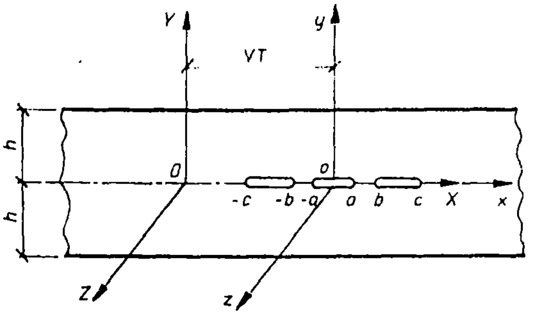

FIG. 1. Geometry and coordinate system.

In dynamic problem of anti-plane shear, the non-vanishing component of displacement W in the Z-direction satisfies the equation of motion

$$W_{,XX} + W_{,YY} = \frac{1}{C_2^2} W_{,TT}, \tag{2.1}$$

where $C_2 = (\mu/\rho)^{1/2}$ is the shear wave velocity, ρ is the material density and $W_{,X}$ represents partial derivatives of W with respect to X.

For cracks moving at constant velocity V in the X-direction it is convenient to introduce the Galilean transformation

$$x = X - VT, \quad y = Y, \quad z = Z, \quad t = T, \tag{2.2}$$

where (x, y, z) represents the moving coordinate system as shown in the Fig. 1.

Let the positions of the coplanar Griffith cracks referred to the coordinates (x, y, z) be $-a < x < a$, $-c < x < -b$ and $b < x < c$ on $y = 0$, and let the uniform shearing stress p be applied to the lateral boundaries $y = \pm h$ of the strip. The equivalent problem involves the application of shear stress $-p$ to the crack faces at $y = 0$. Accordingly, the boundary conditions of the proposed problem are

$$\sigma_{yz}(x, 0) = -p, \quad |x| < a, \quad b < |x| < c, \tag{2.3}$$

$$\sigma_{yz}(x, \pm h) = 0, \quad -\infty < x < \infty, \tag{2.4}$$

$$W(x, 0) = 0, \quad a < |x| < b, \quad |x| > c. \tag{2.5}$$

In the moving coordinate system, the equation of motion becomes independent of time and takes the form

$$s^2 W_{,xx} + W_{,yy} = 0, \tag{2.6}$$

with

$$s = \sqrt{1 - V^2/C_2^2}. \tag{2.7}$$

Due to the symmetry about x, z-plane we need to consider the region $0 < y < h$ only. Introducing the Fourier transforms

$$\overline{W}_C(\xi, y) = \int_0^\infty W(x, y)\cos(\xi x)\,dx, \tag{2.8}$$
$$W(x, y) = \frac{2}{\pi}\int_0^\infty \overline{W}_C(\xi, y)\cos(\xi x)\,d\xi,$$

in Eq. (2.6), the solution of Eq. (2.6) is obtained as

$$W(x, y) = \frac{2}{\pi}\int_0^\infty [C_1(\xi)e^{-\xi ys} + C_3(\xi)e^{\xi ys}]\cos(\xi x)\,d\xi, \tag{2.9}$$

with

$$\sigma_{yz}(x, y) = -\frac{2\mu s}{\pi}\int_0^\infty \xi[C_1(\xi)e^{-\xi ys} - C_3(\xi)e^{\xi ys}]\cos(\xi x)\,d\xi. \tag{2.10}$$

Using the expression for $\sigma_{yz}(x, y)$ given by Eq. (2.10) in Eq. (2.4), it has been found that

$$C_1(\xi) = \frac{C(\xi)}{1 + e^{-2\xi hs}},$$
$$C_3(\xi) = \frac{C(\xi)e^{-2\xi hs}}{1 + e^{-2\xi hs}},$$

where the unknown function $C(\xi)$ is to be determined. From conditions (2.3) and (2.5) it is found that $C(\xi)$ satisfies the following quadruple integral equations:

$$\int_0^\infty \xi C(\xi hs)\,\mathrm{th}(\xi hs)\cos(\xi x)\,d\xi = \frac{\pi p}{2\mu s}, \qquad x \in I_1, I_3, \tag{2.11}$$

and

$$\int_0^\infty C(\xi)\cos(\xi x)d\xi = 0, \qquad x \in I_2, I_4, \tag{2.12}$$

where

$$I_1 = (0, a), \quad I_2 = (a, b), \quad I_3 = (b, c), \quad I_4 = (c, \infty).$$

3. Method of solution

In order to solve the quadruple integral equations (2.11) and (2.12), let us take

$$C(\xi) = \frac{1}{\xi}\int_0^a h(u)\sin(\xi u)\,du + \frac{1}{\xi}\int_b^c g(v^2)\,\mathrm{ch}(cv)\sin(\xi v)\,dv, \tag{3.1}$$

where $h(u)$ and $g(v^2)$ are the unknown functions to be determined from the boundary conditions of the problem considered. Substituting the value of $C(\xi)$ given by Eq. (3.1) into Eq. (2.12) and using the well-known result

$$\int_0^\infty \frac{\sin(x\xi)\cos(y\xi)}{\xi}\,d\xi = \begin{cases} \dfrac{\pi}{2}, & x > y > 0,\\ \dfrac{\pi}{4}, & x = y > 0,\\ 0, & y > x > 0 \end{cases}$$

it is found that this choice of $C(\xi)$ leads to the condition

$$\int_b^c g(v^2)\,\mathrm{ch}(ev)\,dv = 0. \tag{3.2}$$

Rewritting Eq. $(2.11)_1$ in the form

$$\frac{d}{dx}\int_0^\infty C(\xi)\,\mathrm{th}(\xi hs)\sin(\xi x)\,d\xi = \frac{\pi p}{2\mu s}, \quad x \in I_1 \tag{3.3}$$

and inserting the value of $C(\xi)$ from Eq. (3.1) in (3.3) it is found that $h(u)$ is the solution of the following singular integral equation:

$$\int_0^a h(u)\log\left|\frac{\mathrm{sh}(ex)+\mathrm{sh}(eu)}{\mathrm{sh}(ex)-\mathrm{sh}(eu)}\right| du = \pi f(x), \quad x \in I_1 \tag{3.4}$$

with

$$f(x) = \int_0^x \left[\frac{p}{\mu s} - \frac{1}{\pi}\int_b^c \frac{eg(v^2)\,\mathrm{ch}(ex')\,\mathrm{sh}(2ev)}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ex')}\,dv\right] dx',$$

where the following result [10] has been used:

$$\int_0^\infty \mathrm{th}(\xi hs)\frac{\sin(\xi x)\sin(\xi u)}{\xi}\,d\xi = \frac{1}{2}\log\left|\frac{\mathrm{sh}(ex)+\mathrm{sh}(eu)}{\mathrm{sh}(ex)-\mathrm{sh}(eu)}\right|, \quad e = \frac{\pi}{2hs}. \tag{3.5}$$

Now using the Cook's result [9], the solution of Eq. (3.4) has been obtained with the aid of the formula

$$\int_0^a \frac{\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(ex)}\,e\,\mathrm{ch}(ex)\,dx}{[\mathrm{sh}^2(ex)-\mathrm{sh}^2(eu)][\mathrm{sh}^2(ev)-\mathrm{sh}^2(ex)]} = -\frac{\pi}{2\,\mathrm{sh}(ev)}\frac{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eu)}$$

for $u \in I_1$ and $v \in I_3$,

$$h(u) = \frac{-e\,\mathrm{sh}(2eu)}{\pi\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}\left[\frac{p}{\mu s}\int_0^a \sqrt{\frac{\mathrm{sh}^2(ea)-\mathrm{sh}^2(ex)}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eu)}}\,dx + \int_b^c \frac{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eu)}g(v^2)\,\mathrm{ch}(ev)\,dv\right]. \tag{3.6}$$

Substitute now the resulting value of $C(\xi)$, obtained by inserting Eqs. (3.6) into Eq. (3.1), in condition $(2.11)_2$, and make use of the following results:

$$\int_0^a \frac{e\,\mathrm{sh}^2(eu)\,\mathrm{ch}(eu)du}{[\mathrm{sh}^2(eu)-\mathrm{sh}^2(ex)][\mathrm{sh}^2(ev)-\mathrm{sh}^2(eu)]\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}$$
$$= \frac{\pi}{2[\mathrm{sh}^2(ev)-\mathrm{sh}^2(ex)]}\left[\frac{\mathrm{sh}(ev)}{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}} - \frac{\mathrm{sh}(ex)}{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}\right],$$

$$\int_0^a \frac{e\,\mathrm{sh}^2(eu)\,\mathrm{ch}(eu)du}{[\mathrm{sh}^2(eu)-\mathrm{sh}^2(ex)][\mathrm{sh}^2(ey')-\mathrm{sh}^2(eu)]\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}$$
$$= \frac{\pi}{2[\mathrm{sh}^2(ex)-\mathrm{sh}^2(ey')]}\frac{\mathrm{sh}(ex)}{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}, \quad \text{for} \quad x, v \in I_3 \quad \text{and} \quad y' \in I_1.$$

It can be shown that $g(v^2)$ is the solution of the following singular integral equation

$$\int_b^c \frac{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ex)} e g(v^2)\,\mathrm{ch}(ev)\,dv = \frac{\pi p}{\mu s}\Bigg[\frac{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}{\mathrm{sh}(2ex)} + \frac{1}{\pi}\int_0^a \frac{\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(ey')}}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ey')}dy'\Bigg], \quad \text{for} \quad x \in I_3. \tag{3.7}$$

Using finite Hilbert transform technique [3] and the formula

$$\int_b^c \sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eb)}} \frac{\mathrm{sh}(2ex)dx}{[\mathrm{sh}^2(ex)-\mathrm{sh}^2(ey')][\mathrm{sh}^2(ex)-\mathrm{sh}^2(ev)]}$$
$$= -\frac{\pi}{e[\mathrm{sh}^2(ev)-\mathrm{sh}^2(ey')]}\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ey')}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ey')}},$$

the solution of Eq. (3.7) is found to be

$$g(v^2) = -\frac{2cp}{\mu\pi s}\frac{\mathrm{sh}(ev)\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}}{\sqrt{[\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)][\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)]}}\Bigg[\int_b^c \sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eb)}} \times \frac{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ev)}dx - \int_0^a \sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ey')}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ey')}}\frac{\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(ey')}}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ey')}dy'\Bigg]$$
$$+\frac{C_1\,\mathrm{sh}(ev)}{\sqrt{[\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)][\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)][\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)]}}. \tag{3.8}$$

Next, substitution of $g(v^2)$ from Eq. (3.8) in Eq. (3.6) and finally application of the formula

$$\int_b^c \sqrt{\frac{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}} \frac{\mathrm{sh}(2ev)dv}{[\mathrm{sh}^2(ev)-\mathrm{sh}^2(eu)][\mathrm{sh}^2(ex')-\mathrm{sh}^2(ev)]}$$
$$= \frac{\pi}{e[\mathrm{sh}^2(eu)-\mathrm{sh}^2(ex')]}\left[\sqrt{\frac{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}} - \sqrt{\frac{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ex')}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex')}}\right], \quad \text{for} \quad u, x' \in I_1$$

yields $h(u)$ in the form

$$(3.9)\ h(u) = -\frac{2ep}{\mu\pi s}\frac{\mathrm{ch}(eu)\,\mathrm{sh}(eu)\sqrt{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}}{\sqrt{[\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)][\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)]}}\Bigg[\int_0^a\sqrt{\frac{\mathrm{sh}^2(ea)-\mathrm{sh}^2(ey')}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ey')}}$$

$$\times\frac{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ey')}}{\mathrm{sh}^2(ey')-\mathrm{sh}^2(eu)}dy' - \int_b^c\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eb)}}\frac{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eu)}\,dx\Bigg]$$

$$-\frac{C_1\,\mathrm{sh}(eu)\,\mathrm{ch}(eu)}{\sqrt{[\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)][\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)][\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)]}}.$$

Substitution of the value of $g(v^2)$ from Eq. (3.8) in the condition (3.2) yields

$$(3.10)\quad C_1 = -\frac{2ep}{\pi\mu s}\Bigg[\int_b^c\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eb)}}\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}\Bigg\{\frac{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}$$

$$\times\Pi\Big\{\frac{\pi}{2},\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)},q\Big\}/F\Big(\frac{\pi}{2},q\Big)+1\Bigg\}\,dx$$

$$+\int_0^a\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(es)}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(es)}}\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(es)}$$

$$\times\Bigg\{1-\frac{\mathrm{sh}^2(eb)-\mathrm{sh}^2(es)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(es)}\Pi\Big\{\frac{\pi}{2},\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(es)},q\Big\}/F\Big(\frac{\pi}{2},q\Big)\Bigg\}\,ds,$$

where $F(\phi,q)$ and $\Pi(\phi,n,q)$ are elliptic integrals of the first and third kind, respectively, and $q=\sqrt{\dfrac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}$.

The relevant displacement and stress components in the plane of the crack can now be shown to be given by

$$(3.11)\qquad W(x,0) = \int_x^a h(u)\,du, \qquad 0\le x\le a,$$

$$= \int_x^c g(v^2)\,\mathrm{ch}(ev)\,dv, \qquad b\le x\le c,$$

and

$$[\sigma_{yz}(x,0)]_{a<x<b} = \frac{2\mu s}{\pi}\Bigg[\int_0^a\frac{eh(u)\,\mathrm{sh}(eu)\,du}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eu)} - \int_b^c\frac{eg(v^2)\,\mathrm{sh}(ev)\,\mathrm{ch}(ev)}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ex)}\,dv\Bigg]\mathrm{ch}(ex),$$

(3.12)

$$[\sigma_{yz}(x,0)]_{x>c} = \frac{2\mu s}{\pi}\Bigg[\int_0^a\frac{eh(u)\,\mathrm{sh}(eu)\,du}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eu)} + \int_b^c\frac{eg(v^2)\,\mathrm{sh}(ev)\,\mathrm{ch}(ev)}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ev)}\,dv\Bigg]\mathrm{ch}(ex).$$

Now, insertion of the values of $h(u)$ and $g(v^2)$, as given by Eqs. (3.9) and (3.8), in the expressions (3.12) yields (after some algebraic manipulations)

$$[\sigma_{yz}(x,0)]_{a<x<b} = \frac{2pe}{\pi}\Bigg[-\sqrt{\frac{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}\frac{\mathrm{sh}(ex)}{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}$$

$$\left\{\int_0^a F_2(u,x)\,du+\int_b^c F_2(v,x)dv\right\}-\frac{2e[\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)]}{\pi}\left\{\int_0^a F_2(u',x)\,du'\int_0^a F_4(c,u)\right.$$

$$\left.\times F_3(0,x,u)\,du+\int_b^c F_2(v,x)dv\int_0^a F_4(c,u)F_3(v,x,u)\,du\right\}+\frac{\mu s}{ep}C_1\left\{\frac{\pi}{2}\right.$$

$$\left.\times\frac{1-\mathrm{sh}(ex)/\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}{\sqrt{[\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)][\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)]}}+e\int_0^a F_4(c,u)F_5(u,x)\,du\right\}$$

$$+\frac{e[\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)]}{\pi}\left\{\int_b^c F_2(v',x)dv'\int_b^c F_4(a,v)F_6(v',x,v)dv+\int_0^a F_2(u,x)\,du\right.$$

$$\left.\times\int_b^c F_4(a,v)F_6(u,x,v)dv-\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)}\int_0^a F_1(u,x)du\int_0^a F_4(c,u')F_9(u,u')du'\right\}$$

$$-\frac{\mu s}{pe}\frac{C_1}{X_1}\left\{\frac{\pi}{2}\frac{\mathrm{sh}(ec)}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}+e\,\mathrm{sh}^2(ea)\int_b^c F_7(x,v)\,dv\right\}\Bigg]\,\mathrm{ch}(ex),$$

(3.13)
$$[\sigma_{yz}(x,0)]_{x>c} = \frac{2pe}{\pi}\Bigg[-\sqrt{\frac{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}\frac{\mathrm{sh}(ex)}{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}$$

$$\times\left\{\int_0^a F_2(u,x)\,du+\int_b^c F_2(v,x)\,dv\right\}-\frac{2e[\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)]}{\pi}\left\{\int_0^a F_2(u',x)\,du'\int_0^a F_4(c,u)\right.$$

$$\left.\times F_3(0,x,u)\,du+\int_b^c F_2(v,x)\,dv\int_0^a F_4(c,u)F_3(v,x,u)\,du\right\}+\frac{\mu s}{ep}C_1\left\{\frac{\pi}{2}\right.$$

$$\left.\times\frac{1-\mathrm{sh}(ex)/\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}{\sqrt{[\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)][\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)]}}+e\int_0^a F_4(c,u)F_5(u,x)\,du\right\}$$

$$-\frac{e[\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)]}{\pi}\left\{\int_b^c F_2(v',x)\,dv'\int_b^c F_4(a,v)F_8(v',v,x)\,dv+\int_0^a F_2(u,x)\,du\right.$$

$$\left.\times\int_b^c F_4(a,v)F_8(u,v,x)dv+\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)}\int_0^a F_1(u,x)\,du\int_0^a F_4(c,u')F_9(u,u')du'\right\}$$

$$+\frac{\mu sC_1}{peX_1}\left\{\frac{\pi}{2}\frac{\mathrm{sh}(ec)}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}+e\,\mathrm{sh}^2(ea)\int_b^c F_7(x,v)dv\right\}-\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}$$

$$\times\frac{\mathrm{sh}(ex)}{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ec)}}\left\{\int_0^a F_2(u,x)\,du+\int_b^c F_2(v,x)\,dv\right\}\Bigg]\,\mathrm{ch}(ex).$$

In the above formulae

$$F_1(u,x) = \sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}}\,\frac{\mathrm{sh}(eu)}{\mathrm{sh}^2(ex)-\mathrm{sh}^2(eu)},$$

$$F_2(v,x) = \sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}}\,\frac{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ex)},$$

$$F_3(v,x,u) = \frac{\mathrm{sh}(ex)}{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}}\tan^{-1}\left\{\frac{\mathrm{sh}(eu)}{\mathrm{sh}(ex)}\sqrt{\frac{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ea)}{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}\right\}$$
$$-\frac{\mathrm{sh}(ev)}{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}}\tan^{-1}\left\{\frac{\mathrm{sh}(eu)}{\mathrm{sh}(ev)}\sqrt{\frac{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}\right\},$$

$$F_4(\omega,u) = \frac{\mathrm{ch}(eu)\,\mathrm{sh}(eu)}{\sqrt{[\mathrm{sh}^2(e\omega)-\mathrm{sh}^2(eu)]^3[\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)]}},$$

$$F_5(u,x) = [2\,\mathrm{sh}^2(eu)-\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)]\left\{\sin^{-1}\left(\frac{\mathrm{sh}(eu)}{\mathrm{sh}(ea)}\right)-F_3(0,x,u)\right\}, \tag{3.14}$$

$$F_6(u,x,v) = \frac{\mathrm{sh}(ex)}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}}$$
$$\times\log\left|\frac{\mathrm{sh}(ex)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}+\mathrm{sh}(ev)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}}{\mathrm{sh}(ex)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}-\mathrm{sh}(ev)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}}\right|-\frac{\mathrm{sh}(eu)}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}}$$
$$\times\log\left|\frac{\mathrm{sh}(eu)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}+\mathrm{sh}(ev)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}}{\mathrm{sh}(eu)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}-\mathrm{sh}(ev)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}}\right|,$$

$$F_7(x,v) = \tan^{-1}\left(\frac{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}}\,\frac{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}}{\sqrt{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ex)}}\right)$$
$$\times\frac{\mathrm{ch}(ev)}{\sqrt{[\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)]^3}},$$

$$F_8(u,v,x) = -\frac{2\,\mathrm{sh}(ex)}{\sqrt{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ec)}}\tan^{-1}\left\{\frac{\mathrm{sh}(ev)}{\mathrm{sh}(ex)}\sqrt{\frac{\mathrm{sh}^2(ex)-\mathrm{sh}^2(ec)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}}\right\}$$
$$+\frac{\mathrm{sh}(eu)}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}}\log\left|\frac{\mathrm{sh}(eu)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}+\mathrm{sh}(ev)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}}{\mathrm{sh}(eu)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}-\mathrm{sh}(ev)\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}}\right|,$$

$$F_9(u,u') = \log\left|\frac{\mathrm{sh}(eu)\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu')}+\mathrm{sh}(eu')\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}{\mathrm{sh}(eu)\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu')}-\mathrm{sh}(eu')\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}\right|$$

and

$$X_1 = \sqrt{[\mathrm{sh}^2(eb) - \mathrm{sh}^2(ex)][\mathrm{sh}^2(ec) - \mathrm{sh}^2(ex)]}.$$

The dynamic stress intensity factors are defined by

$$\begin{aligned} N_a &= \lim_{x\to a^+} \sqrt{2(x-a)}[\sigma_{yz}(x,0)]_{a<x<b}, \\ N_b &= \lim_{x\to b^-} \sqrt{2(b-x)}[\sigma_{yz}(x,0)]_{a<x<b}, \\ N_c &= \lim_{x\to c^+} \sqrt{2(x-c)}[\sigma_{yz}(x,0)]_{x>c}. \end{aligned} \tag{3.15}$$

Substitution of the results given by Eqs. (3.13) in expressions (3.15) yields

$$\begin{aligned} N_a &= \sqrt{\frac{\mathrm{sh}(2ea)}{e}}\Bigg[-\sqrt{\frac{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}\frac{2pe}{\pi}\left\{\int_0^a F_2(u,a)\,du + \int_b^c F_2(v,a)\,dv\right\} \\ &\qquad - \frac{\mu s C_1}{\sqrt{[\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)][\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)]}}\Bigg], \\ N_b &= -\frac{\mu s C_1}{\sqrt{[\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)][\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)]}}\sqrt{\frac{\mathrm{sh}(2eb)}{e}}, \\ N_c &= \sqrt{\frac{\mathrm{sh}(2ec)}{e}}\Bigg[-\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}\frac{2pe}{\pi}\left\{\int_0^a F_2(u,c)du + \int_b^c F_2(v,c)\,dv\right\} \\ &\qquad + \frac{\mu s C_1}{\sqrt{[\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)][\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)]}}\Bigg]. \end{aligned} \tag{3.16}$$

Again, insertion of the values of $h(u)$ and $g(v^2)$, given by Eqs. (3.8) and (3.9), in the expressions for displacements given by Eqs. (3.11) yields

$$\begin{aligned} [W(x,0)]_{0\le x\le a} &= -\frac{p}{\mu\pi s}\Bigg[\frac{2[\mathrm{sh}^2(eb)-\mathrm{sh}^2(ea)]}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}\Bigg\{\int_b^c \Pi\{\lambda, \frac{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}, q\} \\ &\times\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}}\frac{dv}{\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}} - \int_0^a \Pi\{\lambda, \frac{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}, q\} \\ &\times\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}}\frac{du}{\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}}\Bigg\}\Bigg] - \frac{C_1 F(\lambda,q)}{e\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}}, \end{aligned}$$

and

$$\begin{aligned} [W(x,0)]_{b\le x\le c} &= \Bigg[\frac{2p}{\mu\pi s}\Bigg(\int_b^c \sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}}\sqrt{\mathrm{sh}^2(ev)-\mathrm{sh}^2(ea)}\Bigg\{F(\lambda',q) \\ &+ \frac{\mathrm{sh}^2(ev)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}\Pi\left\{\lambda', \frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ev)}, q\right\}\Bigg\}\,dv + \int_0^a \sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}} \end{aligned}$$

$$\times\sqrt{\mathrm{sh}^2(ea)-\mathrm{sh}^2(eu)}\{F(\lambda',q)-\frac{\mathrm{sh}^2(eb)-\mathrm{sh}^2(eu)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)}\Pi\{\lambda',\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eu)},q\}\}\,du\Big)$$
$$+\frac{C_1}{e}F(\lambda',q)\Big]\frac{1}{\sqrt{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ea)}},$$

where

$$\sin\lambda=\sqrt{\frac{\mathrm{sh}^2(ea)-\mathrm{sh}^2(ex)}{\mathrm{sh}^2(eb)-\mathrm{sh}^2(ex)}},\quad \sin\lambda'=\sqrt{\frac{\mathrm{sh}^2(ec)-\mathrm{sh}^2(ex)}{\mathrm{sh}^2(ec)-\mathrm{sh}^2(eb)}}$$

and $F(\phi,q)$, $\Pi(\phi,n,q)$, and q have been defined earlier.

On putting $b=c$ and simplifying, it may be noted that the results $(3.16)_1$ and $(3.17)_1$ become those given by Eqs. (4.18) and (4.19) of SINGH *et al.* [2], and for $a=0$ the results given by Eqs. $(3.16)_2$, $(3.16)_3$ and $(3.17)_2$ coincide with those given by Eqs. (4.38), (4.39) and (4.35) of DAS and GHOSH [5].

4. Numerical results and discussions

Numerical results for stress intensity factors at the tips of the cracks for different values of crack speed, crack lengths and the separating distance between the cracks have been presented in this section. The dependence of the stress intensity factors on crack lengths and their variations with V/C_2 have been shown in Figs. 2–5. It is seen in Figs. 2–3 that stress intensity factors at the edges of the cracks increase rapidly when $V/C_2\to 1$, and variation of stress intensity factors at the edge $x=a$ is greater than that at the tips $x=b$ and $x=c$ when the length of the inner crack increases.

Variations of stress intensity factors at the edges of the cracks with a/b for different values of c/b and that with b/a for different values of c/a are plotted in Figs. 4–5, respectively. It has been found that when the distance between the inner crack and the outer pair of cracks decreases, the stress intensity factors at the tips $x=a$ and $x=b$ become greater than that at the edge $x=c$.

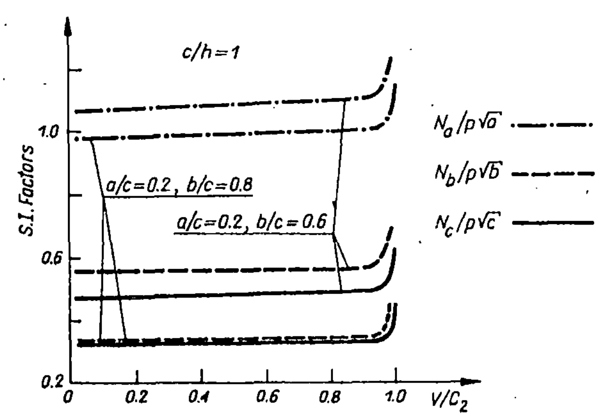

FIG. 2. Variations of stress intensity factors with V/C_2.

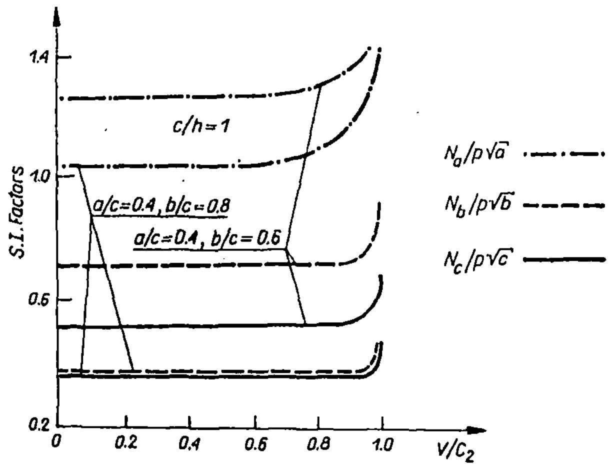

FIG. 3. Variations of stress intensity factors with V/C_2.

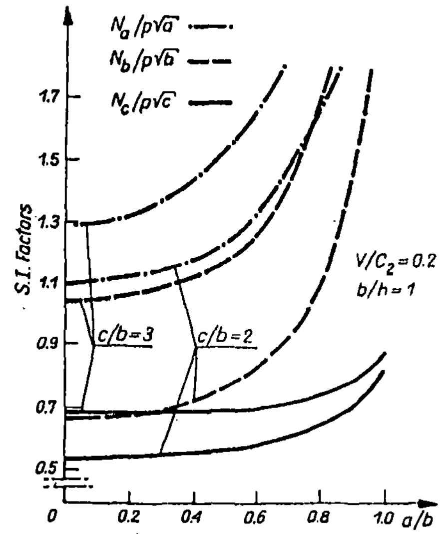

FIG. 4. Stress intensity factors Vs. a/b.

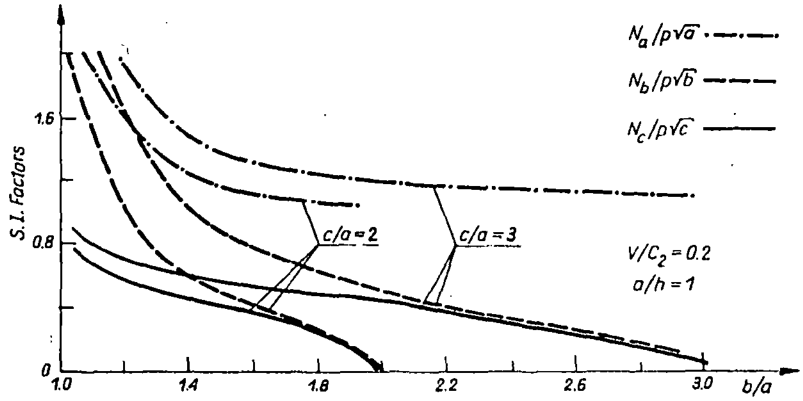

FIG. 5. Stress intensity factors Vs. b/a.

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FOUR CO-PLANAR GRIFFITH CRACKS IN AN INFINITE ELASTIC MEDIUM

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Abstract—The dynamic in-plane problem of determining the stress and displacement due to four co-planar Griffith cracks moving steadily at a subsonic speed in a fixed direction in an infinite, isotropic, homogeneous medium under normal stress has been treated. The static problem of determining the stress and displacement in an infinite isotropic elastic medium has also been considered. In both cases, employing the Fourier integral transform, the problems have been reduced to solving a set of five integral equations. These integral equations have been solved using the finite Hilbert transform technique to obtain the exact form of crack opening displacement and stress intensity factors which are presented in the form of graphs.

INTRODUCTION

In fracture mechanics, scattering of elastic waves by cracks of finite dimension in an infinite elastic medium has been investigated by several investigators. The problem of scattering of elastic waves from an interface crack was solved by Bostrom [1]. Srivastava *et al.* [2] solved the problem of the interaction of an anti-plane shear wave with an interface crack. The problem of diffraction of Love waves by a crack of finite width in the plane interface of a layered composite has been solved by Neerhoff [3]. Itou [4] solved the problem of diffraction of an anti-plane shear wave by two co-planar Griffith cracks in an infinite elastic medium. The scattering of a time harmonic normally incident plane wave by two co-planar Griffith cracks was solved by Jain and Kanwal [5]. Itou [6] also solved the problem of stress concentration around two co-planar Griffith cracks in an infinite elastic medium. Problems on two co-planar Griffith cracks moving along the interface of a layered infinite half-space have also been solved by Das and Ghosh [7] recently.

As regards the crack problem, research has been restricted mainly to the case of a single crack or a pair of cracks because of the severe mathematical complexity encountered in solving the problems of three or more cracks. Recently, Dhawan and Dhaliwal [8] solved the static problem of determining the stress distribution in an infinite transversely isotropic medium containing three co-planar cracks.

To the best knowledge of the authors, the problem of stress distribution around four co-planar Griffith cracks has not been investigated so far. In this paper, we consider two cases regarding the stress distribution around four co-planar Griffith cracks in an infinite homogeneous, isotropic medium. In the first case, cracks are assumed to be moving steadily along a fixed direction with constant velocity V. In the second case, the static problem of determining the stress and displacement in an infinite homogeneous, isotropic medium weakened by four co-planar Griffith cracks has been considered. Using Fourier integral transform both problems have been reduced to solving a set of five integral equations. Employing the finite Hilbert transform technique [9], the integral equations have been solved to derive crack opening displacement and stress intensity factors, which are presented in the form of graphs.

STATEMENT OF PROBLEM I AND ITS FORMULATION

Consider an infinite homogeneous, isotropic material weakened by four co-planar Griffith cracks, moving steadily at a constant velocity V in the X-direction referred to a fixed coordinate system (X, Y, Z), as shown in Fig. 1. In the absence of body force, the equations of motion in terms of displacement are

$$(\lambda + 2\mu)[u_{,XX} + v_{,XY}] + \mu[u_{,YY} - v_{,XY}] = \rho u_{,TT}$$

and

$$(\lambda + 2\mu)[u_{,XY} + v_{,YY}] + \mu[v_{,XX} - u_{,XY}] = \rho u_{,TT} \qquad (1a, b)$$

where u and v denote the displacement components in the X- and Y-directions, λ and μ are Lamé's constants and $u_{,X}$ represents partial derivatives of u with respect to X.

For cracks moving with constant velocity V in the X-direction it is convenient to introduce the Galilean transformation

$$x = X - VT, \quad y = Y, \quad z = Z, \quad t = T \tag{2}$$

where (x, y, z) represents the translating coordinate system as shown in Fig. 1.

In the moving coordinates, the equations of motion (1) become independent of time and take the form

$$(\lambda + 2\mu - \rho V^2)u_{,xx} + (\lambda + \mu)v_{,xy} + \mu u_{,yy} = 0$$
$$(\lambda + 2\mu)v_{,yy} + (\mu - \rho V^2)v_{,xx} + (\lambda + \mu)u_{,xy} = 0. \tag{3a, b}$$

Introducing

$$\bar{u}_s(\xi, y) = \int_0^\infty u(x, y)\sin(\xi x)\,\mathrm{d}x$$
$$\bar{v}_c(\xi, y) = \int_0^\infty v(x, y)\cos(\xi x)\,\mathrm{d}x \tag{4a, b}$$

and

$$u(x, y) = \frac{2}{\pi}\int_0^\infty \bar{u}_s(\xi, y)\sin(\xi x)\,\mathrm{d}\xi$$
$$v(x, y) = \frac{2}{\pi}\int_0^\infty \bar{v}_c(\xi, y)\cos(\xi x)\,\mathrm{d}\xi, \tag{5a, b}$$

in eq. (3) we obtain

$$\mu\bar{u}_{s,yy} - \xi(\lambda + \mu)\bar{v}_{c,y} - \xi^2(\lambda + 2\mu - \rho V^2)\bar{u}_s = 0$$
$$(\lambda + 2\mu)\bar{v}_{c,yy} + \xi(\lambda + \mu)\bar{u}_{s,y} - \xi^2(\mu - \rho V^2)\bar{v}_c = 0. \tag{6a, b}$$

Elimination of $\bar{u}_s$ from (6a, b) yields the following ordinary differential equation:

$$\left[\left\{\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (1 - M^2k^2)\xi^2\right\}\left\{\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (1 - M^2)\xi^2\right\}\right]\bar{v}_c = 0 \tag{7}$$

where $M = V/c_2$, $k = c_2/c_1$.

The solution of the differential equation given by (7), for $y \geqslant 0$, is

$$\bar{v}_c(\xi, y) = A(\xi)\mathrm{e}^{-\xi y\sqrt{(1 - M^2k^2)}} + B(\xi)\mathrm{e}^{-\xi y\sqrt{(1 - M^2)}} \tag{8}$$

where the unknown functions $A(\xi)$ and $B(\xi)$ are to be determined using the boundary conditions of the proposed problem.

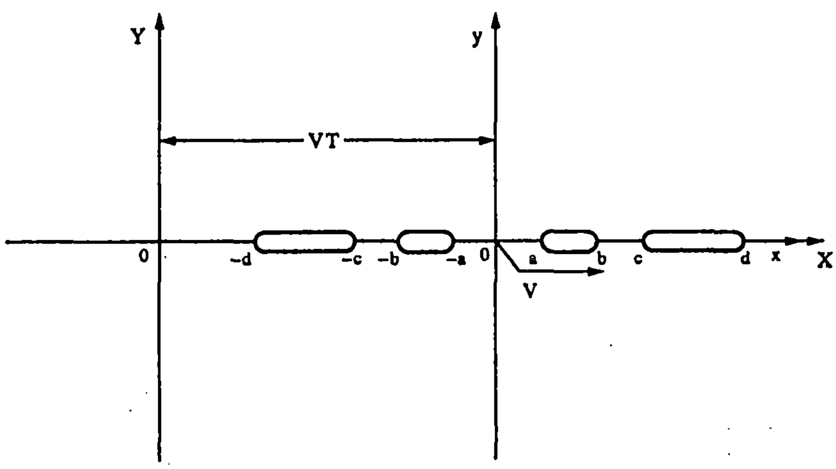

Fig. 1. Geometry and coordinate system.

Employing (8) in eqs (6a, b), we obtain

$$\bar{u}_s(\xi, y) = \frac{A(\xi)}{\sqrt{(1-M^2k^2)}}\,\mathrm{e}^{-\xi y\sqrt{(1-M^2k^2)}} + \sqrt{(1-M^2)}B(\xi)\,\mathrm{e}^{-\xi y\sqrt{(1-M^2)}}, \quad y \geqslant 0. \tag{9}$$

Therefore, the stress components given by

$$\sigma_{yy} = \lambda(u_{,x} + v_{,y}) + 2\mu v_{,y}$$

$$\sigma_{xy} = \mu(u_{,y} + v_{,x}) \tag{10a, b}$$

become

$$\sigma_{yy}(x, y) = -\frac{2\mu}{\pi}\int_0^\infty \xi\left[\frac{2-M^2}{\sqrt{(1-M^2k^2)}}A(\xi)\,\mathrm{e}^{-\xi y\sqrt{(1-M^2k^2)}} + 2\sqrt{(1-M^2)}B(\xi)\,\mathrm{e}^{-\xi y\sqrt{(1-M^2)}}\right]\cdot\cos(\xi x)\,\mathrm{d}\xi$$

$$\sigma_{xy}(x, y) = -\frac{2\mu}{\pi}\int_0^\infty \xi[2A(\xi)\,\mathrm{e}^{-\xi y\sqrt{(1-M^2k^2)}} + (2-M^2)B(\xi)\,\mathrm{e}^{-\xi y\sqrt{(1-M^2)}}]\sin(\xi x)\,\mathrm{d}\xi \tag{11a, b}$$

with

$$u(x, y) = \frac{2}{\pi}\int_0^\infty \left[\frac{A(\xi)}{\sqrt{(1-M^2k^2)}}\,\mathrm{e}^{-\xi y\sqrt{(1-M^2k^2)}} + \sqrt{(1-M^2)}B(\xi)\,\mathrm{e}^{-\xi y\sqrt{(1-M^2)}}\right]\sin(\xi x)\,\mathrm{d}\xi$$

and

$$v(x, y) = \frac{2}{\pi}\int_0^\infty [A(\xi)\,\mathrm{e}^{-\xi y\sqrt{(1-M^2k^2)}} + B(\xi)\,\mathrm{e}^{-\xi y\sqrt{(1-M^2)}}]\cos(\xi x)\,\mathrm{d}\xi. \tag{12a, b}$$

Let four co-planar Griffith cracks of finite length located along the X-axis be moving steadily with velocity V in the X-direction so that their positions, referred to translating coordinates (x, y, z), are $a \leqslant |x| \leqslant b$, $c \leqslant |x| \leqslant d$ on $y = 0$.

The boundary conditions of the proposed problem on account of the symmetry with respect to the y-axis are

$$v(x, 0) = 0, \quad x \in I_1, I_3, I_5 \tag{13a–c}$$

$$\sigma_{xy}(x, 0) = 0, \quad 0 < x < \infty \tag{14}$$

$$\sigma_{yy}(x, 0) = -p, \quad x \in I_2, I_4 \tag{15a, b}$$

where $I_1 = (0, a)$, $I_2 = (a, b)$, $I_3 = (b, c)$, $I_4 = (c, d)$, $I_5 = (d, \infty)$.

Using the condition (14) in (11b) we find that $A(\xi)$ and $B(\xi)$ are related by

$$B(\xi) = -\frac{2}{2-M^2}A(\xi). \tag{16}$$

With the help of the boundary condition (13), we obtain from (12b)

$$\int_0^\infty A(\xi)\cos(\xi x)\,\mathrm{d}\xi = 0, \quad x \in I_1, I_3, I_5. \tag{17a–c}$$

Substitution of (11a) in (15) yields, with the aid of (16)

$$\int_0^\infty \xi A(\xi)\cos(\xi x)\,\mathrm{d}\xi = \frac{P\pi}{2\mu}, \quad x \in I_2, I_4 \tag{18a, b}$$

where

$$P = \frac{p}{K}, \quad K = \frac{(2-M^2)^2 - 4\sqrt{[(1-M^2k^2)(1-M^2)]}}{(2-M^2)\sqrt{(1-M^2k^2)}}.$$

METHOD OF SOLUTION

In order to solve the set of five integral equations given in eqs (17) and (18), we assume

$$A(\xi)=\frac{1}{\xi}\int_a^b h(s^2)\sin(\xi s)\,ds+\frac{1}{\xi}\int_c^d g(t^2)\sin(\xi t)\,dt \tag{19}$$

where $h(s^2)$ and $g(t^2)$ are unknown functions to be determined from the boundary conditions.

Inserting the value of $A(\xi)$ from eq. (19) in eq. (17), it is found that this choice of $A(\xi)$ leads to the equations

$$\int_a^b h(s^2)\,ds=0 \quad\text{and}\quad \int_c^d g(t^2)\,dt=0. \tag{20a, b}$$

Further substituting $A(\xi)$ from eq. (19) in (18a), we obtain

$$\int_a^b \frac{sh(s^2)}{s^2-x^2}\,ds+\int_c^d \frac{tg(t^2)}{t^2-x^2}\,dt=\frac{\pi P}{2\mu},\quad x\in I_2.$$

Rewriting this equation as

$$\int_a^b \frac{sh(s^2)}{s^2-x^2}\,ds=\frac{\pi}{2}F(x),\quad x\in I_2$$

where

$$F(x)=\frac{P}{\mu}-\frac{2}{\pi}\int_c^d \frac{tg(t^2)}{t^2-x^2}\,dt$$

and using finite Hilbert transform technique [9], we obtain

$$h(s^2)=\frac{P}{\mu}\sqrt{\left(\frac{s^2-a^2}{b^2-s^2}\right)}-\frac{2}{\pi}\sqrt{\left(\frac{s^2-a^2}{b^2-s^2}\right)}\int_c^d\sqrt{\left(\frac{t^2-b^2}{t^2-a^2}\right)}\frac{tg(t^2)}{t^2-s^2}\,dt+\frac{C_1}{\sqrt{[(s^2-a^2)(b^2-s^2)]}}, \tag{21}$$

where we have used

$$\int_a^b\sqrt{\left(\frac{b^2-x^2}{x^2-a^2}\right)}\frac{x\,dx}{(s^2-x^2)(t^2-x^2)}=\frac{\pi}{2}\sqrt{\left(\frac{t^2-b^2}{t^2-a^2}\right)}\frac{1}{t^2-s^2}.$$

The constant C_1 is to be determined from eq. (20).

Substituting the value of $h(s^2)$ from (21) in (19) and using the resulting value of $A(\xi)$ in the boundary condition (18b) we obtain, using the results

$$\int_a^b\sqrt{\left(\frac{s^2-a^2}{b^2-s^2}\right)}\frac{s\,ds}{(s^2-x^2)(t^2-s^2)}=\frac{\pi}{2}\left[\sqrt{\left(\frac{t^2-a^2}{t^2-b^2}\right)}-\sqrt{\left(\frac{x^2-a^2}{x^2-b^2}\right)}\right]\frac{1}{t^2-x^2}$$

and

$$\int_a^b\frac{s\,ds}{(s^2-x^2)\sqrt{[(s^2-a^2)(b^2-s^2)]}}=-\frac{\pi}{2\sqrt{[(x^2-a^2)(x^2-b^2)]}}\quad\text{for } x\in I_4,$$

the singular integral equation

$$\int_c^d\sqrt{\left(\frac{t^2-b^2}{t^2-a^2}\right)}\frac{tg(t^2)}{t^2-x^2}\,dt=\frac{\pi}{2}\left[\frac{P}{\mu}+\frac{C_1}{x^2-a^2}\right],\quad x\in I_4.$$

Again using the finite Hilbert transform technique [9], we obtain

$$g(t^2)=\frac{P}{\mu}\sqrt{\left[\frac{(t^2-a^2)(t^2-c^2)}{(t^2-b^2)(d^2-t^2)}\right]}+\sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)}\frac{C_1\sqrt{(t^2-c^2)}}{\sqrt{[(t^2-a^2)(t^2-b^2)(d^2-t^2)]}}$$
$$+\frac{C_2\sqrt{(t^2-a^2)}}{\sqrt{[(t^2-b^2)(t^2-c^2)(d^2-t^2)]}}, \tag{22}$$

where we have used

$$\int_c^d\sqrt{\left(\frac{d^2-x^2}{x^2-c^2}\right)}\frac{x\,dx}{(x^2-a^2)(x^2-t^2)}=-\frac{\pi}{2}\sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)}\frac{1}{t^2-a^2}$$

and the constant C_2 is to be determined using the condition given by eq. (20).

Next, substituting the value of $g(t^2)$ from (22) in eq. (21) and finally using the following results:

$$\int_c^d \sqrt{\left(\frac{t^2-c^2}{d^2-t^2}\right)}\frac{t\,dt}{(t^2-a^2)(t^2-s^2)}=\frac{\pi}{2}\left[\sqrt{\left(\frac{c^2-a^2}{d^2-a^2}\right)}-\sqrt{\left(\frac{c^2-s^2}{d^2-s^2}\right)}\right]\frac{1}{s^2-a^2}$$

$$\int_c^d \frac{t\,dt}{(t^2-s^2)\sqrt{[(t^2-c^2)(d^2-t^2)]}}=\frac{\pi}{2\sqrt{[(c^2-s^2)(d^2-s^2)]}}\quad \text{for } s\in I_2,$$

$h(s^2)$ is derived in the form

$$h(s^2)=\frac{P}{\mu}\sqrt{\left[\frac{(s^2-a^2)(c^2-s^2)}{(b^2-s^2)(d^2-s^2)}\right]}+\sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)}\frac{C_1\sqrt{(c^2-s^2)}}{\sqrt{[(s^2-a^2)(b^2-s^2)(d^2-s^2)]}}-\frac{C_2\sqrt{(s^2-a^2)}}{\sqrt{[(b^2-s^2)(c^2-s^2)(d^2-s^2)]}}. \quad (23)$$

To determine the values of the unknown constants C_1 and C_2, we substitute $g(t^2)$ and $h(s^2)$ given by (22) and (23) in (20) and obtain

$$C_1=\frac{K_{a,b}^{c,d}I_{c,d}^{a,b}+K_{c,d}^{a,b}J_{a,b}^{c,d}}{I_{a,b}^{c,d}I_{c,d}^{a,b}+J_{a,b}^{c,d}J_{c,d}^{a,b}}\frac{P}{\mu}\sqrt{\left(\frac{c^2-a^2}{d^2-a^2}\right)}$$

$$C_2=\frac{K_{c,d}^{a,b}I_{a,b}^{c,d}-K_{a,b}^{c,d}J_{c,d}^{a,b}}{I_{a,b}^{c,d}I_{c,d}^{a,b}+J_{a,b}^{c,d}J_{c,d}^{a,b}}\frac{P}{\mu}$$

where

$$I_{p,q}^{r,s}=\int_p^q \frac{\sqrt{(x^2-r^2)}\,dx}{\sqrt{[(x^2-p^2)(x^2-q^2)(s^2-x^2)]}}$$

$$J_{p,q}^{r,s}=\int_p^q \frac{\sqrt{(x^2-p^2)}\,dx}{\sqrt{[(x^2-q^2)(x^2-r^2)(s^2-x^2)]}}$$

$$K_{p,q}^{r,s}=-\int_p^q \frac{\sqrt{[(x^2-p^2)(x^2-r^2)]}}{\sqrt{[(x^2-q^2)(s^2-x^2)]}}\,dx.$$

The relevant displacement and stress components in the plane of the crack can now be shown to be given by

$$v(x,0)=\int_x^b h(s^2)\,ds,\quad a\leqslant x\leqslant b$$

$$=\int_x^d g(t^2)\,dt,\quad c\leqslant x\leqslant d \quad (24a, b)$$

and

$$[\sigma_{yy}(x,0)]_{0<x<a}=-\frac{2\mu K}{\pi}\left[\int_a^b \frac{sh(s^2)}{s^2-x^2}ds+\int_c^d \frac{tg(t^2)}{t^2-x^2}dt\right]$$

$$[\sigma_{yy}(x,0)]_{b<x<c}=\frac{2\mu K}{\pi}\left[\int_a^b \frac{sh(s^2)}{x^2-s^2}ds-\int_c^d \frac{tg(t^2)}{t^2-x^2}dt\right]$$

$$[\sigma_{yy}(x,0)]_{x>d}=\frac{2\mu K}{\pi}\left[\int_a^b \frac{sh(s^2)}{x^2-s^2}ds-\int_c^d \frac{tg(t^2)}{t^2-x^2}dt\right]. \quad (25a–c)$$

Insertion of the values of $h(s^2)$ and $g(t^2)$ as given by eqs (22) and (23) in the expressions (25) yields, after some algebraic manipulation,

$$[\sigma_{yy}(x,0)]_{0<x<a}=-\frac{2\mu K}{\pi}[F_1(x)+F_2(x)+F_3(x)+F_4(x)+F_6(x)+F_7(x)]$$

$$[\sigma_{yy}(x,0)]_{b<x<c}=-\frac{2\mu K}{\pi}[F_1(x)+F_2(x)+F_3(x)+F_4(x)-F_5(x)-F_8(x)]$$

$$[\sigma_{yy}(x,0)]_{x>d}=-\frac{2\mu K}{\pi}[F_1(x)+F_2(x)+F_3(x)+F_4(x)-F_6(x)-F_7(x)] \quad (26a–c)$$

where

$$F_1(x) = \left[\frac{P}{\mu}(c^2-a^2) - C_2\right]\left[1 - \sqrt{\left(\frac{a^2-x^2}{b^2-x^2}\right)}\right]\frac{\pi}{2\sqrt{[(c^2-a^2)(d^2-a^2)]}}$$

$$F_2(x) = \int_a^b \left[\frac{P}{\mu}(d^2-c^2) - C_2\frac{2u^2-d^2-c^2}{c^2-u^2}\right]\frac{g_1(u,x)}{d^2-u^2}\,\mathrm{d}u$$

$$F_3(x) = \left[\frac{P}{\mu}(c^2-a^2) + C_1\sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)}\right]\left[1 - \sqrt{\left(\frac{c^2-x^2}{d^2-x^2}\right)}\right]\frac{\pi}{2\sqrt{[(c^2-a^2)(c^2-b^2)]}}$$

$$F_4(x) = \int_c^d \left[\frac{P}{\mu}(b^2-a^2) + C_1\sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)}\frac{2u^2-a^2-b^2}{u^2-a^2}\right]\frac{g_2(u,x)}{u^2-b^2}\,\mathrm{d}u$$

$$F_{5,6}(x) = \frac{\pi}{2}\sqrt{\left(\frac{d^2-a^2}{d^2-b^2}\right)}\left[\frac{C_1}{X_1}\sqrt{\left(\frac{c^2-b^2}{c^2-a^2}\right)} \mp \frac{C_2}{X_2}\right]$$

$$F_{7,8}(x) = \frac{C_1}{X_1}\sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)}L^{a,b}_{c,d}(x) \mp \frac{C_2}{X_2}L^{c,d}_{a,b}(x)$$

$$g_1(u,x) = \frac{u}{\sqrt{[(d^2-u^2)(c^2-u^2)]}}\left[\sqrt{\left(\frac{a^2-x^2}{b^2-x^2}\right)}\tan^{-1}\sqrt{\left[\frac{(a^2-x^2)(b^2-u^2)}{(b^2-x^2)(u^2-a^2)}\right]} - \tan^{-1}\sqrt{\left(\frac{b^2-u^2}{u^2-a^2}\right)}\right]$$

$$g_2(u,x) = \frac{u}{\sqrt{[(u^2-b^2)(u^2-a^2)]}}\left[\sqrt{\left(\frac{c^2-x^2}{d^2-x^2}\right)}\tan^{-1}\sqrt{\left[\frac{(c^2-x^2)(d^2-u^2)}{(d^2-x^2)(u^2-c^2)}\right]} - \tan^{-1}\sqrt{\left(\frac{d^2-u^2}{u^2-c^2}\right)}\right]$$

$$X_1 = \sqrt{[(x^2-a^2)(x^2-b^2)]}$$

$$X_2 = \sqrt{[(x^2-c^2)(x^2-d^2)]}$$

$$L^{p,q}_{r,s}(x) = \int_p^q \frac{(s^2-r^2)u\tan^{-1}\sqrt{\left[\frac{(u^2-p^2)(x^2-q^2)}{(q^2-u^2)(x^2-p^2)}\right]}}{\sqrt{[(s^2-u^2)^3(r^2-u^2)]}}\,\mathrm{d}u. \qquad (27a–k)$$

STRESS INTENSITY FACTOR

The dynamic stress intensity factors are given by

$$N_a = \lim_{x\to a^-}\sqrt{[2(a-x)]}[\sigma_{yy}(x,0)]_{0<x<a}$$

$$N_b = \lim_{x\to b^+}\sqrt{[2(x-b)]}[\sigma_{yy}(x,0)]_{b<x<c}$$

$$N_c = \lim_{x\to c^-}\sqrt{[2(c-x)]}[\sigma_{yy}(x,0)]_{b<x<c}$$

$$N_d = \lim_{x\to d^+}\sqrt{[2(x-d)]}[\sigma_{yy}(x,0)]_{x>d}. \qquad (28a–d)$$

Employing (26) in (28) we obtain

$$N_a = -\frac{\mu K C_1}{\sqrt{[a(b^2-a^2)]}}$$

$$N_b = \mu K\left[\frac{P}{\mu}\sqrt{\left[\frac{(b^2-a^2)(c^2-b^2)}{b(d^2-b^2)}\right]} + C_1\sqrt{\left[\frac{(d^2-a^2)(c^2-b^2)}{b(b^2-a^2)(d^2-b^2)(c^2-a^2)}\right]} - C_2\sqrt{\left[\frac{(b^2-a^2)}{b(c^2-b^2)(d^2-b^2)}\right]}\right]$$

$$N_c = -\frac{\mu K C_2}{\sqrt{[c(d^2-c^2)]}}\sqrt{\left(\frac{c^2-a^2}{c^2-b^2}\right)}$$

$$N_d = \mu K\left[\frac{P}{\mu}\sqrt{\left[\frac{(d^2-a^2)(d^2-c^2)}{d(d^2-b^2)}\right]} + C_1\sqrt{\left[\frac{(d^2-c^2)}{d(c^2-a^2)(d^2-b^2)}\right]}\right.$$
$$\left. + C_2\sqrt{\left[\frac{(d^2-a^2)}{d(d^2-c^2)(d^2-b^2)}\right]}\right] \qquad (29a\text{–}d)$$

It is interesting to note that the crack opening displacements depend on the crack velocity V, but in the plane of the cracks the stresses and stress intensity factors are independent of the velocity of the moving cracks in an infinite elastic medium.

STATEMENT OF PROBLEM II AND ITS FORMULATION

In this case, we consider an infinite homogeneous isotropic material with four co-planar Griffith cracks located at $Y = 0$, $a \leqslant |X| \leqslant b$, $c \leqslant |X| \leqslant d$ and subjected to uniform internal pressure q. In the absence of body force, the equations of equilibrium in terms of displacement are

$$(\lambda + 2\mu)[u_{,XX} + v_{,XY}] + \mu[u_{,YY} - v_{,XY}] = 0$$

and

$$(\lambda + 2\mu)[u_{,XY} + v_{,YY}] + \mu[v_{,XX} - u_{,XY}] = 0. \qquad (30a, b)$$

Since the problem exhibits a state of symmetry about $Y = 0$, we can restrict our attention to a single half-space occupying the region $Y \geqslant 0$.

The equations (30) are to be solved subject to the boundary conditions

$$v(X, 0) = 0, \quad |X| \leqslant a, \quad b \leqslant |X| \leqslant c, \quad |X| \geqslant d \qquad (31a\text{–}c)$$

$$\sigma_{XY}(X, 0) = 0, \quad -\infty < X < \infty \qquad (32)$$

$$\sigma_{YY}(X, 0) = -q, \quad a \leqslant |X| \leqslant b, \quad c \leqslant |X| \leqslant d. \qquad (33a, b)$$

In view of the boundary conditions, the appropriate integral solutions of eq. (30) are

$$u(X, Y) = \frac{2}{\pi}\int_0^\infty \left[C(\xi) + D(\xi)\left\{Y - \frac{1}{\xi}\frac{\lambda + 3\mu}{\lambda + \mu}\right\}\right] e^{-\xi Y}\sin(\xi X)\,d\xi$$

and

$$v(X, Y) = \frac{2}{\pi}\int_0^\infty [C(\xi) + YD(\xi)]\, e^{-\xi Y}\cos(\xi X)\,d\xi. \qquad (34a, b)$$

Therefore,

$$\sigma_{YY}(X, Y) = -\frac{4\mu}{\pi}\int_0^\infty \left[\xi C(\xi) + \left\{Y\xi - \frac{\mu}{\lambda + \mu}\right\} D(\xi)\right] e^{-\xi Y}\cos(\xi X)\,d\xi$$

$$\sigma_{XY}(X, Y) = -\frac{4\mu}{\pi}\int_0^\infty \left[\xi C(\xi) + \left\{Y\xi - \frac{\lambda + 2\mu}{\lambda + \mu}\right\} D(\xi)\right] e^{-\xi Y}\sin(\xi X)\,d\xi. \qquad (35a, b)$$

It may be noted that the displacement and stress components given by (34) and (35) cannot be derived from the corresponding expressions of the dynamic problem given in (11) and (12) on setting $M = 0$.

The functions $C(\xi)$ and $D(\xi)$ are to be determined from the boundary conditions (31)–(33), which yield

$$C(\xi) = \frac{1}{\xi}\frac{\lambda + 2\mu}{\lambda + \mu}D(\xi) \qquad (36)$$

and the following set of five integral equations

$$\int_0^\infty C(\xi)\cos(\xi X)\,d\xi = 0, \quad X \in I_1, I_3, I_5 \qquad (37a\text{–}c)$$

$$\int_0^\infty \xi C(\xi)\cos(\xi X)\,d\xi = \frac{Q\pi}{2\mu}, \quad X \in I_2, I_4, \qquad (38a, b)$$

where $Q = (\lambda + 2\mu)/2(\lambda + \mu)q$ and I_j $(j = 1, 2, \ldots, 5)$ are the intervals defined earlier in problem I.

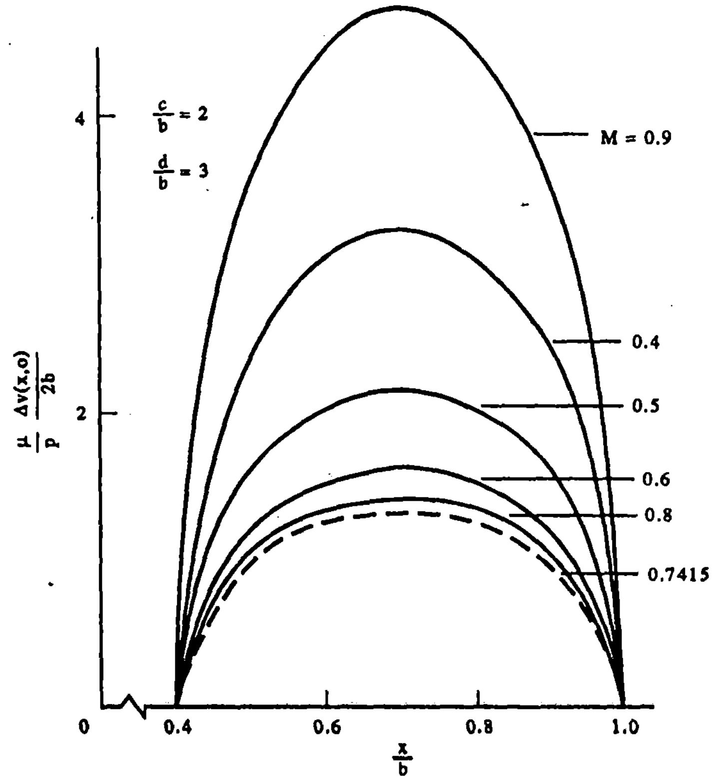

Fig. 2. Variation of crack opening displacement with x/b on the crack of the outer pair for problem I.

METHOD OF SOLUTION AND QUANTITIES OF PHYSICAL INTEREST

The integral equations given by (37) and (38) are found to be the same as those given by eqs (17) and (18) with the exception that P is replaced by Q. Therefore, the same technique as that used in problem I can be employed to obtain

$$v(X,0) = \int_X^b \left[\frac{Q}{\mu}\sqrt{\left[\frac{(s^2-a^2)(c^2-s^2)}{(b^2-s^2)(d^2-s^2)}\right]} + \sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)}\frac{C_1\sqrt{(c^2-s^2)}}{\sqrt{[(s^2-a^2)(b^2-s^2)(d^2-s^2)]}}\right.$$

$$\left. - \frac{C_2\sqrt{(s^2-a^2)}}{\sqrt{[(b^2-s^2)(c^2-s^2)(d^2-s^2)]}}\right]\mathrm{d}s, \quad a \leqslant X \leqslant b$$

$$= \int_X^d \left[\frac{Q}{\mu}\sqrt{\left[\frac{(t^2-a^2)(t^2-c^2)}{(t^2-b^2)(d^2-t^2)}\right]} + \sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)}\frac{C_1\sqrt{(t^2-c^2)}}{\sqrt{[(t^2-a^2)(t^2-b^2)(d^2-t^2)]}}\right.$$

$$\left. + \frac{C_2\sqrt{(t^2-a^2)}}{\sqrt{[(t^2-b^2)(t^2-c^2)(d^2-t^2)]}}\right]\mathrm{d}t, \quad c \leqslant X \leqslant d. \qquad \text{(39a,b)}$$

Stresses in the regions $0 < X < a$, $b < X < c$, $X > d$ are found to be the same as that given in (26), the only change being that P is replaced by Q.

The amounts of energy in opening the cracks $a \leqslant |X| \leqslant b$, $c \leqslant |X| \leqslant d$ are given by $E = 2E_1 + 2E_2$, where

$$E_1 = 2\left|\int_a^b [\sigma_{YY}(X,0)v(X,0)]\,\mathrm{d}X\right|$$

$$E_2 = 2\left|\int_c^d [\sigma_{YY}(X,0)v(X,0)]\,\mathrm{d}X\right|. \qquad (40a, b)$$

Equations (40) can be simplified, with the aid of (33) and (39), to

$$E_1 = -2q\left[\frac{Q}{\mu}M_{a,b}^{c,d} + (c^2-b^2)L_1\Pi\left\{\frac{\pi}{2},\ \frac{b^2-a^2}{c^2-a^2}, r\right\} + \frac{(c^2-a^2)C_2 - c^4\frac{Q}{\mu}}{\sqrt{[(d^2-b^2)(c^2-a^2)]}}F\left(\frac{\pi}{2}, r\right)\right]$$

$$E_2 = 2q\left[\frac{Q}{\mu}M_{c,d}^{a,b} - (d^2-a^2)L_2\Pi\left\{\frac{\pi}{2},\ \frac{c^2-d^2}{c^2-a^2}, r\right\} - \frac{\sqrt{[(c^2-a^2)(d^2-a^2)]}C_1 + a^4\frac{Q}{\mu}}{\sqrt{[(d^2-c^2)(c^2-a^2)]}}F\left(\frac{\pi}{2}, r\right)\right]$$

where

$$L_{1,2} = \frac{\left[(a^2+c^2)\frac{Q}{\mu} \mp C_1\sqrt{\left(\frac{d^2-a^2}{c^2-a^2}\right)} \mp C_2\right]}{\sqrt{[(d^2-b^2)(c^2-a^2)]}},$$

$$r = \sqrt{\left[\frac{(d^2-c^2)(b^2-a^2)}{(d^2-b^2)(c^2-a^2)}\right]}, \quad 2M_{p,q}^{r,s} = \int_{p^2}^{q^2}\frac{z^2\,\mathrm{d}z}{\sqrt{[(z-p^2)(z-q^2)(z-r^2)(s^2-z)]}}$$

and $F(\phi, r)$, $\Pi(\phi, n, r)$ are the elliptic integrals of the first and third kinds respectively.

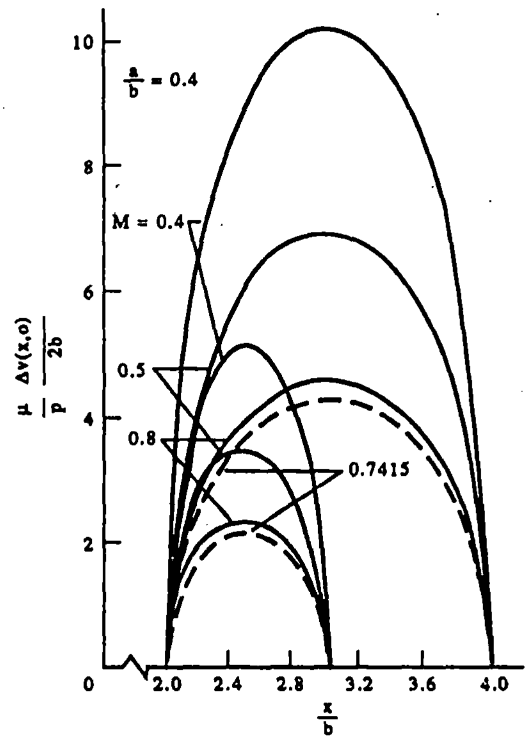

Fig. 3. Variation of crack opening displacement with x/b on the crack of the inner pair for problem I.

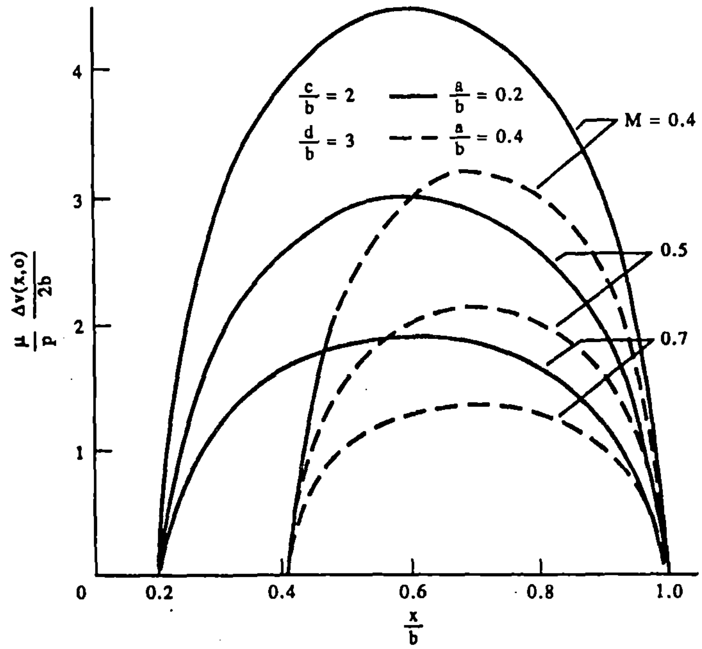

Fig. 4. Variation of crack opening displacement with x/b on the crack of the inner pair for problem I.

NUMERICAL RESULTS AND DISCUSSION

Numerical results for the stress intensity factors and crack opening displacement, defined as $\Delta v(x, 0) = v(x, 0^+) - v(x, 0^-)$, for different values of the parameters are presented in this section. Numerical calculations have been carried out for both the dynamic and static problems. As the crack velocity is less than the Rayleigh wave velocity, it is reasonable to take the value of M as less than 0.9194.

Problem I

Variations of crack opening displacement for different values of crack speed, crack lengths and the separating distance between the cracks have been plotted in Figs 2–4. It is interesting to note from these graphs that the crack opening displacement on both the cracks decreases with increase in the value of M at the onset and takes its minimum value at $M = 0.7415$, after which it increases

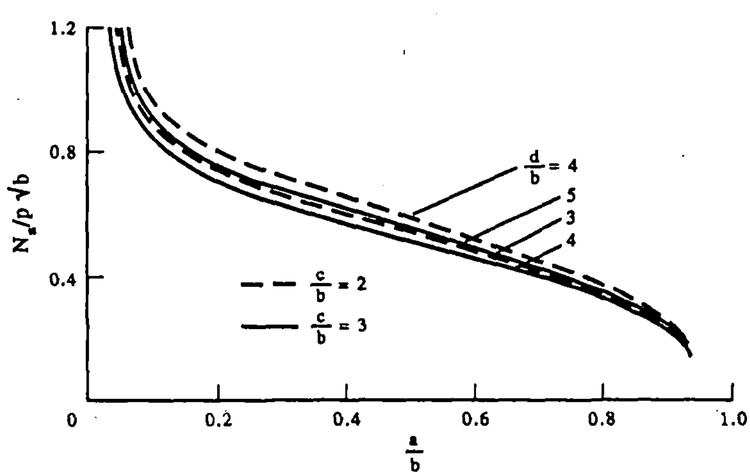

Fig. 5. Stress intensity factor vs a/b at the edge $x = a$.

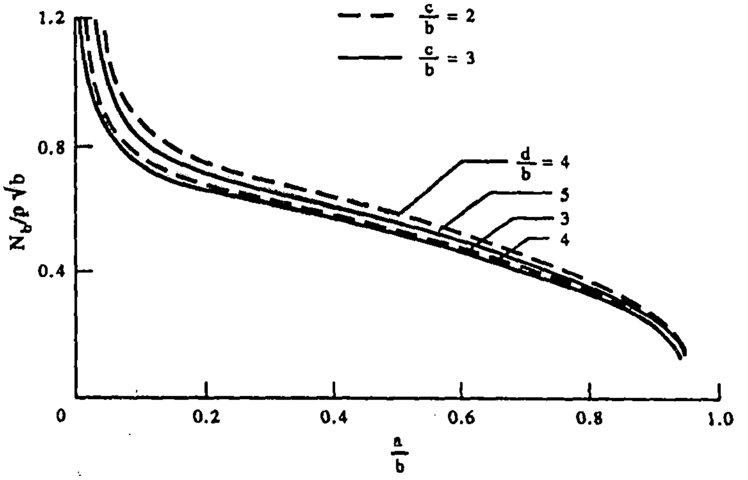

Fig. 6. Stress intensity factor vs a/b at the edge $x = b$.

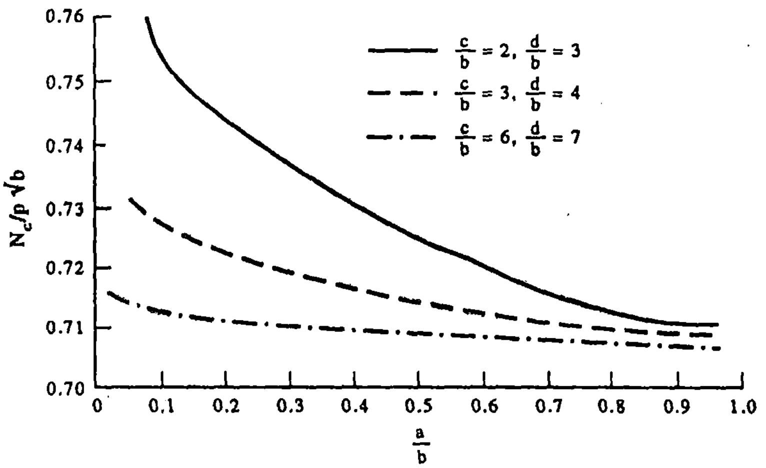

Fig. 7. Stress intensity factor vs a/b at the edge $x = c$.

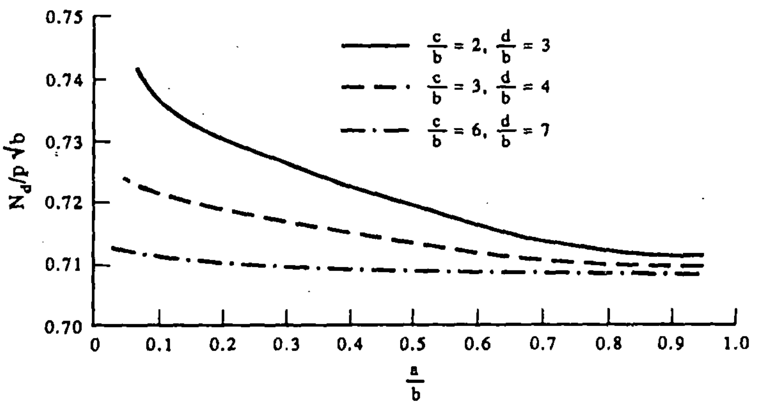

Fig. 8. Stress intensity factor vs a/b at the edge $x = d$.

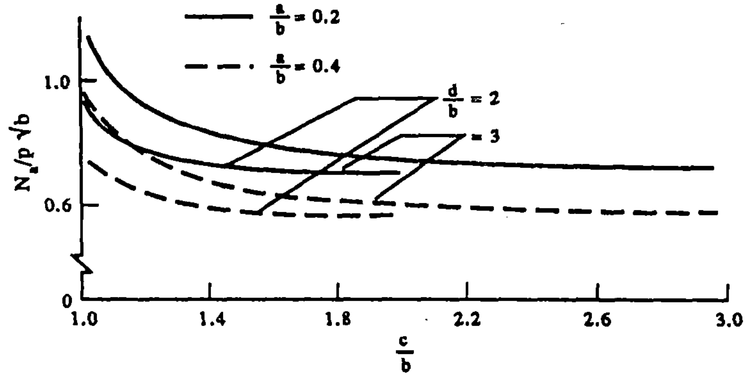

Fig. 9. Stress intensity factor vs c/b at the edge $x = a$.

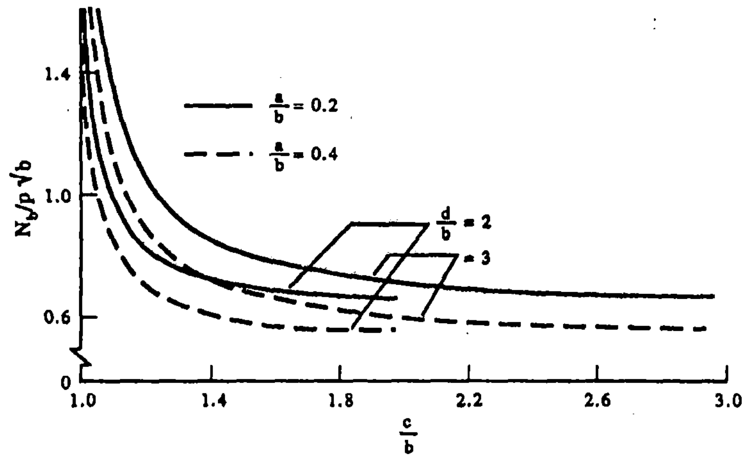

Fig. 10. Stress intensity factor vs c/b at the edge $x = b$.

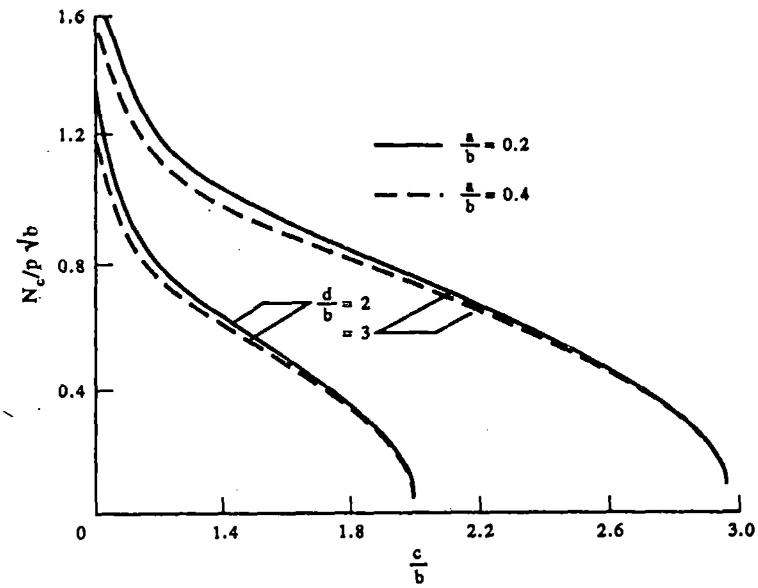

Fig. 11. Stress intensity factor vs c/b at the edge $x = c$.

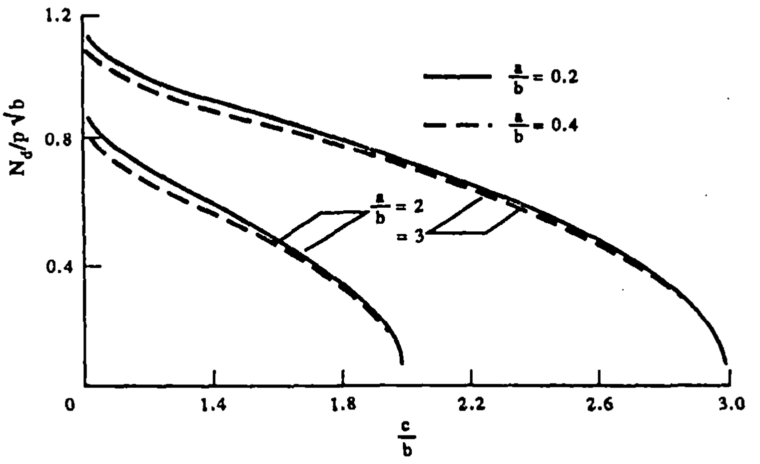

Fig. 12. Stress intensity factor vs c/b at the edge $x = d$.

with increase in the value of M. It has also been depicted in Figs 3 and 4 that on each of the cracks, crack opening displacement decreases as the crack length decreases.

It has been mentioned earlier that the stress intensity factors at the crack tips are independent of crack speed and are found to depend on the crack lengths and the separating distance between the cracks. Variation of stress intensity factors with a/b for different values of c/b, d/b and that with c/b for different values of a/b and d/b are plotted in Figs 5–8 and Figs 9–12 respectively.

It has been found that the effect of variation of the length of either the inner or the outer pair of cracks is more prominent on the stress intensity factors at the edges of the cracks whose lengths are varying compared to its effect on the stress intensity factors at the tips of the cracks whose lengths are kept fixed.

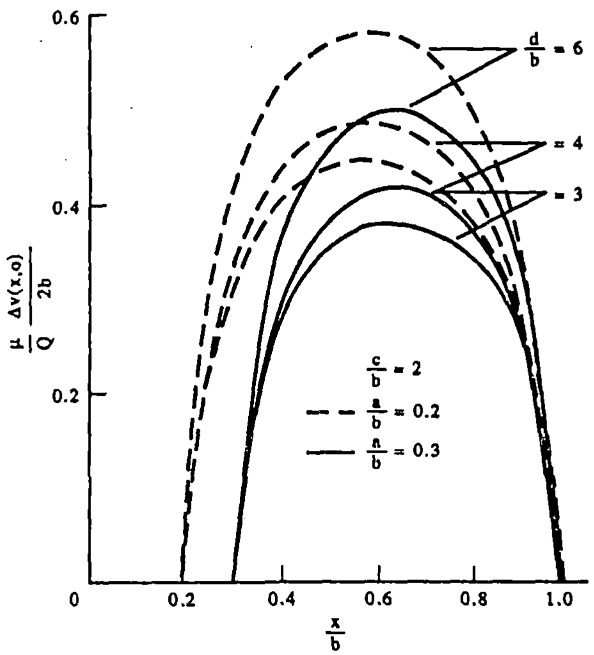

Fig. 13. Variation of crack opening displacement with X/b on the crack of the inner pair for problem II.

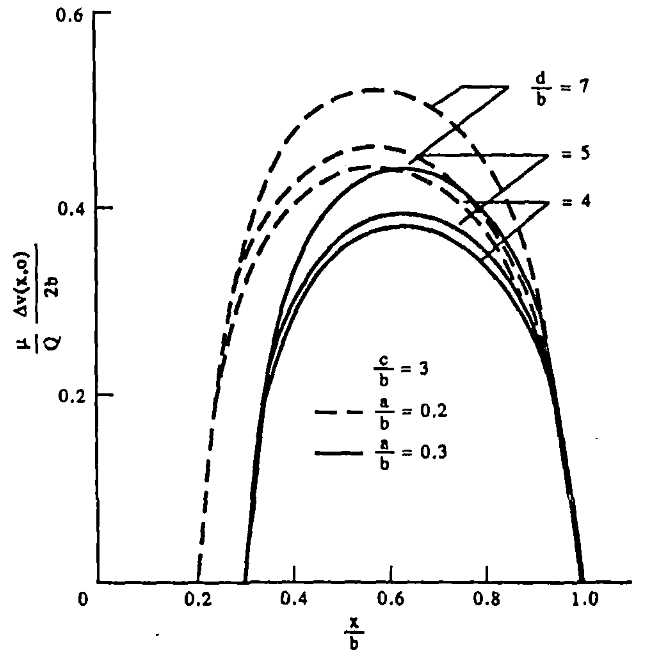

Fig. 14. Variation of crack opening displacement with X/b on the crack of the inner pair for problem II.

Problem II

Figures 13–15 show the variations of crack opening displacement for different values of the parameters a/b, c/b and d/b. They show that crack opening displacement on a crack of fixed length increases with increase in the length of the other crack, as expected from the physical standpoint.

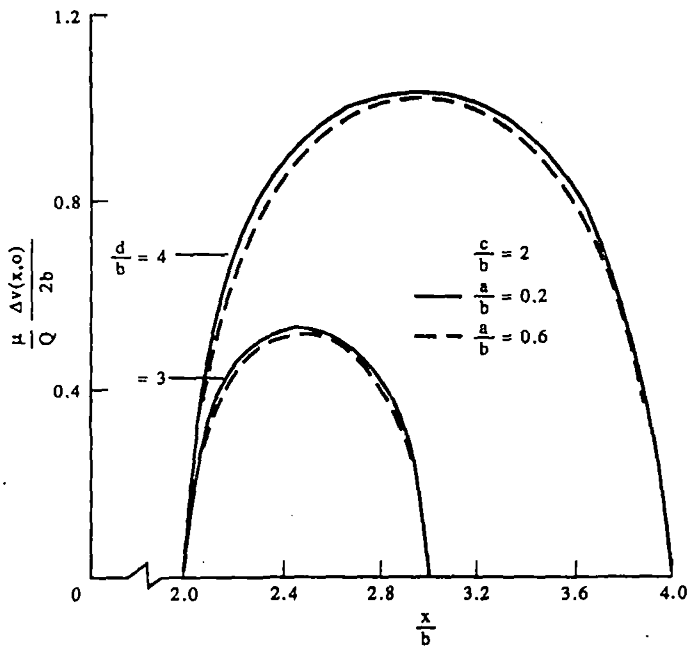

Fig. 15. Variation of crack opening displacement with X/b on the crack of the outer pair for problem II.

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Non-symmetric extension of a plane crack due to plane SH-waves in a prestressed infinite elastic medium

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Abstract. In an infinite isotropic elastic medium initially in a state of uniform anti-plane shear, the problem of non-symmetric extension of an infinitesimal flaw into a plane shear crack due to two identical linearly varying plane SH-waves with non-parallel wave fronts has been analyzed. Fracture is assumed to initiate at a point a finite time after the waves intersect there and the crack is assumed to extend non-symmetrically along the trace of the wave intersection. Following Cherepanov [10], Cherepanov and Afanas'ev [11] the general solution of the problem has been derived in terms of analytic function of complex variable. Numerical results have been presented to illustrate the nature of variation of the stress intensity factors and the rate of energy flux into the crack edges with the speed of the crack tips and also with the time after fracture initiation.

1. Introduction

Since Broberg's [1] investigation of the solution of a crack expanding symmetrically with constant velocity under conditions of plane stress or strain in a homogeneous isotropic elastic medium in a field of spatially and time invariant tensile stress, a number of papers have appeared analyzing different geometrical situations. Craggs [2] later solved the same problem as that done by Broberg but he used the method of homogeneous function to obtain the solution. Achenbach and Brock [3] considered the wave motion generated by a uniformly extending shear crack in a body in a state of uniform anti-plane shear. The case of a crack expanding in an anisotropic medium was considered by Atkinson [4]. This work was later extended by Burridge and Willis [5], who solved the problem of a crack with elliptical cross-section expanding symmetrically with uniform speed in an anisotropic medium. All the problems mentioned above are however self-similar ones with index (0, 0) and are concerned with symmetric expansion of cracks.

Problems involving non-symmetric extension of cracks under uniform loading along the crack surface are seldom found in the literature perhaps due to severe mathematical complexity encountered in solving such problems. Following the method of homogeneous functions developed by Craggs [2], non-symmetric extension of a small flaw into a plane crack under polynomial form of loading was solved by Brock [6]. Following the same procedure, Brock [7] also solved the problem of non-symmetric extension of a crack due to incidence of plane dilatational waves. The problem of determining the dynamic stress field due to a plane dislocation moving in an infinite elastic medium was formulated by Ang and Williams [8] in terms of Fourier integral equation and solved in closed form. Recently, Georgiadis [9] has developed an integral equation approach to self-similar plane elastodynamic problems. He considered the elastodynamic problem of an expanding crack under homogeneous polynomial form loading and reduced it to the solution of a Cauchy integral equation.

In this paper, non-symmetric extension of an infinitesimal flaw into a plane shear crack at a constant rate due to the action of two identical non-parallel plane SH-waves propagating towards each other in an infinite isotropic elastic medium which is initially in a state of uniform anti-plane shear has been treated. A finite time after the crossing of the plane wave fronts, a fracture is assumed to initiate along the line where the wave fronts crossed and the crack edges are then assumed to travel non-symmetrically with different constants speeds. Superposition considerations allow the original problem to be separated into three self-similar problems with (0, 0), (0, 1) and (1, 0) as the index of self-similarity. Following Cherepanov [10], Cherepanov and Afanas'ev [11] the mentioned self-similar problems have all been formulated as some problems of Riemann and Hilbert for half-plane, which are solved easily. Out of all the existing similarity techniques, the method of Smirnov–Sobolev [12] which has been used extensively by Cherepanov [10], Cherepanov and Afanas'ev [11] being the most elegant and straight forward has been used to solve our problem. Analytical expressions for the dynamic stress intensity factors at the crack tips and also the rate of energy flux into the crack edges have been derived. Finally, the nature of the variation of the stress intensity factors and the energy flux rate at the crack tips with the velocities of the crack edges and also with the time after crack initiation have been depicted by means of graphs. The development of a crack initiating at a point being a physically realistic model from the point of view of modelling of earthquake sources, this problem also has application in seismology.

2. Formulation of the problem

Let two identical plane waves defined by

$$\sigma_{yz} = A_0 W_\pm H(W_\pm), \quad \sigma_{xz} = \pm A_0 \cot\theta_0 W_\pm H(W_\pm), \tag{1a,b}$$

referring to coordinate system (x, y, z) where

$$W_\pm = c_2 t \pm y \sin\theta_0 + x\cos\theta_0, \quad 0 \leqslant \theta_0 \leqslant \tfrac{1}{2}\pi$$

and $H(\ )$ is Heaviside's unit function, propagate through the infinite solid which is pre-stressed such that

$$\sigma^0_{yz} = \sigma, \quad \sigma^0_{xz} = 0. \tag{1c}$$

Let us assume that at $t = 0$ the non-parallel plane waves intersect at $x = y = 0$. A micro crack is assumed to appear at $t = t_0$ at $x = y = 0$ which starts to extend bilaterally along the trace of the wave intersection with uniform velocities v_1 and v_2. The expanding crack, the circular wave front associated with its motion and the plane wave front are shown in Fig. 1(a).

In effect crack extension occurs by removing the stresses which would be generated in the crack plane by the combined applied static and dynamic fields if no cracks were present.

Accordingly, both the crack faces are subjected to shear tractions equal to $-\sigma - 2A_0(c_2 t + x\cos\theta_0)$.

The anti-symmetry of this loading about the crack plane implies that it is sufficient to consider the half-plane $y > 0$ with bounding surface $y = 0$. The boundary conditions for this

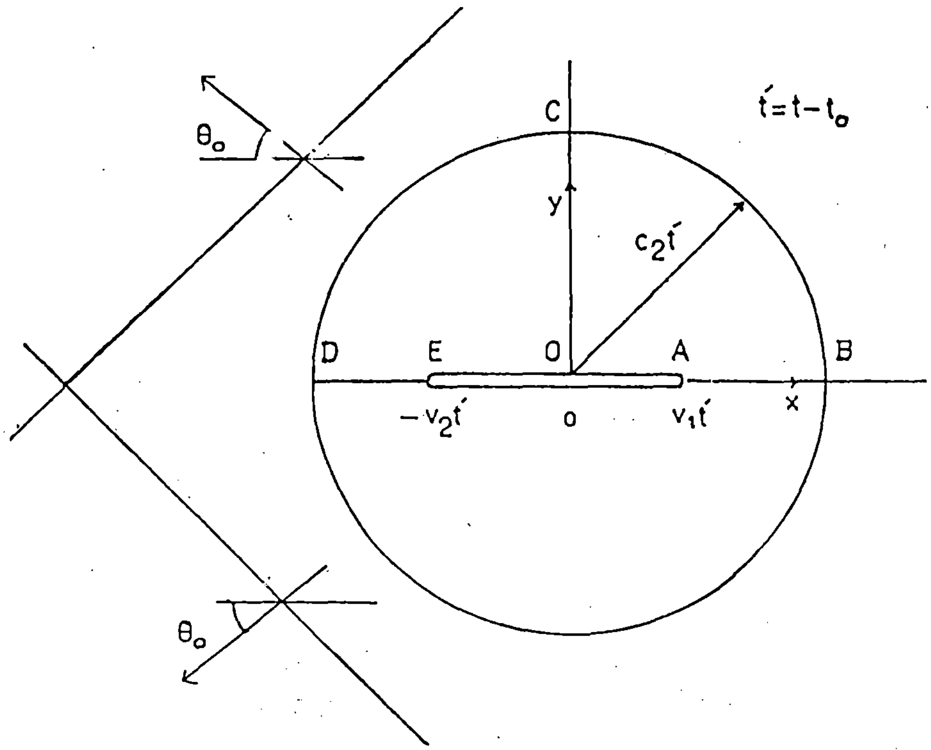

Fig. 1(a). The expanding crack and the pattern of wave front.

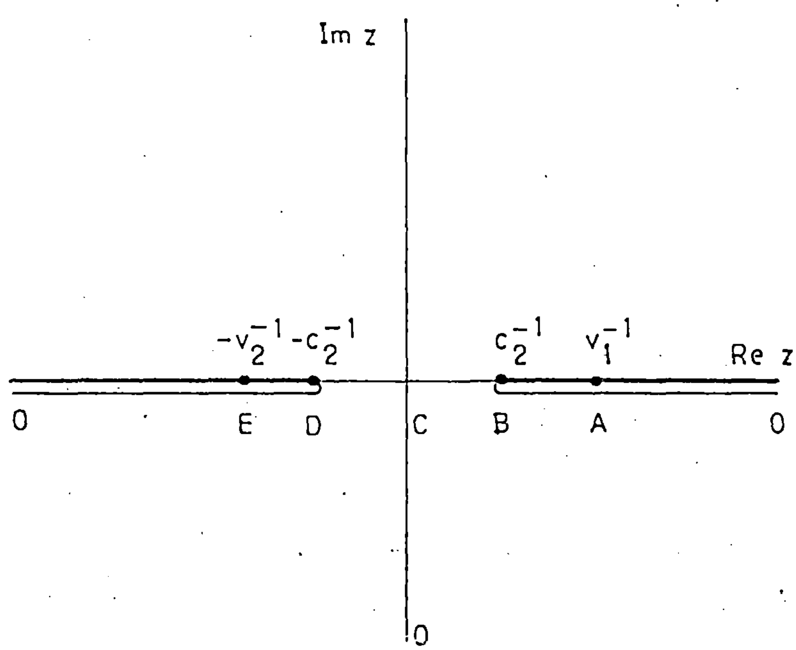

Fig. 1(b). Mapping of the interior of the semi-circle OABCDE in x–y plane on the lower half of th: complex z-plane.

half-plane are then given by

$$y = 0, \quad -v_2 t' < x < v_1 t'; \quad \sigma_{yz} = -\sigma - 2A_0 c_2 t_0 - 2A_0(c_2 t' + x\cos\theta_0),$$
$$y = 0, \quad x > v_1 t', \quad x < -v_2 t'; \quad W = 0, \tag{2a,b}$$

where $t' = t - t_0$.

Equation (2a) shows that invoking superposition principle the proposed problem can be divided into three separate problems of a constant shear traction, a shearing stress linearly varying with time and shear linearly varying with distance along the crack plane.

3. Constant shear traction on the crack faces

The wave motion generated by constant shear tractions on the faces of the crack defined by $y = 0$, $-v_2 t < x < v_1 t$ has been considered in this section and for simplicity t instead of t' has been used. The boundary conditions are

$$y = 0, \quad -v_2 t < x < v_1 t; \quad \sigma_{yz} = -p_0,$$
$$y = 0, \quad x > v_1 t, \quad x < -v_2 t, \quad W = 0, \tag{3a,b}$$

where $p_0 = \sigma + 2A_0 c_2 t_0$.

The displacement W which satisfies the wave equation

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = \frac{1}{c_2^2}\frac{\partial^2 W}{\partial t^2} \tag{4}$$

is to be determined subject to the boundary conditions given by (3). From the boundary conditions we observe that $\partial W/\partial t$ shows dynamic similarity and is a homogeneous function of degree zero in x/t and y/t. Therefore, by the functionally invariant method of Smirnoff and Sobolev [12] we can write

$$\frac{\partial W}{\partial t} = \mathrm{Re}\,\phi_0(z), \tag{5}$$

where

$$t - xz + y\sqrt{c_2^{-2} - z^2} = 0. \tag{6}$$

The sign of the radical is to be fixed by the condition that

$$\text{as } z \to \infty, \quad \sqrt{c_2^{-2} - z^2} = iz + O(z^{-1}). \tag{7}$$

Equation (6) maps the semi-circular region of the cylindrical waves defined by OABCDE to the lower half of the complex cut z-plane given by

$$z = \frac{xt - iy\sqrt{t^2 - c_2^{-2}(x^2 + y^2)}}{x^2 + y^2}, \tag{8}$$

as shown in Fig. 1(b).

In view of (5) and (6) we find

$$\frac{\partial \sigma_{yz}}{\partial t} = \mu \operatorname{Re}\left[\phi_0'(z)\frac{\partial z}{\partial y}\right],$$

so that

$$\frac{\partial \sigma_{yz}}{\partial t}(x, 0, t) = \frac{1}{t}\operatorname{Re}[-\mu z \phi_0'(z)\sqrt{c_2^{-2} - z^2}]. \tag{9}$$

Therefore the boundary conditions (3) are converted to the following conditions in z-plane

$$\operatorname{Im} z = 0, \quad -v_2^{-1} < \operatorname{Re} z < v_1^{-1}, \quad \operatorname{Re}\phi_0(z) = 0, \tag{10}$$

$$\operatorname{Im} z = 0, \quad \operatorname{Re} z < -v_2^{-1}, \quad \operatorname{Re} z > v_1^{-1}, \quad \operatorname{Im}\phi_0'(z) = 0. \tag{11}$$

In order to determine the analytic function $\phi_0(z)$ subject to the conditions (10), (11) it is necessary to know the behavior of the function $\phi_0(z)$ when $z \to v_1^{-1}, -v_2^{-1}$ and $z \to \infty$. The infinite point of the z-plane corresponds to the origin of the coordinate of the physical plane where the displacement W is limited. Hence taking the representation (5) into account, we obtain

$$\operatorname{Re}\phi_0(z) = O(1) \quad \text{as} \quad z \to \infty. \tag{12}$$

Further the condition (11) after integration with respect to z may be put in the form

$$\operatorname{Im} z = 0, \quad \operatorname{Re} z < -v_2^{-1}, \quad \operatorname{Re} z > v_1^{-1}, \quad \operatorname{Im}\phi_0(z) = 0. \tag{13}$$

Moreover, the displacement derivative $\partial W/\partial t$ near the crack tips $x = v_1 t, -v_2 t$ should show square root singularities so that at $z \to v_1^{-1}, -v_2^{-1}$

$$\phi_0(z) = O(1/\sqrt{z - v_1^{-1}}), \quad O(1/\sqrt{z + v_2^{-1}}) \tag{14}$$

respectively.

The above boundary conditions given by (10) and (13) together with the consideration (12) and (14) suggest that

$$\phi_0(z) = \frac{Az + B}{\sqrt{(z - v_1^{-1})(z + v_2^{-1})}}, \tag{15}$$

where A and B are unknown constants to be determined.

Integrating (9) with respect to t it can be easily shown that for $x > 0$

$$\sigma_{yz}(x,0,t) = -\mu\ \mathrm{Re}\left\{[\phi_0(z)\sqrt{c_2^{-2}-z^2}]_{c_2^{-1}}^{t/x} + \int_{c_2^{-1}}^{t/x} \frac{z\phi_0(z)\,\mathrm{d}z}{\sqrt{c_2^{-2}-z^2}}\right\},$$

$$\sigma_{yz}(-x,0,t) = -\mu\ \mathrm{Re}\left\{[\phi_0(z)\sqrt{c_2^{-2}-z^2}]_{-c_2^{-1}}^{-t/x} + \int_{-c_2^{-1}}^{-t/x} \frac{z\phi_0(z)\,\mathrm{d}z}{\sqrt{c_2^{-2}-z^2}}\right\}. \qquad (16a,b)$$

Next using the boundary conditions that

$$\sigma_{yz}(x,0,t) = -p_0, \qquad 0 \leqslant x < v_1 t,$$
$$\sigma_{yz}(-x,0,t) = -p_0, \quad -v_2 t < -x \leqslant 0,$$

in (16a, b) respectively we obtain two linear equations in A and B viz;

$$AI_2(v_1^{-1}, v_2^{-1}) + BI_1(v_1^{-1}, v_2^{-1}) = \frac{p_0}{\mu},$$

$$AI_2(v_2^{-1}, v_1^{-1}) - BI_1(v_2^{-1}, v_1^{-1}) = \frac{p_0}{\mu}, \qquad (17a,b)$$

where

$$I_p(u,v) = \int_{c_2^{-1}}^{u} \frac{z^p\,\mathrm{d}z}{\sqrt{(z^2-c_2^{-2})(u-z)(v+z)}}, \quad (p=1,2).$$

The stress intensity factors at the crack tips $|x| = v_1 t$, $y = 0$ and $|x| = v_2 t$, $y = 0$ defined by

$$N_{01} = \operatorname*{lt}_{x\to v_1 t} \sqrt{x - v_1 t}\ \sigma_{yz}(x,0,t),$$

$$N_{02} = \operatorname*{lt}_{x\to v_2 t} \sqrt{x - v_2 t}\ \sigma_{yz}(-x,0,t),$$

respectively are obtained with the help of (15) and (16) as

$$N_{01} = \frac{\mu}{c_2}\sqrt{\frac{v_2 t}{v_1}}\sqrt{\frac{c_2^2 - v_1^2}{v_1+v_2}}\,(A + Bv_1),$$

$$N_{02} = \frac{\mu}{c_2}\sqrt{\frac{v_1 t}{v_2}}\sqrt{\frac{c_2^2 - v_2^2}{v_1+v_2}}\,(A - Bv_2). \qquad (18a,b)$$

The rate of energy flux into the extending crack edges defined by $\mathrm{d}E/\mathrm{d}t$ is given by [3]

$$\frac{1}{2}\frac{\mathrm{d}E}{\mathrm{d}t} = \int_{-x}^{x} \sigma_{yz}\frac{\partial W}{\partial t}\,\mathrm{d}x, \qquad (19)$$

which is obtained with the aid of (5), (15) and (16) for this case as

$$\frac{\mathrm{d}E_1}{\mathrm{d}t} = -\frac{\mu\pi t}{c_2(v_1+v_2)}[v_2\sqrt{c_2^2-v_1^2}(A+Bv_1)^2 + v_1\sqrt{c_2^2-v_2^2}(A-Bv_2)^2], \tag{20}$$

where while carrying on the integration (19) the following result [13]

$$\frac{H(v)}{\sqrt{v}}\frac{H(-v)}{\sqrt{-v}} = \frac{\pi}{2}\delta(v) \tag{21}$$

has been used.

4. Problem of linearly increasing shear traction with time on the crack faces

For the case of shear tractions on the faces of the crack increasing linearly with time, the boundary conditions are

$$y = 0, \quad -v_2 t < x < v_1 t; \quad \sigma_{yz} = -p_1 t, \tag{22}$$

$$y = 0, \quad x > v_1 t, \quad x < -v_2 t, \quad W = 0, \tag{23}$$

where $p_1 = 2A_0 c_2$.

The second order derivative $\partial^2 W/\partial t^2$ now shows dynamic similarity which can be taken as the real part of the analytic function $\phi_1(z)$ so that

$$\frac{\partial^2 W}{\partial t^2} = \operatorname{Re}\phi_1(z), \tag{24}$$

which implies

$$\frac{\partial^2\sigma_{yz}}{\partial t^2}(x,0,t) = \frac{1}{t}\operatorname{Re}[-\mu z\phi_1'(z)\sqrt{c_2^{-2}-z^2}], \tag{25}$$

where z is given by (8) and $\phi_1(z)$ satisfies the conditions

$$\operatorname{Im} z = 0, \quad -v_2^{-1} < \operatorname{Re} z < v_1^{-1}, \quad \operatorname{Re}\phi_1(z) = 0, \tag{26}$$

$$\operatorname{Im} z = 0, \quad \operatorname{Re} z < -v_2^{-1}, \quad \operatorname{Re} z > v_1^{-1}, \quad \operatorname{Im}\phi_1'(z) = 0. \tag{27}$$

Integrating (24), we obtain

$$W = \tfrac{1}{2}x^2 \operatorname{Re}\int_{v_1^{-1}}^{z} (z-\tau)^2\phi_1'(\tau)\,\mathrm{d}\tau, \tag{28}$$

$$= \tfrac{1}{2}x^2 \operatorname{Re}\int_{v_1^{-1}}^{z} 2(z-\tau)\phi_1(\tau)\,\mathrm{d}\tau, \tag{29}$$

so that

$$\frac{d^2}{dz^2}\left\{\frac{W}{x^2}\right\} = \text{Re}\,\phi_1(z). \tag{30}$$

Taking into consideration the facts that near the crack tips $x = v_1 t, \ -v_2 t;\ y = 0$ the displacement W varies in direct proportion to the factors $\sqrt{v_1 t - x}$, $\sqrt{v_2 t + x}$ respectively and that as $z \to \infty$,

$$\text{Re}\,\phi_1(z) = O(1),$$

we have in view of the conditions (26), (27) and also (30), the result that

$$\phi_1(z) = \frac{d^2}{dz^2}[(Cz + D)\sqrt{(z - v_1^{-1})(z + v_2^{-1})}], \tag{31}$$

where the constants C, D are to be determined from the condition that on the crack surface stress $\sigma_{yz} = -p_1 t$.

From (25) after integration, we derive for $x > 0$.

$$\sigma_{yz}(x, 0, t) = -\mu x\, \text{Re} \int_{c_2^{-1}}^{t/x} \left\{\sqrt{c_2^{-2} - \tau^2} + \frac{\tau(t/x - \tau)}{\sqrt{c_2^{-2} - \tau^2}}\right\}\phi_1(\tau)\, d\tau,$$

$$\sigma_{yz}(-x, 0, t) = \mu x\, \text{Re} \int_{-c_2^{-1}}^{-t/x} \left\{\sqrt{c_2^{-2} - \tau^2} - \frac{\tau(t/x + \tau)}{\sqrt{c_2^{-2} - \tau^2}}\right\}\phi_1(\tau)\, d\tau. \tag{32a,b}$$

Therefore, using the boundary conditions that

$$\sigma_{yz}(x, 0, t) = -p_1 t, \qquad 0 \leqslant x < v_1 t,$$
$$\sigma_{yz}(-x, 0, t) = -p_1 t, \qquad -v_2 t < -x \leqslant 0,$$

we obtain by the help of (32a,b) after simplification

$$C J_1(v_1^{-1}, v_2^{-1}) + D J_2(v_1^{-1}, v_2^{-1}) = \frac{p_1}{\mu},$$

$$C J_1(v_2^{-1}, v_1^{-1}) - D J_2(v_2^{-1}, v_1^{-1}) = \frac{p_1}{\mu}, \tag{33a,b}$$

where

$$J_1(v_1^{-1}, v_2^{-1}) = \int_{c_2^{-1}}^{v_1^{-1}} [\{8\tau + 3(v_2^{-1} - v_1^{-1})\} M(\tau, v_1^{-1}, v_2^{-1}) + N(\tau, v_1^{-1}, v_2^{-1}) \cdot$$
$$\cdot \{4\tau^2 + 3\tau(v_2^{-1} - v_1^{-1}) - 2(v_1 v_2)^{-1}\}]\, d\tau,$$

$$J_2(v_1^{-1}, v_2^{-1}) = \int_{c_2^{-1}}^{v_1^{-1}} [2M(\tau, v_1^{-1}, v_2^{-1}) + N(\tau, v_1^{-1}, v_2^{-1})\{2\tau + (v_2^{-1} - v_1^{-1})\}] \, d\tau,$$

with

$$M(\tau, v_1^{-1}, v_2^{-1}) = \frac{v_1 \tau \sqrt{v_1^{-1} - \tau}}{2\sqrt{(\tau + v_2^{-1})(\tau^2 - c_2^{-2})}},$$

$$N(\tau, v_1^{-1}, v_2^{-1}) = \frac{\tau v_1}{4\sqrt{\tau^2 - c_2^{-2}}} \left[\frac{3}{\sqrt{(\tau + v_2^{-1})(v_1^{-1} - \tau)}} - \frac{\sqrt{v_1^{-1} - \tau}}{(\tau + v_2^{-1})^{3/2}} \right].$$

The stress intensity factors at the crack tips defined by

$$N_{11} = \lim_{x \to v_1 t} \sqrt{x - v_1 t}\, \sigma_{yz}(x, 0, t),$$

$$N_{12} = \lim_{x \to v_2 t} \sqrt{x - v_2 t}\, \sigma_{yz}(-x, 0, t),$$

are found to be

$$N_{11} = \frac{\mu t}{2c_2} \sqrt{\frac{t}{v_1 v_2}} \sqrt{(c_2^2 - v_1^2)(v_1 + v_2)}(C + Dv_1),$$

$$N_{12} = \frac{\mu t}{2c_2} \sqrt{\frac{t}{v_1 v_2}} \sqrt{(c_2^2 - v_2^2)(v_1 + v_2)}(C - Dv_2) \qquad \text{(34a,b)}$$

and in this case the rate of energy flux dE_2/dt into the crack edges defined by (19) is obtained as

$$\frac{dE_2}{dt} = -\frac{\pi \mu t^3 (v_1 + v_2)}{4c_2} [v_2^{-1} \sqrt{c_2^2 - v_1^2}(C + Dv_1)^2 + v_1^{-1} \sqrt{c_2^2 - v_2^2}(C - Dv_2)^2], \qquad (35)$$

where while carrying on the integration (19) the use of the result (21) has again been made.

5. Problem of linearly varying shear traction with distance along the crack plane

Consider the initially undisturbed half-space $y \geqslant 0$ subjected to the shear traction $-p_2 x$ over $y = 0$, $-v_2 t < x < v_1 t$. The boundary conditions are

$$y = 0, \quad -v_2 t < x < v_1 t; \quad \sigma_{yz} = -p_2 x,$$

$$y = 0, \quad x > v_1 t, \quad x < -v_2 t, \quad W = 0, \qquad \text{(36a,b)}$$

where $p_2 = 2A_0 \cos \theta_0$.

In this case, $\partial^2 W/\partial x\,\partial t$ shows dynamic similarity. So, keeping (8) in mind,

$$\frac{\partial^2 W}{\partial x\,\partial t} = \operatorname{Re}\phi_2(z),$$

with

$$\frac{\partial^2 \sigma_{yz}}{\partial x\,\partial t} = -\frac{\mu}{t}\operatorname{Re}[z\phi_2'(z)\sqrt{c_2^{-2} - z^2}], \tag{37a,b}$$

where $\phi_2(z)$ satisfies the conditions

$$\begin{aligned} &\operatorname{Im} z = 0, \quad -v_2^{-1} < \operatorname{Re} z < v_1^{-1}, \quad \operatorname{Re}\phi_2(z) = 0,\\ &\operatorname{Im} z = 0, \quad \operatorname{Re} z < -v_2^{-1}, \quad \operatorname{Re} z > v_1^{-1}, \quad \operatorname{Im}\phi_2'(z) = 0. \end{aligned} \tag{38a,b}$$

From (37a) after integration it is found that

$$W = -x^2 \operatorname{Re}\int_{v_1^{-1}}^{z} \tau^{-1}(z-\tau)\phi_2(\tau)\,d\tau,$$

so that

$$-z^2\frac{d}{dz}\left\{\frac{1}{t}\frac{\partial W}{\partial t}\right\} = \operatorname{Re}\phi_2(z).$$

Since $\partial W/\partial t$ near the crack tips should show square root singularity and also since $\operatorname{Re}\phi_2(z) = O(1)$ as $z \to \infty$, we have in view of the conditions (38)

$$\phi_2(z) = z^2\frac{d}{dz}\left[\frac{Rz^{-1} + L}{\sqrt{(z - v_1^{-1})(z + v_2^{-1})}}\right], \tag{39}$$

where the constants R, L are to be determined.

Equation (37b) can be integrated to obtain for $x > 0$

$$\sigma_{yz}(x,0,t) = \mu x \operatorname{Re}\int_{c_2^{-1}}^{t/x}\left\{\frac{t}{x\tau^2}\sqrt{c_2^{-2}-\tau^2} + \frac{t-\tau x}{x\sqrt{c_2^{-2}-\tau^2}}\right\}\phi_2(\tau)\,d\tau,$$

$$\sigma_{yz}(-x,0,t) = \mu x \operatorname{Re}\int_{-c_2^{-1}}^{-t/x}\left\{\frac{t}{x\tau^2}\sqrt{c_2^{-2}-\tau^2} + \frac{t+\tau x}{x\sqrt{c_2^{-2}-\tau^2}}\right\}\phi_2(\tau)\,d\tau. \tag{40a, b}$$

So using the boundary conditions that

$$\begin{aligned} &\sigma_{yz}(x,0,t) = -p_2 x, \qquad 0 \leqslant x < v_1 t,\\ &\sigma_{yz}(-x,0,t) = p_2 x, \qquad -v_2 t < -x \leqslant 0, \end{aligned}$$

it is found by the help of (39), (40)

$$-RK_1(v_1^{-1}, v_2^{-1}) + LK_2(v_1^{-1}, v_2^{-1}) = \frac{p_2}{\mu},$$

$$RK_1(v_2^{-1}, v_1^{-1}) + LK_2(v_2^{-1}, v_1^{-1}) = \frac{p_2}{\mu}, \tag{41}$$

where

$$K_1(v_1^{-1}, v_2^{-1}) = \int_{c_2^{-1}}^{v_1^{-1}} [P(\tau, v_1^{-1}, v_2^{-1}) - \tau^{-1}Q(\tau, v_1^{-1}, v_2^{-1})]\, d\tau,$$

$$K_2(v_1^{-1}, v_2^{-1}) = \int_{c_2^{-1}}^{v_1^{-1}} Q(\tau, v_1^{-1}, v_2^{-1})]\, d\tau,$$

$$P(\tau, v_1^{-1}, v_2^{-1}) = -\frac{\sqrt{v_1^{-1} - \tau}}{\sqrt{(\tau + v_2^{-1})(\tau^2 - c_2^{-2})}}$$

and

$$Q(\tau, v_1^{-1}, v_2^{-1}) = \frac{\tau^2}{\sqrt{\tau^2 - c_2^{-2}}}\left[\frac{\sqrt{v_1^{-1} - \tau}}{2(\tau + v_2^{-1})^{3/2}} - \frac{(2v_1^{-1} + \tau)}{2\tau\sqrt{(\tau + v_2^{-1})(v_1^{-1} - \tau)}}\right].$$

In this case, the stress intensity factors are obtained as

$$N_{21} = \lim_{x \to v_1 t} \sqrt{x - v_1 t}\, \sigma_{yz}(x, 0, t) = -\frac{\mu t}{c_2}\sqrt{\frac{v_2 t}{v_1}}\sqrt{\frac{c_2^2 - v_1^2}{v_1 + v_2}}\,(Rv_1^2 + Lv_1),$$

$$N_{22} = \lim_{x \to v_2 t} \sqrt{x - v_2 t}\, \sigma_{yz}(-x, 0, t) = -\frac{\mu t}{c_2}\sqrt{\frac{v_1 t}{v_2}}\sqrt{\frac{c_2^2 - v_2^2}{v_1 + v_2}}\,(Rv_2^2 - Lv_2). \tag{42a,b}$$

The rate of energy flux dE_3/dt into the extending crack edges is found to be

$$\frac{dE_3}{dt} = 2\int_{-t}^{t} \sigma_{yz}\frac{\partial W}{\partial t}\, dx = -\frac{\mu\pi t^3}{c_2(v_1 + v_2)}$$

$$\times\, [v_2\sqrt{c_2^2 - v_1^2}(Rv_1^2 + Lv_1)^2 + v_1\sqrt{c_2^2 - v_2^2}(Rv_2^2 - Lv_2)^2], \tag{43}$$

where the result (21) has been used.

6. Particular case: $v_1 = v_2$

If we set $v_1 = v_2 = v$ in all the cases solved above, the following results are obtained

(i) For the case of constant traction $\sigma_{yz} = -p_0$ on the crack faces, we find from (17) that

$$B = 0, \quad A = \frac{v p_0}{\mu E(q)},$$

where $E(q)$ is the complete elliptic integral of second kind and $q = \sqrt{1 - v^2/c_2^2}$. Equations (18) yield the stress intensity factor at the crack tips as

$$N_0 = N_{01} = N_{02} = \frac{A\mu\sqrt{t}}{c_2}\sqrt{\frac{c_2^2 - v^2}{2v}}.$$

Also from (20) we obtain

$$\frac{\mathrm{d}E_1}{\mathrm{d}t} = -\frac{\mu\pi t}{c_2}\sqrt{c_2^2 - v^2}A^2.$$

(ii) For the case of shear traction $\sigma_{yz} = -p_1 t$ on the crack faces increasing linearly with time, it is found from (39) that

$$D = 0, \quad C = \frac{p_1 v}{\mu I},$$

where

$$I = 2E(q) - F(q) + \frac{2c_2^2}{(v + c_2)(v^2 - c_2^2)}\{2v\Pi(r^2, r) + (v + c_2)F(r)\},$$

$F(r), \Pi(r^2, r)$ are complete elliptic integrals of first and third kind respectively and $r = (c_2 - v)/(c_2 + v)$.

In this case, the stress intensity factors and the rate of energy flux into the extending crack tips given by (34) and (35) can be simplified to

$$N_1 = N_{11} = N_{12} = \frac{C\mu t}{c_2}\sqrt{\frac{t}{2v}}\sqrt{c_2^2 - v^2}$$

and

$$\frac{\mathrm{d}E_2}{\mathrm{d}t} = -\frac{\mu\pi t^3}{c_2}\sqrt{c_2^2 - v^2}C^2.$$

(iii) For the case of shear traction $\sigma_{yz} = -p_2 x$ on the crack faces, it is obvious from (41) that

$$R = 0, \quad L = \frac{p_2 v}{\mu J},$$

where

$$J = \frac{2c_2^2}{(v + c_2)(v^2 - c_2^2)}\{2v\Pi(r^2, r) + (v + c_2)F(r)\} - E(q) - F(q)$$

and it is found from (42), (43) that stress intensity factors at the crack tips and the rate of energy flux into the extending crack edges in this case are given by

$$N_2 = N_{21} = -N_{22} = \frac{-\mu t L}{c_2}\sqrt{\frac{vt}{2}}\sqrt{c_2^2 - v^2}$$

and

$$\frac{dE_3}{dt} = -\frac{\mu\pi t^3}{c_2}\sqrt{c_2^2 - v^2}L^2 v^2.$$

7. Numerical results and discussions

The solution of the original crack problem is obtained by taking $p_0 = \sigma + 2A_0 c_2 t_0$, $p_1 = 2A_0 c_2$ and $p_2 = 2A_0 \cos\theta_0$ and superposing the results obtained in Sections 3–5 with the stress fields given by (1). Taking together the results obtained in the Sections 3–5 it is possible to write the stress intensity factors at the edges of the crack and the rate of energy flux into the extending crack edges as

$$S_1 = \frac{N_{01} + N_{11} + N_{21}}{\sigma\sqrt{v_1 t_0}} = \sqrt{\frac{u_2\tau}{u_1 + u_2}}\,\mu H_+(u_1, u_2, \tau),$$

$$S_2 = \frac{N_{02} + N_{12} + N_{22}}{\sigma\sqrt{v_1 t_0}} = \sqrt{\frac{u_2\tau}{u_1 + u_2}}\,\mu H_-(u_2, u_1, \tau) \qquad (44a,b)$$

and

$$En = \frac{\mu}{t_0 c_2^2\sigma^2}\frac{d}{dt}(E_1 + E_2 + E_3) = -\frac{\pi u_2 \mu^2}{u_1 + u_2}\left[G_+(u_1, u_2, \tau) + \frac{u_1}{u_2}G_-(u_2, u_1, \tau)\right], \qquad (45)$$

where

$$H_{\pm}(u_1,u_2,\tau)=\sqrt{1-u_1^2}\left[\frac{1+\Delta}{p_0}\left(\frac{A}{c_2u_1}\pm B\right)+\Delta\tau\left\{\frac{u_1+u_2}{2p_1u_2}\left(\frac{C}{c_2u_1}\pm D\right)-\right.\right.$$

$$\left.\left.-\frac{u_1\cos\theta_0}{p_2}\left(\pm\frac{L}{c_2u_1}+R\right)\right\}\right],$$

$$G_{\pm}(u_1,u_2,\tau)=\tau\sqrt{1-u_1^2}\left[\left(\frac{1+\Delta}{p_0}\right)^2\left(\frac{A}{c_2}\pm Bu_1\right)^2+(\Delta\tau)^2\left\{\frac{(u_1+u_2)^2}{4p_1^2u_2^2}\left(\frac{C}{c_2}\pm Du_1\right)^2+\right.\right.$$

$$\left.\left.+\frac{u_1^2\cos^2\theta_0}{p_2^2}\left(\frac{L}{c_2}\pm Ru_1\right)^2\right\}\right]$$

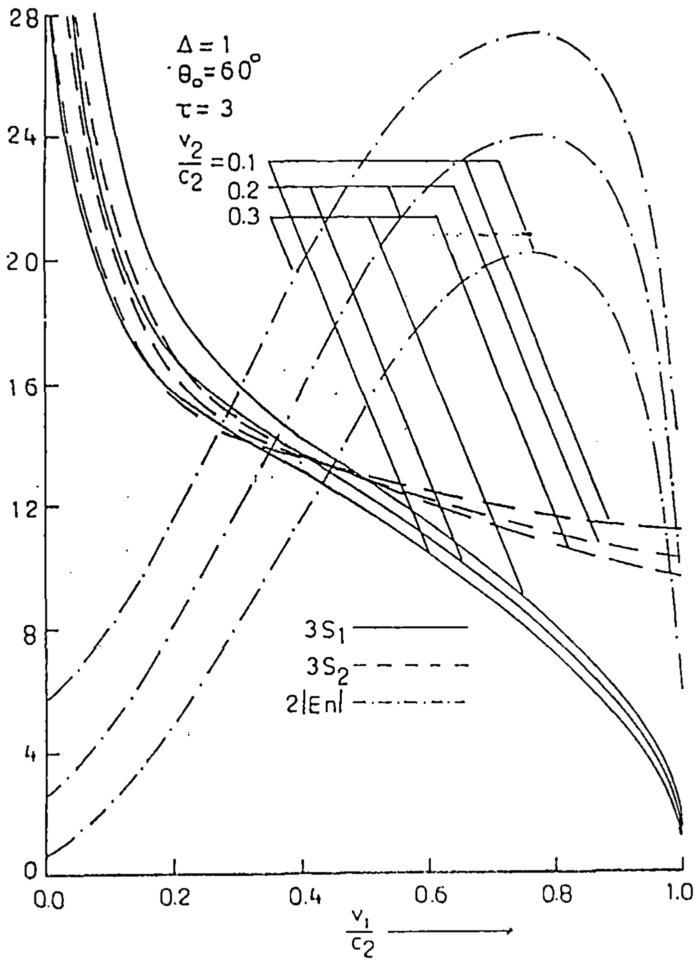

Fig. 2. Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional speed v_1/c_2.

and the parameter $\tau = (t/t_0) - 1$ is the non-dimensionalized time after crack initiation and $\Delta = 2A_0c_2t_0/\sigma$ is the ratio at $x = y = 0$ at initiation of the crack plane stress due to the plane waves and the prestress.

Also u_1, u_2 are the non-dimensional crack tip velocities given by $u_1 = v_1/c_2$ and $u_2 = v_2/c_2$.

The variations of stress intensity factors and energy flux rate given by (44) and (45) respectively with

(i) v_1/c_2 for different values of v_2/c_2 and with
(ii) τ for different values of v_1/c_2 and Δ have been presented in Figs. 2–4.

It has been shown in Fig. 2 that stress intensity factors at the edge $x = v_1t'$, $y = 0$ decrease with the increase in the values of v_1/c_2 but increase with the increase in the values of v_2/c_2 and for $v_1/c_2 < 0.45$, the stress intensity factor at the edge $x = v_2t'$, $y = 0$ increases as v_2/c_2 increases but for $v_1/c_2 > 0.45$, the variation of stress intensity factor at that edge shows an opposite character. It has also been shown in Fig. 2 that the value of energy flux rate $|En|$ increases with the increase

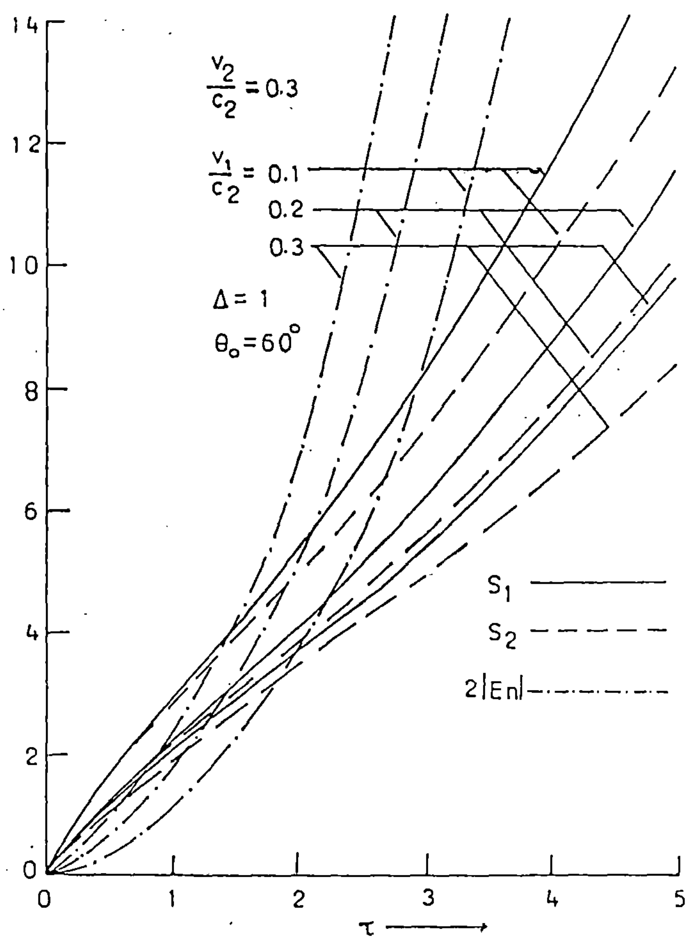

Fig. 3. Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional time after fracture initiation τ.

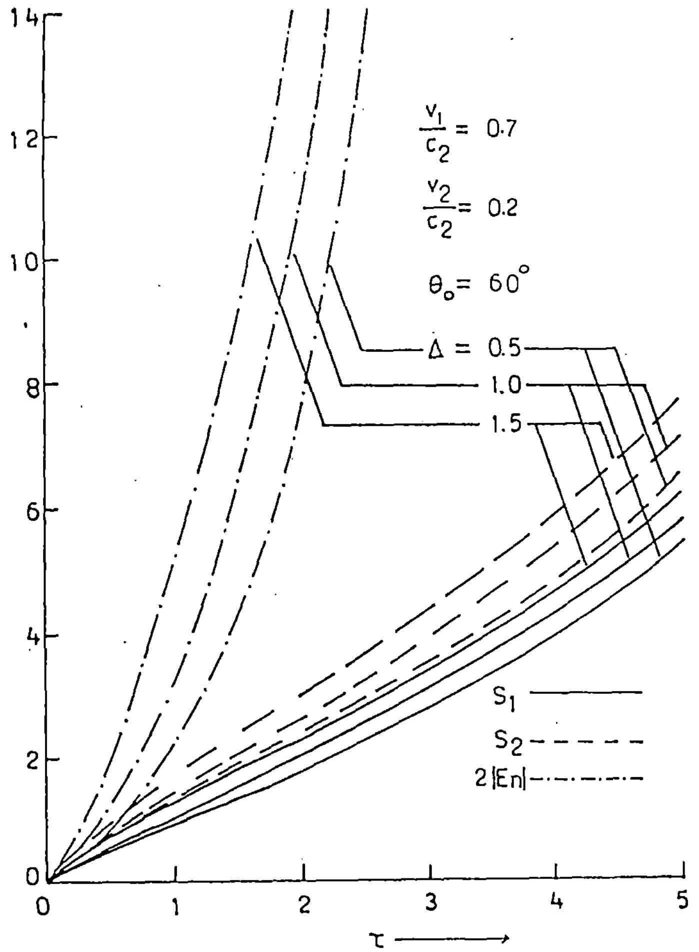

Fig. 4. Variations of non-dimensional stress intensity factors S_1, S_2 and energy flux rate $|En|$ with non-dimensional time after fracture initiation τ.

in the value of v_1/c_2, shows maximum at $v_1/c_2 = 0.8$ after which it decreases with the increase in the value of v_1/c_2.

In Fig. 3, the variations of S_1, S_2 and $|En|$ with τ for various values of $v_1/c_2 \leqslant v_2/c_2$ have been depicted. It may be observed from this figure that $S_1, S_2, |En|$ all increase rapidly with the increase in the value of τ. It may be noted further that for fixed value of v_2/c_2, values of stress intensity factors at the crack tips decrease with the increase in the value of v_1/c_2 whereas energy flux rate $|En|$ increases with the gradual increase in the value of v_1/c_2.

In Fig. 4, S_1, S_2 and $|En|$ are again plotted vs. τ but in this case, crack tip velocities are kept fixed whereas Δ is assumed to vary. It may be seen that increase in the values of Δ produce marked increase in the value of S_1, S_2 and $|En|$ for any fixed value of τ.

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SH-WAVE PROPAGATION ACROSS A VERTICAL STEP IN TWO JOINED ELASTIC HALFSPACES

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A SH-wave propagates in a medium consisting of two welded quarter-spaces of different materials and exhibiting a step-like change in elevation at the interface. The transmitted and reflected waves at a large distance from the step are determined by reducing the problem to an integral equation, using the Green function method and applying the method of steepest descent.

1. Introduction

The problem of propagation of elastic waves in the presence of surface irregularities has been studied by several investigators. ABUBAKAR [1] studied the effect of an irregular surface with an isolated irregularity like a trough or ditch on the incident harmonic P- and SV-waves. Propagation of a Love wave in a elastic layer having an irregular boundary overlying a rigid half-space has been treated by WOLF [2] using the perturbation technique. The transmission of elastic waves across a step-like irregularity in the surface of an elastic half-space is of great importance in seismology in connection with the propagation of waves from the ocean basins to continental regions and vice versa. KNOPOFF and HUDSON [3] studied the transmission of Love waves past a continental margin considering the crust to have an abrupt increase in thickness on the continental side. The transmission of SH-waves across a step-like irregularity at the surface of an elastic half-space was also considered by BOSE [4]. SATO [5] discussed the problem of propagation of Love waves in an elastic layer of variable thickness overlying a semi-infinite elastic medium. Approximate expressions for the transmission and reflection factors are obtained by the application of a method based on the Wiener–Hopf technique.

In this paper, we consider the propagation of SH-wave in a medium consisting of two welded quarter-spaces of different materials and having a step-like change in elevation at the vertical interface. The problem is reduced to an integral equation by using the Fourier transform and Green's function methods and, finally, by applying the method of steepest descent, the transmitted and reflected fields at large distance from the step have been determined. It may be mentioned in this connection that the problem of transient shear wave in a half-space composed of two elastic quarter-spaces of different materials which are subjected to time-dependent shear tractions at the free surface, parallel to plane of junction, has been solved by ACHENBACH [8]. DATTA and MITRA [7] also considered the SH-wave propagation in a composite elastic medium consisting of an elastic quarter-space welded to a uniform layer of different shear wave velocity. Recently, DAS and GHOSH [8] have solved the problem on transmission of time step SH-wave across a rectangular step using integral transform and Green's function technique.

2. Formulation of the Problem

We consider two quarter-spaces of different materials joined along the common boundary $X = 0$ in such a way that there is a step change in elevation at the free surface. We consider the coordinate axes as shown in Fig. 1. Denoting the coordinates of a point in the X–Z plane by (X, Z), we take the incident plane SH-wave as $\exp[i(\omega t - K_2 X)]$ where $K_2 = \omega/c_2$, so that the propagation proceeds from the higher side to the lower side of the step.

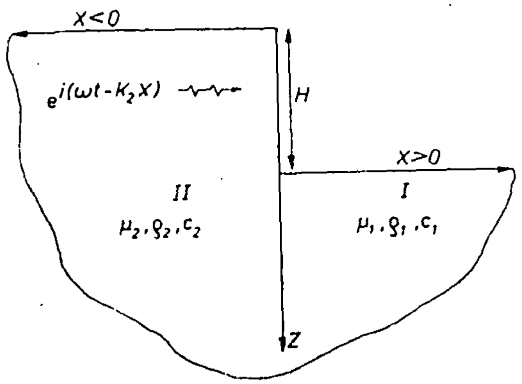

FIG. 1.

The boundaries $Z = 0$, $X < 0$, $Z = H$, $X > 0$ and $X = 0$, $0 \leq Z \leq H$ are assumed to be stress-free. Omitting the time factor $\exp(i\omega t)$ let $V_1(X, z)$, $V_2(X, Z)$ be the SH-wave displacement component in two media (I) and (II), respectively, in Y-direction which is perpendicular to the plane of the paper.

The field equations are wave equations in the two media and boundary conditions are the following: (i) — the outer boundary is stress-free, and (ii) — the displacements and stresses are continuous on the interface $X = 0$, $Z > H$. μ, ρ, c are assumed to be the modulus of rigidity, density and shear wave velocity with appropriate subscript for each of the two media.

Introducing the dimensionless quantities

$$x = \frac{X}{H}, \quad z = \frac{Z}{H},$$

we get from the wave equations and the boundary conditions

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_1^2\right] v_1 = 0, \tag{2.1}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_2^2\right] v_2 = 0;$$

$$\mu_2 \frac{\partial v_2}{\partial x}\bigg|_{x=0} = 0 \quad \text{for } 0 < z < 1, \tag{2.2}$$

$$\mu_2 \frac{\partial v_2}{\partial z}\bigg|_{z=0} = 0 \quad \text{for } x < 0;$$

$$\mu_1 \frac{\partial v_1}{\partial z}\bigg|_{z=1} = 0 \quad \text{for } x > 0, \quad \mu_1 \frac{\partial v_1}{\partial x}\bigg|_{x=0} = \mu_2 \frac{\partial v_2}{\partial x}\bigg|_{x=0} = 0 \quad \text{for } z > 1; \tag{2.3}$$

$$v_1(0, z) = v_2(0, z) \quad \text{for } z > 1, \tag{2.4}$$

where H is the height of the step and $k_i^2 = \omega^2 H^2 / c_i^2$, $V_i(X, Z) = v_i(x, z)$. We represent transverse displacement in the two domains $x < 0$ and $x > 0$ in the form

$$\begin{aligned} v_2 &= 2\cos k_2 x + v_2'(x, z), \; x < 0, \; z > 0, \\ v_1 &= v_1(x, z), \; x > 0, \; z > 1. \end{aligned} \tag{2.5}$$

3. Reduction to Integral Equation and Its Solution

We introduce Green's functions $G_1(x, z : r, s)$ and $G_2(x, z : u, v)$ for the medium (I) and (II), respectively, such that $G_2(x, z : u, v)$ is the solution of

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_2^2\right] G_2(x, z : u, v) = -4\pi\delta(x - u)\delta(z - v)$$

for medium (II) with vanishing normal derivative at $x < 0$, $z = 0$ and at $x = 0$, $z > 0$. Similarly, $G_1(x, z : r, s)$ is the solution of

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_1^2\right] G_1(x, z : r, s) = -4\pi\delta(x - r)\delta(z - s) \tag{3.2}$$

and satisfies the condition of vanishing of the normal derivative at $x > 0$, $z = 1$ and at $x = 0$, $z > 1$.

From Eqs. $(2.1)_2$ and (3.1) we obtain, by applying Green's theorem to the medium (II) and using appropriate boundary condition

$$4\pi v_2'(u, v) = \int_1^\infty G_2(0, z : u, v) \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} dz; \tag{3.3}$$

a similar application of Green's theorem to the medium (I) yields

$$4\pi v_1(r, s) = -\int_1^\infty G_1(0, z : r, s) \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz. \tag{3.4}$$

Substitution of Eqs. (3.3) and (3.4) into Eq. $(2.4)_2$ yields, with the aid of Eq. $(2.4)_1$ and Eq. (2.5), the integral equation

$$\int_1^\infty \left[G_1(0, z : 0, v) + \frac{\mu_1}{\mu_2} G_2(0, z : 0, v)\right] \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz = -8\pi. \tag{3.5}$$

The expression of Green's function for the medium (II) will now be derived using the Fourier cosine transform with respect to z; this reduces the determination of $G_2(x, z : u, v)$ to that of a Green's function for an ordinary differential equation. Accordingly, taking the Fourier cosine transform defined by

$$G_2^c = \int_0^\infty G_2(x, z : u, v) \cos(\alpha z)\, dz,$$

we obtain from Eq. (3.1)

$$\frac{d^2G_2^c}{dx^2} - (\alpha^2 - k_2^2)G_2^c = -4\pi\cos(\alpha v)\delta(x-u)$$

from which we obtain in a straightforward manner

$$\text{(3.6)} \quad G_2(x,z:u,v) = 8\int_0^\infty \frac{e^{\beta_2 u}}{\beta_2}\cosh(\beta_2 x)\cos(\alpha v)\cos(\alpha z)\,d\alpha, \qquad u \le x \le 0,$$

$$= 8\int_0^\infty \frac{e^{\beta_2 x}}{\beta_2}\cosh(\beta_2 u)\cos(\alpha v)\cos(\alpha z)d\alpha, \qquad -\infty < x \le u,$$

where

$$\beta_2^2 = \alpha^2 - k_2^2.$$

Again introducing the Fourier cosine transform defined by

$$G_1^c = \int_0^\infty G_1(x,z:r,s)\cos\alpha(z-1)\,d(z-1),$$

we obtain from Eq. (3.2)

$$\frac{d^2G_1^c}{dx^2} - (\alpha^2 - k_1^2)G_1^c = -4\pi\cos\alpha(s-1)\delta(x-r),$$

from which it follows that

$$\text{(3.7)} \quad G_1(x,z:r,s) = 8\int_0^\infty \frac{e^{-\beta_1 r}}{\beta_1}\cosh(\beta_1 x)\cos\alpha(s-1)\cos\alpha(z-1)\,d\alpha, \qquad 0 \le x \le r$$

$$= 8\int_0^\infty \frac{e^{-\beta_1 x}}{\beta_1}\cosh(\beta_1 r)\cos\alpha(s-1)\cos\alpha(z-1)d\alpha, \qquad r \le x \le \infty,$$

where

$$\beta_1^2 = \alpha^2 - k_1^2.$$

On substituting the values of $G_1(0,z:0,v)$ and $G_2(0,z:0,v)$ from Eqs. (3.7) and (3.6) in Eq. (3.5), we obtain

$$\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz \int_0^\infty \left[\frac{\cos\alpha(v-1)\cos\alpha(z-1)}{\beta_1} + \frac{\mu_1}{\mu_2}\frac{\cos(\alpha v)\cos(\alpha z)}{\beta_2}\right] d\alpha = -\pi,$$

$$\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz \int_0^\infty \frac{\cos\alpha(v-1)\cos\alpha(z-1)}{\beta_1}\,d\alpha = -\pi - \frac{\mu_1}{\mu_2}\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz \times$$

$$\times \int_0^\infty \frac{\cos(\alpha v)\cos(\alpha z)}{\beta_2}\,d\alpha.$$

Taking the inverse Fourier cosine transform with respect to α, we get

$$\text{(3.8)} \quad \int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} \frac{\cos\alpha(z-1)}{\beta_1}\,dz = -2\int_1^\infty \cos\alpha(v-1)\,dv - \frac{2\mu_1}{\pi\mu_2}\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz$$

$$\times \int_0^\infty \frac{\cos(\tau z)}{\beta_2(\tau)}d\tau \int_1^\infty \cos(\tau v)\cos\alpha(v-1)\,dv,$$

where $\beta_2(\tau)$ is obtained from β_2 by replacing α by τ. Next, using the formulae

$$\int_1^\infty \cos\alpha(v-1)\,dv = \pi\delta(\alpha), \tag{3.9}$$

$$\int_1^\infty \sin\alpha(v-1)\,dv = \frac{1}{\alpha} \tag{3.10}$$

it can be easily shown that

$$\int_1^\infty \cos(\tau v)\cos\alpha(v-1)\,dv = \frac{\pi}{2}\cos\tau[\delta(\tau+\alpha)+\delta(\tau-\alpha)] - \frac{\tau\sin\tau}{\tau^2-\alpha^2}, \tag{3.11}$$

where $\delta(x)$ is the Dirac δ-function.

Using these results and after a little algebraic manipulation it can be easily shown that (3.8) reduces to the form

$$\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} \cos\alpha(z-1)\,dz = -\frac{2\pi\mu_2\beta_1\beta_2\delta(\alpha)}{\mu_1\beta_1+\mu_2\beta_2} + \tag{3.12}$$
$$+\frac{\mu_1\beta_1}{\mu_1\beta_1+\mu_2\beta_2}\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} \sin(\alpha z)\sin\alpha\,dz +$$
$$+\frac{2\mu_1\beta_1\beta_2}{\pi(\mu_1\beta_1+\mu_2\beta_2)}\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz \int_0^\infty \frac{\tau\cos(\tau z)\sin\tau}{\beta_2(\tau)(\tau^2-\alpha^2)}d\tau.$$

4. Evaluation of Displacement

Substituting the value of $G_1(0, z : r, s)$ from Eq. (3.7) in Eq. (3.4) and then using the result (3.12), the displacement in the medium I is obtained in the form

$$v_1(x,s) = \frac{2\mu_2 k_2}{\mu_1 k_1+\mu_2 k_2}e^{-ik_1 x} - \frac{2\mu_1}{\pi}\int_0^\infty \frac{e^{-\beta_1 x}\cos\alpha(s-1)}{\mu_1\beta_1+\mu_2\beta_2}d\alpha \times \tag{4.1}$$
$$\times \int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} \sin(\alpha z)\sin\alpha\,dz - \frac{4\mu_1}{\pi^2}\int_0^\infty \frac{\beta_2 e^{-\beta_1 x}\cos\alpha(s-1)}{\mu_1\beta_1+\mu_2\beta_2}d\alpha \times$$
$$\int_1^\infty \left[\frac{\partial v_1}{\partial x}\right]_{x=0} dz \int_0^\infty \frac{\tau\cos(\tau z)\sin\tau}{\beta_2(\tau)(\tau^2-\alpha^2)}d\tau.$$

We can compute $v_1(x, s)$ iteratively solving equations (4.1) and using asymptotic values of integrals appearing the right-hand side of Eq. (4.1) for large values of x.

The first iteration yields

$$v_1(x,s) = \frac{2\mu_2 k_2}{\mu_1 k + \mu_2 k_2}e^{-ik_1 x}, \tag{4.2}$$

which is obviously the displacement in medium (I) in the absence of step change in elevation.

Deriving $\left[\frac{\partial v_1}{\partial x}\right]_{x=0}$ from the first iterate given in Eq. (4.2) and using this at the right-hand side of Eq. (4.1) with the aid of Eq. (3.10) it can be easily shown that the second term at the right-hand side of Eq. (4.1) takes the form

$$(4.3) \qquad I_1 = \frac{2ik_1k_2\mu_1\mu_2}{\pi(\mu_1k_1 + \mu_2k_2)} \int_0^\infty \frac{e^{-\beta_1 x}\cos\alpha(s-1)}{\alpha(\mu_1\beta_1 + \mu_2\beta_2)} \sin 2\alpha d\alpha.$$

For large values of x it can be evaluated asymptotically by the method of steepest descent; therefore, for large x we find

$$(4.4) \qquad I_1 \sim \frac{4\mu_1k_1\mu_2k_2}{(\mu_1k_1 + \mu_2k_2)^2}\left[\frac{k_1}{2\pi x}\right]^{1/2} \exp(\pi i/4 - ik_1x).$$

Similarly, with the aid of Eq. (4.2) and the result given in Eq. (3.9) the third term on the right-hand side of Eq. (4.1) reduces to the form

$$(4.5) \qquad I_2 = \frac{8i\mu_1k_2\mu_2k_2}{\pi^2(\mu_1k_1 + \mu_2k_2)} \int_0^\infty \frac{\sin^2\tau}{\beta_2(\tau)}d\tau \int_0^\infty \frac{\beta_2 e^{-\beta_1 x}\cos\alpha(s-1)}{(\mu_1\beta_1 + \mu_2\beta_2)(\alpha^2 - \tau^2)}d\alpha.$$

In order to evaluate asymptotically, for large values of x, integrals of the type

$$I_1' = \int_0^\infty \frac{f(\alpha)}{(\alpha^2 - \tau^2)}e^{-\beta_1 x}d\alpha,$$

we have to take into account the residue at the singularity $\alpha = \tau$ in addition to the integral along the steepest descent. Thus we get

$$(4.6) \qquad I_1' \sim \pi i\frac{f(\tau)}{2\tau}e^{-\beta_1(\tau)x} - \left[\frac{\pi k_1}{2x}\right]^{1/2}\frac{f(0)}{\tau^2}\exp(\pi i/4 - ik_1x).$$

Using this result in Eq. (4.5), for large values of x, we obtain

$$(4.7) \qquad I_2 - \frac{4\mu_1k_1\mu_2k_2}{(\mu_1k_1 + \mu_2k_2)^2}\left[\frac{k_1}{2\pi x}\right]^{1/2} M\exp(\pi i/4 - ik_1x),$$

where

$$(4.8) \quad M = \frac{2ik_2}{\pi}\int_0^\infty \frac{\sin^2\tau}{\tau^2\beta_2(\tau)}d\tau = \frac{2}{\pi k_2}\int_0^1 \frac{\sin^2 k_2t}{t^2\sqrt{1-t^2}}dt + \frac{2i}{\pi k_2}\int_1^\infty \frac{\sin^2 k_2t}{t^2\sqrt{t^2-1}}dt = J - iY.$$

Following Bose [4] it can be shown that

$$J = \int_0^{2k_2} J_0(z)dz - J_1(2k_2) \quad \text{and} \quad Y = \int_0^{2k_2} Y_0(z)dz - Y_1(2k_2) - \frac{1}{\pi k_2}.$$

Thus from Eqs. (4.1), (4.4) and (4.7), the second iterate, for large x, is

$$(4.9) \qquad v_1(x,z) = \frac{2\mu_2k_2}{\mu_1k_1 + \mu_2k_2}\left[1 + \frac{2(1-M)\mu_1k_1}{\mu_1k_1 + \mu_2k_2}\left\{\frac{k_1}{2\pi x}\right\}^{1/2} e^{\pi i/4}\right]e^{-ik_1x}.$$

If we neglect terms of order $1/x$, the higher order iterates yield the same expression.

Now, in order to find the displacement component due to the reflected wave in the medium (II), we rewrite Eqs. (3.12) with the aid of Eqs. $(2.4)_1$ and $(2.5)_1$ as

$$\int_1^\infty \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} \cos\alpha(z-1)dz = \frac{2\pi\mu_1\beta_1\beta_2\delta(\alpha)}{\mu_1\beta_1+\mu_2\beta_2} + \frac{\mu_1\beta_1}{\mu_1\beta_1+\mu_2\beta_2}\int_1^\infty \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} \times$$
$$\sin(\alpha z)\sin\alpha\, dz + \frac{2\mu_1\beta_1\beta_2}{\pi(\mu_1\beta_1+\mu_2\beta_2)}\int_1^\infty \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} dz \int_0^\infty \frac{\tau\cos(\tau z)\sin\tau}{\beta_2(\tau)(\tau^2-\alpha^2)}d\tau.$$

Taking the inverse Fourier cosine transform with respect to z, we get

$$(4.10)\quad \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} = -\frac{2i\mu_1k_1k_2}{\mu_1k_1+\mu_2k_2} + \frac{2}{\pi}\int_0^\infty \frac{\mu_1\beta_1}{\mu_1\beta_1+\mu_2\beta_2}\times$$
$$\times\cos\alpha(z-1)\,d\alpha \int_1^\infty \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} \sin(\alpha u)\sin\alpha\, du +$$
$$+\frac{4}{\pi^2}\int_0^\infty \frac{\mu_1\beta_1\beta_2}{(\mu_1\beta_1+\mu_2\beta_2)}\cos\alpha(z-1)\,d\alpha \int_1^\infty \left[\frac{dv_2'}{\partial x}\right]_{x=0} du \int_0^\infty \frac{\tau\cos(\tau u)\sin\tau}{\beta_2(\tau)(\tau^2-\alpha^2)}d\tau.$$

Thus substitution of (4.10) in (3.3) with the aid of (3.6) yields

$$(4.11)\quad v_2'(x,v) = \frac{-2\mu_1k_1}{\mu_1k_1+\mu_2k_2}e^{ik_2x} + \frac{4i\mu_1k_1k_2}{\pi(\mu_1k_1+\mu_2k_2)}\times$$
$$\times\int_0^\infty \frac{e^{\beta_2x}\cos(\alpha v)\sin\alpha}{\alpha\beta_2}d\alpha + \frac{1}{\pi}\int_0^\infty \frac{\mu_1\beta_1e^{\beta_2x}\cos(\alpha v)}{\beta_2(\mu_1\beta_1+\mu_2\beta_2)}\times$$
$$\times\sin 2\alpha\, d\alpha \int_1^\infty \left[\frac{dv_2'}{\partial x}\right]_{x=0} \sin(\alpha u)\,du - \frac{4}{\pi^2}\times$$
$$\times\int_0^\infty \frac{\alpha e^{\beta_2x}\cos(\alpha v)\sin\alpha}{\beta_2}d\alpha \int_0^\infty \frac{\beta_1'\mu_1\sin\alpha'}{(\mu_1\beta_1'+\mu_2\beta_2')}\frac{d\alpha'}{(\alpha^2-\alpha'^2)}\times$$
$$\times\int_1^\infty \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} \sin(\alpha' u)du + \frac{4\mu_1}{\pi^2}\int_0^\infty \frac{\beta_1e^{\beta_2x}\cos(\alpha v)}{(\mu_1\beta_1+\mu_2\beta_2)}\cos\alpha\times$$
$$\times d\alpha\int_0^\infty \frac{\tau\sin\tau}{\beta_2(\tau)(\tau^2-\alpha^2)}d\tau \int_1^\infty \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} \cos(\tau u)\,du -$$
$$-\frac{8\mu_1}{\pi^3}\int_0^\infty \frac{\alpha e^{\beta_2x}\cos(\alpha v)\sin\alpha}{\beta_2}d\alpha \int_0^\infty \frac{\beta_1'\beta_2'}{(\mu_1\beta_1'+\mu_2\beta_2')}\frac{d\alpha'}{(\alpha^2-\alpha'^2)}\times$$
$$\times\int_0^\infty \frac{\tau\sin\tau}{\beta_2(\tau)(\tau^2-\alpha'^2)}d\tau \int_1^\infty \left[\frac{\partial v_2'}{\partial x}\right]_{x=0} \cos(\tau u)\,du,$$

where β_1' and β_2' are obtained from β_1 and β_2 by replacing α by α'.

Now, to solve equation (4.11) iteratively, we take the first iteration as

$$(4.12)\quad v_2'(x,v) = \frac{-2\mu_1k_1}{\mu_1k_1+\mu_2k_2}e^{ik_2x} + \frac{4i\mu k_1k_2}{\pi(\mu_1k_1+\mu_2k_2)}\int_0^\infty \frac{e^{\beta_2x}\cos(\alpha v)\sin\alpha}{\alpha\beta_2}d\alpha,$$

(4.13) i.e., $\left[\frac{\partial v_2'}{\partial x}\right]_{x=0} = -\frac{2ik_2\mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2}.$

The asymptotic evaluation of Eq. (4.12) for large values of x by the method of steepest descent yield the first iterate as

$$v_2'(x,v) = \frac{-2\mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2}\left[1 - \left\{\frac{2k_2}{-\pi x}\right\}^{1/2} e^{\pi i/4}\right] e^{ik_2 x}. \tag{4.14}$$

The second iterate is obtained by inserting the value of $\left[\frac{\partial v_2'}{\partial x}\right]_{x=0}$ given in Eq. (4.13) on the right-hand side of Eq. (4.11) and using the results

$$\int_1^\infty \sin(\alpha z)\,dz = \frac{\cos\alpha}{\alpha} \quad \text{and} \quad \int_1^\infty \cos(\alpha z)\,dz = \pi\delta(\alpha) - \frac{\sin\alpha}{\alpha}$$

and then evaluating the integrals on the right-hand side of Eq. (4.11) by the method steepest descent for large value of x. Thus we obtain the second iterate as

$$v_2'(x,v) = \frac{-2\mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2}\left[1 - 2\left(1 - \frac{(1-M)\mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2}\right)\left\{\frac{k_2}{-2\pi x}\right\}^{1/2} e^{\pi i/4}\right] e^{ik_2 x}, \tag{4.15}$$

where M is given in Eq. (4.8).

Thus from Eqs. (2.5) and (4.15) we get

$$v_2(x,v) = e^{-ik_2 x} + \left[\frac{\mu_2 k_2 - \mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2} + \right.$$
$$\left. + \frac{4\mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2}\left(1 - \frac{(1-M)\mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2}\right)\left\{\frac{k_2}{-2\pi x}\right\}^{1/2} e^{\pi i/4}\right] e^{ik_2 x}. \tag{4.16}$$

5. Numerical Results and Discussion

To investigate the nature of the motion, we have evaluated numerically the increment in amplitude due to the step for both the transmitted and reflected waves. The results are shown in the form of graphs showing the variation of $\sqrt{x}\Delta V_{1T}$ with k_2 for different values of μ_2/μ_1 in Fig. 2 for the transmitted wave, and the variation $\sqrt{-x}\Delta V_{2R}$ with k_2 in Fig. 3 for the reflected wave, where

$$\Delta V_{1T} = |v_1| - \frac{2\mu_2 k_2}{\mu_1 k_1 + \mu_2 k_2} \quad \text{and} \quad \Delta V_{2R} = |v_{2R}| - \frac{\mu_2 k_2 - \mu_1 k_1}{\mu_1 k_1 + \mu_2 k_2},$$

and v_{2R} is the reflected part of v_2.

The value of $\sqrt{x}\Delta V_{1T}$ is found to increase gradually with the increase in the value of μ_2/μ_1, and for all values of μ_2/μ_1 it is found that the maximum value of $\sqrt{x}\Delta V_{1T}$ occurs at $k_2 = 0.75$. It is also observed from Fig. 2 that $\sqrt{x}\Delta V_{1T}$ is positive for all values of k_2 and μ_2/μ_1, what means that the amplitude of transmitted wave is always greater than that of the transmitted wave in the absence of the step. Moreover, with the increase in the values of k_2 the graphs show an undulating character and decreasing amplitude of the motion.

From Fig. 3 we see that the value of $\sqrt{-x}\Delta V_{2R}$ gradually decreases with the increase in the value of μ_2/μ_1 and shows a gradual increase as the value of k_2 increases.

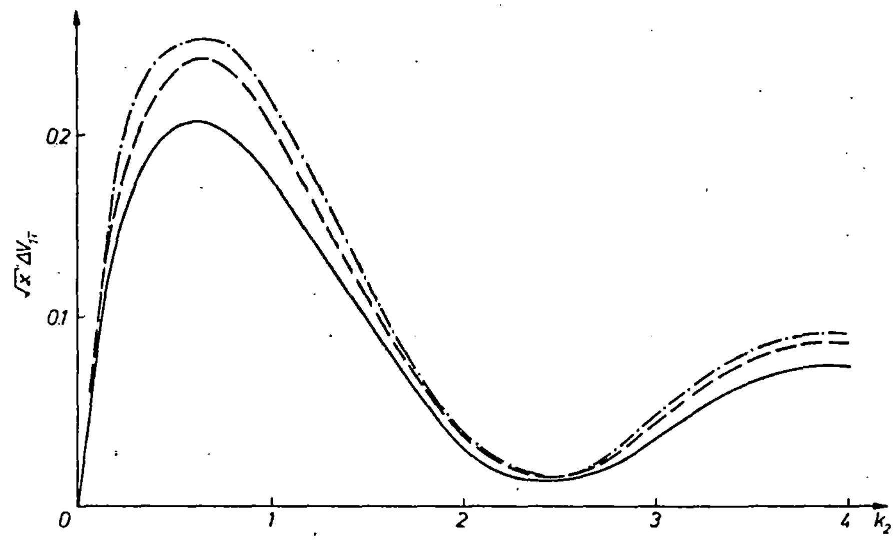

FIG. 2.

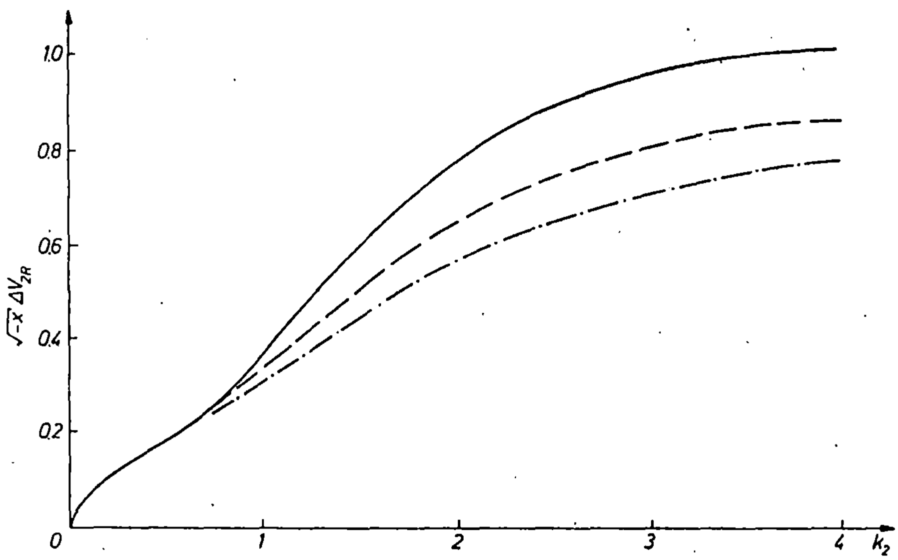

FIG. 3.

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